FRBNY DSGE Model Notes

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1 The FRBNY DSGE model: SW original model

1.1 Model

In this section we describe in detail the Smets and Wouters (2007) model, henceforth SW), and emphasize the differences with the DSSW model.

1.1.1 Intermediate firms

We follow SW and assume the production function to be:

$$Y_t(i) = \max\{e^{\tilde{z}_t} K_t(i)^\alpha \left(L_t(i)e^{\gamma t}\right)^{1-\alpha} - \Phi e^{(\gamma + \frac{\alpha}{1-\alpha}\log \Upsilon)t}, 0\}, \tag{1.1}$$

where

$$\tilde{z}_t = \rho_z \tilde{z}_{t-1} + \sigma_z \epsilon_{z,t},\ \epsilon_{z,t} \sim N(0,1) \tag{1.2}$$

(Note that what SW call "γ" in our notation is e^γ, and that they assume $\Upsilon = 1$.) SW assume that productivity $\tilde{z}_t$ is stationary. Define Z_t as follows:

$$\ln Z_t = \frac{1}{1-\alpha}\tilde{z}_t. \tag{1.3}$$

For $\rho_z \in (0,1)$ the process $\ln Z_t$ is stationary, as in SW. For $\rho_z = 1$ it follows a random walk. This specification accomodates both. Note that we can rewrite the production function as:

$$Y_t(i) = \max\{K_t(i)^\alpha \left(L_t(i)Z_t\right)^{1-\alpha} - \Phi e^{-\frac{1}{1-\alpha}\tilde{z}_t} Z_t e^{(\gamma + \frac{\alpha}{1-\alpha}\log \Upsilon)t}, 0\}. \tag{1.4}$$

Cost minimization subject to 1.4 yields the conditions:

$$\begin{aligned}
&(\partial L_t(i)) \quad \mathcal{V}_t(i)(1-\alpha)Z_t^{1-\alpha}K_t(i)^\alpha L_t(i)^{-\alpha} = W_t \\
&(\partial K_t(i)) \quad \mathcal{V}_t(i)\alpha Z_t^{1-\alpha}K_t(i)^{\alpha-1} L_t(i)^{1-\alpha} = R_t^k
\end{aligned}$$

where $\mathcal{V}_t(i)$ is the Lagrange multiplier associated with 1.1.8. In turn, these conditions imply:

$$\frac{K_t(i)}{L_t(i)} = \frac{\alpha}{1-\alpha}\frac{W_t}{R_t^k}.$$

Note that if we integrate both sides of the equation wrt di and define $K_t = \int K_t(i)di$ and $L_t = \int L_t(i)di$ we obtain a relationship between aggregate labor and capital:

$$K_t = \frac{\alpha}{1-\alpha}\frac{W_t}{R^k_t}L_t. \tag{1.5}$$

Total variable cost is given by

$$\begin{aligned}\text{Variable Costs} &= (W_t + R^k_t\frac{K_t(i)}{L_t(i)})L_t(i)\\ &= (W_t + R^k_t\frac{K_t(i)}{L_t(i)})\tilde{Y}_t(i)Z_t^{-(1-\alpha)}\left(\frac{K_t(i)}{L_t(i)}\right)^{-\alpha},\end{aligned}$$

where $\tilde{Y}_t(i) = Z_t^{1-\alpha}K_t(i)^\alpha L_t(i)^{1-\alpha}$ is the "variable" part of output. The marginal cost MC_t is the same for all firms and equal to:

$$\begin{aligned}MC_t &= (W_t + R^k_t\frac{K_t(i)}{L_t(i)})Z_t^{-(1-\alpha)}\left(\frac{K_t(i)}{L_t(i)}\right)^{-\alpha}\\ &= \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}W_t^{1-\alpha}R^{k\ \alpha}_t Z_t^{-(1-\alpha)}.\end{aligned} \tag{1.6}$$

[TO DO WITH KIMBALL] Prices are sticky as in Calvo (1983). Specifically, each firm can readjust prices with probability $1-\zeta_p$ in each period. We depart rfom Calvo (1983) in assuming that for those firms that cannot adjust prices, $P_t(i)$ will increase at the geometric weighted average (with weigths $1-\iota_p$ and ι_p, respectively) of the steady state rate of inflation π_* and of last period's inflation π_{t-1}. For those firms that can adjust prices, the problem is to choose a price level $\tilde{P}_t(i)$ that maximizes the expected present discounted value of profits in all states of nature where the firm is stuck with that price in the future:

$$\begin{aligned}\max_{\tilde{P}_t(i)} \quad & \Xi^p_t\left(\tilde{P}_t(i) - MC_t\right)Y_t(i)\\ &+ E_t\sum_{s=1}^{\infty}\zeta_p^s\beta^s\Xi^p_{t+s}\left(\tilde{P}_t(i)\left(\Pi_{l=1}^s\pi_{t+l-1}^{\iota_p}\pi_*^{1-\iota_p}\right) - MC_{t+s}\right)Y_{t+s}(i)\\ \text{s.t. } Y_{t+s}(i) = &\left(\frac{\tilde{P}_t(i)\left(\Pi_{l=1}^s\pi_{t+l-1}^{\iota_p}\pi_*^{1-\iota_p}\right)}{P_{t+s}}\right)^{-\frac{1+\lambda_{f,t+s}}{\lambda_{f,t+s}}}Y_{t+s},\end{aligned} \tag{1.7}$$

where $\beta^s\Xi^p_{t+s}$ is today's value of a future dollar for the consumers (Ξ^p_{t+s} is the Lagrange multiplier associated with the consumer's nominal budget constraint - remember there are complete markets so $\beta^s \Xi^p_{t+s}$ is the same for all consumers). The FOC for the firm is:

$$\begin{aligned}
&\Xi^p_t \left(\frac{\tilde{P}_t(i)}{P_t}\right)^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}-1} \frac{1}{\lambda_{f,t} P_t} \left(\tilde{P}_t(i) - (1+\lambda_{f,t}) MC_t\right) Y_t(i) + \\
&E_t \sum_{s=0}^{\infty} \zeta_p^s \beta^s \Xi^p_{t+s} \left(\frac{\tilde{P}_t(i)\left(\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p}\right)}{P_{t+s}}\right)^{-\frac{1+\lambda_{f,t+s}}{\lambda_{f,t+s}}-1} \frac{\left(\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p}\right)}{\lambda_{f,t+s} P_{t+s}} \\
&\left(\tilde{P}_t(i)\left(\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p}\right) - (1+\lambda_{f,t+s}) MC_{t+s}\right) Y_{t+s}(i) = 0
\end{aligned} \tag{1.8}$$

Note that all firms readjusting prices face an indentical problem. We will consider only the symmetric equilibrium in which all firms that can readjust prices will choose the same $\tilde{P}_t(i)$, so we can drop the i index from now on. From 1.1.6 it follows that:

$$P_t = [(1-\zeta_p)\tilde{P}_t^{-\frac{1}{\lambda_f}} + \zeta_p(\pi_{t-1}^{\iota_p} \pi_*^{1-\iota_p} P_{t-1})^{-\frac{1}{\lambda_f}}]^{-\lambda_f}. \tag{1.9}$$

1.1.2 Households

Household j's utility is (as opposed to 1.1.16):

$$I\!E_t \sum_{s=0}^{\infty} \beta^s \left[\frac{1}{1-\sigma_c} \left(C_{t+s}(j) - hC_{t+s-1}\right)^{1-\sigma_c}\right] \exp\left(\frac{\sigma_c - 1}{1+\nu_l} L_{t+s}(j)^{1+\nu_l}\right) \tag{1.10}$$

where $C_t(j)$ is consumption, $L_t(j)$ is labor supply. Three observations are in order regarding this utility function. First, utility is increasing in consumption and leisure regardless of the value of σ_c. Second, there are no "discount rate" or "leisure" shocks in the utility function. Third, SW have external (as opposed to internal) habit.

The household's budget constraint, written in real terms, is given by:

$$\begin{aligned}
&C_{t+s}(j) + I_{t+s}(j) + \frac{B_{t+s}(j)}{b_{t+s} R_{t+s} P_{t+s}} \leq \frac{B_{t+s-1}(j)}{P_{t+s}} \\
&+ \frac{W^h_{t+s}}{P_{t+s}} L_{t+s}(j) + \left(\frac{R^k_{t+s}}{P_{t+s}} u_{t+s}(j) \bar{K}_{t+s-1}(j) - a(u_{t+s}(j)) \Upsilon^{-t} \bar{K}_{t+s-1}(j)\right) + \Pi_{t+s} - T_{t+s},
\end{aligned} \tag{1.11}$$

where $I_t(j)$ is investment, $B_t(j)$ is holdings of government bonds, R_t is the gross nominal interest rate paid on government bonds, Π_t is the per-capita profit the household gets from owning firms (assume household pool their firm shares, T_{t+s} is lump-sum taxes, so that they all receive the same profit) $W_t^h(j)$ is the wage earned by household j. b_t is a "risk premium shock". The term within parenthesis represents the return to owning $\bar{K}_t(j)$ units of capital. Households choose the utilization rate of their own capital, $u_t(j)$, and end up renting to firms in period t an amount of "effective" capital equal to:

$$K_t(j) = u_t(j)\bar{K}_{t-1}(j), \tag{1.12}$$

and getting $R_t^k u_t(j)\bar{K}_{t-1}(j)$ in return. They however have to pay a cost of utilization in terms of the consumption good which is equal to $a(u_t(j))\Upsilon^{-t}\bar{K}_{t-1}(j)$. Households accumulate capital according to the equation:

$$\bar{K}_t(j) = (1-\delta)\bar{K}_{t-1}(j) + \Upsilon^t \mu_t \left(1 - S(\frac{I_t(j)}{I_{t-1}(j)})\right) I_t(j), \tag{1.13}$$

where δ is the rate of depreciation, and $S(\cdot)$ is the cost of adjusting investment, with $S'(\cdot) > 0, S''(\cdot) > 0$. The term μ_t is a stochastic disturbance to the price of investment relative to consumption

Households are all identical, so the j subscript is pretty redundant except for the fact that we have external habits. We will drop the j subsequently.

The FOCs for consumption, bonds, and labor are:

$$(\partial C_t(j)) \qquad (C_t - hC_{t-1})^{-\sigma_c} \exp\left(\frac{\sigma_c - 1}{1+\nu_l} L_t^{1+\nu_l}\right) = \Xi_t \tag{1.14}$$

$$(\partial B_t(j)) \qquad \Xi_t = \beta R_t b_t \mathbb{E}_t[\frac{\Xi_{t+1}}{\pi_{t+1}}] \tag{1.15}$$

$$(\partial L_t(j)) \qquad (C_t - hC_{t-1})^{1-\sigma_c} \exp\left(\frac{\sigma_c - 1}{1+\nu_l} L_t^{1+\nu_l}\right) L_t^{\nu_l} = \Xi_t \frac{W_t^h}{P_t}. \tag{1.16}$$

Note that households take W_t^h as given and maximize with respect to L_t. The wage stickiness part will be discussed below. Using 1.14 we can rewrite 1.16 as:

$$(C_t - hC_{t-1}) L_t^{\nu_l} = \frac{W_t^h}{P_t}. \tag{1.17}$$

Let us now address the capital accumulation/utilization problem. Call Ξ_t^k the Lagrange multiplier associated with constraint 1.13. The FOC with respect to investment, capital, and capital utilization are:

$$(\partial I_t) \qquad \Xi_t^k \Upsilon^t \mu_t \left(1 - S(\frac{I_t}{I_{t-1}}) - S'(\frac{I_t}{I_{t-1}})\frac{I_t}{I_{t-1}}\right)$$

$$+ \beta \mathbb{E}_t[\Xi_{t+1}^k \Upsilon^{t+1} \mu_{t+1} S'(\frac{I_{t+1}}{I_t})(\frac{I_{t+1}}{I_t})^2] = \Xi_t \tag{1.18}$$

$$(\partial \bar{K}_t) \quad \Xi_t^k = \beta \mathbb{E}_t[\Xi_{t+1}(\frac{R_{t+1}^k}{P_{t+1}} u_{t+1} - a(u_{t+1})\Upsilon^{-(t+1)}) + \Xi_{t+1}^k(1-\delta)] \tag{1.19}$$

$$(\partial u_t) \qquad \Upsilon^t \frac{R_t^k}{P_t} = a'(u_t) \tag{1.20}$$

The first FOC is the law of motion for the shadow value of capital. Note that if adjustment cost were absent, the FOC would simply say that $\Xi_t^k \Upsilon^t \mu_t$ is equal to the marginal utility of consumption. In other words, in absence of adjustment costs the shadow cost of taking resources away from consumption equals the shadow benefit (abstracting from $\Upsilon^t \mu_t$) of putting these resources into investment: Tobin's Q is equal to one. The second FOC says that if I buy a unit of capital today I have to pay its price in real terms, Ξ_t^k, but tomorrow I will get the proceeds from renting capital, plus I can sell back the capital that has not depreciated. Define $Q_t^k = \frac{\Xi_t^k}{\Xi_t}$. Q_t^k has the interpretation of the value of installed capital relative to consumption goods (i.e., Tobin's Q). Then condition 1.19 can be rewritten as:

$$Q_t^k = \beta \mathbb{E}_t \left[\frac{\Xi_{t+1}}{\Xi_t}\left(\frac{R_{t+1}^k}{P_{t+1}} u_{t+1} - a(u_{t+1})\Upsilon^{-(t+1)} + Q_{t+1}^k(1-\delta)\right)\right]. \tag{1.21}$$

1.1.3 Government Policies

The central bank follows a nominal interest rate rule by adjusting its instrument in response to deviations of inflation and output from their respective target levels:

$$\frac{R_t}{R^*} = \left(\frac{R_{t-1}}{R^*}\right)^{\rho_R} \left[\left(\frac{\pi_t}{\pi_*}\right)^{\psi_1} \left(\frac{Y_t}{Y_t^f}\right)^{\psi_2}\right]^{1-\rho_R} \left(\frac{Y_t}{Y_{t-1}} \frac{Y_{t-1}^f}{Y_t^f}\right)^{\psi_3} e^{r_t^m} \tag{1.22}$$

where the parameter ρ_R determines the degree of interest rate smoothing, R^* is the steady state nominal rate and Y_t^f is output under flexible/prices and wages. Note that policy reacts to both level differences between Y_t and Y_t^f ($\psi_2(1-\rho_R)$ coefficient) as well as growth differences (ψ_2 coefficient). Note also that the exogenours part of monetary policy is captured by the process r_t^m, which follows an autoregressive process. The central bank supplies the money demanded by the household to support the desired nominal interest rate.

The government budget constraint is of the form

$$P_t G_t + B_{t-1} = P_t T_t + \frac{B_t}{b_t R_t}, \tag{1.23}$$

where T_t are nominal lump-sum taxes (or subsidies) that also appear in household's budget constraint. SW, who assume technology is stationary, express government spending relative to the *deterministic* trend in output:

$$G_t = \tilde{g}_t y_* e^{z_*^* t} \tag{1.24}$$

where y_* is the steady state of detrended output. Since we detrend everything (see below) by Z_t^*, we need to be careful. Define

$$g_t = \frac{G_t}{y_* Z_t^*} = \tilde{g}_t e^{-\frac{1}{1-\alpha}\tilde{z}_t}. \tag{1.25}$$

At steady state $\tilde{g}_* = g_*$. Note the difference with DSSW, where $g_t = \dfrac{Y_t}{Y_t - G_t}$ and $g_* = \dfrac{y_*}{c_* + i_*} > 1$. In SW $g_* \in (0,1)$.

1.1.4 Resource constraints

To obtain the market clearing condition for the final goods market first integrate the HH budge constraint across households, and combine it with the gvmt budget constraint:

$$\begin{aligned} P_t C_t + P_t I_t + P_t G_t &\leq +\Pi_t + \int W_t(j) L_t(j) dj \\ &+ R_t^k \int K_t(j) dj - P_t a(u_t) \Upsilon^{-t} \int \bar{K}_{t-1}(j) dj. \end{aligned}$$

Next, realize that

$$\Pi_t = \int \Pi(i)_t di \; = \; \int P(i)_t Y(i)_t di - W_t L_t - R_t^k K_t,$$

where $L_t = \displaystyle\int L(i)_t di$ is total labor supplied by the labor packers (and demanded by the firms), and $K_t = \displaystyle\int K(i)_t di = \int K_t(j) dj$. Now replace the defintion of Π_t into the

HH budget constraint, realize that by the labor and goods' packers' zero profit condition $W_tL_t = \int W_t(j)L_t(j)dj$, and $P_tY_t = \int P(i)_tY(i)_tdi$ and obtain:

$$C_t + I_t + a(u_t)\Upsilon^{-t}\bar{K}_{t-1} + G_t = Y_t \tag{1.26}$$

where Y_t is defined by (1.1.1). The relationship between output and the aggregate inputs, labor anc capital, is:

$$\begin{aligned}\dot{Y}_t &= \int Z_t^{1-\alpha}K_t(i)^\alpha L_t(i)^{1-\alpha}di - Z_t^*\Phi \\ &= Z_t^{1-\alpha}\int (K/L)^\alpha L(i)di - Z_t^*\Phi \\ &= Z_t^{1-\alpha}K_t^\alpha L_t^{1-\alpha} - Z_t^*\Phi,\end{aligned} \tag{1.27}$$

where I used the fact that the capital labor ratio is constant across firms (also, since $K(i) = (K/L)L(i)$ it must be the case that $\frac{\int K(i)di}{\int L(i)di} = K_t/L_t = (K/L)$). The problem with these resource constraints is that what we observe in the data is $\dot{Y}_t = \int Y_t(i)di$ and $\dot{L}_t = \int L_t(j)dj$, as opposed to Y_t and L_t. But note that from 1.1.5:

$$\begin{aligned}\dot{Y}_t &= Y_tP_t^{\frac{1+\lambda_{f,t}}{\lambda_{f,t}}}\int P(i)_t^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}}di \\ &= Y_tP_t^{\frac{1+\lambda_{f,t}}{\lambda_{f,t}}}\dot{P}_t^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}},\end{aligned}$$

where $\dot{P}_t = \left(\int P_t(i)^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}}di\right)^{-\frac{\lambda_{f,t}}{1+\lambda_{f,t}}}$, and

$$\begin{aligned}\dot{L}_t &= \int L_t(j)dj \\ &= L_tW_t^{\frac{1+\lambda_{w,t}}{\lambda_{w,t}}}\int W(j)_t^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}}di \\ &= L_tW_t^{\frac{1+\lambda_{w,t}}{\lambda_{w,t}}}\dot{W}_t^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}},\end{aligned}$$

where $\dot{W}_t = \left(\int W(j)_t^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}}dj\right)^{-\frac{\lambda_{w,t}}{1+\lambda_{w,t}}}$.

1.1.5 Exogenous Processes

When technology is stationary or has a unit root, its process is given by 1.3, which we report here:

$$\tilde{z}_t = \rho_z\tilde{z}_{t-1} + \sigma_z\varepsilon_{z,t}.$$

We now discuss the process for g_t. SW assume a stationary process for $\hat{\tilde{g}}_t = log(\frac{\tilde{g}_t}{g_*})$, which is correlated with shocks in technology:

$$\hat{\tilde{g}}_t = \rho_g \hat{\tilde{g}}_{t-1} + \sigma_g \varepsilon_{g,t} + \eta_{gz} \sigma_z \varepsilon_{z,t}. \tag{1.28}$$

If technology is not stationary, this process does not make sense since $\hat{\tilde{g}}_t$ is non stationary. Hence we replace it by the assumption that $\hat{g}_t = log(\frac{g_t}{g_*})$ is stationary

$$\hat{g}_t = \rho_g \hat{g}_{t-1} + \sigma_g \varepsilon_{g,t} + \eta_{gz} \sigma_z \varepsilon_{z,t}. \tag{1.29}$$

We express all remaining processes in log deviations from their steady state value, which is assumed to be 1:

$$\hat{b}_t = \rho_b \hat{b}_{t-1} + \sigma_b \varepsilon_{b,t}, \tag{1.30}$$

$$\hat{\mu}_t = \rho_\mu \hat{\mu}_{t-1} + \sigma_\mu \varepsilon_{\mu,t}, \tag{1.31}$$

$$\hat{r}^m_t = \rho_{r^m} \hat{r}^m_{t-1} + \sigma_r \varepsilon_{r^m,t}, \tag{1.32}$$

The mark-up shocks follow ARMA(1,1) processes:

$$\hat{\lambda}_{f,t} = \rho_{\lambda_f} \hat{\lambda}_{f,t-1} + \sigma_{\lambda_f} \varepsilon_{\lambda_f,t} - \eta_{\lambda_f} \sigma_{\lambda_f} \varepsilon_{\lambda_f,t-1}, \tag{1.33}$$

$$\hat{\lambda}_{w,t} = \rho_{\lambda_w} \hat{\lambda}_{w,t-1} + \sigma_{\lambda_w} \varepsilon_{\lambda_w,t} - \eta_{\lambda_w} \sigma_{\lambda_w} \varepsilon_{\lambda_w,t-1}, \tag{1.34}$$

1.2 Detrending

SW detrend the variables by the deterministic trend $e^{\gamma t}$ (or by $e^{(\gamma+\frac{\alpha}{1-\alpha}\log\Upsilon)t}$ if there is a trend in the relative price of capital). We detrend by

$$Z_t^* = Z_t e^{(\gamma+\frac{\alpha}{1-\alpha}\log\Upsilon)t}, \Upsilon > 1. \tag{1.35}$$

Define $z_t^* = \log(Z_t^*/Z_{t-1}^*)$. Denote with $_*$ the steady state values of the variables, and realize that at st.st. $z_*^* = \gamma + \dfrac{\alpha}{1-\alpha}\log\Upsilon$. From 1.2 and 1.3 we see that

$$\hat{z}_t^* = z_t^* - z_*^* = \frac{1}{1-\alpha}(\rho_z - 1)\tilde{z}_{t-1} + \frac{1}{1-\alpha}\sigma_z\epsilon_{z,t}, \tag{1.36}$$

and

$$E_t[\hat{z}_{t+1}^*] = \frac{1}{1-\alpha}(\rho_z - 1)\tilde{z}_t. \tag{1.37}$$

Note that for $\rho_z = 1$ $\tilde{z}_t$ has no impact on $\hat{z}_t$.

Specifically:

$$\begin{aligned}
&c_t = \frac{C_t}{Z_t^*},\ y_t = \frac{Y_t}{Z_t^*},\ i_t = \frac{I_t}{Z_t^*},\ k_t = \Upsilon^{-t}\frac{K_t}{Z_t^*},\ \bar{k}_t = \Upsilon^{-t}\frac{\bar{K}_t}{Z_t^*},\\
&r_t^k = \Upsilon^t\frac{R_t^k}{P_t},\ w_t = \frac{W_t}{P_t Z_t^*},\ w_t^h = \frac{W_t^h}{P_t Z_t^*},\ \tilde{p}_t = \frac{\tilde{P}_t}{P_t},\ \tilde{w}_t = \frac{\tilde{W}_t}{W_t},\\
&\xi_t = \Xi_t Z_t^{*\sigma_c},\ \xi_t^k = \Xi_t^k Z_t^{*\sigma_c}\Upsilon^t,\ q_t^k = Q_t^k\Upsilon^t.
\end{aligned} \tag{1.38}$$

Note that this implies that some of the equilibrium conditions will look different from SW.

Intermediate goods producers

We start by expressing 1.6 in terms of detrended variables:

$$mc_t = \frac{MC_t}{P_t} = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)} w_t^{1-\alpha} r_t^{k\ \alpha}. \tag{1.39}$$

Hence

$$mc_* = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)} w_*^{1-\alpha} r_*^{k\ \alpha}. \tag{1.40}$$

***************** (TO BE DONE) ******************

Expression 1.8 becomes:

$$\frac{\xi_t}{\lambda_{f,t}}\tilde{p}_t^{-\frac{(1+\lambda_{f,t})}{\lambda_{f,t}}-1}\left(\tilde{p}_t-(1+\lambda_{f,t})mc_t\right)y_t(i)$$
$$+\mathbb{E}_t\sum_{s=1}^{\infty}\zeta_p^s\beta^s\frac{\xi_{t+s}}{\lambda_{f,t+s}}\left(\frac{\tilde{p}_t}{\Pi_{l=1}^s\pi_{t+l}}\right)^{\frac{(1+\lambda_{f,t+s})}{\lambda_{f,t+s}}-1}\left(\Pi_{l=1}^s\pi_{t+l-1}^{\iota_p}\pi_*^{1-\iota_p}\right)^{\frac{(1+\lambda_{f,t+s})}{\lambda_{f,t+s}}} \tag{1.41}$$
$$\left(\tilde{p}_t\frac{\Pi_{l=1}^s\pi_{t+l-1}^{\iota_p}\pi_*^{1-\iota_p}}{\Pi_{l=1}^s\pi_{t+l}}-(1+\lambda_{f,t+s})mc_{t+s}\right)y_{t+s}(i)=0$$

this implies that:

$$\tilde{p}_*=(1+\lambda_f)\alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}w_*^{1-\alpha}r_*^{k\ \alpha} \tag{1.42}$$

Expression **??** becomes:

$$1=[(1-\zeta_p)\tilde{p}_t^{-\frac{1}{\lambda_{f,t}}}+\zeta_p(\pi_{t-1}^{\iota_p}\pi_*^{1-\iota_p}\pi_t^{-1})^{-\frac{1}{\lambda_{f,t}}}]^{-\lambda_{f,t}}. \tag{1.43}$$

which means that:

$$\tilde{p}_*=1. \tag{1.44}$$

Recall that aggregate profits are equal to:

$$\Pi_t=P_tY_t-W_tL_t-R_t^kK_t.$$

In terms of detrended variables we then have :

$$\begin{aligned}\frac{\Pi_t}{P_tZ_t^*} &= y_t-w_tL_t-r_t^kk_t\\ &= k_t^\alpha L_t^{1-\alpha}-\Phi-w_tL_t-\frac{\alpha}{1-\alpha}w_tL_t\\ &= \left((\frac{k_t}{L_t}^\alpha-\frac{1}{1-\alpha}w_t\right)L_t-\Phi\\ &= \left((\frac{\alpha}{1-\alpha})^\alpha w_t^\alpha r_t^{k\ -\alpha}-\frac{1}{1-\alpha}w_t\right)L_t-\Phi\end{aligned}$$

At steady state we can use 1.42 to get that st. st. profits are:

$$\frac{\Pi_t}{P_tZ_t^*}=\frac{\lambda_f}{1-\alpha}w_*L_*-\Phi. \tag{1.45}$$

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Equation 1.5 becomes:

$$k_t=\frac{\alpha}{1-\alpha}\frac{w_t}{r_t^k}L_t. \tag{1.46}$$

and at st.st.:

$$k_* = \frac{\alpha}{1-\alpha}\frac{w_*}{r^k_*}L_*. \tag{1.47}$$

Households

Expressions 1.14, 1.15, and 1.17 become:

$$\xi_t = \left(c_t - hc_{t-1}e^{-z^*_t}\right)^{-\sigma_c} \exp\left(\frac{\sigma_c - 1}{1+\nu_l}L_t^{1+\nu_l}\right), \tag{1.48}$$

$$\xi_t = \beta R_t b_t \mathbb{E}_t[\xi_{t+1}e^{-\sigma_c z^*_{t+1}}\pi^{-1}_{t+1}], \tag{1.49}$$

$$\left(c_t - hc_{t-1}e^{-z^*_t}\right)L_t^{\nu_l} = w^h_t, \tag{1.50}$$

respectively. At steady state:

$$\xi_* = c_*^{-\sigma_c}(1-he^{-z^*_*})^{-\sigma_c}\exp\left(\frac{\sigma_c - 1}{1+\nu_l}L_*^{1+\nu_l}\right), \tag{1.51}$$

$$R_* b_* = \beta^{-1}\pi_* e^{\sigma_c z^*_*}, \tag{1.52}$$

$$c_*\left(1 - he^{-z^*_*}\right)L_*^{\nu_l} = w^h_*. \tag{1.53}$$

Equation 1.12 and 1.13 become:

$$k_t = u_t \Upsilon^{-1}e^{-z^*_t}\bar{k}_{t-1}, \tag{1.54}$$

$$\bar{k}_t = (1-\delta)\Upsilon^{-1}e^{-z^*_t}\bar{k}_{t-1} + \mu_t\left(1 - S(\frac{i_t}{i_{t-1}}e^{z^*_t})\right)i_t. \tag{1.55}$$

which deliver the steady state relationships:

$$k_* = e^{-\gamma}\Upsilon^{-\frac{1}{1-\alpha}}\bar{k}_*, \tag{1.56}$$

$$i_* = \mu^{-1}\left(1 - (1-\delta)e^{-\gamma}\Upsilon^{-\frac{1}{1-\alpha}}\right)\bar{k}_*. \tag{1.57}$$

under the assumption that $S(e^{\gamma}\Upsilon^{\frac{\alpha}{1-\alpha}}) = 0$.

Equation 1.18, 1.21, and 1.20 become:

$$\begin{aligned}&\xi^k_t\mu_t\left(1 - S(\frac{i_t}{i_{t-1}}e^{z^*_t}) - S'(\frac{i_t}{i_{t-1}}e^{z^*_t})\frac{i_t}{i_{t-1}}e^{z^*_t}\right)\\ &+ \beta\mathbb{E}_t[e^{-\sigma_c z^*_{t+1}}\xi^k_{t+1}\mu_{t+1}S'(\frac{i_{t+1}}{i_t}e^{z^*_{t+1}})(\frac{i_{t+1}}{i_t}e^{z^*_{t+1}})^2] = \xi_t\end{aligned} \tag{1.58}$$

$$q^k_t = \beta\mathbb{E}_t\left[\Upsilon^{-1}e^{-\sigma_c z^*_{t+1}}\frac{\xi_{t+1}}{\xi_t}\left(r^k_{t+1}u_{t+1} - a(u_{t+1}) + q^k_{t+1}(1-\delta)\right)\right] \tag{1.59}$$

$$r^k_t = a'(u_t). \tag{1.60}$$

Under the assumptions that $S'(e^{\gamma}\Upsilon^{\frac{\alpha}{1-\alpha}}) = 0$, $u_* = 1$ and $a(u_*) = 0$, the above equations at steady state imply

$$\xi_*^k = \xi_* \tag{1.61}$$
$$r_*^k = \beta^{-1} e^{\sigma_c z_*^*} \Upsilon - (1-\delta) \tag{1.62}$$
$$r_*^k = a'(u_*). \tag{1.63}$$

where 1.61 implies $q_*^k = 1$ (note the $a(.)$ function can be normalized so to make $a'(1)$ be whatever the steady state r_*^k is).

Expressed in terms of detrended variables, equation 1.1.37 becomes:

$$I\!E_t \sum_{s=0}^{\infty} (\zeta_w \beta)^s L(j)_{t+s} \xi_{t+s} \Big[-\tilde{\mathcal{X}}_{t,s} \tilde{w}_t w_t + (1+\lambda_w) \frac{b_{t+s}\varphi_{t+s} L_{t+s}(j)^{\nu_l}}{\xi_{t+s}} \Big] = 0, \tag{1.64}$$

where

$$\tilde{\mathcal{X}}_{t,s} = \begin{cases} 1 \;\; \text{if } s = 0 \\ \dfrac{\Pi_{l=1}^{s} (\pi_* e^{\gamma} \Upsilon^{\frac{\alpha}{1-\alpha}})^{1-\iota_w} (\pi_{t+l-1} e^{z_{t+l-1}^*})^{\iota_w}}{\Pi_{l=1}^{s} \pi_{t+l} e^{z_{t+l}^*}} \;\; \text{otherwise} \end{cases}$$

and

$$L_{t+s}(j) = \left(\tilde{w}_t w_t w_{t+s}^{-1} \tilde{\mathcal{X}}_{t,s} \right)^{-\frac{1+\lambda_w}{\lambda_w}} L_{t+s}.$$

Equation 1.1.38 becomes:

$$1 = [(1-\zeta_w) \tilde{w}_t^{\frac{1}{\lambda_w}} + \zeta_w ((\pi_* e^{\gamma} \Upsilon^{\frac{\alpha}{1-\alpha}})^{1-\iota_w} (\pi_{t-1} e^{z_{t-1}^*})^{\iota_w} \frac{w_{t-1}}{w_t} \pi_t^{-1} e^{-z_t^*})^{\frac{1}{\lambda_w}}]^{\lambda_w}. \tag{1.65}$$

which imply at steady state:

$$w_* = (1+\lambda_w) \frac{\varphi L_*^{\nu_l}}{\xi_*}, \tag{1.66}$$
$$\tilde{w}_* = 1. \tag{1.67}$$

Resource constraints

If the technology process is stationary, the resource constraint become:

$$y_* \tilde{g}_t e^{-\frac{1}{1-\alpha}\tilde{z}_t} + c_t + i_t + a(u_t) \bar{e}^{-z_t^*} k_{t-1} = y_t, \tag{1.68}$$

otherwise it becomes:

$$y_* g_t + c_t + i_t + a(u_t) \bar{e}^{-z_t^*} k_{t-1} = y_t. \tag{1.69}$$

Detrended output is also given as a function of inputs by:

$$\dot{y}_t = k_t^{\alpha} L_t^{1-\alpha} - \Phi e^{-\frac{1}{1-\alpha}\tilde{z}_t}. \tag{1.70}$$

$$Y_t = \left(\frac{\dot{P}_t}{P_t}\right)^{\frac{1+\lambda_{f,t}}{\lambda_{f,t}}} \dot{Y}_t$$

becomes

$$y_t = (\dot{p}_t)^{\frac{1+\lambda_{f,t}}{\lambda_{f,t}}} \dot{y}_t \tag{1.71}$$

where

$$\begin{aligned} \dot{p}_t &= \frac{\dot{P}_t}{P_t} \\ &= [(1-\zeta_p)(\frac{\tilde{P}_t}{P_t})^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}} + \zeta_p(\pi_* \frac{\dot{P}_{t-1}}{P_t})^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}}]^{-\frac{\lambda_{f,t}}{1+\lambda_{f,t}}} \\ &= [(1-\zeta_p)\tilde{p}_t^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}} + \zeta_p(\pi_* \dot{p}_{t-1}\pi_t^{-1})^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}}]^{-\frac{\lambda_{f,t}}{1+\lambda_{f,t}}} \end{aligned} \tag{1.72}$$

While

$$L_t = \left(\frac{\dot{W}_t}{W_t}\right)^{\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} \dot{L}_t$$

becomes

$$L_t = (\dot{w}_t)^{\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} \dot{L}_t \tag{1.73}$$

where

$$\begin{aligned} \dot{w}_t &= \frac{\dot{W}_t}{W_t} \\ &= [(1-\zeta_w)(\frac{\tilde{W}_t}{W_t})^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} + \zeta_w(\pi_* e^{\gamma}\Upsilon^{\frac{\alpha}{1-\alpha}} \frac{\dot{W}_{t-1}}{W_t})^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}}]^{-\frac{\lambda_{w,t}}{1+\lambda_{w,t}}} \\ &= [(1-\zeta_w)\tilde{w}_t^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} + \zeta_w(\pi_* e^{\gamma}\Upsilon^{\frac{\alpha}{1-\alpha}}\pi_t^{-1} e^{-z_t^*}\frac{w_{t-1}}{w_t}\dot{w}_{t-1})^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}}]^{-\frac{\lambda_{w,t}}{1+\lambda_{w,t}}} \end{aligned} \tag{1.74}$$

At steady state we have:

$$\frac{1}{1-g_*}(c_* + i_*) = y_*. \tag{1.75}$$

and

$$y_* = k_*^{\alpha} L_*^{1-\alpha} - \Phi. \tag{1.76}$$

and

$$\dot{y}_* = y_*, \ \dot{L}_* = L_*.$$

1.3 Steady State

For now treat L^* as a parameter (we will see that the real variables are all defined as a ratio to L_*, so L_* is just a normalization constant). Define the real rate

$$r_* = \frac{R_*}{\pi_*}, \tag{1.77}$$

then from 1.52 we have:

$$r_* = \beta^{-1} e^{\sigma_c z_*^*} b_*^{-1}. \tag{1.78}$$

From 1.62:

$$r_*^k = b_* r_* \Upsilon - (1-\delta) = \beta^{-1} e^{\sigma_c z_*^*} \Upsilon - (1-\delta). \tag{1.79}$$

From 1.42:

$$w_* = \left(\frac{1}{1+\lambda_f} \alpha^\alpha (1-\alpha)^{(1-\alpha)} r_*^{k\ -\alpha} \right)^{\frac{1}{1-\alpha}} \tag{1.80}$$

From 1.47

$$k_* = \frac{\alpha}{1-\alpha} \frac{w_*}{r_*^k} L_*. \tag{1.81}$$

From 1.56 and 1.57:

$$\bar{k}_* = e^\gamma \Upsilon^{\frac{1}{1-\alpha}} k_*, \tag{1.82}$$

$$i_* = \left(1 - (1-\delta) e^{-\gamma} \Upsilon^{-\frac{1}{1-\alpha}}\right) \bar{k}_*. \tag{1.83}$$

From 1.76:

$$y_* = k_*^\alpha L_*^{1-\alpha} - \Phi. \tag{1.84}$$

SW use the reparameterization $\Phi_p = \dfrac{y_* + \Phi}{y_*}$, implying that steady state output is given by:

$$y_* = \frac{k_*^\alpha L_*^{1-\alpha}}{\Phi_p}. \tag{1.85}$$

From 1.75:

$$c_* = (1-g_*) y_* - i_*, \tag{1.86}$$

(as opposed to $c_* = \dfrac{y_*}{g_*} - i_*$ in DSSW). (Aside: note that 1.53 implies

$$c_*(1 - h e^{-z_*^*}) L_*^{\nu_l} = w_*.$$

Since we already have c_* and w_* it would seem L_* is given. In fact, this is because SW do not use the parameter φ, which would make 1.53 hold for any L_*.)

1.4 Log-linear

1. If technology is stationary, eq. 1.68 becomes:

$$\hat{y}_t = \hat{\tilde{g}}_t - \frac{1}{1-\alpha}\tilde{z}_t + \frac{c_*}{y_*}\hat{c}_t + \frac{i_*}{y_*}\hat{i}_t + \frac{r^k_* k_*}{y_*}\hat{u}_t, \tag{1.87}$$

   If technology has a unit root, eq. 1.69 becomes:

$$\hat{y}_t = \hat{g}_t + \frac{c_*}{y_*}\hat{c}_t + \frac{i_*}{y_*}\hat{i}_t + \frac{r^k_* k_*}{y_*}\hat{u}_t, \tag{1.88}$$

   This is one of the two equilibrium conditions for which we need to write two different versions for the stationary and non-stationary case (the other being the production function). The difference with DSSW (eq. 1.2.77) are due to a different definition of the government spending process which in SW is given by $g_t = \frac{G_t}{y_* Z^*_t}$, with $g_* = 1 - \frac{c_* + i_*}{y_*}$ (in our case, $g_t = \frac{Y_t}{Y_t - G_t}$ and $g_* = \frac{y_*}{c_* + i_*}$). Note that in SW $g_* \in (0,1)$. This is eq. (1) in SW using the reparameterizations

$$c_y = \frac{c_*}{y_*},\ i_y = \frac{i*}{y_*},\ z_y = r^k_* \frac{k_*}{y_*}.$$

2. Eq. 1.49 becomes:

$$\hat{\xi}_t = \hat{R}_t + \hat{b}_t + \mathbb{E}_t[\hat{\xi}_{t+1}] - \mathbb{E}_t[\hat{\pi}_{t+1}] - \sigma_c \mathbb{E}_t[\hat{z}^*_{t+1}], \tag{1.89}$$

   and eq. 1.48 becomes:

$$\hat{\xi}_t = -\sigma_c(1 - he^{-z^*_*})^{-1}\left(\hat{c}_t - he^{-z^*_*}\hat{c}_{t-1} + he^{-z^*_*}z_t\right) + (\sigma_c - 1)L_*^{1+\nu_l}\hat{L}_t,$$

   which becomes using 1.53:

$$\frac{(1 - he^{-z^*_*})}{\sigma_c}\hat{\xi}_t \quad = \quad -\left(\hat{c}_t - he^{-z^*_*}\hat{c}_{t-1} + he^{-z^*_*}z_t\right) \quad + \quad \frac{(\sigma_c - 1)}{\sigma_c}\frac{w_* L_*}{c_*}\hat{L}_t. \tag{1.90}$$

   Putting 1.89 and 1.90 two together we obtain:

$$\begin{aligned}\hat{c}_t = &-\frac{(1 - he^{-z^*_*})}{\sigma_c(1 + he^{-z^*_*})}\left(\hat{R}_t - \mathbb{E}_t[\hat{\pi}_{t+1}] + \hat{b}_t\right) + \frac{he^{-z^*_*}}{(1 + he^{-z^*_*})}\left(\hat{c}_{t-1} - \hat{z}^*_t\right)\\ &+ \frac{1}{(1 + he^{-z^*_*})}\mathbb{E}_t\left[\hat{c}_{t+1} + \hat{z}^*_{t+1}\right] + \frac{(\sigma_c - 1)}{\sigma_c(1 + he^{-z^*_*})}\frac{w_* L_*}{c_*}\left(\hat{L}_t - \mathbb{E}_t[\hat{L}_{t+1}]\right).\end{aligned} \tag{1.91}$$

   This corresponds to eq. (2) in SW, and to the combination of eqs 1.2.68 and 1.2.66 in DSSW. In the code we follow SW's code and use the normalization:

$$\hat{\tilde{b}}_t = -\frac{(1 - he^{-z^*_*})}{\sigma_c(1 + he^{-z^*_*})}\hat{b}_t. \tag{1.92}$$

3. Eq. 1.58 becomes:

$$\hat{i}_t = \frac{1}{S'' e^{2z_*^*}(1+\beta e^{(1-\sigma_c)z_*^*})}\hat{q}_t^k + \frac{1}{1+\beta e^{(1-\sigma_c)z_*^*}}\left(\hat{i}_{t-1} - \hat{z}_t^*\right)$$
$$+ \frac{\beta e^{(1-\sigma_c)z_*^*}}{1+\beta e^{(1-\sigma_c)z_*^*}} I\!E_t\left[\hat{i}_{t+1} + \hat{z}_{t+1}^*\right] + \hat{\mu}_t, \quad (1.93)$$

where we follows SW and renormalize the process $\hat{\mu}_t$ by dividing it for $S'' e^{2z_*^*}(1+\beta e^{(1-\sigma_c)z_*^*})$. This is eq. (3) in SW, and corresponds to eq. 1.2.71 in DSSW (which was expressed in terms of ξ_t^k). The equation can be expressed, perhaps more intuitively, in terms of $\hat{q}_t^k$:

$$\hat{q}_t^k = S'' e^{2z_*^*}(1+\beta e^{(1-\sigma_c)z_*^*})\left(\hat{i}_t - \frac{1}{1+\beta e^{(1-\sigma_c)z_*^*}}\left(\hat{i}_{t-1} - \hat{z}_t^*\right)\right.$$
$$\left. - \frac{\beta e^{(1-\sigma_c)z_*^*}}{1+\beta e^{(1-\sigma_c)z_*^*}} I\!E_t\left[\hat{i}_{t+1} + \hat{z}_{t+1}^*\right] - \hat{\mu}_t\right), \quad (1.94)$$

4. Eq. 1.59 becomes

$$\frac{r_*^k}{r_*^k + (1-\delta)} I\!E_t[r_{t+1}^k] + \frac{1-\delta}{r_*^k + (1-\delta)} I\!E_t[q_{t+1}^k] - \hat{q}_t^k = \hat{R}_t + \hat{b}_t - I\!E_t[\hat{\pi}_{t+1}] \quad (1.95)$$

where we used 1.89. This is eq. (4) in SW (using the value of r_*^k one can see they correspond) and is the same as eq. 1.2.72 in DSSW, except that this was expressed in terms of ξ_t^k. In the code we use the normalization 1.92, consistently with 1.91.

5. Eq. 1.2.78 becomes:

$$\hat{y}_t \;=\; \Phi_p\left(\alpha \hat{k}_t + (1-\alpha)\hat{L}_t\right) + (\Phi_p - 1)\frac{1}{1-\alpha}\tilde{z}_t. \quad (1.96)$$

This is eq. (5) in SW. Note that the last term in 1.96 is non-stationary if $\rho_z = 1$, so it needs to be dropped in that case (which amounts to assuming the fixed costs are proportional to Z_t as opposed to just $e^{\gamma t}$)

6. Eq. 1.2.69 remains the same:

$$\hat{k}_t \;=\; \hat{u}_t - \hat{z}_t^* + \hat{\bar{k}}_{t-1}. \quad (1.97)$$

This is eq. (6) in SW.

7. Equations 1.98 becomes:

$$\frac{1-\psi}{\psi}\hat{r}_t^k = u_t. \tag{1.98}$$

where $\frac{1-\psi}{\psi}$ is simply a reparameterization of the ratio $\frac{r_*^k}{a''}$ that appears in 1.2.73. This is eq. (7) in SW with $z_1 = \frac{1-\psi}{\psi}$, and eq. 1.2.73 in DSSW.

8. Eq. 1.55 becomes:

$$\hat{\bar{k}}_t \;=\; (1 \;-\; \frac{i_*}{\bar{k}_*})\left(\hat{\bar{k}}_{t-1} - \hat{z}_t^*\right) \;+\; \frac{i_*}{\bar{k}_*}\hat{i}_t \;+\; \frac{i_*}{\bar{k}_*}S''e^{2z_*^*}(1 \;+\; \beta e^{(1-\sigma_c)z_*^*})\hat{\mu}_t. \tag{1.99}$$

This is eq. (8) in SW, and corresponds to eq. 1.2.70 in DSSW, except for the renormalization of the exogenous process μ_t. Note that in SW's code the term $(1+\beta e^{(1-\sigma_c)z_*^*})$ is erroneously omitted from the coefficient multiplying $\hat{\mu}_t$.

9. Eq. 1.2.61 remains the same as in DSSW:

$$\hat{m}c_t \;=\; (1-\alpha)\;\hat{w}_t + \alpha\;\hat{r}_t^k. \tag{1.100}$$

This is eq. (9) in SW, where $\hat{\mu}_t^p = -\hat{m}c_t$ and where they used (1.103) to substitute for $\hat{r}_t^k$. That is actually what we also do in the code, obtaining:

$$\hat{m}c_t \;=\; \hat{w}_t + \alpha\hat{L}_t - \alpha\hat{k}_t. \tag{1.101}$$

10. Eq. (TO DO) becomes:

$$\begin{aligned}\hat{\pi}_t = & \frac{(1-\zeta_p\beta e^{(1-\sigma_c)z_*^*})(1-\zeta_p)}{(1+\iota_p\beta e^{(1-\sigma_c)z_*^*})\zeta_p((\Phi_p-1)\epsilon_p+1)}\hat{m}c_t \\ & + \frac{\iota_p}{1+\iota_p\beta e^{(1-\sigma_c)z_*^*}}\hat{\pi}_{t-1} + \frac{\beta e^{(1-\sigma_c)z_*^*}}{1+\iota_p\beta e^{(1-\sigma_c)z_*^*}}\mathbb{E}_t[\hat{\pi}_{t+1}] + \hat{\lambda}_{f,t}\end{aligned} \tag{1.102}$$

This is eq. (10) in SW.

11. Eq. 1.2.65 remains the same:

$$\hat{k}_t \;=\; \hat{w}_t - \hat{r}_t^k + \hat{L}_t. \tag{1.103}$$

This is eq. (11) in SW.

12. Eq. 1.50, which essentially defines the household's marginal rate of subsitution between consumption and labor,

$$\frac{1}{1-he^{-z_*^*}}\left(\hat{c}_t - he^{-z_*^*}\hat{c}_{t-1} + he^{-z_*^*}\hat{z}_t^*\right) + \nu_l \hat{L}_t = \hat{w}_t^h. \tag{1.104}$$

This corresponds to eq. (12) in SW, except that they express it in terms of the markup $\hat{\mu}_t^w = \hat{w}_t - \hat{w}_t^h$, which is what we also do in our code. DSSW did not have this equation as we plugged it the wage Phillips curve directly.

13. Eq. (TO DO) becomes:

$$\begin{aligned}\hat{w}_t = & \frac{(1-\zeta_w \beta e^{(1-\sigma_c)z_*^*})(1-\zeta_w)}{(1+\beta e^{(1-\sigma_c)z_*^*})\zeta_w((\lambda_w - 1)\epsilon_w + 1)}\left(\hat{w}_t^h - \hat{w}_t\right) \\ & - \frac{1+\iota_w \beta e^{(1-\sigma_c)z_*^*}}{1+\beta e^{(1-\sigma_c)z_*^*}}\hat{\pi}_t + \frac{1}{1+\beta e^{(1-\sigma_c)z_*^*}}\left(\hat{w}_{t-1} - \hat{z}_t^* + \iota_w \hat{\pi}_{t-1}\right) \\ & + \frac{\beta e^{(1-\sigma_c)z_*^*}}{1+\beta e^{(1-\sigma_c)z_*^*}}\mathbb{E}_t\left[\hat{w}_{t+1} + \hat{z}_{t+1}^* + \hat{\pi}_{t+1}\right] + \hat{\lambda}_{w,t}\end{aligned} \tag{1.105}$$

This is eq. (13) in SW. In the code we follow SW and replace $\hat{w}_t^h - \hat{w}_t$ with $-\hat{\mu}_t^w$.

14. Eq. 1.2.79 becomes:

$$\begin{aligned}\hat{R}_t \; = \; & \rho_R \hat{R}_{t-1} + (1-\rho_R)\left(\psi_1 \hat{\pi}_t + \psi_2(\hat{y}_t - \hat{y}_t^f)\right) \\ & + \psi_3\left((\hat{y}_t - \hat{y}_t^f) - (\hat{y}_{t-1} - \hat{y}_{t-1}^f)\right) + \hat{r}_t^m\end{aligned} \tag{1.106}$$

where the differences are (1) the use of flexible price/wage output to measure the output gap, (2) the addition of the term $\psi_3\left((\hat{y}_t - \hat{y}_t^f) - (\hat{y}_{t-1} - \hat{y}_{t-1}^f)\right)$; (3) the fact that the residual $\hat{r}_t^m$ is autocorrelated.

15. Additional flexible price/wages equations, where we replace the real rate r_t^f for $\hat{R}_t - \mathbb{E}_t[\hat{\pi}_{t+1}]$:

$$
\begin{aligned}
\hat{y}_t^f &= \hat{g}_t + \frac{c_*}{y_*}\hat{c}_t^f + \frac{i_*}{y_*}\hat{i}_t^f + \frac{r_*^k k_*}{y_*}\hat{u}_t^f, && (1.107)\\
\hat{c}_t^f &= -\frac{(1-he^{-z_*^*})}{\sigma_c(1+he^{-z_*^*})}\left(\hat{r}_t^f + \hat{b}_t\right) + \frac{he^{-z_*^*}}{(1+he^{-z_*^*})}\left(\hat{c}_{t-1}^f - \hat{z}_t^*\right) + \frac{1}{(1+he^{-z_*^*})}\mathbb{E}_t[\hat{c}_{t+1}^f + \hat{z}_{t+1}^*] \\
&\quad + \frac{(\sigma_c - 1)}{\sigma_c(1+he^{-z_*^*})}\frac{w_* L_*}{c_*}\left(\hat{L}_t^f - \mathbb{E}_t[\hat{L}_{t+1}^f]\right), && (1.108)\\
\hat{q}_t^{kf} &= S'' e^{2z_*^*}(1+\beta e^{(1-\sigma_c)z_*^*})\Big(\hat{i}_t^f - \frac{1}{1+\beta e^{(1-\sigma_c)z_*^*}}\left(\hat{i}_{t-1}^f - \hat{z}_t^*\right) \\
&\quad - \frac{\beta e^{(1-\sigma_c)z_*^*}}{1+\beta e^{(1-\sigma_c)z_*^*}}\mathbb{E}_t\left[\hat{i}_{t+1}^f + \hat{z}_{t+1}^*\right] - \hat{\mu}_t\Big), && (1.109)\\
\hat{r}_t^f &= \frac{r_*^k}{r_*^k + (1-\delta)}\mathbb{E}_t[r_{t+1}^{kf}] + \frac{1-\delta}{r_*^k + (1-\delta)}\mathbb{E}_t[q_{t+1}^{kf}] - \hat{q}_t^{kf} - \hat{b}_t && (1.110)\\
\hat{y}_t^f &= \Phi_p\left(\alpha\hat{k}_t^f + (1-\alpha)\hat{L}_t^f\right) + (\Phi_p - 1)\frac{1}{1-\alpha}\tilde{z}_t, && (1.111)\\
\hat{k}_t^f &= \hat{u}_t^f - \hat{z}_t^* + \hat{\bar{k}}_{t-1}^f, && (1.112)\\
u_t^f &= \frac{1-\psi}{\psi}\hat{r}_t^{kf}, && (1.113)\\
\hat{\bar{k}}_t^f &= (1 - \frac{i_*}{\bar{k}_*})\left(\hat{\bar{k}}_{t-1}^f - \hat{z}_t^*\right) + \frac{i_*}{\bar{k}_*}\hat{i}_t^f + \frac{i_*}{\bar{k}_*}S'' e^{2z_*^*}(1+\beta e^{(1-\sigma_c)z_*^*})\hat{\mu}_t. && (1.114)\\
0 &= (1-\alpha)\ \hat{w}_t^f + \alpha\ \hat{r}_t^{kf}. && (1.115)\\
\hat{k}_t^f &= \hat{w}_t^f - \hat{r}_t^{kf} + \hat{L}_t^f, && (1.116)\\
\hat{w}_t^f &= \frac{1}{1-he^{-z_*^*}}\left(\hat{c}_t^f - he^{-z_*^*}\hat{c}_{t-1}^f + he^{-z_*^*}\hat{z}_t^*\right) + \nu_l\hat{L}_t^f. && (1.117)
\end{aligned}
$$

16. The exogenous processes are described in section 1.1.5.

1.5 Adding BGG financial frictions to SW

Amounts to replacing 1.95 with conditions 2.50 and 2.52 (see section 2), which we repeat here for convenience:

$$E_t\left[\hat{\tilde{R}}_{t+1}^k - \hat{R}_t\right] = \hat{b}_t + \zeta_{sp,b}\left(\hat{q}_t^k + \hat{\bar{k}}_t - \hat{n}_t\right) + \tilde{\sigma}_{\omega,t} + \tilde{\mu}_t^e \quad (1.118)$$

$$\hat{\tilde{R}}_t^k - \pi_t = \frac{r_*^k}{r_*^k + (1-\delta)}\hat{r}_t^k + \frac{(1-\delta)}{r_*^k + (1-\delta)}\hat{q}_t^k - \hat{q}_{t-1}^k, \quad (1.119)$$

and adding the eq. condition 2.51 describing the evolution of entrepreneurial net worth

$$
\begin{aligned}
\hat{n}_t \;\; = \;\; & \zeta_{n,\tilde{R}^k}\left(\hat{\tilde{R}}^k_t - \pi_t\right) - \zeta_{n,R}\left(\hat{R}_{t-1} + b_{t-1} - \pi_t\right) + \zeta_{n,qK}\left(\hat{q}^k_{t-1} + \hat{\bar{k}}_{t-1}\right) \\
& + \zeta_{n,n}\hat{n}_{t-1} + \tilde{\gamma}_t + \frac{w^e_*}{n_*}\hat{w}^e_t - \gamma_*\frac{v_*}{n_*}\hat{z}^*_t - \frac{\zeta_{n,\mu^e}}{\zeta_{sp,\mu^e}}\tilde{\mu}^e_{t-1} - \frac{\zeta_{n,\sigma_\omega}}{\zeta_{sp,\sigma_\omega}}\tilde{\sigma}_{\omega,t-1}
\end{aligned}
\tag{1.120}
$$

Note that if $\zeta_{sp,b} = 0$ and the financial friction shocks are zero, 1.95 coincides with 1.118 plus 1.119. In particular, we stick to SW's assumption that returns to deposit are not subject to the same "intermediation cost" shock b_t as government bonds. This assumption mirrors SW's assumption that capital investment was not subject to that transaction cost.

1.6 Anticipated policy shocks

We modify the policy rule (1.106) so to incorporate anticipated policy shocks. In order to do so we add the anticipated shocks to the exogenous component of monetary policy as follows:

$$
\hat{r}^m_t = \rho_{r^m}\hat{r}^m_{t-1} + \sigma_r\varepsilon_{r^m,t} + \sum_{k=1}^{K}\sigma_{k,r}\varepsilon^R_{k,t-k}, \tag{1.121}
$$

where $\varepsilon_{R,t}$ is the usual contemporaneous policy shock and $\varepsilon^R_{k,t-k}$ is a policy shock that is known to agents at time $t-k$, but affects the policy rule k periods later, that is, at time t. We assume as usual that $\varepsilon^R_{k,t-k} \sim N(0,1),\ i.i.d.$.

In order to solve the model we need to express the anticipated shocks in recursive form. For this purpose, we augment the state vector s_t with K additional states ν^R_t,...,ν^R_{t-K} whose law of motion is as follows:

$$
\begin{aligned}
\nu^R_{1,t} &= \nu^R_{2,t-1} + \sigma_{1,r}\varepsilon^R_{1,t} \\
\nu^R_{2,t} &= \nu^R_{3,t-1} + \sigma_{2,r}\varepsilon^R_{2,t} \\
&\vdots \\
\nu^R_{K,t} &= \sigma_{K,r}\varepsilon^R_{K,t}
\end{aligned}
$$

and rewrite expression (1.122) as

$$
\hat{r}^m_t = \rho_{r^m}\hat{r}^m_{t-1} + \sigma_r\varepsilon_{r^m,t} + \nu^R_{1,t-1}, \tag{1.122}
$$

It is easy to verify that $\nu^R_{1,t-1} = \sum_{k=1}^{K} \sigma_{k,r}\varepsilon^R_{k,t-k}$, that is, $\nu^R_{1,t-1}$ is a "bin" that collects all anticipated shocks that affect the policy rule in period t. In the implementation, we assume that these shocks have the same standard deviation as the contemporaneous shock: $\sigma_{k,r} = \sigma_r$.

1.7 Adding long run changes in productivity

We add long run changes in productivity. Specifically we assume that the production function is:

$$Y_t(i) = \max\{e^{\tilde{z}_t} K_t(i)^\alpha \left(L_t(i)e^{\gamma t} Z^p_t\right)^{1-\alpha} - \Phi Z^*_t, 0\}, \tag{1.123}$$

where $\tilde{z}_t$ and $z^p_t = log(Z^p_t / Z^p_{t-1})$ follow AR(1) processes:

$$\tilde{z}_t = \rho_z \tilde{z}_{t-1} + \sigma_z \epsilon_{z,t},\ \epsilon_{z,t} \sim N(0,1), \tag{1.124}$$

$$z^p_t = \rho_{z^p} z^p_{t-1} + \sigma_{z^p}\epsilon_{z^p,t},\ \epsilon_{z^p,t} \sim N(0,1), \tag{1.125}$$

and

$$Z^*_t = Z_t Z^p_t e^{\left(\gamma + \frac{\alpha}{1-\alpha}\log \Upsilon\right)t},\ Z_t = e^{\frac{1}{1-\alpha}\tilde{z}_t}. \tag{1.126}$$

We detrend by Z^*_t as in section 1.2. Define $z^*_t = \log(Z^*_t / Z^*_{t-1})$ with $z^*_* = \gamma + \frac{\alpha}{1-\alpha}\log\Upsilon$. From 1.124, 1.125 and 1.126 we see that

$$\hat{z}^*_t = z^*_t - z^*_* = \frac{1}{1-\alpha}(\rho_z - 1)\tilde{z}_{t-1} + \frac{1}{1-\alpha}\sigma_z\epsilon_{z,t} + z^p_t, \tag{1.127}$$

and

$$E_t[\hat{z}^*_{t+1}] = \frac{1}{1-\alpha}(\rho_z - 1)\tilde{z}_t + \rho_{z^p} z^p_t. \tag{1.128}$$

Note that we can accommodate both cases where $\tilde{z}_t$ is stationary and random walk ($\rho_z = 1$). Regardless, there is a stochastic trend in growth. Therefore in equations 1.87 and 1.96 the term $\frac{1}{1-\alpha}\tilde{z}_t$ needs to be dropped.

1.8 Measurement equation for TFP (version 1)

Assume $\Upsilon = 1$. Total factor productivity takes the following form in our model, when abstracting from the fixed costs:

$$TFP_t = (Z_t^*)^{1-\alpha} = e^{\tilde{z}_t + (1-\alpha)\gamma t} (Z_t^p)^{1-\alpha}.$$

TFP including variable capital utilization, i.e., unadjusted TFP, is given by

$$TFP_t^u = (Z_t^*)^{1-\alpha} u_t^\alpha.$$

Taking log on both sides and dividing by $1-\alpha$, we obtain

$$\frac{1}{1-\alpha} \log TFP_t^u = \log Z_t^* + \frac{\alpha}{1-\alpha} \log u_t$$

and taking differences

$$\begin{aligned} \frac{1}{1-\alpha} \log \left(TFP_t^u / TFP_{t-1}^u \right) &= z_t^* + \frac{\alpha}{1-\alpha} (\log u_t - \log u_{t-1}) \\ &= z_*^* + \hat{z}^* + \frac{\alpha}{1-\alpha} (\hat{u}_t - \hat{u}_{t-1}) \end{aligned}$$

Fernald provides data on total factor productivity growth. We use the utilization unadjusted series and divide it by his estimates of $1-\alpha$ to obtain a series of productivity expressed in labor-augmenting form. The resulting series should correspond in our model to the expression on the right hand side. We demean Fernald's series and add measurement error so that we obtain the measurement equation

$$\begin{aligned} &\text{TFP growth (unadjusted for utilization, demeaned, expressed in labor augmenting terms)} \\ = \quad &\hat{z}^* + \frac{\alpha}{1-\alpha} (\hat{u}_t - \hat{u}_{t-1}) + e_t^{tfp}. \end{aligned}$$

1.9 Measurement equation for TFP (version 2)

We have the linearized production function

$$\hat{y}_t = \Phi_p \left(\alpha \hat{k}_t + (1-\alpha) \hat{L}_t \right)$$

where $\hat{y}_t = \log\left(Y_t/Z_t^*\right) - \log y_*$, $Z_t^* = e^{\frac{1}{1-\alpha}\tilde{z}_t + \left(\gamma + \frac{\alpha}{1-\alpha}\Upsilon\right)t} Z_t^p$, $\hat{L}_t = \log\left(L_t/L_*\right)$, and the effective capital $\hat{k}_t = \log\left(\Upsilon^{-t} K_t/Z_t^*\right) - \log k_*$ relates to installed capital $\hat{\bar{k}}_{t-1} = \log\left(\Upsilon^{-(t-1)} \bar{K}_{t-1}/Z_{t-1}^*\right) - \log\left(e^{\gamma} \Upsilon^{\frac{1}{1-\alpha}} k_*\right)$ as follows

$$\hat{k}_t = \hat{u}_t - \hat{z}_t^* + \hat{\bar{k}}_{t-1}$$

where $\hat{u}_t = \log u_t - \log u_* = \log u_t$, and $\hat{z}_t^* = \log\left(Z_t^*/Z_{t-1}^*\right) - z_*^*$, and $z_*^* = \gamma + \dfrac{\alpha}{1-\alpha} \log \Upsilon$.

We can rewrite this as

$$\begin{aligned}
\log\left(Y_t/Z_t^*\right) - \log y_* &= \Phi_p\left(\alpha\left(\hat{u}_t - \hat{z}^* + \hat{\bar{k}}_{t-1}\right) + (1-\alpha)\hat{L}_t\right) \\
&= \Phi_p\left(\alpha\left(\hat{u}_t - \hat{z}^* + \log\left(\Upsilon^{-(t-1)}\bar{K}_{t-1}/Z_{t-1}^*\right) - \log\left(e^{\gamma}\Upsilon^{\frac{1}{1-\alpha}} k_*\right)\right) + (1-\alpha)(\log L_t -
\end{aligned}$$

Taking first differences

$$\begin{aligned}
\log Y_t - \log Y_{t-1} &= \log Z_t^* - \log Z_{t-1}^* + \Phi_p\alpha\left(\hat{u}_t - \hat{u}_{t-1} - \hat{z}^* + \hat{z}_{t-1}^* + \log \bar{K}_{t-1} - \log \bar{K}_{t-2} - \log Z_{t-1}^* + \log\right. \\
&\quad + \Phi_p\alpha\left(\log \Upsilon^{-(t-1)} - \log \Upsilon^{-(t-2)}\right) + \Phi_p(1-\alpha)\left(\log L_t - \log L_{t-1}\right) \\
&= z_*^* + \hat{z}^* + \Phi_p\alpha\left(\hat{u}_t - \hat{u}_{t-1} - \hat{z}^* - z_*^*\right) + \Phi_p\alpha \log\left(\bar{K}_{t-1}/\bar{K}_{t-2}\right) + \Phi_p\alpha\log\left(\Upsilon^{-1}\right) + \Phi_p(1
\end{aligned}$$

or

$$\begin{aligned}
\log\left(Y_t/Y_{t-1}\right) &= \Phi_p\alpha\log\left(\bar{K}_{t-1}/\bar{K}_{t-2}\right) + \Phi_p\alpha\log\left(\Upsilon^{-1}\right) + \Phi_p(1-\alpha)\log\left(L_t/L_{t-1}\right) \\
&\quad + \Phi_p(1-\alpha)\left[\left(\frac{(1-\Phi_p\alpha)}{\Phi_p(1-\alpha)}\left(z_*^* + \hat{z}^*\right) + \frac{\Phi_p\alpha}{\Phi_p(1-\alpha)}\left(\hat{u}_t - \hat{u}_{t-1}\right)\right)\right].
\end{aligned}$$

Fernald provides data on total factor productivity growth. We use the utilization unadjusted series and divide it by his estimates of $1-\alpha$ to obtain a series of productivity expressed in labor-augmenting form. The resulting series should correspond in our model to the last term in square brackets. We demean Fernald's series and add measurement error so that we obtain the measurement equation

$$\begin{aligned}
&\text{TFP growth (unadjusted for utilization, demeaned, expressed in labor augmenting terms)} \\
&= \frac{(1-\Phi_p\alpha)}{\Phi_p(1-\alpha)}\hat{z}^* + \frac{\alpha}{1-\alpha}\left(\hat{u}_t - \hat{u}_{t-1}\right) + e_t^{tfp}.
\end{aligned}$$

Note that if $\Phi_p = 1$, this reduces to

$$\begin{aligned}
&\text{TFP growth (unadjusted for utilization, demeaned, expressed in labor augmenting terms)} \\
&= \hat{z}^* + \frac{\alpha}{1-\alpha}\left(\hat{u}_t - \hat{u}_{t-1}\right) + e_t^{tfp}.
\end{aligned}$$

1.10 Computing the Initial Level of Stationary Productivity/BN decomposition

Imagine there are no long run changes in productivity and $\rho_z < 1$, so that $\tilde{z}_t$ and therefore $\log(Z_t) = \dfrac{1}{1-\alpha}\tilde{z}_t$ are stationary (with zero mean), and $Z_t^* = Z_t e^{\gamma t}$ is trend-stationary. Then we know that $E\log(Z_\infty) \to 0$. We can use this fact to pin down the initial level of $\log(Z_0)$.

Define (as we did in the previous section) $\hat{z}_t = \log(Z_t) - \log(Z_{t-1})$ as the demeaned growth rate of productivity. Then

$$E_T\left[\log(Z_\infty) - \log(Z_T)\right] = E_T\left[\sum_{j=1}^{\infty} \hat{z}_{T+j}\right]$$

or

$$E_T\left[\log(Z_T)\right] = -E_T\left[\sum_{j=1}^{\infty} \hat{z}_{T+j}\right] \tag{1.129}$$

after imposing $E_T \log(Z_\infty) = 0$. The term $E_T\left[\sum_{j=1}^{\infty} \hat{z}_{T+j}\right]$ can be computed using our state space model. Define the transition equation as

$$s_t = \Phi s_{t-1} + R\epsilon_t,$$

where s_t is the state vector and Ψ_z is a row vector of zeros with 1 corresponding to the position of the state $\hat{z}_t$ (that is, Ψ_z 'picks' the state $\hat{z}_t$). Then

$$\begin{aligned} E_T\left[\sum_{j=1}^{\infty} \hat{z}_{T+j}\right] &= E_T\left[\sum_{j=1}^{\infty} \Psi_z s_{T+j}\right] = E_T\left[\sum_{j=1}^{\infty} \Psi_z \Phi^j s_T\right] \\ &= \Psi_z \Phi \left(\sum_{j=0}^{\infty} \Phi^j\right) E_T\left[s_T\right] = \Psi_z \Phi \left(I - \Phi\right)^{-1} s_{T|T}. \end{aligned} \tag{1.130}$$

We have also computed ("blue" line in the plots)

$$E_T\left[\log(Z_T) - \log(Z_0)\right] = E_T\left[\sum_{t=1}^{T} \hat{z}_t\right] = \sum_{t=1}^{T} \hat{z}_{t|T}.$$

Putting this together with conditions 1.130 and 1.129 we obtain:

$$E_T\left[\log(Z_0)\right] = -\Psi_z \Phi \left(I - \Phi\right)^{-1} s_{T|T} - \sum_{t=1}^{T} \hat{z}_{t|T}.$$

1.11 Adding income as an observable

We add Gross Domestic Income (GDI) as an additional observable in the model. We assume that model output loads on both observed GDP and GDI growth rates as follows:

$$\Delta GDP_t = (\hat{y}_t - \hat{y}_{t-1} + \hat{z}_t) + e_t^{gdp} - e_{t-1}^{gdp} + 100(exp(z_t^*) - 1) \tag{1.131}$$

$$\Delta GDI_t = \Gamma_{gdi}(\hat{y}_t - \hat{y}_{t-1} + \hat{z}_t) + e_t^{gdi} - e_{t-1}^{gdi} + 100(exp(z_t^*) - 1) + \delta_{gdi} \tag{1.132}$$

We set $\delta_{gdi} = 0$ and $\Gamma_{gdi} = 1$. We also introduce correlated measurement error in levels between GDP and GDI:

$$\begin{aligned} e_t^{gdp} &= \rho_{gdp} \cdot e_{t-1}^{gdp} + \sigma_{gdp}\epsilon_t^{gdp},\ \epsilon_t^{gdp} \sim i.i.d.N(0,1) \\ e_t^{gdi} &= \rho_{gdi} \cdot e_{t-1}^{gdi} + \varrho_{gdp} \cdot \sigma_{gdp}\epsilon_t^{gdp} + \sigma_{gdi}\epsilon_t^{gdi},\ \epsilon_t^{gdi} \sim i.i.d.N(0,1). \end{aligned}$$

We also try including the model output level to the “measurement error”. That is, the observed output may be further from the true output during periods of larger output growth/contraction. We augment equations 1.131 and 1.132 as follows:

$$\begin{aligned} GDP_t &= (y_t - y_{t-1} + z_t) + (e_t^{gdp} - e_{t-1}^{gdp}) \\ &\quad + \lambda_{gdp}(y_t - y_{t-1}) + 100(exp(z_t^*) - 1) \end{aligned} \tag{1.133}$$

$$\begin{aligned} GDI_t &= \Gamma_{gdi}(y_t - y_{t-1} + z_t) + (e_t^{gdi} - e_{t-1}^{gdi}) \\ &\quad + \lambda_{gdi}(y_t - y_{t-1}) + 100(exp(z_t^*) - 1) + \delta_{gdi} \end{aligned} \tag{1.134}$$

1.12 Alternative Scenarios

Here we discuss how we implement alternative scenarios. Write the state space system as:

$$s_t = \mathcal{T}(\theta)s_{t-1} + \mathcal{R}(\theta)\epsilon_t \tag{1.1}$$

where s_t is the model's vector of "state" variables, the matrices $\mathcal{T}$ and $\mathcal{R}$ are functions of the vector of all model parameters θ, and ϵ_t is the vector of structural shocks. The vector of observables y_t described below is in turn related to the states according to the system of measurement equations:

$$y_t = \mathcal{D}(\theta) + \mathcal{Z}(\theta)s_t. \tag{1.2}$$

Starting from s_T, roll the system forward to obtain:

$$\begin{aligned}
y_{T+1} &= \mathcal{D} + \mathcal{Z}\mathcal{T}s_T + \mathcal{Z}\mathcal{R}\epsilon_{T+1} \\
y_{T+2} &= \mathcal{D} + \mathcal{Z}\mathcal{T}^2 s_T + \mathcal{Z}\mathcal{T}\mathcal{R}\epsilon_{T+1} + \mathcal{Z}\mathcal{R}\epsilon_{T+2} \\
&\vdots \\
y_{T+\bar{H}} &= \mathcal{D} + \mathcal{Z}\mathcal{T}^{\bar{H}} s_T + \sum_{j=0}^{\bar{H}-1} \mathcal{Z}\mathcal{T}^j\mathcal{R}\epsilon_{T+\bar{H}-j}.
\end{aligned} \tag{1.3}$$

Algorithm 1 Drawing Counterfactual (Alt Scenarios) Forecasts .[1]

1. *Choose a set of targets (all expressed in deviations from steady state/baseline forecasts)* $\bar{\Delta y} = [\bar{\Delta y}_{T+1}, .., \bar{\Delta y}_{T+\bar{H}}]$ *and shocks* $\bar{\epsilon} = [\bar{\epsilon}_{T+1}, .., , \bar{\epsilon}_{T+\bar{H}}]$*, where* $\bar{\Delta y}$ *and* $\bar{\epsilon}$ *must be of the same size* k*. The two are linked by the system of equations:*

$$\begin{aligned}
\bar{\Delta y}_{T+1} &= \mathcal{Z}_{T+1,.}\mathcal{R}_{.,T+1}\bar{\epsilon}_{T+1} \\
\bar{\Delta y}_{T+2} &= \mathcal{Z}_{T+2,.}\mathcal{T}\mathcal{R}_{.,T+1}\bar{\epsilon}_{T+1} + \mathcal{Z}_{T+2,.}\mathcal{R}_{.,T+2}\bar{\epsilon}_{T+2} \\
&\vdots \\
\bar{\Delta y}_{T+\bar{H}} &= \sum_{j=0}^{\bar{H}-1} \mathcal{Z}_{T+\bar{H},.}\mathcal{T}^j\mathcal{R}_{.,T+\bar{H}-j}\bar{\epsilon}_{T+\bar{H}-j}.
\end{aligned} \tag{1.4}$$

[1] Here we focus on the mode of the posterior density for θ. If we want the whole distribution we just repeat the algorithm for each draw of θ.

where $\mathcal{Z}_{T+j,.}$ and $\mathcal{R}_{.,T+j}$ refer to the rows of the matrix $\mathcal{Z}$ and the columns of the matrix $\mathcal{R}$ corresponding to the targets ($\bar{\Delta} y_{T+j}$) and the instruments ($\bar{\epsilon}_{T+j}$) in period $T+j$. This linear system of k equations with k unknowns can be solved for the vector of shocks $\bar{\epsilon}$. Specifically, rewrite the system (1.4) as

$$\bar{\Delta} y = M \bar{\epsilon} \tag{1.5}$$

where

$$M = \begin{bmatrix} \mathcal{Z}_{T+1,.}\mathcal{R}_{.,T+1} & 0 & .. & 0 \\ \mathcal{Z}_{T+2,.}\mathcal{T}\mathcal{R}_{.,T+1} & \mathcal{Z}_{T+2,.}\mathcal{R}_{.,T+2} & .. & 0 \\ \vdots & \vdots & \ddots & \\ \mathcal{Z}_{T+\bar{H},.}\mathcal{T}^{\bar{H}-1}\mathcal{R}_{.,T+1} & \mathcal{Z}_{T+\bar{H},.}\mathcal{T}^{\bar{H}-2}\mathcal{R}_{.,T+2} & \dots & \mathcal{Z}_{T+\bar{H},.}\mathcal{R}_{.,T+\bar{H}} \end{bmatrix}$$

and the solution is $\bar{\epsilon} = M^{-1}\bar{\Delta} y$.

2. *Compute the alternative scenario for the entire set of observables y using*

$$\begin{aligned} \Delta y_{T+1} &= \mathcal{Z}\mathcal{R}_{.,T+1}\bar{\epsilon}_{T+1} \\ \Delta y_{T+2} &= \mathcal{Z}\mathcal{T}\mathcal{R}_{.,T+1}\bar{\epsilon}_{T+1} + \mathcal{Z}_{T+2,.}\mathcal{R}_{.,T+2}\bar{\epsilon}_{T+2} \\ &\vdots \\ \Delta y_{T+\bar{H}} &= \sum_{j=0}^{\bar{H}-1} \mathcal{Z}\mathcal{T}^{j}\mathcal{R}_{.,T+\bar{H}-j}\bar{\epsilon}_{T+\bar{H}-j}. \end{aligned} \tag{1.6}$$

These are the same equations as in (1.4), except that $\mathcal{Z}$ replaces $\mathcal{Z}_{T+j,.}$.

1.12.1 No effect of initial state

The initial state of the forecast and of the shock decomposition has no effect on the forecast under scenario and shock decomposition under scenario values.

The forecast under scenario is computed as the difference of the scenario forecast

$$s_t^{scen} = \mathcal{T} s_{t-1}^{scen} + \mathcal{R}\epsilon_t^0 \tag{1.7}$$

and the baseline, no-shocks, forecast

$$s_t^{base} = \mathcal{T} s_{t-1}^{base} \tag{1.8}$$

where ϵ_t^0 is the solved shock path under the scenario and the initial state vector, s_T, is the same for both forecasts. Note that we switch to the notation ϵ_t^0 from the notation $\bar{\epsilon}_t$ above.

Our "forecast under scenario" is computed by subtracting (1.7) - (1.8), and we then refer to the units of this forecast under scenario as "deviations from baseline":

$$s_t = s_t^{scen} - s_t^{base} \tag{1.9}$$

We can rewrite equation 1.7 as

$$\begin{aligned}
s_{T+1}^{scen} &= \mathcal{T}s_T + \mathcal{R}\epsilon_{T+1}^0 \\
s_{T+2}^{scen} &= \mathcal{T}s_{T+1}^{scen} + \mathcal{R}\epsilon_{T+2}^0 = \mathcal{T}^2 s_T + \mathcal{T}\mathcal{R}\epsilon_{T+1}^0 + \mathcal{R}\epsilon_{T+2}^0 \\
&\ldots \\
s_{T+H}^{scen} &= \mathcal{T}^H s_T + \sum_{h=1}^{H} \mathcal{T}^{h-1}\mathcal{R}\epsilon_h^0 \qquad (1.10)
\end{aligned}$$

and equation 1.8 as

$$\begin{aligned}
s_{T+1}^{base} &= \mathcal{T}s_T \\
s_{T+2}^{base} &= \mathcal{T}s_{T+1}^{base} = \mathcal{T}^2 s_T \\
&\ldots \\
s_{T+H}^{base} &= \mathcal{T}^H s_T \qquad (1.11)
\end{aligned}$$

We can now rewrite equation 1.9 as

$$\begin{aligned}
s_{T+H} &= s_{T+H}^{scen} - s_{T+H}^{base} \\
s_{T+H} &= (\mathcal{T}^H s_T + \sum_{h=1}^{H} \mathcal{T}^{h-1}\mathcal{R}\epsilon_h^0) - (\mathcal{T}^H s_T) \\
s_{T+H} &= \sum_{h=1}^{H} \mathcal{T}^{h-1}\mathcal{R}\epsilon_h^0 \qquad (1.12)
\end{aligned}$$

The important observation is that the forecast under scenario does not depend on the initial state s_T.

Similarly, we compute a shock decomposition of our forecast under scenario. We know that the model is linear and additive, so we can write out the shock decomposition procedure ignoring the fact that we do this for only one shock at a time.

$$\begin{aligned} s_t^{shockdec} &= \mathcal{T} s_{t-1}^{shockdec} + \mathcal{R}\epsilon_t^0 \\ s_T^{shockdec} &= 0 \end{aligned} \tag{1.13}$$

For some period $T + H$, we can then write equation 1.13 as

$$\begin{aligned} s_{T+1}^{shockdec} &= \mathcal{T} s_T^{shockdec} + \mathcal{R}\epsilon_{T+1}^0 = s_{T+1}^{shockdec} = \mathcal{T} \cdot 0 + \mathcal{R}\epsilon_{T+1}^0 = s_{T+1}^{shockdec} = \mathcal{R}\epsilon_{T+1}^0 \\ s_{T+2}^{shockdec} &= \mathcal{T}\mathcal{R}\epsilon_{T+1}^0 + \mathcal{R}\epsilon_{T+2}^0 \\ &\dots \\ s_{T+H}^{shockdec} &= \sum_{h=1}^{H} \mathcal{T}^{h-1}\mathcal{R}\epsilon_h^0 \end{aligned} \tag{1.14}$$

Comparing equations 1.12 and 1.14, we can see that the initial state does not enter into either formulation and that the two should match at each period.

2 The FRBNY DSGE model: Adding BGG-type financial frictions to the SW model

2.0.2 Households

The household's problem is different as households no longer hold the capital stock, and make investment and capital utilization decisions. Rather, they invest in deposits to the banking sector D_t (in addition to government bonds), which pay a gross nominal interest rate R_t^d. Household j's budget constraint is:

$$C_{t+s}(j) + \frac{B_{t+s}(j)}{b_{t+s}R_{t+s}P_{t+s}} + \frac{D_{t+s}(j)}{R^d_{t+s}P_{t+s}} \leq \frac{B_{t+s-1}(j)}{P_{t+s}} + \frac{D_{t+s-1}(j)}{P_{t+s}} + \frac{W^h_{t+s}}{P_{t+s}}L_{t+s}(j) + \Pi_{t+s} - T_{t+s}. \quad (2.1)$$

Households' first order conditions for consumption, bonds, and labor supply are unchanged. The FOC for deposits is

$$\Xi_t = \beta R_t^d \mathbb{E}_t[\Xi_{t+1}\pi_{t+1}^{-1}]$$

which implies that

$$R_t^d = R_t b_t,$$

and at steady state $\dfrac{R_*^d}{\pi_*} = r_* b_* = \beta^{-1} e^{\sigma_c z_*^*}$.

2.0.3 Capital Producers

There is a representative, competitive, capital producer who produces new capital by transforming general output – which is bought from final goods producers at the nominal price Q_t^k – into new capital via the technology:

$$x' = x + \Upsilon^t \mu_t \left(1 - S(\frac{I_t}{I_{t-1}})\right) I_t. \quad (2.2)$$

where x is the initial capital purchased from entrepreneurs in period t, and x' is the new stock of capital, which they sell back to entrepreneurs at the end of the same period.

Their period profits, expressed in terms of consumption goods, are therefore given by:

$$\begin{aligned}\Pi_t^k &= \frac{Q_t^k}{P_t}x' - \frac{Q_t^k}{P_t}x - I_t \\ &= \frac{Q_t^k}{P_t}\Upsilon^t \mu_t \left(1 - S(\frac{I_t}{I_{t-1}})\right) I_t - I_t.\end{aligned} \tag{2.3}$$

Note that these profits do not depend on the initial level of capital x purchased, so effectively the only decision variable for capital producers is I_t. Since they discount profits using the households' discount rate $\beta^t \Xi_t$, their FOC wrt I_t are:

$$\begin{aligned}(\partial I_t) \qquad & \Xi_t \frac{Q_t^k}{P_t}\Upsilon^t \mu_t \left(1 - S(\frac{I_t}{I_{t-1}}) - S'(\frac{I_t}{I_{t-1}})\frac{I_t}{I_{t-1}}\right) \\ & + \beta \mathbb{E}_t[\Xi_{t+1}\frac{Q_{t+1}^k}{P_{t+1}}\Upsilon^{t+1}\mu_{t+1}S'(\frac{I_{t+1}}{I_t})(\frac{I_{t+1}}{I_t})^2] = \Xi_t\end{aligned} \tag{2.4}$$

Note that this FOC is identical to 1.18 if we replace $\frac{Q_t^k}{P_t}$ with Ξ_t^k / Ξ_t.

2.0.4 Entrepreneurs

There is a continuum of entrepreneurs indexed by e. Each entrepreneur buys installed capital $\bar{K}_{t-1}(e)$ from the capital producers at the end of period $t-1$ using her own net worth $N_{t-1}(e)$ and a loan $B_{t-1}^l(e)$ from the banking sector:

$$Q_{t-1}^k \bar{K}_{t-1}(e) = B_{t-1}^l(e) + N_{t-1}(e)$$

where net worth is expressed in nominal terms. In the next period she rents capital out to firms, earning a rental rate R_t^k per unit of effective capital. In period t she is subject to an i.i.d. (across entrepreneurs and over time) shock $\omega(e)_t$ that increases or shrinks her capital, where $log\ \omega(e)_t \sim N(m_{\omega,t-1}, \sigma_{\omega,t-1}^2)$ where $m_{\omega,t-1}$ is such that $\mathbb{E}\omega(e)_t = 1$. Denote by $F_{t-1}(\omega)$ the cumulative distribution function of ω at time t, where the distribution needs to be known at time $t-1$. In addition, after observing the shock she can choose a level of utilization $u(e)_t$ by paying a cost in terms of general output equal to $a(u(e)_t)\Upsilon^{-t}$ per-unit-of-capital. At the end of period t the entrepreneurs sells undepreciated capital to the capital producers. Entrepreneurs' revenues in period t are therefore:

$$\left\{R_t^k u(e)_t + (1-\delta)Q_t^k - P_t a(u(e)_t)\Upsilon^{-t}\right\}\omega(e)_t \bar{K}(e)_{t-1}$$

or equivalently

$$\omega(e)_t \tilde{R}^k(e)_t Q^k_{t-1} \bar{K}(e)_{t-1}$$

where

$$\tilde{R}^k(e)_t = \frac{R^k_t u(e)_t + (1-\delta) Q^k_t - P_t a(u(e)_t) \Upsilon^{-t}}{Q^k_{t-1}} \tag{2.5}$$

is the gross nominal return to capital for entrepreneurs. From the profit function it is clear that the choice of the utilization rate is independent from the amount of capital purchased or the ω shock, and is given by the FOC:

$$\frac{R^k_t}{P_t} = a'(u(e)_t) \Upsilon^{-t}, \tag{2.6}$$

which is the same condition as 1.20. Consequently we can drop the index from the return $\tilde{R}^k_t$.

The debt contract undertaken by the entrepreneur in period $t-1$ consists of the triplet $(B^l(e)_{t-1}, R^d(e)_t, \bar{\omega}(e)_t)$ where $R^l_t(e)$ represents the contractual interest rate, and $\bar{\omega}(e)_t$ the theshold level of $\omega(e)_t$ below which the entrepreneur cannot pay back, which is therefore defined by the equation:

$$\bar{\omega}(e)_t \tilde{R}^k_t Q^k_{t-1} \bar{K}(e)_{t-1} = R^l(e)_t B^l(e)_{t-1}. \tag{2.7}$$

For $\omega(e)_t < \bar{\omega}(e)_t$ the bank monitors the entrepreneurs and extracts a fraction $(1-\mu^e_t)$ of its revenues $\tilde{R}^k_t Q^k_{t-1} \bar{K}(e)_{t-1}$, where μ^e_t represents exogenous bankrupcty costs. The bank's zero profit condition implies that [state by state?]:

$$\left[1 - F_{t-1}(\bar{\omega}(e)_t)\right] R^l(e)_t B^l(e)_{t-1} + (1-\mu^e_{t-1}) \int_0^{\bar{\omega}(e)_t} \omega dF_{t-1}(\omega) \tilde{R}^k_t Q^k_{t-1} \bar{K}(e)_{t-1} = R^d_{t-1} B^l(e)_{t-1}$$

where R^d_{t-1} is the rate paid by the bank to the depositors. If we define leverage as:

$$\varrho(e)_t \equiv \frac{B^l(e)_t}{N(e)_t},$$

use the definitions

$$\begin{aligned} \Gamma_{t-1}(\bar{\omega}_t) \equiv & \quad \bar{\omega}_t \left[1 - F_{t-1}(\bar{\omega}_t)\right] + G_{t-1}(\bar{\omega}_t) \\ G_{t-1}(\bar{\omega}_t) \equiv & \quad \int_0^{\bar{\omega}_t} \omega dF_{t-1}(\omega), \end{aligned}$$

as well as the definiton of $\bar{\omega}(e)_t$, the zero-profit condition can be rewritten as:

$$\left[\Gamma_{t-1}(\bar{\omega}(e)_t) - \mu^e_{t-1} G_{t-1}(\bar{\omega}(e)_t)\right] \frac{\tilde{R}^k_t}{R^d_{t-1}} (1 + \varrho(e)_{t-1}) = \varrho(e)_{t-1}. \tag{2.8}$$

Entrepreneurs' expected profits (before the realization of the shock ω_t) can be written as:

$$\begin{aligned}
&\int_{\bar{\omega}(e)_t}^{\infty} \left[\omega(e)_t \tilde{R}^k(e)_t Q^k_{t-1} \bar{K}(e)_{t-1} - R^l(e)_t B^l(e)_{t-1}\right] dF_{t-1}(\omega(e)_t) = \\
&\left[\int_{\bar{\omega}(e)_t}^{\infty} \omega(e)_t dF_{t-1}(\omega(e)_t) - \bar{\omega}(e)_t [1 - F_{t-1}(\bar{\omega}(e)_t)]\right] \tilde{R}^k(e)_t Q^k_{t-1} \bar{K}(e)_{t-1} = \\
&[1 - \Gamma_{t-1}(\bar{\omega}(e)_t)] \frac{\tilde{R}^k_t}{R^d_{t-1}} [1 + \varrho(e)_{t-1}] R^d_{t-1} N(e)_{t-1}
\end{aligned}$$

The contract that maximizes expected net worth for the entrepreneurs is given by:

$$\max_{\{\varrho(e)_{t-1}, \bar{\omega}(e)_t\}} E_{t-1} \left[\begin{array}{c} [1 - \Gamma_{t-1}(\bar{\omega}(e)_t)] \frac{\tilde{R}^k_t}{R^d_{t-1}} [1 + \varrho(e)_{t-1}] R^d_{t-1} N(e)_{t-1} \\ + \eta_t \left\{ \left[\Gamma_{t-1}(\bar{\omega}(e)_t) - \mu^e_{t-1} G_{t-1}(\bar{\omega}(e)_t)\right] \frac{\tilde{R}^k_t}{R^d_{t-1}} [1 + \varrho(e)_{t-1}] - \varrho(e)_{t-1} \right\} \end{array} \right]$$

so that the FOCs are:

$$\begin{aligned}
\varrho(e)_{t-1} : \quad & 0 = E_{t-1} \left[[1 - \Gamma_{t-1}(\bar{\omega}(e)_t)] \frac{\tilde{R}^k_t}{R^d_{t-1}} \right] R^d_{t-1} N(e)_{t-1} \\
& + E_{t-1} \left[\eta_t \left\{ \left[\Gamma_{t-1}(\bar{\omega}(e)_t) - \mu^e_{t-1} G_{t-1}(\bar{\omega}(e)_t)\right] \frac{\tilde{R}^k_t}{R^d_{t-1}} - 1 \right\} \right] \\
\bar{\omega}(e)_t : \quad & \eta_t = \frac{\Gamma'_{t-1}(\bar{\omega}(e)_t)}{\Gamma'_{t-1}(\bar{\omega}(e)_t) - \mu^e_{t-1} G'_{t-1}(\bar{\omega}(e)_t)} R^d_{t-1} N(e)_{t-1}
\end{aligned}$$

Substituting the second FOC into the first we obtain:

$$\begin{aligned}
E_{t-1} \Bigg[& [1 - \Gamma_{t-1}(\bar{\omega}_t)] \frac{\tilde{R}^k_t}{R^d_{t-1}} + \frac{\Gamma'_{t-1}(\bar{\omega}_t)}{\Gamma'_{t-1}(\bar{\omega}_t) - \mu^e_{t-1} G'_{t-1}(\bar{\omega}_t)} \\
& \left\{ \left[\Gamma_{t-1}(\bar{\omega}_t) - \mu^e_{t-1} G_{t-1}(\bar{\omega}_t)\right] \frac{\tilde{R}^k_t}{R^d_{t-1}} - 1 \right\} \Bigg] = 0.
\end{aligned} \tag{2.9}$$

where we omit the the indicator (e) since the condition implies that $\bar{\omega}(e)_t$ only depends on aggregate variables and is the same across entrepreneurs. From the zero profits condition 2.8 this implies that leverage $\varrho(e)_{t-1}$ is also the same, hence we can rewrite 2.8 as a function of aggregate variables only:

$$\left[\Gamma_{t-1}(\bar{\omega}_t) - \mu^e_{t-1} G_{t-1}(\bar{\omega}_t)\right] \frac{\tilde{R}^k_t}{R^d_{t-1}} = \frac{Q^k_{t-1} \bar{K}_{t-1} - N_{t-1}}{Q^k_{t-1} \bar{K}_{t-1}}. \tag{2.10}$$

Aggregate entrepreneurs' equity evolves according to:

$$\begin{aligned} V_t &= \int_{\bar{\omega}_t}^{\infty} \omega_t \tilde{R}_t^k Q_{t-1}^k \bar{K}(e)_{t-1} dF_{t-1}(\omega_t) - [1 - F_{t-1}(\bar{\omega}_t)] R^l(e)_t B^l(e)_{t-1} \\ &= \tilde{R}_t^k Q_{t-1}^k \bar{K}_{t-1} - \left[R_{t-1}^d + \mu_{t-1}^e G_{t-1}(\bar{\omega}_t) \tilde{R}_t^k \frac{Q_{t-1}^k \bar{K}_{t-1}}{Q_{t-1}^k \bar{K}_{t-1} - N_{t-1}} \right] \left(Q_{t-1}^k \bar{K}_{t-1} - N_{t-1} \right). \end{aligned} \tag{2.11}$$

A fraction $1-\gamma_t$ of entrepreneurs exits the economy and fraction γ_t survives to continue operating for another period. A fraction Θ of the total net worth owned by exiting entrepreneurs is consumed upon exit and the remaining fraction of their networth is transfered as a lump sum to the households. Each period new entrepreneurs enter and receive a net worth transfer W_t^e. Because W_t^e is small, this exit and entry process ensures that entrepreneurs do not accumulate enough net worth to escape the financial frictions. Aggregate entrepreneurs' net worth evolves accordingly as:

$$N_t = \gamma_t V_t + W_t^e. \tag{2.12}$$

2.0.5 Detrending and steady state

We detrend the additional variables introduced by this extension as follows:

$$q_t^k = \frac{Q_t^k}{P_t} \Upsilon^t, \ n_t = \frac{N_t}{P_t Z_t^*}, \ v_t = \frac{V_t}{P_t Z_t^*}, \ w_t^e = \frac{W_t^e}{P_t Z_t^*}. \tag{2.13}$$

All other variables are detrended as in 1.38. Expressions 2.4, 2.5, 2.6, 2.10, 2.11, and 2.12 become

$$\begin{aligned} &\xi_t q_t^k \mu_t \left(1 - S(\frac{i_t}{i_{t-1}} e^{z_t^*}) - S'(\frac{i_t}{i_{t-1}} e^{z_t^*}) \frac{i_t}{i_{t-1}} e^{z_t^*} \right) \\ &\quad + \beta \mathbb{E}_t [e^{-\sigma_c z_{t+1}^*} \xi_{t+1} q_{t+1}^k \mu_{t+1} S'(\frac{i_{t+1}}{i_t} e^{z_{t+1}^*})(\frac{i_{t+1}}{i_t} e^{z_{t+1}^*})^2] = \xi_t \end{aligned} \tag{2.14}$$

$$\tilde{R}_t^k = \frac{r_t^k u_t + (1-\delta) q_t^k - a(u_t)}{q_{t-1}^k \Upsilon} \pi_t \tag{2.15}$$

$$r_t^k = a'(u_t) \tag{2.16}$$

$$\bar{\omega}_t \tilde{R}_t^k = R_t^l \frac{q_{t-1}^k \bar{k}_{t-1} - n_{t-1}}{q_{t-1}^k \bar{k}_{t-1}} \tag{2.17}$$

$$\left[\Gamma_{t-1}(\bar{\omega}_t) - \mu_{t-1}^e G_{t-1}(\bar{\omega}_t) \right] \frac{\tilde{R}_t^k}{R_{t-1}^d} = \frac{q_{t-1}^k \bar{k}_{t-1} - n_{t-1}}{q_{t-1}^k \bar{k}_{t-1}} \tag{2.18}$$

$$v_t e^{z_t^*}\pi_t = \tilde{R}_t^k q_{t-1}^k \bar{k}_{t-1} - \left[R_{t-1}^d + \mu_{t-1}^e G_{t-1}(\bar{\omega}_t)\tilde{R}_t^k \frac{q_{t-1}^k \bar{k}_{t-1}}{q_{t-1}^k \bar{k}_{t-1} - n_{t-1}}\right]\left(q_{t-1}^k \bar{k}_{t-1} - n_{t-1}\right) \tag{2.19}$$

$$n_t = \gamma_t v_t + w_t^e. \tag{2.20}$$

Expression 2.9 is already expressed in terms of detrended variables. Note that expression 2.17 pins down R^l given the other variables, as R^l does not enter anywhere else. Therefore, unless we are interested in R^l, we can ignore this condition. Note also that expression 2.14 is the same as 1.58.

The steady state relationships are:

$$\xi_* q_*^k \mu \left(1 - S(e^{z_*^*}) - S'(e^{z_*^*})e^{z_*^*}\right) + \beta e^{-\sigma_c z_*^*}\xi_* q_*^k \mu S'(e^{z_*^*})(e^{z_*^*})^2 = \xi_* \tag{2.21}$$

which implies since $S(.) = S'(.) = 0$ at steady state that $q_*^k = 1$. We also parameterize a(.) so that $u_* = 1$ and $a(u_*) = 0$. With this information, and after some simplification, we can rewrite the remaining steady state equations as

$$\frac{\tilde{R}_*^k}{\pi_*} = \frac{r_*^k + (1-\delta)}{\Upsilon} \quad \text{(from 2.15)} \tag{2.22}$$

$$\frac{\tilde{R}_*^k}{R_*^d} = \Psi(\bar{\omega}_*, \sigma_{\omega *}, \mu_*^e) \quad \text{(from 2.9)} \tag{2.23}$$

$$\frac{n_*}{\bar{k}_*} = 1 - \left[\Gamma_*(\bar{\omega}_*) - \mu_*^e G_*(\bar{\omega}_*)\right]\frac{\tilde{R}_*^k}{R_*^d} \quad \text{(from 2.18)} \tag{2.24}$$

$$\left(1 - \gamma_* \frac{R_*^d}{\pi_* e^{z_*^*}}\right)\frac{n_*}{\bar{k}_*} = \gamma_* \frac{R_*^d}{\pi_* e^{z_*^*}}\left\{\frac{\tilde{R}_*^k}{R_*^d}\left[1 - \mu_*^e G_*(\bar{\omega}_*)\right] - 1\right\} + \frac{w_*^e}{\bar{k}_*} \quad \text{(from 2.19 and 2.20)} \tag{2.25}$$

$$v_* = \gamma_*^{-1}(n_* - w_*^e). \quad \text{(from 2.20)} \tag{2.26}$$

with

$$\begin{aligned}\Psi(\bar{\omega}_*, \sigma_{\omega *}, \mu_*^e) \quad &\equiv \quad \frac{\Gamma_*'(\bar{\omega}_*)}{\left[1 - \Gamma_*(\bar{\omega}_*)\right]\left[\Gamma_*'(\bar{\omega}_*) - \mu_*^e G_*'(\bar{\omega}_*)\right] + \Gamma_*'(\bar{\omega}_*)\left[\Gamma_*(\bar{\omega}_*) - \mu_*^e G_*(\bar{\omega}_*)\right]} \\ &= \quad \frac{1}{1 - \mu_*^e \frac{G_*'(\bar{\omega}_*)}{\Gamma_*'(\bar{\omega}_*)}\left[1 - \Gamma_*(\bar{\omega}_*)\right] - \mu_*^e G_*(\bar{\omega}_*)}\end{aligned} \tag{2.27}$$

and where $\dfrac{R_*^d}{\pi_* e^{z_*^*}} = \beta^{-1} e^{(\sigma_c - 1) z_*^*}$.

Our strategy for computing the steady state is going to be the following: find a solution for the real return to capital $\frac{\tilde{R}^k_*}{\pi_*}$ and use 2.22 to find r^k_*:

$$r^k_* = \Upsilon \frac{\tilde{R}^k_*}{\pi_*} - (1-\delta). \tag{2.28}$$

Once we have r^k_* we can proceed exactly as in section 1.3 to find the steady state for the other variables. In absence of financial friction $\frac{R^d_*}{\pi_*}$ and $\frac{\tilde{R}^k_*}{\pi_*}$ would be identical, but frictions induce a spread between the two, which we will compute subsequently as a function of the primitives in the economy $(\sigma^2_{\omega,*},\ \mu^e_*,\ \gamma_*,\ w^e_*)$.

We solve for the steady state according to the following steps:

1. Set

   $$F_*\left(\bar{\omega}_*\right) = \bar{F}_* \tag{2.29}$$

   and define

   $$z^\omega_* \equiv \frac{\ln \bar{\omega}_* + \frac{1}{2}\sigma^2_{\omega *}}{\sigma_{\omega *}} = \Phi^{-1}\left(\bar{F}_*\right) \tag{2.30}$$

   which we can use to write

   $$\bar{\omega}\left(\sigma_{\omega *}\right) = \exp\left\{\sigma_{\omega *} z^\omega_* - \frac{1}{2}\sigma^2_{\omega *}\right\} \tag{2.31}$$

2. Given the value for the spread for debt contracts, R^l_*/R^d_*, we can use equation (2.17) to write

   $$\frac{\tilde{R}^k_*}{R^d_*} = \frac{R^l_*}{R^d_*} \frac{1 - \frac{n_*}{q^k_* \bar{k}_*}}{\bar{\omega}_*} \tag{2.32}$$

   (Note: this second step can be skipped if instead we calibrate/estimate $\tilde{R}^k_*/R^d_*$ directly.)

3. Given $\tilde{R}^k_*/R^d_*$, we can use (2.23) to write

   $$\begin{aligned}\left(\frac{\tilde{R}^k_*}{R^d_*}\right)^{-1} &= 1 - \mu^e_* \frac{G'_*\left(\bar{\omega}_*\right)}{\Gamma'_*\left(\bar{\omega}_*\right)}\left[1 - \Gamma_*\left(\bar{\omega}_*\right)\right] - \mu^e_* G_*\left(\bar{\omega}_*\right) \\ &= 1 - \mu^e_* \left\{\frac{G'_*\left(\bar{\omega}_*\right)}{\Gamma'_*\left(\bar{\omega}_*\right)}\left[1 - \Gamma_*\left(\bar{\omega}_*\right)\right] + G_*\left(\bar{\omega}_*\right)\right\}\end{aligned}$$

   which we can use to set

   $$\mu^e_*\left(\sigma_{\omega *}\right) = \frac{1 - \left(\frac{\tilde{R}^k_*}{R^d_*}\right)^{-1}}{\frac{G'_*\left(\bar{\omega}_*\right)}{\Gamma'_*\left(\bar{\omega}_*\right)}\left[1 - \Gamma_*\left(\bar{\omega}_*\right)\right] + G_*\left(\bar{\omega}_*\right)}$$

and plugging in the exact expressions we get

$$\mu_*^e\left(\sigma_\omega\right) = \frac{1 - \left(\frac{\tilde{R}_*^k}{R_*^d}\right)^{-1}}{\frac{1}{\sigma_{\omega *}} \frac{\phi(z_*^\omega)}{1-\bar{F}_*} \left\{1 - \Phi\left(z_*^\omega - \sigma_{\omega *}\right) - \bar{\omega}_*\left(1 - \bar{F}_*\right)\right\} + \Phi\left(z_*^\omega - \sigma_{\omega *}\right)} \tag{2.33}$$

4. Given the above and equation (2.24) we get

$$\frac{n_*}{\bar{k}_*}\left(\sigma_\omega\right) = 1 - \left\{\bar{\omega}_*\left[1 - \bar{F}_*\right] + \left(1 - \mu_*^e\right)\Phi\left(z_*^\omega - \sigma_{\omega *}\right)\right\}\frac{\tilde{R}_*^k}{R_*^d} \tag{2.34}$$

5. Given the elasticity of the spread w.r.t. leverage, $\zeta_{sp,b}$, derived below in equation (2.42), we get the following expression

$$\frac{\frac{\frac{1-\Phi\left(z_*^\omega - \sigma_{\omega *}\right)}{1-\bar{F}_*} - \bar{\omega}_*}{\bar{\omega}_* + (1-\mu_*^e)\frac{\Phi\left(z_*^\omega - \sigma_{\omega *}\right)}{1-\bar{F}_*}}\left[1 - \frac{\mu_*^e}{\sigma_{\omega *}}\frac{\phi(z_*^\omega)}{1-\bar{F}_*}\right] + 1}{\frac{\mu_*^e}{\bar{\omega}_*\sigma_{\omega *}^2}\frac{\frac{\phi(z_*^\omega)}{1-\bar{F}_*} - z_*^\omega}{\left[1 - \frac{\mu_*^e}{\sigma_{\omega *}}\frac{\phi(z_*^\omega)}{1-\bar{F}_*}\right]^2}\frac{\phi(z_*^\omega)}{\left(1-\bar{F}_*\right)^2}}\frac{\frac{\tilde{R}_*^k}{R_*^d}}{\frac{n_*}{\bar{k}_*}} = -1 - \frac{\zeta_{sp,b}^{-1}}{\left(\frac{n_*}{\bar{k}_*}\right)^{-1} - 1} \tag{2.35}$$

which we can solve for $\sigma_{\omega *}$. Once we find this value we can plug back into the previous expressions, that depend on $\sigma_{\omega *}$.

6. Given γ_*, and using equation (2.25) we get

$$\frac{w_*^e}{\bar{k}_*} = \left(1 - \gamma_*\frac{R_*^d}{\pi_* e^{z_*^*}}\right)\frac{n_*}{\bar{k}_*} - \gamma_*\frac{R_*^d}{\pi_* e^{z_*^*}}\left\{\frac{\tilde{R}_*^k}{R_*^d}\left[1 - \mu_*^e\Phi\left(z_*^\omega - \sigma_{\omega *}\right)\right] - 1\right\} \tag{2.36}$$

and from equation (2.26)

$$\frac{v_*}{\bar{k}_*} = \gamma_*^{-1}\left(\frac{n_*}{\bar{k}_*} - \frac{w_*^e}{\bar{k}_*}\right) \tag{2.37}$$

7. We get r_*^k using equation (2.22) to write

$$r_*^k = \Upsilon\frac{\tilde{R}_*^k}{\pi_*} - (1 - \delta) \tag{2.38}$$

2.0.6 Log-linearization

Log-linearization of the FOC w.r.t. leverage (expression 2.9) yields:

$$0 = E_t\left(\hat{\tilde{R}}_{t+1}^k - \hat{R}_t^d\right) + \zeta_{b,\bar{\omega}}E_t\hat{\bar{\omega}}_{t+1} + \zeta_{b,\sigma_\omega}\hat{\sigma}_{\omega,t} + \zeta_{b,\mu^e}\hat{\mu}_t^e \tag{2.39}$$

with

$$\zeta_{b,x} \equiv \frac{\frac{\partial}{\partial x}\left[\left\{[1-\Gamma(\bar{\omega})]+\frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega})-\mu^e G'(\bar{\omega})}\left[\Gamma(\bar{\omega})-\mu^e G(\bar{\omega})\right]\right\}\frac{\tilde{R}^k_*}{R^d_*}-\frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega})-\mu^e G'(\bar{\omega})}\right]}{\left\{[1-\Gamma_*(\bar{\omega}_*)]+\frac{\Gamma'_*(\bar{\omega}_*)}{\Gamma'_*(\bar{\omega}_*)-\mu^e_* G'_*(\bar{\omega}_*)}\left[\Gamma_*(\bar{\omega}_*)-\bar{\mu}^e G_*(\bar{\omega}_*)\right]\right\}\frac{\tilde{R}^k_*}{R^d_*}}\, x$$

defined for $x \in \left\{\bar{\omega}, \sigma^2_\omega, \mu^e\right\}$. Log-linearization of the zero profit condition (expression 2.18) yields:

$$\hat{\tilde{R}}^k_t - \hat{R}^d_{t-1} + \zeta_{z,\bar{\omega}}\hat{\bar{\omega}}_t + \zeta_{z,\sigma_\omega}\hat{\sigma}_{\omega,t-1} + \zeta_{z,\mu^e}\hat{\mu}^e_{t-1} = -\left(\varrho_*\right)^{-1}\left(\hat{n}_{t-1} - \hat{q}^k_{t-1} - \hat{\bar{k}}_{t-1}\right) \tag{2.40}$$

with

$$\zeta_{z,x} \equiv \frac{\frac{\partial}{\partial x}\left[\Gamma(\bar{\omega}) - \mu^e G(\bar{\omega})\right]}{\Gamma_*(\bar{\omega}_*) - \mu^e_* G_*(\bar{\omega}_*)}\, x \tag{2.41}$$

defined for $x \in \left\{\bar{\omega}, \sigma^2_\omega, \mu^e\right\}$. We can further write

$$\hat{\bar{\omega}}_t = -\frac{1}{\zeta_{z,\bar{\omega}}\varrho_*}\left(\hat{n}_{t-1} - \hat{q}^k_{t-1} - \hat{\bar{k}}_{t-1}\right) - \frac{1}{\zeta_{z,\bar{\omega}}}\left(\hat{\tilde{R}}^k_t - \hat{R}^d_{t-1} + \zeta_{z,\sigma_\omega}\hat{\sigma}_{\omega,t-1} + \zeta_{z,\mu^e}\hat{\mu}^e_{t-1}\right)$$

and plug this expression into 2.39 to obtain:

$$\begin{aligned} 0 \;=\;& E_t\left[\hat{\tilde{R}}^k_{t+1} - \hat{R}^d_t\right] + \zeta_{b,\sigma_\omega}\hat{\sigma}_{\omega,t} + \zeta_{b,\mu^e}\hat{\mu}^e_t \\ & -\frac{\zeta_{b,\bar{\omega}}}{\zeta_{z,\bar{\omega}}}\left[\frac{1}{\varrho_*}\left(\hat{n}_t - \hat{q}^k_t - \hat{\bar{k}}_t\right) + E_t\left[\hat{\tilde{R}}^k_{t+1} - \hat{R}^d_t\right] + \zeta_{z,\sigma_\omega}\hat{\sigma}_{\omega,t} + \zeta_{z,\mu^e}\hat{\mu}^e_t\right] \end{aligned}$$

hence

$$E_t\left[\hat{\tilde{R}}^k_{t+1} - \hat{R}^d_t\right] = \zeta_{sp,b}\left(\hat{q}^k_t + \hat{\bar{k}}_t - \hat{n}_t\right) + \zeta_{sp,\sigma_\omega}\hat{\sigma}_{\omega,t} + \zeta_{sp,\mu^e}\hat{\mu}^e_t \tag{2.42}$$

where

$$\begin{aligned} \zeta_{sp,b} &\equiv -\frac{\frac{\zeta_{b,\bar{\omega}}}{\zeta_{z,\bar{\omega}}}}{1 - \frac{\zeta_{b,\bar{\omega}}}{\zeta_{z,\bar{\omega}}}}\frac{1}{\varrho_*} \\ \zeta_{sp,\sigma_\omega} &\equiv \frac{\frac{\zeta_{b,\bar{\omega}}}{\zeta_{z,\bar{\omega}}}\zeta_{z,\sigma_\omega} - \zeta_{b,\sigma_\omega}}{1 - \frac{\zeta_{b,\bar{\omega}}}{\zeta_{z,\bar{\omega}}}} \\ \zeta_{sp,\mu^e} &\equiv \frac{\frac{\zeta_{b,\bar{\omega}}}{\zeta_{z,\bar{\omega}}}\zeta_{z,\mu^e} - \zeta_{b,\mu^e}}{1 - \frac{\zeta_{b,\bar{\omega}}}{\zeta_{z,\bar{\omega}}}} \end{aligned}$$

Log-linearization of the expression 2.20, characterizing net worth, yields:

$$\hat{n}_t = \gamma_*\frac{v_*}{n_*}\left(\hat{\gamma}_t + \hat{v}_t\right) + \frac{w^e_*}{n_*}\hat{w}^e_t. \tag{2.43}$$

Log-linearization of the expression 2.19, characterizing the evolution of entrepreneurial equity, is

$$
\begin{aligned}
\hat{v}_t \;=\; & -\hat{z}_t - \frac{R_*^d}{\pi_* e^{z_*^*}} \frac{\bar{k}_* - n_*}{v_*} \left(\hat{R}_{t-1}^d - \pi_t \right) + \frac{\tilde{R}_*^k}{\pi_* e^{z_*^*}} \frac{\bar{k}_*}{v_*} (1 - \mu_*^e G_* (\bar{\omega}_*)) \left(\hat{\tilde{R}}_t^k - \pi_t \right) + \frac{R_*^d}{\pi_* e^{z_*^*}} \frac{n_*}{v_*} \hat{n}_{t-1} \\
& \left(\frac{\tilde{R}_*^k}{\pi_* e^{z_*^*}} (1 - \mu_*^e G_* (\bar{\omega}_*)) - \frac{R_*^d}{\pi_* e^{z_*^*}} \right) \frac{\bar{k}_*}{v_*} \left(\hat{q}_{t-1}^k + \hat{\bar{k}}_{t-1} \right) \\
& -\mu_*^e G_* (\bar{\omega}_*) \frac{\tilde{R}_*^k}{\pi_* e^{z_*^*}} \frac{\bar{k}_*}{v_*} \left[\hat{\mu}_{t-1}^e + \zeta_{G,\bar{\omega}} \hat{\bar{\omega}}_t + \zeta_{G,\sigma_\omega} \hat{\sigma}_{\omega,t-1} \right],
\end{aligned}
\tag{2.44}
$$

where we used the fact that at steady state $v_* e^{z_*^*} \pi_* = \tilde{R}_*^k \bar{k}_* - R_*^d (\bar{k}_* - n_*) - \mu_*^e G_* (\bar{\omega}_*) \tilde{R}_*^k \bar{k}_*$. Plugging in the expression for $\hat{\bar{\omega}}_t$ we obtain

$$
\begin{aligned}
\hat{v}_t \;=\; & -\hat{z}_t - \frac{R_*^d}{\pi_* e^{z_*^*}} \frac{\bar{k}_* - n_*}{v_*} \left(\hat{R}_{t-1}^d - \pi_t \right) + \frac{\tilde{R}_*^k}{\pi_* e^{z_*^*}} \frac{\bar{k}_*}{v_*} (1 - \mu_*^e G_* (\bar{\omega}_*)) \left(\hat{\tilde{R}}_t^k - \pi_t \right) + \frac{R_*^d}{\pi_* e^{z_*^*}} \frac{n_*}{v_*} \hat{n}_{t-1} \\
& \left(\frac{\tilde{R}_*^k}{\pi_* e^{z_*^*}} (1 - \mu_*^e G_* (\bar{\omega}_*)) - \frac{R_*^d}{\pi_* e^{z_*^*}} \right) \frac{\bar{k}_*}{v_*} \left(\hat{q}_{t-1}^k + \hat{\bar{k}}_{t-1} \right) - \mu_*^e G_* (\bar{\omega}_*) \frac{\tilde{R}_*^k}{\pi_* e^{z_*^*}} \frac{\bar{k}_*}{v_*} \left[\hat{\mu}_{t-1}^e + \zeta_{G,\sigma_\omega} \hat{\sigma}_{\omega,t-1} \right] \\
& -\mu_*^e G_* (\bar{\omega}_*) \frac{\tilde{R}_*^k}{\pi_* e^{z_*^*}} \frac{\bar{k}_*}{v_*} \zeta_{G,\bar{\omega}} \Big[- \frac{1}{\zeta_{z,\bar{\omega}} \varrho_*} \left(\hat{n}_{t-1} - \hat{q}_{t-1}^k - \hat{\bar{k}}_{t-1} \right) \\
& - \frac{1}{\zeta_{z,\bar{\omega}}} \left(\hat{\tilde{R}}_t^k - \hat{R}_{t-1}^d + \zeta_{z,\sigma_\omega} \hat{\sigma}_{\omega,t-1} + \zeta_{z,\mu^e} \hat{\mu}_{t-1}^e \right) \Big]
\end{aligned}
$$

Collecting terms yields

$$
\begin{aligned}
\hat{v}_t \;=\; & -\hat{z}_t + \zeta_{v,\tilde{R}^k} \left(\hat{\tilde{R}}_t^k - \pi_t \right) - \zeta_{v,R} \left(\hat{R}_{t-1}^d - \pi_t \right) + \zeta_{v,qK} \left(\hat{q}_{t-1}^k + \hat{\bar{k}}_{t-1} \right) + \zeta_{v,n} \hat{n}_{t-1} \\
& -\zeta_{v,\mu^e} \hat{\mu}_{t-1}^e - \zeta_{v,\sigma_\omega} \hat{\sigma}_{\omega,t-1}
\end{aligned}
\tag{2.45}
$$

with

$$
\begin{aligned}
\zeta_{v,\tilde{R}^k} &\equiv \frac{\tilde{R}_*^k}{\pi_* e^{z_*^*}} \frac{\bar{k}_*}{v_*} \left[1 - \mu_*^e G_* (\bar{\omega}_*) \left(1 - \frac{\zeta_{G,\bar{\omega}}}{\zeta_{z,\bar{\omega}}} \right) \right] \\
\zeta_{v,R} &\equiv \frac{R_*^d}{\pi_* e^{z_*^*}} \frac{\bar{k}_*}{v_*} \left[1 - \frac{n_*}{\bar{k}_*} + \mu_*^e G_* (\bar{\omega}_*) \frac{\tilde{R}_*^k}{R_*^d} \frac{\zeta_{G,\bar{\omega}}}{\zeta_{z,\bar{\omega}}} \right] \\
\zeta_{v,qK} &\equiv \frac{\tilde{R}_*^k}{\pi_* e^{z_*^*}} \frac{\bar{k}_*}{v_*} \left[1 - \mu_*^e G_* (\bar{\omega}_*) \left(1 + \frac{\zeta_{G,\bar{\omega}}}{\zeta_{z,\bar{\omega}} \varrho_*} \right) \right] - \frac{R_*^d}{\pi_* e^{z_*^*}} \frac{\bar{k}_*}{v_*} \\
\zeta_{v,n} &\equiv \frac{R_*^d}{\pi_* e^{z_*^*}} \frac{n_*}{v_*} + \frac{\tilde{R}_*^k}{\pi_* e^{z_*^*}} \frac{\bar{k}_*}{v_*} \mu_*^e G_* (\bar{\omega}_*) \frac{\zeta_{G,\bar{\omega}}}{\zeta_{z,\bar{\omega}} \varrho_*} \\
\zeta_{v,\mu^e} &\equiv \mu_*^e G_* (\bar{\omega}_*) \frac{\tilde{R}_*^k}{\pi_* e^{z_*^*}} \frac{\bar{k}_*}{v_*} \left(1 - \zeta_{G,\bar{\omega}} \frac{\zeta_{z,\mu^e}}{\zeta_{z,\bar{\omega}}} \right) \\
\zeta_{v,\sigma_\omega} &\equiv \mu_*^e G_* (\bar{\omega}_*) \frac{\tilde{R}_*^k}{\pi_* e^{z_*^*}} \frac{\bar{k}_*}{v_*} \zeta_{G,\bar{\omega}} \left(\frac{\zeta_{G,\sigma_\omega}}{\zeta_{G,\bar{\omega}}} - \frac{\zeta_{z,\sigma_\omega}}{\zeta_{z,\bar{\omega}}} \right)
\end{aligned}
$$

Finally, substituting this expression into 2.43 we get:

$$\begin{aligned}\hat{n}_t &= \gamma_* \frac{v_*}{n_*}\hat{\gamma}_t + \frac{w_*^e}{n_*}\hat{w}_t^e - \gamma_* \frac{v_*}{n_*}\hat{z}_t \\ &+\zeta_{n,\tilde{R}^k}\left(\hat{\tilde{R}}_t^k - \pi_t\right) - \zeta_{n,R}\left(\hat{R}_{t-1}^d - \pi_t\right) + \zeta_{n,qK}\left(\hat{q}_{t-1}^k + \hat{\bar{k}}_{t-1}\right) + \zeta_{n,n}\hat{n}_{t-1} \\ &-\zeta_{n,\mu^e}\hat{\mu}_{t-1}^e - \zeta_{n,\sigma_\omega}\hat{\sigma}_{\omega,t-1}\end{aligned} \tag{2.46}$$

with

$$\begin{aligned}\zeta_{n,\tilde{R}^k} &\equiv \gamma_* \frac{\tilde{R}_*^k}{\pi_* e^{z_*^*}}(1+\varrho_*)\left[1 - \mu_*^e G_*(\bar{\omega}_*)\left(1 - \frac{\zeta_{G,\bar{\omega}}}{\zeta_{z,\bar{\omega}}}\right)\right] \\ \zeta_{n,R} &\equiv \gamma_* \frac{R_*^d}{\pi_* e^{z_*^*}}(1+\varrho_*)\left[1 - \frac{n_*}{\bar{k}_*} + \mu_*^e G_*(\bar{\omega}_*)\frac{\tilde{R}_*^k}{R_*^d}\frac{\zeta_{G,\bar{\omega}}}{\zeta_{z,\bar{\omega}}}\right] \\ \zeta_{n,qK} &\equiv \gamma_* \frac{\tilde{R}_*^k}{\pi_* e^{z_*^*}}(1+\varrho_*)\left[1 - \mu_*^e G_*(\bar{\omega}_*)\left(1 + \frac{\zeta_{G,\bar{\omega}}}{\zeta_{z,\bar{\omega}}\varrho_*}\right)\right] - \gamma_* \frac{R_*^d}{\pi_* e^{z_*^*}}(1+\varrho_*) \\ \zeta_{n,n} &\equiv \gamma_* \frac{R_*^d}{\pi_* e^{z_*^*}} + \gamma_* \frac{\tilde{R}_*^k}{\pi_* e^{z_*^*}}(1+\varrho_*)\mu_*^e G_*(\bar{\omega}_*)\frac{\zeta_{G,\bar{\omega}}}{\zeta_{z,\bar{\omega}}\varrho_*} \\ \zeta_{n,\mu^e} &\equiv \gamma_* \mu_*^e G_*(\bar{\omega}_*)\frac{\tilde{R}_*^k}{\pi_* e^{z_*^*}}(1+\varrho_*)\left(1 - \zeta_{G,\bar{\omega}}\frac{\zeta_{z,\mu^e}}{\zeta_{z,\bar{\omega}}}\right) \\ \zeta_{n,\sigma_\omega} &\equiv \gamma_* \mu_*^e G_*(\bar{\omega}_*)\frac{\tilde{R}_*^k}{\pi_* e^{z_*^*}}(1+\varrho_*)\zeta_{G,\bar{\omega}}\left(\frac{\zeta_{G,\sigma_\omega}}{\zeta_{G,\bar{\omega}}} - \frac{\zeta_{z,\sigma_\omega}}{\zeta_{z,\bar{\omega}}}\right)\end{aligned}$$

Now normalize the shocks,

$$\tilde{\sigma}_{\omega,t} \equiv \zeta_{sp,\sigma_\omega}\hat{\sigma}_{\omega,t} \tag{2.47}$$

$$\tilde{\mu}_t^e \equiv \zeta_{sp,\mu^e}\hat{\mu}_t^e \tag{2.48}$$

$$\tilde{\gamma}_t \equiv \gamma_* \frac{v_*}{n_*}\hat{\gamma}_t \tag{2.49}$$

so that the relevant log-linear equations, (2.42) and (2.46), become:

$$E_t\left[\hat{\tilde{R}}_{t+1}^k - \hat{R}_t^d\right] = \zeta_{sp,b}\left(\hat{q}_t^k + \hat{\bar{k}}_t - \hat{n}_t\right) + \tilde{\sigma}_{\omega,t} + \tilde{\mu}_t^e \tag{2.50}$$

and

$$\begin{aligned}\hat{n}_t &= \zeta_{n,\tilde{R}^k}\left(\hat{\tilde{R}}_t^k - \pi_t\right) - \zeta_{n,R}\left(\hat{R}_{t-1}^d - \pi_t\right) + \zeta_{n,qK}\left(\hat{q}_{t-1}^k + \hat{\bar{k}}_{t-1}\right) + \zeta_{n,n}\hat{n}_{t-1} \\ &+\tilde{\gamma}_t + \frac{w_*^e}{n_*}\hat{w}_t^e - \gamma_* \frac{v_*}{n_*}\hat{z}_t - \frac{\zeta_{n,\mu^e}}{\zeta_{sp,\mu^e}}\tilde{\mu}_{t-1}^e - \frac{\zeta_{n,\sigma_\omega}}{\zeta_{sp,\sigma_\omega}}\tilde{\sigma}_{\omega,t-1}\end{aligned} \tag{2.51}$$

Log-linearization of 2.15 and 2.14 yield:

$$\hat{\tilde{R}}_t^k - \pi_t = \frac{r_*^k}{r_*^k + (1-\delta)}\hat{r}_t^k + \frac{(1-\delta)}{r_*^k + (1-\delta)}\hat{q}_t^k - \hat{q}_{t-1}^k, \tag{2.52}$$

and 1.93.

2.0.7 Log-linear distribution

Consider

$$\ln \omega \sim N\left(m_\omega, \sigma_\omega^2\right) \tag{2.53}$$

which has the properties

$$E[\omega] = e^{m_\omega + \frac{1}{2}\sigma_\omega^2} \tag{2.54}$$

In order to get $E[\omega] = 1$ we need to set

$$m_\omega = -\frac{1}{2}\sigma_\omega^2 \tag{2.55}$$

This implies that the pdf is

$$f(\omega) = \frac{1}{\omega\sigma_\omega\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln\omega + \frac{1}{2}\sigma_\omega^2}{\sigma_\omega}\right)^2} \tag{2.56}$$

Define

$$\phi(z) \equiv \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \tag{2.57}$$

$$\Phi(z) \equiv \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \tag{2.58}$$

for which we can use matlab functions `normpdf` and `normcdf`. Given that $Pr[\omega \leq \bar{\omega}] = Pr[\log\omega \leq \log\bar{\omega}]$ the CDF is

$$F(\bar{\omega}) = \Phi\left(\frac{\ln\bar{\omega} + \frac{1}{2}\sigma_\omega^2}{\sigma_\omega}\right) \tag{2.59}$$

We also need the following expression

$$z = \Phi^{-1}\left(\bar{F}\right) \tag{2.60}$$

for which we can use an inverse cdf function also available in matlab as `norminv`.

The partial expectation obeys

$$E[\omega|\omega > \bar{\omega}] = \Phi\left(\frac{\frac{1}{2}\sigma_\omega^2 - \ln\bar{\omega}}{\sigma_\omega}\right) = 1 - \Phi\left(\frac{\ln\bar{\omega} - \frac{1}{2}\sigma_\omega^2}{\sigma_\omega}\right)$$

which implies that

$$
\begin{aligned}
G(\bar{\omega}) &\equiv \int_0^{\bar{\omega}} \omega f(\omega)\, d\omega = \int_0^{\infty} \omega f(\omega)\, d\omega - \int_{\bar{\omega}}^{\infty} \omega f(\omega)\, d\omega \\
&= \Phi\left(\frac{\ln\bar{\omega} - \frac{1}{2}\sigma_\omega^2}{\sigma_\omega}\right)
\end{aligned}
\tag{2.61}
$$

Finally we define

$$
\begin{aligned}
\Gamma(\bar{\omega}) &\equiv \int_0^{\bar{\omega}} \omega f(\omega)\, d\omega + \bar{\omega}\int_{\bar{\omega}}^{\infty} f(\omega)\, d\omega \\
&= \bar{\omega}\left[1 - \Phi\left(\frac{\ln\bar{\omega} + \frac{1}{2}\sigma_\omega^2}{\sigma_\omega}\right)\right] + G(\bar{\omega})
\end{aligned}
\tag{2.62}
$$

If we define

$$
z^\omega \equiv \frac{\ln\bar{\omega} + \frac{1}{2}\sigma_\omega^2}{\sigma_\omega} \tag{2.63}
$$

then we get

$$
G(\bar{\omega}) = \Phi(z^\omega - \sigma_\omega) \tag{2.64}
$$

and

$$
\Gamma(\bar{\omega}) = \bar{\omega}\left[1 - \Phi(z^\omega)\right] + \Phi(z^\omega - \sigma_\omega) \tag{2.65}
$$

In order to compute the derivatives, first notice that we can write

$$
\phi(z^\omega - \sigma_\omega) = \bar{\omega}\phi(z^\omega) \tag{2.66}
$$

and

$$
\phi'(z) = -z\phi(z)\,, \forall z \tag{2.67}
$$

Using this result we can write the derivatives as follows:

$$
G'(\bar{\omega}) = \frac{1}{\sigma_\omega}\phi(z^\omega) \tag{2.68}
$$

$$
G''(\bar{\omega}) = -\frac{z^\omega}{\bar{\omega}\sigma_\omega}G'(\bar{\omega}) = -\frac{z^\omega}{\bar{\omega}\sigma_\omega^2}\phi(z^\omega) \tag{2.69}
$$

$$
\Gamma'(\bar{\omega}) = \frac{\Gamma(\bar{\omega}) - G(\bar{\omega})}{\bar{\omega}} = 1 - \Phi(z^\omega) \tag{2.70}
$$

$$
\Gamma''(\bar{\omega}) = -\frac{1}{\bar{\omega}}G'(\bar{\omega};\sigma_\omega) = -\frac{1}{\bar{\omega}\sigma_\omega}\phi(z^\omega) \tag{2.71}
$$

and

$$\frac{\partial z^{\omega}}{\partial \sigma_{\omega}} = -\left(\frac{z^{\omega}}{\sigma_{\omega}} - 1\right) \tag{2.72}$$

$$G_{\sigma_{\omega}}(\bar{\omega}) = -\frac{z^{\omega}}{\sigma_{\omega}}\phi\left(z^{\omega} - \sigma_{\omega}\right) \tag{2.73}$$

$$G'_{\sigma_{\omega}}(\bar{\omega}) = -\frac{\phi\left(z^{\omega}\right)}{\sigma_{\omega}^{2}}\left[1 - z^{\omega}\left(z^{\omega} - \sigma_{\omega}\right)\right] \tag{2.74}$$

$$\Gamma_{\sigma_{\omega}}(\bar{\omega}) = -\phi\left(z^{\omega} - \sigma_{\omega}\right) \tag{2.75}$$

$$\Gamma'_{\sigma_{\omega}}(\bar{\omega}) = \left(\frac{z^{\omega}}{\sigma_{\omega}} - 1\right)\phi\left(z^{\omega}\right) \tag{2.76}$$

where we use notation $f'(\bar{\omega}) \equiv \partial f(\bar{\omega})/\partial\bar{\omega}$ and $f_{\sigma_{\omega}}(\bar{\omega}) \equiv \partial f(\bar{\omega})/\partial\sigma_{\omega}$, for $f \in \{G, \Gamma\}$.

2.0.8 Elasticities

First notice that we have several elasticities defined as

$$\zeta_{b,x} \equiv \frac{\frac{\partial}{\partial x}\left[\left\{1 - \Gamma(\bar{\omega}) + \frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega}) - \mu_{*}^{e}G'(\bar{\omega})}\left[\Gamma(\bar{\omega}) - \mu_{*}^{e}G(\bar{\omega})\right]\right\}\frac{\tilde{R}_{*}^{k}}{R_{*}^{d}} - \frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega}) - \mu_{*}^{e}G'(\bar{\omega})}\right]}{\left\{1 - \Gamma(\bar{\omega}) + \frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega}) - \mu_{*}^{e}G'(\bar{\omega})}\left[\Gamma(\bar{\omega}) - \mu_{*}^{e}G(\bar{\omega})\right]\right\}\frac{\tilde{R}_{*}^{k}}{R_{*}^{d}}}x$$

which we can rewrite as

$$\zeta_{b,x} \equiv \frac{\frac{\partial\tilde{\Psi}}{\partial x}x}{\left\{1 - \Gamma(\bar{\omega}) + \frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega}) - \mu_{*}^{e}G'(\bar{\omega})}\left[\Gamma(\bar{\omega}) - \mu_{*}^{e}G(\bar{\omega})\right]\right\}\frac{\tilde{R}_{*}^{k}}{R_{*}^{d}}}$$

with

$$\begin{aligned}\tilde{\Psi} &\equiv \left\{1 - \Gamma(\bar{\omega}) + \frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega}) - \mu_{*}^{e}G'(\bar{\omega})}\left[\Gamma(\bar{\omega}) - \mu_{*}^{e}G(\bar{\omega})\right]\right\}\frac{\tilde{R}_{*}^{k}}{R_{*}^{d}} - \frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega}) - \mu_{*}^{e}G'(\bar{\omega})} \\ &= \left[1 - \Gamma(\bar{\omega})\right]\frac{\tilde{R}_{*}^{k}}{R_{*}^{d}} + \frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega}) - \mu_{*}^{e}G'(\bar{\omega})}\left[\left[\Gamma(\bar{\omega}) - \mu_{*}^{e}G(\bar{\omega})\right]\frac{\tilde{R}_{*}^{k}}{R_{*}^{d}} - 1\right]\end{aligned} \tag{2.77}$$

Elasticities w.r.t. $\bar{\omega}$

First write

$$\begin{aligned}\frac{\partial\tilde{\Psi}}{\partial\bar{\omega}} &= -\Gamma'_{*}(\bar{\omega}_{*})\frac{\tilde{R}_{*}^{k}}{R_{*}^{d}} \\ &\quad + \left[\left[\Gamma_{*}(\bar{\omega}_{*}) - \mu_{*}^{e}G_{*}(\bar{\omega}_{*})\right]\frac{\tilde{R}_{*}^{k}}{R_{*}^{d}} - 1\right]\frac{\Gamma''(\bar{\omega})\left[\Gamma'(\bar{\omega}) - \mu_{*}^{e}G'(\bar{\omega})\right] - \Gamma'(\bar{\omega})\left[\Gamma''(\bar{\omega}) - \mu_{*}^{e}G''(\bar{\omega})\right]}{\left[\Gamma'(\bar{\omega}) - \mu_{*}^{e}G'(\bar{\omega})\right]^{2}} \\ &\quad + \frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega}) - \mu_{*}^{e}G'(\bar{\omega})}\left[\Gamma'(\bar{\omega}) - \mu_{*}^{e}G'(\bar{\omega})\right]\frac{\tilde{R}_{*}^{k}}{R_{*}^{d}}\end{aligned}$$

and simplify to

$$
\begin{aligned}
\frac{\partial \tilde{\Psi}}{\partial \bar{\omega}} &= \mu_*^e \left\{ \left[\Gamma_*\left(\bar{\omega}_*\right) - \mu_*^e G_*\left(\bar{\omega}_*\right)\right] \frac{\tilde{R}_*^k}{R_*^d} - 1 \right\} \frac{G''\left(\bar{\omega}\right)\Gamma'\left(\bar{\omega}\right) - G'\left(\bar{\omega}\right)\Gamma''\left(\bar{\omega}\right)}{\left[\Gamma'\left(\bar{\omega}\right) - \mu_*^e G'\left(\bar{\omega}\right)\right]^2} \\
&= \mu_*^e \frac{n_*}{\bar{k}_*} \frac{\Gamma''\left(\bar{\omega}\right) G'\left(\bar{\omega}\right) - G''\left(\bar{\omega}\right)\Gamma'\left(\bar{\omega}\right)}{\left[\Gamma'\left(\bar{\omega}\right) - \mu_*^e G'\left(\bar{\omega}\right)\right]^2}
\end{aligned}
$$

where we used 2.24. We can plug this expression into the elasticity to obtain

$$
\zeta_{b,\bar{\omega}} = \frac{\mu_*^e \frac{n_*}{\bar{k}_*} \frac{\Gamma_*''(\bar{\omega}_*) G_*'(\bar{\omega}_*) - G_*''(\bar{\omega}_*)\Gamma_*'(\bar{\omega}_*)}{\left[\Gamma_*'(\bar{\omega}_*) - \mu_*^e G_*'(\bar{\omega}_*)\right]^2}}{\left\{1 - \Gamma_*\left(\bar{\omega}_*\right) + \Gamma_*'\left(\bar{\omega}_*\right) \frac{\Gamma_*(\bar{\omega}_*) - \mu_*^e G_*(\bar{\omega}_*)}{\Gamma_*'(\bar{\omega}_*) - \mu_*^e G_*'(\bar{\omega}_*)}\right\} \frac{\tilde{R}_*^k}{R_*^d}} \bar{\omega}_* \tag{2.78}
$$

We also have from 2.41

$$
\zeta_{z,\bar{\omega}} \equiv \frac{\Gamma_*'\left(\bar{\omega}_*\right) - \mu_*^e G_*'\left(\bar{\omega}_*\right)}{\Gamma_*\left(\bar{\omega}_*\right) - \mu_*^e G_*\left(\bar{\omega}_*\right)} \bar{\omega}_* \tag{2.79}
$$

Notice that if we plug everything into

$$
\begin{aligned}
\zeta_{sp,b} = -\frac{\frac{\zeta_{b,\bar{\omega}}}{\zeta_{z,\bar{\omega}}}}{1 - \frac{\zeta_{b,\bar{\omega}}}{\zeta_{z,\bar{\omega}}}} \frac{\frac{n_*}{\bar{k}_*}}{1 - \frac{n_*}{\bar{k}_*}} &= \frac{\left(\frac{n_*}{\bar{k}_*} - 1\right)^{-1}}{1 - \frac{\zeta_{z,\bar{\omega}}}{\zeta_{b,\bar{\omega}}}} \\
&= \frac{\left(\frac{n_*}{\bar{k}_*} - 1\right)^{-1}}{1 - \frac{\Gamma'(\bar{\omega}) - \mu_*^e G'(\bar{\omega})}{\Gamma(\bar{\omega}) - \mu_*^e G(\bar{\omega})} \frac{\left\{1 - \Gamma(\bar{\omega}) + \Gamma'(\bar{\omega}) \frac{\Gamma(\bar{\omega}) - \mu_*^e G(\bar{\omega})}{\Gamma'(\bar{\omega}) - \mu_*^e G'(\bar{\omega})}\right\} \frac{\tilde{R}_*^k}{R_*^d}}{\mu_*^e \frac{\Gamma''(\bar{\omega}) G'(\bar{\omega}) - G''(\bar{\omega})\Gamma'(\bar{\omega})}{\left(\Gamma'(\bar{\omega}) - \mu_*^e G'(\bar{\omega})\right)^2} \frac{n_*}{\bar{k}_*}}}
\end{aligned} \tag{2.80}
$$

this becomes

$$
\zeta_{sp,b} = \frac{\frac{\frac{n_*}{\bar{k}_*}}{1 - \frac{n_*}{\bar{k}_*}}}{1 + \frac{\left\{\frac{\left[1 - \bar{\omega}_*\left[1 - \Phi\left(z_*^\omega\right)\right] - \Phi\left(z_*^\omega - \sigma_{\omega*}\right)\right]}{\bar{\omega}_*\left[1 - \Phi\left(z_*^\omega\right)\right] + \left(1 - \mu_*^e\right)\Phi\left(z_*^\omega - \sigma_{\omega*}\right)} \left[1 - \Phi\left(z_*^\omega\right) - \frac{\mu_*^e}{\sigma_{\omega*}} \phi\left(z_*^\omega\right)\right] + 1 - \Phi\left(z_*^\omega\right)\right\} \frac{\tilde{R}_*^k}{R_*^d}}{\frac{\mu_*^e \phi\left(z_*^\omega\right)}{\bar{\omega}_* \sigma_{\omega*}^2} \frac{\phi\left(z_*^\omega\right) - z_*^\omega\left[1 - \Phi\left(z_*^\omega\right)\right]}{\left[1 - \Phi\left(z_*^\omega\right) - \frac{\mu_*^e}{\sigma_{\omega*}} \phi\left(z_*^\omega\right)\right]^2} \frac{n_*}{\bar{k}_*}}} \tag{2.81}
$$

Elasticity of w.r.t. σ_ω

First we compute the derivative

$$
\begin{aligned}
\frac{\partial \tilde{\Psi}}{\partial \sigma_\omega} &= -\Gamma_{\sigma_\omega}\left(\bar{\omega}\right) \frac{\tilde{R}_*^k}{R_*^d} + \frac{\Gamma'\left(\bar{\omega}\right)}{\Gamma'\left(\bar{\omega}\right) - \mu_*^e G'\left(\bar{\omega}\right)} \left[\Gamma_{\sigma_\omega}\left(\bar{\omega}\right) - \mu_*^e G_{\sigma_\omega}\left(\bar{\omega}\right)\right] \frac{\tilde{R}_*^k}{R_*^d} \\
&\quad + \frac{\Gamma_{\sigma_\omega}'\left(\bar{\omega}\right)\left[\Gamma'\left(\bar{\omega}\right) - \mu_*^e G'\left(\bar{\omega}\right)\right] - \Gamma'\left(\bar{\omega}\right)\left[\Gamma_{\sigma_\omega}'\left(\bar{\omega}\right) - \mu_*^e G_{\sigma_\omega}'\left(\bar{\omega}\right)\right]}{\left[\Gamma'\left(\bar{\omega}\right) - \mu_*^e G'\left(\bar{\omega}\right)\right]^2} \left[\left[\Gamma\left(\bar{\omega}\right) - \mu_*^e G\left(\bar{\omega}\right)\right] \frac{\tilde{R}_*^k}{R_*^d} - 1\right]
\end{aligned}
$$

hence

$$
\frac{\partial \tilde{\Psi}}{\partial \sigma_\omega} = \left(\frac{1 - \mu_*^e \frac{G_{\sigma_\omega}(\bar{\omega})}{\Gamma_{\sigma_\omega}(\bar{\omega})}}{1 - \mu_*^e \frac{G'(\bar{\omega})}{\Gamma'(\bar{\omega})}} - 1\right) \Gamma_{\sigma_\omega}\left(\bar{\omega}\right) \frac{\tilde{R}_*^k}{R_*^d} + \mu_*^e \frac{n_*}{\bar{k}_*} \frac{G'\left(\bar{\omega}\right)\Gamma_{\sigma_\omega}'\left(\bar{\omega}\right) - \Gamma'\left(\bar{\omega}\right) G_{\sigma_\omega}'\left(\bar{\omega}\right)}{\left[\Gamma'\left(\bar{\omega}\right) - \mu_*^e G'\left(\bar{\omega}\right)\right]^2}
$$

so that

$$\zeta_{b,\sigma_\omega} = \frac{\left(\frac{1-\mu_*^e \frac{G_{\sigma_\omega *}(\bar{\omega}_*)}{\Gamma_{\sigma_\omega *}(\bar{\omega}_*)}}{1-\mu_*^e \frac{G'_*(\bar{\omega}_*)}{\Gamma'_*(\bar{\omega}_*)}} - 1\right) \Gamma_{\sigma_\omega *}\left(\bar{\omega}_*\right) \frac{\tilde{R}_*^k}{R_*^d} + \mu_*^e \frac{n_*}{\bar{k}_*} \frac{G'_*(\bar{\omega}_*)\Gamma'_{\sigma_\omega *}(\bar{\omega}_*) - \Gamma'_*(\bar{\omega}_*) G'_{\sigma_\omega *}(\bar{\omega}_*)}{\left[\Gamma'_*(\bar{\omega}_*) - \mu_*^e G'_*(\bar{\omega}_*)\right]^2}}{\left[1-\Gamma_*\left(\bar{\omega}_*\right)\right] \frac{\tilde{R}_*^k}{R_*^d} + \frac{\Gamma'_*(\bar{\omega}_*)}{\Gamma'_*(\bar{\omega}_*) - \mu_*^e G'_*(\bar{\omega}_*)}\left(1 - \frac{n_*}{\bar{k}_*}\right)} \sigma_{\omega *} \tag{2.82}$$

We also have

$$\zeta_{z,\sigma_\omega} = \frac{\Gamma_{\sigma_\omega *}\left(\bar{\omega}_*\right) - \mu_*^e G_{\sigma_\omega *}\left(\bar{\omega}_*\right)}{\Gamma_*\left(\bar{\omega}_*\right) - \mu_*^e G_*\left(\bar{\omega}_*\right)} \sigma_{\omega *}. \tag{2.83}$$

Elasticity of w.r.t. μ^e

First solve

$$\frac{\partial \tilde{\Psi}}{\partial \mu^e} = -\frac{\Gamma'\left(\bar{\omega}\right) G'\left(\bar{\omega}\right)}{\left[\Gamma'\left(\bar{\omega}\right) - \mu_*^e G'\left(\bar{\omega}\right)\right]^2} \frac{n_*}{\bar{k}_*} - \frac{\Gamma'\left(\bar{\omega}\right) G\left(\bar{\omega}\right)}{\Gamma'\left(\bar{\omega}\right) - \mu_*^e G'\left(\bar{\omega}\right)} \frac{\tilde{R}_*^k}{R_*^d}$$

so that

$$\zeta_{b,\mu^e} = -\frac{\frac{\Gamma'_*(\bar{\omega}_*) G'_*(\bar{\omega}_*)}{\Gamma'_*(\bar{\omega}_*) - \mu_*^e G'_*(\bar{\omega}_*)} \frac{n_*}{\bar{k}_*} + \Gamma'_*\left(\bar{\omega}_*\right) G_*\left(\bar{\omega}_*\right) \frac{\tilde{R}_*^k}{R_*^d}}{\left[1-\Gamma_*\left(\bar{\omega}_*\right)\right]\left[\Gamma'_*\left(\bar{\omega}_*\right) - \mu_*^e G'_*\left(\bar{\omega}_*\right)\right] \frac{\tilde{R}_*^k}{R_*^d} + \Gamma'_*\left(\bar{\omega}_*\right)\left(1 - \frac{n_*}{\bar{k}_*}\right)} \mu_*^e \tag{2.84}$$

We also have

$$\zeta_{z,\mu^e} = -\frac{G_*\left(\bar{\omega}_*\right)}{\Gamma_*\left(\bar{\omega}_*\right) - \mu_*^e G_*\left(\bar{\omega}_*\right)} \mu_*^e. \tag{2.85}$$