Reproducing Kernel of Bessel Potential space

The standard definition of Bessel potential space H^s can be found in ([1], [2], [6], [11], [12]). Here the normal splines will be constructed in the Bessel potential space H^s_ε defined as:

$$H^s_\varepsilon(R^n) = \left\{ \varphi | \varphi \in S', (\varepsilon^2 + |\xi|^2)^{s/2} \mathcal{F}[\varphi] \in L_2(R^n) \right\}, \quad \varepsilon > 0, \ s > \frac{n}{2}.$$

where $S'(R^n)$ is space of L. Schwartz tempered distributions, parameter s may be treated as a fractional differentiation order and $\mathcal{F}[\varphi]$ is a Fourier transform of the φ. The parameter ε introduced here may be considered as a "scaling parameter". It allows to control approximation properties of the normal spline which usually are getting better with smaller values of ε, also it may be used to reduce the ill-conditioness of the related computational problem (in traditional theory $\varepsilon = 1$).

Theoretical properties of spaces H^s_ε at $\varepsilon > 0$ are identical — they are Hilbert spaces with inner product

$$\langle \varphi, \psi \rangle_{H^s_\varepsilon} = \int (\varepsilon^2 + |\xi|^2)^s \mathcal{F}[\varphi] \overline{\mathcal{F}[\psi]} \, d\xi$$

and norm

$$\|\varphi\|_{H^s_\varepsilon} = \left(\langle \varphi, \varphi \rangle_{H^s_\varepsilon} \right)^{1/2} = \|(\varepsilon^2 + |\xi|^2)^{s/2} \mathcal{F}[\varphi]\|_{L_2} \ .$$

It is easy to see that all $\|\varphi\|_{H^s_\varepsilon}$ norms are equivalent. It means that space $H^s_\varepsilon(R^n)$ is equivalent to $H^s(R^n) = H^s_1(R^n)$.

Let's describe the Hölder spaces $C^t_b(R^n), t > 0$ ([9], [2]).

Definition 1. We denote the space

$$S(R^n) = \left\{ f | f \in C^\infty(R^n), \sup_{x \in R^n} |x^\alpha D^\beta f(x)| < \infty, \forall \alpha, \beta \in \mathbb{N}^n \right\}$$

as Schwartz space (or space of complex-valued rapidly decreasing infinitely differentiable functions defined on R^n) ([6], [7]).

Below is a definition of Hölder space $C^t_b(R^n)$ [9]:

Definition 2. If $0 < t = [t] + \{t\}, [t]$ is non-negative integer, $0 < \{t\} < 1$, then $C^t_b(R^n)$ denotes the completion of $S(R^n)$ in the norm

$$C^t_b(R^n) = \left\{ f | f \in C^{[t]}_b(R^n), \|f\|_{C^t_b} < \infty \right\},$$
$$\|f\|_{C^t_b} = \|f\|_{C^{[t]}_b} + \sum_{|\alpha| = [t]} \sup_{x \neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x - y|^{\{t\}}} \ ,$$
$$\|f\|_{C^{[t]}_b} = \sup_{x \in R^n} |D^\alpha f(x)|, \ \forall \alpha : |\alpha| \leq [t].$$

Space $C_b^{[t]}(R^n)$ consists of all functions having bounded continuous derivatives up to order $[t]$. It is easy to see that $C_b^t(R^n)$ is Banach space [9].

Connection of Bessel potential spaces $H^s(R^n)$ with the spaces $C_b^t(R^n)$ is expressed in Embedding theorem ([9], [2]).

Embedding Theorem: If $s = n/2 + t$, where t non-integer, $t > 0$, then space $H^s(R^n)$ is continuously embedded in $C_b^t(R^n)$.

Particularly from this theorem follows that if $f \in H_\varepsilon^{n/2+1/2}(R^n)$, corrected if necessary on a set of Lebesgue measure zero, then it is uniformly continuous and bounded. Further if $f \in H_\varepsilon^{n/2+1/2+r}(R^n)$, r — integer non-negative number, then it can be treated as $f \in C^r(R^n)$, where $C^r(R^n)$ is a class of functions with r continuous derivatives.

It can be shown ([3], [11], [8], [4], [5]) that function

$$V_s(\eta, x, \varepsilon) = c_V(n, s, \varepsilon)(\varepsilon|\eta - x|)^{s-\frac{n}{2}} K_{s-\frac{n}{2}}(\varepsilon|\eta - x|) \ ,$$
$$c_V(n, s, \varepsilon) = \frac{\varepsilon^{n-2s}}{2^{s-1}(2\pi)^{n/2}\Gamma(s)}, \ \eta \in R^n, \ x \in R^n, \ \varepsilon > 0, s > \frac{n}{2}$$

is a reproducing kernel of $H_\varepsilon^s(R^n)$ space. Here K_γ is modified Bessel function of the second kind [10]. The exact value of $c_V(n, s, \varepsilon)$ is not important here and will be set to $\sqrt{\frac{2}{\pi}}$ for ease of further calculations.

This reproducing kernel is known as Matérn kernel [4,13].

The kernel K_γ becomes especially simple when γ is half-integer.

$$\gamma = r + \frac{1}{2} \, , (r = 0, 1, \dots).$$

In this case it is expressed via elementary functions (see [10]):

$$K_{r+1/2}(t) = \sqrt{\frac{\pi}{2t}} t^{r+1} \left(-\frac{1}{t}\frac{d}{dt}\right)^{r+1} \exp(-t) \ ,$$
$$K_{r+1/2}(t) = \sqrt{\frac{\pi}{2t}} \exp(-t) \sum_{k=0}^{r} \frac{(r+k)!}{k!(r-k)!(2t)^k} \ , \ (r = 0, 1, \dots) \ .$$

Let $s_r = r + \frac{n}{2} + \frac{1}{2}$, $r = 0, 1, \dots$, then $H_\varepsilon^{s_r}(R^n)$ is continuously embedded in $C_b^r(R^n)$ and its reproducing kernel with accuracy to constant multiplier can be presented as follows

$$V_{r+\frac{n}{2}+\frac{1}{2}}(\eta, x, \varepsilon) = \exp(-\varepsilon|\eta - x|) \sum_{k=0}^{r} \frac{(r+k)!}{2^k k!(r-k)!} (\varepsilon|\eta - x|)^{r-k} \ ,$$
$$(r = 0, 1, \dots) \ .$$

In particular we have:

$$V_{\frac{n}{2}+\frac{1}{2}}(\eta, x, \varepsilon) = \exp(-\varepsilon|\eta - x|) \ ,$$
$$V_{1+\frac{n}{2}+\frac{1}{2}}(\eta, x, \varepsilon) = \exp(-\varepsilon|\eta - x|)(1 + \varepsilon|\eta - x|) \ ,$$
$$V_{2+\frac{n}{2}+\frac{1}{2}}(\eta, x, \varepsilon) = \exp(-\varepsilon|\eta - x|)(3 + 3\varepsilon|\eta - x| + \varepsilon^2|\eta - x|^2) \ .$$

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