

# Singular

A Computer Algebra System for Polynomial Computations

## Manual

Version 4.4.0, 2024

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## Short Contents

1	Preface . . . . .	1
2	Introduction . . . . .	4
3	General concepts . . . . .	15
4	Data types . . . . .	73
5	Functions and system variables . . . . .	156
6	Tricks and pitfalls . . . . .	310
7	Non-commutative subsystem . . . . .	318
	Appendix A Examples . . . . .	699
	Appendix B Polynomial data . . . . .	768
	Appendix C Mathematical background . . . . .	775
	Appendix D SINGULAR libraries . . . . .	794
8	Release Notes . . . . .	2589
9	Index . . . . .	2604

# 1 Preface

SINGULAR version 4.4.0  
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For information on how to cite SINGULAR see

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## Availability

The latest information regarding the status of SINGULAR is always available from <https://www.singular.uni-kl.de>. The program SINGULAR and the above mentioned parts are available via anonymous ftp through the following addresses:

GMP, libreadline

© Free Software Foundation  
<https://gmplib.org>

NTL

© Victor Shoup  
<http://www.shoup.net/ntl>

cdd (C implementation of the Double Description Method of Motzkin et al)

© Komei Fukuda

[http://www-oldurls.inf.ethz.ch/personal/fukudak/cdd\\_home/](http://www-oldurls.inf.ethz.ch/personal/fukudak/cdd_home/)

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gfanlib © Anders Jensen

<https://users-math.au.dk/~jensen/software/gfan/gfan.html>

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<https://www.singular.uni-kl.de/ftp/pub/Math/Singular/Factory>

Singular-libfac

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<ftp://jim.mathematik.uni-kl.de/pub/Math/Singular/Libfac/>

SINGULAR binaries and sources

<ftp://jim.mathematik.uni-kl.de/pub/Math/Singular/> or via a WWW browser  
from <https://www.singular.uni-kl.de/ftp/pub/Math/Singular/>

Cygwin <https://www.cygwin.com/>

Xemacs <https://www.xemacs.org>

Some external programs are optional:

4ti2 (used by sing4ti2.lib, see [Section D.4.33 \[sing4ti2\\_lib\]](#), page 1382)

<https://4ti2.github.io>

gfan (used by tropical.lib, see [Section D.13.6 \[tropical\\_lib\]](#), page 2140)

<https://users-math.au.dk/~jensen/software/gfan/gfan.html>

graphviz (used by resgraph.lib, see [Section D.5.13 \[resgraph\\_lib\]](#), page 1524)

<https://www.graphviz.org/>

normaliz (used by normaliz.lib, see [Section D.4.24 \[normaliz\\_lib\]](#), page 1210)

© Winfried Bruns and Bogdan Ichim

<https://www.normaliz.uni-osnabrueck.de>

surf (used by surf.lib, see [Section D.9.3 \[surf\\_lib\]](#), page 1918)

© Stephan Endrass

<http://surf.sf.net>

surfer (used by surf.lib, see [Section D.9.3 \[surf\\_lib\]](#), page 1918)

<https://imaginary.org/program/surfer>

surfex (used by surfex.lib, see [Section D.9.4 \[surfex\\_lib\]](#), page 1919)

© Oliver Labs (2001-2008), Stephan Holzer (2004-2005)

<https://github.com/Singular/Singular/tree/spielwiese/Singular/LIB/surfex>

TOPCOM (used by polymake.lib, see [Section D.13.4 \[polymake\\_lib\]](#), page 2129)

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<http://www.rambau.wm.uni-bayreuth.de/TOPCOM/>

## Acknowledgements

The development of SINGULAR is directed and coordinated by Wolfram Decker, Gert-Martin Greuel, Gerhard Pfister, and Hans Schönemann.

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The development of SINGULAR has been supported by the Deutsche Forschungsgemeinschaft (DFG), the Stiftung Rheinland-Pfalz für Innovation, the Volkswagen Stiftung, and the European Union. From 2010 to 2016, it was part of the DFG Priority Project SPP 1489. It is currently funded by the DFG transregional collaborative research centre (SFB-TRR) 195 Symbolic Tools in Mathematics and their Application and the European Unions Horizon 2020 research and innovation programme under grant agreement 676541.

## 2 Introduction

### 2.1 Background

SINGULAR is a Computer Algebra system for polynomial computations with emphasis on the special needs of commutative algebra, algebraic geometry, and singularity theory.

SINGULAR's main computational objects are ideals and modules over a large variety of baserings. The baserings are polynomial rings or localizations thereof over a field (e.g., finite fields, the rationals, floats, algebraic extensions, transcendental extensions) or over a limited set of rings, or over quotient rings with respect to an ideal.

SINGULAR features one of the fastest and most general implementations of various algorithms for computing Groebner resp. standard bases. The implementation includes Buchberger's algorithm (if the ordering is a wellordering) and Mora's algorithm (if the ordering is a tangent cone ordering) as special cases. Furthermore, it provides polynomial factorization, resultant, characteristic set and gcd computations, syzygy and free-resolution computations, and many more related functionalities.

Based on an easy-to-use interactive shell and a C-like programming language, SINGULAR's internal functionality is augmented and user-extendible by libraries written in the SINGULAR programming language. A general and efficient implementation of communication links allows SINGULAR to make its functionality available to other programs.

SINGULAR's development started in 1984 with an implementation of Mora's Tangent Cone algorithm in Modula-2 on an Atari computer (K.P. Neuendorf, G. Pfister, H. Schönemann; Humboldt-Universität zu Berlin). The need for a new system arose from the investigation of mathematical problems coming from singularity theory which none of the existing systems was able to handle.

In the early 1990s SINGULAR's "home-town" moved to Kaiserslautern, a general standard basis algorithm was implemented in C and SINGULAR was ported to Unix, MS-DOS, Windows NT, and MacOS.

Continuous extensions (like polynomial factorization, gcd computations, links) and refinements led in 1997 to the release of SINGULAR version 1.0 and in 1998 to the release of version 1.2 (with a much faster standard and Groebner bases computation based on Hilbert series and on an improved implementation of the core algorithms, libraries for primary decomposition, ring normalization, etc.)

For the highlights of the new SINGULAR version 4.4.0, see [Section 8.1 \[News and changes\]](#), page 2589.

### 2.2 How to use this manual

#### For the impatient user

In [Section 2.3 \[Getting started\]](#), page 6, some simple examples explain how to use SINGULAR in a step-by-step manner.

[Appendix A \[Examples\]](#), page 699 should come next for real learning-by-doing or to quickly solve some given mathematical problem without dwelling too deeply into SINGULAR. This chapter contains a lot of real-life examples and detailed instructions and explanations on how to solve mathematical problems using SINGULAR.

## For the systematic user

In [Chapter 3 \[General concepts\]](#), [page 15](#), all basic concepts which are important to use and to understand SINGULAR are developed. But even for users preferring the systematic approach it will be helpful to take a look at the examples in [Section 2.3 \[Getting started\]](#), [page 6](#), every now and then. The topics in the chapter are organized more or less in the natural order in which the novice user is expected to have to deal with them.

- In [Section 3.1 \[Interactive use\]](#), [page 15](#), and its subsections there are some words on entering and exiting SINGULAR, followed by a number of other aspects concerning the interactive user-interface.
- To do anything more than trivial integer computations, one needs to define a basering in SINGULAR. This is explained in detail in [Section 3.3 \[Rings and orderings\]](#), [page 30](#).
- An overview of the algorithms implemented in the kernel of SINGULAR is given in [Section 3.4 \[Implemented algorithms\]](#), [page 37](#).
- In [Section 3.5 \[The SINGULAR language\]](#), [page 41](#), language specific concepts are introduced, such as the notions of names and objects, data types and conversion between them, etc.
- In [Section 3.6 \[Input and output\]](#), [page 48](#), SINGULAR's mechanisms to store and retrieve data are discussed.
- The more complex concepts of procedures and libraries as well as tools for debugging them are considered in the following sections: [Section 3.7 \[Procedures\]](#), [page 50](#), [Section 3.8 \[Libraries\]](#), [page 55](#), and [Section 3.9 \[Debugging tools\]](#), [page 68](#).

[Chapter 4 \[Data types\]](#), [page 73](#), is a complete treatment of SINGULAR's data types in alphabetical order, where each section corresponds to one data type. For each data type, its purpose is explained, the syntax of its declaration is given, related operations and functions are listed, and one or more examples illustrate its usage.

[Chapter 5 \[Functions and system variables\]](#), [page 156](#), is an alphabetically ordered reference list of all of SINGULAR's functions, control structures, and system variables. Each entry includes a description of the syntax and semantics of the item being explained as well as one or more examples on how to use it.

## Miscellaneous

[Chapter 6 \[Tricks and pitfalls\]](#), [page 310](#), is a loose collection of limitations and features which may be unexpected by those who expect the SINGULAR language to be an exact copy of the C programming language or of some other Computer Algebra system's language. Additionally, some mathematical hints are collected there.

[Appendix C \[Mathematical background\]](#), [page 775](#), introduces some of the mathematical notions and definitions used throughout this manual. For example, if in doubt what exactly a "negative degree reverse lexicographical ordering" is in SINGULAR, one should refer to this chapter.

[Appendix D \[SINGULAR libraries\]](#), [page 794](#), lists the libraries which come with SINGULAR, and all functions contained in them.

## Typographical conventions

Throughout this manual, the following typographical conventions are adopted:

- text in **typewriter** denotes SINGULAR input and output as well as reserved names:  
The basering can, e.g., be set using the command **setring**.
- the arrow  $\mapsto$  denotes SINGULAR output:

```
poly p=x+y+z;
p*p;
↦ x2+2xy+y2+2xz+2yz+z2
```

- square brackets are used to denote parts of syntax descriptions which are optional:  
[optional\_text] required\_text
- keys are denoted using typewriter, for example:  
N (press the key N to get to the next node in help mode)  
RETURN (press RETURN to finish an input line)  
CTRL-P (press the control key together with the key P to get the previous input line)

## 2.3 Getting started

SINGULAR is a special purpose system for polynomial computations. Hence, most of the powerful computations in SINGULAR require the prior definition of a ring. Most important rings are polynomial rings over a field, localizations thereof, or quotient rings of such rings modulo an ideal. However, some simple computations with integers (machine integers of limited size) and manipulations of strings can be carried out without the prior definition of a ring.

### 2.3.1 First steps

Once SINGULAR is started, it awaits an input after the prompt `>`. Every statement has to be terminated by `;`.

```
37+5;
↦ 42
```

All objects have a type, e.g., integer variables are defined by the word `int`. An assignment is made using the symbol `=`.

```
int k = 2;
```

Test for equality resp. inequality is done using `==` resp. `!=` (or `<>`), where 0 represents the boolean value FALSE, and any other value represents TRUE.

```
k == 2;
↦ 1
k != 2;
↦ 0
```

The value of an object is displayed by simply typing its name.

```
k;
↦ 2
```

On the other hand, the output is suppressed if an assignment is made.

```
int j;
j = k+1;
```

The last displayed (!) result can be retrieved via the special symbol `_`.

```
2*_; // the value from k displayed above
↦ 4
```

Text starting with `//` denotes a comment and is ignored in calculations, as seen in the previous example. Furthermore SINGULAR maintains a history of the previous lines of input, which may be accessed by CTRL-P (previous) and CTRL-N (next) or the arrows on the keyboard.

The whole manual is available online by typing the command `help;`. Documentation on single topics, e.g., on `intmat`, which defines a matrix of integers, is obtained by



```
help intmat;
```

This will display the text of [Section 4.8 \[intmat\]](#), page 90, in the printed manual.

Next, we define a  $3 \times 3$  matrix of integers and initialize it with some values, row by row from left to right:

```
intmat m[3][3] = 1,2,3,4,5,6,7,8,9;
m;
```

A single matrix entry may be selected and changed using square brackets [ and ].

```
m[1,2]=0;
m;
↪ 1,0,3,
↪ 4,5,6,
↪ 7,8,9
```

To calculate the trace of this matrix, we use a `for` loop. The curly brackets { and } denote the beginning resp. end of a block. If you define a variable without giving an initial value, as the variable `tr` in the example below, SINGULAR assigns a default value for the specific type. In this case, the default value for integers is 0. Note that the integer variable `j` has already been defined above.

```
int tr;
for ( j=1; j <= 3; j++ ) { tr=tr + m[j,j]; }
tr;
↪ 15
```

Variables of type string can also be defined and used without having an active ring. Strings are delimited by " (double quotes). They may be used to comment the output of a computation or to give it a nice format. If a string contains valid SINGULAR commands, it can be executed using the function `execute`. The result is the same as if the commands would have been written on the command line. This feature is especially useful to define new rings inside procedures.

```
"example for strings:";
↪ example for strings:
string s="The element of m ";
s = s + "at position [2,3] is:"; // concatenation of strings by +
s , m[2,3] , ".";
↪ The element of m at position [2,3] is: 6 .
s="m[2,1]=0; m;";
execute(s);
↪ 1,0,3,
↪ 0,5,6,
↪ 7,8,9
```

This example shows that expressions can be separated by , (comma) giving a list of expressions. SINGULAR evaluates each expression in this list and prints all results separated by spaces.

### 2.3.2 Rings and standard bases

In order to compute with objects such as ideals, matrices, modules, and polynomial vectors, a ring has to be defined first.

```
ring r = 0,(x,y,z),dp;
```

The definition of a ring consists of three parts: the first part determines the ground field, the second part determines the names of the ring variables, and the third part determines the monomial ordering to be used. Thus, the above example declares a polynomial ring called `r` with a ground

field of characteristic 0 (i.e., the rational numbers) and ring variables called  $x$ ,  $y$ , and  $z$ . The `dp` at the end determines that the degree reverse lexicographical ordering will be used.

Other ring declarations:

```
ring r1=32003,(x,y,z),dp;
    characteristic 32003, variables x, y, and z and ordering dp.

ring r2=32003,(a,b,c,d),lp;
    characteristic 32003, variable names a, b, c, d and lexicographical ordering.

ring r3=7,(x(1..10)),ds;
    characteristic 7, variable names x(1),...,x(10), negative degree reverse lexicographical
    ordering (ds).

ring r4=(0,a),(mu,nu),lp;
    transcendental extension of  $Q$  by  $a$ , variable names mu and nu, lexicographical ordering.

ring r5=real,(a,b),lp;
    floating point numbers (single machine precision), variable names a and b.

ring r6=(real,50),(a,b),lp;
    floating point numbers with precision extended to 50 digits, variable names a and b.

ring r7=(complex,50,i),(a,b),lp;
    complex floating point numbers with precision extended to 50 digits and imaginary
    unit i, variable names a and b.

ring r8=integer,(a,b),lp;
    the ring of integers (see Section 3.3.4 \[Coefficient rings\], page 37), variable names a and
    b.

ring r9=(integer, 60),(a,b),lp;
    the ring of integers modulo 60 (see Section 3.3.4 \[Coefficient rings\], page 37), variable
    names a and b.

ring r10=(integer, 2, 10),(a,b),lp;
    the ring of integers modulo  $2^{10}$  (see Section 3.3.4 \[Coefficient rings\], page 37), variable
    names a and b.
```

Typing the name of a ring prints its definition. The example below shows that the default ring in SINGULAR is  $\mathbb{Z}/32003[x, y, z]$  with degree reverse lexicographical ordering:

```
ring r11;
r11;
⇒ // coefficients: ZZ/32003
⇒ // number of vars : 3
⇒ //      block   1 : ordering dp
⇒ //              : names   x y z
⇒ //      block   2 : ordering C
```

Defining a ring makes this ring the current active basering, so each ring definition above switches to a new basering. The concept of rings in SINGULAR is discussed in detail in [Section 3.3 \[Rings and orderings\], page 30](#).

The basering is now `r11`. Since we want to calculate in the ring `r`, which we defined first, we need to switch back to it. This can be done using the function `setring`:

```
setring r;
```

Once a ring is active, we can define polynomials. A monomial, say  $x^3$ , may be entered in two ways: either using the power operator `^`, writing `x^3`, or in short-hand notation without operator,

writing `x3`. Note that the short-hand notation is forbidden if a name of the ring variable(s) consists of more than one character (see [Section 6.4 \[Miscellaneous oddities\]](#), page 314 for details). Note, that SINGULAR always expands brackets and automatically sorts the terms with respect to the monomial ordering of the basering.

```
poly f = x3+y3+(x-y)*x2y2+z2;
f;
↪ x3y2-x2y3+x3+y3+z2
```

The command `size` retrieves in general the number of entries in an object. In particular, for polynomials, `size` returns the number of monomials.

```
size(f);
↪ 5
```

A natural question is to ask if a point, e.g.,  $(x,y,z)=(1,2,0)$ , lies on the variety defined by the polynomials `f` and `g`. For this we define an ideal generated by both polynomials, substitute the coordinates of the point for the ring variables, and check if the result is zero:

```
poly g = f^2 *(2x-y);
ideal I = f,g;
ideal J = subst(I,var(1),1);
J = subst(J,var(2),2);
J = subst(J,var(3),0);
J;
↪ J[1]=5
↪ J[2]=0
```

Since the result is not zero, the point  $(1,2,0)$  does not lie on the variety  $V(f,g)$ .

Another question is to decide whether some function vanishes on a variety, or in algebraic terms, if a polynomial is contained in a given ideal. For this we calculate a standard basis using the command `groebner` and afterwards reduce the polynomial with respect to this standard basis.

```
ideal sI = groebner(f);
reduce(g,sI);
↪ 0
```

As the result is 0 the polynomial `g` belongs to the ideal defined by `f`.

The function `groebner`, like many other functions in SINGULAR, prints a protocol during calculations, if desired. The command `option(prot);` enables protocolling whereas `option(noprot);` turns it off. [Section 5.1.111 \[option\]](#), page 234, explains the meaning of the different symbols printed during calculations.

The command `kbase` calculates a basis of the polynomial ring modulo an ideal, if the quotient ring is finite dimensional. As an example we calculate the Milnor number of a hypersurface singularity in the global and local case. This is the vector space dimension of the polynomial ring modulo the Jacobian ideal in the global case resp. of the power series ring modulo the Jacobian ideal in the local case. See [Section A.4.2 \[Critical points\]](#), page 737, for a detailed explanation.

The Jacobian ideal is obtained with the command `jacob`.

```
ideal J = jacob(f);
↪ // ** redefining J **
J;
↪ J[1]=3x2y2-2xy3+3x2
↪ J[2]=2x3y-3x2y2+3y2
↪ J[3]=2z
```

SINGULAR prints the line `// ** redefining J **`. This indicates that we had previously defined a variable with name `J` of type `ideal` (see above).

To obtain a representing set of the quotient vector space we first calculate a standard basis, and then apply the function `kbase` to this standard basis.

```
J = groebner(J);
ideal K = kbase(J);
K;
↪ K[1]=y4
↪ K[2]=xy3
↪ K[3]=y3
↪ K[4]=xy2
↪ K[5]=y2
↪ K[6]=x2y
↪ K[7]=xy
↪ K[8]=y
↪ K[9]=x3
↪ K[10]=x2
↪ K[11]=x
↪ K[12]=1
```

Then

```
size(K);
↪ 12
```

gives the desired vector space dimension  $K[x, y, z]/\text{jacob}(f)$ . As in SINGULAR the functions may take the input directly from earlier calculations, the whole sequence of commands may be written in one single statement.

```
size(kbase(groebner(jacob(f))));
↪ 12
```

When we are not interested in a basis of the quotient vector space, but only in the resulting dimension we may even use the command `vdim` and write:

```
vdim(groebner(jacob(f)));
↪ 12
```

### 2.3.3 Procedures and libraries

SINGULAR offers a comfortable programming language, with a syntax close to C. So it is possible to define procedures which bind a sequence of several commands in a new one. Procedures are defined using the keyword `proc` followed by a name and an optional parameter list with specified types. Finally, a procedure may return a value using the command `return`.

We may e.g. define the following procedure called `Milnor`: (Here the parameter list is `(poly h)` meaning that `Milnor` must be called with one argument which can be assigned to the type `poly` and is referred to by the name `h`.)

```
proc Milnor (poly h)
{
  return(vdim(groebner(jacob(h))));
}
```

Note: if you have entered the first line of the procedure and pressed `RETURN`, SINGULAR prints the prompt `.` (dot) instead of the usual prompt `>`. This shows that the input is incomplete and SINGULAR expects more lines. After typing the closing curly bracket, SINGULAR prints the usual prompt indicating that the input is now complete.

Then we can call the procedure:

```
Milnor(f);
↳ 12
```

Note that the result may depend on the basering as we will see in the next chapter.

The distribution of SINGULAR contains several libraries, each of which is a collection of useful procedures based on the kernel commands, which extend the functionality of SINGULAR. The command `listvar(package);` list all currently loaded libraries. The command `LIB "all.lib";` loads all libraries.

One of these libraries is `sing.lib` which already contains a procedure called `milnor` to calculate the Milnor number not only for hypersurfaces but more generally for complete intersection singularities. Libraries are loaded using the command `LIB`. Some additional information during the process of loading is displayed on the screen, which we omit here.

```
LIB "sing.lib";
```

As all input in SINGULAR is case sensitive, there is no conflict with the previously defined procedure `Milnor`, but the result is the same.

```
milnor(f);
↳ 12
```

The procedures in a library have a help part which is displayed by typing

```
help milnor;
```

as well as some examples, which are executed by

```
example milnor;
```

Likewise, the library itself has a help part, to show a list of all the functions available for the user which are contained in the library.

```
help sing.lib;
```

The output of the help commands is omitted here.

### 2.3.4 Change of rings

To calculate the local Milnor number we have to do the calculation with the same commands in a ring with local ordering. We can define the localization of the polynomial ring at the origin (see [Appendix B \[Polynomial data\], page 768](#), and [Appendix C \[Mathematical background\], page 775](#)).

```
ring rl = 0,(x,y,z),ds;
```

The ordering directly affects the standard basis which will be calculated. Fetching the polynomial defined in the ring `r` into this new ring, helps us to avoid retyping previous input.

```
poly f = fetch(r,f);
f;
↳ z2+x3+y3+x3y2-x2y3
```

Instead of `fetch` we can use the function `imap` which is more general but less efficient. The most general way to fetch data from one ring to another is to use maps, this will be explained in [Section 4.12 \[map\], page 106](#).

In this ring the terms are ordered by increasing exponents. The local Milnor number is now

```
Milnor(f);
↳ 4
```

This shows that `f` has outside the origin in affine 3-space singularities with local Milnor number adding up to  $12 - 4 = 8$ . Using global and local orderings as above is a convenient way to check whether a variety has singularities outside the origin.

The command `jacob` applied twice gives the Hessian of `f`, in our example a 3x3 - matrix.

```

matrix H = jacob(jacob(f));
H;
⇒ H[1,1]=6x+6xy2-2y3
⇒ H[1,2]=6x2y-6xy2
⇒ H[1,3]=0
⇒ H[2,1]=6x2y-6xy2
⇒ H[2,2]=6y+2x3-6x2y
⇒ H[2,3]=0
⇒ H[3,1]=0
⇒ H[3,2]=0
⇒ H[3,3]=2

```

The `print` command displays the matrix in a nicer format.

```

print(H);
⇒ 6x+6xy2-2y3, 6x2y-6xy2, 0,
⇒ 6x2y-6xy2, 6y+2x3-6x2y, 0,
⇒ 0, 0, 2

```

We may calculate the determinant and (the ideal generated by all) minors of a given size.

```

det(H);
⇒ 72xy+24x4-72x3y+72xy3-24y4-48x4y2+64x3y3-48x2y4
minor(H,1); // the 1x1 - minors
⇒ _[1]=2
⇒ _[2]=6y+2x3-6x2y
⇒ _[3]=6x2y-6xy2
⇒ _[4]=6x2y-6xy2
⇒ _[5]=6x+6xy2-2y3

```

The algorithm of the standard basis computation may be affected by the command `option`. For example, a reduced standard basis of the ideal generated by the  $1 \times 1$ -minors of  $H$  is obtained in the following way:

```

option(redSB);
groebner(minor(H,1));
⇒ _[1]=1

```

This shows that 1 is contained in the ideal of the  $1 \times 1$ -minors, hence the corresponding variety is empty.

### 2.3.5 Modules and their annihilator

Now we shall give three more advanced examples.

SINGULAR is able to handle modules over all the rings, which can be defined as a basering. A free module of rank  $n$  is defined as follows:

```

ring rr;
int n = 4;
freemodule(4);
⇒ _[1]=gen(1)
⇒ _[2]=gen(2)
⇒ _[3]=gen(3)
⇒ _[4]=gen(4)
typeof(_);
⇒ module
print(freemodule(4));
⇒ 1,0,0,0,
⇒ 0,1,0,0,

```

```

 $\mapsto 0,0,1,0,$ 
 $\mapsto 0,0,0,1$ 

```

To define a module, we provide a list of vectors generating a submodule of a free module. Then this set of vectors may be identified with the columns of a matrix. For that reason in SINGULAR matrices and modules may be interchanged. However, the representation is different (modules may be considered as sparse matrices).

```

ring r =0,(x,y,z),dp;
module MD = [x,0,x],[y,z,-y],[0,z,-2y];
matrix MM = MD;
print(MM);
 $\mapsto x,y,0,$ 
 $\mapsto 0,z,z,$ 
 $\mapsto x,-y,-2y$ 

```

However the submodule  $MD$  may also be considered as the module of relations of the factor module  $r^3/MD$ . In this way, SINGULAR can treat arbitrary finitely generated modules over the basering (see [Section B.1 \[Representation of mathematical objects\], page 768](#)).

In order to get the module of relations of  $MD$ , we use the command `syz`.

```

syz(MD);
 $\mapsto \_ [1]=x*gen(3)-x*gen(2)+y*gen(1)$ 

```

We want to calculate, as an application, the annihilator of a given module. Let  $M = r^3/U$ , where  $U$  is our defining module of relations for the module  $M$ .

```

module U = [z3,xy2,x3],[yz2,1,xy5z+z3],[y2z,0,x3],[xyz+x2,y2,0],[xyz,x2y,1];

```

Then, by definition, the annihilator of  $M$  is the ideal  $\text{ann}(M) = \{a \mid aM = 0\}$  which is, by definition of  $M$ , the same as  $\{a \mid ar^3 \in U\}$ . Hence we have to calculate the quotient  $U:r^3$ . The rank of the free module is determined by the choice of  $U$  and is the number of rows of the corresponding matrix. This may be retrieved by the function `nrows`. All we have to do now is the following:

```

quotient(U,freemodule(nrows(U)));

```

The result is too big to be shown here.

### 2.3.6 Resolution

There are several commands in SINGULAR for computing free resolutions. The most general command is `res(...,n)` which determines heuristically what method to use for the given problem. It computes the free resolution up to the length  $n$ , where  $n = 0$  corresponds to the full resolution.

Here we use the possibility to inspect the calculation process using the option `prot`.

```

ring R;          // the default ring in char 32003
R;
 $\mapsto //$     characteristic : 32003
 $\mapsto //$     number of vars : 3
 $\mapsto //$           block   1 : ordering dp
 $\mapsto //$           : names   x y z
 $\mapsto //$           block   2 : ordering C
ideal I = x4+x3y+x2yz,x2y2+xy2z+y2z2,x2z2+2xz3,2x2z2+xyz2;
option(prot);
resolution rs = res(I,0);
 $\mapsto$  using lres
 $\mapsto 4(m0)4(m1).5(m1)g.g6(m1) \dots 6(m2) \dots$ 

```

Disable this protocol with

```
option(noprot);
```

When we enter the name of the calculated resolution, we get a pictorial description of the minimized resolution where the exponents denote the rank of the free modules. Note that the calculated resolution itself may not yet be minimal.

```
rs;
↳ 1      4      5      2      0
↳ R  <-- R  <-- R  <-- R  <-- R
↳
↳ 0      1      2      3      4
print(betti(rs),"betti");
↳
↳      0      1      2      3
↳ -----
↳ 0:      1      -      -      -
↳ 1:      -      -      -      -
↳ 2:      -      -      -      -
↳ 3:      -      4      1      -
↳ 4:      -      -      1      -
↳ 5:      -      -      3      2
↳ -----
↳ total:      1      4      5      2
```

In order to minimize the resolution, that is to calculate the maps of the minimal free resolution, we use the command `minres`:

```
rs=minres(rs);
```

A single module in this resolution is obtained (as usual) with the brackets `[` and `]`. The `print` command can be used to display a module in a more readable format:

```
print(rs[3]);
↳ z3,   -xyz-y2z-4xz2+16z3,
↳ 0,    -y2,
↳ -y+4z,48z,
↳ x+2z, 48z,
↳ 0,    x+y-z
```

In this case, the output is to be interpreted as follows: the 3rd syzygy module of  $R/I$ , `rs[3]`, is the rank-2-submodule of  $R^5$  generated by the vectors  $(z^3, 0, -y + 4z, x + 2z, 0)$  and  $(-xyz - y^2z - 4xz^2 + 16z^3, -y^2, 48z, 48z, x + y - z)$ .



## 3 General concepts

### 3.1 Interactive use

In this section, aspects of interactive use are discussed. This includes how to enter and exit SINGULAR, how to interpret its prompt, how to get online help, and so on.

There are a few important notes which one should not forget:

- every command has to be terminated by a `;` (semicolon) followed by a `(RETURN)`
- the online help is accessible by means of the `help` function

#### 3.1.1 How to enter and exit

SINGULAR can either be run in an ASCII-terminal or within Emacs.

To start SINGULAR in its ASCII-terminal user interface, enter `Singular` at the system prompt. The SINGULAR banner appears which, among other data, reports the version and the compilation date.

To start SINGULAR in its Emacs user interface, either enter `ESingular` at the system prompt, or type `M-x singular` within a running Emacs (provided you have loaded the file `singular.el` in your running Emacs, see [Section 3.2.2 \[Running SINGULAR under Emacs\]](#), page 25 for details).

Generally, we recommend to use SINGULAR in its Emacs interface, since this offers many more features and is more convenient to use than the ASCII-terminal interface (see [Section 3.2 \[Emacs user interface\]](#), page 22).

To exit SINGULAR type `quit;`, `exit;` or `$` (or, when running within Emacs preferably type `C-c $`).

SINGULAR and ESingular may also be started with command line options and with filenames as arguments. More generally, the startup syntax is

```
Singular [options] [file1 [file2 ...]]
ESingular [options] [file1 [file2 ...]]
```

See [Section 3.1.6 \[Command line options\]](#), page 19, [Section 3.1.7 \[Startup sequence\]](#), page 22, [Section 3.2.2 \[Running SINGULAR under Emacs\]](#), page 25.

#### 3.1.2 The SINGULAR prompt

The SINGULAR prompt `>` (larger than) asks the user for input of commands. The “continuation” prompt `.` (period) asks the user for input of missing parts of a command (e.g. the semicolon at the end of every command).

SINGULAR does not interpret the semicolon as the end of a command if it occurs inside a string. Also, SINGULAR waits for blocks (sequences of commands enclosed in curly brackets) to be closed before prompting with `>` for more commands. Thus, if SINGULAR does not respond with its regular prompt after typing a semicolon it may wait for a `"` or a `}` first.

Additional semicolons will not harm SINGULAR since they are interpreted as empty statements.

#### 3.1.3 The online help system

The online help system is invoked by the `help` command. `?` may be used as a synonym for `help`. Simply typing `help;` displays the “top” of the help system (i.e., the title page of the SINGULAR manual) which offers a short table of contents. Typing `help topic;` shows the available documentation on the respective topic. Here, `topic` may be either a function name or, more generally, any

index entry of the SINGULAR manual. Furthermore, topic may contain wildcard characters. See [Section 5.1.54 \[help\], page 193](#), for more information.

Online help information can be displayed in various help browsers. The following table lists a summary of the browsers which are always present. Usually, external browsers are much more convenient: A complete, customizable list can be found in the file `LIB/help.cnf`.

Browser	Platform	Description
html	Windows	displays a html version of the manual in your default html browser
builtin	all	simply outputs the help information in plain ASCII format
emacs	Unix, Windows	when running SINGULAR within (X)emacs, displays help inside the (X)emacs info buffer.
dummy	all	displays an error message due to the non-availability of a help browser

External browsers depend on your system and the contents of `LIB/help.cnf`, the default includes:

`htmlview` (displays HTML help pages via `htmlview`),  
`mac` (displays HTML help pages via `open`),  
`mac-net` (displays HTML help pages via `open`),  
`mozilla` (displays HTML help pages via `mozilla`),  
`firefox` (displays HTML help pages via `firefox`),  
`konqueror` (displays HTML help pages via `konqueror`),  
`galeon` (displays HTML help pages via `galeon`),  
`netscape` (displays HTML help pages via `netscape`),  
`safari` (displays HTML help pages on MacOSX via `safari`),  
`tkinfo` (displays INFO help pages via `tkinfo`),  
`xinfo` (displays INFO help pages via `info`),  
`info` (displays INFO help pages via `info`),  
`lynx` (displays HTML help pages via `lynx`).

The browser which is used to display the help information, can be either set at startup time with the command line option (see [Section 3.1.6 \[Command line options\], page 19](#))

```
--browser=<browser>
```

or with the SINGULAR command (see [Section 5.1.155 \[system\], page 275](#))

```
system("--browser", "<browser>");
```

The SINGULAR command

```
system("browsers");
```

lists all available browsers and the command

```
system("--browser");
```

returns the currently used browser.

If no browser is explicitly set by the user, then the first available browser (w.r.t. the order of the browsers in the file `LIB/help.cnf`) is chosen.

The `.singularrrc` (see [Section 3.1.7 \[Startup sequence\], page 22](#)) file is a good place to set your default browser. Recall that if a file `$HOME/.singularrrc` exists on your system, then the content of this file is executed before the first user input. Hence, putting

```

if (! system("--emacs"))
{
    // only set help browser if not running within emacs
    system("--browser", "info");
}
// if help browser is later on set to a web browser,
// allow it to fetch HTML pages from the net
system("--allow-net", 1);

```

in your file `$HOME/.singularrrc` sets your default browser to `info`, unless SINGULAR is run within emacs (in which case the default browser is automatically set to `emacs`).

Obviously, certain external files and programs are required for the SINGULAR help system to work correctly. If something is not available or goes wrong, here are some tips for troubleshooting the help system:

- Under Unix, the environment variable `DISPLAY` has to be set for all X11 browsers to work.
- The help browsers are only available if the respective programs are installed on your system (for `xinfo`, the programs `xterm` and `info` are necessary). You can explicitly specify which program to use, by changing the entry in `LIB/help.cnf`
- If the help browser cannot find the local html pages of the SINGULAR manual (which it will look for at `$RootDir/html` – see [Section 3.8.11 \[Loading a library\], page 67](#) for more info on `$RootDir`) and the (command-line) option `--allow-net` has *explicitly* been set (see [Section 3.1.6 \[Command line options\], page 19](#) and [Section 5.1.155 \[system\], page 275](#) for more info on setting values of command-line options), then it dispatches the html pages from <https://www.singular.uni-kl.de/Manual>. (Note that the non-local net-access of HTML pages is disabled, by default.)

An alternative location of a local directory where the html pages reside can be specified by setting the environment variable `SINGULAR_HTML_DIR`.

- The `info` based help browsers `tkinfo`, `xinfo`, `info`, and `builtin` need the (info) file `singular.info` which will be looked for at `$RootDir/info/singular.info` (see [Section 3.8.11 \[Loading a library\], page 67](#) for more info on `$RootDir`). An alternative location of the info file of the manual can be specified by setting the environment variable `SINGULAR_INFO_FILE`.

[Section 3.1.6 \[Command line options\], page 19](#)

## Info help browsers

The help browsers `tkinfo`, `xinfo` and `info` (so-called info help browsers) are based on the `info` program from the GNU `texinfo` package. See [section “Getting started” in \*The Info Manual\*](#), for more information.

For info help browsers, the online manual is decomposed into “nodes” of information, closely related to the division of the printed manual into sections and subsections. A node contains text describing a specific topic at a specific level of detail. The top line of a node is its “header”. The node’s header tells the name of the current node (`Node:`), the name of the next node (`Next:`), the name of the previous node (`Prev:`), and the name of the upper node (`Up:`).

To move within info, type commands consisting of single characters. Do not type `RETURN`. Do not use cursor keys, either. Using some of the cursor keys by accident might pop to some totally different node. Type `1` to return to the original node. Some of the `info` commands read input from the command line at the bottom. The `TAB` key may be used to complete partially entered input.

The most important commands are:

`q` leaves the online help system

<b>n</b>	goes to the next node
<b>p</b>	goes to the previous node
<b>u</b>	goes to the upper node
<b>m</b>	picks a menu item specified by name
<b>f</b>	follows a cross reference
<b>l</b>	goes to the previously visited node
<b>b</b>	goes to the beginning of the current node
<b>e</b>	goes to the end of the current node
<b>SPACE</b>	scrolls forward a page
<b>DEL</b>	scrolls backward a page
<b>h</b>	invokes info tutorial (use <b>l</b> to return to the manual or <b>CTRL-X 0</b> to remove extra window)
<b>CTRL-H</b>	shows a short overview over the online help system (use <b>l</b> to return to the manual or <b>CTRL-X 0</b> to remove extra window)
<b>s</b>	searches through the manual for a specific string, and selects the node in which the next occurrence is found
<b>1, ..., 9</b>	picks i-th subtopic from a menu

### 3.1.4 Interrupting SINGULAR

On Unix-like operating systems and on Windows NT, typing **CTRL-C** (or, alternatively **C-c C-c**, when running within Emacs), interrupts SINGULAR. SINGULAR prints the current command and the current line and prompts for further action. The following choices are available:

<b>a</b>	returns to the top level after finishing the current (kernel) command. Notice that commands of the SINGULAR kernel (like <b>std</b> ) cannot be aborted, i.e. (a)bort only happens whenever the interpreter is active.
<b>c</b>	continues
<b>q</b>	quits SINGULAR

### 3.1.5 Editing input

The following keys can be used for editing the input and retrieving previous input lines:

<b>TAB</b>	provides command line completion for function names and file names
<b>CTRL-B</b>	moves cursor to the left
<b>CTRL-F</b>	moves cursor to the right
<b>CTRL-A</b>	moves cursor to the beginning of the line
<b>CTRL-E</b>	moves cursor to the end of the line
<b>CTRL-D</b>	deletes the character under the cursor Warning: on an empty line, <b>CTRL-D</b> is interpreted as the EOF character which immediately terminates SINGULAR.

BACKSPACE

DELETE

CTRL-H      deletes the character before the cursor

CTRL-K      kills from cursor to the end of the line

CTRL-U      kills from cursor to the beginning of the line

CTRL-N      saves the current line to history and gives the next line

CTRL-P      saves the current line to history and gives the previous line

RETURN      saves the current line to the history and sends it to the SINGULAR parser for interpretation

When run under a Unix-like operating system and in its ASCII-terminal user interface, SINGULAR tries to dynamically link at runtime with the GNU Readline library. See [section “Command Line Editing” in \*The GNU Readline Library Manual\*](#), for more information. If a shared version of this library can be found on your machine, then additional command-line editing features like history completion are available. In particular, if SINGULAR is able to load that library the input history is stored across sessions using the file given in the environment variable `SINGULARHIST`. If `SINGULARHIST` is not set `.singularhistory` is used. Otherwise, i.e., if the environment variable `SINGULARHIST` is set to the empty string, the history of the last inputs is only available for previous commands of the current session.

### 3.1.6 Command line options

The startup syntax is

```
Singular  [options] [file1 [file2 ...]]
ESingular [options] [file1 [file2 ...]]
```

Options can be given in both their long and short format. The following options control the general behaviour of SINGULAR:

`-d, --sdb`    Enable the use of the source code debugger. See [Section 3.9.3 \[Source code debugger\]](#), [page 69](#).

`-e, --echo[=VAL]`

Set value of variable `echo` to `VAL` (integer in the range 0, . . . , 9). Without an argument, `echo` is set to 1, which echoes all input coming from a file. By default, the value of `echo` is 0. See [Section 5.3.2 \[echo\]](#), [page 303](#).

`-h, --help`

Print a one-line description of each command line option and exit.

`--allow-net`

Allow the help browsers based on a web browser to fetch HTML manual pages over the net from the WWW home-site of SINGULAR. See [Section 3.1.3 \[The online help system\]](#), [page 15](#), for more info.

`--browser="VAL"`

Use `VAL` as browser for the SINGULAR online manual.

`VAL` may be one of the browsers mentioned in `LIB/help.cnf`, for example `html` (Windows only), `mozilla`, `firefox`, `konqueror`, `galeon`, `netscape`, `safari` (OsX only), `xinfo`, `tkinfo`, `info`, `builtin`, or `emacs`. Depending on your platform and local installation, only some browsers might be available. The default browser is `html` for Windows and one based on a web browser for Unix platforms. See [Section 3.1.3 \[The online help system\]](#), [page 15](#), for more info.

- `--no-rc` Do not execute the `.singularrc` file on start-up. By default, this file is executed on start-up. See [Section 3.1.7 \[Startup sequence\]](#), page 22.
- `--no-stdlib` Do not load the library `standard.lib` on start-up. By default, this library is loaded on start-up. See [Section 3.1.7 \[Startup sequence\]](#), page 22.
- `--no-warn` Do not display warning messages.
- `--no-out` Suppress display of all output.
- `--no-shell` Runs Singular in restricted mode to disallow shell escape commands. Objects of type `link` will also be unable to use.
- `-t, --no-tty` Do not redefine the characteristics of the terminal. This option should be used for batch processes.
- `-q, --quiet` Do not print the start-up banner and messages when loading libraries. Furthermore, redirect `stderr` (all error messages) to `stdout` (normal output channel). This option should be used if SINGULAR's output is redirected to a file.
- `-v` Print extended information about the version and configuration of SINGULAR (used optional parts, compilation date, start of random generator etc.). This information should be included if a user reports an error to the authors.  
It also list all the used directories/files (see [Section 8.5 \[Used environment variables\]](#), page 2601).

The following command line options allow manipulations of the timer and the pseudo random generator and enable the passing of commands and strings to SINGULAR:

- `-c, --execute=STRING`  
Execute `STRING` as (a sequence of) SINGULAR commands on start-up after the `.singularrc` file is executed, but prior to executing the files given on the command line. E.g., `Singular -c "help all.lib; quit;"` shows the help for the library `all.lib` and exits.
- `-u, --user-option=STRING`  
Returns `STRING` on `system("--user-option")`. This is useful for passing arbitrary arguments from the command line to the SINGULAR interpreter. E.g., `Singular -u "xxx.dump" -c 'getdump(system("--user-option"))'` reads the file `xxx.dump` at start-up and allows the user to start working with all the objects defined in a previous session.
- `-r, --random=SEED`  
Seed (i.e., set the initial value of) the pseudo random generator with integer `SEED`. If this option is not given, then the random generator is seeded with a time-based `SEED` (the number of seconds since January, 1, 1970, on Unix-like operating systems, to be precise).
- `--min-time=SECS`  
If the `timer` (see [Section 5.3.8 \[timer\]](#), page 305), resp. `rtimer` (see [Section 5.3.10 \[rtimer\]](#), page 308), variable is set, report only times larger than `SECS` seconds (`SECS` needs to be a floating point number greater than 0). By default, this value is set to 0.5

(i.e., half a second). E.g., the option `--min-time=0.01` forces SINGULAR to report all times larger than  $1/100$  of a second.

`--ticks-per-sec=TICKS`

Set unit of timer to TICKS ticks per second (i.e., the value reported by the `timer` and `rtimer` variable divided by TICKS gives the time in seconds). By default, this value is 1.

`--cpus=CPUs`

set the maximal number of CPUs to use.

`--cntrlc=C`

set the default answer for interrupt signals to C which should be a for abort, c for continue or q for quit.

The next three options are of interest for the use with ssi links:

`-b, --batch`

Run in batch mode. Opens a TCP/IP connection with host specified by `--MPhost` at the port specified by `--MPport`. Input is read from and output is written to this connection in the format given by `--link`. See [Section 4.10.5 \[Ssi links\]](#), page 99.

`--MPport=PORT`

Use PORT as default port number for connections (whenever not further specified). This option is mandatory when the `--batch` option is given. See [Section 4.10.5 \[Ssi links\]](#), page 99.

`--MPhost=HOST`

Use HOST as default host for connections (whenever not further specified). This option is mandatory when the `--batch` option is given. See [Section 4.10.5 \[Ssi links\]](#), page 99.

Finally, the following options are only available when running ESingular (see [Section 3.2.2 \[Running SINGULAR under Emacs\]](#), page 25 for details).

`--emacs=EMACS`

Use EMACS as Emacs program to run the SINGULAR Emacs interface, where EMACS may e.g. be emacs or xemacs.

`--emacs-dir=DIR`

Set the singular-emacs-home-directory, which is the directory where singular.el can be found, to DIR.

`--emacs-load=FILE`

Load FILE on Emacs start-up, instead of the default load file.

`--singular=PROG`

Start PROG as SINGULAR program within Emacs

The value of options given to SINGULAR (resp. their default values, if an option was not given), can be checked with the command `system("--long-option_name")`. See [Section 5.1.155 \[system\]](#), page 275.

```
system("--quiet");    // if ‘‘quiet’’ 1, otherwise 0
↪ 1
system("--min-time"); // minimal reported time
↪ 0.5
system("--random");   // seed of the random generator
↪ 12345678
```



Furthermore, the value of options (e.g., `--browser`) can be re-defined while SINGULAR is running using the command `system("--long_option_name_string ", expression)`. See [Section 5.1.155 \[system\]](#), page 275.

```
system("--browser", "builtin"); // sets browser to 'builtin'
system("--ticks-per-sec", 100); // sets timer resolution to 100
```

### 3.1.7 Startup sequence

On start-up, SINGULAR

1. loads the library `standard.lib` (provided the `--no-stdlib` option was not given),
2. searches the current directory and then the home directory of the user, and then all directories contained in the library `SearchPath` (see [Section 3.8.11 \[Loading a library\]](#), page 67 for more info on `SearchPath`) for a file named `.singularrc` and executes it, if found (provided the `--no-rc` option was not given),
3. executes the string specified with the `--execute` command line option,
4. executes the files `file1`, `file2` ... (given on the command line) in that order.

**Note:** `.singularrc` file(s) are an appropriate place for setting some default values of (command-line) options.

For example, a system administrator might remove the locally installed HTML version of the manual and put a `.singularrc` file with the following content

```
if (system("version") >= 1306) // assure backwards-compatibility
{
    system("--allow-net", 1);
}; // the last semicolon is important: otherwise no ">", but "." prompt
```

in the directory containing the SINGULAR libraries, thereby allowing to fetch the HTML on-line help from the WWW home-site of SINGULAR.

On the other hand, a single user might put a `.singularrc` with the following content

```
if (system("version") >= 1306) // assure backwards-compatibility
{
    if (! system("--emacs"))
    {
        // set default browser to info, unless we run within emacs
        system("--browser", "info");
    }
}; // the last semicolon is important: otherwise no ">", but "." prompt
```

in his home directory, which sets the default help browser to `info` (unless SINGULAR is run within emacs) and thereby prevents the execution of the "global" `.singularrc` file installed by the system administrator (since the `.singularrc` file of the user is found before the "global" `.singularrc` file installed by the system administrator).

## 3.2 Emacs user interface

Besides running SINGULAR in an ASCII-terminal, SINGULAR might also be run within Emacs. Emacs (or, XEmacs which is very similar) is a powerful and freely available text editor, which, among others, provides a framework for the implementation of interactive user interfaces. Starting from version 1.3.6, SINGULAR provides such an implementation, the so-called SINGULAR Emacs mode, or Emacs user interface.



Generally, we recommend to use the Emacs interface, instead of the ASCII-terminal interface: The Emacs interface does not only provide everything the ASCII-terminal interface provides, but offers much more. Among others, it offers

- color highlighting
- truncation of long lines
- folding of input and output
- TAB-completion for help topics
- highlighting of matching parentheses
- key-bindings and interactive menus for most user interface commands and for basic SINGULAR commands (such as loading of libraries and files)
- a mode for running interactive SINGULAR demonstrations
- convenient ways to edit SINGULAR input files
- interactive customization of nearly all aspects of the user-interface.

In order to use the SINGULAR-Emacs interface you need to have Emacs version 20 or higher, or XEmacs version 20.3 or higher installed on your system. These editors can be downloaded for most hard- and software platforms, sources from either <http://www.gnu.org/software/emacs/emacs.html> (Emacs), or from <http://www.xemacs.org> (XEmacs). (Download of binaries depend on your OS). The differences between Emacs and XEmacs w.r.t. the SINGULAR-Emacs interface are marginal – which editor to use is mainly a matter of personal preferences.

The simplest way to start-up SINGULAR in its Emacs interface is by running the program `ESingular` which is contained in the Singular distribution. Alternatively, SINGULAR can be started within an already running Emacs – see [Section 3.2.2 \[Running SINGULAR under Emacs\]](#), page 25 for details.

The next section gives a tutorial-like introduction to Emacs. This introductory section is followed by sections which explain the functionality of various aspects of the Emacs user interface in more detail: how to start/restart/kill SINGULAR within Emacs, how to run an interactive demonstration, how to customize the Emacs user interface, etc. Finally, the 20 most important commands of the Emacs interface together with their key bindings are listed.

### 3.2.1 A quick guide to Emacs

This section gives a tutorial-like introduction to Emacs. Especially to users who are not familiar with Emacs, we recommend that they go through this section and try out the described features.

Emacs commands generally involve the **CONTROL** key (sometimes labeled **CTRL** or **CTL**) or the **META** key. On some keyboards, the **META** key is labeled **ALT** or **EDIT** or something else (for example, on Sun keyboards, the diamond key to the left of the space-bar is **META**). If there is no **META** key, the **ESC** key can be used, instead. Rather than writing out **META** or **CONTROL** each time we want to prefix a character, we will use the following abbreviations:

**C-*<chr>*** means hold the **CONTROL** key while typing the character *<chr>*. Thus, **C-f** would be: hold the **CONTROL** key and type **f**.  
**M-*<chr>*** means hold the **META** key down while typing *<chr>*. If there is no **META** key, type **ESC**, release it, then type the character *<chr>*.

For users new to Emacs, we highly recommend that they go through the interactive Emacs tutorial: type **C-h t** to start it.

For others, it is important to understand the following Emacs concepts:

**window** In Emacs terminology, a window refers to separate panes within the same window of the window system, and not to overlapping, separate windows. When using SINGULAR

within Emacs, extra windows may appear which display help or output from certain commands. The most important window commands are:

<b>C-x 1</b>	<b>File-&gt;Un-Split</b>	Un-Split window (i.e., kill other windows)
<b>C-x o</b>		Goto other window, i.e. move cursor into other window.

#### cursor and point

The location of the cursor in the text is also called "point". To paraphrase, the cursor shows on the screen where point is located in the text. Here is a summary of simple cursor-moving operations:

<b>C-f</b>	Move forward a character
<b>C-b</b>	Move backward a character
<b>M-f</b>	Move forward a word
<b>M-b</b>	Move backward a word
<b>C-a</b>	Move to the beginning of line
<b>C-e</b>	Move to the end of line

#### buffer

Any text you see in an Emacs window is always part of some buffer. For example, each file you are editing with Emacs is stored inside a buffer, but also SINGULAR is running inside an Emacs buffer. Each buffer has a name: for example, the buffer of a file you edit usually has the same name as the file, SINGULAR is running in a buffer which has the name *\*singular\** (or, *\*singular<2>\**, *\*singular<3>\**, etc., if you have multiple SINGULAR sessions within the same Emacs).

When you are asked for input to an Emacs command, the cursor moves to the bottom line of Emacs, i.e., to a special buffer, called the "minibuffer". Typing **(RETURN)** within the minibuffer, ends the input, typing **(SPACE)** within the minibuffer, lists all possible input values to the interactive Emacs command.

The most important buffer commands are

<b>C-x b</b>	Switch buffer
<b>C-x k</b>	Kill current buffer

Alternatively, you can switch to or kill buffers using the **Buffer** menu.

#### Executing commands

Emacs commands are executed by typing **M-x <command-name>** (remember that **(SPACE)** completes partial command names). Important and frequently used commands have short-cuts for their execution: Key bindings or even menu entries. For example, a file can be loaded with **M-x load-file**, or **C-x C-f**, or with the **File->Open** menu.

#### How to exit

To end the Emacs (and, SINGULAR) session, type **C-x C-c** (two characters), or use the **File -> Exit** menu.

#### When Emacs hangs

If Emacs stops responding to your commands, you can stop it safely by typing **C-g**, or, if this fails, by typing **C-]**.

#### More help

Nearly all aspects of Emacs are very well documented: type **C-h** and then a character saying what kind of help you want. For example, typing **C-h i** enters the **Info** documentation browser.

#### Using the mouse

Emacs is fully integrated with the mouse. In particular, clicking the right mouse button brings up a pop-up menu which usually contains a few commonly used commands.

### 3.2.2 Running SINGULAR under Emacs

There are two ways to start the SINGULAR Emacs interface: Typing `ESingular` instead of `Singular` on the command shell launches a new Emacs process, initializes the interface and runs SINGULAR within Emacs. The other way is to start the interface in an already running Emacs, by typing `M-x singular` inside Emacs. This initializes the interface and runs SINGULAR within Emacs. Both ways are described in more detail below.

Note: To properly run the Emacs interface, several files are needed which usually reside in the `emacs` subdirectory of your SINGULAR distribution. This directory is called `singular-emacs-home-directory` in the following.

#### Starting the interface using ESingular

As mentioned above, `ESingular` is an "out-of-the-box" solution: You don't have to add special things to your `.emacs` startup file to initialize the interface; everything is done for you in a special file called `.emacs-singular` (which comes along with the SINGULAR distribution and resides in the `singular-emacs-home-directory`) which is automatically loaded on Emacs startup (and the loading of the `.emacs` file is automatically suppressed).

The customizable variables of the SINGULAR Emacs interface are set to defaults which give the novice user a very shell like feeling of the interface. Nevertheless, these default settings can be changed, see [Section 3.2.4 \[Customization of the Emacs interface\], page 27](#). Besides other Emacs initializations, such as fontification or blinking parentheses, a new menu item called `Singular` is added to the main menu, providing menu items for starting SINGULAR. On XEmacs, a button starting SINGULAR is added to the main toolbar.

The SINGULAR interface is started automatically; once you see a buffer called `*singular*` and the SINGULAR prompt, you are ready to start your SINGULAR session.

`ESingular` inherits all `Singular` options. For a description of all these options, see [Section 3.1.6 \[Command line options\], page 19](#). Additionally there are the following options which are special to `ESingular`:

command-line option / environment variable	functionality
<code>--emacs=EMACS</code> <code>ESINGULAR_EMACS</code>	Use <code>EMACS</code> as Emacs program to run the SINGULAR Emacs interface, where <code>EMACS</code> may e.g. be <code>emacs</code> or <code>xemacs</code> .
<code>--emacs-dir=DIR</code> <code>ESINGULAR_EMACS_DIR</code>	Set the <code>singular-emacs-home-directory</code> , which is the directory where <code>singular.el</code> can be found, to <code>DIR</code> .
<code>--emacs-load=FILE</code> <code>ESINGULAR_EMACS_LOAD</code>	Load <code>FILE</code> on Emacs start-up, instead of the default load file.
<code>--singular=PROG</code> <code>ESINGULAR_SINGULAR</code>	Start <code>PROG</code> as SINGULAR program within Emacs

Notice that values of these options can also be given by setting the above mentioned environment variables (where values given as command-line arguments take priority over values given by environment variables).

#### Starting the interface within a running Emacs

If you are a more experienced Emacs user and you already have your own local `.emacs` startup file, you might want to start the interface out of your running Emacs without using `ESingular`. For this, you should add the following lisp code to your `.emacs` file:

```
(setq load-path (cons "<singular-emacs-home-directory>" load-path))
(autoload 'singular "singular"
  "Start Singular using default values." t)
(autoload 'singular-other "singular"
  "Ask for arguments and start Singular." t)
```

Then typing `M-x singular` in a running Emacs session initializes the interface in a new buffer and launches a SINGULAR process. The SINGULAR prompt comes up and you are ready to start your SINGULAR session.

It is a good idea to take a look at the (well documented) file `.emacs-singular` in the `singular-emacs-home-directory`, which comes along with the distribution. In it you find some useful initializations of the SINGULAR interface as well as some lisp code, which, for example, adds a button to the XEmacs toolbar. Some of this code might be useful for your `.emacs` file, too. And if you are an Emacs wizard, it is of course a good idea to take a look at `singular.el` in the `singular-emacs-home-directory`.

## CYGWIN and ESingular

**X11 server** install `xlaunch`, `emacs-X11`, `xterm` and all dependencies. Create with `xlaunch` a startup file for the X-server which also starts the client `xterm`. From that one can start ESingular.

**fork problems** The simplest way to overcome fork problem is to run `/usr/bin/rebase-trigger full`, then stop all Cygwin processes and services, and then run `setup-x86.exe`. The `_autorebase` postinstall script will then take care of the rebase. Occasionally it is necessary to reboot the computer before doing this.

## Starting, interrupting and stopping SINGULAR

There are the following commands to start and stop SINGULAR:

- `singular-other` (or menu `Singular`, item `Start...`)

Starts a SINGULAR process and asks for the following four parameters in the minibuffer area:

1. The SINGULAR executable. This can either be a file name with complete path, e.g., `/local/bin/Singular`. Then exactly this executable is started. The path may contain the character `~` denoting your home directory. Or it can be the name of a command without path, e.g., `Singular`. Then the executable is searched for in your `$PATH` environment variable.
2. The default working directory. This is the path to an existing directory, e.g., `~/work`. The current directory is set to this directory before SINGULAR is started.
3. Command line options. You can set any SINGULAR command line option (see [Section 3.1.6 \[Command line options\]](#), page 19).
4. The buffer name. You can specify the name of the buffer the interface is running in.

- `singular` (or menu `Singular`, item `Start default`)

Starts SINGULAR with default settings for the executable, the working directory, command line switches, and the buffer name. You can customize this default settings, see [Section 3.2.4 \[Customization of the Emacs interface\]](#), page 27.

- `singular-exit-singular` (bound to `C-c $` or menu `Singular`, item `Exit`)

Kills the running SINGULAR process of the current buffer (but does not kill the buffer). Once you have killed a SINGULAR process you can start a new one in the same buffer with the command `singular` (or select the item `Start default` of the `Singular` menu).

- **singular-restart** (bound to `C-c C-r` or menu **Singular**, item **Restart**)  
Kills the running SINGULAR process of the current buffer and starts a new process in the same buffer with exactly the same command line arguments as before.
- **singular-control-c** (bound to `C-c C-c` or menu **Singular**, item **Interrupt**)  
Interrupt the SINGULAR process running in the current buffer. Asks whether to (a)bort the current SINGULAR command, (q)uit or (r)estart the current SINGULAR process, or (c)ontinue without doing anything (default).

Whenever a SINGULAR process is started within the Emacs interface, the contents of a special startup file (by default `~/.emacs-singularrc`) is pasted as input to SINGULAR at the very end of the usual startup sequence (see [Section 3.1.7 \[Startup sequence\]](#), page 22). The name of the startup file can be changed, see [Section 3.2.4 \[Customization of the Emacs interface\]](#), page 27.

### 3.2.3 Demo mode

The Emacs interface can be used to run interactive SINGULAR demonstrations. A demonstration is started by loading a so-called SINGULAR demo file with the Emacs command **singular-demo-load**, bound to `C-c C-d`, or with the menu **Commands->Load Demo**.

A SINGULAR demo file should consist of SINGULAR commands separated by blank lines. When running a demo, the input up to the next blank line is echoed to the screen. Hitting `(RETURN)` executes the echoed commands and shows their output. Hitting `(RETURN)` again, echos the next commands to the screen, and so on, until all commands of the demo file are executed. While running a demo, you can execute other commands on the SINGULAR prompt: the next input from the demo file is then echoed again, if you hit `(RETURN)` on an empty input line.

A SINGULAR demo can prematurely be exited by either starting another demo, or by executing the Emacs command **singular-demo-exit** (menu: **Commands->Exit Demo**).

Some aspects of running SINGULAR demos can be customized. See [Section 3.2.4 \[Customization of the Emacs interface\]](#), page 27, for more info.

### 3.2.4 Customization of the Emacs interface

Emacs provides a convenient interface to customize the behavior of Emacs and the SINGULAR Emacs interface for your own needs. You enter the customize environment by either calling `M-x customize` (on XEmacs you afterwards have to enter **emacs** in the minibuffer area) or by selecting the menu item **Options->Customize->Emacs...** for XEmacs, and the menu item **Help->Customize->Toplevel Customization Group** for Emacs, resp. A brief introduction to the customization mode comes up with the customization buffer. All customizable parameters are hierarchically grouped and you can browse through all these groups and change the values of the parameters using the mouse. At the end you can save your settings to a file making your changes permanent.

To change the settings of the SINGULAR Emacs interface you can either select the item **Preferences** of the **Singular** menu, call `M-x customize-group` and give the argument **singular-interactive** in the minibuffer area, or browse from the top-level customization group through the path **External->Singular->Singular interactive**.

The SINGULAR interface customization buffer is divided into four groups:

- **Singular Faces**  
Here you can specify various faces used if font-lock-mode is enabled (which, by default, is).

- Singular Sections And Foldings

Here you can specify special faces for SINGULAR input and output and change the text used as replacement for folded sections.

For doing this, you also might find handy the function `customize-face-at-point`, which lets you customize the face at the current position of point. This function is automatically defined if you run `ESingular`). Otherwise, you should add its definition (see below) to your personal `.emacs` file.

- Singular Interactive Miscellaneous

Here you can specify various things such as the behavior of the cursor keys, the name of the special SINGULAR startup file, the appearance of the help window, or the default values for the `singular` command.

- Singular Demo Mode

Here you can specify how chunks of the demo file are divided, or specify a default directory for demo files.

When you run `ESingular`, the settings of customized variables are saved in the file `$HOME/.emacs-singular-cust`. Otherwise, the settings are appended to your `.emacs` file. Among others, this means that the customized settings of `ESingular` are not automatically taken over by a "normal" Emacs, and vice versa.

### 3.2.5 Editing SINGULAR input files with Emacs

Since SINGULAR's programming language is similar to C, you should use the Emacs C/C++-mode to edit SINGULAR input files and SINGULAR libraries. Among others, this Emacs mode provides automatic indentation, line-breaking and keyword highlighting.

When running `ESingular`, the C/C++-mode is automatically turned on whenever a file with the suffix `.sing`, or `.lib` is loaded.

For Emacs sessions which were not started by `ESingular`, you should add the following to your `.emacs` file:

```
;; turn on c++-mode for files ending in ".sing" and ".lib"
(setq auto-mode-alist (cons '("\\.sing\\\\" . c++-mode) auto-mode-alist))
(setq auto-mode-alist (cons '("\\.lib\\\\" . c++-mode) auto-mode-alist))
;; turn-on fontification for c++-mode
(add-hook 'c++-mode-hook
  (function (lambda () (font-lock-mode 1))))
;; turn on aut-new line and hungry-delete
(add-hook 'c++-mode-hook
  (function (lambda () (c-toggle-auto-hungry-state 1))))
;; a handy function for customization
(defun customize-face-at-point ()
  "Customize face which point is at."
  (interactive)
  (let ((face (get-text-property (point) 'face)))
    (if face
        (customize-face face)
        (message "No face defined at point"))))
```

Notice that you can change the default settings for source-code highlighting (colors, fonts, etc.) by customizing the respective faces using the `Customize` feature of Emacs. For doing this, you might find handy the above given function `customize-face-at-point`, which lets you customize the face of the current position of point (this function is automatically defined if you run `ESingular`).



### 3.2.6 Top 20 Emacs commands

Here is a list of the 20 probably most useful commands when using the SINGULAR Emacs interface. Starting and stopping of SINGULAR:

- `singular` (menu `Singular->Start Default...`): starts SINGULAR using default arguments.
- `singular-other` (menu `Singular->Start`): starts SINGULAR asking for several arguments in the minibuffer area.
- `singular-exit` (key `C-c $` or menu `Singular->Exit`): kills the SINGULAR process running in the current buffer (but does not kill the buffer).
- `singular-restart` (key `C-c C-r` or menu `Singular->Restart`): kills the SINGULAR process running in the current buffer and starts a new SINGULAR process with exactly the same arguments as before.

Editing input and output:

- `singular-beginning-of-line` (key `C-a`): moves point to beginning of line, then skips past the SINGULAR prompt, if any.
- `singular-toggle-truncate-lines` (key `C-c C-t` or menu `Commands->Truncate lines`): toggles whether long lines should be truncated or not. If lines are not truncated, the commands `singular-scroll-left` and `singular-scroll-right` are useful to scroll left and right, resp.
- `singular-dynamic-complete` (key `TAB`): performs context specific completion. If point is inside a string, file name completion is done. If point is at the end of a help command (i.e., `help` or `?`), completion on SINGULAR help topics is done. If point is at the end of an example command (i.e., `example`), completion is done on SINGULAR examples. In all other cases, completion on SINGULAR commands is done.
- `singular-folding-toggle-fold-latest-output` (key `C-c C-o` or menu `Commands->Fold/Unfold Latest Output`): toggles folding of the latest output section. If your last SINGULAR command produced a huge output, simply type `C-c C-o` and it will be replaced by a single line.
- `singular-folding-toggle-fold-at-point` (key `C-c C-f` or menu `Commands->Fold/Unfold At Point`): toggles folding of the section the point currently is in.
- `singular-folding-fold-all-output` (menu `Commands->Fold All Output`): folds all SINGULAR output, replacing each output section by a single line.
- `singular-folding-unfold-all-output` (menu `Commands->Unfold All Output`): unfolds all SINGULAR output sections showing their true contents.

Loading of files and SINGULAR demo mode:

- `singular-load-library` (key `C-c C-l` or menu `Commands->Libraries->other...`): asks for a standard library name or a library file in the minibuffer (hit `TAB` for completion) and loads the library into SINGULAR. The submenu `Libraries` of the `Commands` menu also provides a separate menu item for each standard library.
- `singular-load-file` (key `C-c <` or menu `Commands->Load File...`): asks for a file name in the minibuffer (which is expanded using `expand-file-name` if given a prefix argument) and loads the file into SINGULAR.
- `singular-demo-load` (key `C-c C-d` or menu `Commands->Load Demo...`): asks for a file name of a SINGULAR demo file in the minibuffer area (hit `SPACE` for completion) and enters the SINGULAR demo mode showing the first chunk of the demo.
- `singular-demo-exit` (menu `Commands->Exit Demo`): exits from SINGULAR demo mode and cleans up everything that is left from the demo.

Help and Customization:

- **singular-help** (key `C-h C-s` or menu **Singular->Singular Help**): asks for a SINGULAR help topic in the minibuffer (hit TAB for completion) and shows the help text in a separate buffer.
- **singular-example** (key `C-c C-e` or menu **Singular->Singular Example**): asks for a SINGULAR command in the minibuffer (hit TAB for completion) and executes the example of this command in the current SINGULAR buffer.
- **customize-group** (menu **Singular->Preferences**): enters the customization group of the SINGULAR Emacs interface. (If called via `M-x customize-group` give argument **singular-interactive** in the minibuffer area.)

### 3.3 Rings and orderings

All non-trivial algorithms in SINGULAR require the prior definition of a ring. Such a ring can be

1. a polynomial ring over a field,
2. a polynomial ring over a ring
3. a localization of 1.
4. a quotient ring by an ideal of 1. or 2.,
5. a tensor product of 1. or 2.

Except for quotient rings, all of these rings are realized by choosing a coefficient field, ring variables, and an appropriate global or local monomial ordering on the ring variables. See [Section 3.3.3 \[Term orderings\]](#), page 34, [Appendix C \[Mathematical background\]](#), page 775.

The coefficient field of the rings may be

1. the field of rational numbers  $Q$  (QQ),
2. finite fields  $Z/p$ ,  $p$  a prime  $\leq 2147483647$ ,
3. finite fields  $GF(p^n)$  with  $p^n$  elements,  $p$  a prime,  $p^n \leq 2^{16}$ ,
4. transcendental extension of  $Q$  or  $Z/p$ ,
5. simple algebraic extension of  $Q$  or  $Z/p$ ,
6. the field of real numbers represented by floating point numbers of a user defined precision,
7. the field of complex numbers represented by (pairs of) floating point numbers of a user defined precision,
8. the ring of integers (ZZ),
9. finite rings  $Z/m$  with  $m \in Z$ .

In case of coefficient rings, which are not fields (i.e.  $Z$  and  $Z/ma$ ), only the following functions are guaranteed to work:

- basic polynomial arithmetic, i.e. addition, multiplication, exponentiation
- std, i.e. computing standard bases (and related: syz, etc.)
- interred
- reduce

Throughout this manual, the current active ring in SINGULAR is called basering. The reserved name **basing** in SINGULAR is an alias for the current active ring. The basering can be set by declaring a new ring as described in the following subsections or by using the commands **setring** and **keepring**. See [Section 5.2.11 \[keepring\]](#), page 298, [Section 5.1.141 \[setring\]](#), page 260.

Objects of ring dependent types are local to a ring. To access them after a change of the basering they have to be mapped using **map** or by the functions **imap** or **fetch**. See [Section 3.5.4 \[Objects\]](#),



page 46, [Section 5.1.38 \[fetch\]](#), page 182, [Section 5.1.59 \[imap\]](#), page 198, [Section 4.12 \[map\]](#), page 106.

All changes of the basering in a procedure are local to this procedure unless a `keepring` command is used as the last statement of the procedure. See [Section 3.7 \[Procedures\]](#), page 50, [Section 5.2.11 \[keepring\]](#), page 298.

### 3.3.1 Examples of ring declarations

The exact syntax of a ring declaration is given in the next two subsections; this subsection lists some examples first. Note that the chosen ordering implies that a unit-elements of the ring will be among the elements with leading monomial 1. For more information, see [Section B.2 \[Monomial orderings\]](#), page 769.

Every floating point number in a ring consists of two parts, which may be chosen by the user. The leading part represents the number and the rest is for numerical stability. Two numbers with a difference only in the rest will be regarded equal.

- the ring  $Z/32003[x, y, z]$  with degree reverse lexicographical ordering. The exact ring declaration may be omitted in the first example since this is the default ring:

```
ring r1;
ring r2 = 32003, (x, y, z), dp;
ring r3 = (ZZ/32003) [x, y, z];
ring r4 = (ZZ/32003), (x, y, z), dp;
```

- similar examples with indexed variables. The ring variables of `r1` are going to be `x(1)..x(10)`; in `r2` they will be `x(1)(1)`, `x(1)(2)`, ..., `x(1)(8)`, `x(2)(1)`, ..., `x(5)(8)`:

```
ring r1 = 32003, (x(1..10)), dp;
ring r2 = 32003, (x(1..5)(1..8)), dp;
ring r3 = (ZZ/32003) [x(1..5)(1..8)];
ring r4 = (ZZ/32003), (x(1..5)(1..8)), dp;
```

- the ring  $Q[a, b, c, d]$  with lexicographical ordering:

```
ring r1 = 0, (a, b, c, d), lp;
ring r2 = QQ, (a, b, c, d), lp;
```

- the ring  $Z/7[x, y, z]$  with local degree reverse lexicographical ordering. The non-prime 10 is converted to the next lower prime in the second example:

```
ring r1 = 7, (x, y, z), ds;
ring r2 = 10, (x, y, z), ds;
ring r3 = (ZZ/7), (x, y, z), ds;
```

- the ring  $Z/7[x_1, \dots, x_6]$  with lexicographical ordering for  $x_1, x_2, x_3$  and degree reverse lexicographical ordering for  $x_4, x_5, x_6$ :

```
ring r1 = 7, (x(1..6)), (lp(3), dp);
ring r2 = (ZZ/7), (x(1..6)), (lp(3), dp);
```

- the localization of  $(Q[a, b, c])[x, y, z]$  at the maximal ideal

$(x, y, z)$  :

```
ring r1 = 0, (x, y, z, a, b, c), (ds(3), dp(3));
ring r2 = QQ, (x, y, z, a, b, c), (ds(3), dp(3));
```

- the ring  $Q[x, y, z]$  with weighted reverse lexicographical ordering. The variables  $x$ ,  $y$ , and  $z$  have the weights 2, 1, and 3, respectively, and vectors are first ordered by components (in descending order) and then by monomials:

```
ring r1 = 0, (x, y, z), (c, wp(2, 1, 3));
ring r2 = QQ, (x, y, z), (c, wp(2, 1, 3));
```

For ascending component order, the component ordering `C` has to be used.

- the ring  $K[x, y, z]$ , where  $K = \mathbb{Z}/7(a, b, c)$  denotes the transcendental extension of  $\mathbb{Z}/7$  by  $a$ ,  $b$  and  $c$  with degree lexicographical ordering:

```
ring r = (7,a,b,c),(x,y,z),Dp;
```

- the ring  $K[x, y, z]$ , where  $K = \mathbb{Z}/7[a]$  denotes the algebraic extension of degree 2 of  $\mathbb{Z}/7$  by  $a$ . In other words,  $K$  is the finite field with 49 elements. In the first case,  $a$  denotes an algebraic element over  $\mathbb{Z}/7$  with minimal polynomial  $\mu_a = a^2 + a + 3$ , in the second case,  $a$  refers to some generator of the cyclic group of units of  $K$ :

```
ring r = (7,a),(x,y,z),dp; minpoly = a^2+a+3;
ring r = (7^2,a),(x,y,z),dp;
```

- the ring  $R[x, y, z]$ , where  $R$  denotes the field of real numbers represented by simple precision floating point numbers. This is a special case:

```
ring r = real,(x,y,z),dp;
```

- the ring  $R[x, y, z]$ , where  $R$  denotes the field of real numbers represented by floating point numbers of 50 valid decimal digits and the same number of digits for the rest:

```
ring r = (real,50),(x,y,z),dp;
```

- the ring  $R[x, y, z]$ , where  $R$  denotes the field of real numbers represented by floating point numbers of 10 valid decimal digits and with 50 digits for the rest:

```
ring r = (real,10,50),(x,y,z),dp;
```

- the ring  $R(j)[x, y, z]$ , where  $R$  denotes the field of real numbers represented by floating point numbers of 30 valid decimal digits and the same number for the rest.  $j$  denotes the imaginary unit.

```
ring r = (complex,30,j),(x,y,z),dp;
```

- the ring  $R(i)[x, y, z]$ , where  $R$  denotes the field of real numbers represented by floating point numbers of 6 valid decimal digits and the same number for the rest.  $i$  is the default for the imaginary unit.

```
ring r = complex,(x,y,z),dp;
```

- the quotient ring  $\mathbb{Z}/7[x, y, z]$  modulo the square of the maximal ideal  $(x, y, z)$ :

```
ring R = 7,(x,y,z), dp;
qring r = std(maxideal(2));
```

- the ring  $\mathbb{Z}[x, y, z]$ :

```
ring R = integer,(x,y,z), dp;
```

- the ring  $\mathbb{Z}/6^3[x, y, z]$ :

```
ring R = (integer, 6, 3),(x,y,z), dp;
```

- the ring  $\mathbb{Z}/100[x, y, z]$ :

```
ring R = (integer, 100),(x,y,z), dp;
```

### 3.3.2 General syntax of a ring declaration

#### Rings

**Syntax:** `ring name = (coefficients), ( names_of_ring_variables ), ( ordering );` or  
`ring name = cring [ names_of_ring_variables ]`

**Default:** `(ZZ/32003)[x,y,z];`

**Purpose:** declares a ring and sets it as the current basering. The second form sets the ordering to `(dp,C)`. `cring` stands currently for `QQ` (the rationals), `ZZ` (the integers) or `(ZZ/m)` (the field ( $m$  prime and  $<2147483648$ ) resp. ring of the integers modulo  $m$ ).

The coefficients (for the first form) are given by one of the following:

1. a **cring** as given above
2. a non-negative `int_expression` less than 2147483648 ( $2^{31}$ ).  
The `int_expression` should either be 0, specifying the field of rational numbers  $\mathbb{Q}$ , or a prime number  $p$ , specifying the finite field with  $p$  elements. If it is not a prime number, `int_expression` is converted to the next lower prime number.
3. an `expression_list` of an `int_expression` and one or more names.  
The `int_expression` specifies the characteristic of the coefficient field as described above. The names are used as parameters in transcendental or algebraic extensions of the coefficient field. Algebraic extensions are implemented for one parameter only. In this case, a minimal polynomial has to be defined by an assignment to `minpoly`. See [Section 5.3.3 \[minpoly\]](#), page 303.
4. an `expression_list` of an `int_expression` and a name.  
The `int_expression` has to be a prime number  $p$  to the power of a positive integer  $n$ . This defines the Galois field  $\text{GF}(p^n)$  with  $p^n$  elements, where  $p^n$  has to be less than or equal to  $2^{15}$ . The given name refers to a primitive element of  $\text{GF}(p^n)$  generating the multiplicative group. Due to a different internal representation, the arithmetic operations in these coefficient fields are faster than arithmetic operations in algebraic extensions as described above.
5. an `expression_list` of the name **real** and two optional `int_expressions` determining the precision in decimal digits and the size for the stabilizing rest. The default for the rest is the same size as for the representation. An exception is the name **real** without any integers. These numbers are implemented as machine floating point numbers of single precision. Note that computations over all these fields are not exact.
6. an `expression_list` of the name **complex**, two optional `int_expression` and a name. This specifies the field of complex numbers represented by floating point numbers with a precision similar to **real**. An `expression_list` without `int_expression` defines a precision and rest with length 6. The name of the imaginary unit is given by the last parameter. Note that computations over these fields are not exact.
7. an `expression_list` with the name **integer**. This specifies the ring of integers.
8. an `expression_list` with the name **integer** and one subsequent `int_expression`. This specifies the ring of integers modulo the given `int_expression`.
9. an `expression_list` with the name **integer** and two `int_expressions` **b** and **e**. This specifies the ring of integers modulo  $b^e$ . If  $b = 2$  and  $e < \text{int\_bit\_size}$  an optimized implementation is used.

'`names_of_ring_variables`' is a list of names or indexed names.

'`ordering`' is a list of block orderings where each block ordering is either

1. **lp**, **dp**, **Dp**, **ls**, **ds**, or **Ds** optionally followed by a size parameter in parentheses.
2. **wp**, **Wp**, **ws**, **Ws**, or **a** followed by a weight vector given as an `intvec_expression` in parentheses.
3. **M** followed by an `intmat_expression` in parentheses.
4. **c** or **C**.

For the definition of the orderings, see [Section B.2 \[Monomial orderings\]](#), page 769.

If one of coefficients, `names_of_ring_variables`, and `ordering` consists of only one entry, the parentheses around this entry may be omitted.

## Quotient rings

**Syntax:**     `qring name = ideal_expression ;`

**Default:** none

**Purpose:** declares a quotient ring as the basering modulo `ideal_expression`, and sets it as current basering.

`ideal_expression` has to be represented by a standard basis.

The most convenient way to map objects from a ring to its quotient ring and vice versa is to use the `fetch` function (see [Section 5.1.38 \[fetch\]](#), page 182).

SINGULAR computes in a quotient ring as long as possible with the given representative of a polynomial, say, `f`. I.e., it usually does not reduce `f` w.r.t. the quotient ideal. This is only done when necessary during standard bases computations or by an explicit reduction using the command `reduce(f, std(0))` (see [Section 5.1.131 \[reduce\]](#), page 251).

Operations based on standard bases (e.g. `std,groebner`, etc., `reduce`) and functions which require a standard basis (e.g. `dim,hilb`, etc.) operated with the residue classes; all others on the polynomial objects.

**Example:**

```
// definition and usage:
ring r=(ZZ/32003)[x,y];
poly f=x3+yx2+3y+4;
qring q=std(maxideal(2));
basing;
↳ // coefficients: ZZ/32003
↳ // number of vars : 2
↳ //      block 1 : ordering dp
↳ //      : names  x y
↳ //      block 2 : ordering C
↳ // quotient ring from ideal
↳ _[1]=y2
↳ _[2]=xy
↳ _[3]=x2
poly g=fetch(r, f);
g;
↳ x3+x2y+3y+4
reduce(g,std(0));
↳ 3y+4
// polynomial and residue class:
ring R=QQ[x,y];
qring Q=std(y);
poly p1=x;
poly p2=x+y;
// comparing polynomial objects:
p1==p2;
↳ 0
// comparing residue classes:
reduce(p1,std(0))==reduce(p2,std(0));
↳ 1
```

### 3.3.3 Term orderings

Any polynomial (resp. vector) in SINGULAR is ordered w.r.t. a term ordering (or, monomial ordering), which has to be specified together with the declaration of a ring. SINGULAR stores and displays a polynomial (resp. vector) w.r.t. this ordering, i.e., the greatest monomial (also called the

leading monomial) is the first one appearing in the output polynomial, and the smallest monomial is the last one.

**Remark:** The novice user should generally use the ordering `dp` for computations in the polynomial ring  $K[x_1, \dots, x_n]$ , resp. `ds` for computations in the localization  $\text{Loc}_{(x)} K[x_1, \dots, x_n]$ . For more details, see [Appendix B \[Polynomial data\]](#), page 768.

In a ring declaration, SINGULAR offers the following orderings (but see also [Section B.2 \[Monomial orderings\]](#), page 769):

#### 1. Global orderings

- `lp`           lexicographical ordering
- `rp`           inverse lexicographical ordering, i.e. a lexicographical ordering from the right with  $1 < x_{-1} < \dots < x_{-n}$   
(should not be used as it reverses the "natural"  $x_{-1} > \dots > x_{-n}$ , reorder the variables instead)
- `dp`           degree reverse lexicographical ordering
- `Dp`           degree lexicographical ordering
- `wp( intvec_expression )`  
                weighted reverse lexicographical ordering; the weight vector is expected to consist of positive integers only.
- `Wp( intvec_expression )`  
                weighted lexicographical ordering; the weight vector is expected to consist of positive integers only.
- `Ip`           degree inverse lexicographical ordering, i.e. sorting by degree, followed by a lexicographical ordering from the right with  $1 < x_{-1} < \dots < x_{-n}$   
(should not be used as it reverses the "natural"  $x_{-1} > \dots > x_{-n}$ , reorder the variables instead)

Global orderings are well-orderings, i.e.,  $1 < x$  for each ring variable  $x$ . They are denoted by a `p` as the second character in their name.

#### 2. Local orderings

- `ls`           negative lexicographical ordering
- `rs`           negative inverse lexicographical ordering, i.e. a lexicographical ordering from the right  
(should not be used as it reverses the "natural"  $x_{-1} < \dots < x_{-n}$ , reorder the variables instead)
- `ds`           negative degree reverse lexicographical ordering
- `Ds`           negative degree lexicographical ordering
- `ws( intvec_expression )`  
                (general) weighted reverse lexicographical ordering; the first element of the weight vector has to be non-zero.
- `Ws( intvec_expression )`  
                (general) weighted lexicographical ordering; the first element of the weight vector has to be non-zero.

Local orderings are not well-orderings. They are denoted by an `s` as the second character in their name.

## 3. Matrix orderings

`M( intmat_expression )`

`intmat_expression` has to be an invertible square matrix

Using matrix orderings, SINGULAR can compute standard bases w.r.t. any monomial ordering which is compatible with the natural semi-group structure on the monomials. In practice, the predefined global and local orderings together with the block orderings should be sufficient in most cases. These orderings are faster than their corresponding matrix orderings since evaluation of a matrix ordering is more time consuming.

## 4. Extra weight vector

`a( intvec_expression )`

`aa( intvec_expression )`

`am( intvec_expression )`

an extra weight vector `a( intvec_expression )`, may precede any monomial ordering.

`aa` allows larger degrees.

`am` allows degrees for module generators (see example)

Example for ordering `am`

```
ring r=QQ,(x,y,z),(am(1,1,1,4,1,6),dp);
r;
↳ // coefficients: QQ
↳ // number of vars : 3
↳ //          block  1 : ordering am
↳ //                      : names    x y z
↳ //                      : weights  1 1 1
↳ //                      : 3 module weights  4 1 6
↳ //          block  2 : ordering dp
↳ //                      : names    x y z
↳ //          block  3 : ordering C
[x,y,z];
↳ z*gen(3)+x*gen(1)+y*gen(2)
deg([x]);
↳ 5
deg([0,y]);
↳ 2
deg([0,0,z]);
↳ 7
deg([0,0,0,x]);
↳ 1
```

## 5. Product ordering

`( ordering [ ( int_expression ) ], ... )`

any of the above orderings and the extra weight vector may be combined to yield product or block orderings

The orderings `lp`, `dp`, `Dp`, `ls`, `ds`, `Ds` and `rp` may be followed by an `int_expression` in parentheses giving the size of the block. For the last block the size is calculated automatically. For weighted orderings, the size of the block is given by the size of the weight vector. The same holds analogously for matrix orderings.

## 6. Module orderings

( ordering, ..., C )

( ordering, ..., c )

sort polynomial vectors by the monomial ordering first, then by components

( C, ordering, ... )

( c, ordering, ... )

sort polynomial vectors by components first, then by the monomial ordering

Here a capital **C** sorts generators in ascending order, i.e.,  $\text{gen}(1) < \text{gen}(2) < \dots$ . A small **c** sorts in descending order, i.e.,  $\text{gen}(1) > \text{gen}(2) > \dots$ . It is not necessary to specify the module ordering explicitly since ( ordering, ..., C ) is the default.

In fact, **c** or **C** may be specified anywhere in a product ordering specification, not only at its beginning or end. All monomial block orderings preceding the component ordering have higher precedence, all monomial block orderings following after it have lower precedence.

For a mathematical description of these orderings, see [Appendix B \[Polynomial data\]](#), page 768.

### 3.3.4 Coefficient rings

SINGULAR supports coefficient ranges which are not fields, i.e. the integers  $Z$  and the finite rings  $Z/n$  for a number **n**. These coefficient rings were implemented in SINGULAR 3.0.5 and at the moment only limited functionality is available.

#### p-adic numbers

The p-adic integers  $Z_p$  are the projective limit of the finite rings  $Z/p^n$  for **n** to infinity. Therefore, computations in this ring can be approximated by computations in  $Z/p^n$  for large **n**.

## 3.4 Implemented algorithms

The basic algorithm in SINGULAR is a general standard basis algorithm for any monomial ordering which is compatible with the natural semi-group structure of the exponents. This includes well-orderings (Buchberger algorithm to compute a Groebner basis) and tangent cone orderings (Mora algorithm) as special cases.

Nonetheless, there are a lot of other important algorithms:

- Algorithms to compute the standard operations on ideals and modules: intersection, ideal quotient, elimination, etc.
- Different Syzygy algorithms and algorithms to compute free resolutions of modules.
- Combinatorial algorithms to compute dimensions, Hilbert series, multiplicities, etc.
- Algorithms for univariate and multivariate polynomial factorization, resultant and gcd computations.

## Commands to compute standard bases

**facstd**     [Section 5.1.34 \[facstd\]](#), page 179

computes a list of Groebner bases via the Factorizing Groebner Basis Algorithm, i.e., their intersection has the same radical as the original ideal. It need not be a Groebner basis of the given ideal.

The intersection of the zero-sets is the zero-set of the given ideal.

<b>fglm</b>	<a href="#">Section 5.1.39 [fglm], page 183</a> computes a Groebner basis provided that a reduced Groebner basis w.r.t. another ordering is given. Implements the so-called FGLM (Faugere, Gianni, Lazard, Mora) algorithm. The given ideal must be zero-dimensional.
<b>groebner</b>	<a href="#">Section 5.1.53 [groebner], page 191</a> computes a standard resp. Groebner basis using a heuristically chosen method. This is the preferred method to compute a standard resp. Groebner bases.
<b>mstd</b>	<a href="#">Section 5.1.100 [mstd], page 227</a> computes a standard basis and a minimal set of generators.
<b>std</b>	<a href="#">Section 5.1.151 [std], page 271</a> computes a standard resp. Groebner basis.
<b>stdfglm</b>	<a href="#">Section 5.1.152 [stdfglm], page 272</a> computes a Groebner basis in a ring with a “difficult” ordering (e.g., lexicographical) via <b>std</b> w.r.t. a “simple” ordering and <b>fglm</b> . The given ideal must be zero-dimensional.
<b>stdhilb</b>	<a href="#">Section 5.1.153 [stdhilb], page 273</a> computes a Groebner basis in a ring with a “difficult” ordering (e.g., lexicographical) via <b>std</b> w.r.t. a “simple” ordering and a <b>std</b> computation guided by the Hilbert series.

## Further processing of standard bases

The next commands require the input to be a standard basis.

<b>degree</b>	<a href="#">Section 5.1.20 [degree], page 171</a> computes the (Krull) dimension, codimension and the multiplicity. The result is only displayed on the screen.
<b>dim</b>	<a href="#">Section 5.1.25 [dim], page 174</a> computes the dimension of the ideal resp. module.
<b>highcorner</b>	<a href="#">Section 5.1.55 [highcorner], page 194</a> computes the smallest monomial not contained in the ideal resp. module. The ideal resp. module has to be finite dimensional as a vector space over the ground field.
<b>hilb</b>	<a href="#">Section 5.1.56 [hilb], page 195</a> computes the first, and resp. or, second Hilbert series of an ideal resp. module.
<b>kbase</b>	<a href="#">Section 5.1.69 [kbase], page 205</a> computes a vector space basis (consisting of monomials) of the quotient of a ring by an ideal resp. of a free module by a submodule. The ideal resp. module has to be finite dimensional as a vector space over the ground field and has to be represented by a standard basis w.r.t. the ring ordering.
<b>mult</b>	<a href="#">Section 5.1.101 [mult], page 228</a> computes the degree of the monomial ideal resp. module generated by the leading monomials of the input.
<b>reduce</b>	<a href="#">Section 5.1.131 [reduce], page 251</a> reduces a polynomial, vector, ideal or module to its normal form with respect to an ideal or module represented by a standard basis.



**vdim**      [Section 5.1.168 \[vdim\], page 286](#)  
 computes the vector space dimension of a ring (resp. free module) modulo an ideal (resp. module).

## Commands to compute resolutions

**res**      [Section 5.1.134 \[res\], page 253](#)  
 computes a free resolution of an ideal or module using a heuristically chosen method. This is the preferred method to compute free resolutions of ideals or modules.

**fres**      [Section 5.1.48 \[fres\], page 188](#)  
 improved version of [Section 5.1.149 \[sres\], page 269](#), computes a free resolution of an ideal or module using Schreyer's method. The input has to be a standard basis.

**lres**      [Section 5.1.83 \[lres\], page 215](#)  
 computes a free resolution of an ideal or module with LaScala's method. The input needs to be homogeneous.

**mres**      [Section 5.1.98 \[mres\], page 225](#)  
 computes a minimal free resolution of an ideal or module with the Syzygy method.

**sres**      [Section 5.1.149 \[sres\], page 269](#)  
 computes a free resolution of an ideal or module with Schreyer's method. The input has to be a standard basis.

**nres**      [Section 5.1.106 \[nres\], page 232](#)  
 computes a free resolution of an ideal or module with the standard basis method.

**syz**      [Section 5.1.156 \[syz\], page 280](#)  
 computes the first Syzygy (i.e., the module of relations of the given generators).

## Further processing of resolutions

**betti**      [Section 5.1.4 \[beti\], page 159](#)  
 computes the graded Betti numbers of a module from a free resolution.

**minres**      [Section 5.1.93 \[minres\], page 223](#)  
 minimizes a free resolution of an ideal or module.

**regularity**  
[Section 5.1.132 \[regularity\], page 252](#)  
 computes the regularity of a homogeneous ideal resp. module from a given minimal free resolution.

## Processing of polynomials

**char\_series**  
[Section 5.1.6 \[char\\_series\], page 161](#)  
 computes characteristic sets of polynomial ideals.

**extgcd**      [Section 5.1.33 \[extgcd\], page 178](#)  
 computes the extended gcd of two polynomials.  
 This is implemented as extended Euclidean Algorithm, and applicable for univariate polynomials only.

**factorize**

[Section 5.1.36 \[factorize\], page 180](#)

computes factorization of univariate and multivariate polynomials into irreducible factors.

The most basic algorithm is univariate factorization in prime characteristic. The Cantor-Zassenhaus Algorithm is used in this case. For characteristic 0, a univariate Hensel-lifting is done to lift from prime characteristic to characteristic 0. For multivariate factorization in any characteristic, the problem is reduced to the univariate case first, then a multivariate Hensel-lifting is used to lift the univariate factorization.

Factorization of polynomials over algebraic extensions is provided by factoring the norm for univariate polynomials  $f$  (the gcd of  $f$  and the factors of the norm is a factorization of  $f$ ) resp. by the extended Zassenhaus algorithm for multivariate polynomials.

**gcd**

[Section 5.1.50 \[gcd\], page 190](#)

computes greatest common divisors of univariate and multivariate polynomials.

In the univariate case NTL is used. For prime characteristic, a subresultant gcd is used. In characteristic 0, the EZGCD is used, except for a special case where a modular algorithm is used.

**resultant**

[Section 5.1.136 \[resultant\], page 255](#)

computes the resultant of two univariate polynomials using the subresultant algorithm.

Multivariate polynomials are considered as univariate polynomials in the main variable (which has to be specified by the user).

**vandermonde**

[Section 5.1.164 \[vandermonde\], page 284](#)

interpolates a polynomial from its values at several points

**Matrix computations****bareiss**

[Section 5.1.3 \[bareiss\], page 158](#)

implements sparse Gauss-Bareiss method for elimination (matrix triangularization) in arbitrary integral domains.

**det**

[Section 5.1.23 \[det\], page 172](#)

computes the determinant of a square matrix.

For matrices with integer entries a modular algorithm is used. For other domains the Gauss-Bareiss method is used.

**minor**

[Section 5.1.92 \[minor\], page 221](#)

computes all minors (=subdeterminants) of a given size for a matrix.

**Numeric computations****laguerre**

[Section 5.1.74 \[laguerre\], page 208](#)

computes all (complex) roots of a univariate polynomial

**uressolve**

[Section 5.1.163 \[uressolve\], page 283](#)

finds all roots of a 0-dimensional ideal with multivariate resultants

## Controlling computations

**option**      [Section 5.1.111 \[option\], page 234](#)

allows setting of options for manipulating the behaviour of computations (such as reduction strategies) and for showing protocol information indicating the progress of a computation.

## 3.5 The SINGULAR language

SINGULAR interprets commands given interactively on the command line as well as given in the context of user-defined procedures. In fact, SINGULAR makes no distinction between these two cases. Thus, SINGULAR offers a powerful programming language as well as an easy-to-use command line interface without differences in syntax or semantics.

In the following, the basic language concepts such as commands, expressions, names, objects, etc., are discussed. See [Section 3.7 \[Procedures\], page 50](#), and [Section 3.8 \[Libraries\], page 55](#), for the concepts of procedures and libraries.

In many aspects, the SINGULAR language is similar to the C programming language. For a description of some of the subtle differences, see [Section 6.3 \[Major differences to the C programming language\], page 311](#).

### Elements of the language

The major building blocks of the SINGULAR language are expressions, commands, and control structures. The notion of expressions in the SINGULAR and the C programming language are identical, whereas the notion of commands and control structures only roughly corresponds to C statements.

- An “expression” is a sequence of operators, functions, and operands that specifies a computation. An expression always results in a value of a specific type. See [Chapter 4 \[Data types\], page 73](#), and its subsections (e.g., [Section 4.17.2 \[poly expressions\], page 121](#)), for information on how to build expressions.
- A “command” is either a declaration, an assignment, a call to a function without return value, or a print command. For detailed information, see [Section 3.5.1 \[General command syntax\], page 41](#).
- “Control structures” determine the execution sequence of commands. SINGULAR provides control structures for conditional execution (`if ... else`) and iteration (`for` and `while`). Commands may be grouped in pairs of `{ }` (curly brackets) to form blocks. See [Section 5.2 \[Control structures\], page 290](#), for more information.

### Other notational conventions

For user-defined functions, the notions of “procedure” and “function” are synonymous.

As already mentioned above, functions without return values are called commands. Furthermore, whenever convenient, the term “command” is used for a function, even if it does return a value.

#### 3.5.1 General command syntax

In SINGULAR a command is either a declaration, an assignment, a call to a function without return value, or a print command. The general form of a command is described in the following subsections.

## Declaration

1. `type name = expression ;`  
declares a variable with the given name of the given type and assigns the expression as initial value to it. Expression is an expression of the specified type or one that can be converted to that type. See [Section 3.5.5 \[Type conversion and casting\]](#), page 46.
2. `alias type name`  
Introduces name as an alternative, read-only name for another variable\_name. Can only be used in procedure headings to avoid copying large data.
3. `type name_list = expression_list ;`  
declares variables with the given names and assigns successively each expression of expression\_list to the corresponding name of name\_list. Both lists must be of the same length. Each expression in expression\_list is an expression of the specified type or one that can be converted to that type. See [Section 3.5.5 \[Type conversion and casting\]](#), page 46.
4. `type name ;`  
declares a variable with the given name of the given type and assigns the default value of the specific type to it.

See [Section 3.5.3 \[Names\]](#), page 44, for more information on declarations. See [Chapter 4 \[Data types\]](#), page 73, for a description of all data types known to SINGULAR.

```
ring r;                // the default ring
poly f,g = x^2+y^3,xy+z^2; // the polynomials f=x^2+y^3 and g=x*y+z^2
ideal I = f,g;         // the ideal generated by f and g
matrix m[3][3];        // a 3 x 3 zero matrix
int i=2;               // the integer i=2
```

## Assignment

4. `name = expression ;`  
assigns expression to name.
5. `name_list = expression_list ;`  
assigns successively each expression of expression\_list to the corresponding name of name\_list. Both lists must be of the same length. This is not a simultaneous assignment. Thus, `f, g = g, f;` does not swap the values of f and g, but rather assigns g to both f and g.

A type conversion of the type of expression to the type of name must be possible. See [Section 3.5.5 \[Type conversion and casting\]](#), page 46.

An assignment itself does not yield a value. Hence, compound assignments like `i = j = k;` are not allowed and result in an error.

```
f = x^2 + y^2 ;        // overrides the old value of f
I = jacob(f);
f,g = I[1],x^2+y^2 ; // overrides the old values of f and g
```

## Function without return value

6. `function_name [ ( argument_list ) ] ;`  
calls function function\_name with arguments argument\_list.

The function may have output (not to be confused with a return value of type string). See [Section 5.1 \[Functions\]](#), page 156. Functions without a return value are specified there to have a return type 'none'.

Some of these functions have to be called without parentheses, e.g., `help`, `LIB`.

```

ring r;
ideal i=x2+y2,x;
i=std(i);
degree(i);          // degree has no return value but prints output
⇒ // dimension (proj.) = 0
⇒ // degree (proj.) = 2

```

## Print command

7. expression ;  
prints the value of an expression, for example, of a variable.

Use the function `print` (or the procedure `show` from `inout.lib`) to get a pretty output of various data types, e.g., matrix or `intmat`. See [Section 5.1.120 \[print\]](#), page 242.

```

int i=2;
i;
⇒ 2
intmat m[2][2]=1,7,10,0;
print(m);
⇒      1      7
⇒     10      0

```

### 3.5.2 Special characters

The following characters and operators have special meanings:

=	assignment
{, }	parentheses for block programming
(, )	in expressions, for indexed names and for argument lists
[, ]	access operator for strings, integer vectors, ideals, matrices, polynomials, resolutions, and lists. Used to build vectors of polynomials. Example: <code>s[3]</code> , <code>m[1,3]</code> , <code>i[1..3]</code> , <code>[f,g+x,0,0,1]</code> .
+	addition operator
++	increment operator
-	subtraction operator
--	decrement operator
*	multiplication operator
/	division operator. See <a href="#">Section 6.4 [Miscellaneous oddities]</a> , page 314, for the difference between the division operators <code>/</code> and <code>div</code> .
%	modulo operator ( <code>mod</code> is an alias to <code>%</code> ): result is always non-negative
^ or **	exponentiation operator
==	comparison operator equal
!= or <>	comparison operator not equal
>=	comparison operator larger than or equal to
>	comparison operator larger
<=	comparison operator smaller than or equal to

<	comparison operator smaller. Also used for file input. See <a href="#">Section 5.1.41 [filecmd]</a> , <a href="#">page 184</a> .
!	boolean operator not
&&	boolean operator and
	boolean operator or
"	delimiter for string constants
'	delimiter for name substitution
?	synonym for <code>help</code>
//	comment delimiter. Comment extends to the end of the line.
/*	comment delimiter. Starts a comment which ends with */.
*/	comment delimiter. Ends a comment which starts with /*.
;	statement separator
,	separator for expression lists and function arguments
\	escape character for " and \ within strings
..	interval specifier returning intvec. E.g., <code>1..3</code> which is equivalent to the intvec <code>1, 2, 3</code> .
:	repeated entry. E.g., <code>3:5</code> generates an intvec of length 5 with constant entries 3, i.e., <code>(3, 3, 3, 3, 3)</code> .
::	accessor for package members. E.g., <code>MyPackage::i</code> accesses variable <code>i</code> in package <code>MyPackage</code> .
_	value of expression displayed last
~	breakpoint in procedures
#	list of parameters in procedures without explicit parameter list
\$	terminates SINGULAR

### 3.5.3 Names

SINGULAR is a strongly typed language. This means that all names (= identifiers) have to be declared prior to their use. For the general syntax of a declaration, see the description of declaration commands (see [Section 3.5.1 \[General command syntax\]](#), [page 41](#)).

See [Chapter 4 \[Data types\]](#), [page 73](#), for a description of SINGULAR's data types. See [Section 5.1.161 \[typeof\]](#), [page 282](#), for a short overview of possible types. To get information on a name and the object named by it, the `type` command may be used (see [Section 5.1.160 \[type\]](#), [page 282](#)).

It is possible to redefine an already existing name if doing so does not change its type. A redefinition first sets the variable to the default value and then computes the expression. The difference between redefining and overriding a variable is shown in the following example:

```

int i=3;
i=i+1;      // overriding
i;
↳ 4
int i=i+1;  // redefinition
↳ // ** redefining i ( int i=i+1;    // redefinition) ./examples/Names.sin\
g:4

```

```
i;
↳ 1
```

User defined names should start with a letter and consist of letters and digits only. As an exception to this rule, the characters @, and \_ may be used as part of a name, too (@ as the first letter is reserved for purposes of library routines). Capital and small letters are distinguished. Indexed names are built as a name followed by an int\_expression in parentheses. A list of indexed names can be built as a name followed by an intvec\_expression in parentheses. For multi-indices, append an int\_expression in parentheses to an indexed name. An alternative multi-index construction is `name_prefix( index_1, index_2,... )` where the `name_prefix` must be an undefined name.

```
ring R;
int n=3;
ideal j(3);
ideal j(n);      // is equivalent to the above
↳ // ** redefining j(3) ( ideal j(n);      // is equivalent to the above) .\
  /examples/Names_1.sing:4
ideal j(2)=x;
j(2..3);
↳ j(2)[1]=x j(3)[1]=0
ring r=0,(x(1..2)(1..3)(1..2)),dp;
r;
↳ // coefficients: QQ
↳ // number of vars : 12
↳ //          block 1 : ordering dp
↳ //          : names  x(1)(1)(1) x(1)(1)(2) x(1)(2)(1) x(1)(2)(2)\
  ) x(1)(3)(1) x(1)(3)(2) x(2)(1)(1) x(2)(1)(2) x(2)(2)(1) x(2)(2)(2) x(2)(\
  3)(1) x(2)(3)(2)
↳ //          block 2 : ordering C
int i(1,2),i(2,3);
i(2,3);
↳ 0
```

Names must not coincide with reserved names (keywords). Type `reservedName()`; to get a list of the reserved names. See [Section 5.1.135 \[reservedName\], page 254](#). Names should not interfere with names of ring variables or, more generally, with monomials. See [Section 6.5 \[Identifier resolution\], page 316](#).

The command `listvar` provides a list of the names in use (see [Section 5.1.82 \[listvar\], page 213](#)).

The most recently printed expression is available under the special name `_`, e.g.,

```
ring r;
ideal i=x2+y3,y3+z4;
std(i);
↳ _[1]=y3+x2
↳ _[2]=z4-x2
ideal k=_;
k*k+x;
↳ _[1]=y6+2x2y3+x4
↳ _[2]=y3z4+x2z4-x2y3-x4
↳ _[3]=z8-2x2z4+x4
↳ _[4]=x
size(_[3]);
↳ 3
```

A string\_expression enclosed in ‘...’ (back ticks) evaluates to the value of the variable given by the string\_expression. This feature is referred to as name substitution.



```

int foo(1)=42;
string bar="foo";
'bar+"(1)";
↦ 42

```

### 3.5.4 Objects

Every object in SINGULAR has a type and a value. In most cases it has also a name and in some cases an attribute list. The value of an object may be examined simply by printing it with a print command: `object;`. The type of an object may be determined by means of the `typeof` function, the attributes by means of the `attrib` function ([Section 5.1.161 \[typeof\]](#), [page 282](#), [Section 5.1.2 \[attrib\]](#), [page 156](#)):

```

ring r=0,x,dp;
typeof(10);
↦ int
typeof(1000000000000000000);
↦ bigint
typeof(r);
↦ ring
attrib(x);
↦ no attributes
attrib(std(ideal(x)));
↦ attr:isSB, type int

```

Each object of type `poly`, `ideal`, `vector`, `module`, `map`, `matrix`, `number`, or `resolution` belongs to a specific ring. This is also true for `list`, if at least one of the objects contained in the list belongs to a ring. These objects are local to the ring. Their names can be duplicated for other objects in other rings. Objects from one ring can be mapped to another ring using maps or the commands `fetch` or `imap`. See [Section 4.12 \[map\]](#), [page 106](#), [Section 5.1.38 \[fetch\]](#), [page 182](#), [Section 5.1.59 \[imap\]](#), [page 198](#).

All other types do not belong to a ring and can be accessed within every ring and across rings. They can be declared even if there is no active basering.

### 3.5.5 Type conversion and casting

#### Type conversion

Assignments convert the type of the right-hand side to the type of the left-hand side of the assignment, if possible. Operators and functions which require certain types of operands can also implicitly convert the type of an expression. It is, for example, possible to multiply a polynomial by an integer because the integer is automatically converted to a polynomial. Type conversions do not act transitively. Possible conversions are:

- |    |                     |                       |
|----|---------------------|-----------------------|
| 1. | <code>intvec</code> | ↦ <code>intmat</code> |
| 2. | <code>poly</code>   | ↦ <code>ideal</code>  |
| 3. | <code>bigint</code> | ↦ <code>ideal</code>  |
| 4. | <code>int</code>    | ↦ <code>ideal</code>  |
| 5. | <code>intmat</code> | ↦ <code>matrix</code> |
| 6. | <code>ideal</code>  | ↦ <code>matrix</code> |
| 7. | <code>module</code> | ↦ <code>matrix</code> |
| 8. | <code>number</code> | ↦ <code>matrix</code> |
| 9. | <code>poly</code>   | ↦ <code>matrix</code> |

10.	<code>vector</code>	$\mapsto$ <code>matrix</code>
11.	<code>bigint</code>	$\mapsto$ <code>matrix</code>
12.	<code>int</code>	$\mapsto$ <code>matrix</code>
13.	<code>intvec</code>	$\mapsto$ <code>matrix</code>
14.	<code>ideal</code>	$\mapsto$ <code>module</code>
15.	<code>matrix</code>	$\mapsto$ <code>module</code>
16.	<code>vector</code>	$\mapsto$ <code>module</code>
17.	<code>bigint</code>	$\mapsto$ <code>number</code>
18.	<code>int</code>	$\mapsto$ <code>number</code>
19.	<code>number</code>	$\mapsto$ <code>poly</code>
20.	<code>bigint</code>	$\mapsto$ <code>poly</code>
21.	<code>int</code>	$\mapsto$ <code>poly</code>
22.	<code>list</code>	$\mapsto$ <code>resolution</code>
23.	<code>poly</code>	$\mapsto$ <code>vector</code> ( $p \mapsto p*\text{gen}(1)$ )
24.	<code>bigint</code>	$\mapsto$ <code>vector</code>
25.	<code>int</code>	$\mapsto$ <code>vector</code> ( $i \mapsto i*\text{gen}(1)$ )
26.	<code>int</code>	$\mapsto$ <code>bigint</code>
27.	<code>int</code>	$\mapsto$ <code>intvec</code>
28.	<code>string</code>	$\mapsto$ <code>link</code>
29.	<code>resolution</code>	$\mapsto$ <code>list</code>
30.	<code>intvec</code>	$\mapsto$ <code>bigintvec</code>

## Type casting

An expression can be casted to another type by using a type cast expression:  
`type ( expression )`.

Possible type casts are:

to	from
<code>bigint</code>	expression <code>int</code> , <code>number</code> , <code>poly</code>
<code>bigintvec</code>	expression <code>int</code> , <code>bigint</code>
<code>ideal</code>	expression lists of <code>int</code> , <code>number</code> , <code>poly</code>
<code>ideal</code>	<code>int</code> , <code>matrix</code> , <code>module</code> , <code>number</code> , <code>poly</code> , <code>vector</code>
<code>int</code>	<code>number</code> , <code>poly</code>
<code>intvec</code>	expression lists of <code>int</code> , <code>intmat</code> , <code>bigintvec</code>
<code>intmat</code>	<code>intvec</code> (see <a href="#">Section 4.8.3 [intmat type cast]</a> , page 92)
<code>list</code>	expression lists of any type
<code>matrix</code>	<code>module</code> , <code>ideal</code> , <code>vector</code> , <code>matrix</code> .
	There are two forms to convert something to a matrix: if <code>matrix( expression )</code> is used then the size of the matrix is determined by the size of expression. But <code>matrix( expression , m , n )</code> may also be used - the result is a $m \times n$ matrix (see <a href="#">Section 4.13.3 [matrix type cast]</a> , page 110)
<code>module</code>	expression lists of <code>int</code> , <code>number</code> , <code>poly</code> , <code>vector</code>
<code>module</code>	<code>ideal</code> , <code>matrix</code> , <code>vector</code>
<code>number</code>	<code>poly</code>
<code>poly</code>	<code>int</code> , <code>number</code>
<code>ring</code>	<code>list</code> (the inverse of <code>ringlist</code> )
<code>string</code>	any type (see <a href="#">Section 4.22.3 [string type cast]</a> , page 131)

**Example:**

```
ring r=0,x,(c,dp);
number(3x);
```

```

⇒ 0
    number(poly(3));
⇒ 3
    ideal i=1,2,3,4,5,6;
    print(matrix(i));
⇒ 1,2,3,4,5,6
    print(matrix(i,3,2));
⇒ 1,2,
⇒ 3,4,
⇒ 5,6
    vector v=[1,2];
    print(matrix(v));
⇒ 1,
⇒ 2
    module(matrix(i,3,2));
⇒ _[1]=[1,3,5]
⇒ _[2]=[2,4,6]
    // generators are columns of a matrix

```

### 3.5.6 Flow control

A block is a sequence of commands surrounded by { and }.

```

{
    command;
    ...
}

```

Blocks are used whenever SINGULAR is used as a structured programming language. The `if` and `else` structures allow conditional execution of blocks (see [Section 5.2.9 \[if\]](#), page 296, [Section 5.2.5 \[else\]](#), page 292). `for` and `while` loops are available for a repeated execution of blocks (see [Section 5.2.8 \[for\]](#), page 295, [Section 5.2.15 \[while\]](#), page 301). In procedure definitions, the main part and the example section are blocks as well (see [Section 4.18 \[proc\]](#), page 124).

## 3.6 Input and output

SINGULAR's input and output (short, I/O) are realized using links. Links are the communication channels of SINGULAR, i.e., something SINGULAR can write to and read from. In this section, a short overview of the usage of links and of the different link types is given.

For loading of libraries, see [Section 5.1.79 \[LIB\]](#), page 211. For executing program scripts, see [Section 5.1.41 \[filecmd\]](#), page 184.

### Monitoring

A special form of I/O is monitoring. When monitoring is enabled, SINGULAR makes a typescript of everything printed on your terminal to a file. This is useful to create a protocol of a SINGULAR session. The `monitor` command enables and disables this feature (see [Section 5.1.95 \[monitor\]](#), page 224).

### How to use links

Recall that links are the communication channels of SINGULAR, i.e., something SINGULAR can write to and read from using the functions `write` and `read`. There are furthermore the functions `dump`

and `getdump` which store resp. retrieve the content of an entire SINGULAR session to, resp. from, a link. The `dump` and `getdump` commands are not available for DBM links.

For more information, see [Section 5.1.174 \[write\]](#), page 289, [Section 5.1.130 \[read\]](#), page 250, [Section 5.1.27 \[dump\]](#), page 175, [Section 5.1.52 \[getdump\]](#), page 191.

**Example:**

```

ring r; poly p = x+y;
dump(":w test.sv"); // dump the session to the file test.sv
kill r;           // kill the basering
listvar();        // no output after killing the ring
getdump(":r test.sv");// read the dump from the file
listvar();
⇒ // r                [0] *ring
⇒ //      p           [0] poly

```

Specifying a link can be as easy as specifying a filename as a string. Links do not even need to be explicitly opened or closed before, resp. after, they are used. To explicitly open or close a link, the `open`, resp. `close`, commands may be used (see [Section 5.1.110 \[open\]](#), page 234, [Section 5.1.10 \[close\]](#), page 163).

Links have various properties which can be queried using the `status` function (see [Section 5.1.150 \[status\]](#), page 270).

**Example:**

```

link l = "ssi:fork";
l;
⇒ // type : ssi
⇒ // mode : fork
⇒ // name :
⇒ // open : no
⇒ // read : not open
⇒ // write: not open
open(l);
status(l, "open");
⇒ yes
close(l);
status(l, "open");
⇒ no

```

## ASCII links

Data that can be converted to a string can be written into files for storage or communication with other programs. The data are written in plain ASCII format. Reading from an ASCII link returns a string — conversion into other data is up to the user. This can be done, for example, using the command `execute` (see [Section 5.1.32 \[execute\]](#), page 178).

ASCII links should primarily be used for storing small amounts of data, especially if it might become necessary to manually inspect or manipulate the data.

See [Section 4.10.4 \[ASCII links\]](#), page 98, for more information.

**Example:**

```

// (over)write file test.ascii, link is specified as string
write(":w test.ascii", "int i =", 3, ";");
// reading simply returns the string
read("test.ascii");
⇒ int i =

```

```

↳ 3
↳ ;
↳
  // but now test.ascii is "executed"
  execute(read("test.ascii"));
  i;
↳ 3

```

## Ssi links

Data is communicated with other processes (e.g., SINGULAR processes) which may run on the same computer or on different ones. Data exchange is accomplished using TCP/IP links in the ssi format. Reading from an ssi link returns the written expressions (i.e., not a string, in general).

Ssi links should primarily be used for communicating with other programs or for parallel computations (see, for example, [Section A.1.8 \[Parallelization with ssi links\]](#), page 708).

See [Section 4.10.5 \[Ssi links\]](#), page 99, for more information.

### Example:

```

ring r;
link l = "ssi:tcp localhost:"+system("Singular"); // declare a link explicitly
open(l); // needs an open, launches another SINGULAR as a server
write(l, x+y);
kill r;
def p = read(l);
typeof(p); p;
↳ poly
↳ x+y
close(l); // shuts down SINGULAR server

```

## DBM links

Data is stored in and accessed from a data base. Writing is accomplished by a key and a value and associates the value with the key in the specified data base. Reading is accomplished w.r.t. a key, the value associated to it is returned. Both the key and the value have to be specified as strings. Hence, DBM links may be used only for data which may be converted to or from strings.

DBM links should primarily be used when data needs to be accessed not in a sequential way (like with files) but in an associative way (like with data bases).

See [Section 4.10.7 \[DBM links\]](#), page 102, for more information.

### Example:

```

ring r;
// associate "x+y" with "mykey"
write("DBM:w test.dbm", "mykey", string(x+y));
// get from data base what is stored under "mykey"
execute(read("DBM: test.dbm", "mykey"));
↳ x+y

```

## 3.7 Procedures

Procedures contain sequences of commands in the SINGULAR language. They are used to extend the set of commands by user defined commands. In a SINGULAR session, procedures are defined by either typing them on the command line or by loading them from a library file with the LIB or

`load` command (see [Section 3.8 \[Libraries\]](#), page 55). A procedure is invoked like normal built-in commands, i.e., by typing its name followed by the list of arguments in parentheses. The invocation then executes the sequence of commands constituting the procedure. All procedures defined in a SINGULAR session can be displayed by entering `listvar(proc);`.

See also [Section 3.8.6 \[Procedures in a library\]](#), page 58.

### 3.7.1 Procedure definition

**Syntax:** `[static] proc proc_name [(<parameter_list>)]`  
`[<help_string>]`  
`{`  
`<procedure_body>`  
`}`  
`[example`  
`{`  
`<sequence_of_commands>`  
`}]`

**Purpose:**

- Defines a new function, the `proc proc_name`.
- The help string, the parameter list, and the example section are optional. They are, however, mandatory for the procedures listed in the header of a library. The help string is ignored and no example section is allowed if the procedure is defined interactively, i.e., if it is not loaded from a file by the `LIB` or `load` command (see [Section 5.1.79 \[LIB\]](#), page 211 and see [Section 5.2.12 \[load\]](#), page 299 ).
- Once loaded from a file into a SINGULAR session, the information provided in the help string will be displayed upon entering `help proc_name;`, while the `example` section will be executed upon entering `example proc_name;`. See [Section 3.7.2 \[Parameter list\]](#), page 52, [Section 3.7.3 \[Help string\]](#), page 54, and the example in [Section 3.8.6 \[Procedures in a library\]](#), page 58.
- In the body of a library, each procedure not meant to be accessible by users should be declared static. See [Section 3.8.6 \[Procedures in a library\]](#), page 58.

**Example of an interactive procedure definition and its execution:**

```
proc milnor_number (poly p)
{
    ideal i= std(jacob(p));
    int m_nr=vdim(i);
    if (m_nr<0)
    {
        "// not an isolated singularity";
    }
    return(m_nr);          // the value of m_nr is returned
}
ring r1=0,(x,y,z),ds;
poly p=x^2+y^2+z^5;
milnor_number(p);
↪ 4
```

### Example of a procedure definition in a library:

First, we define the library (and store it as `sample.lib`):

```
// Example of a user accessible procedure
proc tab (int n)
"USAGE:    tab(n);  n integer
RETURNS:   string of n space tabs
EXAMPLE:   example tab; shows an example"
{ return(internal_tab(n)); }
example
{
  "EXAMPLE: "; echo=2;
  for(int n=0; n<=4; n=n+1)
    { tab(4-n)+"*"+tab(n)+" "+tab(n)+"*"; }
}

// Example of a static procedure
static proc internal_tab (int n)
{ return(" "[1,n]); }
```

Now, we load the library and execute its procedures:

```
LIB "sample.lib";          // load the library sample.lib
example tab;               // show an example
⇒ // proc tab from lib sample.lib
⇒ EXAMPLE:
⇒   for(int n=0; n<=4; n=n+1)
⇒   { tab(4-n)+"*"+tab(n)+" "+tab(n)+"*"; }
⇒   ***
⇒   * + *
⇒   * + *
⇒   * + *
⇒   * + *
⇒   *
⇒   "*" + tab(3) + "*";          // use the procedure tab
⇒ * *
⇒ // the static procedure internal_tab is not accessible
⇒ "*" + internal_tab(3) + "*";
⇒   ? 'internal_tab(3)' is not defined
⇒   ? error occurred in or before ./examples/Example_of_a_procedure_defini\
tion_in_a_library:.sing line 5: ' "*" + internal_tab(3) + "*"; '
⇒ // show the help section for tab
⇒ help tab;
⇒ // ** Could not get 'IdxFile'.
⇒ // ** Either set environment variable 'SINGULAR_IDX_FILE' to 'IdxFile',
⇒ // ** or make sure that 'IdxFile' is at "%D/singular/singular.idx"
⇒ // ** Displaying help in browser 'dummy'.
⇒ // ** Use 'system("--browser", <browser>);' to change browser,
⇒ // ** where <browser> can be: "dummy", "emacs".
⇒   ? No functioning help browser available.
⇒   ? error occurred in or before ./examples/Example_of_a_procedure_defini\
tion_in_a_library:.sing line 7: ' help tab; '
```

### 3.7.2 Parameter list



**Syntax:**     ( )  
               ( parameter\_definition )

**Purpose:**

- Defines the number, type and names of the arguments of a procedure.
- The parameter\_list is optional.
- Adding `list #` as argument to a parameter list means to allow optional parameters. Furthermore, `(list #)` is the default for a parameter list (in case no list is explicitly given). Inside the procedure body, the arguments of `list #` are referenced by `#[1]`, `#[2]`, etc.
- If a procedure has optional parameters, the attribute `default_arg` gives the default values for the optional arguments. This provides in particular the possibility to also change the behaviour of all procedures nested inside the given procedure.

**Example:**

```

proc x0
{
    // can be called with
    ... // any number of arguments of any type: #[1], #[2],...
    // number of arguments: size(#)
}

proc x1 ()
{
    ... // can only be called without arguments
}

proc x2 (ideal i, int j)
{
    ... // can only be called with 2 arguments,
        // which can be converted to ideal resp. int
}

proc x3 (i,j)
{
    ... // can only be called with 2 arguments
        // of any type
        // (i,j) is the same as (def i,def j)
}

proc x5 (i,list #)
{
    ... // can only be called with at least 1 argument
        // number of arguments: size(#)+1
}

attrib(x5,"default_arg",3);
x5(2); // is equivalent to
x5(2,3);

```

**Note:**

The parameter\_list may stretch across multiple lines.  
 A parameter may have any type (including the types `proc` and `ring`).

If a parameter is of type `ring`, then it can only be specified by name, but not with a type. For instance:

```
proc x6 (r)
{
... // this is correct, r may be of any type, even of type ring
}

proc x7 (ring r)
{
... // this is NOT CORRECT
}
```

### 3.7.3 Help string

**Syntax:** `string_constant;`

**Purpose:** Constitutes the help text of a procedure.

**Format:**

```
USAGE:    <proc_name>(<parameter list>);  <explanation of parameters>
ASSUME:   <description of assumptions made>
RETURN:   <description of what is returned>
SIDE EFFECTS: <description of global objects generated or manipulated,
but not returned>
REMARKS:  <information on theory and implemented algorithms, references>
NOTE:     <particularities, limitations, additional details>
KEYWORDS: <semicolon-separated phrases of index keys>
SEE ALSO: <comma-separated names of related procedures/cross references>
EXAMPLE:  example <proc_name>; shows an example
```

**NOTE:**

- ASSUME, SIDE EFFECTS, KEYWORDS, and SEE ALSO are optional. No help string is required for static procedures.
- EXAMPLE: refers to the example section of the procedure. In a SINGULAR session, the example will be carried out upon entering `example <proc_name>`; if the procedure is loaded from a file by the `LIB` or `load` command (see [Section 5.1.79 \[LIB\]](#), page 211 and see [Section 5.2.12 \[load\]](#), page 299 ). No example section is allowed if the procedure is defined interactively.
- See [Section 3.8.10 \[Typesetting of help and info strings\]](#), page 64 for help strings in the SINGULAR documentation.
- See the example in [Section 3.8.6 \[Procedures in a library\]](#), page 58 for an illustration.

### 3.7.4 Names in procedures

- All variables defined inside a procedure are local to the procedure and their names cannot interfere with names in other procedures. Without further action, they are automatically deleted after leaving the procedure.
- To keep local variables and their value after leaving the procedure, they have to be exported (i.e. made global) by a command like `export` or `exportto` (see [Section 5.2.6 \[export\]](#), page 292, see [Section 5.2.7 \[exportto\]](#), page 293, see [Section 5.2.10 \[importfrom\]](#), page 296; see [Section 4.16 \[package\]](#), page 119). To return the value of a local variable, use the `return` command (see [Section 5.2.14 \[return\]](#), page 301).

**Example:**

```

proc xxx
{
  int k=4;          //defines a local variable k
  int result=k+2;
  export(result);  //defines the global variable "result".
}
xxx();
listvar(all);
⇨ // result                [0] int 6

```

Note that the variable `result` became a global variable after the execution of `xxx`.

### 3.7.5 Procedure-specific commands

A few commands should only be used inside a procedure. They either make local objects global ones or return results to the level from where the procedure was called.

See [Section 5.2.6 \[export\], page 292](#); [Section 5.2.7 \[exportto\], page 293](#); [Section 5.2.14 \[return\], page 301](#).

## 3.8 Libraries

- A library is a collection of SINGULAR procedures in a file.
- To load a library into a SINGULAR session, use the `LIB` or `load` command. Having loaded a library, its procedures can be used like any built-in SINGULAR function, and information on the library is obtained by entering `help libname.lib`;
- See [Appendix D \[SINGULAR libraries\], page 794](#), for all libraries currently distributed with SINGULAR.
- When writing your own library, it is important to comply with the guidelines described in this section. Otherwise, due to potential parser errors, it may not be possible to load the library.
- Each library consists of a header and a body. The first line of a library must start with a double slash `//`.
- The library header consists of a version string, a category string, an info string, and `LIB` commands. The strings are mandatory. `LIB` commands are meant to load the additional libraries used by the library under consideration.
- The library body collects the procedures (declared static or not).
- No line of a library should consist of more than 60 characters.

### 3.8.1 Libraries in the SINGULAR Documentation

- The typesetting language in which the SINGULAR documentation is written is `texinfo`. The info string of a library included in the SINGULAR distribution will be parsed and automatically translated to the `texinfo` format. The same applies to the help string of each procedure listed in the PROCEDURE: section of the info string.
- Based on various tools, `info`, `dvi`, `p`, and `html` versions of the `texinfo` documentation are generated.
- For `texinfo` markup elements and other information facilitating optimal typesetting, see [Section 3.8.10 \[Typesetting of help and info strings\], page 64](#).
- For the convenience of users checking directly the source code, the `texinfo` tools should be used economically. That is, the info and help texts should be well readable verbatim.

- The example of each procedure listed in the PROCEDURE: section of the info string is computed and its output is included in the documentation.

### 3.8.2 Version string

A version string is part of the header of a library.

**Syntax:**     `version = string_constant;`

**Purpose:**     Defines the version number of a library. It is displayed when the library is loaded.

**Example:**   `version="version sample.lib 4.0.0.0 Dec_2013 ";`

**Note:**       Syntax: `version<space><filename><space><version><space><date><space>`

### 3.8.3 Category string

A category string is part of the header of a library.

**Syntax:**     `category = string_constant;`

**Purpose:**     Defines the category of a library.

**Example:**   `category="Algebraic geometry";`

**Note:**       Reserved for sorting the libraries into categories.

### 3.8.4 Info string

**Syntax:**     `info = string_constant;`

**Purpose:**     Constitutes the help text of a library. Will be displayed in a SINGULAR session upon entering `help libname.lib;` . Will be part of the SINGULAR documentation if the library is distributed with SINGULAR. See [Section 3.8.1 \[Libraries in the SINGULAR Documentation\]](#), page 55.

**Format:**

```
info="
LIBRARY: <library_name> <one line description of the purpose>
AUTHOR:  <name, and email address of author>
OVERVIEW: <concise, additional information on what is implemented>
REFERENCES: <references for further information>
KEYWORDS: <semicolon-separated phrases of index keys>
SEE ALSO: <comma-separated words of cross references>
PROCEDURES:
    <proc_name_1>();      <one line description of the purpose>
    .
    .
    <proc_name_N>();      <one line description of the purpose>
";
```

**NOTE:**

- In the documentation, the one line description of the purpose following LIBRARY: will be printed in its own line, starting with the prefix PURPOSE: .
- REFERENCES, KEYWORDS, and SEE ALSO are optional.

- Only non-static procedures should be listed in the PROCEDURES: section. A procedure parameter should be included between the brackets () only if the corresponding one line description of the purpose refers to it. See [Section 3.8.6 \[Procedures in a library\]](#), page 58.
- In the documentation, separate nodes (subsections in printed documents) are created precisely for those procedures of the library appearing in the PROCEDURES: section (that is, for some if not all non-static procedures of the library).

**Example:**

```

info="
LIBRARY: absfact.lib    Absolute factorization for characteristic 0
AUTHORS: Wolfram Decker,      decker at math.uni-sb.de
          Gregoire Lecerf,     lecerf at math.uvsq.fr
          Gerhard Pfister,     pfister at mathematik.uni-kl.de

OVERVIEW:
A library for computing the absolute factorization of multivariate
polynomials f with coefficients in a field K of characteristic zero.
Using Trager's idea, the implemented algorithm computes an absolutely
irreducible factor by factorizing over some finite extension field L
(which is chosen such that V(f) has a smooth point with coordinates in L).
Then a minimal extension field is determined making use of the
Rothstein-Trager partial fraction decomposition algorithm.

REFERENCES:
G. Cheze, G. Lecerf: Lifting and recombination techniques for absolute
                    factorization. Journal of Complexity, 23(3):380-420, 2007

KEYWORDS: factorization; absolute factorization.
SEE ALSO: factorize

PROCEDURES:
  absFactorize();      absolute factorization of poly
";

```

To see how this infostring appears in the documentation after typesetting, check [Section D.4.1 \[absfact.lib\]](#), page 1002:

**3.8.5 LIB commands**

LIB commands are part of the header of a library.

**Syntax:**    LIB "lib\_1.lib";  
              ...  
              LIB "lib\_r.lib";

**Purpose:**    Loads libraries used by the library under consideration.

**Example:**

```

LIB "primdec.lib";
LIB "normal.lib";

```

**Note:**       The keyword LIB must be followed by at least one space.

### 3.8.6 Procedures in a library

Here are hints and requirements on how procedures contained in a library should be implemented. For more on procedures, see [Section 3.7 \[Procedures\]](#), page 50.

1. Each procedure not meant to be accessible by users should be declared static.
2. The header of each procedure not declared static must comply with the guidelines described in [Section 3.7.1 \[Procedure definition\]](#), page 51 and [Section 3.7.3 \[Help string\]](#), page 54. In particular, it must have a help and example section, and assumptions made should be carefully explained. If the assumptions are checked by the procedure on run-time, errors may be reported using the [Section 5.1.30 \[ERROR\]](#), page 177 function.
3. Names of procedures should not be shorter than 4 characters and should not contain any special characters. In particular, the use of `_` in names of procedures is discouraged. If the name of the procedure is composed of more than one word, each new word should start with a capital letter, all other letters should be lower case (e.g. `linearMapKernel`).
4. No procedures should be defined within the body of another procedure.
5. A procedure may print out comments, for instance to explain results or to display intermediate computations. This is often helpful when calling the procedure directly, but it may also cause confusions in cases where the procedure is called by another procedure. The SINGULAR solution to this problem makes use of the function `dbprint` (see [Section 5.1.17 \[dbprint\]](#), page 169) and the reserved variables `printlevel` and `voice` (see [Section 5.3.6 \[printlevel\]](#), page 304 and see [Section 5.3.11 \[voice\]](#), page 308). Note that `printlevel` is a predefined, global variable whose value can be changed by the user, while `voice` is an internal variable, representing the nesting level of procedures. Accordingly, the value of [Section 5.3.11 \[voice\]](#), page 308 is 1 on the top level, 2 inside the first procedure, and so on. The default value of `printlevel` is 0, but `printlevel` can be set to any integer value by the user.

**Example:** If the procedure `Test` below is called directly from the top level, then ‘comment1’ is displayed, but not ‘comment2’. By default, nothing is displayed if `Test` is called from within any other procedure. However, if `printlevel` is set to a value `k` with `k>0`, then ‘comment1’ (resp. ‘comment2’) is displayed – provided `Test` is called from another procedure with nesting level at most `k` (resp. `k-1`).

The example part of a procedure behaves in this respect like the procedure on top level (the nesting level is 1, that is, the value of `voice` is 2). Therefore, due to the command `printlevel=1;`, ‘comment1’ will be displayed when entering `example Test;`. However, since `printlevel` is a global variable, it should be reset to its old value at the end of the example part.

The predefined variable `echo` controls whether input lines are echoed or not. Its default is 0, but it can be reset by the user. Input is echoed if `echo>=voice`. At the beginning of the example part, `echo` is set to the value 2. In this way, the input lines of the example will be displayed when entering `example Test;`.

```
proc Test
"USAGE:    ...
          ...
EXAMPLE: example Test; shows an example
"
{
    ...
    int p = printlevel - voice + 3;
    ...
    dbprint(p,"comment1");
    dbprint(p-1,"comment2");
}
```

```

        // dbprint prints only if p > 0
        ...
    }
    example
    { "EXAMPLE:"; echo = 2;
      int p = printlevel;    //store old value of printlevel
      printlevel = 1;        //assign new value to printlevel
      ...
      Test();
      printlevel = p;        //reset printlevel to old value
    }

```

**Note:** SINGULAR functions such as `pause` or `read` allow and require interactive user-input. They are, thus, in particular useful for debugging purposes. If such a command is used inside the procedure of a library to be distributed with SINGULAR, the example section of the procedure has to be written with some care – the procedure should only be called from within the example if the value of `printlevel` is 0. Otherwise, the automatic build process of SINGULAR will not run through since the examples are carried out during the build process. They are, thus, tested against changes in the code.

### 3.8.7 template.lib

First, we show the source-code of a template library:

```

////////////////////////////////////
version="version template.lib 4.1.2.0 Feb_2019 "; // $Id: 4d4a314bcbeaaaf113c4c4687b
category="Miscellaneous";
// summary description of the library
info="
LIBRARY:    template.lib  A Template for a Singular Library
AUTHOR:     Olaf Bachmann, email: obachman@mathematik.uni-kl.de

SEE ALSO:   standard_lib, Libraries,
            Typesetting of help and info strings

KEYWORDS:   library, template.lib; template.lib; library, info string

PROCEDURES:
    mdouble(int)          return double of int argument
    mtriple(int)         return three times int argument
    msum([int,..,int])    sum of int arguments
";
////////////////////////////////////
proc mdouble(int i)
"USAGE:    mdouble(i); i int
RETURN:    int: i+i
NOTE:      Help string is in pure ASCII.
           This line starts on a new line since previous line is short.
           No new line here.
SEE ALSO:  msum, mtriple, Typesetting of help and info strings
KEYWORDS:  procedure, ASCII help
EXAMPLE:   example mdouble; shows an example"
{
    return (i + i);
}

```



```

}
example
{ "EXAMPLE:"; echo = 2;
  mdouble(0);
  mdouble(-1);
}
////////////////////////////////////////////////////////////////
proc mtriple(int i)
"@c we do texinfo here
@table @asis
@item @strong{Usage:}
@code{mtriple(i)}; @code{i} int

@item @strong{Return:}
int: @math{i+i+i}
@item @strong{Note:}
Help is in pure Texinfo.
@*This help string is written in texinfo, which enables you to use,
among others, the @math command for mathematical typesetting
(for instance, to print @math{\alpha, \beta}).
@*Texinfo also gives more control over the layout, but is, admittedly,
more cumbersome to write.
@end table
@c use @c ref contstuct for references
@cindex procedure, texinfo help
@c ref
@strong{See also:}
@ref{mdouble}, @ref{msum}, @ref{Typesetting of help and info strings}
@c ref
"
{
  return (i + i + i);
}
example
{ "EXAMPLE:"; echo = 2;
  mtriple(0);
  mtriple(-1);
}
////////////////////////////////////////////////////////////////
proc msum(list #)
"USAGE:  msum([i_1,...,i_n]); @code{i_1,...,i_n} def
RETURN:  Sum of int arguments
NOTE:    This help string is written in a mixture of ASCII and texinfo.
        @* Use @ref for references (e.g., @pxref{mtriple}).
        @* Use @code for typewriter font (e.g., @code{i_1}).
        @* Use @math for simple math mode typesetting (e.g., @math{i_1}).
        @* Warning: Parenthesis like } are not allowed inside @math and @code.
        @* Use @example for indented, preformatted text typesetting in
        typewriter font:
@example
    this --> that
@end example
    Use @format for preformatted text typesetting in normal font:

```

```

@format
    this --> that
@end format
    Use @texinfo for text in pure texinfo:
@texinfo
@expansion{}
@tex
i{1,1}$
@end tex

@end texinfo
    Note that
        automatic linebreaking is still in affect (like in this line).
SEE ALSO: mdouble, mtriple, Typesetting of help and info strings
KEYWORDS: procedure, ASCII/Texinfo help
EXAMPLE: example msum; shows an example"
{
    if (size(#) == 0) { return (0);}
    if (size(#) == 1) { return ([1]);}
    int i;
    def s = [1];
    for (i=2; i<=size(#); i++)
    {
        s = s + [i];
    }
    return (s);
}
example
{ "EXAMPLE:"; echo = 2;
    msum();
    msum(4);
    msum(1,2,3,4);
}

```

Second, we show how the library appears in the documentation after typesetting (with one subsection for each procedure):

**Library:** template.lib

**Purpose:** A Template for a Singular Library

**Author:** Olaf Bachmann, email: obachman@mathematik.uni-kl.de

**Procedures:** See also: [Section 3.8 \[Libraries\]](#), page 55; [Section 3.8.10 \[Typesetting of help and info strings\]](#), page 64; [Section D.1 \[standard\\_lib\]](#), page 794.

### 3.8.7.1 mdouble

Procedure from library `template.lib` (see [Section 3.8.7 \[template\\_lib\]](#), page 59).

**Usage:** mdouble(i); i int

**Return:** int: i+i

**Note:** Help string is in pure ASCII.

This line starts on a new line since previous line is short. No new line here.

**Example:**

```
LIB "template.lib";
mdouble(0);
↦ 0
mdouble(-1);
↦ -2
```

See also: [Section 3.8.10 \[Typesetting of help and info strings\]](#), page 64; [Section 3.8.7.3 \[msum\]](#), page 62; [Section 3.8.7.2 \[mtriple\]](#), page 62.

### 3.8.7.2 mtriple

Procedure from library `template.lib` (see [Section 3.8.7 \[template\\_lib\]](#), page 59).

**Usage:** `mtriple(i); i int`

**Return:** `int:  $i + i + i$`

**Note:** Help is in pure Texinfo.

This help string is written in texinfo, which enables you to use, among others, the `@math` command for mathematical typesetting (for instance, to print  $\alpha, \beta$  ).

Texinfo also gives more control over the layout, but is, admittedly, more cumbersome to write.

**See also:**

**Example:**

```
LIB "template.lib";
mtriple(0);
↦ 0
mtriple(-1);
↦ -3
```

### 3.8.7.3 msum

Procedure from library `template.lib` (see [Section 3.8.7 \[template\\_lib\]](#), page 59).

**Usage:** `msum([i_1,...,i_n]); i_1,...,i_n def`

**Return:** Sum of int arguments

**Note:** This help string is written in a mixture of ASCII and texinfo.

Use `@ref` for references (e.g., see [Section 3.8.7.2 \[mtriple\]](#), page 62).

Use `@code` for typewriter font (e.g., `i_1` ).

Use `@math` for simple math mode typesetting (e.g.,  $i_1$  ).

Warning: Parenthesis like `}` are not allowed inside `@math` and `@code`.

Use `@example` for indented, preformatted text typesetting in typewriter font:

```
      this --> that
```

Use `@format` for preformatted text typesetting in normal font:

```
      this -> that
```

Use `@texinfo` for text in pure texinfo:

```
↦  $i_{1,1}$ 
```

Note that

automatic linebreaking is still in affect (like in this line).

**Example:**

```
LIB "template.lib";
msum();
  ↦ 0
msum(4);
  ↦ 4
msum(1,2,3,4);
  ↦ 10
```

See also: [Section 3.8.10 \[Typesetting of help and info strings\]](#), page 64; [Section 3.8.7.1 \[mdouble\]](#), page 61; [Section 3.8.7.2 \[mtriple\]](#), page 62.

### 3.8.8 Formal Checker

There is a formal library checker for SINGULAR which can be used online: see <https://www.singular.uni-kl.de/index.php/new-libraries/formal-library-checker.html>.

After uploading your library file, you will receive an output of hints, warnings, and errors which may help you to improve your library.

### 3.8.9 Documentation Tool

lib2doc is a utility to generate the stand-alone documentation for a SINGULAR library in various formats.

The lib2doc utility should be used by developers of SINGULAR libraries to check the generation of the documentation of their libraries.

lib2doc can be downloaded from

<ftp://jim.mathematik.uni-kl.de/pub/Math/Singular/misc/lib2doc.tar.gz> (mirror at <https://www.mathematik.uni-kl.de/ftp/pub/Math/Singular/misc/lib2doc.tar.gz>)

#### Important:

To use lib2doc, you need to have `perl` (version 5 or higher), `texinfo` (version 3.12 or higher) and `Singular` and `libparse` (version 1-3-4 or higher) installed on your system.

To generate the documentation for a library, follow these steps:

1. Unpack lib2doc.tar.gz
 

```
gzip -dc lib2doc.tar.gz | tar -pxf -
```

 and
 

```
cd lib2doc
```
2. Edit the beginning of the file `Makefile`, filling in the values for `SINGULAR` and `LIBPARSE`. Check also the values of `PERL` and `LATEX2HTML`.
3. Copy your library to the current directory:
 

```
cp <path-where-your-lib-is>/mylib.lib .
```
4. Now you can run the following commands:

```
make mylib.hlp
    Generates the file mylib.hlp – the info file for the documentation of mylib.lib.
    This file can be viewed using
        info -f mylib.hlp
```

```
make mylib.dvi
    Generates the file mylib.dvi – the dvi file for the documentation of mylib.lib.
    This file can be viewed using
```

```

        xdvi mylib.dvi

make mylib.ps
    Generates the file mylib.ps – the PostScript file for the documentation of
    mylib.lib. This file can be viewed using (for example)
        ghostview mylib.dvi

make mylib.html
    Generates the file mylib.html – the HTML file for the documentation of
    mylib.lib. This file can be viewed using (for example)
        firefox mylib.html

make clean
    Deletes all generated files.

```

Note that you can safely ignore messages complaining about undefined references.

### 3.8.10 Typesetting of help and info strings

The info strings of the libraries which are included in the distribution of SINGULAR and the help strings of the corresponding procedures are parsed and automatically converted into the `texinfo` format (the typesetting language in which the documentation of SINGULAR is written).

The illustrative example given in [Section 3.8.7 \[template\\_lib\], page 59](#) should provide sufficient information on how this works. For more details, check the following items:

- Users familiar with `texinfo` may write help and info strings directly in the `texinfo` format. The string should, then, start with the `@` sign. In this case, no parsing will be done.
- Help and info strings are typeset within a `@table @asis` environment (which is similar to the `latex description` environment).
- If a line starts with uppercase words up to a colon, then the text up to the colon is taken to be the description-string of an item, and the text following the colon is taken to be the content of the item.
- If the description-string of an item matches

**SEE ALSO** then the content of the item is assumed to consist of comma-separated words which are valid references to other `texinfo` nodes of the manual (e.g., all procedure and command names are also `texinfo` nodes).

**KEYWORDS** then the content of the item is assumed to be a semicolon-separated list of phrases which are taken as keys for the index of the manual (the name of a procedure/library is automatically added to the index keys).

- If the description-string of an item in the **info string of a library** matches

**LIBRARY** then the content of the item is assumed to be a one-line description of the library. If this one-line description consists of uppercase characters only, then it is typeset in lowercase characters (otherwise it is left as is).

**PROCEDURES**

then the content of the item is assumed to consist of lines of type

```
<proc_name>();    <one line description of the purpose>
```

Separate `texinfo` nodes (subsections in printed documents) are created precisely for those procedures of the library appearing here (that is, for some if not all non-static procedures of the library).

With respect to the content of an item, the following `texinfo` markup elements are recognized:

**@\*** Enforces a line-break.

Example:   old line **@\*** new line  
                $\mapsto$   
               old line  
               new line

**@ref{...}** For references to other parts of the SINGULAR manual, use one of the following **@ref{node}** constructs. Here, **node** must be the name of a section of the SINGULAR manual. In particular, it may be the name of a function, library or procedure in a library.

**@xref{node}**  
               for a reference to the node **node** at the beginning of a sentence.

**@ref{node}**  
               for a reference to the node **node** at the end of a sentence.

**@pxref{node}**  
               for a reference to the node **node** within parentheses.

Example:   **@xref{Hurricanes}, for more info.**  
                $\mapsto$ \*Note Hurricanes::, for more info.  
                $\mapsto$ See Section 3.1 [Hurricanes], page 24, for more info.  
               **For more information, see @ref{Hurricanes}.**  
                $\mapsto$ For more information, see \*Note Hurricanes::  
                $\mapsto$ For more information, see Section 3.1 [Hurricanes], page 24.  
               ... storms cause flooding (**@pxref{Hurricanes}**) ...  
                $\mapsto$ ... storms cause flooding (\*Note Hurricanes::) ...  
                $\mapsto$ ... storms cause flooding (see Section 3.1 [Hurricanes], page 24)

**@math{...}** Typeset short mathematical expressions in LaTeX math-mode syntax (short: does not cause expansion over multiple lines).

Example:   **@math{\alpha}**  
                $\mapsto$   
                $\alpha$

Note:       The mathematical expressions inside **@math{...}** must not contain the characters **{,}**, and **@**.

**@code{...}** Typeset short strings in typewriter font (short: does not cause expansion over multiple lines).

Example:   **@code{typewriter font}**  
                $\mapsto$   
               typewriter font

Note:       The string inside **@code{...}** must not contain the characters **{,}**, and **@**.

Typeset pre-formatted text in typewriter font.

```
@example
...
@end example
```

Example:

```
before example
@example
in example
notice escape of special characters like @{,@},@@
```

```

@end example
after example

↪
before example
  in example
  notice escape of special characters like {,},@
after example

```

Note: Inside an @example environment, the characters {,},@ have to be escaped by an @ sign.

Typeset pre-formatted text in normal font.

```

@format
...
@end format

```

Example:

```

before format
@format
in format
notice escape of special characters like @{,@},@@
@end format
after format

↪
before format
in format
escape of special characters like {,},@
after format

```

Note: Inside an @format environment, the characters {,},@ have to be escaped by an @ sign.

Write text in pure texinfo.

```

@texinfo
...
@end texinfo

```

Example:

```

@texinfo
Among others, within a texinfo environment,
one can use the tex environment to typeset
more complex mathematical items like
@tex
i1,1 $
@tex
@end texinfo

```

```

↪
Among others, within a texinfo environment, one can use the tex environment to
typeset more complex mathematical items like i1,1

```

Furthermore, a line-break is inserted before each line whose previous line is shorter than 60 characters and does not contain any of the above described recognized texinfo markup elements.



### 3.8.11 Loading a library

Libraries can be loaded with the `LIB` or the `load` command (see [Section 5.1.79 \[LIB\]](#), page 211 and see [Section 5.2.12 \[load\]](#), page 299).

**Syntax:**     `LIB string_expression ;`  
               `load string_expression ;`

**Type:**       none

**Purpose:**     Reads a library from a file. If the given filename does not start with `.` or `/` and if the file cannot be located in the current directory, the `SearchPath` is checked for a directory containing a file with this name.

**Note on SearchPath:**

The `SearchPath` for a library is constructed at SINGULAR start-up time as follows:

1. the directories contained in the environment variable `SINGULARPATH` are appended.
2. the directories `$BinDir/LIB`, `$RootDir/LIB`, `$RootDir/../LIB`, `$DefaultDir/LIB`, `$DefaultDir/../LIB` are appended, where
  - `$BinDir` is the value of the environment variable `SINGULAR_BIN_DIR`, if set, or, if not set, the directory in which the SINGULAR program resides
  - `$RootDir` is the value of the environment variable `SINGULAR_ROOT_DIR`, if set, or, if not set, `$BinDir/..`.
  - `$DefaultDir` is the value of the environment variable `SINGULAR_DEFAULT_DIR`, if set, or `/usr/local`.
3. all directories which do not exist are removed from the `SearchPath`.

For setting environment variables, see [Section 5.1.155 \[system\]](#), page 275, or consult the manual of your shell.

The library `SearchPath` can be examined by starting up SINGULAR with the option `-v`, or by issuing the command `system("--version");`.

**Note on standard.lib:**

Unless SINGULAR is started with the `--no-stdlib` option, the library `standard.lib` is automatically loaded at start-up time.

Following a `LIB` or `load` command, only the names of the procedures in the library are loaded. The body of a particular procedure is only read upon the first call of the procedure. This minimizes memory consumption by unused procedures. Starting a SINGULAR session with the `-q` or `--quiet` option unsets the option `loadLib` and inhibits, thus, the monitoring of library loading (see option).

All libraries loaded in a SINGULAR session are displayed upon entering `listvar(package);` :

```
option(loadLib);    // show loading of libraries;
                   // standard.lib is loaded

listvar(package);
⇒ // Singmathic           [0] package Singmathic (C,singmathic.s\
   o)
⇒ // Standard             [0] package Standard (S,standard.lib)
⇒ // Top                  [0] package Top (T)

                        // the names of the procedures of inout.lib
LIB "inout.lib";    // are now known to Singular
⇒ // ** loaded inout.lib (4.1.2.0, Feb_2019)
listvar(package);
```

$\mapsto$ // Inout	[0] package Inout (S,inout.lib)
$\mapsto$ // Singmathic o)	[0] package Singmathic (C,singmathic.s\
$\mapsto$ // Standard	[0] package Standard (S,standard.lib)
$\mapsto$ // Top	[0] package Top (T)

See [Section 3.1.6 \[Command line options\]](#), page 19; [Section 5.1.79 \[LIB\]](#), page 211; [Section 2.3.3 \[Procedures and libraries\]](#), page 10; [Appendix D \[SINGULAR libraries\]](#), page 794; [Section 4.18 \[proc\]](#), page 124; [Section D.1 \[standard.lib\]](#), page 794; [Section 4.22 \[string\]](#), page 130; [Section 5.1.155 \[system\]](#), page 275.

## 3.9 Debugging tools

If SINGULAR does not come back to the prompt while calling a user defined procedure, probably a bracket or a " is missing. The easiest way to leave the procedure is to type some brackets or " and then `(RETURN)` .

### 3.9.1 ASSUME

**Syntax:** ASSUME ( int\_constant , expression )

**Purpose:** Tests the expression for correctness if the int\_constant is smaller as a variable `assumeLevel`. If no such variable exist the int expression is compared against 0. It is possible to define an individual `assumeLevel` for each library and/or procedure. If the expression is evaluated and not true (i.e. does not evaluate to `int(0)`) an error is raised.

**Note:** ASSUME shall be used for documentation and debugging, production code of a library must never define `assumeLevel`.

**Example:**

```

ASSUME(0,2==2); // always tested
ASSUME(1,1==2); // not evaluated
int assumeLevel=2;
ASSUME(1,1==2);
 $\mapsto$  ? ASSUME failed: ASSUME(1,1==2);
 $\mapsto$  ? error occurred in or before ./examples/ASSUME.sing line 4: ' ASSU
(1,1==2);'
// setting a different assumeLevel for poly.lib:
int Poly::assumeLevel=2;
 $\mapsto$  Poly of type 'ANY'. Trying load.
 $\mapsto$  ? 'Poly' no such package
 $\mapsto$  ? error occurred in or before ./examples/ASSUME.sing line 6: ' int l
ly::assumeLevel=2;'
 $\mapsto$  ? wrong type declaration. type 'help int;'
```

### 3.9.2 Tracing of procedures

Setting the TRACE variable to 1 (resp. 3) results in reporting of all procedure entries and exits (resp. together with line numbers). If TRACE is set to 4, Singular displays each line before its interpretation and waits for the `(RETURN)` key being pressed. See [Section 5.3.9 \[TRACE var\]](#), page 306.

**Example:**

```

proc t1
{
  int i=2;
  while (i>0)
  { i=i-1; }
}
TRACE=3;
t1();
⇒
⇒ entering t1 (level 0)
⇒ {1}{2}{3}{4}{5}{4}{5}{6}{7}{4}{5}{6}{7}{4}{6}{7}{8}
⇒ leaving t1 (level 0)

```

### 3.9.3 Source code debugger

The source code debugger (sdb) is an experimental feature, its interface may change in future versions of SINGULAR.

To enable the use of the source code debugger SINGULAR has to be started with the option `-d` or `--sdb` (see [Section 3.1.6 \[Command line options\]](#), page 19).

#### sdb commands

Each sdb command consists of one character which may be followed by a parameter.

<b>b</b>	print backtrace of calling stack
<b>c</b>	continue
<b>e</b>	edit the current procedure and reload it (current call will be aborted) only available on UNIX systems
<b>h,?</b>	display help screen
<b>n</b>	execute current line, sdb break at next line
<b>p &lt;identifier&gt;</b>	display type and value of the variable given by <identifier>
<b>Q</b>	quit this SINGULAR session
<b>q &lt;flags&gt;</b>	quit debugger, set debugger flags(0,1,2) 0: continue, disable the debugger 1: continue 2: throw an error, return to toplevel

### Syntactical errors in procedures

If SINGULAR was started using the command line option `-d` or `--sdb`, a syntactical error in a procedure will start the source code debugger instead of returning to the top level with an error message. The commands `q 1` and `q 2` are equivalent in this case.

### SDB breakpoints in procedures

Up to seven SDB breakpoints can be set. To set a breakpoint at a procedure use `breakpoint`. (See [Section 5.2.3 \[breakpoint\]](#), page 291).

These breakpoints can be cleared with the command `d breakpoint_no` from within the debugger or with `breakpoint( proc_name ,-1);`.

### 3.9.4 Break points

A break point can be put into a proc by inserting the command `~`. If **Singular** reaches a break point it asks for lines of commands (line-length must be less than 80 characters) from the user. It returns to normal execution if given an empty line. See [Section 5.2.16 \[~\]](#), page 302.

**Example:**

```
proc t
{
  int i=2;
  ~;
  return(i+1);
}
t();
⇒ -- break point in t --
⇒ -- 0: called      from STDIN --
i;                // here local variables of the procedure can be accessed
⇒ 2
⇒ -- break point in t --
⇒ 3
```

### 3.9.5 Printing of data

The procedure `dbprint` is useful for optional output of data: it takes 2 arguments and prints the second argument, if the first argument is positive; otherwise, it does nothing. See [Section 5.1.17 \[dbprint\]](#), page 169; [Section 5.3.11 \[voice\]](#), page 308.

### 3.9.6 libparse

`libparse` is a stand-alone program contained in the **SINGULAR** distribution (at the place where the **SINGULAR** executable program resides), which cannot be called inside **SINGULAR**. It is a debugging tool for libraries which performs exactly the same checks as the `load` command in **SINGULAR**, but generates more output during parsing. `libparse` is useful if an error occurs while loading the library, but the whole block around the line specified seems to be correct. In these situations the real error might have occurred hundreds of lines earlier in the library.

**Usage:**

`libparse [options] singular-library`

**Options:**

`-d Debuglevel`

increases the amount of output during parsing, where `Debuglevel` is an integer between 0 and 4. Default is 0.

`-s`

turns on reporting about violations of unenforced syntax rules

The following syntax checks are performed in any case:

- counting of pairs of brackets `{,}`, `[,]` and `(,)` (number of `{` has to match number of `}`, same for `[,]` and `(,)`).
- counting of `"` ( number of `"` must be even ).
- general library syntax ( only `LIB`, `static`, `proc` (with parameters, help, body and example) and comments, i.e `//` and `/* ... */`, are allowed).

Its output lists all procedures that have been parsed successfully:

```
$ libparse sample.lib
Checking library 'sample.lib'
  Library      function      line,start-eod line,body-eob  line,example-eoe
Version:0.0.0;
g Sample      tab line      9,  149-165   13,  271-298   14,  300-402
l Sample      internal_tab line 24,  450-475  25,  476-496   0,    0-496
```

where the following abbreviations are used:

- g: global procedure (default)
- l: static procedure, i.e., local to the library.

each of the following is the position of the byte in the library.

- start: begin of 'proc'
- eod: end of parameters
- body: start of procedurebody '{'
- eob: end of procedurebody '}'
- example: position of 'example'
- eoe: end of example '}'

Hence in the above example, the first procedure of the library sample.lib is user-accessible and its name is tab. The procedure starts in line 9, at character 149. The head of the procedure ends at character 165, the body starts in line 13 at character 271 and ends at character 298. The example section extends from line 14 character 300 to character 402.

The following example shows the result of a missing close-bracket } in line 26 of the library sample.lib.

```
LIB "sample.lib";
⇒ ? Library sample.lib: ERROR occurred: in line 26, 497.
⇒ ? missing close bracket '}' at end of library in line 26.
⇒ ? Cannot load library,... aborting.
⇒ ? error occurred in STDIN line 1: 'LIB "sample.lib";'
```

### 3.9.7 option(warn)

If this option is set some constructs which **may** lead to bug will result in a warning. While there are legitimate uses for them and they are **not errors** is is worth thinking about it.

change of options during a procedure call: is this side effect intended?

use of **def**: avoids type checking, but useful if a procedure handles several types at once

**ASSUME** outside of procedures: while a failed **ASSUME** aborts the current procedures and return to the top level - what should it do at top level?

See [Section 5.1.111 \[option\]](#), page 234.

## 3.10 Dynamic loading

In addition to the concept of libraries, it is also possible to dynamically extend the functionality by loading functions written in C/C++ or some other higher programming language. A collection of such functions is called a dynamic module and can be loaded by the command **LIB** or **load**. It is basically handled in the same way as a library: upon loading, a new **package** is created which holds the contents of the dynamic module. General information about the loaded module can be

displayed by the command `help package_name`. After loading the dynamic module, its functions can be used exactly like the built-in SINGULAR functions.

To have the full functionality of a built-in function, dynamic modules need to comply with certain requirements on their internal structure. As this would be beyond the scope of the **Singular** manual, a separate, more detailed guide on how to write and use dynamic modules is available.

## 4 Data types

This chapter explains all data types of SINGULAR in alphabetical order. For every type, there is a description of the declaration syntax as well as information about how to build expressions of certain types.

The term expression list in SINGULAR refers to any comma separated list of expressions.

For the general syntax of a declaration see [Section 3.5.1 \[General command syntax\]](#), page 41.

### 4.1 cring

Variables of type cring represent the ring of coefficients (see [Section 4.15 \[number\]](#), page 115)

#### 4.1.1 cring declarations

**Syntax:** `cring name = cring-expression ;`

**Purpose:** defines a new coefficient ring resp. field to be used for a ring definition (see [Section 4.20 \[ring\]](#), page 127). Most objects of this type are predefined.

**Default:** none

**Example:**

```
ZZ;
↦ ZZ
ZZ/3;
↦ ZZ/3
```

#### 4.1.2 cring expressions

A cring expression is:

1. an identifier of type cring:  
 $\mathbb{Q}\mathbb{Q}$  - the rational numbers  
 $\mathbb{Z}\mathbb{Z}$  - the integers
2. a function returning cring
3. an expression involving crings and the arithmetic operations  $/$ .

**Example:**

```
ZZ/3;
↦ ZZ/3
```

See [Section 4.20 \[ring\]](#), page 127.

#### 4.1.3 cring operations

$/$             residue class ring

**Example:**

```
ZZ/101;
↦ ZZ/101
```

### 4.1.4 cring related functions

**crossprod**

cross product of several objects of type cring (see [Section 5.1.15 \[crossprod\]](#), page 168)

**Float**

several variants of Floating point (inexact) real and complex numbers (see [Section 5.1.45 \[Float\]](#), page 186).

**flintQ**

multivariate rational functions over  $\mathbb{Q}$  (via flint, requires  $\geq 2.5.3$ ) (see [Section 5.1.44 \[flintQ\]](#), page 186).

See [Section 5.1.45 \[Float\]](#), page 186; [Section 5.1.15 \[crossprod\]](#), page 168; [Section 5.1.44 \[flintQ\]](#), page 186.

## 4.2 bigint

Variables of type bigint represent the arbitrary long integers. They can only be constructed from other types (int, number).

### 4.2.1 bigint declarations

**Syntax:** `bigint name = int_expression ;`

**Purpose:** defines a long integer variable

**Default:** 0

**Example:**

```

    bigint i = 42;
    ring r=0,x,dp;
    number n=2;
    bigint j = i + bigint(n)^50; j;
    ↪ 1125899906842666

```

### 4.2.2 bigint expressions

A bigint expression is:

1. an identifier of type bigint
2. a function returning bigint
3. an expression involving bigints and the arithmetic operations `+`, `-`, `*`, `div`, `% (mod)`, or `^`
4. a type cast to bigint.

**Example:**

```

// Note: 11*13*17*100*200*2000*503*1111*222222
// returns a machine integer:
11*13*17*100*200*2000*503*1111*222222;
↪ // ** int overflow(*), result may be wrong
↪ -6869239595516308480
// using the type cast number for a greater allowed range
bigint(11)*13*17*100*200*2000*503*1111*222222;
↪ 12075748128684240000000

```

See [Section 3.5.5 \[Type conversion and casting\]](#), page 46; [Section 4.7 \[int\]](#), page 85; [Section 4.15 \[number\]](#), page 115.



### 4.2.3 bigint operations

+	addition
-	negation or subtraction
*	multiplication
div	integer division (omitting the remainder $\geq 0$ )
mod, %	integer modulo (the remainder of the division <code>div</code> )
^, **	exponentiation (exponent must be non-negative)
<, >, <=, >=, ==, <>	comparators

**Example:**

```
bigint(5)*2, bigint(2)^100-10;
↦ 10 1267650600228229401496703205366
bigint(-5) div 2, bigint(-5) mod 2;
↦ -3 1
```

### 4.2.4 bigint related functions

gcd	greatest common divisor (see <a href="#">Section 5.1.50 [gcd]</a> , page 190)
memory	memory usage (see <a href="#">Section 5.1.89 [memory]</a> , page 219)

See [Section 5.1.89 \[memory\]](#), page 219.

## 4.3 bigintmat

Big integer matrices are matrices with big integer entries. No basering definition is required to use bigint matrices, for they do not belong to a ring. Bigintmat entries can have any size because of the use of bigint.

### 4.3.1 bigintmat declarations

**Syntax:** `bigintmat name = bigintmat_expression ;`  
`bigintmat name [ rows ] [ cols ] = bigintmat_expression ;`  
`bigintmat name [ rows ] [ cols ] = list_of_int_and_bigint_expressions ;`  
rows and cols must be positive int expressions.

**Purpose:** defines a bigintmat variable.  
Given a list of (big) integers, the matrix is filled up with the first row from the left to the right, then the second one and so on. If the (big-)int\_list contains less than rows\*cols elements, the remaining ones are set to zero; if it contains more elements, only the first rows\*cols ones are considered.

**Default:** empty (1x0 matrix)

**Example:**

```

bigintmat bim[4][3]=2, 5, 224553233465, 232444, 434, 0, 0, 4544232222;
bim;
↳      2,          5,224553233465,
↳ 232444,          434,          0,
↳      0,4544232222,          0,
↳      0,          0,          0
bim[2, 1];
↳ 232444

```

### 4.3.2 bigintmat expressions

A bigintmat expression is:

1. an identifier of type bigintmat
2. a function returning bigintmat
3. a bigintmat operation involving (big-)ints and int operations (+, -, \*)
4. an expression involving bigintmats and the operations (+, -, \*)
5. a type cast to bigintmat (see [Section 4.3.3 \[bigintmat type cast\]](#), page 76)

**Example:**

```

bigintmat m1[2][2]=1, 2, 6, 3;
m1*3;
↳ 3,6,
↳ 18,9
intmat im[3][2] = intmat(m1*3);
bigintmat m2 = bigintmat(im); // cast intmat im to bigintmat
m2;
↳ 3,6,
↳ 18,9
m2*m1+m2;
↳ 42,30,
↳ 90,72
_+4;
↳ 46, 0,
↳ 0,76

```

See [Section 3.5.5 \[Type conversion and casting\]](#), page 46; [Section 4.3 \[bigintmat\]](#), page 75.

### 4.3.3 bigintmat type cast

**Syntax:** `bigintmat ( expression )`

**Type:** `bigintmat`

**Purpose:** Converts expression to a bigintmat, where expression must be of type intmat, bigintvec, or bigintmat. The size (resp. dimension) of the created bigintmat equals the size (resp. dimension) of the expression.

**Example:**

```

intmat im[2][1]=2, 3;
bigintmat(im);
↳ 2,
↳ 3
bigintmat(_);

```

```

    ↪ 2,
    ↪ 3
    bigintmat(intmat(intvec(1,2,3,4), 2, 2)); //casts at first to intmat, th
    ↪ 1,2,
    ↪ 3,4

```

See [Section 3.5.5 \[Type conversion and casting\]](#), page 46; [Section 4.3 \[bigintmat\]](#), page 75; [Section 4.8.3 \[intmat type cast\]](#), page 92.

#### 4.3.4 bigintmat operations

- +** addition with intmat, int, or bigint. In case of (big-)int, it is added to every entry of the matrix.
- negation or subtraction with intmat, int, or bigint. In case of (big-)int, it is subtracted from every entry of the matrix.
- \*** multiplication with intmat, int, or bigint; In case of (big-)int, every entry of the matrix is multiplied by the (big-)int
- <>, ==** comparators

`bigintmat_expression [ int, int ]`

is a bigintmat entry, where the first index indicates the row and the second the column

##### Example:

```

    bigintmat m[3][4] = 3,3,6,3,5,2,2,7,0,0,45,3;
    m;
    ↪ 3,3, 6,3,
    ↪ 5,2, 2,7,
    ↪ 0,0,45,3
    m[1,3];                // show entry at [row 1, col 3]
    ↪ 6
    m[1,3] = 10;           // set entry at [row 1, col 3] to 10
    m;
    ↪ 3,3,10,3,
    ↪ 5,2, 2,7,
    ↪ 0,0,45,3
    size(m);               // number of entries
    ↪ 12
    bigintmat n[2][3] = 2,6,0,4,0,5;
    n * m;
    ↪ 36,18, 32,48,
    ↪ 12,12,265,27
    typeof(_);
    ↪ bigintmat
    -m;
    ↪ -3,-3,-10,-3,
    ↪ -5,-2, -2,-7,
    ↪ 0, 0,-45,-3
    bigintmat o;
    o=n-10;
    o;
    ↪ -8, 0,0,

```

```

⇒ 0,-10,0
   m*2;           // double each entry of m
⇒ 6,6,20, 6,
⇒ 10,4, 4,14,
⇒ 0,0,90, 6
   o-2*m;
⇒      ? bigintmat/cmatrix not compatible
⇒      ? error occurred in or before ./examples/bigintmat_operations.sing lin\
      e 15: ' o-2*m;'
```

## 4.4 bigintvec

Big integer vectors are vectors with big integer entries. No basering definition is required to use bigint vectors, for they do not belong to a ring. Bigintvec entries can have any size because of the use of bigint.

### 4.4.1 bigintvec declarations

**Syntax:**    `bigintvec name = bigintvec_expression ;`  
              `bigintvec name ] = list_of_int_and_bigint_expressions ;`

**Purpose:**    defines a bigintvec variable.

**Default:**    empty

**Example:**

```

      bigintvec v=2, 5, 224553233465, 232444, 434, 0, 0, 4544232222;
      v;
⇒ 2,5,224553233465,232444,434,0,0,4544232222
      v[2];
⇒ 5
```

### 4.4.2 bigintvec expressions

A bigintvec expression is:

1. an identifier of type bigintvec
2. a function returning bigintvec
3. a type cast to bigintvec (see [Section 4.4.3 \[bigintvec type cast\]](#), page 78)

**Example:**

```

      bigintmat m1[2][2]=1, 2, 6, 3;
      bigintvec v=m1[1,1..2];
      v;
⇒ 1,2
```

See [Section 3.5.5 \[Type conversion and casting\]](#), page 46; [Section 4.4 \[bigintvec\]](#), page 78.

### 4.4.3 bigintvec type cast

**Syntax:**    `bigintvec ( expression )`

**Type:**        bigintvec

**Purpose:** Converts expression to a bigintvec, where expression must be of type intvec, or bigint-mat. The size of the created bigintvec equals the size of the expression.

**Example:**

```
// TODO
```

See [Section 3.5.5 \[Type conversion and casting\]](#), page 46; [Section 4.4 \[bigintvec\]](#), page 78.

#### 4.4.4 bigintvec operations

- +** addition with intvec, int, or bigint. In case of (big-)int, it is added to every entry of the vector.
- negation or subtraction with intvec, int, or bigint. In case of (big-)int, it is subtracted from every entry of the matrix.
- \*** multiplication with int, or bigint; every entry of the vector is multiplied by the (big-)int
- <>, ==** comparators

bigintvec\_expression [ int ]  
is a bigintvec entry.

**Example:**

```
bigintvec v = 3,3,6,3,5,2,2,7,0,0,45,3;
v;
⇒ 3,3,6,3,5,2,2,7,0,0,45,3
v[3];           // show entry at [3]
⇒ 6
v[3] = 10;      // set entry at [3] to 10
v;
⇒ 3,3,10,3,5,2,2,7,0,0,45,3
size(v);        // number of entries
⇒ 12
-v;
⇒ -3,-3,-10,-3,-5,-2,-2,-7,0,0,-45,-3
v-10;
⇒ -7,0,0,0,0,0,0,0,0,0,0,0
v*2;            // double each entry of v
⇒ 6,6,20,6,10,4,4,14,0,0,90,6
```

## 4.5 def

Objects may be defined without a specific type: they inherit their type from the first assignment to them. E.g., `ideal i=x,y,z; def j=i^2;` defines the ideal  $i^2$  with the name `j`.

**Note:** Unlike other assignments a ring as an untyped object is not a copy but another reference to the same (possibly unnamed) ring. This means that entries in one of these rings appear also in the other ones. The following defines a ring `s` which is just another reference (or name) for the basering `r`. The name `basering` is an alias for the current ring.

```
ring r=32003,(x,y,z),dp;
poly f = x;
def s=basering;
setring s;
```

```

    nameof(basering);
  ↪ s
    listvar();
  ↪ // s                                [0] *ring
  ↪ //      f                            [0] poly
  ↪ // r                                [0] ring(*)
    poly g = y;
    kill f;
    listvar(r);
  ↪ // r                                [0] ring(*)
  ↪ // g                                [0] poly
    ring t=32003,(u,w),dp;
    def rt=r+t;
    rt;
  ↪ // coefficients: ZZ/32003
  ↪ // number of vars : 5
  ↪ //      block  1 : ordering dp
  ↪ //              : names    x y z
  ↪ //      block  2 : ordering dp
  ↪ //              : names    u w
  ↪ //      block  3 : ordering C

```

This reference to a ring with `def` is useful if the basering is not local to the procedure (so it cannot be accessed by its name) but one needs a name for it (e.g., for a use with `setring` or `map`). `setring r`; does not work in this case, because `r` may not be local to the procedure.

### 4.5.1 def declarations

**Syntax:**    `def name = expression ;`

**Purpose:**    defines an object of the same type as the right-hand side.

**Default:**    none

**Note:**       This is useful if the right-hand side may be of variable type as a consequence of a computation (e.g., ideal or module or matrix). It may also be used in procedures to give the basering a name which is local to the procedure.

**Example:**

```

    def i=2;
    typeof(i);
  ↪ int

```

See [Section 5.1.161 \[typeof\], page 282](#).

## 4.6 ideal

Ideals are represented as lists of polynomials which generate the ideal. Like polynomials they can only be defined or accessed with respect to a basering.

**Note:** `size` counts only the non-zero generators of an ideal whereas `ncols` counts all generators; see [Section 5.1.144 \[size\], page 264](#), [Section 5.1.104 \[ncols\], page 231](#).

### 4.6.1 ideal declarations

**Syntax:**    `ideal name = list_of_poly_and_ideal_expressions ;`  
               `ideal name = ideal_expression ;`

**Purpose:**    defines an ideal.

**Default:**    0

**Example:**

```

ring r=0,(x,y,z),dp;
poly s1 = x2;
poly s2 = y3;
poly s3 = z;
ideal i = s1, s2-s1, 0,s2*s3, s3^4;
i;
↪ i[1]=x2
↪ i[2]=y3-x2
↪ i[3]=0
↪ i[4]=y3z
↪ i[5]=z4
size(i);
↪ 4
ncols(i);
↪ 5

```

### 4.6.2 ideal expressions

An ideal expression is:

1. an identifier of type ideal
2. a function returning an ideal
3. a combination of ideal expressions by the arithmetic operations + or \*
4. a power of an ideal expression (operator ^ or \*\*)

Note that the computation of the product `i*i` involves all products of generators of `i` while `i^2` involves only the different ones, and is therefore faster.

5. a type cast to ideal

**Example:**

```

ring r=0,(x,y,z),dp;
ideal m = maxideal(1);
m;
↪ m[1]=x
↪ m[2]=y
↪ m[3]=z
poly f = x2;
poly g = y3;
ideal i = x*y*z , f-g, g*(x-y) + f^4 ,0, 2x-z2y;
ideal M = i + maxideal(10);
timer =0;
i = M*M;
timer;
↪ 0

```

```

    ncols(i);
    ↪ 505
    timer =0;
    i = M^2;
    ncols(i);
    ↪ 505
    timer;
    ↪ 0
    i[ncols(i)];
    ↪ x20
    vector v = [x,y-z,x2,y-x,x2yz2-y];
    ideal j = ideal(v);

```

### 4.6.3 ideal operations

- +** addition (concatenation of the generators and simplification)
- \*** multiplication (with ideal, poly, vector, module; simplification in case of multiplication with ideal)
- ^** exponentiation (by a non-negative integer)

ideal\_expression [ intvec\_expression ]

are polynomial generators of the ideal, index 1 gives the first generator.

**Note:** For simplification of an ideal, see also [Section 5.1.143 \[simplify\]](#), page 263.

**Example:**

```

    ring r=0,(x,y,z),dp;
    ideal I = 0,x,0,1;
    I;
    ↪ I[1]=0
    ↪ I[2]=x
    ↪ I[3]=0
    ↪ I[4]=1
    I + 0; // simplification
    ↪ _[1]=1
    ideal J = I,0,x,x-z;;
    J;
    ↪ J[1]=0
    ↪ J[2]=x
    ↪ J[3]=0
    ↪ J[4]=1
    ↪ J[5]=0
    ↪ J[6]=x
    ↪ J[7]=x-z
    I * J; // multiplication with simplification
    ↪ _[1]=1
    I*x;
    ↪ _[1]=0
    ↪ _[2]=x2
    ↪ _[3]=0
    ↪ _[4]=x
    vector V = [x,y,z];

```



```

    print(V*I);
    ↪ 0,x2,0,x,
    ↪ 0,xy,0,y,
    ↪ 0,xz,0,z
    ideal m = maxideal(1);
    m^2;
    ↪ _[1]=x2
    ↪ _[2]=xy
    ↪ _[3]=xz
    ↪ _[4]=y2
    ↪ _[5]=yz
    ↪ _[6]=z2
    ideal II = I[2..4];
    II;
    ↪ II[1]=x
    ↪ II[2]=0
    ↪ II[3]=1

```

#### 4.6.4 ideal related functions

##### char\_series

irreducible characteristic series (see [Section 5.1.6 \[char\\_series\]](#), page 161)

**coeffs** matrix of coefficients (see [Section 5.1.12 \[coeffs\]](#), page 165)

**contract** contraction by an ideal (see [Section 5.1.13 \[contract\]](#), page 167)

**diff** partial derivative (see [Section 5.1.24 \[diff\]](#), page 173)

**degree** multiplicity, dimension and codimension of the ideal of leading terms (see [Section 5.1.20 \[degree\]](#), page 171)

**dim** Krull dimension of basering modulo the ideal of leading terms (see [Section 5.1.25 \[dim\]](#), page 174)

##### eliminate

elimination of variables (see [Section 5.1.28 \[eliminate\]](#), page 176)

**facstd** factorizing Groebner basis algorithm (see [Section 5.1.34 \[facstd\]](#), page 179)

##### factorize

ideal of factors of a polynomial (see [Section 5.1.36 \[factorize\]](#), page 180)

**fglm** Groebner basis computation from a Groebner basis w.r.t. a different ordering (see [Section 5.1.39 \[fglm\]](#), page 183)

**finduni** computation of univariate polynomials lying in a zero dimensional ideal (see [Section 5.1.43 \[finduni\]](#), page 185)

**fres** free resolution of a standard basis (see [Section 5.1.48 \[fres\]](#), page 188)

**groebner** Groebner basis computation (a wrapper around `std`, `stdhilb`, `stdfglm`,...) (see [Section 5.1.53 \[groebner\]](#), page 191)

##### highcorner

the smallest monomial not contained in the ideal. The ideal has to be zero-dimensional. (see [Section 5.1.55 \[highcorner\]](#), page 194)

**homog** homogenization with respect to a variable (see [Section 5.1.57 \[homog\]](#), page 196)

<code>hilb</code>	Hilbert series of a standard basis (see <a href="#">Section 5.1.56 [hilb]</a> , page 195)
<code>indepSet</code>	sets of independent variables of an ideal (see <a href="#">Section 5.1.61 [indepSet]</a> , page 199)
<code>interred</code>	interreduction of an ideal (see <a href="#">Section 5.1.64 [interred]</a> , page 201)
<code>intersect</code>	ideal intersection (see <a href="#">Section 5.1.65 [intersect]</a> , page 202)
<code>jacob</code>	ideal of all partial derivatives resp. jacobian matrix (see <a href="#">Section 5.1.66 [jacob]</a> , page 203)
<code>jet</code>	Taylor series up to a given order (see <a href="#">Section 5.1.68 [jet]</a> , page 204)
<code>kbase</code>	vector space basis of basering modulo ideal of leading terms (see <a href="#">Section 5.1.69 [kbase]</a> , page 205)
<code>koszul</code>	Koszul matrix (see <a href="#">Section 5.1.73 [koszul]</a> , page 207)
<code>lead</code>	leading terms of a set of generators (see <a href="#">Section 5.1.75 [lead]</a> , page 209)
<code>lift</code>	lift-matrix (see <a href="#">Section 5.1.80 [lift]</a> , page 211)
<code>liftstd</code>	standard basis and transformation matrix computation (see <a href="#">Section 5.1.81 [liftstd]</a> , page 212)
<code>lres</code>	free resolution for homogeneous ideals (see <a href="#">Section 5.1.83 [lres]</a> , page 215)
<code>maxideal</code>	power of the maximal ideal at 0 (see <a href="#">Section 5.1.88 [maxideal]</a> , page 219)
<code>minbase</code>	minimal generating set of a homogeneous ideal, resp. module, or an ideal, resp. module, in a local ring (see <a href="#">Section 5.1.91 [minbase]</a> , page 221)
<code>minor</code>	set of minors of a matrix (see <a href="#">Section 5.1.92 [minor]</a> , page 221)
<code>modulo</code>	representation of $(h1 + h2)/h1 \cong h2/(h1 \cap h2)$ (see <a href="#">Section 5.1.94 [modulo]</a> , page 223)
<code>mres</code>	minimal free resolution of an ideal resp. module w.r.t. a minimal set of generators of the given ideal resp. module (see <a href="#">Section 5.1.98 [mres]</a> , page 225)
<code>mstd</code>	standard basis and minimal generating set of an ideal (see <a href="#">Section 5.1.100 [mstd]</a> , page 227)
<code>mult</code>	multiplicity, resp. degree, of the ideal of leading terms (see <a href="#">Section 5.1.101 [mult]</a> , page 228)
<code>ncols</code>	number of columns (see <a href="#">Section 5.1.104 [ncols]</a> , page 231)
<code>nres</code>	a free resolution of an ideal resp. module M which is minimized from the second free module on (see <a href="#">Section 5.1.106 [nres]</a> , page 232)
<code>preimage</code>	preimage under a ring map (see <a href="#">Section 5.1.117 [preimage]</a> , page 240)
<code>qhweight</code>	quasihomogeneous weights of an ideal (see <a href="#">Section 5.1.124 [qhweight]</a> , page 246)
<code>quotient</code>	ideal quotient (see <a href="#">Section 5.1.127 [quotient]</a> , page 248)
<code>reduce</code>	normalform with respect to a standard basis (see <a href="#">Section 5.1.131 [reduce]</a> , page 251)
<code>res</code>	free resolution of an ideal resp. module but not changing the given ideal resp. module (see <a href="#">Section 5.1.134 [res]</a> , page 253)
<code>simplify</code>	simplification of a set of polynomials (see <a href="#">Section 5.1.143 [simplify]</a> , page 263)
<code>size</code>	number of non-zero generators (see <a href="#">Section 5.1.144 [size]</a> , page 264)
<code>slimgb</code>	Groebner basis computation with slim technique (see <a href="#">Section 5.1.145 [slimgb]</a> , page 265)

<code>sortvec</code>	permutation for sorting ideals resp. modules (see <a href="#">Section 5.1.146 [sortvec]</a> , page 266)
<code>sres</code>	free resolution of a standard basis (see <a href="#">Section 5.1.149 [sres]</a> , page 269)
<code>std</code>	standard basis computation (see <a href="#">Section 5.1.151 [std]</a> , page 271)
<code>stdfglm</code>	standard basis computation with fglm technique (see <a href="#">Section 5.1.152 [stdfglm]</a> , page 272)
<code>stdhilb</code>	Hilbert driven standard basis computation (see <a href="#">Section 5.1.153 [stdhilb]</a> , page 273)
<code>subst</code>	substitution of a ring variable (see <a href="#">Section 5.1.154 [subst]</a> , page 274)
<code>syz</code>	computation of the first syzygy module (see <a href="#">Section 5.1.156 [syz]</a> , page 280)
<code>vdim</code>	vector space dimension of basering modulo ideal of leading terms (see <a href="#">Section 5.1.168 [vdim]</a> , page 286)
<code>weight</code>	optimal weights (see <a href="#">Section 5.1.172 [weight]</a> , page 288)

## 4.7 int

Variables of type `int` represent the machine integers and are, therefore, limited in their range (e.g., the range is between -2147483647 and 2147483647 on 32-bit machines). They are mainly used to count things (dimension, rank, etc.), in loops (see [Section 5.2.8 \[for\]](#), page 295), and to represent boolean values (`FALSE` is represented by 0, every other value means `TRUE`, see [Section 4.7.5 \[boolean expressions\]](#), page 89).

Integers consist of a sequence of digits, possibly preceded by a sign. A space is considered as a separator, so it is not allowed between digits. A sequence of digits outside the allowed range is converted to the type `bigint`, see [Section 4.2 \[bigint\]](#), page 74.

### 4.7.1 int declarations

**Syntax:** `int name = int_expression ;`

**Purpose:** defines an integer variable.

**Default:** 0

**Example:**

```

    int i = 42;
    int j = i + 3; j;
    ↪ 45
    i = i * 3 - j; i;
    ↪ 81
    int k;    // assigning the default value 0 to k
    k;
    ↪ 0

```

### 4.7.2 int expressions

An `int` expression is:

1. a sequence of digits (if the number represented by this sequence is too large to fit into the range of integers it is automatically converted to the type `number`, if a basering is defined)
2. an identifier of type `int`
3. a function returning `int`

4. an expression involving ints and the arithmetic operations  $+$ ,  $-$ ,  $*$ ,  $\text{div } (/)$ ,  $\% \text{ (mod)}$ , or  $\wedge$
5. a boolean expression
6. a type cast to int

**Note:** Variables of type int represent the compiler integers and are, therefore, limited in their range (see [Section 6.1 \[Limitations\]](#), page 310). If this range is too small the expression must be converted to the type number over a ring with characteristic 0.

**Example:**

```

12345678901; // too large
⇒ 12345678901
typeof(_);
⇒ bigint
ring r=0,x,dp;
12345678901;
⇒ 12345678901
typeof(_);
⇒ bigint
// Note: 11*13*17*100*200*2000*503*1111*222222
// returns a machine integer:
11*13*17*100*200*2000*503*1111*222222;
⇒ // ** int overflow(*), result may be wrong
⇒ -6869239595516308480
// using the type cast number for a greater allowed range
number(11)*13*17*100*200*2000*503*1111*222222;
⇒ 12075748128684240000000
ring rp=32003,x,dp;
12345678901;
⇒ 12345678901
typeof(_);
⇒ bigint
intmat m[2][2] = 1,2,3,4;
m;
⇒ 1,2,
⇒ 3,4
m[2,2];
⇒ 4
typeof(_);
⇒ int
det(m);
⇒ -2
m[1,1] + m[2,1] == trace(m);
⇒ 0
! 0;
⇒ 1
1 and 2;
⇒ 1
intvec v = 1,2,3;
def d =transpose(v)*v; // scalarproduct gives an 1x1 intvec
typeof(d);
⇒ intvec
int i = d[1]; // access the first (the only) entry in the intvec

```

```

ring rr=31,(x,y,z),dp;
poly f = 1;
i = int(f);           // cast to int
// Integers may be converted to constant polynomials by an assignment,
poly g=37;
// define the constant polynomial g equal to the image of
// the integer 37 in the actual coefficient field, here it equals 6
g;
↳ 6

```

See [Section 3.5.5 \[Type conversion and casting\]](#), page 46; [Section 4.15 \[number\]](#), page 115.

### 4.7.3 int operations

++	changes its operand to its successor, is itself no int expression
--	changes its operand to its predecessor, is itself no int expression
+	addition
-	negation or subtraction
*	multiplication
div	integer division (omitting the remainder), rounding toward 0
%, mod	integer modulo (the remainder of the division)
^, **	exponentiation (exponent must be non-negative)
<, >, <=, >=, ==, <>	comparators

**Note:** An assignment `j=i++`; or `j=i--`; is not allowed, in particular it does not change the value of `j`, see [Section 6.1 \[Limitations\]](#), page 310.

#### Example:

```

int i=1;
int j;
i++; i; i--; i;
↳ 2
↳ 1
// ++ and -- do not return a value as in C, cannot assign
j = i++;
↳ // ** right side is not a datum, assignment ignored
↳ // ** in line >> j = i++;<<
// the value of j is unchanged
j; i;
↳ 0
↳ 2
i+2, 2-i, 5^2;
↳ 4 0 25
5 div 2, 8%3;
↳ 2 2
-5 div 2, -5 mod 2, -5 % 2;
↳ -2 -1 -1
1<2, 2<=2;
↳ 1 1

```

#### 4.7.4 int related functions

<code>char</code>	characteristic of the coefficient field of a ring (see <a href="#">Section 5.1.5 [char]</a> , page 161)
<code>deg</code>	degree of a polynomial resp. vector (see <a href="#">Section 5.1.19 [deg]</a> , page 170)
<code>det</code>	determinant (see <a href="#">Section 5.1.23 [det]</a> , page 172)
<code>dim</code>	Krull dimension of basering modulo ideal of leading terms, resp. dimension of module of leading terms (see <a href="#">Section 5.1.25 [dim]</a> , page 174)
<code>extgcd</code>	Bezout representation of gcd (see <a href="#">Section 5.1.33 [extgcd]</a> , page 178)
<code>find</code>	position of a substring in a string (see <a href="#">Section 5.1.42 [find]</a> , page 185)
<code>gcd</code>	greatest common divisor (see <a href="#">Section 5.1.50 [gcd]</a> , page 190)
<code>koszul</code>	Koszul matrix (see <a href="#">Section 5.1.73 [koszul]</a> , page 207)
<code>memory</code>	memory usage (see <a href="#">Section 5.1.89 [memory]</a> , page 219)
<code>mult</code>	multiplicity of an ideal, resp. module, of leading terms (see <a href="#">Section 5.1.101 [mult]</a> , page 228)
<code>ncols</code>	number of columns (see <a href="#">Section 5.1.104 [ncols]</a> , page 231)
<code>npars</code>	number of ring parameters (see <a href="#">Section 5.1.105 [npars]</a> , page 231)
<code>nrows</code>	number of rows of a matrix, resp. the rank of the free module where the vector or module lives (see <a href="#">Section 5.1.107 [nrows]</a> , page 232)
<code>nvars</code>	number of ring variables (see <a href="#">Section 5.1.109 [nvars]</a> , page 233)
<code>ord</code>	degree of the leading term of a polynomial resp. vector (see <a href="#">Section 5.1.112 [ord]</a> , page 238)
<code>par</code>	n-th parameter of the basering (see <a href="#">Section 5.1.114 [par]</a> , page 239)
<code>pardeg</code>	degree of a number considered as a polynomial in the ring parameters (see <a href="#">Section 5.1.115 [pardeg]</a> , page 239)
<code>prime</code>	the next lower prime (see <a href="#">Section 5.1.118 [prime]</a> , page 241)
<code>random</code>	a pseudo random integer between the given limits (see <a href="#">Section 5.1.128 [random]</a> , page 249)
<code>regularity</code>	regularity of a resolution (see <a href="#">Section 5.1.132 [regularity]</a> , page 252)
<code>rvar</code>	test, if the given expression or string is a ring variable (see <a href="#">Section 5.1.139 [rvar]</a> , page 258)
<code>size</code>	number of elements in an object (see <a href="#">Section 5.1.144 [size]</a> , page 264)
<code>trace</code>	trace of an integer matrix (see <a href="#">Section 5.1.158 [trace]</a> , page 281)
<code>var</code>	n-th ring variable of the basering (see <a href="#">Section 5.1.165 [var]</a> , page 284)
<code>vdim</code>	vector space dimension of basering modulo ideal of leading terms, resp. of freemodule modulo module of leading terms (see <a href="#">Section 5.1.168 [vdim]</a> , page 286)

### 4.7.5 boolean expressions

A boolean expression is an int expression used in a logical context:

An int expression  $\neq 0$  evaluates to *TRUE* (represented by 1), 0 evaluates to *FALSE* (represented by 0).

The following is the list of available comparisons of objects of the same type.

**Note:** There are no comparisons for ideals and modules, resolutions and maps.

1. integer comparisons:

```
i == j
i != j    // or    i <> j
i <= j
i >= j
i > j
i < j
```

2. number comparisons:

```
m == n
m != n    // or    m <> n
m < n
m > n
m <= n
m >= n
```

For numbers from  $\mathbb{Z}/p$  or from field extensions not all operations are useful:

- 0 is always the smallest element,
- in  $\mathbb{Z}/p$  the representatives in the range  $-(p-1)/2..(p-1)/2$  when  $p>2$  resp. 0 and 1 for  $p=2$  are used for comparisons,
- in field extensions the last two operations ( $\geq, \leq$ ) yield always *TRUE* (1) and the  $<$  and  $>$  are equivalent to  $\neq$ .

3. polynomial or vector comparisons:

```
f == g
f != g    // or    f <> g
f <= g    // comparing the leading term w.r.t. the monomial order
f < g
f >= g
f > g
```

4. intmat or matrix comparisons:

```
v == w
v != w    // or    v <> w
```

5. intvec or string comparisons:

```
f == g
f != g    // or    f <> g
f <= g    // comparing lexicographically
f >= g    // w.r.t. the order specified by ASCII
f > g
f < g
```

6. boolean expressions combined by boolean operations (*and*, *or*, *not*)

**Note:** All arguments of a logical expression are first evaluated and then the value of the logical expression is determined. For example, the logical expression  $(a \ || \ b)$  is evaluated by first evalu-

ating **a** and **b**, even though the value of **b** has no influence on the value of **(a || b)**, if **a** evaluates to true.

Note that this evaluation is different from the left-to-right, conditional evaluation of logical expressions (as found in most programming languages). For example, in these other languages, the value of **(1 || b)** is determined without ever evaluating **b**.

See [Section 6.3 \[Major differences to the C programming language\]](#), page 311.

### 4.7.6 boolean operations

**and**            logical **and**, may also be written as **&&**

**or**            logical **or**, may also be written as **||**

**not**           logical **not**, may also be written as **!**

The precedence of the boolean operations is:

1. parentheses
2. comparisons
3. not
4. and
5. or

**Example:**

```
(1>2) and 3;
⇒ 0
1 > 2 and 3;
⇒ 0
! 0 or 1;
⇒ 1
!(0 or 1);
⇒ 0
```

## 4.8 intmat

Integer matrices are matrices with integer entries. For the range of integers see [Section 6.1 \[Limitations\]](#), page 310. Integer matrices do not belong to a ring, they may be defined without a basering being defined. An **intmat** can be multiplied by and added to an **int**; in this case the **int** is converted into an **intmat** of the right size with the integer on the diagonal. The integer 1, for example, is converted into the unit matrix.

### 4.8.1 intmat declarations

**Syntax:**    **intmat** name = intmat\_expression ;  
               **intmat** name [ rows ] [ cols ] = intmat\_expression ;  
               **intmat** name [ rows ] [ cols ] = list\_of\_int\_and\_intvec\_and\_intmat\_expressions ;  
               rows and cols must be positive int expressions.

**Purpose:**    defines an **intmat** variable.  
               Given a list of integers, the matrix is filled up with the first row from the left to the right, then the second row and so on. If the **int\_list** contains less than **rows\*cols** elements, the matrix is filled up with zeros; if it contains more elements, only the first **rows\*cols** elements are used.



**Default:** 0 (1 x 1 matrix)

**Example:**

```

intmat im[3][5]=1,3,5,7,8,9,10,11,12,13;
im;
⇒ 1,3,5,7,8,
⇒ 9,10,11,12,13,
⇒ 0,0,0,0,0
im[3,2];
⇒ 0
intmat m[2][3] = im[1..2,3..5]; // defines a submatrix
m;
⇒ 5,7,8,
⇒ 11,12,13

```

### 4.8.2 intmat expressions

An intmat expression is:

1. an identifier of type intmat
2. a function returning intmat
3. an intmat operation involving ints and int operations (+, -, \*, div, %)
4. an expression involving intmats and the operations (+, -, \*)
5. a type cast to intmat (see [Section 4.8.3 \[intmat type cast\]](#), page 92)

**Example:**

```

intmat Idm[2][2];
Idm +1; // add the unit intmat
⇒ 1,0,
⇒ 0,1
intmat m1[3][2] = _,1,-2; // take entries from the last result
m1;
⇒ 1,0,
⇒ 0,1,
⇒ 1,-2
intmat m2[2][3]=1,0,2,4,5,1;
transpose(m2);
⇒ 1,4,
⇒ 0,5,
⇒ 2,1
intvec v1=1,2,4;
intvec v2=5,7,8;
m1=v1,v2; // fill m1 with v1 and v2
m1;
⇒ 1,2,
⇒ 4,5,
⇒ 7,8
trace(m1*m2);
⇒ 56

```

See [Section 3.5.5 \[Type conversion and casting\]](#), page 46; [Section 4.15 \[number\]](#), page 115.

### 4.8.3 intmat type cast

**Syntax:**     `intmat ( expression )`  
               `intmat ( expression, int_n, int_m )`

**Type:**       `intmat`

**Purpose:**     Converts expression to an `intmat`, where expression must be of type `intvec`, `intmat`, or `bigintmat`. If `int_n` and `int_m` are supplied, then they specify the dimension of the `intmat`. Otherwise, the size (resp. dimensions) of the `intmat` are determined by the size (resp. dimensions) of the expression. If expression is a `bigintmat` containing an entry larger than the limit of `int`, it is set to 0 in the returning `intmat`.

**Example:**

```

    intmat(intvec(1));
    ↪ 1
    intmat(intvec(1), 1, 2);
    ↪ 1,0
    intmat(intvec(1,2,3,4), 2, 2);
    ↪ 1,2,
    ↪ 3,4
    intmat(_, 2, 3);
    ↪ 1,2,3,
    ↪ 4,0,0
    intmat(_, 2, 1);
    ↪ 1,2
    bigintmat bim[2][3]=34, 64, 345553234, 35553, 6434, 6563335675;
    intmat(bim);
    ↪ 34,64,345553234,
    ↪ 35553,6434,-2026598917

```

See [Section 3.5.5 \[Type conversion and casting\]](#), page 46; [Section 4.8 \[intmat\]](#), page 90; [Section 4.13.3 \[matrix type cast\]](#), page 110.

### 4.8.4 intmat operations

+	addition with <code>intmat</code> or <code>int</code> ; the <code>int</code> is converted into a diagonal <code>intmat</code>
-	negation or subtraction with <code>intmat</code> or <code>int</code> ; the <code>int</code> is converted into a diagonal <code>intmat</code>
*	multiplication with <code>intmat</code> , <code>intvec</code> , or <code>int</code> ; the <code>int</code> is converted into a diagonal <code>intmat</code>
<code>div,/</code>	division of entries in the integers (omitting the remainder)
<code>%, mod</code>	entries modulo <code>int</code> (remainder of the division)
<code>&lt;&gt;, ==</code>	comparators

`intmat_expression [ intvec_expression, intvec_expression ]`  
           is an `intmat` entry, where the first index indicates the row and the second the column

**Example:**

```

    intmat m[2][4] = 1,0,2,4,0,1,-1,0,3,2,1,-2;
    m;
    ↪ 1,0,2,4,
    ↪ 0,1,-1,0

```

```

    m[2,3];          // entry at row 2, col 3
    ↪ -1
    size(m);         // number of entries
    ↪ 8
    intvec v = 1,0,-1,2;
    m * v;
    ↪ 7,1
    typeof(_);
    ↪ intvec
    intmat m1[4][3] = 0,1,2,3,v,1;
    intmat m2 = m * m1;
    m2;              // 2 x 3 intmat
    ↪ -2,5,4,
    ↪ 4,-1,-1
    m2*10;           // multiply each entry of m with 10;
    ↪ -20,50,40,
    ↪ 40,-10,-10
    -m2;
    ↪ 2,-5,-4,
    ↪ -4,1,1
    m2 % 2;
    ↪ 0,1,0,
    ↪ 0,1,1
    m2 div 2;
    ↪ -1,2,2,
    ↪ 2,-1,-1
    m2[2,1];         // entry at row 2, col 1
    ↪ 4
    m1[2..3,2..3];   // submatrix
    ↪ 1 0 2 1
    m2[nrows(m2),ncols(m2)]; // the last entry of intmat m2
    ↪ -1

```

#### 4.8.5 intmat related functions

<b>beti</b>	Betti numbers of a free resolution (see <a href="#">Section 5.1.4 [beti]</a> , page 159)
<b>det</b>	determinant (see <a href="#">Section 5.1.23 [det]</a> , page 172)
<b>ncols</b>	number of cols (see <a href="#">Section 5.1.104 [ncols]</a> , page 231)
<b>nrows</b>	number of rows (see <a href="#">Section 5.1.107 [nrows]</a> , page 232)
<b>random</b>	pseudo random intmat (see <a href="#">Section 5.1.128 [random]</a> , page 249)
<b>size</b>	total number of entries (see <a href="#">Section 5.1.144 [size]</a> , page 264)
<b>transpose</b>	transpose of an intmat (see <a href="#">Section 5.1.159 [transpose]</a> , page 281)
<b>trace</b>	trace of an intmat (see <a href="#">Section 5.1.158 [trace]</a> , page 281)

#### 4.9 intvec

Variables of type `intvec` are lists of integers. For the range of integers see [Section 6.1 \[Limitations\]](#), page 310. They may be used for simulating sets of integers (and other sets if the `intvec` is used as

an index set for other objects). Addition and subtraction of an intvec with an int or an intvec is done element-wise.

### 4.9.1 intvec declarations

**Syntax:**     intvec name = intvec\_expression ;  
                   intvec name = list\_of\_int\_and\_intvec\_expressions ;

**Purpose:**     defines an intvec variable.  
                   An intvec consists of an ordered list of integers.

**Default:**    0

**Example:**

```
intvec iv=1,3,5,7,8;
iv;
↳ 1,3,5,7,8
iv[4];
↳ 7
iv[3..size (iv)];
↳ 5 7 8
```

### 4.9.2 intvec expressions

An intvec expression is:

1. a range: int expression .. int expression
2. a repeated entry: int expression : positive int expression  
 (a:b generates an intvec of length b>0 with identical entries a)
3. a function returning intvec
4. an expression involving intvec operations with int (+, -, \*, /, %)
5. an expression of intvecs involving intvec operations (+, -)
6. an expression involving an intvec operation with intmat (\*)
7. a type cast to intvec

**Example:**

```
intvec v=-1,2;
intvec w=v,v;           // concatenation
w;
↳ -1,2,-1,2
w=2:3;                  // repetition
w;
↳ 2,2,2
int k = 3;
v = 7:k;
v;
↳ 7,7,7
v=-1,2;
w=-2..2,v,1;
w;
↳ -2,-1,0,1,2,-1,2,1
intmat m[3][2] = 0,1,2,-2,3,1;
```



```

    iv < iw;
    ↦ 1
    iv != iw;
    ↦ 1
    iv[2];
    ↦ 3
    iw = 4,1,2;
    iv[iw];
    ↦ 7 1 3

```

#### 4.9.4 intvec related functions

<b>hilb</b>	Hilbert series as intvec (see <a href="#">Section 5.1.56 [hilb]</a> , page 195)
<b>indepSet</b>	sets of independent variables of an ideal (see <a href="#">Section 5.1.61 [indepSet]</a> , page 199)
<b>leadexp</b>	the exponent vector of the leading monomial (see <a href="#">Section 5.1.77 [leadexp]</a> , page 210)
<b>monomial</b>	the power product corresponding to the exponent vector (see <a href="#">Section 5.1.96 [monomial]</a> , page 225)
<b>nrows</b>	number of rows (see <a href="#">Section 5.1.107 [nrows]</a> , page 232)
<b>qhweight</b>	quasihomogeneous weights (see <a href="#">Section 5.1.124 [qhweight]</a> , page 246)
<b>size</b>	length of the intvec (see <a href="#">Section 5.1.144 [size]</a> , page 264)
<b>sortvec</b>	permutation for sorting ideals/modules (see <a href="#">Section 5.1.146 [sortvec]</a> , page 266)
<b>transpose</b>	transpose of an intvec, returns an intmat (see <a href="#">Section 5.1.159 [transpose]</a> , page 281)
<b>weight</b>	weights for the weighted ecart method (see <a href="#">Section 5.1.172 [weight]</a> , page 288)

### 4.10 link

Links are the communication channels of SINGULAR, i.e., something SINGULAR can write to and/or read from. Currently, SINGULAR supports four different link types:

- ASCII links (see [Section 4.10.4 \[ASCII links\]](#), page 98)
- ssi links (see [Section 4.10.5 \[Ssi links\]](#), page 99)
- pipe links (see [Section 4.10.6 \[Pipe links\]](#), page 101)
- DBM links (see [Section 4.10.7 \[DBM links\]](#), page 102)

#### 4.10.1 link declarations

**Syntax:** link name = string-expression ;

**Purpose:** defines a new communication link.

**Default:** none

**Example:**

```

link l=":w example.txt";
int i=22;           // cf. ASCII links for explanation
string s="An int follows:";
write(l,s,i);
l;

```

```

↳ // type : ASCII
↳ // mode : w
↳ // name : example.txt
↳ // open : yes
↳ // read : not ready
↳ // write: ready
  close(1);          //
  read(1);
↳ An int follows:
↳ 22
↳
  close(1);

```

### 4.10.2 link expressions

A link expression is:

1. an identifier of type link
2. a string describing the link

A link is described by a string which consists of two parts: a property string followed by a name string. The property string describes the type of the link (`ASCII`, `ssi` or `DBM`) and the mode of the link (e.g., open for read, write or append). The name string describes the filename of the link, resp. a network connection for ssi links.

For a detailed format description of the link describing string see:

- for ASCII links: [Section 4.10.4 \[ASCII links\], page 98](#)
- ssi links (see [Section 4.10.5 \[Ssi links\], page 99](#))
- pipe links (see [Section 4.10.6 \[Pipe links\], page 101](#))
- for DBM links: [Section 4.10.7 \[DBM links\], page 102](#)

### 4.10.3 link related functions

<code>close</code>	closes a link (see <a href="#">Section 5.1.10 [close], page 163</a> )
<code>dump</code>	generates a dump of all variables and their values (see <a href="#">Section 5.1.27 [dump], page 175</a> )
<code>getdump</code>	reads a dump (see <a href="#">Section 5.1.52 [getdump], page 191</a> )
<code>open</code>	opens a link (see <a href="#">Section 5.1.110 [open], page 234</a> )
<code>read</code>	reads from a link (see <a href="#">Section 5.1.130 [read], page 250</a> )
<code>status</code>	gets the status of a link (see <a href="#">Section 5.1.150 [status], page 270</a> )
<code>write</code>	writes to a link (see <a href="#">Section 5.1.174 [write], page 289</a> )
<code>kill</code>	closes and kills a link (see <a href="#">Section 5.1.71 [kill], page 206</a> )
<code>waitall</code>	wait till all links of a list of links become ready (only ssi:tcp links) (see <a href="#">Section 5.1.169 [waitall], page 286</a> )
<code>waitfirst</code>	wait till at least one link of a list of links become ready (only ssi:tcp links) (see <a href="#">Section 5.1.170 [waitfirst], page 287</a> )

#### 4.10.4 ASCII links

Via ASCII links data that can be converted to a string can be written into files for storage or communication with other programs. The data is written in plain ASCII format. The output format of polynomials is done w.r.t. the value of the global variable `short` (see [Section 5.3.7 \[short\]](#), page 305). Reading from an ASCII link returns a string — conversion into other data is up to the user. This can be done, for example, using the command `execute` (see [Section 5.1.32 \[execute\]](#), page 178).

The ASCII link describing string has to be one of the following:

1. "ASCII: " + filename  
the mode (read or append) is set by the first `read` or `write` command.
2. "ASCII:r " + filename  
opens the file for reading.
3. "ASCII:w " + filename  
opens the file for overwriting.
4. "ASCII:a " + filename  
opens the file for appending.

There are the following default values:

- the type `ASCII` may be omitted since ASCII links are the default links.
- if non of `r`, `w`, or `a` is specified, the mode of the link is set by the first `read` or `write` command on the link. If the first command is `write`, the mode is set to `a` (append mode).
- if the filename is omitted, `read` reads from `stdin` and `write` writes to `stdout`.

Using these default rules, the string `":r temp"` describes a link which is equivalent to the link `"ASCII:r temp"`: an ASCII link to the file `temp` which is opened for reading. The string `"temp"` describes an ASCII link to the file `temp`, where the mode is set by the first `read` or `write` command. See also the example below.

Note that the filename may contain a path. On Microsoft Windows (resp. MS-DOS) platforms, names of a drive can precede the filename, but must be started with a `//` (as in `//c/temp/ex`). An ASCII link can be used either for reading or for writing, but not for both at the same time. A `close` command must be used before a change of I/O direction. Types without a conversion to `string` cannot be written.

##### Example:

```

ring r=32003,(x,y,z),dp;
link l=":w example.txt";    // type is ASCII, mode is overwrite
l;
⇒ // type : ASCII
⇒ // mode : w
⇒ // name : example.txt
⇒ // open : no
⇒ // read : not ready
⇒ // write: not ready
status(l, "open", "yes");    // link is not yet opened
⇒ 0
ideal i=x2,y2,z2;
write (l,1,";"",2,";"", "ideal i=" ,i,";");
status(l, "open", "yes");    // now link is open
⇒ 1

```



```

    status(l, "mode");          // for writing
⇨ w
    close(l);                  // link is closed
    write("example.txt","int j=5;");// data is appended to file
    read("example.txt");        // data is returned as string
⇨ 1
⇨ ;
⇨ 2
⇨ ;
⇨ ideal i=
⇨ x2,y2,z2;
⇨ int j=5;
⇨
    execute(read(l));          // read string is executed
⇨ 1
⇨ 2
⇨ // ** redefining i (ideal i=) ./examples/ASCII_links.sing:14
    close(l);                  // link is closed

```

#### 4.10.5 Ssi links

Ssi (simple singular interface) links give the possibility to store and communicate data between Singular processes: Read and write access is very fast compared to ASCII links. Ssi links can be established using files or using TCP sockets. For ring-dependent data, a ring description is written together with the data. Reading from an Ssi link returns an expression (not a string) which was evaluated after the read operation. If the expression read from an Ssi link is not from the same ring as the current ring, then a **read** changes the current ring.

Currently under development - not everything is implemented.

##### 4.10.5.1 Ssi file links

Ssi file links provide the possibility to store data in a file using the ssi format. For storing large amounts of data, ssi file links should be used instead of ASCII links. Unlike ASCII links, data read from ssi file links is returned as expressions one at a time.

The ssi file link describing string has to be one of the following:

1. "ssi:r " + filename  
opens the file for reading.
2. "ssi:w " + filename  
opens the file for overwriting.
3. "ssi:a " + filename  
opens the file for appending.

Note that the filename may contain a path. An ssi file link can be used either for reading or for writing, but not for both at the same time. A **close** command must be used before a change of I/O direction.

#### Example:

```

    ring r;
    link l="ssi:w example.ssi"; // type=ssi, mode=overwrite
    l;
⇨ // type : ssi

```

```

⇒ // mode : w
⇒ // name : example.ssi
⇒ // open : no
⇒ // read : not open
⇒ // write: not open
  ideal i=x2,y2,z2;
  write (1,1, i, "hello world");// write three expressions
  write(1,4);                    // append one more expression
  close(1);                      // link is closed
  // open the file for reading now
  read(1);                      // only first expression is read
⇒ 1
  kill r;                       // no basering active now
  def i = read(1);              // second expression
  // notice that current ring was set, the name was assigned
  // automatically
  listvar(ring);
⇒ // ssiRing0                   [0]  *ring
⇒ // ZZ                        [0]  cring
⇒ // QQ                        [0]  cring
  def s = read(1);              // third expression
  listvar();
⇒ // s                         [0]  string hello world
⇒ // ssiRing0                 [0]  *ring
⇒ //      i                   [0]  ideal, 3 generator(s)
⇒ // 1                         [0]  link
  close(1);                    // link is closed

```

#### 4.10.5.2 Ssi tcp links

Ssi tcp links give the possibility to exchange data between two processes which may run on the same or on different computers. Ssi tcp links can be opened in four different modes:

**tcp**        SINGULAR acts as a server.

**connect**    SINGULAR acts as a client.

**tcp <host>:<program>**

SINGULAR acts as a client, launching an application as server. This requires `ssh/sshd` to be installed on the computers (and preferably an automatic login via `.ssh/authorized_keys`).

**fork**        SINGULAR acts as a client, forking another SINGULAR as server.

The Ssi tcp link describing string has to be

- tcp mode:

1. "ssi:tcp"

SINGULAR becomes a server and waits at the first free port (>1024) for a connect call.

- connect mode:

2. "ssi:connect " + host:port

SINGULAR becomes a client and connects to a server waiting at the host and port.

- launch mode:

4. "ssi:tcp" + host:application

SINGULAR becomes a client and starts (launches) the application using `ssh` on a (possibly) different host which then acts as a server.

- fork mode:

8. `"ssi:fork"`

SINGULAR becomes a client and forks another SINGULAR on the same host which acts as a server.

To open an ssi tcp link in launch mode, the application to launch must either be given with an absolute pathname, or must be in a directory contained in the search path. The launched application acts as a server, whereas the SINGULAR that actually opened the link acts as a client. The client "listens" at the some free port until the server application does a connect call.

If the ssi tcp link is opened in fork mode a child of the current SINGULAR is forked. All variables and their values are inherited by the child. The child acts as a server whereas the SINGULAR that actually opened the link acts as a client.

To arrange the evaluation of an expression by a server, the expression must be quoted using the command `quote` (see [Section 5.1.126 \[quote\], page 247](#)), so that a local evaluation is prevented. Otherwise, the expression is evaluated first, and the result of the evaluation is written, instead of the expression which is to be evaluated.

If SINGULAR is in server mode, the value of the variable `link_11` is the ssi link connecting to the client and SINGULAR is in an infinite read-eval-write loop until the connection is closed from the client side (by closing its connecting link). Reading and writing is done to the link `link_11`: After an expression is read, it is evaluated and the result of the evaluation is written back. That is, for each expression which was written to the server, there is exactly one expression written back. This might be an "empty" expression, if the evaluation on the server side does not return a value.

Ssi tcp links should explicitly be opened before being used. Ssi tcp links are bidirectional, i.e. can be used for both, writing and reading. Reading from an ssi tcp link blocks until data was written to that link. The `status` command can be used to check whether there is data to read.

#### Example:

```
int i=7;
link l = "ssi:fork";      // fork link declaration
open(l); l;
⇒ // type : ssi
⇒ // mode : fork
⇒ // name :
⇒ // open : yes
⇒ // read : not ready
⇒ // write: ready

write(l,quote(i)); // Child inherited vars and their values
read(l);
⇒ 7
close(l);           // shut down forked child
```

#### 4.10.6 Pipe links

Pipe links provide access to stdin and stdout of any program. Pipe links are bidirectional. **Syntax:**  
`"| : " + string_for_system`

The `string_for_system` will be passed to `system` after connecting the input and output to the corresponding stdout and stdin.

**Example:**

```

    link l="|: date";
    open(l); l;
    ↪ // type : pipe
    ↪ // mode :
    ↪ // name : date
    ↪ // open : yes
    ↪ // read : not ready
    ↪ // write: ready
    read(l);
    ↪ Di 9. Apr 07:47:28 2024
    l;
    ↪ // type : pipe
    ↪ // mode :
    ↪ // name : date
    ↪ // open : yes
    ↪ // read : not ready
    ↪ // write: ready
    close(l);

```

**4.10.7 DBM links**

DBM links provide access to data stored in a data base. Each entry in the data base consists of a (key\_string, value\_string) pair. Such a pair can be inserted with the command `write(link, key_string, value_string)`. By calling `write(link, key_string)`, the entry with key `key_string` is deleted from the data base. The value of an entry is returned by the command `read(link, key_string)`. With only one argument, `read(link)` returns the next key in the data base. Using this feature a data base can be scanned in order to access all entries of the data base.

If a data base with name `name` is opened for writing for the first time, two files (`name.pag` and `name.dir`), which contain the data base, are automatically created.

The DBM link describing string has to be one of the following:

1. "DBM: " + name  
opens the data base for reading (default mode).
2. "DBM:r " + name  
opens the data base for reading.
3. "DBM:rw " + name  
opens the data base for reading and writing.

Note that `name` must be given without the suffix `.pag` or `.dir`. The name may contain an (absolute) path.

**Example:**

```

    link l="DBM:rw example";
    write(l,"1","abc");
    write(l,"3","XYZ");
    write(l,"2","ABC");
    l;
    ↪ // type : DBM
    ↪ // mode : rw
    ↪ // name : example
    ↪ // open : yes

```

```

⇒ // read : ready
⇒ // write: ready
  close(l);
  // read all keys (till empty string):
  read(l);
⇒ 1
  read(l);
⇒ 3
  read(l);
⇒ 2
  read(l);
⇒
  // read data corresponding to key "1"
  read(l,"1");
⇒ abc
  // read all data:
  read(l,read(l));
⇒ abc
  read(l,read(l));
⇒ XYZ
  read(l,read(l));
⇒ ABC
  // close
  close(l);

```

## 4.11 list

Lists are arrays whose elements can be of different types (including ring). If one element belongs to a ring the whole list belongs to that ring. This applies also to the special list `#`. The expression `list()` is the empty list.

Note that a list stores the objects itself and not the names. Hence, if `L` is a list, `L[1]` for example has no name. A name, say `R`, can be created for `L[1]` by `def R=L[1];`. To store also the name of an object, say `r`, it can be added to the list with `nameof(r);`. Rings may be objects of a list.

**Note:** Unlike other assignments a ring as an element of a list is not a copy but another reference to the same ring.

### 4.11.1 list declarations

**Syntax:**    `list name = expression_list;`  
              `list name = list_expression;`

**Purpose:**    defines a list (of objects of possibly different types).

**Default:**    empty list

**Example:**

```

      list l=1,"str";
      l[1];
⇒ 1
      l[2];
⇒ str
      ring r;
      listvar(r);

```

```

⇒ // r
    ideal i = x^2, y^2 + z^3;
    l[3] = i;
    l;
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ str
⇒ [3]:
⇒ _[1]=x2
⇒ _[2]=z3+y2
    listvar(r);    // the list l belongs now to the ring r
⇒ // r
⇒ // l
⇒ // i

```

[0] \*ring  
 [0] list, size: 3  
 [0] ideal, 2 generator(s)

### 4.11.2 list expressions

A list expression is:

1. the empty list `list()`
2. an identifier of type list
3. a function returning list
4. list expressions combined by the arithmetic operation `+`
5. a type cast to list

See [Section 3.5.5 \[Type conversion and casting\]](#), page 46.

#### Example:

```

    list l = "hello",1;
    l;
⇒ [1]:
⇒ hello
⇒ [2]:
⇒ 1
    l = list();
    l;
⇒ empty list
    ring r =0,x,dp;
    factorize((x+1)^2);
⇒ [1]:
⇒ _[1]=1
⇒ _[2]=x+1
⇒ [2]:
⇒ 1,2
    list(1,2,3);
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 2
⇒ [3]:
⇒ 3

```

### 4.11.3 list operations

**+** concatenation

**delete** deletes one element from list, returns new list

**insert** inserts or appends a new element to list, returns a new list

**list\_expression [ int\_expression ]**  
is a list entry; the index 1 gives the first element.

**Example:**

```
list l1 = 1,"hello",list(-1,1);
list l2 = list(1,5,7);
l1 + l2;           // a new list
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ hello
⇒ [3]:
⇒ [1]:
⇒ -1
⇒ [2]:
⇒ 1
⇒ [4]:
⇒ 1
⇒ [5]:
⇒ 5
⇒ [6]:
⇒ 7
l2 = delete(l2, 2); // delete 2nd entry
l2;
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 7
```

### 4.11.4 list related functions

**bareiss** returns a list of a matrix (lower triangular) and of an intvec (permutations of columns, see [Section 5.1.3 \[bareiss\]](#), page 158)

**betti** Betti numbers of a resolution (see [Section 5.1.4 \[betti\]](#), page 159)

**delete** deletion of an element from a list (see [Section 5.1.21 \[delete\]](#), page 171)

**facstd** factorizing Groebner basis algorithm (see [Section 5.1.34 \[facstd\]](#), page 179)

**factorize**  
list of factors of a polynomial (see [Section 5.1.36 \[factorize\]](#), page 180)

**insert** insertion of a new element into a list (see [Section 5.1.62 \[insert\]](#), page 200)

**minres** minimization of a free resolution (see [Section 5.1.93 \[minres\]](#), page 223)

**names** list of all user-defined variable names (see [Section 5.1.103 \[names\]](#), page 229)

**size**            number of entries (see [Section 5.1.144 \[size\]](#), page 264)

**conversion from resolution**  
(see [Section 4.19 \[resolution\]](#), page 125)

## 4.12 map

Maps are ring maps from a preimage ring into the basering.

**Note:**

- The target of a map is **ALWAYS** the actual basering
- The preimage ring has to be stored "by its name", that means, maps can only be used in such contexts, where the name of the preimage ring can be resolved (this has to be considered in subprocedures). See also [Section 6.5 \[Identifier resolution\]](#), page 316, [Section 3.7.4 \[Names in procedures\]](#), page 54.

Maps between rings with different coefficient fields are possible and listed below.

Canonically realized are

- $Q \rightarrow Q(a, \dots)$  (  $Q$  : the rational numbers)
- $Q \rightarrow R$  (  $R$  : the real numbers)
- $Q \rightarrow C$  (  $C$  : the complex numbers)
- $Z/p \rightarrow (Z/p)(a, \dots)$  (  $Z$  : the integers)
- $Z/p \rightarrow GF(p^n)$  (  $GF$  : the Galois field)
- $Z/p \rightarrow R$
- $R \rightarrow C$

Possible are furthermore

- $Z/p \rightarrow Q$ ,  $[i]_p \mapsto i \in [-p/2, p/2] \subseteq Z$
- $Z/p \rightarrow Z/p'$ ,  $[i]_p \mapsto i \in [-p/2, p/2] \subseteq Z$ ,  $i \mapsto [i]_{p'} \in Z/p'$
- $C \rightarrow R$ , by taking the real part

Finally, in SINGULAR we allow the mapping from rings with coefficient field  $Q$  to rings whose ground fields have finite characteristic:

- $Q \rightarrow Z/p$
- $Q \rightarrow (Z/p)(a, \dots)$

In these cases the denominator and the numerator of a number are mapped separately by the usual map from  $Z$  to  $Z/p$ , and the image of the number is built again afterwards by division. It is thus not allowed to map numbers whose denominator is divisible by the characteristic of the target ground field, or objects containing such numbers. We, therefore, strongly recommend using such maps only to map objects with integer coefficients.

### 4.12.1 map declarations

**Syntax:**    `map name = preimage_ring_name , ideal_expression ;`  
               `map name = preimage_ring_name , list_of_poly_and_ideal_expressions ;`  
               `map name = map_expression ;`

**Purpose:**    defines a ring map from `preimage_ring` to basering.  
               Maps the variables of the preimage ring to the generators of the ideal. If the ideal contains less elements than variables in the `preimage_ring` the remaining variables are



mapped to 0, if the ideal contains more elements these are ignored. The image ring is always the current basering. For the mapping of coefficients from different fields see [Section 4.12 \[map\], page 106](#).

**Default:** none

**Note:** There are standard mappings for maps which are close to the identity map: `fetch` and `imap`.

The name of a map serves as the function which maps objects from the `preimage_ring` into the basering. These objects must be defined by names (no evaluation in the preimage ring is possible).

**Example:**

```
ring r1=32003,(x,y,z),dp;
ideal i=x,y,z;
ring r2=32003,(a,b),dp;
map f=r1,a,b,a+b;
// maps from r1 to r2,
// x -> a
// y -> b
// z -> a+b
f(i);
↳ _[1]=a
↳ _[2]=b
↳ _[3]=a+b
// operations like f(i[1]) or f(i*i) are not allowed
ideal i=f(i);
// objects in different rings may have the same name
map g = r2,a2,b2;
map phi = g(f);
// composition of map f and g
// maps from r1 to r2,
// x -> a2
// y -> b2
// z -> a2+b2
phi(i);
↳ _[1]=a2
↳ _[2]=b2
↳ _[3]=a2+b2
```

See [Section 5.1.38 \[fetch\], page 182](#); [Section 4.6.2 \[ideal expressions\], page 81](#); [Section 5.1.59 \[imap\], page 198](#); [Section 4.12 \[map\], page 106](#); [Section 4.20 \[ring\], page 127](#).

### 4.12.2 map expressions

A map expression is:

1. an identifier of type map
2. a function returning map
3. map expressions combined by composition using parentheses `( , )`

### 4.12.3 map operations

- ( )      composition of maps. If, for example,  $f$  and  $g$  are maps, then  $f(g)$  is a map expression giving the composition  $f \circ g$  of  $f$  and  $g$ , provided the target ring of  $g$  is the basering of  $f$ .

map\_expression [ int\_expressions ]  
                   is a map entry (the image of the corresponding variable)

**Example:**

```

ring r=0,(x,y),dp;
map f=r,y,x;      // the map f permutes the variables
f;
↪ f[1]=y
↪ f[2]=x
poly p=x+2y3;
f(p);
↪ 2x3+y
map g=f(f);      // the map g defined as f^2 is the identity
g;
↪ g[1]=x
↪ g[2]=y
g(p) == p;
↪ 1

```

#### 4.12.4 map related functions

- fetch**      the identity map between rings (see [Section 5.1.38 \[fetch\]](#), page 182)
- imap**      a convenient map procedure for inclusions and projections of rings (see [Section 5.1.59 \[imap\]](#), page 198)
- preimage**   preimage under a ring map (see [Section 5.1.117 \[preimage\]](#), page 240)
- subst**      substitution of a ring variable (see [Section 5.1.154 \[subst\]](#), page 274)

See also the libraries [Section D.4.2 \[algebra.lib\]](#), page 1004 and [Section D.2.12 \[ring\\_lib\]](#), page 943, which contain more functions, related to maps.

### 4.13 matrix

Objects of type matrix are matrices with polynomial entries. Like polynomials they can only be defined or accessed with respect to a basering. In order to compute with matrices having integer or rational entries, define a ring with characteristic 0 and at least one variable.

A matrix can be multiplied by and added to a poly; in this case the polynomial is converted into a matrix of the right size with the polynomial on the diagonal.

If  $A$  is a matrix then the assignment `module M=A;` or `module M=module(A);` creates a module generated by the columns of  $A$ . Note that the trailing zero columns of  $A$  may be deleted by module operations with  $M$ .

#### 4.13.1 matrix declarations

**Syntax:**    `matrix name[rows][cols] = list_of_poly_expressions ;`  
                   `matrix name = matrix_expression ;`

**Purpose:** defines a matrix (of polynomials).

The given `poly_list` fills up the matrix beginning with the first row from the left to the right, then the second row and so on. If the `poly_list` contains less than `rows*cols` elements, the matrix is filled up with zeros; if it contains more elements, then only the first `rows*cols` elements are used. If the right-hand side is a matrix expression the matrix on the left-hand side gets the same size as the right-hand side, otherwise the size is determined by the left-hand side. If the size is omitted a 1x1 matrix is created.

**Default:** 0 (1 x 1 matrix)

**Example:**

```

int ro = 3;
ring r = 32003, (x,y,z), dp;
poly f=xyz;
poly g=z*f;
ideal i=f,g,g^2;
matrix m[ro][3] = x3y4, 0, i, f ; // a 3 x 3 matrix
m;
↪ m[1,1]=x3y4
↪ m[1,2]=0
↪ m[1,3]=xyz
↪ m[2,1]=xyz2
↪ m[2,2]=x2y2z4
↪ m[2,3]=xyz
↪ m[3,1]=0
↪ m[3,2]=0
↪ m[3,3]=0
print(m);
↪ x3y4,0,      xyz,
↪ xyz2,x2y2z4,xyz,
↪ 0,    0,      0
matrix A; // the 1 x 1 zero matrix
matrix B[2][2] = m[1..2, 2..3]; //defines a submatrix
print(B);
↪ 0,      xyz,
↪ x2y2z4,xyz
matrix C=m; // defines C as a 3 x 3 matrix equal to m
print(C);
↪ x3y4,0,      xyz,
↪ xyz2,x2y2z4,xyz,
↪ 0,    0,      0

```

### 4.13.2 matrix expressions

A matrix expression is:

1. an identifier of type matrix
2. a function returning matrix
3. matrix expressions combined by the arithmetic operations `+`, `-` or `*`
4. a type cast to matrix (see [Section 4.13.3 \[matrix type cast\]](#), page 110)

**Example:**

```

ring r=0,(x,y),dp;
poly f= x3y2 + 2x2y2 +2;
matrix H = jacob(jacob(f));    // the Hessian of f
matrix mc = coef(f,y);
print(mc);
↳ y2,      1,
↳ x3+2x2, 2
module MD = [x+y,1,x],[x+y,0,y];
matrix M = MD;
print(M);
↳ x+y,x+y,
↳ 1,   0,
↳ x,   y

```

### 4.13.3 matrix type cast

**Syntax:**    `matrix ( expression )`  
               `matrix ( expression, int_n, int_m )`

**Type:**        `matrix`

**Purpose:**    Converts expression to a matrix, where expression must be of type `int`, `intmat`, `intvec`, `number`, `poly`, `ideal`, `vector`, `module`, or `matrix`. If `int_n` and `int_m` are supplied, then they specify the dimension of the matrix. Otherwise, the size (resp. dimensions) of the matrix is determined by the size (resp. dimensions) of the expression.

**Example:**

```

ring r=32003,(x,y,z),dp;
matrix(x);
↳ _[1,1]=x
matrix(x, 1, 2);
↳ _[1,1]=x
↳ _[1,2]=0
matrix(intmat(intvec(1,2,3,4), 2, 2));
↳ _[1,1]=1
↳ _[1,2]=2
↳ _[2,1]=3
↳ _[2,2]=4
matrix(_, 2, 3);
↳ _[1,1]=1
↳ _[1,2]=2
↳ _[1,3]=0
↳ _[2,1]=3
↳ _[2,2]=4
↳ _[2,3]=0
matrix(_, 2, 1);
↳ _[1,1]=1
↳ _[2,1]=3

```

See [Section 3.5.5 \[Type conversion and casting\]](#), page 46; [Section 4.8.3 \[intmat type cast\]](#), page 92; [Section 4.13 \[matrix\]](#), page 108.

### 4.13.4 matrix operations

+            addition with matrix or poly; the polynomial is converted into a diagonal matrix  
 -            negation or subtraction with matrix or poly (the first operand is expected to be a matrix); the polynomial is converted into a diagonal matrix  
 \*            multiplication with matrix or poly; the polynomial is converted into a diagonal matrix  
 /            division by poly  
 ==, <>, !=   comparators  
 matrix\_expression [ int\_expression, int\_expression ]  
               is a matrix entry, where the first index indicates the row and the second the column

**Example:**

```

ring r=32003,x,dp;
matrix A[3][3] = 1,3,2,5,0,3,2,4,5; // define a matrix
print(A); // nice printing of small matrices
⇒ 1,3,2,
⇒ 5,0,3,
⇒ 2,4,5
  A[2,3]; // matrix entry
⇒ 3
  A[2,3] = A[2,3] + 1; // change entry
  A[2,1..3] = 1,2,3; // change 2nd row
  print(A);
⇒ 1,3,2,
⇒ 1,2,3,
⇒ 2,4,5
  matrix E[3][3]; E = E + 1; // the unit matrix
  matrix B =x*E - A;
  print(B);
⇒ x-1,-3, -2,
⇒ -1, x-2,-3,
⇒ -2, -4, x-5
  // the same (but x-A does not work):
  B = -A+x;
  print(B);
⇒ x-1,-3, -2,
⇒ -1, x-2,-3,
⇒ -2, -4, x-5
  det(B); // the characteristic polynomial of A
⇒ x3-8x2-2x-1
  A*A*A - 8 * A*A - 2*A == E; // Cayley-Hamilton
⇒ 1
  vector v =[x,-1,x2];
  A*v; // multiplication of matrix and vector
⇒ _[1,1]=2x2+x-3
⇒ _[2,1]=3x2+x-2
⇒ _[3,1]=5x2+2x-4
  matrix m[2][2]=1,2,3;
  print(m-transpose(m));
⇒ 0,-1,
⇒ 1,0

```

### 4.13.5 matrix related functions

<code>bareiss</code>	Gauss-Bareiss algorithm (see <a href="#">Section 5.1.3 [bareiss]</a> , page 158)
<code>coef</code>	matrix of coefficients and monomials (see <a href="#">Section 5.1.11 [coef]</a> , page 164)
<code>coeffs</code>	matrix of coefficients (see <a href="#">Section 5.1.12 [coeffs]</a> , page 165)
<code>det</code>	determinant (see <a href="#">Section 5.1.23 [det]</a> , page 172)
<code>diff</code>	partial derivative (see <a href="#">Section 5.1.24 [diff]</a> , page 173)
<code>jacob</code>	Jacobi matrix (see <a href="#">Section 5.1.66 [jacob]</a> , page 203)
<code>koszul</code>	Koszul matrix (see <a href="#">Section 5.1.73 [koszul]</a> , page 207)
<code>lift</code>	lift-matrix (see <a href="#">Section 5.1.80 [lift]</a> , page 211)
<code>liftstd</code>	standard basis and transformation matrix computation (see <a href="#">Section 5.1.81 [liftstd]</a> , page 212)
<code>minor</code>	set of minors of a matrix (see <a href="#">Section 5.1.92 [minor]</a> , page 221)
<code>ncols</code>	number of columns (see <a href="#">Section 5.1.104 [ncols]</a> , page 231)
<code>nrows</code>	number of rows (see <a href="#">Section 5.1.107 [nrows]</a> , page 232)
<code>print</code>	nice print format (see <a href="#">Section 5.1.120 [print]</a> , page 242)
<code>size</code>	number of matrix entries (see <a href="#">Section 5.1.144 [size]</a> , page 264)
<code>subst</code>	substitute a ring variable (see <a href="#">Section 5.1.154 [subst]</a> , page 274)
<code>trace</code>	trace of a matrix (see <a href="#">Section 5.1.158 [trace]</a> , page 281)
<code>transpose</code>	transposed matrix (see <a href="#">Section 5.1.159 [transpose]</a> , page 281)
<code>wedge</code>	wedge product (see <a href="#">Section 5.1.171 [wedge]</a> , page 287)

See also the library [Section D.3.1 \[matrix\\_lib\]](#), page 968, which contains more matrix-related functions.

## 4.14 module

Modules are submodules of a free module over the basering with basis `gen(1)`, `gen(2)`,  $\dots$ . They are represented by lists of vectors which generate the submodule. Like vectors they can only be defined or accessed with respect to a basering.

If  $R$  is the basering, and  $M$  is a submodule of  $R^n$

generated by vectors  $v_1, \dots, v_k$ , then  $v_1, \dots, v_k$

may be considered as the generators of relations of  $R^n/M$  between the canonical generators `gen(1), ..., gen(n)`. Hence any finitely generated  $R$ -module can be represented in SINGULAR by its module of relations. The assignments `module M=v1,...,vk; matrix A=M;` create the presentation matrix of size  $n \times k$  for  $R^n/M$ , i.e., the columns of  $A$  are the vectors  $v_1, \dots, v_k$  which generate  $M$  (cf. [Section B.1 \[Representation of mathematical objects\]](#), page 768).

### 4.14.1 module declarations

**Syntax:**    `module name = list_of_vector_expressions ;`  
               `module name = module_expression ;`

**Purpose:**    defines a module.

**Default:**    `[0]`

**Example:**

```
ring r=0,(x,y,z),(c,dp);
vector s1 = [x2,y3,z];
vector s2 = [xy,1,0];
vector s3 = [0,x2-y2,z];
poly f = xyz;
module m = s1, s2-s1,f*(s3-s1);
m;
↳ m[1]=[x2,y3,z]
↳ m[2]=[-x2+xy,-y3+1,-z]
↳ m[3]=[-x3yz,-xy4z+x3yz-xy3z]
// show m in matrix format (columns generate m)
print(m);
↳ x2,-x2+xy,-x3yz,
↳ y3,-y3+1, -xy4z+x3yz-xy3z,
↳ z, -z, 0
```

### 4.14.2 module expressions

A module expression is:

1. an identifier of type module
2. a function returning module
3. module expressions combined by the arithmetic operation `+`
4. multiplication of a module expression with an ideal or a poly expression: `*`
5. a type cast to module

See [Section 3.5.5 \[Type conversion and casting\]](#), page 46; [Section 4.6 \[ideal\]](#), page 80; [Section 4.17 \[poly\]](#), page 120; [Section 4.23 \[vector\]](#), page 134.

### 4.14.3 module operations

- `+`            addition (concatenation of the generators and simplification)
- `*`            multiplication with ideal or poly (but not ‘module’ \* ‘module’!)

`module_expression [ int_expression , int_expression ]`

is a module entry, where the first index indicates the row and the second the column

`module_expressions [ int_expression ]`

is a vector, where the index indicates the column (generator)

**Example:**

```
ring r=0,(x,y,z),dp;
module m=[x,y],[0,0,z];
```

```

    print(m*(x+y));
    ↦ x2+xy,0,
    ↦ xy+y2,0,
    ↦ 0,    xz+yz
    // this is not distributive:
    print(m*x+m*y);
    ↦ x2,0, xy,0,
    ↦ xy,0, y2,0,
    ↦ 0, xz,0, yz

```

#### 4.14.4 module related functions

<code>coeffs</code>	matrix of coefficients (see <a href="#">Section 5.1.12 [coeffs]</a> , page 165)
<code>degree</code>	multiplicity, dimension and codimension of the module of leading terms (see <a href="#">Section 5.1.20 [degree]</a> , page 171)
<code>diff</code>	partial derivative (see <a href="#">Section 5.1.24 [diff]</a> , page 173)
<code>dim</code>	Krull dimension of free module over the basering modulo the module of leading terms (see <a href="#">Section 5.1.25 [dim]</a> , page 174)
<code>eliminate</code>	elimination of variables (see <a href="#">Section 5.1.28 [eliminate]</a> , page 176)
<code>freemodule</code>	the free module of given rank (see <a href="#">Section 5.1.47 [freemodule]</a> , page 188)
<code>fres</code>	free resolution of a standard basis (see <a href="#">Section 5.1.48 [fres]</a> , page 188)
<code>groebner</code>	Groebner basis computation (a wrapper around <code>std</code> , <code>stdhilb</code> , <code>stdfglm</code> ,...) (see <a href="#">Section 5.1.53 [groebner]</a> , page 191)
<code>hilb</code>	Hilbert function of a standard basis (see <a href="#">Section 5.1.56 [hilb]</a> , page 195)
<code>homog</code>	homogenization with respect to a variable (see <a href="#">Section 5.1.57 [homog]</a> , page 196)
<code>interred</code>	interreduction of a module (see <a href="#">Section 5.1.64 [interred]</a> , page 201)
<code>intersect</code>	module intersection (see <a href="#">Section 5.1.65 [intersect]</a> , page 202)
<code>jet</code>	Taylor series up to a given order (see <a href="#">Section 5.1.68 [jet]</a> , page 204)
<code>kbase</code>	vector space basis of free module over the basering modulo the module of leading terms (see <a href="#">Section 5.1.69 [kbase]</a> , page 205)
<code>lead</code>	initial module (see <a href="#">Section 5.1.75 [lead]</a> , page 209)
<code>lift</code>	lift-matrix (see <a href="#">Section 5.1.80 [lift]</a> , page 211)
<code>liftstd</code>	standard basis and transformation matrix computation (see <a href="#">Section 5.1.81 [liftstd]</a> , page 212)
<code>lres</code>	free resolution (see <a href="#">Section 5.1.83 [lres]</a> , page 215)
<code>minbase</code>	minimal generating set of a homogeneous ideal, resp. module, or an ideal, resp. module, over a local ring
<code>modulo</code>	represents $(h1 + h2)/h1 = h2/(h1 \cap h2)$ (see <a href="#">Section 5.1.94 [modulo]</a> , page 223)
<code>mres</code>	minimal free resolution of an ideal resp. module w.r.t. a minimal set of generators of the given module (see <a href="#">Section 5.1.98 [mres]</a> , page 225)



<code>mult</code>	multiplicity, resp. degree, of the module of leading terms (see <a href="#">Section 5.1.101 [mult]</a> , page 228)
<code>nres</code>	computation of a free resolution of an ideal resp. module M which is minimized from the second free module on (see <a href="#">Section 5.1.106 [nres]</a> , page 232)
<code>ncols</code>	number of columns (see <a href="#">Section 5.1.104 [ncols]</a> , page 231)
<code>nrows</code>	number of rows (see <a href="#">Section 5.1.107 [nrows]</a> , page 232)
<code>print</code>	nice print format (see <a href="#">Section 5.1.120 [print]</a> , page 242)
<code>prune</code>	minimization of the embedding into a free module (see <a href="#">Section 5.1.122 [prune]</a> , page 245)
<code>qhweight</code>	quasihomogeneous weights of an ideal, resp. module (see <a href="#">Section 5.1.124 [qhweight]</a> , page 246)
<code>quotient</code>	module quotient (see <a href="#">Section 5.1.127 [quotient]</a> , page 248)
<code>reduce</code>	normalform with respect to a standard basis (see <a href="#">Section 5.1.131 [reduce]</a> , page 251)
<code>res</code>	free resolution of an ideal, resp. module, but not changing the given ideal, resp. module (see <a href="#">Section 5.1.134 [res]</a> , page 253)
<code>simplify</code>	simplification of a set of vectors (see <a href="#">Section 5.1.143 [simplify]</a> , page 263)
<code>size</code>	number of non-zero generators (see <a href="#">Section 5.1.144 [size]</a> , page 264)
<code>sortvec</code>	permutation for sorting ideals/modules (see <a href="#">Section 5.1.146 [sortvec]</a> , page 266)
<code>sres</code>	free resolution of a standard basis (see <a href="#">Section 5.1.149 [sres]</a> , page 269)
<code>std</code>	standard basis computation (see <a href="#">Section 5.1.151 [std]</a> , page 271, <a href="#">Section 5.1.81 [liftstd]</a> , page 212)
<code>subst</code>	substitution of a ring variable (see <a href="#">Section 5.1.154 [subst]</a> , page 274)
<code>syz</code>	computation of the first syzygy module (see <a href="#">Section 5.1.156 [syz]</a> , page 280)
<code>vdim</code>	vector space dimension of free module over the basering modulo module of leading terms (see <a href="#">Section 5.1.168 [vdim]</a> , page 286)
<code>weight</code>	"optimal" weights (see <a href="#">Section 5.1.172 [weight]</a> , page 288)

## 4.15 number

Numbers are elements from the coefficient ring (or ground ring). They can only be defined or accessed with respect to a basering which determines the coefficient field. See [Section 4.20.2 \[ring declarations\]](#), page 127 for declarations of coefficient fields.

**Warning:** Beware of the special meaning of the letter `e` (immediately following a sequence of digits) if the field is real (or complex), [Section 6.4 \[Miscellaneous oddities\]](#), page 314.

### 4.15.1 number declarations

**Syntax:** `number name = number_expression ;`

**Purpose:** defines a number.

**Default:** 0





```

poly g = (3t2 +1)*x2 +1;
leadcoef(g);
↪ (3t2+1)
typeof(_);
↪ number
par(1);
↪ (t)
typeof(_);
↪ number

```

See [Section 3.5.5 \[Type conversion and casting\]](#), page 46; [Section 4.20 \[ring\]](#), page 127.

### 4.15.3 number operations

+	addition
-	negation or subtraction
*	multiplication
/	division
%, mod	modulo
^, **	power, exponentiation (by an integer)
<=, >=, ==, <>	comparison

mod integer modulo (the remainder of the division `div`), always non-negative

**Note:** Quotient and exponentiation is only recognized as a number expression if it is already a number, see [Section 6.4 \[Miscellaneous oddities\]](#), page 314.

For the behavior of comparison operators in rings with ground field different from real or the rational numbers, see [Section 4.7.5 \[boolean expressions\]](#), page 89.

#### Example:

```

ring r=0,x,dp;
number n = 1/2 +1/3;
n;
↪ 5/6
n/2;
↪ 5/12
1/2/3;
↪ 1/6
1/2 * 1/3;
↪ 1/6
n = 2;
n^-2;
↪ 1/4
// the following oddities appear here
2/(2+3);
↪ // ** int division with '/': use 'div' instead in line >> 2/(2+3);<<
↪ 0
number(2)/(2+3);
↪ 2/5
2^-2; // for int's exponent must be non-negative

```

```

↳      ? exponent must be non-negative
↳      ? error occurred in or before ./examples/number_operations.sing line 1\
2: '  2^-2; // for int's exponent must be non-negative'
  number(2)^-2;
↳ 1/4
  3/4>=2/5;
↳ 1
  2/6==1/3;
↳ 1

```

#### 4.15.4 number related functions

**cleardenom**

cancellation of denominators of numbers in polyomial and divide it by its content (see [Section 5.1.9 \[cleardenom\]](#), page 163)

**impart** imaginary part of a complex number, 0 otherwise (see [Section 5.1.60 \[impart\]](#), page 199, [Section 5.1.133 \[repart\]](#), page 253)

**numerator, denominator**

the numerator/denominator of a rational number (see [Section 5.1.108 \[numerator\]](#), page 233, [Section 5.1.22 \[denominator\]](#), page 172)

**leadcoef** coefficient of the leading term (see [Section 5.1.76 \[leadcoef\]](#), page 209)

**par** n-th parameter of the basering (see [Section 5.1.114 \[par\]](#), page 239)

**pardeg** degree of a number in ring parameters (see [Section 5.1.115 \[pardeg\]](#), page 239)

**parstr** string form of ring parameters (see [Section 5.1.116 \[parstr\]](#), page 240)

**repart** real part of a complex number (see [Section 5.1.60 \[impart\]](#), page 199, [Section 5.1.133 \[repart\]](#), page 253)

### 4.16 package

The data type package is used to group identifiers into collections. It is mainly used as an internal means to avoid collisions of names of identifiers in libraries with variable names defined by the user. The most important package is the toplevel package, called **Top**. It contains all user defined identifiers as well as all user accessible library procedures. Identifiers which are local to a library are contained in a package whose name is obtained from the name of the library, where the first letter is converted to uppercase, the remaining ones to lowercase. Another reserved package name is **Current** which denotes the current package name in use. See also [Section 3.8 \[Libraries\]](#), page 55.

#### 4.16.1 package declarations

**Syntax:** package name ;

**Purpose:** defines a package (Only relevant in very special situations).

**Example:**

```

package Test;
int i=3; exportto(Test,i);
Test::i+2;
↳ 5
i;

```

```

↳      ? 'i' is undefined
↳      ? error occurred in or before ./examples/package_declarations.sing 1.
      e 4: ' i;'
      listvar();
      listvar(Test);
↳ // Test                                [0] package Test (N)
↳ // ::i                                [0] int 3
      package dummy = Test;
      kill Test;
      listvar(dummy);
↳ // dummy                                [0] package dummy (N)
↳ // ::i                                [0] int 3

```

### 4.16.2 package related functions

**exportto** transfer an identifier to the specified package (see [Section 5.2.7 \[exportto\]](#), page 293)

**importfrom**

generate a copy of an identifier from the specified package in the current package (see [Section 5.2.10 \[importfrom\]](#), page 296)

**listvar** list variables currently defined in a given package (see [Section 5.1.82 \[listvar\]](#), page 213)

**load** load a library or dynamic module (see [Section 5.2.12 \[load\]](#), page 299)

**LIB** load a library or dynamic module (see [Section 5.1.79 \[LIB\]](#), page 211)

## 4.17 poly

Polynomials are the basic data for all main algorithms in SINGULAR. They consist of finitely many terms (coefficient\*monomial) which are combined by the usual polynomial operations (see [Section 4.17.2 \[poly expressions\]](#), page 121). Polynomials can only be defined or accessed with respect to a basering which determines the coefficient type, the names of the indeterminates and the monomial ordering.

```

ring r=32003,(x,y,z),dp;
poly f=x3+y5+z2;

```

### 4.17.1 poly declarations

**Syntax:** poly name = poly\_expression ;

**Purpose:** defines a polynomial.

**Default:** 0

**Example:**

```

ring r = 32003,(x,y,z),dp;
poly s1 = x3y2+151x5y+186xy6+169y9;
poly s2 = 1*x^2*y^2*z^2+3z8;
poly s3 = 5/4x4y2+4/5*x*y^5+2x2y2z3+y7+11x10;
int a,b,c,t=37,5,4,1;
poly f=3*x^a+x*y^(b+c)+t*x^a*y^b*z^c;
f;
↳ x37y5z4+3x37+xy9
short = 0;

```

```
f;
↦ x^37*y^5*z^4+3*x^37+x*y^9
```

[Section 5.3.7 \[short\], page 305](#)

### 4.17.2 poly expressions

A polynomial expression is (optional parts in square brackets):

1. a monomial (there are NO spaces allowed inside a monomial)

[coefficient] ring\_variable [ exponent] [ring\_variable [exponent] ...].

Monomials which contain an indexed ring variable must be built from `ring_variable` and `coefficient` with the operations `*` and `^`

2. an identifier of type `poly`
3. a function returning `poly`
4. polynomial expressions combined by the arithmetic operations `+`, `-`, `*`, `/`, or `^`
5. an `int` expression (see [Section 3.5.5 \[Type conversion and casting\], page 46](#))
6. a type cast to `poly`

**Example:**

```
ring S=0,(x,y,z,a(1)),dp;
2x, x3, 2x2y3, xyz, 2xy2; // are monomials
2*x, x^3, 2*x^2*y^3, x*y*z, 2*x*y^2; // are poly expressions
2*a(1); // is a valid polynomial expression (a(1) is a name of a variable),
        // but not 2a(1) (is a syntax error)
2*x^3; // is a valid polynomial expression equal to 2x3 (a valid monomial)
        // but not equal to 2x^3 which will be interpreted as (2x)^3
        // since 2x is a monomial
ring r=0,(x,y),dp;
poly f = 10x2y3 +2x2y2-2xy+y -x+2;
lead(f);
↦ 10x2y3
leadmonom(f);
↦ x2y3
simplify(f,1); // normalize leading coefficient
↦ x2y3+1/5x2y2-1/5xy-1/10x+1/10y+1/5
poly g = 1/2x2 + 1/3y;
cleardenom(g);
↦ 3x2+2y
int i = 102;
poly(i);
↦ 102
typeof(_);
↦ poly
```

See [Section 3.5.5 \[Type conversion and casting\], page 46](#); [Section 4.20 \[ring\], page 127](#).

### 4.17.3 poly operations

- |   |                         |
|---|-------------------------|
| + | addition                |
| - | negation or subtraction |

<code>*</code>	multiplication
<code>/, div</code>	division by a polynomial, ignoring the remainder (only implemented for polynomials over $\mathbb{Q}\mathbb{Q}$ , $\mathbb{Z}\mathbb{Z}/p$ and field extensions of them) (See also <a href="#">Section 5.1.127 [quotient]</a> , page 248, <a href="#">Section 5.1.26 [division]</a> , page 174, <a href="#">Section 5.1.131 [reduce]</a> , page 251)
<code>%, mod</code>	the remainder from the division by a polynomial (only implemented for polynomials over $\mathbb{Q}\mathbb{Q}$ , $\mathbb{Z}\mathbb{Z}/p$ and field extensions of them) (See also <a href="#">Section 5.1.127 [quotient]</a> , page 248, <a href="#">Section 5.1.26 [division]</a> , page 174, <a href="#">Section 5.1.131 [reduce]</a> , page 251)
<code>^, **</code>	power by a positive integer
<code>&lt;, &lt;=, &gt;, &gt;=, ==, &lt;&gt;</code>	comparators (considering leading monomials w.r.t. monomial ordering)
<code>poly_expression [ intvec_expression ]</code>	the sum of monomials at the indicated places w.r.t. the monomial ordering

**Example:**

```

ring R=0,(x,y),dp;
poly f = x3y2 + 2x2y2 + xy - x + y + 1;
f;
↪ x3y2+2x2y2+xy-x+y+1
f + x5 + 2;
↪ x5+x3y2+2x2y2+xy-x+y+3
f * x2;
↪ x5y2+2x4y2+x3y-x3+x2y+x2
(x+y)/x;
↪ 1
f/3x2;
↪ 1/3xy2+2/3y2
x5 > f;
↪ 1
x<=y;
↪ 0
x>y;
↪ 1
ring r=0,(x,y),ds;
poly f = fetch(R,f);
f;
↪ 1-x+y+xy+2x2y2+x3y2
x5 > f;
↪ 0
f[2..4];
↪ -x+y+xy
size(f);
↪ 6
f[size(f)+1]; f[-1]; // monomials out of range are 0
↪ 0
↪ 0
intvec v = 6,1,3;
f[v]; // the polynom built from the 1st, 3rd and 6th monomial of f
↪ 1+y+x3y2

```



### 4.17.4 poly related functions

<code>cleardenom</code>	cancellation of denominators of numbers in polynomial and divide it by its content (see <a href="#">Section 5.1.9 [cleardenom]</a> , page 163; <a href="#">Section D.2.8.14 [content]</a> , page 895)
<code>coef</code>	matrix of coefficients and monomials (see <a href="#">Section 5.1.11 [coef]</a> , page 164)
<code>coeffs</code>	matrix of coefficients (see <a href="#">Section 5.1.12 [coeffs]</a> , page 165)
<code>deg</code>	degree (see <a href="#">Section 5.1.19 [deg]</a> , page 170)
<code>diff</code>	partial derivative (see <a href="#">Section 5.1.24 [diff]</a> , page 173)
<code>extgcd</code>	Bezout representation of gcd (see <a href="#">Section 5.1.33 [extgcd]</a> , page 178)
<code>factorize</code>	factorization of polynomial (see <a href="#">Section 5.1.36 [factorize]</a> , page 180)
<code>finduni</code>	univariate polynomials in a zero-dimensional ideal (see <a href="#">Section 5.1.43 [finduni]</a> , page 185)
<code>gcd</code>	greatest common divisor (see <a href="#">Section 5.1.50 [gcd]</a> , page 190)
<code>homog</code>	homogenization (see <a href="#">Section 5.1.57 [homog]</a> , page 196)
<code>jacob</code>	ideal, resp. matrix, of all partial derivatives (see <a href="#">Section 5.1.66 [jacob]</a> , page 203)
<code>lead</code>	leading term (see <a href="#">Section 5.1.75 [lead]</a> , page 209)
<code>leadcoef</code>	coefficient of the leading term (see <a href="#">Section 5.1.76 [leadcoef]</a> , page 209)
<code>leadexp</code>	the exponent vector of the leading monomial (see <a href="#">Section 5.1.77 [leadexp]</a> , page 210)
<code>leadmonom</code>	leading monomial (see <a href="#">Section 5.1.78 [leadmonom]</a> , page 210)
<code>jet</code>	monomials of degree at most k (see <a href="#">Section 5.1.68 [jet]</a> , page 204)
<code>ord</code>	degree of the leading monomial (see <a href="#">Section 5.1.112 [ord]</a> , page 238)
<code>qhweight</code>	quasihomogeneous weights (see <a href="#">Section 5.1.124 [qhweight]</a> , page 246)
<code>reduce</code>	normal form with respect to a standard base (see <a href="#">Section 5.1.131 [reduce]</a> , page 251)
<code>rvar</code>	test for ring variable (see <a href="#">Section 5.1.139 [rvar]</a> , page 258)
<code>simplify</code>	normalization of a polynomial (see <a href="#">Section 5.1.143 [simplify]</a> , page 263)
<code>size</code>	number of monomials (see <a href="#">Section 5.1.144 [size]</a> , page 264)
<code>subst</code>	substitution of a ring variable (see <a href="#">Section 5.1.154 [subst]</a> , page 274)
<code>trace</code>	trace of a matrix (see <a href="#">Section 5.1.158 [trace]</a> , page 281)
<code>var</code>	the indicated variable of the ring (see <a href="#">Section 5.1.165 [var]</a> , page 284)
<code>varstr</code>	variable(s) in string form (see <a href="#">Section 5.1.167 [varstr]</a> , page 285)

## 4.18 proc

Procedures are sequences of SINGULAR commands in a special format. They are used to extend the set of SINGULAR commands with user defined commands. Once a procedure is defined it can be used as any other SINGULAR command. Procedures may be defined by either typing them on the command line or by loading them from a file. For a detailed description on the concept of procedures in SINGULAR see [Section 3.7 \[Procedures\]](#), page 50. A file containing procedure definitions which comply with certain syntax rules is called a library. Such a file is loaded using the command LIB. For more information on libraries see [Section 3.8 \[Libraries\]](#), page 55.

### 4.18.1 proc declaration

**Syntax:**     [static] proc proc\_name [([<parameter\\_list>](#))  
                  [\[<help\\_string>](#)]  
              {  
                [<procedure\\_body>](#)  
              }  
              [example  
              {  
                [<sequence\\_of\\_commands>](#)  
              }]

**Purpose:**     Defines a new function, the proc proc\_name. Once loaded in a SINGULAR session, the information provided in the help string will be displayed upon entering `help proc_name;`, while the `example` section will be executed upon entering `example proc_name;`. See [Section 3.7.2 \[Parameter list\]](#), page 52, [Section 3.7.3 \[Help string\]](#), page 54, and the example in [Section 3.8.6 \[Procedures in a library\]](#), page 58.

The help string, the parameter list, and the example section are optional. They are, however, mandatory for the procedures listed in the header of a library. The help string is ignored and no example section is allowed if the procedure is defined interactively, i.e., if it is not loaded from a file by the LIB or load command (see [Section 5.1.79 \[LIB\]](#), page 211 and see [Section 5.2.12 \[load\]](#), page 299 ).

In the body of a library, each procedure not meant to be accessible by users should be declared static. See [Section 3.8.6 \[Procedures in a library\]](#), page 58.

**Example:**

```
proc milnor_number (poly p)
{
  ideal i= std(jacob(p));
  int m_nr=vdim(i);
  if (m_nr<0)
  {
    "// not an isolated singularity";
  }
  return(m_nr);           // the value of m_nr is returned
}
ring r1=0,(x,y,z),ds;
poly p=x^2+y^2+z^5;
milnor_number(p);
⇒ 4
```

See [Section 5.1.79 \[LIB\]](#), page 211; [Section 3.8 \[Libraries\]](#), page 55; [Section 5.2.1 \[apply\]](#), page 290.

### 4.18.2 proc expression

**Syntax:** `variable_name -> { expression(s) }`

**Purpose:** Defines a new function, within `apply` or for assigning.

**Example:**

```
    apply(1..3,x->{x**2});
    ↪ 1 4 9
```

See [Section 5.2.1 \[apply\], page 290](#); [Section 4.18 \[proc\], page 124](#).

### 4.18.3 procs with different argument types

**Syntax:** `branchTo ( string_expression , ... proc_name )`

**Purpose:** branch to the given procedure if the argument types matches the types given as strings (which may be empty - matching the empty argument list). The main procedure (`p` in the example) must be defined without an argument list, and `branchTo` statement must be the first statement within the procedure body.

**Example:**

```
proc p1(int i) { "int:",i; }
proc p21(string s) { "string:",s; }
proc p22(string s1, string s2) { "two strings:",s1,s2; }
proc p()
{ branchTo("int",p1);
  branchTo("string","string",p22);
  branchTo("string",p21);
  ERROR("not defined for these argument types");
}
p(1);
↪ int: 1
p("hu");
↪ string: hu
p("ha","ha");
↪ two strings: ha ha
p(1,"hu");
↪ ? not defined for these argument types
↪ ? leaving ::p (0)
```

See [Section 4.18 \[proc\], page 124](#).

## 4.19 resolution

The type resolution is intended as an intermediate representation which internally retains additional information obtained during computation of resolutions. It furthermore enables the use of partial results to compute, for example, Betti numbers or minimal resolutions. Like ideals and modules, a resolution can only be defined w.r.t. a basering (see [Section C.3 \[Syzygies and resolutions\], page 776](#)).

**Note:** To access the elements of a resolution, it has to be assigned to a list. This assignment also completes computations and may therefore take time, (resp. an access directly with the brackets `[ , ]` causes implicitly a cast to a list).

### 4.19.1 resolution declarations

**Syntax:** `resolution name = resolution_expression ;`

**Purpose:** defines a resolution.

**Default:** none

**Example:**

```

ring R;
ideal i=z2,x;
resolution re=res(i,0);
re;
↪ 1      2      1
↪ R <--  R <--  R
↪
↪ 0      1      2
↪
    betti(re);
↪ 1,1,0,
↪ 0,1,1
    list l = re;
    l;
↪ [1]:
↪   _[1]=x
↪   _[2]=z2
↪ [2]:
↪   _[1]=-z2*gen(1)+x*gen(2)
↪ [3]:
↪   _[1]=0

```

### 4.19.2 resolution expressions

A resolution expression is:

1. an identifier of type resolution
2. a function returning a resolution
3. a type cast to resolution from a list of ideals, resp. modules..

See [Section 3.5.5 \[Type conversion and casting\]](#), page 46.

### 4.19.3 resolution related functions

<b>betti</b>	Betti numbers of a resolution (see <a href="#">Section 5.1.4 [betti]</a> , page 159)
<b>fres</b>	free resolution of a standard basis (see <a href="#">Section 5.1.48 [fres]</a> , page 188)
<b>lres</b>	free resolution (see <a href="#">Section 5.1.83 [lres]</a> , page 215)
<b>minres</b>	minimize a free resolution (see <a href="#">Section 5.1.93 [minres]</a> , page 223)
<b>mres</b>	minimal free resolution of an ideal, resp. module and a minimal set of generators of the given ideal, resp. module (see <a href="#">Section 5.1.98 [mres]</a> , page 225)
<b>res</b>	free resolution of an ideal, resp. module, but not changing the given ideal, resp. module (see <a href="#">Section 5.1.134 [res]</a> , page 253)
<b>sres</b>	free resolution of a standard basis (see <a href="#">Section 5.1.149 [sres]</a> , page 269)

## 4.20 ring

Rings are used to describe properties of polynomials, ideals etc. Almost all computations in SINGULAR require a basering. For a detailed description of the concept of rings see [Section 3.3 \[Rings and orderings\]](#), page 30.

### 4.20.1 qring

SINGULAR offers the opportunity to calculate in quotient rings (factor rings), i.e., rings modulo an ideal. The ideal has to be given as a standard basis. For a detailed description of the concept of rings and quotient rings see [Section 3.3 \[Rings and orderings\]](#), page 30. Beside the construction, an object describing a quotient ring is of type `ring`.

See [Section 4.20.5 \[qring declaration\]](#), page 129.

### 4.20.2 ring declarations

**Syntax:**     `ring name = ( coefficients ), ( names_of_ring_variables ), ( ordering );` or  
                   `ring name = cring [ names_of_ring_variables ]`

**Default:**    `(ZZ/32003)[x,y,z]`

**Purpose:**     declares a ring and sets it as the actual basering. The second form sets the ordering to `(dp,C)`.

For the second form: `cring` stands currently for `QQ` (the rationals), `ZZ` (the integers) or `(ZZ/m)` (the field ( $m$  prime and  $<2147483648$ ) resp. ring of the integers modulo  $m$ ).

The coefficients for the first form are given by one of the following:

1. a `cring` as given above
2. a non-negative `int_expression` less or equal 2147483647.
3. an `expression_list` of an `int_expression` and one or more names.
4. the name `real`
5. an `expression_list` of the name `real` and an `int_expression`.
6. an `expression_list` of the name `complex`, an optional `int_expression` and a name.
7. an `expression_list` of the name `ZZ`.
8. an `expression_list` of the name `integer` and following `int_expressions`.
9. an `expression_list` of the name `integer` and two `int_expressions`.

For the definition of the 'coefficients', see [Section 3.3 \[Rings and orderings\]](#), page 30.

'names\_of\_ring\_variables' must be a list of names or (multi-)indexed names.

'ordering' is a list of block orderings where each block ordering is either

1. `lp`, `dp`, `Dp`, `rp`, `ls`, `ds`, `Ds`, or `rs` optionally followed by a size parameter in parentheses.
2. `wp`, `Wp`, `ws`, `Ws`, `am`, `aa`, or `a` followed by a weight vector given as an `intvec_expression` in parentheses.
3. `M` followed by an `intmat_expression` in parentheses.
4. `c` or `C`.

For the definition of the orderings, see [Section 3.3.3 \[Term orderings\]](#), page 34, [Section B.2 \[Monomial orderings\]](#), page 769.

If one of coefficients, names\_of\_ring\_variables, and ordering consists of only one entry, the parentheses around this entry may be omitted.

See also [Section 3.3.1 \[Examples of ring declarations\]](#), page 31; [Section 4.20 \[ring\]](#), page 127; [Section 5.1.137 \[ringlist\]](#), page 255.

### 4.20.3 ring related functions

<code>charstr</code>	description of the coefficient field of a ring (see <a href="#">Section 5.1.7 [charstr]</a> , page 162)
<code>keepring</code>	move ring to next upper level (see <a href="#">Section 5.2.11 [keepring]</a> , page 298)
<code>npars</code>	number of ring parameters (see <a href="#">Section 5.1.105 [npars]</a> , page 231)
<code>nvars</code>	number of ring variables (see <a href="#">Section 5.1.109 [nvars]</a> , page 233)
<code>ordstr</code>	monomial ordering of a ring (see <a href="#">Section 5.1.113 [ordstr]</a> , page 239)
<code>parstr</code>	names of all ring parameters or the name of the n-th ring parameter (see <a href="#">Section 5.1.116 [parstr]</a> , page 240)
<code>qring</code>	quotient ring (see <a href="#">Section 4.20.1 [qring]</a> , page 127)
<code>ringlist</code>	decomposition of a ring into a list of its components (see <a href="#">Section 5.1.137 [ringlist]</a> , page 255)
<code>setring</code>	setting of a new basering (see <a href="#">Section 5.1.141 [setring]</a> , page 260)
<code>varstr</code>	names of all ring variables or the name of the n-th ring variable (see <a href="#">Section 5.1.167 [varstr]</a> , page 285)

### 4.20.4 ring operations

<code>+</code>	construct a new ring $k[X, Y]$ from $k_1[X]$ and $k_2[Y]$ . (The sets of variables must be distinct).
<code>==,&lt;&gt;</code>	compare two rings

**Note:** Concerning the ground fields  $k_1$  and  $k_2$  take the following guide lines into consideration:

- Neither  $k_1$  nor  $k_2$  may be  $R$  or  $C$  .
- If the characteristic of  $k_1$  and  $k_2$  differs, then one of them must be  $Q$  .
- At most one of  $k_1$  and  $k_2$  may have parameters.
- If one of  $k_1$  and  $k_2$  is an algebraic extension of  $Z/p$  it may not be defined by a `charstr` of type  $(p^n, a)$ .

**Example:**

```

ring R1=0,(x,y),dp;
ring R2=32003,(a,b),dp;
def R=R1+R2;
R;
⇒ // coefficients: ZZ/32003
⇒ // number of vars : 4
⇒ //      block   1 : ordering dp
⇒ //                : names   x y
⇒ //      block   2 : ordering dp
⇒ //                : names   a b
⇒ //      block   3 : ordering C

```

[Section D.2.12 \[ring\\_lib\]](#), page 943

### 4.20.5 qring declaration

**Syntax:** `qring name = ideal_expression ;`

**Default:** `none`

**Purpose:** declares a quotient ring as the basering modulo `ideal_expression` and sets it as current basering.

Operations based on standard bases (e.g. `std,groebner`, etc., `reduce`) and functions which require a standard basis (e.g. `dim,hilb`, etc.) operated with the residue classes; all others on the polynomial objects.

**Example:**

```
ring r=0,(x,y,z),dp;
ideal i=xy;
qring q=std(i);
basing;
↪ // coefficients: QQ
↪ // number of vars : 3
↪ //      block 1 : ordering dp
↪ //      : names x y z
↪ //      block 2 : ordering C
↪ // quotient ring from ideal
↪ _[1]=xy
// simplification is not immediate:
(x+y)^2;
↪ x2+2xy+y2
reduce(_,std(0));
↪ x2+y2
// polynomial and residue class:
ring R=0,(x,y),dp;
qring Q=std(y);
poly p1=x;
poly p2=x+y;
// comparing polynomial objects:
p1==p2;
↪ 0
// comparing residue classes:
reduce(p1,std(0))==reduce(p2,std(0));
↪ 1
```

### 4.21 smatrix

An experimental type:

Objects of type `smatrix` are (sparse) matrices with polynomial entries. Like polynomials they can only be defined or accessed with respect to a basering.

Objects of type `smatrix` can be converted to and from `matrix` and `module`.

Operations are `+`, `-`, `*`, `==`, `<>`.

Functions are `chinrem`, `farey`, `ncols`, `nrows`, `std`, `transpose`, `tensor`. Additional `flatten(m)` and `system("unflatten",m,col)`.

Resizing can be done via `smatrix(m,r,c)` where `m` is of type `module` or `smatrix`.

Access to single entries: `m[i,j]`

See [Section 5.1.8 \[chinrem\]](#), page 162; [Section 5.1.37 \[farey\]](#), page 181; [Section D.3.1.5 \[flatten\]](#), page 971; [Section 4.13 \[matrix\]](#), page 108; [Section 4.14 \[module\]](#), page 112; [Section 5.1.104 \[ncols\]](#), page 231; [Section 5.1.107 \[nrows\]](#), page 232; [Section 4.20 \[ring\]](#), page 127; [Section 5.1.151 \[std\]](#), page 271; [Section 5.1.157 \[tensor\]](#), page 280; [Section 5.1.159 \[transpose\]](#), page 281.

## 4.22 string

Variables of type `string` are used for output (almost every type can be "converted" to `string`) and for creating new commands at runtime see [Section 5.1.32 \[execute\]](#), page 178. They are also return values of certain interpreter related functions (see [Section 5.1 \[Functions\]](#), page 156). String constants consist of a sequence of ANY characters (including newline!) between a starting `"` and a closing `"`. There is also a string constant `newline`, which is the newline character. The `+` sign "adds" strings, `""` is the empty string (hence strings form a semigroup). Strings may be used to comment the output of a computation or to give it a nice format. Strings may also be used for intermediate conversion of one type into another.

```

string s="Hi";
string s1="a string with new line at the end"+newline;
string s2="another string with new line at the end
";
s;s1;s2;
⇒ Hi
⇒ a string with new line at the end
⇒
⇒ another string with new line at the end
⇒
ring r; ideal i=std(ideal(x,y^3));
"dimension of i =",dim(i)," , multiplicity of i =",mult(i);
⇒ dimension of i = 1 , multiplicity of i = 3
"dimension of i = "+string(dim(i))+" , multiplicity of i = "+string(mult(i));
⇒ dimension of i = 1, multiplicity of i = 3
"a"+"b","c";
⇒ ab c

```

A comma between two strings makes an expression list out of them (such a list is printed with a separating blank in between), while a `+` concatenates strings.

### 4.22.1 string declarations

**Syntax:**     `string name = string_expression ;`  
               `string name = list_of_string-expressions ;`

**Purpose:**     defines a string variable.

**Default:**    `""` (the empty string)

**Example:**

```

string s1="Now I know";
string s2="how to encode a \" in a string...";
string s=s1+" "+s2; // concatenation of 3 strings
s;
⇒ Now I know how to encode a " in a string...
s1,s2; // 2 strings, separated by a blank in the output:
⇒ Now I know how to encode a " in a string...

```



### 4.22.2 string expressions

A string expression is:

1. a sequence of characters between two unescaped quotes (")
2. an identifier of type string
3. a function returning string
4. a substring (using the bracket operator)
5. a type cast to string (see [Section 4.22.3 \[string type cast\], page 131](#))
6. string expressions combined by the operation +.

**Example:**

```
// string_expression[start, length] : a substring
// (possibly filled up with blanks)
// the substring of s starting at position 2
// with a length of 4
string s="123456";
s[2,4];
⇒ 2345
"abcd"[2,2];
⇒ bc
// string_expression[position] : a character from a string
s[3];
⇒ 3
// string_expression[position..position] :
// a substring starting at the first position up to the second
// given position
s[2..4];
⇒ 2 3 4
// a function returning a string
typeof(s);
⇒ string
```

See [Section 3.5.5 \[Type conversion and casting\], page 46](#)

### 4.22.3 string type cast

**Syntax:**     string ( expression [, expression\_2, ... expression\_n] )

**Type:**       string

**Purpose:**     Converts each expression to a string, where expression can be of any type. The concatenated string of all converted expressions is returned.

The elements of intvec, intmat, ideal, module, matrix, and list, are separated by a comma. No newlines are inserted.

Not defined elements of a list are omitted.

For link, the name of the link is used.

For map, the ideal defining the mapping is converted.

**Note:**       When applied to a list, elements of type intvec, intmat, ideal, module, matrix, and list become indistinguishable.

**Example:**

```

    string("1+1=", 2);
    ↪ 1+1=2
    string(intvec(1,2,3,4));
    ↪ 1,2,3,4
    string(intmat(intvec(1,2,3,4), 2, 2));
    ↪ 1,2,3,4
    ring r;
    string(r);
    ↪ (ZZ/32003),(x,y,z),(dp(3),C)
    string(ideal(x,y));
    ↪ x,y
    qring R = std(ideal(x,y));
    string(R);
    ↪ (ZZ/32003),(x,y,z),(dp(3),C)
    map phi = r, ideal(x,z);
    string(phi);
    ↪ x,z
    list l;
    string(l);
    ↪
    l[3] = 1;
    string(l); // notice that l[1],l[2] are omitted
    ↪ 1
    l[2] = 1;
    l;
    ↪ [2]:
    ↪   [3]:
    ↪     1
    ↪ [3]:
    ↪   1
    string(l); // notice that lists of list is flattened
    ↪ 1,1
    l[1] = intvec(1,2,3);
    l;
    ↪ [1]:
    ↪   1,2,3
    ↪ [2]:
    ↪   [3]:
    ↪     1
    ↪ [3]:
    ↪   1
    string(l); // notice that intvec elements are not distinguishable
    ↪ 1,2,3,1,1

```

See [Section 3.5.5 \[Type conversion and casting\]](#), page 46; [Section 5.1.120 \[print\]](#), page 242; [Section 4.22 \[string\]](#), page 130.

#### 4.22.4 string operations

+ concatenation

<=, >=, ==, <>

comparison (lexicographical with respect to the ASCII encoding)

`string_expression [ int_expression ]`

is a character of the string; the index 1 gives the first character.

`string_expression [ int_expression, int_expression ]`

is a substring, where the first argument is the start index and the second is the length of the substring, filled up with blanks if the length exceeds the total size of the string

`string_expression [ intvec_expression ]`

is a expression list of characters from the string

#### Example:

```

    string s="abcde";
    s[2];
    ↪ b
    s[3,2];
    ↪ cd
    ">>" + s[1,10] + "<<";
    ↪ >>abcde      <<
    s[2]="BC"; s;
    ↪ aBcde
    intvec v=1,3,5;
    s=s[v]; s;
    ↪ ace
    s="654321"; s=s[3..5]; s;
    ↪ 432

```

### 4.22.5 string related functions

<code>charstr</code>	description of the coefficient field of a ring (see <a href="#">Section 5.1.7 [charstr]</a> , page 162)
<code>execute</code>	executing string as command (see <a href="#">Section 5.1.32 [execute]</a> , page 178)
<code>find</code>	position of a substring in a string (see <a href="#">Section 5.1.42 [find]</a> , page 185)
<code>names</code>	list of strings of all user-defined variable names (see <a href="#">Section 5.1.103 [names]</a> , page 229)
<code>nameof</code>	name of an object (see <a href="#">Section 5.1.102 [nameof]</a> , page 228)
<code>option</code>	lists all defined options (see <a href="#">Section 5.1.111 [option]</a> , page 234)
<code>ordstr</code>	monomial ordering of a ring (see <a href="#">Section 5.1.113 [ordstr]</a> , page 239)
<code>parstr</code>	names of all ring parameters or the name of the n-th ring parameter (see <a href="#">Section 5.1.116 [parstr]</a> , page 240)
<code>read</code>	read a file (see <a href="#">Section 5.1.130 [read]</a> , page 250)
<code>size</code>	length of a string (see <a href="#">Section 5.1.144 [size]</a> , page 264)
<code>sprintf</code>	string formatting (see <a href="#">Section 5.1.148 [sprintf]</a> , page 267)
<code>typeof</code>	type of an object (see <a href="#">Section 5.1.161 [typeof]</a> , page 282)
<code>varstr</code>	names of all ring variables or the name of the n-th ring variable (see <a href="#">Section 5.1.167 [varstr]</a> , page 285)

## 4.23 vector

Vectors are elements of a free module over the basering with basis `gen(1)`, `gen(2)`, `...`. Like polynomials they can only be defined or accessed with respect to the basering. Each vector belongs to a free module of rank equal to the biggest index of a generator with non-zero coefficient. Since generators with zero coefficients need not be written any vector may be considered also as an element of a free module of higher rank. (E.g., if `f` and `g` are polynomials then `f*gen(1)+g*gen(3)+gen(4)` may also be written as `[f,0,g,1]` or as `[f,0,g,1,0]`.) Note that the elements of a vector have to be surrounded by square brackets `[ , ]` (cf. [Section B.1 \[Representation of mathematical objects\]](#), [page 768](#)).

### 4.23.1 vector declarations

**Syntax:** `vector name = vector_expression ;`

**Purpose:** defines a vector of polynomials (an element of a free module).

**Default:** `[0]`

**Example:**

```
ring r=0,(x,y,z),(c,dp);
poly s1 = x2;
poly s2 = y3;
poly s3 = z;
vector v = [s1, s2-s1, s3-s1]+ s1*gen(5);
// v is a vector in the free module of rank 5
v;
↦ [x2,y3-x2,-x2+z,0,x2]
```

### 4.23.2 vector expressions

A vector expression is:

1. an identifier of type vector
2. a function returning vector
3. a polynomial expression (via the canonical embedding  $p \mapsto p*\text{gen}(1)$ )
4. vector expressions combined by the arithmetic operations `+` or `-`
5. a polynomial expression and a vector expression combined by the arithmetic operation `*`
6. a type cast to vector using the brackets `[ , ]`

**Example:**

```
// ordering gives priority to components:
ring rr=0,(x,y,z),(c,dp);
vector v=[x2+y3,2,0,x*y]+gen(6)*x6;
v;
↦ [y3+x2,2,0,xy,0,x6]
vector w=[z3-x,3y];
v-w;
↦ [y3-z3+x2+x,-3y+2,0,xy,0,x6]
v*(z+x);
↦ [xy3+y3z+x3+x2z,2x+2z,0,x2y+xyz,0,x7+x6z]
// ordering gives priority to monomials:
```

```
// this results in a different output
ring r=0,(x,y,z),(dp,c);
imap(rr,v);
↦ x6*gen(6)+y3*gen(1)+x2*gen(1)+xy*gen(4)+2*gen(2)
```

See [Section 3.5.5 \[Type conversion and casting\]](#), page 46; [Section 4.20 \[ring\]](#), page 127.

### 4.23.3 vector operations

+            addition  
 -            negation or subtraction  
 /            division by a monomial, not divisible terms yield 0  
 <, <=, >, >=, ==, <>            comparators (considering leading terms w.r.t. monomial ordering)  
 vector\_expression [ int\_expressions ]  
               is a vector entry; the index 1 gives the first entry.

#### Example:

```
ring R=0,(x,y),(c,dp);
[x,y]-[1,x];
↦ [x-1,-x+y]
[1,2,x,4][3];
↦ x
```

### 4.23.4 vector related functions

cleardenom            quotient of a vector by its content (see [Section 5.1.9 \[cleardenom\]](#), page 163)  
 coeffs            matrix of coefficients (see [Section 5.1.12 \[coeffs\]](#), page 165)  
 deg            degree (see [Section 5.1.19 \[deg\]](#), page 170)  
 diff            partial derivative (see [Section 5.1.24 \[diff\]](#), page 173)  
 gen            i-th generator (see [Section 5.1.51 \[gen\]](#), page 191)  
 homog            homogenization (see [Section 5.1.57 \[homog\]](#), page 196)  
 jet            k-jet: monomials of degree at most k (see [Section 5.1.68 \[jet\]](#), page 204)  
 lead            leading term (see [Section 5.1.75 \[lead\]](#), page 209)  
 leadcoef            leading coefficient (see [Section 5.1.76 \[leadcoef\]](#), page 209)  
 leadexp            the exponent vector of the leading monomial (see [Section 5.1.77 \[leadexp\]](#), page 210)  
 leadmonom            leading monomial (see [Section 5.1.78 \[leadmonom\]](#), page 210)  
 nrows            number of rows (see [Section 5.1.107 \[nrows\]](#), page 232)  
 ord            degree of the leading monomial (see [Section 5.1.112 \[ord\]](#), page 238)  
 reduce            normal form with respect to a standard base (see [Section 5.1.131 \[reduce\]](#), page 251)  
 simplify            normalize a vector (see [Section 5.1.143 \[simplify\]](#), page 263)  
 size            number of monomials (see [Section 5.1.144 \[size\]](#), page 264)  
 subst            substitute a ring variable (see [Section 5.1.154 \[subst\]](#), page 274)

## 4.24 User defined types

User defined types are (non-empty) lists with a fixed size whose element can be accessed by names (and not indices). These elements have a predefined type (which can also be a user defined type). If these elements depend on a ring they can only be accessed if their base ring is the current base ring. In contrast to usual lists the elements of a user defined type may belong to different rings.

### 4.24.1 Definition of a user defined type

**Syntax:**     `newstruct( name , string-expression );`  
                  `newstruct( name , name , string-expression );`

**Purpose:**     defines a new type with elements given by the last argument (string-expression). The name of the new type is the first argument (of type string) and must be longer than one character.

The second name (of type string) is an already defined type which should be extended by the new type.

The last argument (of type string) must be an comma separated list of a type followed by a name. If there are duplicate member names, the last one wins.

(User defined) member names are restricted to alphanumeric characters and must start with a letter.

**Operations:**

the only operations of user defined types are:

assignment (between objects of the same or extended type)

`typeof`

`string` and printing

operator `.` to access the elements

**Example:**

```
newstruct("nt","int a,poly b,string c");
nt A;
nt B;
A.a=3;
A.c=string(A.a);
B=A;
newstruct("t2","nt","string c");
t2 C; C.c="t2-c";
A=C;
typeof(A);
↳ t2
A;
↳ c=t2-c
↳ c=
↳ b=?
↳ a=0
// a motivating example -----
newstruct("IDEAL","ideal I,proc prettyprint");
newstruct("HOMOGENEOUS_IDEAL","IDEAL","intvec weights,proc prettyprint")
proc IDEAL_pretty_print(IDEAL I)
{
    "ideal generated by";
```

```

        I.I;
    }
    proc H_IDEAL_pretty_print(HOMOGENEOUS_IDEAL I)
    {
        "homogeneous ideal generated by";
        I.I;
        "with weights";
        I.weights;
    }
    proc p_print(IDEAL I) { I.prettyprint(I); }
    ring r;
    IDEAL I;
    I.I=ideal(x+y^2,z);
    I.prettyprint=H_IDEAL_pretty_print;
    HOMOGENEOUS_IDEAL H;
    H.I=ideal(x,y,z);
    H.prettyprint=H_IDEAL_pretty_print;
    H.weights=intvec(1,1,1);
    p_print(I);
    ↪ ideal generated by
    ↪ _[1]=y^2+x
    ↪ _[2]=z
    p_print(H);
    ↪ homogeneous ideal generated by
    ↪ _[1]=x
    ↪ _[2]=y
    ↪ _[3]=z
    ↪ with weights
    ↪ 1,1,1

```

#### 4.24.2 Declaration of objects of a user defined type

**Example:**

```

newstruct("nt","int a,poly b,string c");
nt A;
// as long as there is no value assigned to A.b, no ring is needed
nt B=A;

```

#### 4.24.3 Access to elements of a user defined type

Access to elements of a user defined type via `.: <object>.<element_name>`. The `<element_names>` are from the definition of the type. Additional, all (potentially) ring dependent elements have an additional entry `r_<element_name>` for the corresponding ring.

**Example:**

```

newstruct("nt","int a,poly b,string c");
nt A;
3+A.a;
↪ 3
A.c="example string";
ring r;
A.b=poly(1); // assignment: expression must be of the given type
A;

```

```

⇒ c=example string
⇒ b=1
⇒ a=0
  A.r_b;
⇒ // coefficients: ZZ/32003
⇒ // number of vars : 3
⇒ //          block  1 : ordering dp
⇒ //                      : names    x y z
⇒ //          block  2 : ordering C

```

#### 4.24.4 Commands for user defined types

User defined types are normal data types (which do not belong to a ring, even if they have ring dependent parts), so they can be passed as argument to procedures, and received as result from procedures.

In order to apply kernel commands to these types (like `string`, `+`), provide a usual procedure (say `proc p.`) for that task and install it via `system("install", user_type , kernel_command ,p, number_of_args )`; The `user_type` and `kernel_command` have to be given as strings. For `kernel_command` having a variable number of arguments (internal `CMD_M`) use 4 independent of the number of really supplied arguments.

List of available kernel commands and the required `number_of_args`, some accept several variants and appear therefore at several places:

inplace binary operands: `+, -, *, /, div, %, &, |, [,` `number_of_args:2`

unary functions: `attrib, bareiss, betti, char, char_series, charstr, cleardenom, close, convhull, defined, deg, degree, denominator, det, dim, dump, ERROR, envelope, execute, facstd, factorize, finduni, gen, getdump, hilb, impart, indepSet, interred, jacob, janet, kbase, killattrib, lead, leadcoef, leadexp, leadmonom, load, ludecomp, maxideal, memory, minbase, minres, monitor, monomial, mult, mstd, nameof, ncols, npars, nrows, numerator, nvars, open, opposite, ord, ordstr, par, pardeg, parstr, preimage, prime, primefactors, prune, qhweight, rank, read, regularity, repart, ringlist, rvar, sba, size, slimgb, sortvec, sqrfree, syz, trace, transpose, twostd, typeof, univariate, var, variables, varstr, vdim, waitfirst, waitall, weight`

functions with 2 arguments: `attrib, betti, bracket, chinrem, coeffs, contract, deg, delete, diff, dim, extgcd, eliminate, exportto, facstd, factorize, farey, fetch, fglm, fglmquot, find, fres, frwalk, gcd, hilb, homog, hres, imap, importfrom, indepSet, insert, interpolation, janet, kbase, kernel, killattrib, koszul, lift, liftstd, load, lres, modulo, mpresmat, mres, newstruct, nc_algebra, nres, oppose, parstr, primefactors, quotient, random, rank, read, sba, simplify, sqrfree, sres, varstr, waitfirst, waitall, wedge`

functions with 3 arguments: `attrib, bareiss, coeffs, eliminate, find, fres, frwalk, hilb, homog, insert, koszul, laguerre, lift, liftstd, newstruct, preimage, random, resultant, sba, vandermonde`

functions with variable number of arguments arguments (`number_of_args:4`): `breakpoint, coef, dbprint, division, factmodd, intersect, jet, luinverse, lusolve, minor, names, option, qrds, reduce, reservedName, simplex, status, std, subst, system, test, uressolve, write`

**Example:**



```

newstruct("nt","int a,poly b,string c");
nt A;
A;
 $\mapsto$  c=
 $\mapsto$  b=??
 $\mapsto$  a=0
ring r;
// a pretty print routine for nt:
proc pretty_print(nt A)
{
    "nt with string c:"+A.c+" and poly:"+string(A.b);
}
system("install","nt","print",pretty_print,1); // default printing uses print
A;
 $\mapsto$  nt with string c: and poly:0
 $\mapsto$ 
// a custom add for nt:
proc nt_add(nt A,nt B)
{
    nt C;
    C.a=A.a+B.a; C.b=A.b+B.b; C.c=A.c+B.c;
    return(C);
}
system("install","nt","+",nt_add,2);
A.b=x;
nt B; B.c="B"; B.b=y;
A+B;
 $\mapsto$  nt with string c:B and poly:x+y
 $\mapsto$ 

```

#### 4.24.5 Assignments for user defined types

By default, only objects of the same (user defined) type can be assigned, there is no automatic type conversion as for the kernel data types.

But the operator = can be overridden in order to write custom constructors (the custom constructor does not apply to assignments of the same type): via `system("install", user_type, "=", p, 1);`. The user\_type has to be given as a string.

##### Example:

```

newstruct("wrapping","poly p");
proc wrap(poly p)
{
    wrapping w; w.p = p;
    return (w);
}
system("install", "wrapping", "=", wrap, 1);
ring r = 0,x,dp;
wrapping w = x+1;
w;
 $\mapsto$  p=x+1
w = int(1); // via conversion int->poly
w;

```

```

↳ p=1
  w=number(2); // via conversion number->poly
  w;
↳ p=2

```

The user defined procedure `for =` provides also generic type conversions: `hh A=hh(b);` is equivalent to `hh tmp=b; hh A=tmp; kill tmp;`.

## 4.25 cone

In order to use convex objects in Singular, Singular has to be build from sources together with `gfanlib`, a C++ library for convex geometry by Anders N. Jensen. Please check the readme file for installation instructions.

This version of SINGULAR does not support `cone`.

## 4.26 fan

Not supported in this version of SINGULAR

## 4.27 polytope

Not supported in this version of SINGULAR

Not supported in this version of SINGULAR

## 4.28 pyobject

The `pyobject` is a black box data type in SINGULAR for handling objects from the programming language `python`. It needs the `python` support of SINGULAR to be installed.

Together with some basic operations and functions, `pyobject` instances access `python` functionality from within SINGULAR and store the results for re-use:

Note that this feature is automatically loaded on demand when initializing an object of type `pyobject`. For accessing `pyobject`-related functions before using any `python` object, please type `LIB("pyobject.so");` at the SINGULAR prompt.

```

pyobject pystr = "Hello";
pyobject pyint = 2;
string singstr = string(pystr + " World!");
singstr;
↳ 'Hello World!'
pystr + pyint; // Error: not possible
↳ ? pyobject error occurred
↳ ? cannot concatenate 'str' and 'int' objects
↳ ? error occurred in or before ./examples/pyobject.sing line 5: 'pystr \
  + pyint; // Error: not possible'
pystr * pyint; // But this is allowed,
↳ 'HelloHello'
pystr * 3; // as well as this;
↳ 'HelloHelloHello'

python_run("def newfunc(*args): return list(args)"); // syncs contexts!
newfunc(1, 2, 3); // newfunc also knowd to SINGULAR
↳ [1, 2, 3]

```

```

def pylst = python_eval("[3, 7, 1]");
proc(attrib(pylst, "sort"))(); // Access python member routines as attributes
pylst.sort();                 // <- equivalent short-notation
pylst."sort"();               // <- alternative short-notation
pylst;
⇨ [1, 3, 7]

python_import("os");          // Gets stuff from python module 'os'
name;                         // The identifier of the operating system
⇨ 'posix'

```

### 4.28.1 pyobject declarations

**Syntax:** `pyobject name = pyobject_expression ;`

**Purpose:** defines a python object.

**Default:** `None`

**Example:**

```

pyobject empty;
empty;
⇨ None

pyobject pystr = "Hello World!";
pyobject pyone = 17;
pyobject pylst = list(pystr, pyone);
pylst;
⇨ ['Hello World!', 17]

```

### 4.28.2 pyobject expressions

A pyobject expression is (optional parts in square brackets):

1. an identifier of type pyobject
2. a function returning pyobject
3. pyobject expressions combined by the arithmetic operations `+`, `-`, `*`, `/`, or `^`, and the member-of operators `.` and `::`
4. an list expression with elements made of pyobject expressions (see [Section 3.5.5 \[Type conversion and casting\]](#), page 46)
5. an string expression (see [Section 3.5.5 \[Type conversion and casting\]](#), page 46)
6. an int expression (see [Section 3.5.5 \[Type conversion and casting\]](#), page 46)

**Example:**

```

pyobject pystr = "python string ";
pystr;
⇨ 'python string '
pyobject pyint = 2;
pyint;
⇨ 2
pyobject pylst = list(pystr, pyint);
pylst;

```

```

↳ ['python string ', 2]
  pyint + pyint;
↳ 4
  pyint * pyint;
↳ 4
  pystr + pystr;
↳ 'python string python string '
  pystr * pyint;
↳ 'python string python string '
  python_eval("17 + 4");
↳ 21
  typeof(_);
↳ pyobject

```

### 4.28.3 pyobject operations

+            addition  
 -            negation or subtraction  
 \*            multiplication  
 /            division  
 ^, \*\*        power by a positive integer  
 <, <=, >, >=, ==, <>        comparators (considering leading monomials w.r.t. monomial ordering)  
 pyobject\_expression [ int\_expression ]  
               get the item from the pyobject by index  
 pyobject\_expression ( pyobject\_expression\_sequence )  
               call the pyobject with a sequence of python arguments (the latter may be empty)  
 pyobject\_expression . ( string\_expression | name ),  
               pyobject\_expression :: ( string\_expression | name ) get attribute (class member) of a  
               python object

#### Example:

```

pyobject two = 2;
pyobject three = 3;

two + three;
↳ 5
two - three;
↳ -1
two * three;
↳ 6
two / three;
↳ 0
two ^ three;
↳ 8
two ** three;
↳ 8

```

```

three < two;
↳ 0
two < three;
↳ 1
three <= two;
↳ 0
two <= three;
↳ 1
two == three;
↳ 0
two == two;
↳ 1
three > two;
↳ 1
two > three;
↳ 0
three >= two;
↳ 1
two >= three;
↳ 0
two != two;
↳ 0
two != three;
↳ 1

pyobject pystr = "Hello";
pystr + " World!";
↳ 'Hello World!'
pystr * 3;
↳ 'HelloHelloHello'
pystr[1];
↳ 'e'

python_run("def newfunc(*args): return list(args)");
newfunc();
↳ []
newfunc(two, three);
↳ [2, 3]

newfunc.__class__;
↳ <type 'function'>
newfunc.__class__;
↳ <type 'function'>
newfunc.func_name;
↳ 'newfunc'
newfunc.__func_name;
↳ 'newfunc'

```

#### 4.28.4 pyobject related functions

**attrib** list, get and set attributes (class members) of a pyobject (see [Section 5.1.2 \[attrib\]](#), [page 156](#))

**Example:**

```

pyobject pystr = "Kublai Khan";

// Additional functionality through attrib
attrib(pystr, "__doc__");
↳ "str(object='') -> string\n\nReturn a nice string representation of the
    bject.\nIf the argument is a string, the return value is the same object."

proc(attrib(pystr, "count"))("K");
↳ 2

pystr.__doc__; // <- Short notations
↳ "str(object='') -> string\n\nReturn a nice string representation of the
    bject.\nIf the argument is a string, the return value is the same object."

pystr.count("a"); // Even shorter (if attribute's name is valid)
↳ 2

python_run("def func(): return 17");
attrib(func);
↳ ['__call__', '__class__', '__closure__', '__code__', '__defaults__', '__delattr__', '__dict__', '__doc__', '__format__', '__get__', '__getattr__', '__hash__', '__init__', '__module__', '__name__', '__new__', '__reduce__', '__reduce_ex__', '__repr__', '__setattr__', '__sizeof__', '__str__', '__subclasshook__', 'func_closure', 'func_code', 'func_defaults', 'func_dict', 'func_doc', 'func_globals', 'func_name']
attrib(func, "func_name");
↳ 'func'
attrib(func, "func_name", "byAnyOtherName");
attrib(func, "func_name");
↳ 'byAnyOtherName'

```

**killattrib**

deletes an attribute from a pyobject (see [Section 5.1.72 \[killattrib\]](#), page 207)

**Example:**

```

LIB("pyobject.so");
python_run("def new_pyobj(): pass");
attrib(new_pyobj, "new_attr", "something");
attrib(new_pyobj, "new_attr");
↳ 'something'
attrib(new_pyobj);
↳ ['__call__', '__class__', '__closure__', '__code__', '__defaults__', '__delattr__', '__dict__', '__doc__', '__format__', '__get__', '__getattr__', '__hash__', '__init__', '__module__', '__name__', '__new__', '__reduce__', '__reduce_ex__', '__repr__', '__setattr__', '__sizeof__', '__str__', '__subclasshook__', 'func_closure', 'func_code', 'func_defaults', 'func_dict', 'func_doc', 'func_globals', 'func_name', 'new_attr']

killattrib(new_pyobj, "new_attr");
attrib(new_pyobj);

```



### 4.28.7 python\_run

**Syntax:** `python_run ( string-expression )`

**Type:** `none`

**Purpose:** Executes python commands (given as a string) in `python` context and syncs the contexts afterwards.

**Example:**

```
LIB("pyobject.so");
python_run("def newfunc(*args): return list(args)");
newfunc(1, 2, 3);           // newfunc also known to SINGULAR now
↳ [1, 2, 3]

python_run("import os");
os;
↳ <module 'os' from '/usr/lib64/python2.7/os.pyc'>
  attrib(os, "name");
↳ 'posix'
```

## 4.29 reference and shared (experimental)

The black box data types `reference` and `shared` in SINGULAR allow for concurrently accessing SINGULAR object data. Copying such object will only add an additional handle which allows you to define multiple identifiers for the same object instance.

Both experimental features are hidden by default, please activate them by typing `system("reference");` or `system("shared");`, respectively, at the SINGULAR prompt.

You must initialize a `reference` using a named identifier or a subexpression of the latter. The resulting object can be stored to gain read and write access from sophisticated data structures.

```
system("reference"); system("shared");
int i = 17;
reference ref = i;

ref;
↳ 17
↳
ref = 19;
ref;
↳ 19
↳
i;           // original handle changed!
↳ 19

kill ref;
i;           // 'i' stays alive
↳ 19

reference uninitialized;
uninitialized; // not initialized
↳ <unassigned reference or shared memory>
// error: not a named identifier:
uninitialized = 17;
```



```

⇒      ? Can only take reference from identifier
⇒      ? error occurred in or before ./examples/reference_and_shared__experim\
      ental_.sing line 16: 'uninitialized = 17;'

// but subexpressions of named identifiers will do
list ll = list(3,4,5);
reference ref = ll[2];
ref;
⇒ 4
⇒
ref = 12;
ref;
⇒ 12
⇒
ll;
⇒ [1]:
⇒ 3
⇒ [2]:
⇒ 12
⇒ [3]:
⇒ 5

```

In contrast, the type `shared` can be used to avoid the initial identifier definition. Each copy has equal rights for manipulating the data.

```

system("reference"); system("shared");
shared ll= list(2,3);

ll[1];
⇒ 2
⇒
ll[1]= 17;
ll;
⇒ [1]:
⇒ 17
⇒ [2]:
⇒ 3
⇒

```

In most cases the value look-up is done automatically, but sometimes you have to disambiguate the input.

```

system("reference"); system("shared");
int i = 0;
reference ref = i;
shared sh = 12;

ref + sh;    // automated 'dereferencing'
⇒ 12
ref + 4;
⇒ 4
4 + sh;
⇒ 16

list ll = list(ref, ref, ref, ref, ref, ref, ref);
string(ll);

```

```

↳ 0,0,0,0,0,0,0
ref = 1;
string(ll); // all one now
↳ 1,1,1,1,1,1,1

ll[3] = 0;
string(ll); // only third element changed
↳ 1,1,0,1,1,1,1

reference(ll[1]) = 9;
string(ll); // all others changed
↳ 9,9,0,9,9,9,9

def(ll[1]) = 11; // alternative (generic) syntax
string(ll);
↳ 11,11,0,11,11,11,11

```

The previous example had shown that **reference** and **shared** objects can store highly structured without duplicating data all over again. As an additional feature, you can use **reference** objects for implementing procedures having side-effects.

```

system("reference"); system("shared");
list changeme;
changeme;
↳ empty list

proc setfirst(reference ll, def arg) { ll[1] = arg; }

setfirst(changeme, 17);
changeme;
↳ [1]:
↳ 17

```

If you do not need write-access to **proc** parameters, your code will usually perform better using the **alias** statement in the parameter list, see [Section 4.18 \[proc\]](#), page 124.

### 4.29.1 reference declarations

**Syntax:** reference name = identifier ;

**Purpose:** defines a **reference** object.

**Default:** None

**Example:**

```

system("reference"); system("shared");
reference empty;
empty;
↳ <unassigned reference or shared memory>

string str = "Hello World!";
reference ref = str;
ref;
↳ Hello World!
↳
ref = 17; // cannot change type of 'i'

```

```

↳      ? 'string'(str) = 'int' is not supported
↳      ? expected 'string' = 'string'
↳      ? error occurred in or before ./examples/reference_declarations.sing
      ine 8: '  ref = 17;    // cannot change type of 'i','
      list ll= list(4, 5, 6);
      reference lref = ll[2];
      lref;
↳ 5
↳
      lref = str; // change list element
      ll;
↳ [1]:
↳ 4
↳ [2]:
↳ Hello World!
↳ [3]:
↳ 6

```

### 4.29.2 reference expressions

A reference expression:

1. any identifier
2. any subexpression of an identifier
3. an object of type **reference** (result will reference the original identifier, too)

**Example:**

```

system("reference"); system("shared");
  int i = 17;
  reference ref = i; // new reference
  ref;
↳ 17
↳
  reference second = ref;
  second;
↳ 17
↳
  second = 9;      // also tied to 'i'
  i;
↳ 9
  typeof(ref);
↳ reference

  list ll = list(1, 2, 3);
  reference lref = ll[1];
  lref;
↳ 1
↳
  lref = 12;
  ll;
↳ [1]:
↳ 12
↳ [2]:

```

```

↳      2
↳ [3]:
↳      3

```

### 4.29.3 shared declarations

**Syntax:**    shared name = expression ;

**Purpose:**    defines a shared object.

**Default:**   None

**Example:**

```

system("reference"); system("shared");
shared empty;
empty;
↳ ' _ '
↳

shared str = "Hello World!";
str;
↳ Hello World!
↳

shared ll= list(4, 5, 6);
ll;
↳ [1]:
↳      4
↳ [2]:
↳      5
↳ [3]:
↳      6
↳

ll[2] = str; // change list element
ll;
↳ [1]:
↳      4
↳ [2]:
↳      Hello World!
↳ [3]:
↳      6
↳

```

### 4.29.4 shared expressions

shared expression:

1. any expression
2. an object of type `shared` (result will reference the same data)

**Example:**

```

system("reference"); system("shared");
shared sh = 17; // new shared
shared second = sh;
second;

```

```

↳ 17
↳
    second = 9;          // also tied to 'sh'
    sh;
↳ 9
↳
    typeof(sh);
↳ shared

    shared ll = list(1, 2, 3);
    shared lref = ll[1];
    lref;
↳ 1
↳
    lref = 12;
    ll;
↳ [1]:
↳ 12
↳ [2]:
↳ 2
↳ [3]:
↳ 3
↳

```

#### 4.29.5 reference and shared operations

All operations of the underlying objects are forwarded by **reference** and **shared** objects. This kind of dereferencing is done automatically in most cases:

**Example:**

```

system("reference"); system("shared");
int i = 2;
reference two = i;
shared three = 3;

two * three;
↳ 6
two ^ three;
↳ 8
two ** three;
↳ 8

two + two;
↳ 4
two - two;
↳ 0

ring r = 0, (x,y,z), dp;
poly p = x + y + z;
reference ref = p;
shared zvar =z;
subst(ref, x,1, y,2, zvar,3);
↳ 6

```

In some cases **references** have to be dereferenced explicitly. For instance, this is the case for n-ary function calls not starting with a **reference** or **shared** object. You can use the **link** operator or a type cast to work around this. In contrast, some constructs like left-hand subexpressions prematurely evaluate. You can avoid this by using the **def** operator or by explicitly type casting to **reference**.

```

system("reference"); system("shared");
ring r = 0, (x,y,z), dp;
poly p = x + y + z;
shared xsh = x;
subst(p, xsh,1, y,2, z,3);          // fails
⇒ ? subst('poly','shared','int') failed
⇒ ? expected subst('poly','poly','poly')
⇒ ? expected subst('matrix','poly','int')
⇒ ? error occurred in or before ./examples/reference_and_shared_operatio\
    ns_1.sing line 5: 'subst(p, xsh,1, y,2, z,3);          // fails'
subst(p, poly(xsh),1, y,2, z,3);    // good
⇒ 6
subst(p, link(xsh),1, y,2, z,3);    // fine
⇒ 6

list ll = list(xsh, xsh, xsh);
ll[1] = y;          // replaced only first entry
ll;
⇒ [1]:
⇒ y
⇒ [2]:
⇒ x
⇒
⇒ [3]:
⇒ x
⇒
shared(ll[2]) = z;    // replaces the others
ll;
⇒ [1]:
⇒ y
⇒ [2]:
⇒ z
⇒
⇒ [3]:
⇒ z
⇒
def(ll[2]) = x;      // generic alternative
ll;
⇒ [1]:
⇒ y
⇒ [2]:
⇒ x
⇒
⇒ [3]:
⇒ x
⇒

```

In particular, explicit dereferencing is useful to distinguish between typecasting and nested constructions.

```

system("reference"); system("shared");
shared shl = list(1);
shl;
↳ [1]:
↳ 1
↳
list(shl); // wraps 'shl' by a list
↳ [1]:
↳ [1]:
↳ 1
↳
link(shl); // extract the list in 'shl'
↳ [1]:
↳ 1

```

#### 4.29.6 reference and shared related functions

**def** explicitly type casts to **reference** or **shared**, respectively. (Note: For the **def** declaration, see [Section 4.5 \[def\]](#), page 79.)

**Example:**

```

system("reference"); system("shared");
int i = 1;
reference ref = i;
shared sh = 17;
list ll = list(ref, ref, ref, sh, sh);
ll[1] = 2; // replace only one entry
ll;
↳ [1]:
↳ 2
↳ [2]:
↳ 1
↳
↳ [3]:
↳ 1
↳
↳ [4]:
↳ 17
↳
↳ [5]:
↳ 17
↳
def(ll[2]) = 3; // change the others
ll;
↳ [1]:
↳ 2
↳ [2]:
↳ 3
↳
↳ [3]:

```

```

↳      3
↳
↳ [4]:
↳      17
↳
↳ [5]:
↳      17
↳
def(l1[4]) = 19;      // same here
l1;
↳ [1]:
↳      2
↳ [2]:
↳      3
↳
↳ [3]:
↳      3
↳
↳ [4]:
↳      19
↳
↳ [5]:
↳      19
↳

```

**link** explicitly dereference a **reference** or **shared** object. (Note: For the **link** declaration, see [Section 4.10 \[link\]](#), page 96.)

**Example:**

```

system("reference"); system("shared");
ring r = 0, (x,y,z), dp;
poly p = x + y + z;
def x_=x;
reference xref=x_;
xref;
↳ x
↳
subst(p, xref,1, y,2, z,3);          // fails
↳ ? subst('poly','reference','int') failed
↳ ? expected subst('poly','poly','poly')
↳ ? expected subst('matrix','poly','int')
↳ ? error occurred in or before ./examples/reference_and_shared_related
    functions_1.sing line 7: 'subst(p, xref,1, y,2, z,3);          // fails'
subst(p, link(xref),1, y,2, z,3); // fine
↳ 6

```

**system** The **reference** and **shared** objects overload the **system** command to gain extended features, see **system(ref, "help")** for more details. (Note: For the general **system** command, see [Section 5.1.155 \[system\]](#), page 275.)

**Example:**

```

system("reference"); system("shared");
shared sh;

```



```
system(sh, "help");
```

⇒ `system(<ref>, ...)`: extended functionality for reference/shared data <ref>

⇒ `system(<ref>, count)` - number of references pointing to <ref>

⇒ `system(<ref>, enumerate)` - unique number for identifying <ref>

⇒ `system(<ref>, undefined)` - checks whether <ref> had been assigned

⇒ `system(<ref>, "help")` - prints this information message

⇒ `system(<ref>, "typeof")` - actual type referenced by <ref>

⇒ `system(<ref1>, same, <ref2>)` - tests for identic reference objects

## 5 Functions and system variables

### 5.1 Functions

This section gives a complete reference of all functions, commands and special variables of the SINGULAR kernel (i.e., all built-in commands). See [Section D.1 \[standard\\_lib\], page 794](#), for those functions from the `standard.lib` (this library is automatically loaded at start-up time) which extend the functionality of the kernel and are written in the SINGULAR programming language.

The general syntax of a function is

[target =] function\_name (<arguments>);

If no target is specified, the result is printed. In some cases (e.g., `export`, `keepring`, `setring`, `type`) the brackets are optional. For the commands `kill`, `help`, `break`, `quit`, `exit` and `LIB` no brackets are allowed.

#### 5.1.1 align

**Syntax:**     `align ( vector_expression, int_expression )`  
               `align ( module_expression, int_expression )`

**Type:**       type of the first argument

**Purpose:**     maps module generators `gen(i)` to `gen(i+s)` for all `i`.

**Example:**

```
ring r=0,(x,y,z),(c,dp);
align([1,2,3],3);
↪ [0,0,0,1,2,3]
align([0,0,1,2,3],-1);
↪ [0,1,2,3]
align(freemodule(2),1);
↪ _[1]=[0,1]
↪ _[2]=[0,0,1]
```

#### 5.1.2 attrib

**Syntax:**     `attrib ( name )`

**Type:**       none

**Purpose:**     displays the attribute list of the object called `name`.

**Example:**

```
ring r=0,(x,y,z),dp;
ideal I=std(maxideal(2));
attrib(I);
↪ attr:isSB, type int
```

**Syntax:**     `attrib ( name , string_expression )`

**Type:**       any

**Purpose:**     returns the value of the attribute `string_expression` of the variable `name`. If the attribute is not defined for this variable, `attrib` returns the empty string.

**Example:**

```

ring r=0,(x,y,z),dp;
ideal I=std(maxideal(2));
attrib(I,"isSB");
↪ 1
// maxideal(2) is a standard basis,
// SINGULAR does know it for maxideal:
attrib(maxideal(2), "isSB");
↪ 1

```

**Syntax:** `attrib ( name, string-expression, expression )`

**Type:** none

**Purpose:** sets the attribute string-expression of the variable name to the value expression.

**Example:**

```

ring r=0,(x,y,z),dp;
ideal I=maxideal(2); // the attribute "isSB" is not set
vdim(I);
↪ 4
attrib(I,"isSB",0); // the standard basis attribute is reset here
vdim(I);
↪ // ** I is no standard basis
↪ 4

```

**Remark:** An attribute may be described by any string-expression. Some of these are used by the kernel of SINGULAR and referred to as reserved attributes. Non-reserved attributes may be used, however, in procedures and can considerably speed up computations.

#### Reserved attributes:

(`cf_class`, `global`, `isSB`, `isHomog`, `rank`, `ring_cf`, `rowShift` are used by the kernel, the other are used by libraries)

`cf_class` (for ring)

the internal type of the coefficients (see `n_coeffType`)

`global` (for ring)

1, if the ordering is global

`isSB` (for ideal, module)

the standard basis property is set by all commands computing a standard basis like `groebner`, `std`, `stdhilb` etc.; used by `lift`, `dim`, `degree`, `mult`, `hilb`, `vdim`, `kbase`

`isHomog` (for ideal, module)

the weight vector of module generators for homogeneous or quasihomogeneous ideals/modules, used by `betti`, `degree`, `highcorner`, `hilbert`, `homog`, `prune`, `prune_map`, `sba`, `slimgb`, `std`, `syz`, `kbase`, `modulo`, `mres`, `mres_map`, `nres`, `stdhilb`.

`isCI` complete intersection property

`isCM` Cohen-Macaulay property

`maxExp` (for ring/list from ringlist)

limit for each exponent (32767 by default)

`rank` (for module)

set/get the rank of a module (see [Section 5.1.107 \[nrows\]](#), page 232)

`ring_cf` (for `ring`)  
     the coefficients of the polynomial ring are considered to be a ring  
`withSB`     value of type `ideal`, resp. `module`, is `std`  
`withHilb`   value of type `intvec` is `hilb(.,1)` (see [Section 5.1.56 \[hilb\]](#), page 195)  
`withRes`     value of type `list` is a free resolution  
`withDim`     value of type `int` is the dimension (see [Section 5.1.25 \[dim\]](#), page 174)  
`withMult`    value of type `int` is the multiplicity (see [Section 5.1.101 \[mult\]](#), page 228)

See [Section 5.1.72 \[killattrib\]](#), page 207.

### 5.1.3 bareiss

qcindex Gauss

**Syntax:**   `bareiss ( module_expression )`  
           `bareiss ( matrix_expression )`  
           `bareiss ( module_expression, int_expression, int_expression )`  
           `bareiss ( matrix_expression, int_expression, int_expression )`

**Type:**     list of `module` and `intvec`

**Purpose:**   applies the sparse Gauss-Bareiss algorithm (see [Section C.9 \[References\]](#), page 792, Lee and Saunders) to a `module` (or with type conversion to a `matrix`) with an 'optimal' pivot strategy. The vectors of the `module` are the columns of the `matrix`, hence elimination takes place w.r.t. rows.

With only one parameter a complete elimination is done. Result is a list: the first entry is a `module` with a minimal independent set of vectors (as a `matrix` lower triangular), the second entry an `intvec` with the permutation of the rows w.r.t. the original `matrix`, that is, a `k` at position `l` indicates that row `k` was carried over to the row `l`.

The further parameters control the algorithm. `bareiss(M,i,j)` does not attempt to diagonalize the last `i` rows in the elimination procedure and stops computing when the remaining number of vectors (columns) to reduce is at most `j`.

**Example:**

```

ring r=0,(x,y,z),(c,dp);
module mm;
// ** generation of the module mm **
int d=7;
int b=2;
int db=d-b;
int i;
for(i=d;i>0;i--){ mm[i]=3*x*gen(i); }
for(i=db;i;i--){ mm[i]=mm[i]+7*y*gen(i+b); }
for(i=d;i>db;i--){ mm[i]=mm[i]+7*y*gen(i-db); }
for(i=d;i>b;i--){ mm[i]=mm[i]+11*z*gen(i-b); }
for(i=b;i;i--){ mm[i]=mm[i]+11*z*gen(i+db); }
// ** the generating matrix of mm **
print(mm);
↪ 3x, 0, 11z, 0, 0, 7y, 0,
↪ 0, 3x, 0, 11z, 0, 0, 7y,
↪ 7y, 0, 3x, 0, 11z, 0, 0,
↪ 0, 7y, 0, 3x, 0, 11z, 0,

```

```

⇒ 0, 0, 7y, 0, 3x, 0, 11z,
⇒ 11z, 0, 0, 7y, 0, 3x, 0,
⇒ 0, 11z, 0, 0, 7y, 0, 3x
// complete elimination
list ss=bareiss(mm);
print(ss[1]);
⇒ 7y, 0, 0, 0, 0, 0, 0,
⇒ 3x, -33xz, 0, 0, 0, 0, 0,
⇒ 11z, -121z2, 1331z3, 0, 0, 0, 0,
⇒ 0, 0, 0, 9317yz3, 0, 0, 0,
⇒ 0, 21xy, _[5,3], 14641z4, -43923xz4, 0, 0,
⇒ 0, 0, 0, 0, 65219y2z3, _[6,6], 0,
⇒ 0, 49y2, _[7,3], 3993xz3, _[7,5], _[7,6], _[7,7]
ss[2];
⇒ 2,7,5,1,4,3,6
// elimination up to 3 vectors
ss=bareiss(mm,0,3);
print(ss[1]);
⇒ 7y, 0, 0, 0, 0, 0, 0,
⇒ 3x, -33xz, 0, 0, 0, 0, 0,
⇒ 11z, -121z2, 1331z3, 0, 0, 0, 0,
⇒ 0, 0, 0, 9317yz3, 0, 0, 0,
⇒ 0, 0, 0, 0, 27951xyz3, 102487yz4, 65219y2z3,
⇒ 0, 21xy, _[6,3], 14641z4, _[6,5], _[6,6], -43923xz4,
⇒ 0, 49y2, _[7,3], 3993xz3, _[7,5], _[7,6], _[7,7]
ss[2];
⇒ 2,7,5,1,3,4,6
// elimination without the last 3 rows
ss=bareiss(mm,3,0);
print(ss[1]);
⇒ 7y, 0, 0, 0, 0, 0, 0,
⇒ 0, 77yz, 0, 0, 0, 0, 0,
⇒ 0, 0, 231xyz, 0, 0, 0, 0,
⇒ 0, 0, 0, 1617xy2z, 0, 0, 0,
⇒ 11z, 21xy, -1331z3, 14641z4, _[5,5], _[5,6], _[5,7],
⇒ 0, 0, 539y2z, _[6,4], _[6,5], _[6,6], -3773y3z,
⇒ 3x, 49y2, -363xz2, 3993xz3, _[7,5], _[7,6], _[7,7]
ss[2];
⇒ 2,3,4,1

```

See [Section 5.1.23 \[det\]](#), page 172; [Section 4.13 \[matrix\]](#), page 108.

### 5.1.4 betti

**Syntax:** `betti ( list_expression )`  
`betti ( resolution_expression )`  
`betti ( list_expression , int_expression )`  
`betti ( resolution_expression , int_expression )`

**Type:** `intmat`

**Purpose:** with 1 argument: computes the graded Betti numbers of a minimal resolution of  $R^n/M$ , if  $R$  denotes the basering,  $M$  is a homogeneous submodule of  $R^n$  and the argument represents a resolution of  $R^n/M$ .

The entry  $d$  of the `intmat` at place  $(i,j)$  is the minimal number of generators in degree  $i+j$  of the  $j$ -th syzygy module (= module of relations) of  $R^n/M$ , i.e. the 0th (resp. 1st) syzygy module of  $R^n/M$  is  $R^n$  (resp.  $M$ ). The argument is considered to be the result of a `res/fres/sres/mres/mres_map/nres/lres` command. This implies that a zero is only allowed (and counted) as a generator in the first module.

For the computation `betti` uses only the initial monomials. This could lead to confusing results for a non-homogeneous input.

If the optional second argument is non-zero, the Betti numbers will be minimized.

`betti` sets the attribute `rowShift`.

### Example:

```

ring r=32003,(a,b,c,d),dp;
ideal j=bc-ad,b3-a2c,c3-bd2,ac2-b2d;
list T=mres(j,0); // 0 forces a full resolution
// a minimal set of generators for j:
print(T[1]);
↪ bc-ad,
↪ c3-bd2,
↪ ac2-b2d,
↪ b3-a2c
// second syzygy module of r/j which is the first
// syzygy module of j (minimal generating set):
print(T[2]);
↪ bd,c2,ac,b2,
↪ -a,-b,0, 0,
↪ c, d, -b,-a,
↪ 0, 0, -d,-c
// the second syzygy module (minimal generating set):
print(T[3]);
↪ -b,
↪ a,
↪ -c,
↪ d
print(T[4]);
↪ 0
betti(T);
↪ 1,0,0,0,
↪ 0,1,0,0,
↪ 0,3,4,1
// most useful for reading off the graded Betti numbers:
print(betti(T),"beti");
↪          0      1      2      3
↪ -----
↪    0:      1      -      -      -
↪    1:      -      1      -      -
↪    2:      -      3      4      1
↪ -----
↪ total:      1      4      4      1
↪

```

Hence,

- the 0th syzygy module of  $r/j$  (which is  $r$ ) has 1 generator in degree 0 (which is 1),

- the 1st syzygy module  $T[1]$  (which is  $j$ ) has 4 generators (one in degree 2 and three in degree 3),
- the 2nd syzygy module  $T[2]$  has 4 generators (all in degree 4),
- the 3rd syzygy module  $T[3]$  has 1 generator in degree 5,

where the generators are the columns of the displayed matrix and degrees are assigned such that the corresponding maps have degree 0:

$$0 \longleftarrow r/j \longleftarrow r(1) \xleftarrow{T[1]} r(2) \oplus r^3(3) \xleftarrow{T[2]} r^4(4) \xleftarrow{T[3]} r(5) \longleftarrow 0 \quad .$$

See [Section C.3 \[Syzygies and resolutions\]](#), page 776; [Section 5.1.48 \[fres\]](#), page 188; [Section 5.1.58 \[hres\]](#), page 197; [Section 5.1.83 \[lres\]](#), page 215; [Section 5.1.98 \[mres\]](#), page 225; [Section 5.1.99 \[mres\\_map\]](#), page 226; [Section 5.1.120 \[print\]](#), page 242; [Section 5.1.134 \[res\]](#), page 253; [Section 4.19 \[resolution\]](#), page 125; [Section 5.1.149 \[sres\]](#), page 269.

### 5.1.5 char

**Syntax:** `char ( ring_name )`

**Type:** `int`

**Purpose:** returns the characteristic of the coefficient field of a ring.

**Example:**

```

ring r=32003,(x,y),dp;
char(r);
↪ 32003
ring s=0,(x,y),dp;
char(s);
↪ 0
ring ra=(7,a),(x,y),dp;
minpoly=a^3+a+1;
char(ra);
↪ 7
ring rp=(49,a),(x,y),dp;
char(rp);
↪ 7
ring rr=real,x,dp;
char(rr);
↪ 0

```

See [Section 5.1.7 \[charstr\]](#), page 162; [Section 4.20 \[ring\]](#), page 127.

### 5.1.6 char\_series

**Syntax:** `char_series ( ideal_expression )`

**Type:** `matrix`

**Purpose:** the rows of the matrix represent the irreducible characteristic series of the ideal with respect to the current ordering of variables.  
One application is the decomposition of the zero-set.

**Example:**

```

ring r=32003,(x,y,z),dp;
print(char_series(ideal(xyz,xz,y)));
↦ y,z,
↦ x,y

```

See [Section C.4 \[Characteristic sets\]](#), page 777.

### 5.1.7 charstr

**Syntax:** `charstr ( ring_name )`

**Type:** string

**Purpose:** returns the description of the coefficient field of a ring. (Tests for certain types of coefficients should use the routines from `ring.lib` as the string representation may change.)

**Example:**

```

ring r=32003,(x,y),dp;
charstr(r);
↦ ZZ/32003
ring s=0,(x,y),dp;
charstr(s);
↦ QQ
ring ra=(7,a),(x,y),dp;
minpoly=a^3+a+1;
charstr(ra);
↦ 7,a
ring rp=(49,a),(x,y),dp;
charstr(rp);
↦ 49,a
ring rr=real,x,dp;
charstr(rr);
↦ Float()

```

See [Section 5.1.5 \[char\]](#), page 161; [Section 5.1.113 \[ordstr\]](#), page 239; [Section 4.20 \[ring\]](#), page 127; [Section D.2.12 \[ring\\_lib\]](#), page 943; [Section 5.1.167 \[varstr\]](#), page 285.

### 5.1.8 chinrem

**Syntax:** `chinrem ( list, intvec )`  
`chinrem ( list, list )`  
`chinrem ( intvec, intvec )`

**Type:** the same type as the elements of the first argument  
 If the elements of the first argument are lists again, `chinrem` is applied recursively.

**Purpose:** applies chinese remainder theorem to the first argument w.r.t. the moduli given in the second. The elements in the first list must be of same type which can be `bigint/int`, `poly`, `ideal`, `module`, `smatrix` or `matrix`. The moduli, if given by a list, must be of type `bigint` or `int`.

If data depending on a ring are involved, the coefficient field must be  $\mathbb{Q}$ .

**Example:**



```

    chinrem(intvec(2,-3),intvec(7,11));
↳ 30
    chinrem(list(2,-3),list(7,11));
↳ 30
    ring r=0,(x,y),dp;
    ideal i1=5x+2y,x2+3y2+xy;
    ideal i2=2x-3y,2x2+4y2+5xy;
    chinrem(list(i1,i2),intvec(7,11));
↳ _[1]=-9x+30y
↳ _[2]=-20x2-6xy-18y2
    chinrem(list(i1,i2),list(bigint(7),bigint(11)));
↳ _[1]=-9x+30y
↳ _[2]=-20x2-6xy-18y2
    chinrem(list(list(i1,i2),list(i1,i2)),list(bigint(7),bigint(11)));
↳ [1]:
↳   _[1]=-9x+30y
↳   _[2]=-20x2-6xy-18y2
↳ [2]:
↳   _[1]=-9x+30y
↳   _[2]=-20x2-6xy-18y2

```

See [Section D.4.16 \[modstd\\_lib\]](#), page 1147.

### 5.1.9 cleardenom

**Syntax:**    `cleardenom ( poly_expression )`  
              `cleardenom ( vector_expression )`

**Type:**     same as the input type

**Purpose:**   multiplies a polynomial, resp. vector, by a suitable constant to cancel all denominators from its coefficients and then divide it by its content.

**Example:**

```

    ring r=0,(x,y,z),dp;
    poly f=(3x+6y)^5;
    f/5;
↳ 243/5x5+486x4y+1944x3y2+3888x2y3+3888xy4+7776/5y5
    cleardenom(f/5);
↳ x5+10x4y+40x3y2+80x2y3+80xy4+32y5
    vector w= [4x2+20,6x+2,0,8]; // application to a vector
    print(cleardenom(w));
↳ [2x2+10,3x+1,0,4]

```

See [Section D.2.8.14 \[content\]](#), page 895.

### 5.1.10 close

**Syntax:**    `close ( link_expression )`

**Type:**     none

**Purpose:**    closes a link.

**Example:**

```
link l="ssi:tcp localhost:"+system("Singular");
open(l); // start SINGULAR "server" on localhost in batchmode
close(l); // shut down SINGULAR server
```

See [Section 4.10 \[link\], page 96](#); [Section 5.1.110 \[open\], page 234](#).

### 5.1.11 coef

**Syntax:**    `coef ( poly_expression, product_of_ringvars )`  
               `coef ( ideal_expression, product_of_ringvars )`

**Type:**     matrix

**Syntax:**    `coef ( vector_expression, product_of_ringvars, matrix_name, matrix_name )`

**Type:**     none

**Purpose:**    determines the monomials in  $f$  divisible by a ring variable of  $m$  (where  $f$  is the first argument and  $m$  the second argument) and the coefficients of these monomials as polynomials in the remaining variables. First case: returns a  $2 \times n$  matrix  $M$ ,  $n$  being the number of the determined monomials. The first row consists of these monomials, the second row of the corresponding coefficients of the monomials in  $f$ . Thus,  $f = M[1,1] \cdot M[2,1] + \dots + M[1,n] \cdot M[2,n]$ .

Second case: apply to all generators of the ideal and combine the results into one matrix.

Third case: the second matrix (i.e., the 4th argument) contains the monomials, the first matrix (i.e., the 3rd argument) the corresponding coefficients of the monomials in the vector.

**Note:**      `coef` considers only monomials which really occur in  $f$  (i.e., which are not 0), while `coeffs` (see [Section 5.1.12 \[coeffs\], page 165](#)) returns the coefficient 0 at the appropriate place if a monomial is not present.

**Example:**

```
ring r=32003,(x,y,z),dp;
poly f=x5+5x4y+10x2y3+y5;
matrix m=coef(f,y);
print(m);
↪ y5,y3, y, 1,
↪ 1, 10x2,5x4,x5
f=x20+xyz+xy+x2y+z3;
print(coef(f,xy));
↪ x20,x2y,xy, 1,
↪ 1, 1, z+1,z3
print(coef(maxideal(3),yz));
↪ y3,y2z,yz2,z3,y2,yz,z2,y, z, 1,
↪ 0, 0, 0, 1, 0, 0, 0, 0, 0, 0,
↪ 0, 0, 1, 0, 0, 0, 0, 0, 0, 0,
↪ 0, 1, 0, 0, 0, 0, 0, 0, 0, 0,
↪ 1, 0, 0, 0, 0, 0, 0, 0, 0, 0,
↪ 0, 0, 0, 0, 0, 0, x, 0, 0, 0,
↪ 0, 0, 0, 0, 0, x, 0, 0, 0, 0,
↪ 0, 0, 0, 0, x, 0, 0, 0, 0, 0,
↪ 0, 0, 0, 0, 0, 0, 0, 0, x2,0,
↪ 0, 0, 0, 0, 0, 0, 0, x2,0, 0,
```

```

↳ 0, 0, 0, 0, 0, 0, 0, 0, 0, x3
vector v=[f,zy+77+xy];
print(v);
↳ [x20+x2y+xyz+z3+xy,xy+yz+77]
matrix mc; matrix mm;
coef(v,y,mc,mm);
print(mc);
↳ x2+xz+x,x20+z3,
↳ x+z, 77
print(mm);
↳ y,1,
↳ y,1

```

See [Section 5.1.12 \[coeffs\]](#), page 165.

### 5.1.12 coeffs

**Syntax:** `coeffs ( poly_expression , ring_variable )`  
`coeffs ( ideal_expression, ring_variable )`  
`coeffs ( vector_expression, ring_variable )`  
`coeffs ( module_expression, ring_variable )`  
`coeffs ( poly_expression, ring_variable, matrix_name )`  
`coeffs ( ideal_expression, ring_variable, matrix_name )`  
`coeffs ( vector_expression, ring_variable, matrix_name )`  
`coeffs ( module_expression, ring_variable, matrix_name )`

**Type:** matrix

**Syntax:** `coeffs ( ring_expression )`

**Type:** cring

**Purpose:** develops each polynomial of the first argument J as a univariate polynomial in the given ring\_variable z, and returns the coefficients as a matrix M.

With e denoting the maximal z-degree occurring in the polynomials of J, and d:=e+1,  $M = (m_{ij})$  satisfies the following conditions:

- (i) If J is a single polynomial f, then M is a  $(d \times 1)$ -matrix and  $m_{i+1,j}, 0 \leq i \leq e$ , is the coefficient of  $z^i$  in f.
- (ii) If J is an ideal with generators  $f_1, f_2, \dots, f_k$  then M is a  $(d \times k)$ -matrix and  $m_{i+1,j}, 0 \leq i \leq e, 1 \leq j \leq k$ , is the coefficient of  $z^i$  in  $f_j$ .
- (iii) If J is a k-dimensional vector with entries  $f_1, f_2, \dots, f_k$  then M is a  $(dk \times 1)$ -matrix and  $m_{(j-1)d+i+1,1}, 0 \leq i \leq e, 1 \leq j \leq k$ , is the coefficient of  $z^i$  in  $f_j$ .
- (iV) If J is a module generated by s vectors  $v_1, v_2, \dots, v_s$  of dimension k then M is a  $(dk \times s)$ -matrix and  $m_{(j-1)d+i+1,r}, 0 \leq i \leq e, 1 \leq j \leq k, 1 \leq r \leq s$ , is the coefficient of  $z^i$  in the j-th entry of  $v_r$ .

The optional third argument T can be used to return the matrix of powers of z such that  $\text{matrix}(J) = T * M$  holds in each of the previous four cases.

**Note:** `coeffs` returns the coefficient 0 at the appropriate matrix entry if a monomial is not present, while `coef` considers only monomials which actually occur in the given expression.

**Example:**

```

ring r;
poly f = (x+y)^3;
poly g = xyz+z10y4;
ideal i = f, g;
matrix M = coeffs(i, y);
print(M);
↪ x3, 0,
↪ 3x2,xz,
↪ 3x, 0,
↪ 1, 0,
↪ 0, z10
vector v = [f, g];
M = coeffs(v, y);
print(M);
↪ x3,
↪ 3x2,
↪ 3x,
↪ 1,
↪ 0,
↪ 0,
↪ xz,
↪ 0,
↪ 0,
↪ z10

```

**Syntax:** `coeffs ( ideal_expression, ideal_expression )`  
`coeffs ( module_expression, module_expression )`  
`coeffs ( ideal_expression, ideal_expression, product_of_ringvars )`  
`coeffs ( module_expression, module_expression, product_of_ringvars )`

**Type:** matrix

**Purpose:** expresses each polynomial of the first argument M as a sum  $\sum_{i=1}^k m_i \cdot a_i \cdot x^{e_i}$ , where the  $m_i$  come from a specified set of monomials, the  $a_i$  are from the underlying coefficient ring (or field), and the  $x^{e_i}$  are powers of a specified ring variable x.

The second parameter K provides the set of monomials which should be sufficient to generate all entries of M.

Both M and K can be thought of as the matrices obtained by `matrix(M)` and `matrix(K)`, respectively. (If M and K are given by ideals, then this matrix has just one row.)

The optional parameter `product_of_ringvars` determines the variable x: It is expected to be either the product of all ring variables (then x is 1, and each polynomial will be expressed as  $\sum_{i=1}^k m_i \cdot a_i$ , or `product_of_ringvars` is the product of all ring variables except one variable (which then determines x). If `product_of_ringvars` is omitted then  $x = 1$  as default.

If K contains all monomials that are necessary to express the entries of M, then the returned matrix A satisfies  $K \cdot A = M$ . Otherwise only a subset of entries of  $K \cdot A$  and M will coincide. In this case, the valid entries start at `M[1,1]` and run from left to right, top to bottom.

**Note:** Note that in general not all entries of  $K \cdot A$  and M will coincide, depending on the set of monomials provided by K.

**Example:**

```
ring r=32003,(x,y,z),dp;
```

```

module M = [y3+x2z, xy], [-xy, y2+x2z];
print(M);
↦ y3+x2z,-xy,
↦ xy,      x2z+y2
module K = [x2, xy], [y3, xy], [xy, x];
print(K);
↦ x2,y3,xy,
↦ xy,xy,x
matrix A = coeffs(M, K, xy); // leaving z as variable of interest
print(A); // attention: only the first row of M is reproduced by K*A
↦ z,0,
↦ 1,0,
↦ 0,-1

```

**Syntax:** `coeffs ( ring_expression )`

**Type:** `cring`

**Purpose:** return the coefficient ring of the argument

**Example:**

```

ring R=QQ,x,dp;
coeffs(R);
↦ QQ

```

See [Section 5.1.11 \[coef\]](#), page 164; [Section 5.1.69 \[kbase\]](#), page 205.

### 5.1.13 contract

**Syntax:** `contract ( ideal_expression, ideal_expression )`

**Type:** `matrix`

**Purpose:** contracts each of the  $n$  elements of the second ideal  $J$  by each of the  $m$  elements of the first ideal  $I$ , producing an  $m \times n$  matrix.  
Contraction is defined on monomials by:

$$\text{contract}(x^A, x^B) := \begin{cases} x^{(B-A)}, & \text{if } B \geq A \text{ componentwise} \\ 0, & \text{otherwise.} \end{cases}$$

where  $A$  and  $B$  are the multiexponents of the ring variables represented by  $x$ . `contract` is extended bilinearly to all polynomials.

**Example:**

```

ring r=0,(a,b,c,d),dp;
ideal I=a2,a2+bc,abc;
ideal J=a2-bc,abcd;
print(contract(I,J));
↦ 1,0,
↦ 0,ad,
↦ 0,d

```

See [Section 5.1.24 \[diff\]](#), page 173.

### 5.1.14 create\_ring

Procedure from library `standard.lib` (see [Section D.1 \[standard.lib\]](#), page 794).

**Usage:**     `create_ring(l1, l2, l3[, l4, "no_minpoly"])`;  
               l1 int or list, l2 list or string, l3 list or string, l4 ideal

**Return:**    `ring(list(l1, l2, l3, l4))`

**Note:**     l1, l2, l3, l4 are assumed to be the four entries of `ring_list(R)` where R is the ring to be returned.

Optional arguments: If l4 is not given, it is assumed to be `ideal(0)`. If "no\_minpoly" is given, then the minimal polynomial in l1, if present, is set to 0.

Shortcuts: Strings such as "0", "(32003)" or "(0,a,b,c)" can be given as l1. Indexed parameters as in "(0,a(1..3))" are not supported. Strings such as "(x,y,z)" can be given as l2. Indexed variables as in "(x(1..3),y,z)" are not supported. Strings representing orderings such as "dp" or "(lp(3), ds(2))" can be given as l3, except matrix orderings given by

"M([intmat\_expression])".

**Example:**

```

ring R = (0,a), x, lp;
ring_list(R);
⇒ [1]:
⇒ 0,a
⇒ [2]:
⇒ [1]:
⇒ x
⇒ [3]:
⇒ [1]:
⇒ [1]:
⇒ lp
⇒ [2]:
⇒ 1
⇒ [2]:
⇒ [1]:
⇒ C
⇒ [2]:
⇒ 0
⇒ [4]:
⇒ _[1]=0
minpoly = a^2+1;
qring Q = ideal(x^3-2);
ring S = create_ring(ring_list(Q)[1], "(x,y,t)", "dp", "no_minpoly");
basing;
⇒ // coefficients: QQ[a]/(a2+1)
⇒ // number of vars : 3
⇒ //      block  1 : ordering dp
⇒ //      : names  x y t
⇒ //      block  2 : ordering C

```

### 5.1.15 crossprod

**Syntax:**    `crossprod ( cring_expression, ... )`

**Type:** cring

**Purpose:** crooss product of several objects of type cring

**Example:**

```
crossprod(ZZ/32003,Float());
↪ ZZ/32003 x Float()
```

See [Section 4.1 \[cring\]](#), page 73.

### 5.1.16 datetime

Procedure from library `standard.lib` (see [Section D.1 \[standard.lib\]](#), page 794).

**Syntax:** datetime ()

**Return:** string

**Purpose:** return the current date and time as a string

**Example:**

```
datetime();
↪ Di 9. Apr 07:46:54 2024
```

### 5.1.17 dbprint

**Syntax:** dbprint ( int\_expression, expression\_list )

**Type:** none

**Purpose:** applies the print command to each expression in the expression\_list if int\_expression is positive. dbprint may also be used in procedures in order to print results subject to certain conditions.

**Syntax:** dbprint ( expression )

**Type:** none

**Purpose:** The print command is applied to the expression if `printlevel` ≥ `voice`.

**Note:** See [Section 3.8 \[Libraries\]](#), page 55, for an example how this is used for displaying comments while procedures are executed.

**Example:**

```
int debug=0;
intvec i=1,2,3;
dbprint(debug,i);
debug=1;
dbprint(debug,i);
↪ 1,
↪ 2,
↪ 3
voice;
↪ 1
printlevel;
↪ 0
dbprint(i);
```

See [Section 3.9 \[Debugging tools\]](#), page 68; [Section 5.1.120 \[print\]](#), page 242; [Section 5.3.6 \[print-level\]](#), page 304; [Section 5.3.11 \[voice\]](#), page 308.

### 5.1.18 defined

**Syntax:**     `defined ( name )`

**Type:**       `int`

**Purpose:**     returns a value  $\neq 0$  (TRUE) if there is a user-defined object with this name, and 0 (FALSE) otherwise.

A non-zero return value is the level where the object is defined (level 1 denotes the top level, level 2 the level of a first procedure, level 3 the level of a procedure called by a first procedure, etc.). For ring variables and other constants, -1 is returned.

**Note:**       A local object `m` may be identified by `if (defined(m)==voice)`.

**Example:**

```

ring r=(0,t),(x,y),dp;
matrix m[5][6]=x,y,1,2,0,x+y;
defined(mm);
↳ 0
defined(r) and defined(m);
↳ 1
defined(m)==voice;    // m is defined in the current level
↳ 1
defined(x);
↳ -1
defined(z);
↳ 0
defined("z");
↳ -1
defined(t);
↳ -1
defined(42);
↳ -1

```

See [Section 5.1.139 \[rvar\]](#), page 258; [Section 5.3.11 \[voice\]](#), page 308.

### 5.1.19 deg

**Syntax:**     `deg ( poly_expression )`  
               `deg ( vector_expression )`  
               `deg ( poly_expression , intvec_expression )`  
               `deg ( vector_expression , intvec_expression )`

**Type:**       `int`

**Purpose:**     returns the maximal (weighted) degree of the terms of a polynomial or a vector;  
`deg(0)` is -1.

The optional second argument gives the weight vector, otherwise weight 1 is used for lex orderings and block ordering, the default weights of the base ring are used for orderings consisting of one block.

**Example:**

```

ring r=0,(x,y,z),lp;
deg(0);
↳ -1
deg(x3+y4+xyz3);

```



```

↳ 5
  ring rr=7,(x,y),wp(2,3);
  poly f=x2+y3;
  deg(f);
↳ 9
  ring R=7,(x,y),ws(2,3);
  poly f=x2+y3;
  deg(f);
↳ 9
  vector v=[x2,y];
  deg(v);
↳ 4

```

See [Section 5.1.68 \[jet\]](#), page 204; [Section 5.1.112 \[ord\]](#), page 238; [Section 4.17 \[poly\]](#), page 120; [Section 4.23 \[vector\]](#), page 134.

### 5.1.20 degree

**Syntax:**    `degree ( ideal_expression )`  
               `degree ( module_expression )`

**Type:**     `string`

**Purpose:**    computes the (Krull) dimension and the multiplicity of the ideal, resp. module, generated by the leading monomials of the input and prints it. This is equal to the dimension and multiplicity of the ideal, resp. module, if the input is a standard basis with respect to a degree ordering.

**Example:**

```

ring r3=32003,(x,y,z),ds;
int a,b,c,t=11,10,3,1;
poly f=x^a+y^b+z^(3*c)+x^(c+2)*y^(c-1)+x^(c-1)*y^(c-1)*z3
      +x^(c-2)*y^c*(y2+t*x)^2;
ideal i=jacob(f);
ideal i0=std(i);
degree(i0);
↳ // dimension (local)   = 0
↳ // multiplicity = 314

```

See [Section 5.1.25 \[dim\]](#), page 174; [Section 4.6 \[ideal\]](#), page 80; [Section 5.1.101 \[mult\]](#), page 228; [Section 5.1.151 \[std\]](#), page 271; [Section 5.1.168 \[vdim\]](#), page 286.

### 5.1.21 delete

**Syntax:**    `delete ( list_expression, int_expression )`  
               `delete ( intvec_expression, int_expression )`  
               `delete ( ideal_expression, int_expression )`  
               `delete ( module_expression, int_expression )`  
               `delete ( list_expression, intvec_expression )`  
               `delete ( ideal_expression, intvec_expression )`  
               `delete ( module_expression, intvec_expression )`

**Type:**     type of the first argument

**Purpose:**    deletes the element(s) with the given index/indices from a list/intvec/ideal/module (the input is not changed).

**Example:**

```

list l="a","b","c";
list l1=delete(l,2);l1;
↳ [1]:
↳ a
↳ [2]:
↳ c
l;
↳ [1]:
↳ a
↳ [2]:
↳ b
↳ [3]:
↳ c
delete(1..5,2);
↳ 1,3,4,5
ring r=0,(x,y,z),dp;
delete(maxideal(1),1);
↳ _[1]=y
↳ _[2]=z
delete(maxideal(1),1..2);
↳ _[1]=z

```

See [Section 4.6 \[ideal\]](#), page 80; [Section 5.1.62 \[insert\]](#), page 200; [Section 4.9 \[intvec\]](#), page 93; [Section 4.11 \[list\]](#), page 103; [Section 4.14 \[module\]](#), page 112.

### 5.1.22 denominator

**Syntax:** `denominator ( number_expression )`

**Type:** number

**Purpose:** returns the denominator of a number.

**Example:**

```

ring r = 0, x, dp;
number n = 3/2;
denominator(n);
↳ 2

```

See [Section 5.1.9 \[cleardenom\]](#), page 163; [Section D.2.8.14 \[content\]](#), page 895; [Section 5.1.108 \[numerator\]](#), page 233.

### 5.1.23 det

**Syntax:** `det ( intmat_expression )`  
`det ( matrix_expression )`  
`det ( smatrix_expression )`  
`det ( matrix_expression , string_expression )`  
`det ( smatrix_expression , string_expression )`

**Type:** int, resp. poly

**Purpose:** returns the determinant of a square matrix. The applied algorithms depend on type of input or the optional second argument.

The optional second argument specifies the algorithm to use. Possible values are "Bareiss", "SBareiss", "Mu" and "Factory".

**Example:**

```
ring r=7,(x,y),wp(2,3);
matrix m[3][3]=1,2,3,4,5,6,7,8,x;
det(m);
↦ -3x-1
```

See [Section 4.8 \[intmat\]](#), page 90; [Section 4.13 \[matrix\]](#), page 108; [Section 5.1.92 \[minor\]](#), page 221.

### 5.1.24 diff

**Syntax:** diff ( poly\_expression, ring\_variable )  
diff ( vector\_expression, ring\_variable )  
diff ( ideal\_expression, ring\_variable )  
diff ( module\_expression, ring\_variable )  
diff ( matrix\_expression, ring\_variable )

**Type:** the same as the type of the first argument

**Syntax:** diff ( ideal\_expression, ideal\_expression )

**Type:** matrix

**Syntax:** diff ( number\_expression, ring\_parameter )

**Type:** number

**Purpose:** computes the partial derivative of a polynomial object by a ring variable (first forms) respectively differentiates each polynomial (1..n) of the second ideal by the differential operator corresponding to each polynomial (1..m) in the first ideal, producing an m x n matrix.

respectively if the coefficient ring is a transcendental field extension, differentiates a number (that is, a rational function) by a transcendental variable (ring parameter).

**Example:**

```
ring r=0,(x,y,z),dp;
poly f=2x3y+3z5;
diff(f,x);
↦ 6x2y
vector v=[f,y2+z];
diff(v,z);
↦ 15z4*gen(1)+gen(2)
ideal j=x2-yz,xyz;
ideal i=x2,x2+yz,xyz;
// corresponds to differential operators
// d2/dx2, d2/dx2+d2/dydz, d3/dxdydz:
print(diff(i,j));
↦ 2,0,
↦ 1,x,
↦ 0,1
// differentiation of rational functions:
ring R=(0,t),(x),dp;
number f = t^2/(1-t)^2;
diff(f,t);
↦ (-2t)/(t3-3t2+3t-1)
```

See [Section 5.1.13 \[contract\]](#), page 167; [Section 4.6 \[ideal\]](#), page 80; [Section 5.1.66 \[jacob\]](#), page 203; [Section 4.13 \[matrix\]](#), page 108; [Section 4.14 \[module\]](#), page 112; [Section 4.17 \[poly\]](#), page 120; [Section 5.1.165 \[var\]](#), page 284; [Section 4.23 \[vector\]](#), page 134.

### 5.1.25 dim

**Syntax:** `dim ( ideal_expression )`  
`dim ( module_expression )`  
`dim ( resolution_expression )`  
`dim ( ideal_expression , ideal_expression )`  
`dim ( module_expression , ideal_expression )`

**Type:** `int`

**Purpose:** computes the dimension of the ideal, resp. module, generated by the leading monomials of the given generators of the ideal, resp. module. This is also the dimension of the ideal if it is represented by a standard basis.

`dim(I,J)` is the dimension of  $I/J$ .

`dim( res )` computes the cohomological dimension of `res[1]`.

**Note:** The dimension of an ideal  $I$  means the Krull dimension of the basering modulo  $I$ . The dimension of a module is the dimension of its annihilator ideal. In the case of ideal  $(1)$ , `-1` is returned.

**Example:**

```
ring r=32003,(x,y,z),dp;
ideal I=x2-y,x3;
dim(std(I));
↪ 1
dim(std(ideal(1)));
↪ -1
```

See [Section 5.1.20 \[degree\]](#), page 171; [Section 4.6 \[ideal\]](#), page 80; [Section 5.1.101 \[mult\]](#), page 228; [Section 5.1.151 \[std\]](#), page 271; [Section 5.1.168 \[vdim\]](#), page 286.

### 5.1.26 division

**Syntax:** `division ( ideal_expression, ideal_expression )`  
`division ( module_expression, module_expression )`  
`division ( ideal_expression, ideal_expression, int_expression )`  
`division ( module_expression, module_expression, int_expression )`  
`division ( ideal_expression, ideal_expression, int_expression, intvec_expression )`  
`division ( module_expression, module_expression, int_expression, intvec_expression )`

**Type:** `list`

**Purpose:** `division` computes a division with remainder. For two ideals resp. modules  $M$  (first argument) and  $N$  (second argument), it returns a list  $T,R,U$  where  $T$  is a matrix,  $R$  is an ideal resp. a module, and  $U$  is a diagonal matrix of units such that  $\text{matrix}(M)*U=\text{matrix}(N)*T+\text{matrix}(R)$  is a standard representation for the normal form  $R$  of  $M$  with respect to a standard basis of  $N$ . `division` uses different algorithms depending on whether  $N$  is represented by a standard basis. For a polynomial basering, the matrix  $U$  is the identity matrix. A matrix  $T$  as above is also computed by `lift`.

For additional arguments  $n$  (third argument) and  $w$  (fourth argument), `division` returns a list  $T, R$  as above such that  $\text{matrix}(M) = \text{matrix}(N) * T + \text{matrix}(R)$  is a standard representation for the normal form  $R$  of  $M$  with respect to  $N$  up to weighted degree  $n$  with respect to the weight vector  $w$ . The weighted degree of  $T$  and  $R$  respect to  $w$  is at most  $n$ . If the weight vector  $w$  is not given, `division` uses the standard weight vector  $w = 1, \dots, 1$ .

**Example:**

```
ring R=0,(x,y),ds;
poly f=x5+x2y2+y5;
division(f,jacob(f)); // automatic conversion: poly -> ideal
↳ [1]:
↳   _[1,1]=1/5x
↳   _[2,1]=3/10y
↳ [2]:
↳   _[1]=-1/2y5
↳ [3]:
↳   _[1,1]=1
division(f^2,jacob(f));
↳ [1]:
↳   _[1,1]=1/20x6-9/80xy5-5/16x7y+5/8x2y6
↳   _[2,1]=1/8x2y3+1/5x5y+1/20y6-3/4x3y4-5/4x6y2-5/16xy7
↳ [2]:
↳   _[1]=0
↳ [3]:
↳   _[1,1]=1/4-25/16xy
division(ideal(f^2),jacob(f),10);
↳ // ** _ is no standard basis
↳ [1]:
↳   _[1,1]=-75/8y9
↳   _[2,1]=1/2x2y3+x5y-1/4y6-3/2x3y4+15/4xy7+375/16x2y8
↳ [2]:
↳   _[1]=x10+9/4y10
```

See [Section 4.6 \[ideal\]](#), page 80; [Section 5.1.80 \[lift\]](#), page 211; [Section 4.14 \[module\]](#), page 112; [Section 4.17.3 \[poly operations\]](#), page 121; [Section 5.1.131 \[reduce\]](#), page 251.

### 5.1.27 dump

**Syntax:** `dump ( link_expression )`

**Type:** `none`

**Purpose:** dumps (i.e., writes in a "message" or "block") the state of the SINGULAR session (i.e., all defined variables and their values) to the specified link (which must be either an ASCII or ssi link) such that a `getdump` can retrieve it later on.

**Example:**

```
ring r;
// write the whole session to the file dump.ascii
// in ASCII format
dump(":w dump.ascii");
kill r; // kill the basering
// reread the session from the file
```

```

// redefining everything which was not explicitly killed before
getdump("dump.ascii");
r;
↪ // coefficients: ZZ/32003
↪ // number of vars : 3
↪ //      block 1 : ordering dp
↪ //      : names x y z
↪ //      block 2 : ordering C

```

**Restrictions:**

For ASCII links, integer matrices contained in lists are dumped as integer list elements (and not as integer matrices), and lists of lists are dumped as one flattened list. Furthermore, links themselves are not dumped.

See [Section 5.1.52 \[getdump\], page 191](#); [Section 4.10 \[link\], page 96](#); [Section 5.1.174 \[write\], page 289](#).

**5.1.28 eliminate**

**Syntax:** `eliminate ( ideal-expression, product-of-ring-variables )`  
`eliminate ( module-expression, product-of-ring-variables )`  
`eliminate ( ideal-expression, intvec-expression )`  
`eliminate ( module-expression, intvec-expression )`  
`eliminate ( ideal-expression, product-of-ring-variables, bigintvec_hilb )`  
`eliminate ( module-expression, product-of-ring-variables, bigintvec_hilb )`

**Type:** the same as the type of the first argument

**Purpose:** eliminates variables occurring as factors/entries of the second argument from an ideal (resp. a submodule of a free module), by intersecting it (resp. each component of the submodule) with the subring not containing these variables.  
`eliminate` does not need a special ordering nor a standard basis as input.

**Note:** Since elimination is expensive, for homogeneous input it might be useful first to compute the Hilbert function of the ideal (first argument) with a fast ordering (e.g., `dp`). Then make use of it to speed up the computation: a Hilbert-driven elimination uses the `intvec` provided as the third argument.  
If the ideal (resp. module) is not homogeneous with weights 1, this `intvec` will be silently ignored.

**Example:**

```

ring r=32003,(x,y,z),dp;
ideal i=x2,xy,y5;
eliminate(i,x);
↪ _[1]=y5
ring R=0,(x,y,t,s,z),dp;
ideal i=x-t,y-t2,z-t3,s-x+y3;
eliminate(i,ts);
↪ _[1]=y2-xz
↪ _[2]=xy-z
↪ _[3]=x2-y
ideal j=x2,xy,y2;
bigintvec v=hilb(std(j),1);
eliminate(j,y,v);
↪ _[1]=x2

```

See [Section 5.1.56 \[hilb\]](#), page 195; [Section 4.6 \[ideal\]](#), page 80; [Section 4.14 \[module\]](#), page 112; [Section 5.1.151 \[std\]](#), page 271.

### 5.1.29 eval

**Syntax:** `eval ( expression )`

**Type:** none

**Purpose:** evaluates (quoted) expressions. Within a quoted expression, the quote can be "undone" by an `eval` (i.e., each `eval` "undoes" the effect of exactly one quote). Used only when receiving a quoted expression from an ssi link, with `quote` and `write` to prevent local evaluations when writing to an ssi link.

**Example:**

```
link l="ssi:w example.ssi";
ring r=0,(x,y,z),ds;
ideal i=maxideal(3);
ideal j=x7+x3,x2,z;
// compute i+j before writing, but not std
// this writes 'std(ideal(x3,...,z))'
write (l, quote(std(eval(i+j))));
option(prot);
close(l);
// now read it in again and evaluate
// read(l) forces to compute 'std(ideal(x3,...,z))'
read(l);
↪ _[1]=z
↪ _[2]=x2
↪ _[3]=xy2
↪ _[4]=y3
close(l);
```

See [Section 4.10.5 \[Ssi links\]](#), page 99; [Section 5.1.126 \[quote\]](#), page 247; [Section 5.1.174 \[write\]](#), page 289.

### 5.1.30 ERROR

**Syntax:** `ERROR ( string_expression )`

**Type:** none

**Purpose:** Immediately interrupts the current computation, returns to the top-level, and displays the argument `string_expression` as error message.

**Note:** This should be used as an emergency, resp. failure, exit within procedures.

**Example:**

```
int i=1;
proc myError() {ERROR("Need to leave now");i=2;}
myError();
↪ ? Need to leave now
↪ ? leaving ::myError (0)
i;
↪ 1
```

### 5.1.31 example

**Syntax:** `example topic ;`

**Purpose:** computes an example for `topic`. Examples are available for all SINGULAR kernel and library functions. Where available (e.g., within Emacs), use <TAB> completion for a list of all available example `topics`.

**Example:**

```
example prime;
example intvec_declarations;
```

[Section 5.1.54 \[help\], page 193](#)

### 5.1.32 execute

**Syntax:** `execute ( string_expression )`

**Type:** none

**Purpose:** executes a string containing a sequence of SINGULAR commands.

**Note:** The command `return` cannot appear in the string. `execute` should be avoided in procedures whenever possible, since it may give rise to name conflicts. Moreover, such procedures cannot be precompiled (a feature which SINGULAR will provide in the future).

**Example:**

```
ring r=32003,(x,y,z),dp;
ideal i=x+y,z3+22y;
write(":w save_i",i);
ring r0=0,(x,y,z),Dp;
string s="ideal k="+read("save_i")+";";
s;
↪ ideal k=x+y,z3+22y;
   execute(s); // define the ideal k
   k;
↪ k[1]=x+y
↪ k[2]=z3+22y
```

### 5.1.33 extgcd

**Syntax:** `extgcd ( int_expression, int_expression )`  
`extgcd ( bigint_expression, bigint_expression )`  
`extgcd ( poly_expression, poly_expression )`

**Type:** list of 3 objects of the same type as the type of the arguments

**Purpose:** computes extended gcd: the first element is the greatest common divisor of the two arguments, the second and third are factors such that if `list L=extgcd(a,b);` then `L[1]=a*L[2]+b*L[3]`.

**Note:** Polynomials must be univariate (in the same variable) to apply `extgcd`.

**Example:**



```

      extgcd(24,10);
    ↪ [1]:
    ↪      2
    ↪ [2]:
    ↪     -2
    ↪ [3]:
    ↪      5
      ring r=0,(x,y),lp;
      extgcd(x4-x6,(x2+x5)*(x2+x3));
    ↪ [1]:
    ↪     x5+x4
    ↪ [2]:
    ↪    1/2x2+1/2x+1/2
    ↪ [3]:
    ↪     1/2

```

See [Section 5.1.50 \[gcd\]](#), page 190; [Section 4.7 \[int\]](#), page 85.

### 5.1.34 facstd

**Syntax:** `facstd ( ideal_expression )`  
`facstd ( ideal_expression, ideal_expression )`

**Type:** list of ideals

**Purpose:** returns a list of ideals computed by the factorizing Groebner basis algorithm. The intersection of these ideals has the same zero-set as the input, i.e., the radical of the intersection coincides with the radical of the input ideal. In many (but not all!) cases this is already a decomposition of the radical of the ideal. (Note however that in general, no inclusion between the input and output ideals holds.) The second, optional argument gives a list of polynomials which define non-zero constraints: those ideals which contain one of the constraint polynomials are omitted from the output list. Thus the zero set of the intersection of the output ideals is contained in the zero set  $V$  of the first input ideal and contains the complement in  $V$  of the zero set of the second input ideal.

**Note:** Not implemented for baserings over real ground fields and Galois fields (that is, only implemented for ground fields for which [Section 5.1.36 \[factorize\]](#), page 180 is implemented).

**Example:**

```

      ring r=32003,(x,y,z),(c,dp);
      ideal I=xyz,x2z;
      facstd(I);
    ↪ [1]:
    ↪    _[1]=z
    ↪ [2]:
    ↪    _[1]=x
      facstd(I,x);
    ↪ [1]:
    ↪    _[1]=z

```

See [Section 4.6 \[ideal\]](#), page 80; [Section 4.20 \[ring\]](#), page 127; [Section 5.1.151 \[std\]](#), page 271.

### 5.1.35 factmodd

**Syntax:** `factmodd ( poly_expression, int_expression  
[, poly_expression, poly_expression ]  
[, int_expression, int_expression ]  
)`

**Type:** list of polys

**Purpose:** Computes a factorization of a polynomial  $h(x, y)$  in  $K[[x]][y]$  up to a certain degree in  $x$ , whenever a factorization of  $h(0, y)$  is provided or can be computed. The algorithm is based on Hensel's lemma: Let  $h(x, y)$  denote a monic polynomial in  $y$  of degree  $m + n$  with coefficients in  $K[[x]]$ . Suppose there are two monic factors  $f_0(y)$  (of degree  $n$ ) and  $g_0(y)$  of degree  $(m)$  such that  $h(0, y) = f_0(y) * g_0(y)$  and  $\langle f_0, g_0 \rangle = K[y]$ . Fix an integer  $d \geq 0$ . Then there are monic polynomials in  $y$  with coefficients in  $K[[x]]$ , namely  $f(x, y)$  of degree  $n$  and  $g(x, y)$  of degree  $m$  such that  $h(x, y) = f(x, y) * g(x, y)$  modulo  $\langle x^{(d+1)} \rangle$  (\*). The function's six arguments are  $h, d, f_0, g_0, xIndex$ , and  $yIndex$ , where  $xIndex$  and  $yIndex$  denote indices of ring variables that are to play the roles of  $x$  and  $y$  as above.  $h$  must be provided as an element of  $K[x, y]$  since all terms of  $h$  with  $x$ -degree larger than  $d$  can be ignored due to (\*). If  $f_0$  and  $g_0$  are not given, the algorithm computes the factorization of  $h(0, y)$  and is expected to find exactly two distinct factors (which may appear with multiplicities larger than 1) and uses these as  $f_0$  and  $g_0$ . If  $xIndex$  and  $yIndex$  are missing they will be expected to be 1 and 2, respectively.

**Note:** The function expects the ground ring to contain at least two variables.

**Example:**

```
ring r = 0, (x,y), dp;
poly f0 = y240; poly g0 = y102+1;
poly h = y342+14x260+7x140y110+2x120y130+y240;
int d = 260;
list L = factmodd(h, d, f0, g0); L;
⇒ [1]:
⇒ -14x260y204-4x240y224-14x260y102-7x140y212-2x120y232+14x260+7x140y110
  2x120y130+y240
⇒ [2]:
⇒ 42x260y66+8x240y86+7x140y74+2x120y94+y102+1
  // check result: next output should be zero
  reduce(h - L[1] * L[2], std(x^(d+1)));
⇒ 0
```

See [Section 5.1.36 \[factorize\]](#), page 180.

### 5.1.36 factorize

**Syntax:** `factorize ( poly_expression )  
factorize ( poly_expression, 0 )  
factorize ( poly_expression, 2 )`

**Type:** list of ideal and intvec

**Syntax:** `factorize ( poly_expression, 1 )`

**Type:** ideal

**Purpose:** computes the irreducible factors (as an ideal) of the polynomial together with or without the multiplicities (as an intvec) depending on the second argument:

0: returns factors and multiplicities, first factor is a constant.

May also be written with only one argument.

1: returns non-constant factors (no multiplicities).

2: returns non-constant factors and multiplicities.

**Note:** Not implemented for the coefficient fields real, finite fields of type  $(p^n, a)$  and  $\mathbb{Z}/m$ .

**Example:**

```

ring r=32003,(x,y,z),dp;
factorize(9*(x-1)^2*(y+z));
⇒ [1]:
⇒   _[1]=9
⇒   _[2]=y+z
⇒   _[3]=x-1
⇒ [2]:
⇒   1,1,2
factorize(9*(x-1)^2*(y+z),1);
⇒ _[1]=y+z
⇒ _[2]=x-1
factorize(9*(x-1)^2*(y+z),2);
⇒ [1]:
⇒   _[1]=y+z
⇒   _[2]=x-1
⇒ [2]:
⇒   1,2
ring rQ=0,x,dp;
poly f = x2+1;           // irreducible in Q[x]
factorize(f);
⇒ [1]:
⇒   _[1]=1
⇒   _[2]=x2+1
⇒ [2]:
⇒   1,1
ring rQi = (0,i),x,dp;
minpoly = i2+1;
poly f = x2+1;           // splits into linear factors in Q(i)[x]
factorize(f);
⇒ [1]:
⇒   _[1]=1
⇒   _[2]=x+(-i)
⇒   _[3]=x+(i)
⇒ [2]:
⇒   1,1,1

```

See [Section D.4.1.1 \[absFactorize\]](#), page 1002; [Section 4.17 \[poly\]](#), page 120.

### 5.1.37 farey

- Syntax:**    `farey ( bigint_expression , bigint_expression )`  
               `farey ( ideal_expression , bigint_expression )`  
               `farey ( module_expression , bigint_expression )`  
               `farey ( matrix_expression , bigint_expression )`  
               `farey ( smatrix_expression , bigint_expression )`  
               `farey ( list_expression , bigint_expression )`
- Type:**     type of the first argument (unless it is `list`)
- Purpose:**    lift the first argument modulo the second to the rationals.  
               The (coefficients of the) result  $a/b$  is the best approximation under the condition  $|a|, |b| \leq \sqrt{(N-1)/2}$  `farey(list(a,b,...),B)` is equivalent to `list(farey(a,B),farey(b,B),...)`.
- Note:**     The current coefficient field must be the rationals.
- Example:**

```
ring r=0,x,dp;
farey(2,32003);
↦ 2
```

See [Section 5.1.8 \[chinrem\]](#), page 162.

### 5.1.38 fetch

- Syntax:**    `fetch ( ring_name, name )`  
               `fetch ( ring_name, name, intvec_expression )`  
               `fetch ( ring_name, name, intvec_expression, intvec_expression )`
- Type:**     number, poly, vector, ideal, module, matrix or list (the same type as the second argument)
- Purpose:**    maps objects between rings. `fetch` is the identity map between rings and q rings, in the first case the  $i$ -th variable of the source ring is mapped to the  $i$ -th variable of the basering. If the basering has less variables than the source ring these variables are mapped to zero. In the 3rd and 4th arguments the `intvec` describes the permutation of the variables: an  $i$  at position  $j$  maps the variable `var(j)` of the source to the variable `var(i)` of the destination. Negative numbers (and the fourth argument) describe mapping of parameters.  
               A zero means that that variable/parameter is mapped to 0.  
               The coefficient fields must be compatible. (See [Section 4.12 \[map\]](#), page 106 for a description of possible mappings between different ground fields).  
               `fetch` offers a convenient way to change variable names or orderings, or to map objects from a ring to a quotient ring of that ring or vice versa.  
               `option(Imap)`; reports the mapping.
- Note:**     Compared with `imap`, `fetch` uses the position of the ring variables, not their names.
- Example:**

```
ring r=0,(x,y,z),dp;
ideal i=maxideal(2);
ideal j=std(i);
poly f=x+y2+z3;
vector v=[f,1];
qring q=j;
poly f=fetch(r,f);
```

```

f;
↪ z3+y2+x
vector v=fetch(r,v);
v;
↪ z3*gen(1)+y2*gen(1)+x*gen(1)+gen(2)
ideal i=fetch(r,i);
i;
↪ i[1]=z2
↪ i[2]=yz
↪ i[3]=y2
↪ i[4]=xz
↪ i[5]=xy
↪ i[6]=x2
ring rr=0,(a,b,c),lp;
poly f=fetch(q,f);
f;
↪ a+b2+c3
vector v=fetch(r,v);
v;
↪ a*gen(1)+b2*gen(1)+c3*gen(1)+gen(2)
ideal k=fetch(q,i);
k;
↪ k[1]=c2
↪ k[2]=bc
↪ k[3]=b2
↪ k[4]=ac
↪ k[5]=ab
↪ k[6]=a2
fetch(q,i,1..nvars(q)); // equivalent to fetch(q,i)
↪ _[1]=c2
↪ _[2]=bc
↪ _[3]=b2
↪ _[4]=ac
↪ _[5]=ab
↪ _[6]=a2

```

See [Section 5.1.59 \[imap\], page 198](#); [Section 4.12 \[map\], page 106](#); [Section 5.1.111 \[option\], page 234](#); [Section 4.20.1 \[qring\], page 127](#); [Section 4.20 \[ring\], page 127](#).

### 5.1.39 fglm

**Syntax:** fglm ( ring\_name, ideal\_name )

**Type:** ideal

**Purpose:** computes for the given ideal in the given ring a reduced Groebner basis in the current ring, by applying the so-called FGLM (Faugere, Gianni, Lazard, Mora) algorithm. The main application is to compute a lexicographical Groebner basis from a reduced Groebner basis with respect to a degree ordering. This can be much faster than computing a lexicographical Groebner basis directly.

**Assume:** The ideal must be zero-dimensional and given as a reduced Groebner basis in the given ring. The monomial ordering must be global.

**Note:** The only permissible differences between the given ring and the current ring are the monomial ordering and a permutation of the variables, resp. parameters.

**Example:**

```

ring r=0,(x,y,z),dp;
ideal i=y3+x2, x2y+x2, x3-x2, z4-x2-y;
option(redSB); // force the computation of a reduced SB
i=std(i);
vdim(i);
↪ 28
ring s=0,(z,x,y),lp;
ideal j=fglm(r,i);
j;
↪ j[1]=y4+y3
↪ j[2]=xy3-y3
↪ j[3]=x2+y3
↪ j[4]=z4+y3-y

```

See [Section 5.1.40 \[fglmquot\]](#), page 184; [Section 5.1.111 \[option\]](#), page 234; [Section 4.20.1 \[qring\]](#), page 127; [Section 4.20 \[ring\]](#), page 127; [Section 5.1.151 \[std\]](#), page 271; [Section 5.1.152 \[stdfglm\]](#), page 272; [Section 5.1.168 \[vdim\]](#), page 286.

**5.1.40 fglmquot**

**Syntax:** `fglmquot ( ideal_expression, poly_expression )`

**Type:** `ideal`

**Purpose:** computes a reduced Groebner basis of the ideal quotient  $I:p$  of a zero-dimensional ideal  $I$  and a polynomial  $p$  using FGLM-techniques.

**Assume:** The ideal must be zero-dimensional and given as a reduced Groebner basis in the given ring. The polynomial must be reduced with respect to the ideal.

**Example:**

```

ring r=0,(x,y,z),lp;
ideal i=y3+x2,x2y+x2,x3-x2,z4-x2-y;
option(redSB); // force the computation of a reduced SB
i=std(i);
poly p=reduce(x+yz2+z10,i);
ideal j=fglmquot(i,p);
j;
↪ j[1]=z12
↪ j[2]=yz4-z8
↪ j[3]=y2+y-z8-z4
↪ j[4]=x+y-z10-z6-z4

```

See [Section 5.1.39 \[fglm\]](#), page 183; [Section 5.1.111 \[option\]](#), page 234; [Section 5.1.127 \[quotient\]](#), page 248; [Section 4.20 \[ring\]](#), page 127; [Section 5.1.151 \[std\]](#), page 271; [Section 5.1.168 \[vdim\]](#), page 286.

**5.1.41 files, input from**

**Syntax:** `< "filename"`

**Type:** `none`

**Purpose:** Read and execute the content of the file `filename`. Shorthand for `execute(read(filename))`.

**Example:**

```
< "example"; //read in the file example and execute it
```

See [Section 5.1.32 \[execute\]](#), page 178; [Section 5.1.130 \[read\]](#), page 250.

**5.1.42 find**

**Syntax:** `find ( string-expression, substring-expression )`  
`find ( string-expression, substring-expression, int-expression )`

**Type:** `int`

**Purpose:** returns the first position of the substring in the string or 0 (if not found), starts the search at the position given in the 3rd argument.

**Example:**

```
find("Aac","a");
↳ 2
find("abab","a"+"b");
↳ 1
find("abab","a"+"b",2);
↳ 3
find("abab","ab",3);
↳ 3
find("0123","abcd");
↳ 0
```

See [Section 4.22 \[string\]](#), page 130.

**5.1.43 finduni**

**Syntax:** `finduni ( ideal-expression )`

**Type:** `ideal`

**Purpose:** returns an ideal which is contained in the ideal-expression, such that the i-th generator is a univariate polynomial in the i-th ring variable.  
The polynomials have minimal degree w.r.t. this property.

**Assume:** The ideal must be zero-dimensional and given as a reduced Groebner basis in the current ring.

**Example:**

```
ring r=0,(x,y,z), dp;
ideal i=y3+x2,x2y+x2,z4-x2-y;
option(redSB); // force computation of reduced basis
i=std(i);
ideal k=finduni(i);
print(k);
↳ x4-x2,
↳ y4+y3,
↳ z12
```

See [Section 5.1.111 \[option\]](#), page 234; [Section 4.20 \[ring\]](#), page 127; [Section 5.1.151 \[std\]](#), page 271; [Section 5.1.168 \[vdim\]](#), page 286.

### 5.1.44 flintQ

**Syntax:** `flintQ ( list_of_names )`

**Type:** `cring`

**Purpose:** returns a coefficient ring of multivariate rational functions over  $\mathbb{Q}$  to be used in ring definitions. Require `flint >=2.5.3`.

**Example:**

```
LIB "flint.so";
ring R1=flintQ(a,b),(x,y),dp;
R1;
⇒ // coefficients: flintQQ(a,b)
⇒ // number of vars : 2
⇒ //      block 1 : ordering dp
⇒ //      : names x y
⇒ //      block 2 : ordering C
```

See [Section 4.1 \[cring\], page 73](#); [Section 4.20 \[ring\], page 127](#).

### 5.1.45 Float

**Syntax:** `Float ( )`  
`Float ( int_expression )`  
`Float ( int_expression , int_expression )`

**Type:** `cring`

**Purpose:** returns a coefficient ring of floating point (inexact) real number to be used in ring definitions.

**Example:**

```
ring R1=Float(),(x,y),dp;
R1;
⇒ // coefficients: Float()
⇒ // number of vars : 2
⇒ //      block 1 : ordering dp
⇒ //      : names x y
⇒ //      block 2 : ordering C
ring R2=Float(10,20),(a,b),dp;
R2;
⇒ // coefficients: Float(10,20)
⇒ // number of vars : 2
⇒ //      block 1 : ordering dp
⇒ //      : names a b
⇒ //      block 2 : ordering C
```

See [Section 4.1 \[cring\], page 73](#); [Section 4.20 \[ring\], page 127](#).

### 5.1.46 fprintf

Procedure from library `standard.lib` (see [Section D.1 \[standard.lib\], page 794](#)).

**Syntax:** `fprintf ( link_expression, string_expression [, any_expressions] )`

**Return:** `none`



**Purpose:** `fprintf(l,fmt,...)`; performs output formatting. The second argument is a format control string. Additional arguments may be required, depending on the content of the control string. A series of output characters is generated as directed by the control string; these characters are written to the link `l`. The control string `fmt` is simply text to be copied, except that the string may contain conversion specifications. Type `help print`; for a listing of valid conversion specifications. As an addition to the conversions of `print`, the `%n` and `%2` conversion specification does not consume an additional argument, but simply generates a newline character.

**Note:** If one of the additional arguments is a list, then it should be enclosed once more into a `list()` command, since passing a list as an argument flattens the list by one level.

**Example:**

```

ring r=0,(x,y,z),dp;
module m=[1,y],[0,x+z];
intmat M=betti(mres(m,0));
list l=r,m,M;
link li=""; // link to stdout
fprintf(li,"s:%s,l:%l",1,2);
⇒ s:1,l:int(2)
fprintf(li,"s:%s",l);
⇒ s:(QQ),(x,y,z),(dp(3),C)
fprintf(li,"s:%s",list(l));
⇒ s:(QQ),(x,y,z),(dp(3),C),y*gen(2)+gen(1),x*gen(2)+z*gen(2),1,1
fprintf(li,"2l:%2l",list(l));
⇒ 2l:list("(QQ),(x,y,z),(dp(3),C)",
⇒ module(y*gen(2)+gen(1),
⇒ x*gen(2)+z*gen(2)),
⇒ intmat(intvec(1,1 ),1,2))
⇒
fprintf(li,"%p",list(l));
⇒ [1]:
⇒ // coefficients: QQ
⇒ // number of vars : 3
⇒ //          block  1 : ordering dp
⇒ //          : names  x y z
⇒ //          block  2 : ordering C
⇒ [2]:
⇒ _[1]=y*gen(2)+gen(1)
⇒ _[2]=x*gen(2)+z*gen(2)
⇒ [3]:
⇒ 1,1
fprintf(li,"%;",list(l));
⇒ [1]:
⇒ // coefficients: QQ
⇒ // number of vars : 3
⇒ //          block  1 : ordering dp
⇒ //          : names  x y z
⇒ //          block  2 : ordering C
⇒ [2]:
⇒ _[1]=y*gen(2)+gen(1)
⇒ _[2]=x*gen(2)+z*gen(2)
⇒ [3]:

```

```

↳      1,1
↳
↳ fprintf(li,"%b",M);
↳           0      1
↳ -----
↳      0:      1      1
↳ -----
↳ total:      1      1
↳

```

See also: [Section 5.1.120 \[print\]](#), page 242; [Section 5.1.121 \[printf\]](#), page 244; [Section 5.1.148 \[sprintf\]](#), page 267; [Section 4.22 \[string\]](#), page 130.

### 5.1.47 freemodule

**Syntax:** `freemodule ( int_expression )`

**Type:** module

**Purpose:** creates the free module of rank `n` generated by `gen(1), ..., gen(n)`.

**Example:**

```

      ring r=32003,(x,y),(c,dp);
      freemodule(3);
↳  _[1]=[1]
↳  _[2]=[0,1]
↳  _[3]=[0,0,1]
      matrix m=freemodule(3); // generates the 3x3 unit matrix
      print(m);
↳  1,0,0,
↳  0,1,0,
↳  0,0,1

```

See [Section 5.1.51 \[gen\]](#), page 191; [Section 4.14 \[module\]](#), page 112.

### 5.1.48 fres

**Syntax:** `fres ( ideal_expression/module_expression , int_expression , [ string_expression ] )`

**Type:** resolution

**Purpose:** computes a (not necessarily minimal) free resolution of the input ideal/module, using Schreyer's algorithm, see reference.

If the second argument is `n > 0`, then the resolution is computed up to step `n`. If it is 0, `fres` computes the whole resolution.

The optional third argument can be set to

- "complete" (default) to compute the whole syzygy module in each step,
- "frame" to compute only the so-called frame,
- "extended frame" to compute only the first two terms of each generator w.r.t. the induced monomial ordering, or
- "single module" to return only the frame of each module except the last one and to return the last module in its entirety. This option can be used to reduce the amount of memory needed for the computation.

**Note:** The input ideal/module must be a standard basis.

**Reference:**

B. Erocal, O. Motsak, F.-O. Schreyer, A. Steenpass: Refined Algorithms to Compute Syzygies. J. Symb. Comput. 74 (2016), 308-327. <http://arxiv.org/abs/1502.01654>

**Example:**

```

ring r = 0, (w,x,y,z), dp;
ideal I = w2-xz, wx-yz, x2-wy, xy-z2, y2-wz;
attrib(I, "isSB", 1);
resolution s = fres(I, 0);
s;
↳ 1      5      6      2
↳ r <--  r <--  r <--  r
↳
↳ 0      1      2      3
↳ resolution not minimized yet
↳
print(betti(s, 0), "betti");
↳          0      1      2      3
↳ -----
↳ 0:      1      -      -      -
↳ 1:      -      5      5      1
↳ 2:      -      -      1      1
↳ -----
↳ total:    1      5      6      2
↳
list l = s;
print(l[1]);
↳ w2-xz,
↳ wx-yz,
↳ x2-wy,
↳ xy-z2,
↳ y2-wz
print(l[2]);
↳ -x,y, 0, -z,0, -y2+wz,
↳ w, -x,-y,0, z, z2,
↳ -z,w, 0, -y,0, 0,
↳ 0, 0, w, x, -y,-yz,
↳ 0, 0, -z,-w,x, w2
print(l[3]);
↳ 0, -y2+wz,
↳ y, z2,
↳ -x,-wy,
↳ w, yz,
↳ -z,-w2,
↳ 1, x

```

See [Section A.3.4 \[Free resolution\]](#), page 720; [Section 5.1.93 \[minres\]](#), page 223; [Section 5.1.134 \[res\]](#), page 253; [Section 5.1.149 \[sres\]](#), page 269; [Section 5.1.156 \[syz\]](#), page 280.

**5.1.49 frwalk**

**Syntax:**    `frwalk ( ring_name, ideal_name )`  
               `frwalk ( ring_name, ideal_name , int_expression )`

**Type:** ideal

**Purpose:** computes for the ideal `ideal_name` in the ring `ring_name` a Groebner basis in the current ring, by applying the fractal walk algorithm.  
The main application is to compute a lexicographical Groebner basis from a reduced Groebner basis with respect to a degree ordering. This can be much faster than computing a lexicographical Groebner basis directly.

**Note:** When calling `frwalk`, the only permissible difference between the ring `ring_name` and the active base ring is the monomial ordering.

**Example:**

```
ring r=0,(x,y,z),dp;
ideal i=y3+x2, x2y+x2, x3-x2, z4-x2-y;
i=std(i);
ring s=0,(x,y,z),lp;
ideal j=frwalk(r,i);
j;
↪ j[1]=z12
↪ j[2]=yz4-z8
↪ j[3]=y2+y-z8-z4
↪ j[4]=xy-xz4-y+z4
↪ j[5]=x2+y-z4
```

See [Section 5.1.39 \[fglm\], page 183](#); [Section 5.1.53 \[groebner\], page 191](#); [Section 4.20.1 \[qring\], page 127](#); [Section 4.20 \[ring\], page 127](#); [Section 5.1.151 \[std\], page 271](#).

### 5.1.50 gcd

**Syntax:** `gcd ( int_expression, int_expression )`  
`gcd ( bigint_expression, bigint_expression )`  
`gcd ( number_expression, number_expression )`  
`gcd ( poly_expression, poly_expression )`

**Type:** the same as the type of the arguments

**Purpose:** computes the greatest common divisor.

**Note:** Not implemented for the coefficient fields real and finite fields of type  $(p^n, a)$ .  
The gcd of two numbers is their gcd as integer numbers or polynomials, otherwise it is not defined.

**Example:**

```
gcd(2,3);
↪ 1
gcd(bigint(2)^20,bigint(3)^23);    // also applicable for bigints
↪ 1
typeof(_);
↪ bigint
ring r=0,(x,y,z),lp;
gcd(3x2*(x+y),9x*(y2-x2));
↪ x2+xy
gcd(number(6472674604870),number(878646537247372));
↪ 2
```

See [Section 4.2 \[bigint\], page 74](#); [Section 5.1.33 \[extgcd\], page 178](#); [Section 4.7 \[int\], page 85](#); [Section 4.15 \[number\], page 115](#).

### 5.1.51 gen

**Syntax:** `gen ( int_expression )`

**Type:** vector

**Purpose:** returns the i-th free generator of a free module.

**Example:**

```

ring r=32003,(x,y,z),(c,dp);
gen(3);
↪ [0,0,1]
vector v=gen(5);
poly f=xyz;
v=v+f*gen(4); v;
↪ [0,0,0,xyz,1]
ring rr=32003,(x,y,z),dp;
fetch(r,v);
↪ xyz*gen(4)+gen(5)

```

See [Section 5.1.47 \[freemodule\]](#), page 188; [Section 4.7 \[int\]](#), page 85; [Section 4.23 \[vector\]](#), page 134.

### 5.1.52 getdump

**Syntax:** `getdump ( link_expression )`

**Type:** none

**Purpose:** reads the content of the entire file, resp. link, and restores all variables from it. For ASCII links, `getdump` is equivalent to an `execute(read( link ))` command. For ssi links, `getdump` should only be used on data which were previously `dump`'ed.

**Example:**

```

int i=3;
dump(":w example.txt");
kill i;
option(noredefine);
getdump("example.txt");
i;
↪ 3

```

**Restrictions:**

`getdump` is not supported for DBM links, or for a link connecting to `stdin` (standard input).

See [Section 5.1.27 \[dump\]](#), page 175; [Section 4.10 \[link\]](#), page 96; [Section 5.1.130 \[read\]](#), page 250.

### 5.1.53 groebner

Procedure from library `standard.lib` (see [Section D.1 \[standard.lib\]](#), page 794).

**Syntax:** `groebner ( ideal_expression )`

`groebner ( module_expression )`

`groebner ( ideal_expression, list of string_expressions )`

`groebner ( ideal_expression, list of string_expressions and int_expression )`

**Type:** type of the first argument

**Purpose:** computes a standard basis of the first argument  $I$  (ideal or module) by a heuristically chosen method (default) or by a method specified by further arguments of type string. Possible methods are:

- the direct methods "std" or "slimgb" without conversion,
- conversion methods "hilb" or "fglm" where a Groebner basis is first computed with an "easy" ordering and then converted to the ordering of the basering by the Hilbert driven Groebner basis computation or by linear algebra. The actual computation of the Groebner basis can be specified by "std" or by "slimgb" (not for all orderings implemented).
- "HC": using the "high corner" from char p in char 0, finding a SB for 0-dimensional ideals in local orderings faster.

A further string "par2var" converts parameters to an extra block of variables before a Groebner basis computation (and afterwards back). `option(prot)` informs about the chosen method.

**Hint:** Since there exists no uniform best method for computing standard bases, and since the difference in performance of a method on different examples can be huge, it is recommended to test, for hard examples, first various methods on a simplified example (e.g. use characteristic 32003 instead of 0 or substitute a subset of parameters/variables by integers, etc.).

**Example:**

```

intvec opt = option(get);
option(prot);
ring r = 0,(a,b,c,d),dp;
ideal i = a+b+c+d,ab+ad+bc+cd,abc+abd+acd+bcd,abcd-1;
groebner(i);
⇒ std in (QQ),(a,b,c,d),(dp(4),C)
⇒ [65535:2]1(3)s2(2)s3s4-s5ss6-s7--
⇒ product criterion:8 chain criterion:5
⇒ _[1]=a+b+c+d
⇒ _[2]=b2+2bd+d2
⇒ _[3]=bc2+c2d-bd2-d3
⇒ _[4]=bcd2+c2d2-bd3+cd3-d4-1
⇒ _[5]=bd4+d5-b-d
⇒ _[6]=c3d2+c2d3-c-d
⇒ _[7]=c2d4+bc-bd+cd-2d2
ring s = 0,(a,b,c,d),lp;
ideal i = imap(r,i);
groebner(i,"hilb");
⇒ compute hilbert series with std in ring (QQ),(a,b,c,d,@),(dp(5),C)
⇒ weights used for hilbert series: 1,1,1,1,1
⇒ [1048575:2]1(3)s2(2)s3s4-s5ss6-s7--
⇒ product criterion:8 chain criterion:5
⇒ std with hilb in (QQ),(a,b,c,d,@),(lp(4),dp(1),C)
⇒ [1048575:2]1(6)s2(5)s3(4)s4-s5sshh6(3)shhhhh8shh
⇒ product criterion:9 chain criterion:8
⇒ hilbert series criterion:9
⇒ dehomogenization
⇒ simplification
⇒ imap to ring (QQ),(a,b,c,d),(lp(4),C)
⇒ _[1]=c2d6-c2d2-d4+1

```

```

⇒ _[2]=c3d2+c2d3-c-d
⇒ _[3]=bd4-b+d5-d
⇒ _[4]=bc-bd5+c2d4+cd-d6-d2
⇒ _[5]=b2+2bd+d2
⇒ _[6]=a+b+c+d
ring R = (0,a),(b,c,d),lp;
minpoly = a2+1;
ideal i = a+b+c+d,ab+ad+bc+cd,abc+abd+acd+bcd,d2-c2b2;
groebner(i,"par2var","slimgb");
⇒ //add minpoly to input
⇒ compute hilbert series with slimgb in ring (QQ),(b,c,d,a,@),(dp(5),C)
⇒ weights used for hilbert series: 1,1,1,1,1
⇒ slimgb in ring (QQ),(b,c,d,a,@),(dp(5),C)
⇒ CC2M[2,2](2)C3M[1,1](2)4M[2,e1](2)C5M[2,e2](3)C6M[1,1](0)
⇒ NF:8 product criterion:15, ext_product criterion:3
⇒ std with hilb in (QQ),(b,c,d,a,@),(lp(3),dp(1),dp(1),C)
⇒ [1048575:2]1(7)s2(6)s(5)s3(4)s4-s5sshh6(3)shhhhh
⇒ product criterion:15 chain criterion:5
⇒ hilbert series criterion:7
⇒ dehomogenization
⇒ simplification
⇒ imap to ring (QQ),(b,c,d,a),(lp(3),dp(1),C)
⇒ //simplification
⇒ (S:4)rtrtrtr
⇒ //imap to original ring
⇒ _[1]=d2
⇒ _[2]=c+(a)
⇒ _[3]=b+c+d+(a)
groebner(i,"fglm"); //computes a reduced standard basis
⇒ std in (0,a),(b,c,d),(dp(3),C)
⇒ [1048575:2]1(3)s2(2)s3s4-s5ss6-s7
⇒ (S:2)--
⇒ product criterion:9 chain criterion:1
⇒ ..+++--
⇒ vdim= 2
⇒ ..++-+-
⇒ _[1]=d2
⇒ _[2]=c+(a)
⇒ _[3]=b+d
option(set,opt);
ring Rt = (0,t),(x,y,z),ds;
poly F = y10+(t2)*x7y7+x15+x9y6+(2t)*x6y9+x6y6z3+x5y11+z21;
ideal I = jacob(F);
I=groebner(I,"HC","prot");
⇒ computing HC in char 32003
⇒ found HC in char 32003: x7y2z38
⇒ computing std with HC

```

See also: [Section D.4.9 \[ffmodstd.lib\]](#), page 1081; [Section D.4.16 \[modstd.lib\]](#), page 1147; [Section D.4.20 \[nfmmodstd.lib\]](#), page 1175; [Section 5.1.145 \[slimgb\]](#), page 265; [Section 5.1.151 \[std\]](#), page 271; [Section 5.1.152 \[stdfglm\]](#), page 272; [Section 5.1.153 \[stdhilb\]](#), page 273.

### 5.1.54 help

**Syntax:** `help;`  
`help topic ;`

**Type:** none

**Purpose:** displays online help information for `topic` using the currently set help browser. If no `topic` is given, the title page of the manual is displayed.

**Note:**

- `?` may be used instead of `help`.
- `topic` can be an index entry of the SINGULAR manual or the name of a (loaded) procedure which has a help section.
- `topic` may contain wildcard characters (i.e., `*` characters).
- If a (possibly "wildcarded") `topic` cannot be found (or uniquely matched) a warning is displayed and no help information is provided.
- If `topic` is the name of a (loaded) procedure whose help section has changed w.r.t. the help available in the manual then, instead of displaying the respective help section of the manual in the help browser, the "newer" help section of the procedure is simply printed to the terminal.
- The browser in which the help information is displayed can be either set with the command-line option `--browser=<browser>` (see [Section 3.1.6 \[Command line options\]](#), page 19), or with the command `system("--browser", "<browser>")`. Use the command `system("browsers")`; for a list of all available browsers. See [Section 3.1.3 \[The online help system\]](#), page 15, for more details about help browsers.

**Example:**

```
help;          // display title page of manual
help ring;    // display help for 'ring'
?ring;       // equivalent to 'help ring;'
⇒ // ** No help for topic 'ring' (not even for '*ring*')
⇒ // ** Try '?' for general help
⇒ // ** or '?Index;' for all available help topics
?ring*;
⇒ // ** No unique help for 'ring*'
⇒ // ** Try one of
⇒ ?Rings and orderings; ?Rings and standard bases; ?ring;
⇒ ?ring declarations; ?ring operations; ?ring related functions;
⇒ ?ring.lib; ?ring_lib; ?ringtensor; ?ringweights;
help Rings and orderings;
help standard.lib; // displays help for library 'standard.lib'
```

See [Section 3.1.6 \[Command line options\]](#), page 19; [Section 3.8 \[Libraries\]](#), page 55; [Section 3.7.1 \[Procedure definition\]](#), page 51; [Section 3.1.3 \[The online help system\]](#), page 15; [Section 5.1.155 \[system\]](#), page 275.

### 5.1.55 highcorner

**Syntax:** `highcorner ( ideal-expression )`  
`highcorner ( module-expression )`

**Type:** poly, resp. vector



- Purpose:** returns the smallest monomial not contained in the ideal, resp. module, generated by the initial terms of the given generators. If the generators are a standard basis, this is also the smallest monomial not contained in the ideal, resp. module.  
If the ideal, resp. module, is not zero-dimensional, 0 is returned.  
The command works also in global orderings, but is not very useful there.
- Note:** Let the ideal  $I$  be given by a standard basis. Then `highcorner(I)` returns 0 if and only if  $\dim(I) > 0$  or  $\dim(I) = -1$ . Otherwise it returns the smallest monomial  $m$  not in  $I$  which has the following properties (with  $x_i$  the variables of the basering):
- if  $x_i > 1$  then  $x_i$  does not divide  $m$  (hence,  $m=1$  if the ordering is global)
  - given any set of generators  $f_1, \dots, f_k$  of  $I$ , let  $f'_i$  be obtained from  $f_i$  by deleting the terms divisible by  $x_i \cdot m$  for all  $i$  with  $x_i < 1$ . Then  $f'_1, \dots, f'_k$  generate  $I$ .

**Example:**

```
ring r=0,(x,y),ds;
ideal i=x3,x2y,y3;
highcorner(std(i));
  ↦ xy2
highcorner(std(ideal(1)));
  ↦ 0
```

See [Section 5.1.25 \[dim\]](#), page 174; [Section 5.1.53 \[groebner\]](#), page 191; [Section 5.1.151 \[std\]](#), page 271; [Section 5.1.168 \[vdim\]](#), page 286.

### 5.1.56 hilb

- Syntax:** `hilb ( ideal_expression )`  
`hilb ( module_expression )`  
`hilb ( ideal_expression, int_expression )`  
`hilb ( module_expression, int_expression )`  
`hilb ( ideal_expression, int_expression , intvec_expression )`  
`hilb ( module_expression, int_expression , intvec_expression )`
- Type:** none (if called with one argument)  
 intvec (if called with two or three arguments)
- Purpose:** computes the (weighted) Hilbert series of the base ring  $R$  modulo the ideal, resp.  $R^k$  modulo the module, defined by the leading terms of the generators of the given ideal, resp. module.  
 If `hilb` is called with one argument, then the first and second Hilbert series together with some additional information are displayed.  
 If `hilb` is called with two arguments, then the  $n$ -th Hilbert series is returned as an intvec, where  $n = 1, 2$  is the second argument.  
 If a weight vector  $w$  is given as 3rd argument, then the Hilbert series is computed w.r.t. these weights  $w$  (by default all weights are set to 1).
- Caution:** The last entry of the returned intvec is not part of the actual Hilbert series, but is used in the Hilbert driven standard basis computation (see [Section 5.1.153 \[stdhilb\]](#), page 273). (It is the minimum weight of the module generators or 0).
- Syntax:** `hilb ( intvec_expression )`
- Type:** intvec
- Purpose:** computes the second Hilbert series from the first, i.e. if `intvec v=hilb(I,1)`; then `hilb(v)` yields the same result as `hilb(I,2)`.

- Syntax:** `hilb ( ideal_expression, ring, string_expression )`  
`hilb ( module_expression, ring, string_expression )`
- Type:** `none`
- Purpose:** computes the (weighted) Hilbert series of the base ring  $R$  modulo the ideal, resp.  $R^k$  modulo the module, defined by the leading terms of the generators of the given ideal, resp. module.  
The series is stored as `poly` under the name give as string in the given ring.
- Note:** If the input is homogeneous w.r.t. the weights and a standard basis, the result is the (weighted) Hilbert series of the original ideal, resp. module.

**Example:**

```

ring Qt=QQ,t,dp;
ring R=32003,(x,y,z),dp;
ideal i=x2,y2,z2;
ideal s=std(i);
hilb(s);
↦ (-t6+3t4-3t2+1) / (1-t)^3
↦ (t3+3t2+3t+1) / (1-t)^0
↦ // dimension (affine) = 0
↦ // degree (affine) = 8
hilb(s,1);
↦ 1,0,-3,0,3,0,-1,0
hilb(s,2);
↦ 1,3,3,1,0
intvec w=2,2,2;
hilb(s,1,w);
↦ 1,0,0,0,-3,0,0,0,3,0,0,0,-1,0
hilb(s,Qt,"h");
setring Qt;h;
↦ -t6+3t4-3t2+1

```

See [Section C.2 \[Hilbert function\]](#), page 775; [Section 4.6 \[ideal\]](#), page 80; [Section 4.9 \[intvec\]](#), page 93; [Section 4.14 \[module\]](#), page 112; [Section 5.1.151 \[std\]](#), page 271; [Section 5.1.153 \[stdhilb\]](#), page 273.

**5.1.57 homog**

- Syntax:** `homog ( ideal_expression )`  
`homog ( module_expression )`
- Type:** `int`
- Purpose:** tests for homogeneity: returns 1 for homogeneous input, 0 otherwise.
- Note:** If the current ring has a weighted monomial ordering, `homog` tests for weighted homogeneity w.r.t. the given weights.
- Syntax:** `homog ( ideal_expression, intvec_expression )`
- Type:** `int`
- Purpose:** tests for homogeneity wrt. the given weight vector: returns 1 for homogeneous input, 0 otherwise.

- Syntax:** `homog ( polynomial_expression, ring_variable )`  
`homog ( vector_expression, ring_variable )`  
`homog ( ideal_expression, ring_variable )`  
`homog ( module_expression, ring_variable )`
- Type:** same as first argument
- Purpose:** homogenizes polynomials, vectors, ideals, or modules by multiplying each monomial with a suitable power of the given ring variable.
- Note:** If the current ring has a weighted monomial ordering, `homog` computes the weighted homogenization w.r.t. the given weights.  
The homogenizing variable must have weight 1.

**Example:**

```

ring r=32003,(x,y,z),ds;
poly s1=x3y2+x5y+3y9;
poly s2=x2y2z2+3z8;
poly s3=5x4y2+4xy5+2x2y2z3+y7+11x10;
ideal i=s1,s2,s3;
homog(s2,z);
↪ x2y2z4+3z8
homog(i,z);
↪ _[1]=3y9+x5yz3+x3y2z4
↪ _[2]=x2y2z4+3z8
↪ _[3]=11x10+y7z3+5x4y2z4+4xy5z4+2x2y2z6
homog(i);
↪ 0
homog(homog(i,z));
↪ 1

```

See [Section 4.6 \[ideal\], page 80](#); [Section 4.14 \[module\], page 112](#); [Section 4.17 \[poly\], page 120](#); [Section 4.23 \[vector\], page 134](#).

**5.1.58 hres**

- Syntax:** `hres ( ideal_expression, int_expression )`
- Type:** resolution
- Purpose:** computes a free resolution of an ideal using the Hilbert-driven algorithm.  
More precisely, let  $R$  be the basering and  $I$  be the given ideal. Then `hres` computes a minimal free resolution of  $R/I$

$$\dots \longrightarrow F_2 \xrightarrow{A_2} F_1 \xrightarrow{A_1} R \longrightarrow R/I \longrightarrow 0.$$

If the `int_expression`  $k$  is not zero then the computation stops after  $k$  steps and returns a list of modules  $M_i = \text{module}(A_i)$ ,  $i=1..k$ .

`list L=hres(I,0);` returns a list  $L$  of  $n$  modules (where  $n$  is the number of variables of the basering) such that  $L[i] = M_i$  in the above notation.

- Note:** The `ideal_expression` has to be homogeneous.  
Accessing single elements of a resolution may require some partial computations to be finished. Therefore, it may take some time.

**Example:**

```

ring r=0,(x,y,z),dp;
ideal I=xz,yz,x3-y3;
def L=hres(I,0);
print(betti(L),"beti");
↪          0      1      2
↪ -----
↪    0:      1      -      -
↪    1:      -      2      1
↪    2:      -      1      1
↪ -----
↪ total:      1      3      2
↪
    L[2];      // the first syzygy module of r/I
↪ _[1]=-x*gen(1)+y*gen(2)
↪ _[2]=-x2*gen(2)+y2*gen(1)+z*gen(3)

```

See [Section 5.1.4 \[beti\]](#), page 159; [Section 5.1.48 \[fres\]](#), page 188; [Section 4.6 \[ideal\]](#), page 80; [Section 4.7 \[int\]](#), page 85; [Section 5.1.83 \[lres\]](#), page 215; [Section 5.1.93 \[minres\]](#), page 223; [Section 4.14 \[module\]](#), page 112; [Section 5.1.98 \[mres\]](#), page 225; [Section 5.1.99 \[mres\\_map\]](#), page 226; [Section 5.1.134 \[res\]](#), page 253; [Section 5.1.149 \[sres\]](#), page 269.

### 5.1.59 imap

**Syntax:** `imap ( ring_name, name )`

**Type:** number, poly, vector, ideal, module, matrix or list (the same type as the second argument)

**Purpose:** identity map on common subrings. `imap` is the map between rings and qrings with compatible ground fields which is the identity on variables and parameters of the same name and 0 otherwise. (See [Section 4.12 \[map\]](#), page 106 for a description of possible mappings between different ground fields). Useful for mapping from a homogenized ring to the original ring or for mappings from/to rings with/without parameters. Compared with `fetch`, `imap` uses the names of variables and parameters. Unlike `map` and `fetch` `imap` can map parameters to variables. Mapping rational functions which are not polynomials to polynomials is undefined (i.e. the result depends on the version).

**Example:**

```

ring r=0,(x,y,z,a,b,c),dp;
ideal i=xy2z3a4b5+1,homog(xy2z3a4b5+1,c); i;
↪ i[1]=xy2z3a4b5+1
↪ i[2]=xy2z3a4b5+c15
ring r1=0,(a,b,x,y,z),lp;
ideal j=imap(r,i); j;
↪ j[1]=a4b5xy2z3+1
↪ j[2]=a4b5xy2z3
ring r2=(0,a,b),(x,y,z),ls;
ideal j=imap(r,i); j;
↪ j[1]=1+(a4b5)*xy2z3
↪ j[2]=(a4b5)*xy2z3

```

See [Section 5.1.38 \[fetch\]](#), page 182; [Section 5.1.57 \[homog\]](#), page 196; [Section 4.12 \[map\]](#), page 106; [Section 4.20.1 \[qring\]](#), page 127; [Section 4.20 \[ring\]](#), page 127.

### 5.1.60 impart

**Syntax:** `impart ( number_expression )`

**Type:** `number`

**Purpose:** returns the imaginary part of a number in a complex ground field, returns 0 otherwise.

**Example:**

```
ring r=(complex,i),x,dp;
impart(1+2*i);
↪ 2
```

See [Section 5.1.133 \[repart\]](#), page 253.

### 5.1.61 indepSet

**Syntax:** `indepSet ( ideal_expression )`

**Type:** `intvec`

**Purpose:** computes a maximal set  $U$  of independent variables (in the sense defined in the note below) of the ideal given by a standard basis. If  $v$  is the result then  $v[i]$  is 1 if and only if the  $i$ -th variable of the ring,  $x(i)$ , is an independent variable. Hence, the set  $U$  consisting of all variables  $x(i)$  with  $v[i]=1$  is a maximal independent set.

**Note:**  $U$  is a set of independent variables for  $I$  if and only if  $I \cap K[U] = (0)$ , i.e., eliminating the remaining variables gives  $(0)$ .  $U$  is maximal if  $\dim(I)=\#U$ .

**Syntax:** `indepSet ( ideal_expression, int_expression )`

**Type:** `list`

**Purpose:** computes a list of all maximal independent sets of the leading ideal (if the flag is 0), resp. of all those sets of independent variables of the leading ideal which cannot be enlarged.

**Example:**

```
ring r=32003,(x,y,u,v,w),dp;
ideal I=xyw,yvw,uyw,xv;
attrib(I,"isSB",1);
indepSet(I);
↪ 1,1,1,0,0
eliminate(I,vw);
↪ _[1]=0
indepSet(I,0);
↪ [1]:
↪ 1,1,1,0,0
↪ [2]:
↪ 0,1,1,1,0
↪ [3]:
↪ 1,0,1,0,1
↪ [4]:
↪ 0,0,1,1,1
indepSet(I,1);
↪ [1]:
```

```

⇒      1,1,1,0,0
⇒ [2]:
⇒      0,1,1,1,0
⇒ [3]:
⇒      1,0,1,0,1
⇒ [4]:
⇒      0,0,1,1,1
⇒ [5]:
⇒      0,1,0,0,1
      eliminate(I,xuv);
⇒ _[1]=0

```

See [Section 4.6 \[ideal\]](#), page 80; [Section 5.1.151 \[std\]](#), page 271.

### 5.1.62 insert

**Syntax:**    `insert ( list_expression, expression )`  
               `insert ( list_expression, expression, int_expression )`

**Type:**     `list`

**Purpose:**    inserts a new element (expression) into a list at the beginning, or (if called with 3 arguments) after the given position (the input is not changed).

**Example:**

```

      list L=1,2;
      insert(L,4,2);
⇒ [1]:
⇒      1
⇒ [2]:
⇒      2
⇒ [3]:
⇒      4
      insert(L,4);
⇒ [1]:
⇒      4
⇒ [2]:
⇒      1
⇒ [3]:
⇒      2

```

See [Section 5.1.21 \[delete\]](#), page 171; [Section 4.11 \[list\]](#), page 103.

### 5.1.63 interpolation

**Syntax:**    `interpolation ( list, intvec )`

**Type:**     `ideal`

**Purpose:**    `interpolation(l,v)` computes the reduced Groebner basis of the intersection of ideals  $l[1]^v[1], \dots, l[N]^v[N]$  by applying linear algebra methods.

**Assume:**   Every ideal from the list `l` must be a maximal ideal of a point and should have the following form: `variable_1-coordinate_1, ..., variable_n-coordinate_n`, where `n` is the number of variables in the ring.

The ring should be a polynomial ring over  $\mathbb{Z}_p$  or  $\mathbb{Q}$  with global ordering.

**Example:**

```

ring r=0,(x,y),dp;
ideal p_1=x,y;
ideal p_2=x+1,y+1;
ideal p_3=x+2,y-1;
ideal p_4=x-1,y+2;
ideal p_5=x-1,y-3;
ideal p_6=x,y+3;
ideal p_7=x+2,y;
list l=p_1,p_2,p_3,p_4,p_5,p_6,p_7;
intvec v=2,1,1,1,1,1,1;
ideal j=interpolation(l,v);
// generator of degree 3 gives the equation of the unique
// singular cubic passing
// through p_1,...,p_7 with singularity at p_1
j;
⇒ j[1]=-4x3-4x2y-2xy2+y3-8x2-4xy+3y2
⇒ j[2]=-y4+8x2y+6xy2-2y3+10xy+3y2
⇒ j[3]=-xy3+2x2y+xy2+4xy
⇒ j[4]=-2x2y2-2x2y-2xy2+y3-4xy+3y2
// computes values of generators of j at p_4, results should be 0
subst(j,x,1,y,-2);
⇒ _[1]=0
⇒ _[2]=0
⇒ _[3]=0
⇒ _[4]=0
// computes values of derivatives d/dx of generators at (0,0)
subst(diff(j,x),x,0,y,0);
⇒ _[1]=0
⇒ _[2]=0
⇒ _[3]=0
⇒ _[4]=0

```

See [Section 5.1.24 \[diff\]](#), page 173; [Section 5.1.39 \[fglm\]](#), page 183; [Section 5.1.65 \[intersect\]](#), page 202; [Section 5.1.151 \[std\]](#), page 271; [Section 5.1.154 \[subst\]](#), page 274.

**5.1.64 interred**

**Syntax:** `interred ( ideal_expression )`  
`interred ( module_expression )`

**Type:** the same as the input type

**Purpose:** interreduces a set of polynomials/vectors.

Input:  $f_1, \dots, f_n$

Output:  $g_1, \dots, g_s$  with  $s \leq n$  and the properties

- $(f_1, \dots, f_n) = (g_1, \dots, g_s)$ ,
- $L(g_i) \neq L(g_j)$  for all  $i \neq j$ ,
- in the case of a global ordering (polynomial ring) and `option(redSB);`:  
 $L(g_i)$  does not divide  $m$  for all monomials  $m$  of  $\{g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_s\}$ ,
- in the case of a local ordering (localization of polynomial ring) and `option(redSB);`:  
if  $L(g_i) | L(g_j)$  for any  $i \neq j$ , then  $ecart(g_i) > ecart(g_j)$ .

Here,  $L(g)$  denotes the leading term of  $g$  and  $ecart(g) := deg(g) - deg(L(g))$ .

**Example:**

```

ring r=0,(x,y,z),dp;
ideal i=zx+y3,z+y3,z+xy;
interred(i);
↦ _[1]=xz-z
↦ _[2]=xy+z
↦ _[3]=y3+xz
ring R=0,(x,y,z),ds;
ideal i=zx+y3,z+y3,z+xy;
interred(i);
↦ _[1]=z+xy
↦ _[2]=xy-y3
↦ _[3]=x2y-y3

```

See [Section 4.6 \[ideal\]](#), page 80; [Section 4.14 \[module\]](#), page 112; [Section 5.1.151 \[std\]](#), page 271.

### 5.1.65 intersect

**Syntax:** `intersect ( expression_list of ideal_expression )`  
`intersect ( expression_list of module_expression )`

**Type:** ideal, resp. module

**Purpose:** computes the intersection of ideals, resp. modules.

**Note:** If the option `prot` is enabled then the result the used method (elimination/syzygies) is displayed.  
 An optional last argument specifies the Groebner base algorithm to use. Possible values are `"std"` and `"slimgb"`.

**Example:**

```

ring R=0,(x,y),dp;
ideal i=x;
ideal j=y;
intersect(i,j);
↦ _[1]=xy
ring r=181,(x,y,z),(c,ls);
ideal id1=maxideal(3);
ideal id2=x2+xyz,y2-z3y,z3+y5xz;
ideal id3=intersect(id1,id2,ideal(x,y));
ideal id4=intersect(id1,id2,"slimgb");
id3;
↦ id3[1]=yz3+xy6z
↦ id3[2]=yz4-y2z
↦ id3[3]=y2z3-y3
↦ id3[4]=xz3+x2y5z
↦ id3[5]=xyz2+x2z
↦ id3[6]=xy2+x2z2
↦ id3[7]=xy2z+x2y
↦ id3[8]=x2yz+x3
id4;
↦ id4[1]=xyz2+x2z
↦ id4[2]=xy2z+x2y

```



```

↳ id4[3]=x2yz+x3
↳ id4[4]=-yz4+y2z
↳ id4[5]=-y2z3+y3
↳ id4[6]=-xyz3+xy2
↳ id4[7]=z3+xy5z

```

See [Section 4.6 \[ideal\]](#), page 80; [Section 4.14 \[module\]](#), page 112; [Section 5.1.111 \[option\]](#), page 234.

### 5.1.66 jacob

**Syntax:**    `jacob ( poly_expression )`  
               `jacob ( ideal_expression )`  
               `jacob ( module_expression )`

**Type:**     ideal, if the input is a polynomial  
               matrix, if the input is an ideal  
               module, if the input is a module

**Purpose:**    computes the Jacobi ideal, resp. Jacobi matrix, generated by all partial derivatives of the input.

**Note:**     In a ring with  $n$  variables, `jacob` of a module or an ideal (considered as matrix with a single a row) or a polynomial (considered as a matrix with a single entry) is the matrix consisting of horizontally concatenated blocks (in this order): `diff(MT,var(1))`, ... , `diff(MT,var(n))`, where `MT` is the transposed input argument considered as a matrix.

**Example:**

```

ring R;
poly f = x2yz + xy3z + xyz5;
ideal i = jacob(f); i;
↳ i[1]=yz5+y3z+2xyz
↳ i[2]=xz5+3xy2z+x2z
↳ i[3]=5xyz4+xy3+x2y
matrix m = jacob(i);
print(m);
↳ 2yz,          z5+3y2z+2xz, 5yz4+y3+2xy,
↳ z5+3y2z+2xz,6xyz,          5xz4+3xy2+x2,
↳ 5yz4+y3+2xy,5xz4+3xy2+x2,20xyz3
print(jacob(m));
↳ 0, 2z,          2y,          2z,          6yz,5z4+3y2+2x,2y,          5z4+3y2+2x,
20yz3,
↳ 2z,6yz,          5z4+3y2+2x,6yz,          6xz,6xy,          5z4+3y2+2x,6xy,
20xz3,
↳ 2y,5z4+3y2+2x,20yz3,          5z4+3y2+2x,6xy,20xz3,          20yz3,          20xz3,
60xyz2

```

See [Section 5.1.24 \[diff\]](#), page 173; [Section 4.6 \[ideal\]](#), page 80; [Section 4.14 \[module\]](#), page 112; [Section 5.1.109 \[nvars\]](#), page 233.

### 5.1.67 janet

**Syntax:**    `janet ( ideal_expression )`  
               `janet ( ideal_expression , int_expression )`

**Type:**     ideal

**Purpose:** computes the Janet basis of the given ideal, resp. the standard basis if 1 is given as the second argument.

**Remark:** It works only with global orderings.

**Example:**

```
ring r=0,(x,y,z),dp;
ideal i=x*y*z-1,x+y+z,x*y+x*z+y*z; // cyclic 3
janet(i);
↳ Length of Janet basis: 4
↳ _[1]=x+y+z
↳ _[2]=y2+yz+z2
↳ _[3]=z3-1
↳ _[4]=yz3-y
```

See [Section 5.1.53 \[groebner\], page 191](#); [Section 4.6 \[ideal\], page 80](#); [Section 5.1.151 \[std\], page 271](#).

### 5.1.68 jet

**Syntax:** `jet ( poly_expression , int_expression )`  
`jet ( vector_expression , int_expression )`  
`jet ( ideal_expression , int_expression )`  
`jet ( module_expression , int_expression )`  
`jet ( poly_expression , int_expression , intvec_expression )`  
`jet ( vector_expression , int_expression , intvec_expression )`  
`jet ( ideal_expression , int_expression , intvec_expression )`  
`jet ( module_expression , int_expression , intvec_expression )`  
`jet ( poly_expression , poly_expression , int_expression , intvec_expression )`  
`jet ( vector_expression , poly_expression , int_expression , intvec_expression )`  
`jet ( ideal_expression , matrix_expression , int_expression , intvec_expression )`  
`jet ( module_expression , matrix_expression , int_expression , intvec_expression )`  
`jet ( poly_expression , poly_expression , int_expression , intvec_expression )`  
`jet ( vector_expression , poly_expression , int_expression )`  
`jet ( ideal_expression , matrix_expression , int_expression )`  
`jet ( module_expression , matrix_expression , int_expression )`

**Type:** the same as the type of the first argument

**Purpose:** deletes from the first argument all terms of degree bigger than the second argument. If a third/fourth argument `w` of type `intvec` is given, the degree is replaced by the weighted degree defined by `w`. If a second argument `u` of type `poly` or `matrix` is given, the first argument `p` is replaced by `p/u`. In this case, the coefficient must be from a field.

**Example:**

```
ring r=32003,(x,y,z),(c,dp);
jet(1+x+x2+x3+x4,3);
↳ x3+x2+x+1
poly f=1+x+x2+xz+y2+x3+y3+x2y2+z4;
jet(f,3);
↳ x3+y3+x2+y2+xz+x+1
intvec iv=2,1,1;
jet(f,3,iv);
↳ y3+y2+xz+x+1
```

```

// the part of f with (total) degree >3:
f-jet(f,3);
↦ x2y2+z4
// the homogeneous part of f of degree 2:
jet(f,2)-jet(f,1);
↦ x2+y2+xz
// the part of maximal degree:
jet(f,deg(f))-jet(f,deg(f)-1);
↦ x2y2+z4
// the absolute term of f:
jet(f,0);
↦ 1
// now for other types:
ideal i=f,x,f*f;
jet(i,2);
↦ _[1]=x2+y2+xz+x+1
↦ _[2]=x
↦ _[3]=3x2+2y2+2xz+2x+1
vector v=[f,1,x];
jet(v,1);
↦ [x+1,1,x]
jet(v,0);
↦ [1,1]
v=[f,1,0];
module m=v,v,[1,x2,z3,0,1];
jet(m,2);
↦ _[1]=[x2+y2+xz+x+1,1]
↦ _[2]=[x2+y2+xz+x+1,1]
↦ _[3]=[1,x2,0,0,1]
ring rs=0,x,ds;
// 1/(1+x) till degree 5
jet(1,1+x,5);
↦ 1-x+x2-x3+x4-x5

```

See [Section 5.1.19 \[deg\]](#), page 170; [Section 4.6 \[ideal\]](#), page 80; [Section 4.7 \[int\]](#), page 85; [Section 4.9 \[intvec\]](#), page 93; [Section 4.14 \[module\]](#), page 112; [Section 4.17 \[poly\]](#), page 120; [Section 4.23 \[vector\]](#), page 134.

### 5.1.69 kbase

**Syntax:**    `kbase ( ideal_expression )`  
               `kbase ( module_expression )`  
               `kbase ( ideal_expression, int_expression )`  
               `kbase ( module_expression, int_expression )`

**Type:**     the same as the input type of the first argument

**Purpose:**    With one argument: computes a vector space basis (consisting of monomials) of the quotient ring by the ideal, resp. of a free module by the module, in case it is finite dimensional and if the input is a standard basis with respect to the ring ordering.

Note that, if the input is not a standard basis, the leading terms of the input are used and the result may have no meaning.

With two arguments: computes the part of a vector space basis of the respective quotient with degree of the monomials equal to the second argument. Here, the quotient

does not need to be finite dimensional. If an attribute `isHomog` (of type `intvec`) is present, it is used as module weight.

**Example:**

```

ring r=32003,(x,y,z),ds;
ideal i=x2,y2,z;
kbase(std(i));
↳ _[1]=xy
↳ _[2]=y
↳ _[3]=x
↳ _[4]=1
i=x2,y3,xyz; // quotient not finite dimensional
kbase(std(i),2);
↳ _[1]=z2
↳ _[2]=yz
↳ _[3]=xz
↳ _[4]=y2
↳ _[5]=xy

```

See [Section 4.6 \[ideal\]](#), page 80; [Section 4.14 \[module\]](#), page 112; [Section 5.1.168 \[vdim\]](#), page 286.

### 5.1.70 kernel

**Syntax:** `kernel ( ring_name, map_name )`

**Type:** `ideal`

**Purpose:** returns the kernel of a given map.

The second argument has to be a map from the basering to the given ring (or an ideal defining such a map).

**Example:**

```

ring r1=32003,(x,y,z,w),lp;
ring r=32003,(x,y,z),dp;
ideal i=x,y,z;
map f=r1,i;
setring r1;
// the kernel of f
kernel(r,f);
↳ _[1]=w

```

See [Section D.4.2.5 \[alg\\_kernel\]](#), page 1008; [Section D.4.11.15 \[hom\\_kernel\]](#), page 1110; [Section 4.6 \[ideal\]](#), page 80; [Section 4.12 \[map\]](#), page 106; [Section 5.1.94 \[modulo\]](#), page 223; [Section 5.1.117 \[preimage\]](#), page 240; [Section 4.20 \[ring\]](#), page 127.

### 5.1.71 kill

**Syntax:** `kill name`  
`kill list_of_names`

**Type:** `none`

**Purpose:** deletes objects.

**Example:**

```

int i=3;
ring r=0,x,dp;
poly p;
listvar();
↳ // r                                [0] *ring
↳ //      p                            [0] poly
↳ // i                                [0] int 3
kill i,r;
// the variable 'i' does not exist any more
i;
↳ ? 'i' is undefined
↳ ? error occurred in or before ./examples/kill.sing line 7: ' i;'
listvar();

```

See [Section 5.1.18 \[defined\]](#), page 170; [Section D.2.3 \[general\\_lib\]](#), page 806.

### 5.1.72 killattrib

**Syntax:** killattrib ( name )  
killattrib ( name, string\_expression )

**Type:** none

**Purpose:** deletes all attributes respective the attribute given as the second argument.

**Example:**

```

ring r=32003,(x,y),lp;
ideal i=maxideal(1);
attrib(i,"isSB",1);
attrib(i);
↳ attr:isSB, type int
killattrib(i,"isSB");
attrib(i);
↳ no attributes
attrib(i,"isSB",1);
killattrib(i);
attrib(i);
↳ no attributes

```

See [Section 5.1.2 \[attrib\]](#), page 156; [Section 5.1.111 \[option\]](#), page 234.

### 5.1.73 koszul

**Syntax:** koszul ( int\_expression, int\_expression )  
koszul ( int\_expression, ideal\_expression )  
koszul ( int\_expression, int\_expression, ideal\_expression )

**Type:** matrix

**Purpose:** koszul(d,n) computes a matrix of the Koszul relations of degree d of the first n ring variables.

koszul(d,id) computes a matrix of the Koszul relations of degree d of the generators of the ideal id.

koszul(d,n,id) computes a matrix of the Koszul relations of degree d of the first n generators of the ideal id.

**Note:** `koszul(1,id)`, `koszul(2,id)`, ... form a complex, that is, the product of the matrices `koszul(i,id)` and `koszul(i+1,id)` equals zero.

**Example:**

```

ring r=32003,(x,y,z),dp;
print(koszul(2,3));
↪ -y,-z,0,
↪ x, 0, -z,
↪ 0, x, y
ideal I=xz2+yz2+z3,xyz+y2z+yz2,xy2+y3+y2z;
print(koszul(1,I));
↪ xz2+yz2+z3,xyz+y2z+yz2,xy2+y3+y2z
print(koszul(2,I));
↪ -xyz-y2z-yz2,-xy2-y3-y2z,0,
↪ xz2+yz2+z3, 0, -xy2-y3-y2z,
↪ 0, xz2+yz2+z3, xyz+y2z+yz2
print(koszul(2,I)*koszul(3,I));
↪ 0,
↪ 0,
↪ 0

```

See [Section 4.7 \[int\]](#), page 85; [Section 4.13 \[matrix\]](#), page 108.

### 5.1.74 laguerre

**Syntax:** `laguerre ( poly_expression )`  
`laguerre ( poly_expression , int_expression , int_expression )`

**Type:** list

**Purpose:** In characteristic 0:  
 computes all complex roots of a univariate polynomial using Laguerre's algorithm. The second argument defines the precision of the fractional part if the ground field is the field of rational numbers, otherwise it will be ignored (default: 10). third argument (can be 0, 1 or 2) gives the number of extra runs for Laguerre's algorithm (with corrupted roots), leading to better results (default: 1).  
 In characteristic p:  
 computes all roots of a univariate polynomial using factorization

**Note:** If the ground field is the field of complex numbers, the elements of the list are of type number, otherwise of type string.

**Example:**

```

ring rs1=0,(x,y),lp;
poly f=15x5+x3+x2-10;
laguerre(f);
↪ [1]:
↪ 0.8924637479
↪ [2]:
↪ (-0.7392783383+I*0.5355190078)
↪ [3]:
↪ (-0.7392783383-I*0.5355190078)
↪ [4]:
↪ (0.2930464644-I*0.9003002396)
↪ [5]:

```

```

⇒      (0.2930464644+I*0.9003002396)
laguerre(f,30,2);
⇒ [1]:
⇒      0.892463747926447366650738925573
⇒ [2]:
⇒      (-0.739278338347345779837700592458+I*0.53551900780423973942049588296)
⇒ [3]:
⇒      (-0.739278338347345779837700592458-I*0.53551900780423973942049588296)
⇒ [4]:
⇒      (0.293046464384122096512331129672-I*0.90030023962350314542842479521)
⇒ [5]:
⇒      (0.293046464384122096512331129672+I*0.90030023962350314542842479521)

```

### 5.1.75 lead

**Syntax:**    `lead ( poly_expression )`  
               `lead ( vector_expression )`  
               `lead ( ideal_expression )`  
               `lead ( module_expression )`

**Type:**     the same as the input type

**Purpose:**    returns the leading (or initial) term(s) of a polynomial, a vector, resp. of the generators of an ideal or module with respect to the monomial ordering.

**Note:**      IN may be used instead of `lead`.

**Example:**

```

ring r=32003,(x,y,z),(c,ds);
poly f=2x2+3y+4z3;
vector v=[2x10,f];
ideal i=f,z;
module m=v,[0,0,2+x];
lead(f);
⇒ 3y
lead(v);
⇒ [2x10]
lead(i);
⇒ _[1]=3y
⇒ _[2]=z
lead(m);
⇒ _[1]=[2x10]
⇒ _[2]=[0,0,2]
lead(0);
⇒ 0

```

See [Section 4.6 \[ideal\], page 80](#); [Section 5.1.76 \[leadcoef\], page 209](#); [Section 5.1.77 \[leadexp\], page 210](#); [Section 5.1.78 \[leadmonom\], page 210](#); [Section 4.14 \[module\], page 112](#); [Section 4.17 \[poly\], page 120](#); [Section 4.23 \[vector\], page 134](#).

### 5.1.76 leadcoef

**Syntax:**    `leadcoef ( poly_expression )`  
               `leadcoef ( vector_expression )`

**Type:** number

**Purpose:** returns the leading (or initial) coefficient of a polynomial or a vector with respect to the monomial ordering.

**Example:**

```
ring r=32003,(x,y,z),(c,ds);
poly f=x2+y+z3;
vector v=[2*x^10,f];
leadcoef(f);
↳ 1
leadcoef(v);
↳ 2
leadcoef(0);
↳ 0
```

See [Section 5.1.75 \[lead\]](#), page 209; [Section 5.1.77 \[leadexp\]](#), page 210; [Section 5.1.78 \[leadmonom\]](#), page 210; [Section 4.17 \[poly\]](#), page 120; [Section 4.23 \[vector\]](#), page 134.

### 5.1.77 leadexp

**Syntax:** leadexp ( poly\_expression )  
leadexp ( vector\_expression )

**Type:** intvec

**Purpose:** returns the exponent vector of the leading monomial of a polynomial or a vector. In the case of a vector the last component is the index in the vector. (The inverse to monomial.)

**Example:**

```
ring r=32003,(x,y,z),(c,ds);
poly f=x2+y+z3;
vector v=[2*x^10,f];
leadexp(f);
↳ 0,1,0
leadexp(v);
↳ 10,0,0,1
leadexp(0);
↳ 0,0,0
```

See [Section 4.9 \[intvec\]](#), page 93; [Section 5.1.75 \[lead\]](#), page 209; [Section 5.1.76 \[leadcoef\]](#), page 209; [Section 5.1.78 \[leadmonom\]](#), page 210; [Section 5.1.96 \[monomial\]](#), page 225; [Section 4.17 \[poly\]](#), page 120; [Section 4.23 \[vector\]](#), page 134.

### 5.1.78 leadmonom

**Syntax:** leadmonom ( poly\_expression )  
leadmonom ( vector\_expression )

**Type:** the same as the input type

**Purpose:** returns the leading monomial of a polynomial or a vector as a polynomial or vector whose coefficient is one.

**Example:**



```

ring r=32003,(x,y,z),(c,ds);
poly f=2x2+3y+4z3;
vector v=[0,2x10,f];
leadmonom(f);
↳ y
leadmonom(v);
↳ [0,x10]
leadmonom(0);
↳ 0

```

See [Section 4.9 \[intvec\]](#), page 93; [Section 5.1.75 \[lead\]](#), page 209; [Section 5.1.76 \[leadcoef\]](#), page 209; [Section 5.1.77 \[leadexp\]](#), page 210; [Section 4.17 \[poly\]](#), page 120; [Section 4.23 \[vector\]](#), page 134.

### 5.1.79 LIB

**Syntax:** LIB string\_expression;

**Type:** none

**Purpose:** reads a library of procedures from a file. In contrast to the command `load`, the procedures from the library are added to the package `Top` as well as the package corresponding to the library. If the given filename does not start with `.` or `/` and cannot be located in the current directory, each directory contained in the library `SearchPath` is searched for file of this name. See [Section 3.8.11 \[Loading a library\]](#), page 67, for more info on `SearchPath`.

**Note on standard.lib:**

Unless SINGULAR is started with the `--no-stdlib` option, the library `standard.lib` is automatically loaded at start-up time.

**Example:**

```

option(loadLib); // show loading of libraries

// the names of the procedures of inout.lib
LIB "inout.lib"; // are now known to Singular
↳ // ** loaded inout.lib (4.1.2.0, Feb_2019)

```

See [Section 3.1.6 \[Command line options\]](#), page 19; [Section 2.3.3 \[Procedures and libraries\]](#), page 10; [Appendix D \[SINGULAR libraries\]](#), page 794; [Section 5.2.12 \[load\]](#), page 299; [Section 4.16 \[package\]](#), page 119; [Section 4.18 \[proc\]](#), page 124; [Section D.1 \[standard.lib\]](#), page 794; [Section 4.22 \[string\]](#), page 130; [Section 5.1.155 \[system\]](#), page 275.

### 5.1.80 lift

**Syntax:** lift ( ideal\_expression, subideal\_expression )  
 lift ( module\_expression, submodule\_expression )  
 lift ( ideal\_expression, subideal\_expression, matrix\_name )  
 lift ( module\_expression, submodule\_expression, matrix\_name )  
 lift ( ideal\_expression, subideal\_expression, matrix\_name, string\_expression )  
 lift ( module\_expression, submodule\_expression, matrix\_name, string\_expression )

**Type:** matrix

**Purpose:** computes the transformation matrix which expresses the generators of a submodule in terms of the generators of a module. Depending on which algorithm is used, modules

are represented by a standard basis, or not.

More precisely, if  $m$  is the module (or ideal),  $sm$  the submodule (or ideal), and  $T$  the transformation matrix returned by `lift`, then `matrix(sm)*U = matrix(m)*T` and `module(sm*U) = module(matrix(m)*T)` (resp. `ideal(sm) = ideal(matrix(m)*T)`), where  $U$  is a diagonal matrix of units.

$U$  is always the identity if the basering is a polynomial ring (not power series ring).  $U$  is stored in the optional third argument.

**Note:** Gives a warning if  $sm$  is not a submodule.

An optional 4th argument specifies the Groebner base algorithm to use. Possible values are "std" and "slimgb".

**Example:**

```
ring r=32003,(x,y,z),(dp,C);
ideal m=3x2+yz,7y6+2x2y+5xz;
poly f=y7+x3+xyz+z2;
ideal i=jacob(f);
matrix T=lift(i,m);
matrix(m)-matrix(i)*T;
⇒ _[1,1]=0
⇒ _[1,2]=0
```

See [Section 5.1.26 \[division\]](#), page 174; [Section 4.6 \[ideal\]](#), page 80; [Section 5.1.81 \[liftstd\]](#), page 212; [Section 4.14 \[module\]](#), page 112; [Section 5.1.151 \[std\]](#), page 271; [Section 5.1.156 \[syz\]](#), page 280.

### 5.1.81 liftstd

**Syntax:** `liftstd ( ideal_expression, matrix_name[, module_name][, string_expression ][, ideal_expression ])`  
`liftstd ( module_expression, matrix_name[, module_name][, string_expression ][, module_expression ])`

**Type:** ideal or module

**Purpose:** returns a standard basis of an ideal or module and the transformation matrix from the given ideal, resp. module, to the standard basis.

That is, if  $m$  is the ideal or module,  $sm$  the standard basis returned by `liftstd`, and  $T$  the transformation matrix (`sm=liftstd(m,T)`) then `matrix(sm)=matrix(m)*T` and `sm=ideal(matrix(m)*T)`, resp. `sm=module(matrix(m)*T)`. If working in a quotient ring, then `matrix(sm)=reduce(matrix(m)*T,0)` and `sm=reduce(ideal(matrix(m)*T),0)`.

If a module name is given as a third argument, the syzygy module will be returned. (`sm=liftstd(m,T,s)` then additional `matrix(m)*matrix(s)=0`).

An optional string argument specifies the Groebner base algorithm to use. Possible values are "std" and "slimgb".

Given an optional last argument (say  $n$ ), the algorithm computes a standard bases of  $(m+n)$ , syzygies of  $m$  modulo  $n$ , and the transformation matrix only for  $m$ . These are relative transformation matrix resp. the syzygy module of  $n$  modulo  $m$ . (For syzygies, the same can be achieved using [Section 5.1.94 \[modulo\]](#), page 223.)

**Example:**

```
ring R=0,(x,y,z),dp;
poly f=x3+y7+z2+xyz;
ideal i=jacob(f);
```

```

matrix T;
ideal sm=liftstd(i,T);
sm;
↪ sm[1]=xy+2z
↪ sm[2]=3x2+yz
↪ sm[3]=yz2+3048192z3
↪ sm[4]=3024xz2-yz2
↪ sm[5]=y2z-6xz
↪ sm[6]=3097158156288z4+2016z3
↪ sm[7]=7y6+xz
print(T);
↪ 0,1,T[1,3], T[1,4],y, T[1,6],0,
↪ 0,0,-3x+3024z,3x, 0, T[2,6],1,
↪ 1,0,T[3,3], T[3,4],-3x,T[3,6],0
matrix(sm)-matrix(i)*T;
↪ _[1,1]=0
↪ _[1,2]=0
↪ _[1,3]=0
↪ _[1,4]=0
↪ _[1,5]=0
↪ _[1,6]=0
↪ _[1,7]=0
module s;
sm=liftstd(i,T,s);
print(s);
↪ -xy-2z,0, s[1,3],s[1,4],xyz+2z2, -14y5z+x2z,
↪ 0, -xy-2z,s[2,3],s[2,4],-3x2y-6xz,-3x3+2z2,
↪ 3x2+yz,7y6+xz,7y6+xz,s[3,4],21xy6-yz2,21x2y5-xz2
matrix(i)*matrix(s);
↪ _[1,1]=0
↪ _[1,2]=0
↪ _[1,3]=0
↪ _[1,4]=0
↪ _[1,5]=0
↪ _[1,6]=0

```

See [Section 5.1.26 \[division\]](#), page 174; [Section 4.6 \[ideal\]](#), page 80; [Section 5.1.80 \[lift\]](#), page 211; [Section 4.13 \[matrix\]](#), page 108; [Section 5.1.94 \[modulo\]](#), page 223; [Section 5.1.111 \[option\]](#), page 234; [Section 4.20 \[ring\]](#), page 127; [Section 5.1.151 \[std\]](#), page 271; [Section 5.1.156 \[syz\]](#), page 280.

### 5.1.82 listvar

**Syntax:** `listvar ( [package] )`  
`listvar ( [package,] type )`  
`listvar ( [package,] ring_name )`  
`listvar ( [package,] name )`  
`listvar ( [package,] all )`

**Type:** none

**Purpose:** lists all (user-)defined names:

- `listvar()`: all currently visible names except procedures in the current namespace,
- `listvar(type)`: all currently visible names of the given type,

- `listvar(ring_name)`: all names which belong to the given ring,
- `listvar(name)`: the object with the given name,
- `listvar(all)`: all names except procedures in the current and Top namespace.

The current basering is marked with a \*. The nesting level of variables in procedures is shown in square brackets.

package can be `Current`, `Top` or any other identifier of type package.

**Example:**

```

proc t1 { }
proc t2 { }
ring s;
poly ss;
ring r;
poly f=x+y+z;
int i=7;
ideal I=f,x,y;
listvar();
⇒ // i                [0] int 7
⇒ // r                [0] *ring
⇒ //      I           [0] ideal, 3 generator(s)
⇒ //      f           [0] poly
⇒ // s                [0] ring
    listvar(r);
⇒ // r                [0] *ring
⇒ // I                [0] ideal, 3 generator(s)
⇒ // f                [0] poly
    listvar(t1);
⇒ // t1                [0] proc
    listvar(proc);
⇒ // t2                [0] proc
⇒ // t1                [0] proc
⇒ // mathicgb_prOrder  [0] proc from singmathic.so (C)
⇒ // mathicgb          [0] proc from singmathic.so (C)
⇒ // create_ring       [0] proc from standard.lib
⇒ // min               [0] proc from standard.lib
⇒ // max               [0] proc from standard.lib
⇒ // datetime          [0] proc from standard.lib
⇒ // weightKB          [0] proc from standard.lib
⇒ // fprintf           [0] proc from standard.lib
⇒ // printf            [0] proc from standard.lib
⇒ // sprintf           [0] proc from standard.lib
⇒ // quotient4         [0] proc from standard.lib
⇒ // quotient5         [0] proc from standard.lib
⇒ // quotient3         [0] proc from standard.lib
⇒ // quotient2         [0] proc from standard.lib
⇒ // quotient1         [0] proc from standard.lib
⇒ // quot              [0] proc from standard.lib
⇒ // res               [0] proc from standard.lib
⇒ // groebner          [0] proc from standard.lib
⇒ // qslimb            [0] proc from standard.lib
⇒ // hilbRing          [0] proc from standard.lib
⇒ // par2varRing       [0] proc from standard.lib

```

```

⇒ // quotientList                [0]  proc from standard.lib
⇒ // stdhilb                     [0]  proc from standard.lib
⇒ // stdfglm                     [0]  proc from standard.lib
⇒ // Float                      [0]  proc from kernel (C)
⇒ // crossprod                  [0]  proc from kernel (C)
LIB "polylib.lib";
listvar(Poly);
⇒      ? Poly is undefined
⇒      ? error occurred in or before ./examples/listvar.sing line 14: ' li
var(Poly);'

```

See [Section 3.5.3 \[Names\]](#), page 44; [Section 3.7.4 \[Names in procedures\]](#), page 54; [Section 5.1.18 \[defined\]](#), page 170; [Section 5.1.103 \[names\]](#), page 229; [Section 4.16 \[package\]](#), page 119; [Section 5.1.160 \[type\]](#), page 282.

### 5.1.83 lres

**Syntax:** `lres ( ideal_expression, int_expression )`

**Type:** resolution

**Purpose:** computes a free resolution of an ideal using LaScala's algorithm.

More precisely, let  $R$  be the basering and  $I$  be the given ideal. Then `lres` computes a minimal free resolution of  $R/I$

$$\dots \longrightarrow F_2 \xrightarrow{A_2} F_1 \xrightarrow{A_1} R \longrightarrow R/I \longrightarrow 0.$$

If the `int_expression`  $k$  is not zero then the computation stops after  $k$  steps and returns a list of modules  $M_i = \text{module}(A_i)$ ,  $i=1..k$ .

`list L=lres(I,0)`; returns a list  $L$  of  $n$  modules (where  $n$  is the number of variables of the basering) such that  $L[i] = M_i$  in the above notation.

**Note:** The `ideal_expression` has to be homogeneous.

Accessing single elements of a resolution may require that some partial computations have to be finished and may therefore take some time.

**Example:**

```

ring r=0,(x,y,z),dp;
ideal I=xz,yz,x3-y3;
def L=lres(I,0);
print(betti(L),"beti");
⇒          0      1      2
⇒  -----
⇒    0:      1      -      -
⇒    1:      -      2      1
⇒    2:      -      1      1
⇒  -----
⇒ total:      1      3      2
⇒
L[2];      // the first syzygy module of r/I
⇒ _[1]=-x*gen(1)+y*gen(2)
⇒ _[2]=-x2*gen(2)+y2*gen(1)+z*gen(3)

```

See [Section 5.1.4 \[beti\]](#), page 159; [Section 5.1.48 \[fres\]](#), page 188; [Section 5.1.58 \[hres\]](#), page 197; [Section 4.6 \[ideal\]](#), page 80; [Section 4.7 \[int\]](#), page 85; [Section 5.1.93 \[minres\]](#), page 223; [Section 4.14 \[module\]](#), page 112; [Section 5.1.98 \[mres\]](#), page 225; [Section 5.1.99 \[mres\\_map\]](#), page 226; [Section 5.1.134 \[res\]](#), page 253; [Section 5.1.149 \[sres\]](#), page 269.

### 5.1.84 ludecomp

qcindex Gauss

**Syntax:** `ludecomp ( matrix_expression )`

**Type:** list

**Purpose:** Computes the LU-decomposition of an  $(m \times n)$  matrix.

The matrix,  $A$  say, must consist of numbers, only. This means that when the basering represents some  $K[x_1, x_2, \dots, x_r]$ , then all entries of  $A$  must come from the ground field  $K$ .

The LU-decomposition of  $A$  is a triple of matrices  $P$ ,  $L$ , and  $U$  such that

- $P * A = L * U$ ,
- $P$  is an  $(m \times m)$  permutation matrix, i.e., its rows/columns form the standard basis of  $K^m$ ,
- $L$  is an  $(m \times m)$  matrix in lower triangular form with all diagonal entries equal to 1, and
- $U$  is an  $(m \times n)$  matrix in upper row echelon form.

From these conditions, it easily follows that also  $A = P * L * U$  holds, since  $P$  is self-inverse.

`list L=ludecomp(A)`; fills a list  $L$  with the three above entries  $P$ ,  $L$ , and  $U$ .

**Example:**

```
ring r=0,(x),dp;
matrix A[3][4]=1,2,3,4,1,1,1,1,2,2,1,1;
list plu = ludecomp(A);
print(plu[3]);                // the matrix U of the decomposition
⇨ 1,2, 3, 4,
⇨ 0,-1,-2,-3,
⇨ 0,0, -1,-1
print(plu[1]*A-plu[2]*plu[3]); // should be the zero matrix
⇨ 0,0,0,0,
⇨ 0,0,0,0,
⇨ 0,0,0,0
```

See `system("rref",A)`; . See [Section 5.1.85 \[luinverse\], page 216](#); [Section 5.1.86 \[lusolve\], page 217](#); [Section 5.1.155 \[system\], page 275](#).

### 5.1.85 luinverse

qcindex Gauss

**Syntax:** `luinverse ( matrix_expression )`

**Type:** matrix

**Syntax:** `luinverse ( matrix_expression, matrix_expression, matrix_expression )`

**Type:** matrix

**Purpose:** Computes the inverse of a matrix  $A$ , if  $A$  is invertible.

The matrix  $A$  must be given either directly, or by its LU-decomposition. In the latter case, three matrices  $P$ ,  $L$ , and  $U$  are expected, in this order, which satisfy

- $P * A = L * U$ ,
- $P$  is an  $(m \times m)$  permutation matrix, i.e., its rows/columns form the standard basis

of  $K^m$ ,

- L is an  $(m \times m)$  matrix in lower triangular form with all diagonal entries equal to 1, and

- U is an  $(m \times m)$  matrix in upper row echelon form.

Then, the inverse of A exists if and only if U is invertible, and one has  $A^{-1} = U^{-1} \cdot L^{-1} \cdot P$ , since P is self-inverse.

In the case of A being given directly, `luinverse` first computes its LU-decomposition, and then proceeds as in the case when P, L, and U are provided.

`list L=luinverse(A)`; fills the list L with either one entry = 0 (signaling that A is not invertible), or with the two entries 1,  $A^{-1}$ . Thus, in either case the user may first check the condition `L[1]==1` to find out whether A is invertible.

**Note:** The method will give a warning for any non-quadratic matrix A.

**Example:**

```
ring r=0,(x),dp;
matrix A[3][3]=1,2,3,1,1,1,2,2,1;
list L = luinverse(A);
if (L[1] == 1)
{
  print(L[2]);
  "----- next should be the (3 x 3)-unit matrix:";
  print(A*L[2]);
}
↪ -1,4, -1,
↪ 1, -5,2,
↪ 0, 2, -1
↪ ----- next should be the (3 x 3)-unit matrix:
↪ 1,0,0,
↪ 0,1,0,
↪ 0,0,1
```

See `system("rref",A)`; . See [Section 5.1.84 \[ludecomp\]](#), page 216; [Section 5.1.86 \[lusolve\]](#), page 217; [Section 5.1.155 \[system\]](#), page 275.

### 5.1.86 lusolve

qcindex Gauss

**Syntax:** `lusolve ( matrix_expression, matrix_expression, matrix_expression, matrix_expression )`

**Type:** matrix

**Purpose:** Computes all solutions of a linear equation system  $A \cdot x = b$ , if solvable  
 The  $(m \times n)$  matrix A must be given by its LU-decomposition, that is, by three matrices P, L, and U, in this order, which satisfy

- $P \cdot A = L \cdot U$ ,
- P is an  $(m \times m)$  permutation matrix, i.e., its rows/columns form the standard basis of  $K^m$ ,
- L is an  $(m \times m)$  matrix in lower triangular form with all diagonal entries equal to 1, and
- U is an  $(m \times n)$  matrix in upper row echelon form.

The fourth argument, b, is expected to be an  $(m \times 1)$  matrix.

`list Q=lusolve(P,L,U,b)`; fills the list `Q` with either one entry  $= 0$  (signaling that  $A*x=b$  has no solution), or with the three entries  $1, x, H$ , where  $x$  is any  $(n \times 1)$  solution of the given linear system, and  $H$  is a matrix the columns of which span the solution space of the homogeneous linear system. (I.e., `ncols(H)` is the dimension of the solution space.)

If there is exactly one solution, then  $H$  is the  $1 \times 1$  matrix with entry zero.

**Note:** The method will give a warning if the matrices violate the above conditions regarding row and column numbers, or if the number of rows of the vector `b` does not equal `m`. The method expects matrices with entries coming from the ground field of the given polynomial ring, only.

**Example:**

```
ring r=0,(x),dp;
matrix A[4][4]=1,1,1,0,1,2,3,1,1,3,5,2,1,4,7,3;
matrix b[4][1]=2,5,8,11;
list L=ludecomp(A);
list Q=lusolve(L[1],L[2],L[3],b);
if (Q[1] == 1)
{
  "one solution:";
  print(Q[2]);
  "check whether result is correct (iff next is zero vector):";
  print(A*Q[2]-b);
  if ((nrows(Q[3])==1) and (ncols(Q[3])==1) and (Q[3][1,1]==0))
  { "printed solution is the only solution to given linear system" }
  else
  {
    "homogeneous solution space is spanned by columns of:";
    print(Q[3]);
  }
}
↪ one solution:
↪ -1,
↪ 3,
↪ 0,
↪ 0
↪ check whether result is correct (iff next is zero vector):
↪ 0,
↪ 0,
↪ 0,
↪ 0
↪ homogeneous solution space is spanned by columns of:
↪ -1,-1,
↪ 1, 2,
↪ 0, -1,
↪ -1,0
```

See `system("rref",A)`; see [Section 5.1.84 \[ludecomp\]](#), page 216; [Section 5.1.85 \[luinverse\]](#), page 216; [Section 5.1.155 \[system\]](#), page 275.

### 5.1.87 max

Procedure from library `standard.lib` (see [Section D.1 \[standard\\_lib\]](#), page 794).



**Syntax:** `max (i_1, ..., i_k)`

**Type:** same as type of `i_1, ..., i_k` resp.

**Purpose:** returns the maximum for any arguments of a type for which '`>`' is defined

**Example:**

```
// biggest int
max(2,3);
↳ 3
max(1,4,3);
↳ 4
// lexicographically biggest intvec
max(intvec(1,2),intvec(0,1),intvec(1,1));
↳ 1,2
// polynomial with biggest leading monomial
ring r = 0,x,dp;
max(x+1,x2+x);
↳ x2+x
```

See also: [Section 5.1.90 \[min\]](#), page 220.

### 5.1.88 maxideal

**Syntax:** `maxideal ( int_expression )`

**Type:** ideal

**Purpose:** returns the power given by `int_expression` of the maximal ideal generated by all ring variables (`maxideal(i)=1` for `i<=0`).

**Example:**

```
ring r=32003,(x,y,z),dp;
maxideal(2);
↳ _[1]=z2
↳ _[2]=yz
↳ _[3]=y2
↳ _[4]=xz
↳ _[5]=xy
↳ _[6]=x2
```

See [Section 4.6 \[ideal\]](#), page 80; [Section 4.20 \[ring\]](#), page 127.

### 5.1.89 memory

**Syntax:** `memory ( int_expression )`

**Type:** bigint

**Purpose:** returns statistics concerning the memory management:

- `memory(0)` is the number of active (used) bytes,
- `memory(1)` is the number of bytes allocated from the operating system,
- `memory(2)` is the maximal number of bytes ever allocated from the operating system during the current SINGULAR session.

**Note:** To monitor the memory usage during ongoing computations the option `mem` should be set (using the command `option(mem);`; see also [Section 5.1.111 \[option\]](#), page 234).

**Example:**

```
ring r=0,(x(1..500)),dp;
poly p=(x(1)+x(500))^50;
proc ReportMemoryUsage()
{ "Memory currently used by SINGULAR      :",memory(0),"Byte (",
  int(memory(0) div 1024), "KByte)" +newline+
  "Memory currently allocated from system:",memory(1), "Byte (",
  int(memory(1) div 1024), "KByte)";
  "Maximal memory allocated from system  :",memory(2), "Byte (",
  int(memory(2) div 1024), "KByte)";
}
ReportMemoryUsage();
↪ Memory currently used by SINGULAR      : 154560 Byte ( 151 KByte)
↪ Memory currently allocated from system: 2232320 Byte ( 2180 KByte)
↪ Maximal memory allocated from system  : 2232320 Byte ( 2180 KByte)
kill p;
ReportMemoryUsage(); // less memory used: p killed
↪ Memory currently used by SINGULAR      : 83992 Byte ( 82 KByte)
↪ Memory currently allocated from system: 2232320 Byte ( 2180 KByte)
↪ Maximal memory allocated from system  : 2232320 Byte ( 2180 KByte)
kill r;
ReportMemoryUsage(); // even less memory: r killed
↪ Memory currently used by SINGULAR      : 71752 Byte ( 70 KByte)
↪ Memory currently allocated from system: 2232320 Byte ( 2180 KByte)
↪ Maximal memory allocated from system  : 2232320 Byte ( 2180 KByte)
```

See [Section 5.1.111 \[option\]](#), page 234; [Section 5.1.155 \[system\]](#), page 275.

### 5.1.90 min

Procedure from library `standard.lib` (see [Section D.1 \[standard.lib\]](#), page 794).

**Syntax:** `max (i_1, ..., i_k)`

**Type:** same as type of `i_1, ..., i_k` resp.

**Purpose:** returns the maximum for any arguments of a type for which `'>'` is defined

**Example:**

```
// biggest int
max(2,3);
↪ 3
max(1,4,3);
↪ 4
// lexicographically biggest intvec
max(intvec(1,2),intvec(0,1),intvec(1,1));
↪ 1,2
// polynomial with biggest leading monomial
ring r = 0,x,dp;
max(x+1,x2+x);
↪ x2+x
```

See also: [Section 5.1.90 \[min\]](#), page 220.

### 5.1.91 minbase

**Syntax:**    `minbase ( ideal_expression )`  
               `minbase ( module_expression )`

**Type:**     the same as the type of the argument

**Purpose:**    returns a minimal set of generators of an ideal, resp. module, if the input is either homogeneous or if the ordering is local.

**Note:**      this command is not available over coefficient rings.

**Example:**

```
ring r=181,(x,y,z),(c,ls);
ideal id2=x2+xyz,y2-z3y,z3+y5xz;
ideal id4=maxideal(3)+id2;
size(id4);
↳ 13
minbase(id4);
↳ _[1]=x2
↳ _[2]=xyz+x2
↳ _[3]=xz2
↳ _[4]=y2
↳ _[5]=yz2
↳ _[6]=z3
```

See [Section 5.1.100 \[mstd\], page 227](#).

### 5.1.92 minor

**Syntax:**    `minor ( matrix_expression M, int_expression mSize,`  
               `[ ideal_expression I ],`  
               `[ int_expression k ],`  
               `[ string_expression algorithm ],`  
               `[ int_expression cachedP ],`  
               `[ int_expression cachedM ] )`

**Type:**     ideal

**Purpose:**    returns the specified set of (mSize x mSize)-minors (= subdeterminants) of the given matrix M. These minors form the list of generators of the returned ideal.

If the optional ideal I is given, it is assumed to capture a standard basis. In this case, all computations will be performed modulo I.

If k is not given, all minors will be computed. Otherwise, if  $k > 0$ , the first k non-zero minors will be computed; for  $k < 0$ , the first  $|k|$  minors will be computed regardless whether they are zero or not. Here, "first k minors" is with respect to a fixed ordering among all minors. (To understand the ordering, run the below example, type `minor(m,2,i,18)`; and inspect the ordering among the returned 18 minors. Note that this ordering is only enforced when some  $k \neq 0$  is provided. Otherwise, no ordering among the returned minors can be guaranteed. This is due to the fact that in this case, `minor` may call a specially tuned implementation of Bareiss's algorithm.)

If no algorithm is given, a heuristic will pick the best-suited algorithm among Bareiss's algorithm (which is only applicable over integral domains), Laplace's algorithm, and Laplace's algorithm combined with caching of subdeterminantes. In the heuristic setting, `cacheP` and `cacheM` must also be absent.

If the argument `algorithm` is present it must be one of `B/bareiss`, `L/laplace`, and `C/cache`. For, `B/bareiss` and `L/laplace` the optional arguments `cacheP` and `cacheM` must again be absent, whereas for `C/cache`, they may be provided: `cachedP` determines the maximum number of cached subdeterminantes (=polynomials), and `cachedM` the total number of cached monomials (counted over all cached polynomials). If, for `algorithm = C/cache` `cachedP` and `cachedM` are not provided by the user, the values 200 and 100000, respectively, will be used as defaults.

**Note:** If `mSize` is larger than the given matrix, `minor` returns 0, if `mSize` is smaller than 1, `minor` returns 1.

**Example:**

```

ring r=0,(a,b,c,d,e,f,g,h,s,t,u,v),ds;
matrix m[3][4]=a,b,c,d,e,f,g,h,s,t,u,v;
print(m);
↳ a,b,c,d,
↳ e,f,g,h,
↳ s,t,u,v
// let's compute all non-zero minors;
// here we do not guarantee any ordering:
minor(m,2);
↳ _[1]=-hu+gv
↳ _[2]=-ht+fv
↳ _[3]=-hs+ev
↳ _[4]=-du+cv
↳ _[5]=-dt+bv
↳ _[6]=-ds+av
↳ _[7]=gt-fu
↳ _[8]=gs-eu
↳ _[9]=ct-bu
↳ _[10]=cs-au
↳ _[11]=-fs+et
↳ _[12]=-bs+at
↳ _[13]=-dg+ch
↳ _[14]=-df+bh
↳ _[15]=-de+ah
↳ _[16]=cf-bg
↳ _[17]=ce-ag
↳ _[18]=-be+af
ideal i=a,c; i=std(i);
// here come the first 4 non-zero minors mod I;
// this time, a fixed ordering is guaranteed:
minor(m,2,i,4);
↳ _[1]=-be
↳ _[2]=bg
↳ _[3]=-de
↳ _[4]=-df+bh
// and here the first 4 minors mod I (possibly zero)
// using Laplace's algorithm,
// again, the fixed ordering is guaranteed:
minor(m,2,i,-4,"Laplace");
↳ _[1]=-be
↳ _[2]=0
↳ _[3]=bg

```

$\mapsto$  `_ [4] = -de`

See [Section 5.1.23 \[det\]](#), page 172.

### 5.1.93 minres

**Syntax:** `minres ( list_expression )`

**Type:** `list`

**Syntax:** `minres ( resolution_expression )`

**Type:** `resolution`

**Purpose:** minimizes a free resolution of an ideal or module given by the `list_expression`, resp. `resolution_expression`.

**Example:**

```

ring r1=32003,(x,y),dp;
ideal i=x5+xy4,x3+x2y+xy2+y3;
resolution rs=lres(i,0);
rs;
 $\mapsto$  1      2      1
 $\mapsto$  r1 <-- r1 <-- r1
 $\mapsto$ 
 $\mapsto$  0      1      2
 $\mapsto$ 
list(rs);
 $\mapsto$  [1]:
 $\mapsto$  _ [1]=x3+x2y+xy2+y3
 $\mapsto$  _ [2]=xy4
 $\mapsto$  [2]:
 $\mapsto$  _ [1]=xy4*gen(1)-x3*gen(2)-x2y*gen(2)-xy2*gen(2)-y3*gen(2)
minres(rs);
 $\mapsto$  1      2      1
 $\mapsto$  r1 <-- r1 <-- r1
 $\mapsto$ 
 $\mapsto$  0      1      2
 $\mapsto$ 
list(rs);
 $\mapsto$  [1]:
 $\mapsto$  _ [1]=x3+x2y+xy2+y3
 $\mapsto$  _ [2]=xy4
 $\mapsto$  [2]:
 $\mapsto$  _ [1]=xy4*gen(1)-x3*gen(2)-x2y*gen(2)-xy2*gen(2)-y3*gen(2)
```

See [Section 5.1.48 \[fres\]](#), page 188; [Section 5.1.98 \[mres\]](#), page 225; [Section 5.1.99 \[mres\\_map\]](#), page 226; [Section 5.1.134 \[res\]](#), page 253; [Section 5.1.149 \[sres\]](#), page 269.

### 5.1.94 modulo

**Syntax:** `modulo ( ideal_expression, ideal_expression )`

`modulo ( module_expression, module_expression )`

`modulo ( ideal_expression, ideal_expression, string_expression )`

`modulo ( module_expression, module_expression, string_expression )`

`modulo ( ideal_expression, ideal_expression, matrix_name )`

**Type:** module

**Purpose:** `modulo(h1,h2)` represents  $h_1/(h_1 \cap h_2) \cong (h_1 + h_2)/h_2$  where  $h_1$  and  $h_2$  are considered as submodules of the same free module  $R^l$  ( $l=1$  for ideals). Let  $H_1$ , resp.  $H_2$ , be the matrices of size  $l \times k$ , resp.  $l \times m$ , having the generators of  $h_1$ , resp.  $h_2$ , as columns. Then  $h_1/(h_1 \cap h_2) \cong R^k/\ker(\overline{H_1})$  where  $\overline{H_1} : R^k \rightarrow R^l/Im(H_2) = R^l/h_2$  is the induced map.

`modulo(h1,h2)` returns generators of the kernel of this induced map.

An optional third string argument give the GB algorithm to use. (default: "std", else: "groebner", "modstd", "slimgb").

An optional third argument (a name of a matrix) gives the matrix to store the transformation of the input to the GB (see [Section 5.1.81 \[liftstd\]](#), [page 212](#)).

**Note:** If for at least one of `h1` or `h2` the attribute "isHomog" is set, `modulo(h1,h2)` also sets the attribute "isHomog" (if possible, that is, if the weights are compatible).

**Example:**

```
ring r;
ideal h1=x,y,z;
ideal h2=x;
module m=modulo(h1,h2);
print(m);
↪ 1,0, 0,0,
↪ 0,-z,x,0,
↪ 0,y, 0,x
```

See [Section D.4.11.15 \[hom\\_kernel\]](#), [page 1110](#); [Section 5.1.156 \[syz\]](#), [page 280](#).

## 5.1.95 monitor

**Syntax:** `monitor ( link_expression )`  
`monitor ( link_expression, string_expression )`

**Type:** none

**Purpose:** controls the recording of all user input and/or program output into a file. The second argument describes what to log: "i" means input, "o" means output, "io" for both. The default for the second argument is "i".

Each `monitor` command closes a previous monitor file and opens the file given by the first string expression.

`monitor ("")` turns off recording.

**Example:**

```
monitor("doe.tmp","io"); // log input and output to doe.tmp
ring r;
poly f=x+y+z;
int i=7;
ideal I=f,x,y;
monitor(""); // stop logging:
// doe.tmp contains now all input and output from the example above
```

See [Section 4.10.2 \[link expressions\]](#), [page 97](#).

### 5.1.96 monomial

**Syntax:** `monomial ( intvec_expression )`

**Type:** poly resp. vector

**Purpose:** converts an integer vector to a power product (the inverse to `leadexp`).  
Returns a **vector** iff the length of the argument is number of variables +1.

**Example:**

```
ring r=0,(x,y,z),dp;
monomial(intvec(2,3));
↦ x2y3
monomial(intvec(2,3,0,1));
↦ x2y3*gen(1)
leadexp(monomial(intvec(2,3,0,1)));
↦ 2,3,0,1
```

See [Section 4.9 \[intvec\]](#), page 93; [Section 5.1.77 \[leadexp\]](#), page 210.

### 5.1.97 mpresmat

**Syntax:** `mpresmat ( ideal_expression, int_expression )`

**Type:** module

**Purpose:** computes the multipolynomial resultant matrix of the input system. Uses the sparse resultant matrix method of Gelfand, Kapranov and Zelevinsky (second parameter = 0) or the resultant matrix method of Macaulay (second parameter = 1).

**Note:** When using the resultant matrix method of Macaulay the input system must be homogeneous. The number of elements in the input system must be the number of variables in the basering plus one.

**Example:**

```
ring rsq=(0,s,t,u),(x,y),lp;
ideal i=s+tx+uy,x2+y2-10,x2+xy+2y2-16;
module m=mpresmat(i,0);
print(m);
↦ -16,0, -10,0, (s),0, 0, 0, 0, 0,
↦ 0, -16,0, -10,(u),(s),0, 0, 0, 0,
↦ 2, 0, 1, 0, 0, (u),0, 0, 0, 0,
↦ 0, 2, 0, 1, 0, 0, 0, 0, 0, 0,
↦ 0, 0, 0, 0, (t),0, -10,(s),0, -16,
↦ 1, 0, 0, 0, 0, (t),0, (u),(s),0,
↦ 0, 1, 0, 0, 0, 0, 1, 0, (u),2,
↦ 1, 0, 1, 0, 0, 0, 0, 0, (t),0, 0,
↦ 0, 1, 0, 1, 0, 0, 0, 0, (t),1,
↦ 0, 0, 0, 0, 0, 0, 1, 0, 0, 1
```

See [Section 5.1.163 \[uressolve\]](#), page 283.

### 5.1.98 mres

**Syntax:** `mres ( ideal_expression, int_expression )`

`mres ( module_expression, int_expression )`

**Type:** resolution

**Purpose:** computes a minimal free resolution of an ideal or module  $M$  with the standard basis method. More precisely, let  $A=\text{matrix}(M)$ , then  $\text{mres}$  computes a free resolution of  $\text{coker}(A) = F_0/M$

$$\dots \longrightarrow F_2 \xrightarrow{A_2} F_1 \xrightarrow{A_1} F_0 \longrightarrow F_0/M \longrightarrow 0,$$

where the columns of the matrix  $A_1$  are a minimal set of generators of  $M$  if the basering is local or if  $M$  is homogeneous. If the int expression  $k$  is not zero, then the computation stops after  $k$  steps and returns a list of modules  $M_i = \text{module}(A_i)$ ,  $i=1\dots k$ .

$\text{mres}(M,0)$  returns a resolution consisting of at most  $n+2$  modules, where  $n$  is the number of variables of the basering. Let  $\text{list } L=\text{mres}(M,0)$ ; then  $L[1]$  consists of a minimal set of generators of the input,  $L[2]$  consists of a minimal set of generators for the first syzygy module of  $L[1]$ , etc., until  $L[p+1]$ , such that  $L[i] \neq 0$  for  $i \leq p$ , but  $L[p+1]$ , the first syzygy module of  $L[p]$ , is 0 (if the basering is not a qring).

**Note:** Accessing single elements of a resolution may require some partial computations to be finished and may therefore take some time.

**Example:**

```
ring r=31991,(t,x,y,z,w),ls;
ideal M=t2x2+tx2y+x2yz,t2y2+ty2z+y2zw,
        t2z2+tz2w+xz2w,t2w2+txw2+xyw2;
resolution L=mres(M,0);
L;
↦ 1      4      15      18      7      1
↦ r <--  r <--  r <--  r <--  r <--  r
↦
↦ 0      1      2      3      4      5
↦
// projective dimension of M is 5
```

See [Section 5.1.48 \[fres\]](#), page 188; [Section 5.1.58 \[hres\]](#), page 197; [Section 4.6 \[ideal\]](#), page 80; [Section 5.1.83 \[lres\]](#), page 215; [Section 4.14 \[module\]](#), page 112; [Section 5.1.134 \[res\]](#), page 253; [Section 5.1.149 \[sres\]](#), page 269.

### 5.1.99 mres\_map

**Syntax:**  $\text{mres\_map}(\text{ideal\_expression}, \text{int\_expression})$   
 $\text{mres\_map}(\text{module\_expression}, \text{int\_expression})$

**Type:** resolution

**Purpose:** computes a minimal free resolution of an ideal or module  $M$  with the standard basis method. More precisely, let  $A=\text{matrix}(M)$ , then  $\text{mres\_map}$  computes a free resolution of  $\text{coker}(A) = F_0/M$

$$\dots \longrightarrow F_2 \xrightarrow{A_2} F_1 \xrightarrow{A_1} F_0 \longrightarrow F_0/M \longrightarrow 0,$$

where the columns of the matrix  $A_1$  are a minimal set of generators of  $M$  if the basering is local or if  $M$  is homogeneous. If the int expression  $k$  is not zero, then the computation stops after  $k$  steps and returns a list of modules  $M_i = \text{module}(A_i)$ ,  $i=1\dots k$ .

$\text{mres\_map}(M,0,T)$  returns a resolution consisting of at most  $n+2$  modules, where  $n$  is



the number of variables of the basering. Let `list L=mres_map(M,0,T)`; then `L[1]` consists of a minimal set of generators of the input, `L[2]` consists of a minimal set of generators for the first syzygy module of `L[1]`, etc., until `L[p+1]`, such that `L[i] ≠ 0` for  $i \leq p$ , but `L[p+1]`, the first syzygy module of `L[p]`, is 0 (if the basering is not a qring).

**Note:** Accessing single elements of a resolution may require some partial computations to be finished and may therefore take some time.

**Example:**

```
ring r=31991,(t,x,y,z,w),ls;
ideal M=t2x2+tx2y+x2yz,t2y2+ty2z+y2zw,
        t2z2+tz2w+xz2w,t2w2+txw2+xyw2;
smatrix T;
resolution L=mres_map(M,0,T);
L;
↪ 1      4      15      18      7      1
↪ r <--  r <--  r <--  r <--  r <--  r
↪
↪ 0      1      2      3      4      5
↪
    // projective dimension of M is 5
print(T);
↪ 0,0,0,1,
↪ 1,0,0,0,
↪ 0,1,0,0,
↪ 0,0,1,0
```

See [Section 5.1.48 \[fres\]](#), page 188; [Section 5.1.58 \[hres\]](#), page 197; [Section 4.6 \[ideal\]](#), page 80; [Section 5.1.83 \[lres\]](#), page 215; [Section 4.14 \[module\]](#), page 112; [Section 5.1.98 \[mres\]](#), page 225; [Section 5.1.134 \[res\]](#), page 253; [Section 5.1.149 \[sres\]](#), page 269.

### 5.1.100 mstd

**Syntax:** `mstd ( ideal_expression )`  
`mstd ( module_expression )`

**Type:** list

**Purpose:** returns a list whose first entry is a standard basis for the ideal, resp. module, whose second entry is a generating set for the ideal, resp. module. If the coefficient ring is a field and either the ideal/module is homogeneous or the ordering is local, this second entry is a minimal generating set.

**Example:**

```
ring r=0,(x,y,z,t),dp;
poly f=x3+y4+z6+xyz;
ideal j=jacob(f),f;
j=homog(j,t);j;
↪ j[1]=3x2+yz
↪ j[2]=4y3+xzt
↪ j[3]=6z5+xyt3
↪ j[4]=0
↪ j[5]=z6+y4t2+x3t3+xyzt3
mstd(j);
```

```

⇒ [1]:
⇒   _[1]=3x2+yz
⇒   _[2]=4y3+xzt
⇒   _[3]=6z5+xyt3
⇒   _[4]=xyzt3
⇒   _[5]=y2z2t3
⇒   _[6]=yz3t4
⇒   _[7]=xz3t4
⇒   _[8]=yz2t7
⇒   _[9]=xz2t7
⇒   _[10]=y2zt7
⇒   _[11]=xy2t7
⇒ [2]:
⇒   _[1]=3x2+yz
⇒   _[2]=4y3+xzt
⇒   _[3]=6z5+xyt3
⇒   _[4]=xyzt3

```

See [Section 4.6 \[ideal\]](#), page 80; [Section 5.1.91 \[minbase\]](#), page 221; [Section 4.14 \[module\]](#), page 112; [Section 5.1.151 \[std\]](#), page 271.

### 5.1.101 mult

**Syntax:**    `mult ( ideal_expression )`  
              `mult ( module_expression )`

**Type:**     `int`

**Purpose:**    computes the degree of the monomial ideal, resp. module, generated by the leading monomials of the input.

If the input is a standard basis of a homogeneous ideal then it returns the degree of this ideal.

If the input is a standard basis of an ideal in a (local) ring with respect to a local degree ordering then it returns the multiplicity of the ideal (in the sense of Samuel, with respect to the maximal ideal).

**Example:**

```

ring r=32003,(x,y),ds;
poly f=(x3+y5)^2+x2y7;
ideal i=std(jacob(f));
mult(i);
⇒ 46
mult(std(f));
⇒ 6

```

See [Section 5.1.20 \[degree\]](#), page 171; [Section 5.1.25 \[dim\]](#), page 174; [Section 4.6 \[ideal\]](#), page 80; [Section 5.1.151 \[std\]](#), page 271; [Section 5.1.168 \[vdim\]](#), page 286.

### 5.1.102 nameof

**Syntax:**    `nameof ( expression )`

**Type:**     `string`

**Purpose:**    returns the name of an expression as string.

**Example:**

```

int i=9;
string s=nameof(i);
s;
↳ i
  nameof(s);
↳ s
  nameof(i+1); //returns the empty string:
↳
  nameof(basering);
↳ basering
  basering;
↳ ? 'basering' is undefined
↳ ? error occurred in or before ./examples/nameof.sing line 7: ' base
    ng;'
  ring r;
  nameof(basering);
↳ r

```

See [Section 5.1.103 \[names\], page 229](#); [Section 5.1.135 \[reservedName\], page 254](#); [Section 5.1.161 \[typeof\], page 282](#).

**5.1.103 names**

**Syntax:**    `names ( )`  
              `names ( ring_name )`  
              `names ( package_name )`  
              `names ( level )`

**Type:**     list of strings

**Purpose:**    returns the names of all user-defined variables which are ring independent (this includes the names of procedures) or, in the second case, which belong to the given ring. The third case restricts the variables to the given level.  
              package\_name can be **Current**, **Top** or any other identifier of type package.

**Example:**

```

int i=9;
ring r;
poly f;
package p;
int j; exportto(p,j);
poly g;
setring r;
list l=names();
l[1..3];
↳ l p r
  names(r);
↳ [1]:
↳ g
↳ [2]:
↳ f
  names(p);
↳ [1]:

```

```
↳ j
   names(0);
↳ [1]:
↳ 1
↳ [2]:
↳ p
↳ [3]:
↳ r
↳ [4]:
↳ i
↳ [5]:
↳ mathicgb_prOrder
↳ [6]:
↳ mathicgb
↳ [7]:
↳ Singmathic
↳ [8]:
↳ create_ring
↳ [9]:
↳ min
↳ [10]:
↳ max
↳ [11]:
↳ datetime
↳ [12]:
↳ weightKB
↳ [13]:
↳ fprintf
↳ [14]:
↳ printf
↳ [15]:
↳ sprintf
↳ [16]:
↳ quotient4
↳ [17]:
↳ quotient5
↳ [18]:
↳ quotient3
↳ [19]:
↳ quotient2
↳ [20]:
↳ quotient1
↳ [21]:
↳ quot
↳ [22]:
↳ res
↳ [23]:
↳ groebner
↳ [24]:
↳ qslimgb
↳ [25]:
↳ hilbRing
↳ [26]:
```

```

↳      par2varRing
↳ [27]:
↳      quotientList
↳ [28]:
↳      stdhilb
↳ [29]:
↳      stdfglm
↳ [30]:
↳      Standard
↳ [31]:
↳      Float
↳ [32]:
↳      crossprod
↳ [33]:
↳      ZZ
↳ [34]:
↳      QQ
↳ [35]:
↳      Top

```

See [Section 5.1.102 \[nameof\]](#), page 228; [Section 5.1.135 \[reservedName\]](#), page 254.

### 5.1.104 ncols

**Syntax:**    `ncols ( matrix_expression )`  
              `ncols ( smatrix_expression )`  
              `ncols ( intmat_expression )`  
              `ncols ( ideal_expression )`

**Type:**     `int`

**Purpose:**    returns the number of columns of a matrix, an intmat, or the number of given generators of the ideal, including zeros.

**Note:**      `size(ideal)` counts the number of generators which are different from zero. (Use `nrows` to get the number of rows of a given matrix or intmat.)

**Example:**

```

      ring r;
      matrix m[5][6];
      ncols(m);
↳ 6
      ideal i=x,0,y;
      ncols(i);
↳ 3
      size(i);
↳ 2

```

See [Section 4.13 \[matrix\]](#), page 108; [Section 5.1.107 \[nrows\]](#), page 232; [Section 5.1.144 \[size\]](#), page 264; [Section 4.21 \[smatrix\]](#), page 129.

### 5.1.105 npars

**Syntax:**    `npars ( ring_name )`

**Type:**     `int`

**Purpose:** returns the number of parameters of a ring.

**Example:**

```
ring r=(23,t,v),(x,a(1..7)),lp;
// the parameters are t,v
npars(r);
↪ 2
```

See [Section 5.1.114 \[par\]](#), page 239; [Section 5.1.116 \[parstr\]](#), page 240; [Section 4.20 \[ring\]](#), page 127.

### 5.1.106 nres

**Syntax:** `nres ( ideal_expression, int_expression )`  
`nres ( module_expression, int_expression )`

**Type:** resolution

**Purpose:** computes a free resolution of an ideal or module M which is minimized from the second module on (by the standard basis method).

More precisely, let  $A_1 = \text{matrix}(M)$ , then `nres` computes a free resolution of  $\text{coker}(A_1) = F_0/M$

$$\dots \longrightarrow F_2 \xrightarrow{A_2} F_1 \xrightarrow{A_1} F_0 \longrightarrow F_0/M \longrightarrow 0,$$

where the columns of the matrix  $A_1$  are the given set of generators of M. If the int expression k is not zero then the computation stops after k steps and returns a list of modules  $M_i = \text{module}(A_i)$ ,  $i = 1, \dots, k$ .

`nres(M,0)` returns a list of n modules where n is the number of variables of the basering. Let list  $L = \text{nres}(M,0)$ ; then  $L[1] = M$  is identical to the input,  $L[2]$  is a minimal set of generators for the first syzygy module of  $L[1]$ , etc. ( $L[i] = M_i$  in the notations from above).

**Example:**

```
ring r=31991,(t,x,y,z,w),ls;
ideal M=t2x2+tx2y+x2yz,t2y2+ty2z+y2zw,
        t2z2+tz2w+xz2w,t2w2+txw2+xyw2;
resolution L=nres(M,0);
L;
↪ 1      4      15      18      7      1
↪ r <--  r <--  r <--  r <--  r <--  r
↪
↪ 0      1      2      3      4      5
↪ resolution not minimized yet
↪
```

See [Section 5.1.48 \[fres\]](#), page 188; [Section 5.1.58 \[hres\]](#), page 197; [Section 4.6 \[ideal\]](#), page 80; [Section 5.1.83 \[lres\]](#), page 215; [Section 4.14 \[module\]](#), page 112; [Section 5.1.98 \[mres\]](#), page 225; [Section 5.1.134 \[res\]](#), page 253; [Section 4.19 \[resolution\]](#), page 125; [Section 5.1.149 \[sres\]](#), page 269.

### 5.1.107 nrows

**Syntax:** `nrows ( matrix_expression )`  
`nrows ( smatrix_expression )`  
`nrows ( intmat_expression )`

```
nrows ( intvec_expression )
nrows ( module_expression )
nrows ( vector_expression )
```

**Type:** int

**Purpose:** returns the number of rows of a matrix, an intmat or an intvec, resp. the minimal rank of a free module in which the given module or vector lives (the index of the last non-zero component).

**Note:** Use `ncols` to get the number of columns of a given matrix or intmat.

**Example:**

```
ring R;
matrix M[2][3];
nrows(M);
↳ 2
nrows(freemodule(4));
↳ 4
module m=[0,0,1];
nrows(m);
↳ 3
nrows([0,x,0]);
↳ 2
```

See [Section 5.1.51 \[gen\]](#), page 191; [Section 4.13 \[matrix\]](#), page 108; [Section 4.14 \[module\]](#), page 112; [Section 5.1.104 \[ncols\]](#), page 231; [Section 4.21 \[smatrix\]](#), page 129; [Section 4.23 \[vector\]](#), page 134.

### 5.1.108 numerator

**Syntax:** `numerator ( number_expression )`

**Type:** number

**Purpose:** returns the numerator of a number.

**Example:**

```
ring r = 0, x, dp;
number n = 3/2;
numerator(n);
↳ 3
```

See [Section 5.1.9 \[cleardenom\]](#), page 163; [Section D.2.8.14 \[content\]](#), page 895; [Section 5.1.22 \[denominator\]](#), page 172.

### 5.1.109 nvars

**Syntax:** `nvars ( ring_name )`

**Type:** int

**Purpose:** returns the number of variables of a ring.

**Example:**

```
ring r=(23,t,v),(x,a(1..7)),ls;
// the variables are x,a(1),...,a(7)
nvars(r);
↳ 8
```

See [Section 5.1.105 \[npars\]](#), page 231; [Section 4.20 \[ring\]](#), page 127; [Section 5.1.165 \[var\]](#), page 284; [Section 5.1.167 \[varstr\]](#), page 285.

### 5.1.110 open

**Syntax:**     `open ( link_expression )`

**Type:**       none

**Purpose:**     opens a link.

**Example:**

```
link l="ssi:tcp localhost:"+system("Singular");
open(l); // start SINGULAR "server" on localhost in batchmode
close(l); // shut down SINGULAR server
```

See [Section 5.1.10 \[close\]](#), page 163; [Section 4.10 \[link\]](#), page 96.

### 5.1.111 option

**Syntax:**     `option ( )`

**Type:**       string

**Purpose:**     lists all defined options.

**Syntax:**     `option ( option_name )`

**Type:**       none

**Purpose:**     sets an option.

**Note:**       To disable an option, use the prefix `no`.

**Syntax:**     `option ( get )`

**Type:**       intvec

**Purpose:**     dumps the state of all options to an intvec.

**Syntax:**     `option ( set, intvec_expression )`

**Type:**       none

**Purpose:**     restores the state of all options from an intvec (produced by `option(get)`).

**Values:**     The following options are used to manipulate the behavior of computations and act like boolean switches. Use the prefix `no` to disable an option. Notice that some options are ring dependent and reset to their default values on a change of the current basering.

`none`         turns off all options (including the `prompt` option).

`warn`         be aware of pitfalls. See [Section 3.9.7 \[option\(warn\)\]](#), page 71.

`returnSB`     the functions `syz`, `intersect` (2 arguments), `quotient` return a standard base instead of a generating set if `returnSB` is set. This option should not be used for `lift`.



<b>fastHC</b>	tries to find the highest corner of the staircase (HC) as fast as possible during a standard basis computation (only used for local orderings).
<b>infRedTail</b>	local normal form computations will not use the ecart to avoid possibly infinite tail reductions: should only be used with extreme care. By default, it is only set in the case of a zero-dimensional ideal.
<b>intStrategy</b>	avoids division of coefficients during standard basis computations. This option is ring dependent. By default, it is set for rings with characteristic 0 and not set for all other rings.
<b>lazy</b>	uses a more lazy approach in std computations, which was used in SINGULAR version before 2-0 (and which may lead to faster or slower computations, depending on the example)
<b>length</b>	select shorter reducers in std computations,
<b>notRegularity</b>	disables the regularity bound for <b>res</b> and <b>mres</b> (see <a href="#">Section 5.1.132 [regularity]</a> , page 252).
<b>notSugar</b>	turns off sugar strategy during standard basis computation and reduction.
<b>notBuckets</b>	disables the bucket representation of polynomials during standard basis computations. This option usually decreases the memory consumption but increases the computation time. It should only be set for memory-critical standard basis computations.
<b>prot</b>	shows protocol information indicating the progress during the following computations: <b>facstd</b> , <b>fglm</b> , <b>groebner</b> , <b>intersect</b> , <b>lres</b> , <b>mres</b> , <b>minres</b> , <b>mstd</b> , <b>res</b> , <b>slimgb</b> , <b>sres</b> , <b>std</b> , <b>stdfglm</b> , <b>stdhilb</b> , <b>syz</b> . See below for more details.
<b>qringNF</b>	simplifies modulo the current <b>qring</b> in all assignments.
<b>redSB</b>	computes a reduced standard basis in any standard basis computation (in rings with global ior local orderings, See <a href="#">Section 5.1.64 [interred]</a> , page 201 for the discussion of reduced for local orderings))
<b>redTail</b>	reduction of the tails of polynomials during standard basis computations. This option is ring dependent. By default, it is set for rings with global degree orderings and not set for all other rings. This option changes the reduction strategy and may decrease/increase time and memory consumption - it does not ensure tail reduction on the result - use <b>redSB</b> for that.
<b>redThrough</b>	for inhomogeneous input, polynomial reductions during standard basis computations are never postponed, but always finished through. This option is ring dependent. By default, it is set for rings with global degree orderings and not set for all other rings. This option changes the reduction strategy and may decrease/increase time and memory consumption.

<code>sugarCrit</code>	uses criteria similar to the homogeneous case to keep more pairs which would be excluded by other criteria but which may be useful for downstream computations. This option changes the strategy for criteria and selection and may decrease/increase time and memory consumption.
<code>weightM</code>	automatically computes suitable weights for the weighted ecart and the weighted sugar method.
<code>cancelunit</code>	avoids to divide polynomials by non-constant units in <code>std</code> in the local case. Should usually not be used.
<code>contentSB</code>	avoids to divide by the content of a polynomial in <code>std</code> and related algorithms. Should usually not be used.
<code>intersectElim</code>	prefers elimination to compute intersections (experimental, will be removed in the next release). Should usually not be used.
<code>intersectSyz</code>	prefers syzygy methods to compute intersections (experimental, will be removed in the next release). Should usually not be used.

The following options, which also control computations, are special, since they are not manipulated by the `option` command but by a direct assignment of a value. Reset the option by assigning the value 0; the command `option(none)` will not reset them! If there is a non-zero value assigned, the command `option()` prints the option.

<code>multBound</code>	a multiplicity bound is set (see <a href="#">Section 5.3.4 [multBound]</a> , page 304).
<code>degBound</code>	a degree bound is set (see <a href="#">Section 5.3.1 [degBound]</a> , page 302).

The last set of options controls the output of SINGULAR:

<code>Imap</code>	shows the mapping of variables with the fetch commands.
<code>debugLib</code>	warns about syntax errors when loading a library.
<code>defRes</code>	shows the names of the syzygy modules while converting <code>resolution</code> to <code>list</code>
<code>loadLib</code>	shows loading of libraries (set by default).
<code>loadProc</code>	shows loading of procedures from libraries.
<code>mem</code>	shows memory usage in square brackets (see <a href="#">Section 5.1.89 [memory]</a> , page 219).
<code>notWarnSB</code>	do not warn about using a generating set instead of a standard basis.
<code>prompt</code>	shows prompt ( <code>&gt;</code> , resp. <code>.</code> ) if ready for input (default).
<code>reading</code>	shows the number of characters read from a file.
<code>redefine</code>	warns about variable redefinitions (set by default).

`usage` shows correct usage in error messages (set by default).

**Example:**

```

    option(prot);
    option();
    ↪ //options: prot redefine usage prompt
    option(notSugar);
    option();
    ↪ //options: prot notSugar redefine usage prompt
    option(noprot);
    option();
    ↪ //options: notSugar redefine usage prompt
    option(none);
    option();
    ↪ //options: none
    ring r=0,x,dp;
    degBound=22;
    option();
    ↪ //options: degBound redTail redThrough intStrategy
    intvec i=option(get);
    option(none);
    option(set,i);
    option();
    ↪ //options: degBound redTail redThrough intStrategy

```

The output reported on `option(prot)` has the following meaning:

(command)	(character)	(meaning)
<code>facstd</code>	<code>F</code>	found a new factor
		all other characters: like the output of <code>std</code> and <code>reduce</code>
<code>fglm</code>	<code>.</code>	basis monomial found
	<code>+</code>	edge monomial found
	<code>-</code>	border monomial found
<code>groebner</code>		all characters: like the output of <code>std/slimgb</code>
<code>lres</code>	<code>.</code>	minimal syzygy found
	<code>n</code>	slanted degree, i.e., row of Betti matrix
	<code>(mn)</code>	calculate in module <code>n</code>
	<code>g</code>	pair found giving reductum and syzygy
<code>mres</code>	<code>[d]</code>	computations of the <code>d</code> -th syzygy module
		all other characters: like the output of <code>std</code>
<code>minres</code>	<code>[d]</code>	minimizing of the <code>d</code> -th syzygy module
<code>mstd</code>		all characters: like the output of <code>std</code>
<code>reduce</code>	<code>r</code>	reduced a leading term
	<code>t</code>	reduced a non-leading term

<b>res</b>	[d]	computations of the d-th syzygy module all other characters: like the output of <b>std</b>
<b>slimgb</b>	M[n,m]	parallel reduction of n elements with m non-zero output elements
	v	candidate for postponing, need to canonicalize
	.	postponed a reduction of a pair/S-polynomial
	b	exchange of a reductor by a 'better' one
	e	a new reductor with non-minimal leading term
	r	redTail reduction
	n	no redTail reduction
	B	resort pairs
	C	slimgb_alg::cleanDegs
	(n)	n critical pairs are still to be reduced
	d	the maximal degree of the leading terms is currently d
<b>sres</b>	.	syzygy found
	(n)	n elements remaining
	[n]	finished module n
<b>std</b>	[m:n]	internal ring change to polynomial representation with exponent bound m and n words in exponent vector
	s	found a new element of the standard basis
	-	reduced a pair/S-polynomial to 0
	.	postponed a reduction of a pair/S-polynomial
	h	used Hilbert series criterion
	H(d)	found a 'highest corner' of degree d, no need to consider higher degrees
	(n)	n critical pairs are still to be reduced
	(S:n)	doing complete reduction of n elements
	d	the degree of the leading terms is currently d
<b>stdfglm</b>		all characters in first part: like the output of <b>std</b> all characters in second part: like the output of <b>fglm</b>
<b>stdhilb</b>		all characters: like the output of <b>std</b>
<b>syz</b>		all characters: like the output of <b>std</b>

See [Section 5.3.1 \[degBound\]](#), page 302; [Section 5.3.4 \[multBound\]](#), page 304; [Section 5.1.151 \[std\]](#), page 271.

### 5.1.112 ord

**Syntax:**    `ord ( poly_expression )`  
               `ord ( vector_expression )`

**Type:**     `int`

**Purpose:**    returns the (weighted) degree of the initial term of a polynomial or a vector; the weights are the weights used for the first block of the ring ordering.

**Note:**      `ord(0)` is -1.  
               In a global degree ordering **ord** is the same as **deg**.

**Example:**

```

ring r=7,(x,y),wp(2,3);
ord(0);
↳ -1
poly f=x2+y3; // weight on y is 3
ord(f),deg(f);
↳ 9 9
ring R=7,(x,y),ws(2,3);
poly f=x2+y3;
ord(f),deg(f);
↳ 4 9
vector v=[x2,y];
ord(v),deg(v);
↳ 3 4

```

See [Section 5.1.19 \[deg\]](#), page 170; [Section 4.17 \[poly\]](#), page 120; [Section 4.23 \[vector\]](#), page 134.

**5.1.113 ordstr**

**Syntax:** `ordstr ( ring_name )`

**Type:** string

**Purpose:** returns the description of the monomial ordering of the ring.

**Example:**

```

ring r=7,(x,y),wp(2,3);
ordstr(r);
↳ wp(2,3),C

```

See [Section 5.1.7 \[charstr\]](#), page 162; [Section 5.1.116 \[parstr\]](#), page 240; [Section 4.20 \[ring\]](#), page 127; [Section 5.1.167 \[varstr\]](#), page 285.

**5.1.114 par**

**Syntax:** `par ( int_expression )`

**Type:** number

**Purpose:** `par(n)`; returns the n-th parameter of the basering.

**Example:**

```

ring r=(0,a,b,c),(x,y,z),dp;
char(r); // char to get the characteristic
↳ 0
par(2); // par to get the n-th parameter
↳ (b)

```

See [Section 5.1.5 \[char\]](#), page 161; [Section 5.1.105 \[npars\]](#), page 231; [Section 5.1.116 \[parstr\]](#), page 240; [Section 4.20 \[ring\]](#), page 127; [Section 5.1.165 \[var\]](#), page 284.

**5.1.115 pardeg**

**Syntax:** `pardeg ( number_expression )`

**Type:** int

**Purpose:** returns the degree of a number considered as a polynomial in the ring parameters.

**Example:**

```
ring r=(0,a,b,c),(x,y,z),dp;
pardeg(a^2*b);
↳ 3
```

See [Section 5.1.19 \[deg\]](#), page 170; [Section 4.15 \[number\]](#), page 115; [Section 4.20 \[ring\]](#), page 127; [Section 5.1.165 \[var\]](#), page 284.

### 5.1.116 parstr

**Syntax:** `parstr ( ring_name )`  
`parstr ( int_expression )`  
`parstr ( ring_name, int_expression )`

**Type:** string

**Purpose:** returns the list of parameters of the ring as a string or the name of the n-th parameter where n is given by the int\_expression.  
 If the ring\_name is omitted, the basering is used, thus `parstr(n)` is equivalent to `parstr(basing,n)`.

**Example:**

```
ring r=(7,a,b,c),(x,y),wp(2,3);
parstr(r);
↳ a,b,c
parstr(2);
↳ b
parstr(r,3);
↳ c
```

See [Section 5.1.7 \[charstr\]](#), page 162; [Section 5.1.105 \[npars\]](#), page 231; [Section 5.1.113 \[ordstr\]](#), page 239; [Section 4.20 \[ring\]](#), page 127; [Section 5.1.167 \[varstr\]](#), page 285.

### 5.1.117 preimage

**Syntax:** `preimage ( map )`  
`preimage ( ring_name, map_name, ideal_name )`  
`preimage ( ring_name, ideal_expression, ideal_name )`

**Type:** ring  
 ideal

**Purpose:** returns the source ring of a map (in the first case) or returns the preimage of an ideal under a given map.  
 The second argument has to be a map from the basering to the given ring (or an ideal defining such a map), and the ideal has to be an ideal in the given ring.

**Note:** As `preimage` is handling ideals (not polynomials), the result of a preimage calculation of a principal ideal is (the closure of) the preimage of the ideal, not that of the polynomial.

**Example:**

```
ring r1=32003,(x,y,z,w),lp;
ring r=32003,(x,y,z),dp;
ideal i=x,y,z;
```

```

ideal i1=x,y;
ideal i0=0;
map f=r1,i;
nameof (preimage (f));
↪ r1
setring r1;
ideal i1=preimage(r,f,i1);
i1;
↪ i1[1]=w
↪ i1[2]=y
↪ i1[3]=x
// the kernel of f
preimage(r,f,i0);
↪ _[1]=w
// or, use:
kernel(r,f);
↪ _[1]=w

```

See [Section 4.6 \[ideal\]](#), page 80; [Section 5.1.70 \[kernel\]](#), page 206; [Section 4.12 \[map\]](#), page 106; [Section 4.20 \[ring\]](#), page 127.

### 5.1.118 prime

**Syntax:** `prime ( int_expression )`

**Type:** `int`

**Purpose:** returns the largest prime less than or equal to the argument; returns 2 for all arguments smaller than 3.

**Example:**

```

prime(320000);
↪ 319993
prime(32004);
↪ 32003
prime(0);
↪ 2
prime(-1);
↪ 2

```

See [Section D.2.3 \[general.lib\]](#), page 806; [Section 4.7 \[int\]](#), page 85.

### 5.1.119 primefactors

**Syntax:** `primefactors ( int/bigint/number_expression )`  
`primefactors ( int/bigint/number_expression , int_expression )`

**Type:** `list`

**Purpose:** returns the prime factorisation up to an optionally given bound, b, on the prime factors. When called with `int(s)/bigint(s)`, no ring needs to be active. When called with numbers these are assumed to be integers in a polynomial ring over  $\mathbb{Q}$ . The method finds all prime factors of an integer n. n' will contain the sign, be zero, or the rest (when a bound is given) respectively. The returned list contains the following

information: The returned list contains the following information:

$L[1][i]$  =  $i$ -th prime factor (in ascending order),

$L[2][i]$  = multiplicity of  $L[1][i]$ ,

$L[3] = n$

**Example:**

```
bigint n = bigint(7)^12 * bigint(37)^6 * 121;
primefactors(n);
⇒ [1]:
⇒ [1]:
⇒ 7
⇒ [2]:
⇒ 11
⇒ [3]:
⇒ 37
⇒ [2]:
⇒ [1]:
⇒ 12
⇒ [2]:
⇒ 2
⇒ [3]:
⇒ 6
⇒ [3]:
⇒ 1
primefactors(n,25);
⇒ [1]:
⇒ [1]:
⇒ 7
⇒ [2]:
⇒ 11
⇒ [2]:
⇒ [1]:
⇒ 12
⇒ [2]:
⇒ 2
⇒ [3]:
⇒ 2565726409
```

See [Section 5.1.118 \[prime\]](#), page 241.

### 5.1.120 print

**Syntax:** `print ( expression )`  
`print ( expression, "betti" )`  
`print ( expression, format_string )`

**Type:** string

**Purpose:** The first form prints the expression.

The second form prints the graded Betti numbers from a matrix. The Betti numbers are printed in a matrix-like format where the entry  $d$  in row  $i$  and column  $j$  is the minimal number of generators in degree  $i + j$  of the  $j$ -th syzygy module of  $R^n/M$  (the 0th and 1st syzygy module of  $R^n/M$  is  $R^n$  and  $M$ , resp.).

The last form returns the printed output as a string depending on the format string which determines the format to use to generate the string.



The following format strings are supported:

<code>"%s"</code>	returns <code>string( expression )</code> ,
<code>"%2s"</code>	similar to <code>"%s"</code> , except that newlines are inserted after every comma and at the end,
<code>"%1"</code>	similar to <code>"%s"</code> , except that each object is embraced by its type such that it can be directly used for "cutting and pasting",
<code>"%21"</code>	similar to <code>"%1"</code> , except that newlines are inserted after every comma and at the end,
<code>"%;"</code>	returns the string equivalent to typing <code>expression;</code>
<code>"%t"</code>	returns the string equivalent to typing <code>type expression;</code>
<code>"%p"</code>	returns the string equivalent to typing <code>print(expression);</code>
<code>"%b"</code>	returns the string equivalent to typing <code>print(expression, "beti");</code>
<code>"beti"</code>	is not a format string.

**Example:**

```

ring r=0,(x,y,z),dp;
module m=[1,y],[0,x+z];
m;
↪ m[1]=y*gen(2)+gen(1)
↪ m[2]=x*gen(2)+z*gen(2)
  print(m); // the columns generate m
↪ 1,0,
↪ y,x+z
  string s=print(m,"%s"); s;
↪ y*gen(2)+gen(1),x*gen(2)+z*gen(2)
  s=print(m,"%2s"); s;
↪ y*gen(2)+gen(1),
↪ x*gen(2)+z*gen(2)
↪
  s=print(m,"%1"); s;
↪ module(y*gen(2)+gen(1),x*gen(2)+z*gen(2))
  s=print(m,"%;"); s;
↪ m[1]=y*gen(2)+gen(1)
↪ m[2]=x*gen(2)+z*gen(2)
↪
  s=print(m,"%t"); s;
↪ // m module , rk 2
↪ m[1]=y*gen(2)+gen(1)
↪ m[2]=x*gen(2)+z*gen(2)
  s=print(m,"%p"); s;
↪ 1,0,
↪ y,x+z
  intmat M=betti(mres(m,0));
  print(M,"beti");
↪
↪          0      1
↪ -----
↪    0:      1      1
↪ -----
↪ total:      1      1

```

```

↳
  list l=r,M;
  s=print(l,"%s"); s;
↳ (QQ),(x,y,z),(dp(3),C),1,1
  s=print(l,"%2s"); s;
↳ (QQ),(x,y,z),(dp(3),C),
↳ 1,1
↳
  s=print(l,"%l"); s;
↳ list("(QQ),(x,y,z),(dp(3),C)",intmat(intvec(1,1 ),1,2))

```

See [Section 3.5.5 \[Type conversion and casting\]](#), page 46; [Section 5.1.4 \[beti\]](#), page 159; [Section 5.1.17 \[dbprint\]](#), page 169; [Section 5.1.46 \[fprintf\]](#), page 186; [Section 5.1.121 \[printf\]](#), page 244; [Section 5.3.7 \[short\]](#), page 305; [Section 5.1.148 \[sprintf\]](#), page 267; [Section 4.22.3 \[string type cast\]](#), page 131; [Section 5.1.160 \[type\]](#), page 282.

### 5.1.121 printf

Procedure from library `standard.lib` (see [Section D.1 \[standard\\_lib\]](#), page 794).

**Syntax:** `printf ( string_expression [, any_expressions] )`

**Return:** none

**Purpose:** `printf(fmt,...)`; performs output formatting. The first argument is a format control string. Additional arguments may be required, depending on the content of the control string. A series of output characters is generated as directed by the control string; these characters are displayed (i.e., printed to standard out).

The control string `fmt` is simply text to be copied, except that the string may contain conversion specifications.

Type `help print`; for a listing of valid conversion specifications. As an addition to the conversions of `print`, the `%n` and `%2` conversion specification does not consume an additional argument, but simply generates a newline character.

**Note:** If one of the additional arguments is a list, then it should be enclosed once more into a `list()` command, since passing a list as an argument flattens the list by one level.

**Example:**

```

  ring r=0,(x,y,z),dp;
  module m=[1,y],[0,x+z];
  intmat M=betti(mres(m,0));
  list l=r,m,matrix(M);
  printf("s:%s,l:%l",1,2);
↳ s:1,l:int(2)
  printf("s:%s",l);
↳ s:(QQ),(x,y,z),(dp(3),C)
  printf("s:%s",list(l));
↳ s:(QQ),(x,y,z),(dp(3),C),y*gen(2)+gen(1),x*gen(2)+z*gen(2),1,1
  printf("2l:%2l",list(l));
↳ 2l:list("(QQ),(x,y,z),(dp(3),C)",
↳ module(y*gen(2)+gen(1),
↳ x*gen(2)+z*gen(2)),
↳ matrix(ideal(1,
↳ 1),1,2))
↳

```

```

printf("%p",matrix(M));
↳ 1,1
printf(";",matrix(M));
↳ _[1,1]=1
↳ _[1,2]=1
↳
printf("%b",M);
↳          0      1
↳ -----
↳    0:      1      1
↳ -----
↳ total:      1      1
↳

```

See also: [Section 5.1.46 \[fprintf\], page 186](#); [Section 5.1.120 \[print\], page 242](#); [Section 5.1.148 \[sprintf\], page 267](#); [Section 4.22 \[string\], page 130](#).

### 5.1.122 prune

**Syntax:** `prune ( module_expression )`

**Type:** `module`

**Purpose:** returns the module minimally embedded in a free module such that the corresponding factor modules are isomorphic.

**Note:** If for the input module the attribute "isHomog" is set, `prune` also sets the attribute "isHomog".  
For non-global orderings, only reduction steps with constant units are performed.  
Hence, the returned module does not need to be minimal.

**Note:** The coefficients must be a field.

**Example:**

```

ring r=0,(x,y,z),dp;
module m=gen(1),gen(3),[x,y,0,z],[x+y,0,0,0,1];
print(m);
↳ 1,0,x,x+y,
↳ 0,0,y,0,
↳ 0,1,0,0,
↳ 0,0,z,0,
↳ 0,0,0,1
print(prune(m));
↳ y,
↳ z

```

See [Section 4.14 \[module\], page 112](#).

### 5.1.123 prune\_map

**Syntax:** `prune_map( module_expression , smatrix_name )`

**Type:** `module`

**Purpose:** returns the module minimally embedded in a free module such that the corresponding factor modules are isomorphic together with the map (as matrix)

**Note:** If for the input module the attribute "isHomog" is set, `prune` also sets the attribute "isHomog".  
For non-global orderings, only reduction steps with constant units are performed. Hence, the returned module does not need to be minimal.

**Example:**

```

ring r=0,(x,y,z),dp;
module m=gen(1),gen(3),[x,y,0,z],[x+y,0,0,0,1];
smatrix M;
print(m);
↳ 1,0,x,x+y,
↳ 0,0,y,0,
↳ 0,1,0,0,
↳ 0,0,z,0,
↳ 0,0,0,1
print(prune_map(m,M)); // pruned module
↳ v[1]:1 v[2]:1 v[3]:2 v[4]:2 v[5]:3
↳ y,
↳ z
print(m*M); // apply map
↳ 0,
↳ y,
↳ 0,
↳ z,
↳ 0
print(M); // map
↳ -x,
↳ 0,
↳ 1,
↳ 0

```

See [Section 4.14 \[module\]](#), page 112.

### 5.1.124 qhweight

**Syntax:** `qhweight ( ideal_expression )`

**Type:** `intvec`

**Purpose:** computes the weight vector of the variables for a quasihomogeneous ideal. If the input is not weighted homogeneous, an `intvec` of zeros is returned.

**Example:**

```

ring h1=32003,(t,x,y,z),dp;
ideal i=x4+y3+z2;
qhweight(i);
↳ 0,3,4,6

```

See [Section 4.6 \[ideal\]](#), page 80; [Section 4.9 \[intvec\]](#), page 93; [Section 5.1.172 \[weight\]](#), page 288.

### 5.1.125 qrds

**Syntax:** `qrds ( matrix_expression, number_expression, number_expression, number_expression )`

**Type:** `list`

**Purpose:** computes all eigenvalues with multiplicities of the given matrix by performing the numeric QR double shift algorithm involving Hessenberg form and householder transformations.

This method expects the ground field to be the complex numbers, and all matrix entries to be real numbers, i.e., elements of this ground field with the imaginary part equal to zero.

If the algorithm works, then it returns a list with two entries which are again lists of the same size:

\_ $[1][i]$  is the  $i$ -th mutually distinct eigenvalue that was found,

\_ $[2][i]$  is the (int) multiplicity of  $_{[1][i]}$ .

If the algorithm does not work (due to an ill-posed matrix), a list with the single entry (int)0 is returned.

The first number argument is used for detection of deflation in the actual QR double shift algorithm. The second number argument is used for ending Heron's iteration whenever square roots are being computed. And the third number argument is used to distinguish between distinct eigenvalues: When the Euclidean distance between two computed eigenvalues is less than this number, then they will be regarded equal, resulting in a higher multiplicity of the corresponding eigenvalue. (A good choice for all three number arguments is a small value like e.g.  $10^{-100}$ .)

**Example:**

```

ring r=(complex,50),(dummy),dp;
matrix A[3][3]=-10,37,-5,-14,51,-10,-29,99,-18;
bigint b = bigint(10)^100; number t = 1/b;
list L=qrds(A,t,t,t); L;
⇒ [1]:
⇒ [1]:
⇒ (3+i*2)
⇒ [2]:
⇒ (3-i*2)
⇒ [3]:
⇒ 17
⇒ [2]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ 1

```

### 5.1.126 quote

**Syntax:** quote ( expression )

**Type:** none

**Purpose:** prevents expressions from evaluation. Used only in connections with write to ssi links, prevents evaluation of an expression before sending it to an other SINGULAR process. Within a quoted expression, the quote can be "undone" by an eval (i.e., each eval "undoes" the effect of exactly one quote).

**Example:**

```
link l="ssi:w example.ssi";
```

```

ring r=0,(x,y,z),ds;
ideal i=maxideal(3);
ideal j=x7,x2,z;
option(prot);
// compute i+j before writing, but not std
write (1, quote(std(eval(i+j))));
close(1);
// now read it in again and evaluate:
read(1);
↪ [1048575:1]1(12)s2(11)s3(10)--s(7)s(6)-----7-
↪ product criterion:4 chain criterion:0
↪ _[1]=z
↪ _[2]=x2
↪ _[3]=xy2
↪ _[4]=y3
close(1);

```

See [Section 4.10.5 \[Ssi links\]](#), page 99; [Section 5.1.29 \[eval\]](#), page 177; [Section 5.1.174 \[write\]](#), page 289.

### 5.1.127 quotient

**Syntax:**    `quotient ( ideal-expression, ideal-expression )`  
              `quotient ( module-expression, module-expression )`

**Type:**     ideal

**Syntax:**    `quotient ( module-expression, ideal-expression )`

**Type:**     module

**Purpose:**    computes the ideal quotient, resp. module quotient. Let  $R$  be the basering,  $I, J$  ideals and  $M$  a module in  $R^n$ . Then

$$\begin{aligned}\text{quotient}(I, J) &= \{a \in R \mid aJ \subset I\}, \\ \text{quotient}(M, J) &= \{b \in R^n \mid bJ \subset M\}.\end{aligned}$$

**Example:**

```

ring r=181,(x,y,z),(c,ls);
ideal id1=maxideal(3);
ideal id2=x2+xyz,y2-z3y,z3+y5xz;
ideal id6=quotient(id1,id2);
id6;
↪ id6[1]=z
↪ id6[2]=y
↪ id6[3]=x
quotient(id2,id1);
↪ _[1]=z2
↪ _[2]=yz
↪ _[3]=y2
↪ _[4]=xz
↪ _[5]=xy
↪ _[6]=x2
module m=x*freemodule(3),y*freemodule(2);
ideal id3=x,y;
quotient(m,id3);

```

```

    ↪ _[1]=[1]
    ↪ _[2]=[0,1]
    ↪ _[3]=[0,0,x]

```

See [Section 5.1.40 \[fglmod\]](#), page 184; [Section 4.6 \[ideal\]](#), page 80; [Section D.4.14.1 \[modQuotient\]](#), page 1121; [Section 4.14 \[module\]](#), page 112.

### 5.1.128 random

**Syntax:** `random ( int_expression , int_expression )`

**Type:** `int`

**Purpose:** returns a random integer between the integer given by the first `int_expression` and the one given by the second `int_expression`.

**Syntax:** `random ( int_expression , int_expression , int_expression )`

**Type:** `intmat`

**Purpose:** returns a random `intmat` where the size is given by the second (number of rows) and third argument (number of columns). The absolute value of the entries of the matrix is smaller than or equal to the integer given as the first argument.

**Note:** The random generator can be set to a startvalue with the function `system`, resp. by a command line option. The current value of the random generator is `system("random")`. Internally a random generator with values in 1 to  $2^{31} - 2$  and a full period is used, max-min may not be larger than  $2^{31}-2$ .

**Example:**

```

    random(1,1000);
    ↪ 35
    random(1,2,3);
    ↪ 0,0,0,
    ↪ 1,1,-1
    system("random",210); // start random generator with 210
    random(-1000,1000);
    ↪ 707
    random(-1000,1000);
    ↪ 284
    system("random",210);
    random(-1000,1000); // the same random values again
    ↪ 707

```

See [Section 3.1.6 \[Command line options\]](#), page 19; [Section 4.7 \[int\]](#), page 85; [Section 4.8 \[intmat\]](#), page 90; [Section 5.1.155 \[system\]](#), page 275.

### 5.1.129 rank

**Syntax:** `rank ( matrix_expression ) #*rank ( matrix_expression ,1)`

**Type:** `int`

**Purpose:** returns the rank of a given matrix which is filled with elements of the ground field. The first variant uses a LU-decomposition, the second a row-echelon form.

**Note:** The function works by computing the row echelon form of the matrix using the same algorithm as for `ludecomp`.

**Example:**

```
ring s = 0, x, dp;
matrix A[100][100];
int i; int j; int r;
for (i = 1; i <= 100; i++)
{
    for (j = 1; j <= 100; j++)
    {
        A[i, j] = random(-10, 10);
    }
}
r = rank(A); r;
↳ 100
```

See [Section 5.1.84 \[ludecomp\]](#), page 216.

### 5.1.130 read

**Syntax:** `read ( link_expression )`  
for DBM links:  
`read ( link_expression )`  
`read ( link_expression, string_expression )`

**Type:** any

**Purpose:** reads data from a link.  
For ASCII links, the content of the entire file is returned as a string. If the ASCII link is the empty string, `read` reads from standard input.  
For ssi links, one expression is read from the link and returned after evaluation. See [Section 4.10.5 \[Ssi links\]](#), page 99.  
For ssi links the `read` command blocks as long as there is no data to be read from the link. The `status` command can be used to check whether or not there is data to be read.  
For DBM links, a `read` with one argument returns the value of the next entry in the data base, and a `read` with two arguments returns the value to the key given as the second argument from the data base. See [Section 4.10.7 \[DBM links\]](#), page 102.

**Example:**

```
ring r=32003,(x,y,z),dp;
ideal i=x+y,z3+22y;
// write the ideal i to the file save_i
write(":w save_i",i);
ring r0=0,(x,y,z),Dp;
// create an ideal k equal to the content
// of the file save_i
string s="ideal k="+read("save_i")+";";
execute(s);
k;
↳ k[1]=x+y
↳ k[2]=z3+22y
```

See [Section 5.1.32 \[execute\]](#), page 178; [Section 5.1.52 \[getdump\]](#), page 191; [Section 4.10 \[link\]](#), page 96; [Section 5.1.150 \[status\]](#), page 270; [Section 5.1.174 \[write\]](#), page 289.



### 5.1.131 reduce

**Syntax:**

```

reduce ( poly_expression, ideal_expression )
reduce ( poly_expression, ideal_expression, int_expression )
reduce ( poly_expression, poly_expression, ideal_expression )
reduce ( vector_expression, ideal_expression )
reduce ( vector_expression, ideal_expression, int_expression )
reduce ( vector_expression, module_expression )
reduce ( vector_expression, module_expression, int_expression )
reduce ( vector_expression, poly_expression, module_expression )
reduce ( ideal_expression, ideal_expression )
reduce ( ideal_expression, ideal_expression, int_expression )
reduce ( ideal_expression, matrix_expression, ideal_expression )
reduce ( module_expression, ideal_expression )
reduce ( module_expression, ideal_expression, int_expression )
reduce ( module_expression, module_expression )
reduce ( module_expression, module_expression, int_expression )
reduce ( module_expression, matrix_expression, module_expression )
reduce ( poly/vector/ideal/module, ideal/module, int, intvec )
reduce ( ideal, matrix, ideal, int )
reduce ( poly, poly, ideal, int )
reduce ( poly, poly, ideal, int, intvec )

```

**Type:** the type of the first argument

**Purpose:** reduces a polynomial, vector, ideal or module to its normal form with respect to an ideal or module represented by a standard basis. Returns 0 if and only if the polynomial (resp. vector, ideal, module) is an element (resp. subideal, submodule) of the ideal (resp. module). The result may have no meaning if the second argument is not a standard basis.

The third (optional) argument of type int modifies the behavior:

0 default

1 consider only the leading term and do no tail reduction.

2 tail reduction:n the local/mixed ordering case: reduce also with bad ecart

4 reduce without division, return possibly a non-zero constant multiple of the remainder

If a second argument  $u$  of type poly or matrix is given, the first argument  $p$  is replaced by  $p/u$ . This works only for zero dimensional ideals (resp. modules) in the third argument and gives, even in a local ring, a reduced normal form which is the projection to the quotient by the ideal (resp. module). One may give a degree bound in the fourth argument with respect to a weight vector in the fifth argument in order to have a finite computation. If some of the weights are zero, the procedure may not terminate!

**Note:** The commands `reduce` and `NF` are synonymous.

**Example:**

```

ring r1 = 0, (z,y,x), ds;
poly s1=2x5y+7x2y4+3x2yz3;
poly s2=1x2y2z2+3z8;
poly s3=4xy5+2x2y2z3+11x10;

```

```

ideal i=s1,s2,s3;
ideal j=std(i);
reduce(3z3yx2+7y4x2+yx5+z12y2x2,j);
↪ -yx5+2401/81y14x2+2744/81y11x5+392/27y8x8+224/81y5x11+16/81y2x14
reduce(3z3yx2+7y4x2+yx5+z12y2x2,j,1);
↪ -yx5+z12y2x2
// 4 arguments:
ring rs=0,x,ds;
// normalform of 1/(1+x) w.r.t. (x3) up to degree 5
reduce(poly(1),1+x,ideal(x3),5);
↪ // ** _ is no standard basis
↪ 1-x+x2

```

See [Section 5.1.26 \[division\]](#), page 174; [Section 4.6 \[ideal\]](#), page 80; [Section 4.14 \[module\]](#), page 112; [Section 4.17.3 \[poly operations\]](#), page 121; [Section 5.1.151 \[std\]](#), page 271; [Section 4.23 \[vector\]](#), page 134.

### 5.1.132 regularity

**Syntax:**    `regularity ( list_expression )`  
               `regularity ( resolution_expression )`

**Type:**     `int`

**Purpose:**    computes the regularity of a homogeneous ideal, resp. module, from a minimal resolution given by the argument.

Let  $0 \rightarrow \bigoplus_a K[x]e_{a,n} \rightarrow \dots \rightarrow \bigoplus_a K[x]e_{a,0} \rightarrow I \rightarrow 0$  be a minimal resolution of  $I$  considered with homogeneous maps of degree 0. The regularity is the smallest number  $s$  with the property  $\deg(e_{a,i}) \leq s + i$  for all  $i$ .

**Note:**      If applied to a non minimal resolution only an upper bound is returned.  
               If the input to the commands `res` and `mres` is homogeneous the regularity is computed and used as a degree bound during the computation unless `option(notRegularity);` is given.

**Example:**

```

ring rh3=32003,(w,x,y,z),(dp,C);
poly f=x11+y10+z9+x5y2+x2y2z3+xy3*(y2+x)^2;
ideal j=homog(jacob(f),w);
def jr=res(j,0);
regularity(jr);
↪ 25
// example for upper bound behaviour:
list jj=jr;
regularity(jj);
↪ 25
jj=nres(j,0);
regularity(jj);
↪ 27
jj=minres(jj);
regularity(jj);
↪ 25

```

See [Section 5.1.48 \[fres\]](#), page 188; [Section 4.11 \[list\]](#), page 103; [Section 5.1.93 \[minres\]](#), page 223; [Section 5.1.98 \[mres\]](#), page 225; [Section 5.1.111 \[option\]](#), page 234; [Section 5.1.134 \[res\]](#), page 253; [Section 4.19 \[resolution\]](#), page 125; [Section 5.1.149 \[sres\]](#), page 269.

### 5.1.133 repart

**Syntax:** `repart ( number_expression )`

**Type:** `number`

**Purpose:** returns the real part of a number from a complex ground field, returns its argument otherwise.

**Example:**

```
ring r=(complex,i),x,dp;
repart(1+2*i);
↦ 1
```

See [Section 5.1.60 \[impart\]](#), page 199.

### 5.1.134 res

Procedure from library `standard.lib` (see [Section D.1 \[standard.lib\]](#), page 794).

**Syntax:** `res ( ideal_expression, int_expression [, any_expression ] )`  
`res ( module_expression, int_expression [, any_expression ] )`

**Type:** `resolution`

**Purpose:** computes a (possibly minimal) free resolution of an ideal or module using a heuristically chosen method.

The second (int) argument (say `k`) specifies the length of the resolution. If it is not positive then `k` is assumed to be the number of variables of the basering.

If a third argument is given, the returned resolution is minimized.

Depending on the input, the returned resolution is computed using the following methods:

**quotient rings:**

`nres` (classical method using syzygies), see [Section 5.1.106 \[nres\]](#), page 232.

**homogeneous ideals and `k=0`:**

`lres` (La'Scala's method), see [Section 5.1.83 \[lres\]](#), page 215.

**not minimized resolution and (homogeneous input with `k` not 0, or local rings):**

`sres` (Schreyer's method), see [Section 5.1.149 \[sres\]](#), page 269.

**all other inputs:**

`mres` (classical method), see [Section 5.1.98 \[mres\]](#), page 225.

**Note:** Accessing single elements of a resolution may require some partial computations to be finished and may therefore take some time.

See also [Section 5.1.4 \[beti\]](#), page 159; [Section 5.1.48 \[fres\]](#), page 188; [Section 5.1.58 \[hres\]](#), page 197; [Section 4.6 \[ideal\]](#), page 80; [Section 5.1.83 \[lres\]](#), page 215; [Section 5.1.93 \[minres\]](#), page 223; [Section 4.14 \[module\]](#), page 112; [Section 5.1.98 \[mres\]](#), page 225; [Section 5.1.106 \[nres\]](#), page 232; [Section 4.19 \[resolution\]](#), page 125; [Section 5.1.149 \[sres\]](#), page 269.

**Example:**

```
ring r=0,(x,y,z),dp;
ideal i=xz,yz,x3-y3;
def l=res(i,0); // homogeneous ideal: uses lres
l;
```

```

⇒ 1      3      2
⇒ r <--  r <--  r
⇒
⇒ 0      1      2
⇒
print(betti(1), "beti"); // input to betti may be of type resolution
⇒          0      1      2
⇒ -----
⇒ 0:      1      -      -
⇒ 1:      -      2      1
⇒ 2:      -      1      1
⇒ -----
⇒ total:   1      3      2
⇒
l[2];          // element access may take some time
⇒ _[1]=-x*gen(1)+y*gen(2)
⇒ _[2]=-x2*gen(2)+y2*gen(1)+z*gen(3)
i=i,x+1;
l=res(i,0);    // inhomogeneous ideal: uses mres
l;
⇒ 1      3      3      1
⇒ r <--  r <--  r <--  r
⇒
⇒ 0      1      2      3
⇒ resolution not minimized yet
⇒
ring rs=0,(x,y,z),ds;
ideal i=imap(r,i);
def l=res(i,0); // local ring not minimized: uses sres
l;
⇒ 1      1
⇒ rs <--  rs
⇒
⇒ 0      1
⇒ resolution not minimized yet
⇒
res(i,0,0);    // local ring and minimized: uses mres
⇒ 1      1
⇒ rs <--  rs
⇒
⇒ 0      1
⇒

```

### 5.1.135 reservedName

**Syntax:** reservedName ()

**Type:** none

**Syntax:** reservedName ( string\_expression )

**Type:** int

**Purpose:** prints a list of all reserved identifiers (first form) or tests whether the string is a reserved identifier (second form). This includes blackbox/newstruct types.

**Example:**

```
reservedName();
↳ ... // output skipped
reservedName("ring");
↳ 1
reservedName("xyz");
↳ 0
```

See [Section 5.1.103 \[names\]](#), page 229; [Section 4.22 \[string\]](#), page 130.

**5.1.136 resultant**

**Syntax:** `resultant ( poly_expression, poly_expression, ring_variable )`

**Type:** `poly`

**Purpose:** computes the resultant of the first and second argument with respect to the variable given as the third argument.

**Example:**

```
ring r=32003,(x,y,z),dp;
poly f=3*(x+2)^3+y;
poly g=x+y+z;
resultant(f,g,x);
↳ 3y3+9y2z+9yz2+3z3-18y2-36yz-18z2+35y+36z-24
```

See [Section 4.17 \[poly\]](#), page 120; [Section 4.20 \[ring\]](#), page 127.

**5.1.137 ringlist**

**Syntax:** `ringlist ( ring_expression )`

**Type:** `list`

**Purpose:** decomposes a ring/qrings into a list of 4 (or 6 in the non-commutative case, see [Section 7.3.24 \[ringlist \(plural\)\]](#), page 358) components. It is identical to `ring_list` with the exception of the first list entry.

1. the field description in the following format:
  - for  $\mathbb{Q}$ ,  $\mathbb{Z}/p$ : the characteristic, type `int` (0 or prime number)
  - for real, complex: a list of:
    - the characteristic, type `int` (always 0)
    - the precision, type `list` (2 integers: external, internal precision)
    - the name of the imaginary unit, type `string`
  - for transcendental or algebraic extensions: described as a `ringlist` (that is, as list `L` with 4 entries: `L[1]` the characteristic, `L[2]` the names of the parameters, `L[3]` the monomial ordering for the ring of parameters (default: `lp`), `L[4]` the minimal polynomial (as ideal))
  - for  $\mathbb{Z}$ ,  $\mathbb{Z}/n$ ,  $\mathbb{Z}/n^m$  a list `["integer", [n, m]]` with:
    - the base `n` is of type `int` or `bigint` (if not given `n = 0`,  $\mathbb{Z}/0 = \mathbb{Z}$ )
    - the exponent `m` is of type `int` (if not given `m = 1`)
2. the names of the variables (a list `L` of strings: `L[i]` is the name of the *i*-th variable)
3. the monomial ordering (a list `L` of lists): each block `L[i]` consists of

the name of the ordering ( string )

parameters specifying the ordering and the size of the block ( intvec : typically the weights for the variables [default: 1] )

4. the quotient ideal.

From a list of such structure, a new ring may be defined by the command `ring` ( see the following example ). If the attribute "maxExp" of the ring is different from the default 32767, it is also set for the list.

**Note:** All data which depends on a ring belong to the current ring, not to a ring which can be constructed from a modified list. These data will be mapped via `fetch` to the ring to be constructed.

**Example:**

```

ring r = 0,(x(1..3)),dp;
list l = ringlist(r);
l;
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ [1]:
⇒ x(1)
⇒ [2]:
⇒ x(2)
⇒ [3]:
⇒ x(3)
⇒ [3]:
⇒ [1]:
⇒ [1]:
⇒ dp
⇒ [2]:
⇒ 1,1,1
⇒ [2]:
⇒ [1]:
⇒ C
⇒ [2]:
⇒ 0
⇒ [4]:
⇒ _[1]=0
// Now change l and create a new ring, by
//- changing the base field to the function field with parameter a,
//- introducing one extra variable y,
//- defining the block ordering (dp(2),wp(3,4)).
//- define the minpoly after creating the function field
l[1]=list(0,list("a"),list(list("lp",1)),ideal(0));
l[2][size(l[2])+1]="y";
l[3][3]=l[3][2]; // save the module ordering
l[3][1]=list("dp",intvec(1,1));
l[3][2]=list("wp",intvec(3,4));
attrib(l,"maxExp",100); // and lower the limit for exponents to 100
def ra = ring(l); //creates the newring
ra; setring ra;
⇒ // coefficients: QQ(a)
⇒ // number of vars : 4

```

```

⇒ //      block  1 : ordering dp
⇒ //      : names  x(1) x(2)
⇒ //      block  2 : ordering wp
⇒ //      : names  x(3) y
⇒ //      : weights 3 4
⇒ //      block  3 : ordering C
  attrib(ra,"maxExp");
⇒ 65535
  list lra = ringlist(ra);
  lra[1][4]=ideal(a2+1);
  def Ra = ring(lra);
  setring Ra; Ra;
⇒ // coefficients: QQ[a]/(a^2+1)
⇒ // number of vars : 4
⇒ //      block  1 : ordering dp
⇒ //      : names  x(1) x(2)
⇒ //      block  2 : ordering wp
⇒ //      : names  x(3) y
⇒ //      : weights 3 4
⇒ //      block  3 : ordering C

```

See [Section 4.20.1 \[qring\], page 127](#); [Section 4.20 \[ring\], page 127](#); [Section 5.1.138 \[ring\\_list\], page 257](#).

### 5.1.138 ring\_list

**Syntax:**    `ring_list ( ring_expression )`  
              `ring_list ( cring_expression )`

**Type:**     list

**Purpose:**    decomposes a ring/qring/coefficient ring into a list of 4 (or 6 in the non-commutative case, see [Section 7.3.24 \[ringlist \(plural\)\], page 358](#)) or 1/4 (for `cring`) components. It is identical to `ringlist` with the exception of the first list entry.

1. the field description as `cring`
2. the names of the variables (a list `L` of strings: `L[i]` is the name of the  $i$ -th variable)
3. the monomial ordering (a list `L` of lists): each block `L[i]` consists of
  - the name of the ordering ( string )
  - parameters specifying the ordering and the size of the block ( intvec : typically the weights for the variables [default: 1] )
4. the quotient ideal.

From a list of such structure, a new ring may be defined by the command `ring` ( see the following example ).

**Note:** All data which depends on a ring belong to the current ring, not to a ring which can be constructed from a modified list. These data will be mapped via `fetch` to the ring to be constructed.

**Example:**

```

ring r = 0, (x(1..3)), dp;
list l = ring_list(r);
ring_list(l[1]);

```

```

⇒ 0
  1;
⇒ [1]:
⇒ QQ
⇒ [2]:
⇒   [1]:
⇒   x(1)
⇒   [2]:
⇒   x(2)
⇒   [3]:
⇒   x(3)
⇒ [3]:
⇒   [1]:
⇒   [1]:
⇒   dp
⇒   [2]:
⇒   1,1,1
⇒   [2]:
⇒   [1]:
⇒   C
⇒   [2]:
⇒   0
⇒ [4]:
⇒   _[1]=0
  // Now change l and create a new ring, by
  //- changing the base field to ZZ/32003
  //- introducing one extra variable y,
  //- defining the block ordering (dp(2),wp(3,4)).
  //- define the minpoly after creating the function field
  l[1]=ZZ/32003;
  l[2][size(l[2])+1]="y";
  l[3][3]=l[3][2]; // save the module ordering
  l[3][1]=list("dp",intvec(1,1));
  l[3][2]=list("wp",intvec(3,4));
  def ra = ring(l); //creates the newring
  ra; setring ra;
⇒ // coefficients: ZZ/32003
⇒ // number of vars : 4
⇒ //      block  1 : ordering dp
⇒ //      : names  x(1) x(2)
⇒ //      block  2 : ordering wp
⇒ //      : names  x(3) y
⇒ //      : weights 3 4
⇒ //      block  3 : ordering C

```

See [Section 4.20.1 \[qring\], page 127](#); [Section 4.20 \[ring\], page 127](#); [Section 5.1.137 \[ringlist\], page 255](#).

### 5.1.139 rvar

**Syntax:**    rvar ( name )  
              rvar ( poly-expression )  
              rvar ( string-expression )

**Type:**     int



**Purpose:** returns the number of the variable if the name/polynomial is a ring variable of the basering or if the string is the name of a ring variable of the basering; returns 0 if not. Hence the return value of `rvar` can also be used in a boolean context to check whether the variable exists.

**Example:**

```

ring r=29,(x,y,z),lp;
rvar(x);
↪ 1
rvar(r);
↪ 0
rvar(y);
↪ 2
rvar(var(3));
↪ 3
rvar("x");
↪ 1

```

See [Section 5.1.18 \[defined\]](#), page 170; [Section 4.20 \[ring\]](#), page 127; [Section 5.1.165 \[var\]](#), page 284; [Section 5.1.167 \[varstr\]](#), page 285.

### 5.1.140 sba

**Syntax:** `sba ( ideal_expression )`  
`sba ( ideal_expression, int_expression, int_expression )`

**Type:** ideal

**Purpose:** returns a standard basis of an ideal with respect to the monomial ordering of the basering. A standard basis is a set of generators such that the leading terms generate the leading ideal, resp. module.

Use optional second and third arguments of type `int` to determine the respective variant of the signature-based standard basis algorithm:

The second argument specifies the internal module order `sba` uses:

- 0: induced Schreyer order on the signatures, non-incremental computation of the basis
- 1: position over term order, incremental computation of the basis
- 2: term over position order, non-incremental computation
- 3: Schreyer-weighted degree over index over leading term

The third argument specifies the rewrite order `sba` uses:

- 0: using the rewrite order described in <http://dx.doi.org/10.1016/j.jsc.2010.06.019>
- 1: using the rewrite order described in <http://dx.doi.org/10.1016/j.jsc.2011.05.004>

The standard call of `sba(i)` corresponds to `sba(i,0,1)`.

**Note:** The standard basis is computed with an optimized version of known signature-based algorithms like Faugere's F5 Algorithm. Whereas the correctness of the algorithms is only guaranteed for global orderings, timings for pure lexicographical orderings can be slow. In this situation you should try to compute the basis w.r.t. the graded reverse-lexicographic ordering and then convert to a basis for the lexicographical ordering using

other methods ( see [Section 5.1.39 \[fglm\]](#), page 183 and see [Section D.4.10 \[grwalk\\_lib\]](#), page 1088). If the algorithms tend to use too much memory, you should try the other implemented standard basis algorithms ( see [Section 5.1.151 \[std\]](#), page 271, see [Section 5.1.53 \[groebner\]](#), page 191, and see [Section 5.1.145 \[slimgb\]](#), page 265).

Note that the behaviour of `sba` on an example can be rather different depending on which variant you choose (second and third argument).

**Example:**

```
// incremental F5 computation
ring r=32003,(x,y,z),dp;
poly s1=1x2y+151xyz10+169y21;
poly s2=1xz14+6x2y4+3z24;
poly s3=5y10z10x+2y20z10+y10z20+11x3;
ideal i=s1,s2,s3;
ideal j=sba(i,1,0);
// non-incremental F5 computation
ring rhom=32003,(x,y,z,h),dp;
ideal i=homog(imap(r,i),h);
ideal j=sba(i,0,0);
// non-incremental signature-based computation
ring whom=32003,(x,y,z),dp;
ideal i=fetch(r,i);
ideal j=sba(i);
```

See [Section 5.1.39 \[fglm\]](#), page 183; [Section 5.1.53 \[groebner\]](#), page 191; [Section 4.6 \[ideal\]](#), page 80; [Section 4.20 \[ring\]](#), page 127; [Section 5.1.145 \[slimgb\]](#), page 265; [Section 5.1.151 \[std\]](#), page 271.

### 5.1.141 setring

**Syntax:**     `setring ring_name`

**Type:**       none

**Purpose:**     changes the basering to another (already defined) ring.

**Example:**

```
ring r1=0,(x,y),lp;
// the basering is r1
ring r2=32003,(a(1..8)),ds;
// the basering is r2
setring r1;
// the basering is again r1
nameof(basering);
↳ r1
listvar();
↳ // r2
↳ // r1
```

[0]	ring
[0]	*ring

**Use in procedures:**

All changes of the basering by a definition of a new ring or a `setring` command in a procedure are local to this procedure. Use `keepiring` to move a ring, which is local to a procedure, up by one nesting level.

See [Section 5.2.11 \[keepiring\]](#), page 298; [Section 4.20.1 \[qring\]](#), page 127; [Section 4.20 \[ring\]](#), page 127.

### 5.1.142 simplex

**Syntax:** `simplex ( matrix_expression, int_expression, int_expression, int_expression, int_expression, int_expression )`

**Type:** list

**Purpose:** perform the simplex algorithm for the tableau given by the input, e.g. `simplex (M, m, n, m1, m2, m3 )`:

M matrix of numbers :

first row describing the objective function (maximize problem), the remaining rows describing constraints;

m, n, m1, m2, m3 int :

n = number of variables; m = total number of constraints; m1 = number of inequalities " $\leq$ " (rows 2 ... m1+1 of M); m2 = number of inequalities " $\geq$ " (rows m1+2 ... m1+m2+1 of M); m3 = number of equalities.

The following assumptions are made:

\* ground field is of type `(real,N)`,  $N \geq 4$ ;

\* the matrix M is of size m x n;

\*  $m = m1 + m2 + m3$ ;

\* the entries  $M[2,1], \dots, M[m+1,1]$  are non-negative;

\* the variables  $x(i)$  are non-negative;

\* a row b,  $a(1), \dots, a(n)$  corresponds to  $b + a(1)x(1) + \dots + a(n)x(n)$ ;

\* for a  $\leq$ ,  $\geq$ , or  $=$  constraint: add "in mind"  $\geq 0$ ,  $\leq 0$ , or  $= 0$ .

The output is a list L with

\*  $L[1]$  = matrix

\*  $L[2]$  = int:

0 = finite solution found; 1 = unbounded; -1 = no solution; -2 = error occurred;

\*  $L[3]$  = intvec :

$L[3][k]$  = number of variable which corresponds to row k+1 of  $L[1]$ ;

\*  $L[4]$  = intvec :

$L[4][j]$  = number of variable which is represented by column j+1 of  $L[1]$  ("non-basis variable");

\*  $L[5]$  = int :

number of constraints (= m);

\*  $L[6]$  = int :

number of variables (= n).

The solution can be read off the first column of  $L[1]$  as it is done by the procedure [Section D.8.4.8 \[simplexOut\]](#), page 1852 in `solve.lib`.

**Example:**

```
ring r = (real,10),(x),lp;

// consider the max. problem:
//
```

```

//      maximize  x(1) + x(2) + 3*x(3) - 0.5*x(4)
//
//  with constraints:  x(1) +          2*x(3)          <= 740
//                      2*x(2)          - 7*x(4) <= 0
//                      x(2) - x(3) + 2*x(4) >= 0.5
//                      x(1) + x(2) + x(3) + x(4) = 9
//
matrix sm[5][5]=( 0, 1, 1, 3,-0.5,
                  740,-1, 0,-2, 0,
                  0, 0,-2, 0, 7,
                  0.5, 0,-1, 1,-2,
                  9,-1,-1,-1,-1);

int n = 4; // number of constraints
int m = 4; // number of variables
int m1= 2; // number of <= constraints
int m2= 1; // number of >= constraints
int m3= 1; // number of == constraints
simplex(sm, n, m, m1, m2, m3);
⇒ [1]:
⇒ _[1,1]=17.025
⇒ _[1,2]=-0.95
⇒ _[1,3]=-0.05
⇒ _[1,4]=1.95
⇒ _[1,5]=-1.05
⇒ _[2,1]=730.55
⇒ _[2,2]=0.1
⇒ _[2,3]=-0.1
⇒ _[2,4]=-1.1
⇒ _[2,5]=0.9
⇒ _[3,1]=3.325
⇒ _[3,2]=-0.35
⇒ _[3,3]=-0.15
⇒ _[3,4]=0.35
⇒ _[3,5]=0.35
⇒ _[4,1]=0.95
⇒ _[4,2]=-0.1
⇒ _[4,3]=0.1
⇒ _[4,4]=0.1
⇒ _[4,5]=0.1
⇒ _[5,1]=4.725
⇒ _[5,2]=-0.55
⇒ _[5,3]=0.05
⇒ _[5,4]=0.55
⇒ _[5,5]=-0.45
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ 5,2,4,3
⇒ [4]:
⇒ 1,6,8,7
⇒ [5]:
⇒ 4

```

```

      ↦ [6]:
      ↦    4

```

See [Section D.8.4.8 \[simplexOut\]](#), page 1852.

### 5.1.143 simplify

**Syntax:**    `simplify ( poly_expression, int_expression )`  
               `simplify ( vector_expression, int_expression )`  
               `simplify ( ideal_expression, int_expression )`  
               `simplify ( module_expression, int_expression )`

**Type:**     the type of the first argument

**Purpose:**    returns the "simplified" first argument depending on the simplification rules specified by the second argument. The simplification rules are the following functions:

- 1            normalize (divide by leading coefficient if this is a unit of the ground field/ring).
- 2            erase zero generators/columns.
- 4            erase copies of earlier listed generators/columns.
- 8            erase generators/columns which a scalar multiples (w.r.t. ground field/ring) of earlier listed generators/columns.
- 16           erase generators/columns whose leading monomials are copies of leading monomials of earlier listed generators/columns such that the coefficients of both leading terms are units in the ground field/ring.
- 32           erase generators/columns whose leading terms are divisible by leading terms of other (not necessarily earlier) listed generators/columns.
- 64           normalize each coefficient of every monomial (of every polynomial)

**Example:**

```

ring r=0,(x,y,z),(c,dp);
ideal i=0,2x,2x,4x,3x+y,5x2;
simplify(i,1);
↦ _[1]=0
↦ _[2]=x
↦ _[3]=x
↦ _[4]=x
↦ _[5]=x+1/3y
↦ _[6]=x2
simplify(i,2);
↦ _[1]=2x
↦ _[2]=2x
↦ _[3]=4x
↦ _[4]=3x+y
↦ _[5]=5x2
simplify(i,4);
↦ _[1]=0
↦ _[2]=2x
↦ _[3]=0
↦ _[4]=4x
↦ _[5]=3x+y

```

```

⇒ _[6]=5x2
simplify(i,8);
⇒ _[1]=0
⇒ _[2]=2x
⇒ _[3]=0
⇒ _[4]=0
⇒ _[5]=3x+y
⇒ _[6]=5x2
simplify(i,16);
⇒ _[1]=0
⇒ _[2]=2x
⇒ _[3]=0
⇒ _[4]=0
⇒ _[5]=0
⇒ _[6]=5x2
simplify(i,32);
⇒ _[1]=0
⇒ _[2]=2x
⇒ _[3]=0
⇒ _[4]=0
⇒ _[5]=0
⇒ _[6]=0
simplify(i,32+2+1);
⇒ _[1]=x
matrix A[2][3]=x,0,2x,y,0,2y;
simplify(A,2+8); // by automatic conversion to module
⇒ _[1]=[x,y]

```

See [Section 4.6 \[ideal\], page 80](#); [Section 4.14 \[module\], page 112](#); [Section 4.17 \[poly\], page 120](#); [Section 4.23 \[vector\], page 134](#).

### 5.1.144 size

**Syntax:**    `size ( string_expression )`  
               `size ( bigint_expression )`  
               `size ( number_expression )`  
               `size ( intvec_expression )`  
               `size ( bigintvec_expression )`  
               `size ( intmat_expression )`  
               `size ( poly_expression )`  
               `size ( vector_expression )`  
               `size ( ideal_expression )`  
               `size ( module_expression )`  
               `size ( matrix_expression )`  
               `size ( bigintmat_expression )`  
               `size ( list_expression )`  
               `size ( resolution_expression )`  
               `size ( ring_expression )`

**Type:**     `int`

**Purpose:**    depends on the type of argument:

ideal or module

returns the number of (non-zero) generators.

string, intvec, bigintvec, list or resolution  
     returns the length, i.e., the number of characters, entries or elements.

poly or vector  
     returns the number of monomials.

matrix, bigintmat or intmat  
     returns the number of entries (rows\*columns).

ring  
     returns the number of elements in the ground field (for  $\mathbb{Z}/p$  and algebraic extensions) or -1

number or bigint  
     returns 0 for 0 or the number of words

**Example:**

```

string s="hello";
size(s);
↪ 5
intvec iv=1,2;
size(iv);
↪ 2
ring r=0,(x,y,z),lp;
poly f=x+y+z;
size(f);
↪ 3
vector v=[x+y,0,0,1];
size(v);
↪ 3
ideal i=f,y;
size(i);
↪ 2
module m=v,[0,1],[0,0,1],2*v;
size(m);
↪ 4
matrix mm[2][2];
size(mm);
↪ 4
ring r1=(2,a),x,dp;
minpoly=a4+a+1;
size(r1);
↪ 16

```

See [Section 4.6 \[ideal\]](#), page 80; [Section 4.8 \[intmat\]](#), page 90; [Section 4.9 \[intvec\]](#), page 93; [Section 4.14 \[module\]](#), page 112; [Section 5.1.104 \[ncols\]](#), page 231; [Section 5.1.107 \[nrows\]](#), page 232; [Section 4.17 \[poly\]](#), page 120; [Section 4.22 \[string\]](#), page 130; [Section 4.23 \[vector\]](#), page 134.

**5.1.145 slimgb****Syntax:**

```

slimgb ( ideal_expression )
slimgb ( module_expression )

```

**Type:** ideal or module

**Purpose:** [Section A.2.3 \[slim Groebner bases\]](#), page 715

Returns a Groebner basis of an ideal or module with respect to the monomial ordering of the basering (which has to be global).

**Note:** The algorithm is designed to keep polynomials slim (short with small coefficients). For details see [https://www.singular.uni-kl.de/reports/35/paper\\_35\\_full.ps.gz](https://www.singular.uni-kl.de/reports/35/paper_35_full.ps.gz). A reduced Groebner basis is returned if `option(redSB)` is set (see [\[option\(redSB\)\]](#), page 235). To view the progress of long running computations, use `option(prot)` (see [\[option\(prot\)\]](#), page 235).

**Warning:** Groebner basis computations with inexact coefficients can not be trusted due to rounding errors.

**Example:**

```
ring r=2,(x,y,z),lp;
poly s1=z*(x*y+1);
poly s2=x2+x;
poly s3=y2+y;
ideal i=s1,s2,s3;
slingb(i);
↪ _[1]=y2+y
↪ _[2]=x2+x
↪ _[3]=yz+z
↪ _[4]=xz+z
```

See [Section 5.1.53 \[groebner\]](#), page 191; [Section 4.6 \[ideal\]](#), page 80; [Section 5.1.111 \[option\]](#), page 234; [Section 4.20 \[ring\]](#), page 127; [Section 5.1.151 \[std\]](#), page 271.

### 5.1.146 sortvec

**Syntax:** `sortvec ( ideal_expression )`  
`sortvec ( module_expression )`

**Type:** `intvec`

**Purpose:** computes the permutation `v` which orders the ideal, resp. module, `I` by its initial terms, starting with the smallest, that is,  $I(v[i]) < I(v[i+1])$  for all `i`.

**Example:**

```
ring r=0,(x,y,z),dp;
ideal I=y,z,x,x3,xz;
sortvec(I);
↪ 2,1,3,5,4
```

See [Section D.2.3 \[general\\_lib\]](#), page 806.

### 5.1.147 sqrfree

**Syntax:** `sqrfree ( poly_expression )`  
`sqrfree ( poly_expression, 0 )`  
`sqrfree ( poly_expression, 2 )`

**Type:** list of ideal and `intvec`

**Syntax:** `sqrfree ( poly_expression, 1 )`



**Type:** ideal

**Syntax:** `sqrfree ( poly_expression, 3 )`

**Type:** poly

**Purpose:** computes the squarefree factors (as an ideal) of the polynomial together with or without the multiplicities (as an intvec) depending on the second argument:

0: returns factors and multiplicities, first factor is a constant.

May also be written with only one argument.

1: returns non-constant factors (no multiplicities).

2: returns non-constant factors and multiplicities.

3: returns the product of non-constant factors, i.e. squarefree part

**Note:** Not implemented for the coefficient fields real and finite fields of type  $(p^n, a)$ .

**Example:**

```

ring r=3,(x,y,z),dp;
poly f=(x-y)^3*(x+z)*(y-z);
sqrfree(f);
↳ [1]:
↳   _[1]=1
↳   _[2]=-xy+xz-yz+z2
↳   _[3]=-x+y
↳ [2]:
↳   1,1,3
sqrfree(f,1);
↳ _[1]=-xy+xz-yz+z2
↳ _[2]=-x+y
sqrfree(f,2);
↳ [1]:
↳   _[1]=-xy+xz-yz+z2
↳   _[2]=-x+y
↳ [2]:
↳   1,3
sqrfree(f,3);
↳ x2y-xy2-x2z-xyz-y2z-xz2+yz2

```

See [Section 5.1.36 \[factorize\]](#), page 180.

### 5.1.148 sprintf

Procedure from library `standard.lib` (see [Section D.1 \[standard.lib\]](#), page 794).

**Syntax:** `sprintf ( string_expression [, any_expressions ] )`

**Return:** string

**Purpose:** `sprintf(fmt,...)`; performs output formatting. The first argument is a format control string. Additional arguments may be required, depending on the content of the control string. A series of output characters is generated as directed by the control string; these characters are returned as a string.

The control string `fmt` is simply text to be copied, except that the string may contain conversion specifications.

Type `help print`; for a listing of valid conversion specifications. As an addition to the conversions of `print`, the `%n` and `%2` conversion specification does not consume an additional argument, but simply generates a newline character.

**Note:** If one of the additional arguments is a list, then it should be wrapped in an additional `list()` command, since passing a list as an argument flattens the list by one level.

**Example:**

```

ring r=0,(x,y,z),dp;
module m=[1,y],[0,x+z];
intmat M=betti(mres(m,0));
list l = r, m, M;
string s = sprintf("s:%s,%n l:%l", 1, 2); s;
↳ s:1,
↳ l:int(2)
s = sprintf("s:%n%s", l); s;
↳ s:
↳ (QQ),(x,y,z),(dp(3),C)
s = sprintf("s:%2%s", list(l)); s;
↳ s:
↳ (QQ),(x,y,z),(dp(3),C),y*gen(2)+gen(1),x*gen(2)+z*gen(2),1,1
s = sprintf("2l:%n%2l", list(l)); s;
↳ 2l:
↳ list("(QQ),(x,y,z),(dp(3),C)",
↳ module(y*gen(2)+gen(1),
↳ x*gen(2)+z*gen(2)),
↳ intmat(intvec(1,1 ),1,2))
↳
s = sprintf("%p", list(l)); s;
↳ [1]:
↳ // coefficients: QQ
↳ // number of vars : 3
↳ //          block 1 : ordering dp
↳ //          : names   x y z
↳ //          block 2 : ordering C
↳ [2]:
↳ _[1]=y*gen(2)+gen(1)
↳ _[2]=x*gen(2)+z*gen(2)
↳ [3]:
↳ 1,1
s = sprintf("%;", list(l)); s;
↳ [1]:
↳ // coefficients: QQ
↳ // number of vars : 3
↳ //          block 1 : ordering dp
↳ //          : names   x y z
↳ //          block 2 : ordering C
↳ [2]:
↳ _[1]=y*gen(2)+gen(1)
↳ _[2]=x*gen(2)+z*gen(2)
↳ [3]:
↳ 1,1
↳
s = sprintf("%b", M); s;
↳          0      1
↳ -----
↳      0:      1      1

```

```

↳ -----
↳ total:      1      1
↳

```

See also: [Section 5.1.46 \[fprintf\], page 186](#); [Section 5.1.120 \[print\], page 242](#); [Section 5.1.121 \[fprintf\], page 244](#); [Section 4.22 \[string\], page 130](#).

### 5.1.149 sres

**Syntax:** `sres ( ideal_expression, int_expression )`  
`sres ( module_expression, int_expression )`

**Type:** resolution

**Purpose:** computes a free resolution of an ideal or module with Schreyer's method. The ideal, resp. module, has to be a standard basis. More precisely, let  $M$  be given by a standard basis and  $A_1 = \text{matrix}(M)$ . Then `sres` computes a free resolution of  $\text{coker}(A_1) = F_0/M$

$$\dots \longrightarrow F_2 \xrightarrow{A_2} F_1 \xrightarrow{A_1} F_0 \longrightarrow F_0/M \longrightarrow 0.$$

If the int expression  $k$  is not zero then the computation stops after  $k$  steps and returns a list of modules (given by standard bases)  $M_i = \text{module}(A_i)$ ,  $i=1..k$ .

`sres(M,0)` returns a list of  $n$  modules where  $n$  is the number of variables of the basering.

Even if `sres` does not compute a minimal resolution, the `betti` command gives the true betti numbers! In many cases of interest `sres` is much faster than any other known method. Let `list L=sres(M,0)`; then `L[1]=M` is identical to the input, `L[2]` is a standard basis with respect to the Schreyer ordering of the first syzygy module of `L[1]`, etc. (`L[i] = M_i` in the notations from above.)

**Note:** Accessing single elements of a resolution may require some partial computations to be finished and may therefore take some time.

**Example:**

```

ring r=31991,(t,x,y,z,w),ls;
ideal M=t2x2+tx2y+x2yz,t2y2+ty2z+y2zw,
        t2z2+tz2w+xz2w,t2w2+txw2+xyw2;
M=std(M);
resolution L=sres(M,0);
L;
↳ 1      35      141      209      141      43      4
↳ r <--  r <--  r <--  r <--  r <--  r <--  r
↳
↳ 0      1      2      3      4      5      6
↳ resolution not minimized yet
↳
print(betti(L),"betti");
↳
↳      0      1      2      3      4      5
↳ -----
↳ 0:      1      -      -      -      -      -
↳ 1:      -      -      -      -      -      -
↳ 2:      -      -      -      -      -      -
↳ 3:      -      4      -      -      -      -
↳ 4:      -      -      -      -      -      -
↳ 5:      -      -      -      -      -      -
↳ 6:      -      -      6      -      -      -

```

```

↪      7:      -      -      9      16      2      -
↪      8:      -      -      -      2      5      1
↪ -----
↪ total:      1      4      15      18      7      1
↪

```

See [Section 5.1.4 \[betti\]](#), page 159; [Section 5.1.48 \[fres\]](#), page 188; [Section 5.1.58 \[hres\]](#), page 197; [Section 4.6 \[ideal\]](#), page 80; [Section 4.7 \[int\]](#), page 85; [Section 5.1.83 \[ires\]](#), page 215; [Section 5.1.93 \[minres\]](#), page 223; [Section 4.14 \[module\]](#), page 112; [Section 5.1.98 \[mres\]](#), page 225; [Section 5.1.134 \[res\]](#), page 253; [Section 5.1.156 \[syz\]](#), page 280.

### 5.1.150 status

**Syntax:** `status ( link_expression, string_expression )`

**Type:** string

**Syntax:** `status ( link_expression, string_expression, string_expression )`

**Type:** int

**Purpose:** returns the status of the link as asked for by the second argument. If a third argument is given, the result of the comparison to the status string is returned: `(status(l,s1)==s2)` is equivalent to `status(l,s1,s2)`.

The following string expressions are allowed:

"name"	the name string given by the definition of the link (usually the filename)
"type"	returns "ASCII", "DBM" or "ssi"
"open"	returns "yes" or "no"
"openread"	returns "yes" or "no"
"openwrite"	returns "yes" or "no"
"read"	returns "ready" or "not ready"
"write"	returns "ready" or "not ready"
"mode"	returns (depending on the type of the link and its status) "", "w", "a", "r" or "rw"
"exists"	returns "yes" or "no": existence of the filename for ASCII/ssi links

**Syntax:** `status ( list_expression, int_expression )`

**Type:** int

**Purpose:** the list should be a list L of links, the second argument a timeout in 1/10 seconds. Returns

-2	select returns an error
-1	all links are closed/at eof
0	timeout
>0	(at least) L[i] is ready

**Example:**

```

    link l=":w example.txt";
    status(l,"write");
    ↪ not ready
    open(l);
    status(l,"write","ready");
    ↪ 1
    close(l);

```

See [Section 4.10 \[link\]](#), page 96; [Section 5.1.110 \[open\]](#), page 234; [Section 5.1.130 \[read\]](#), page 250; [Section 5.1.174 \[write\]](#), page 289.

### 5.1.151 std

**Syntax:**    `std ( ideal_expression )`  
               `std ( module_expression )`  
               `std ( smatrix_expression )`  
               `std ( ideal_expression, bigintvec_expression )`  
               `std ( module_expression, bigintvec_expression )`  
               `std ( ideal_expression, bigintvec_expression, intvec_expression )`  
               `std ( module_expression, bigintvec_expression, intvec_expression )`  
               `std ( ideal_expression, poly_expression )`  
               `std ( module_expression, vector_expression )`  
               `std ( ideal_expression, ideal_expression )`  
               `std ( module_expression, module_expression )`  
               `std ( ideal_expression, poly_expression, intvec_expression, intvec_expression )`  
               `std ( module_expression, poly_expression, intvec_expression, intvec_expression )`

**Type:**     ideal, module or smatrix

**Purpose:**    returns a standard basis of an ideal or module with respect to the monomial ordering of the basering. For Letterplace rings, a twosided Groebner basis is computed. A standard basis is a set of generators such that the leading terms generate the leading ideal, resp. module.

Use an optional second argument of type `intvec` as Hilbert series (result of `hilb(i,1)`, see [Section 5.1.56 \[hilb\]](#), page 195), if the ideal, resp. module, is homogeneous (Hilbert driven standard basis computation, [Section 5.1.153 \[stdhilb\]](#), page 273). If the ideal is quasihomogeneous with some weights `w` and if the Hilbert series is computed w.r.t. to these weights, then use `w` as third argument.

Use an optional second argument of type `poly`/`vector`/`ideal`, resp. `module`, to construct the standard basis from an already computed one (given as the first argument) and additional generator(s) (the second argument).

4 arguments `G,p,hv,w` are the combination of the above: standard basis `G`, additional generator `p`, hilbert function `hv` w.r.t. weights `w`.

**Warning:**   Groebner basis computations with inexact coefficients can not be trusted due to rounding errors.

**Note:**        The standard basis is computed with a (more or less) straight-forward implementation of the classical Buchberger (resp. Mora) algorithm. For global orderings, use the `groebner` command instead (see [Section 5.1.53 \[groebner\]](#), page 191), which heuristically chooses the "best" algorithm to compute a Groebner basis.

To view the progress of long running computations, use `option(prot)` (see [\[option\(prot\)\]](#), page 235).

**Example:**

```

// local computation
ring r=32003,(x,y,z),ds;
poly s1=1x2y+151xyz10+169y21;
poly s2=1xz14+6x2y4+3z24;
poly s3=5y10z10x+2y20z10+y10z20+11x3;
ideal i=s1,s2,s3;
ideal j=std(i);
degree(j);
⇒ // dimension (local) = 0
⇒ // multiplicity = 1512
// Hilbert driven elimination (standard)
ring rhom=32003,(x,y,z,h),dp;
ideal i=homog(imap(r,i),h);
ideal j=std(i);
bigintvec iv=hilb(j,1);
ring rlex=32003,(x,y,z,h),lp;
ideal i=fetch(rhom,i);
ideal j=std(i,iv);
j=subst(j,h,1);
j[1];
⇒ z64
// Hilbert driven elimination (ideal is quasihomogeneous)
intvec w=10,1,1;
ring whom=32003,(x,y,z),wp(w);
ideal i=fetch(r,i);
ideal j=std(i);
bigintvec iw=hilb(j,1,w);
ring wlex=32003,(x,y,z),lp;
ideal i=fetch(whom,i);
ideal j=std(i,iw,w);
j[1];
⇒ z64

```

See [Section 5.1.34 \[facstd\]](#), page 179; [Section 5.1.39 \[fglm\]](#), page 183; [Section 5.1.53 \[groebner\]](#), page 191; [Section 4.6 \[ideal\]](#), page 80; [Section 4.14 \[module\]](#), page 112; [Section 5.1.100 \[mstd\]](#), page 227; [Section 5.1.111 \[option\]](#), page 234; [Section 4.20 \[ring\]](#), page 127; [Section 4.21 \[smatrix\]](#), page 129; [Section 5.1.152 \[stdfglm\]](#), page 272; [Section 5.1.153 \[stdhilb\]](#), page 273.

**5.1.152 stdfglm**

Procedure from library `standard.lib` (see [Section D.1 \[standard\\_lib\]](#), page 794).

**Syntax:**     **stdfglm** ( ideal\_expression )  
               **stdfglm** ( ideal\_expression, string\_expression )

**Type:**       ideal

**Purpose:**     computes the standard basis of the ideal in the basering via `fglm` from the ordering given as the second argument to the ordering of the basering. If no second argument is given, "dp" is used. The standard basis for the given ordering (resp. for "dp") is computed via the command `groebner` except if a further argument "std" or "slimgb" is given in which case `std` resp. `slimgb` is used.

**Example:**

```

ring r = 0,(x,y,z),lp;
ideal i = y3+x2,x2y+x2,x3-x2,z4-x2-y;
stdfglm(i); //uses fglm from "dp" (with groebner) to "lp"
⇨ _[1]=z12
⇨ _[2]=yz4-z8
⇨ _[3]=y2+y-z8-z4
⇨ _[4]=xy-xz4-y+z4
⇨ _[5]=x2+y-z4
stdfglm(i,"std"); //uses fglm from "dp" (with std) to "lp"
⇨ _[1]=z12
⇨ _[2]=yz4-z8
⇨ _[3]=y2+y-z8-z4
⇨ _[4]=xy-xz4-y+z4
⇨ _[5]=x2+y-z4
ring s = (0,x),(y,z,u,v),lp;
minpoly = x2+1;
ideal i = u5-v4,zv-u2,zu3-v3,z2u-v2,z3-uv,yv-zu,yu-z2,yz-v,y2-u,u-xy2;
weight(i);
⇨ 2,3,4,5
stdfglm(i,"(a(2,3,4,5),dp)"); //uses fglm from "(a(2,3,4,5),dp)" to "lp"
⇨ _[1]=v2
⇨ _[2]=u
⇨ _[3]=zv
⇨ _[4]=z2
⇨ _[5]=yv
⇨ _[6]=yz-v
⇨ _[7]=y2

```

See also: [Section 5.1.39 \[fglm\]](#), page 183; [Section 5.1.53 \[groebner\]](#), page 191; [Section 5.1.145 \[slimgb\]](#), page 265; [Section 5.1.151 \[std\]](#), page 271; [Section 5.1.153 \[stdhilb\]](#), page 273.

### 5.1.153 stdhilb

Procedure from library `standard.lib` (see [Section D.1 \[standard\\_lib\]](#), page 794).

**Syntax:**     `stdhilb ( ideal_expression )`  
               `stdhilb ( module_expression )`  
               `stdhilb ( ideal_expression, intvec_expression )`  
               `stdhilb ( module_expression, intvec_expression )`  
               `stdhilb ( ideal_expression, list of string_expressions, and intvec_expression )`

**Type:**       type of the first argument

**Purpose:**     Compute a Groebner basis of the ideal/module in the basering by using the Hilbert driven Groebner basis algorithm. If an argument of type string, stating "std" resp. "slimgb", is given, the standard basis computation uses `std` or `slimgb`, otherwise a heuristically chosen method (default)  
               If an optional second argument `w` of type `intvec` is given, `w` is used as variable weights. If `w` is not given, it is computed as  $w[i] = \deg(\text{var}(i))$ . If the ideal is homogeneous w.r.t. `w` then the Hilbert series is computed w.r.t. to these weights.

**Theory:**     If the ideal is not homogeneous compute first a Groebner basis of the homogenization [w.r.t. the weights `w`] of the ideal/module, then the Hilbert function and, finally, a Groebner basis in the original ring by using the computed Hilbert function. If the

given  $w$  does not coincide with the variable weights of the basering, the result may not be a groebner basis in the original ring.

**Note:** 'Homogeneous' means weighted homogeneous with respect to the weights  $w[i]$  of the variables  $\text{var}(i)$  of the basering. Parameters are not converted to variables.

**Example:**

```

ring r = 0,(x,y,z),lp;
ideal i = y3+x2,x2y+x2z2,x3-z9,z4-y2-xz;
ideal j = stdhilb(i); j;
↳ j[1]=z10
↳ j[2]=yz9
↳ j[3]=2y2z4-z8
↳ j[4]=2y3z3-2y2z5-yz7
↳ j[5]=y4+y3z2
↳ j[6]=xz+y2-z4
↳ j[7]=xy2-xz4-y3z
↳ j[8]=x2+y3
ring r1 = 0,(x,y,z),wp(3,2,1);
ideal i = y3+x2,x2y+x2z2,x3-z9,z4-y2-xz; //ideal is homogeneous
ideal j = stdhilb(i,"std"); j;
↳ j[1]=y2+xz-z4
↳ j[2]=x2-xyz+yz4
↳ j[3]=2xz5-z8
↳ j[4]=2xyz4-yz7+z9
↳ j[5]=z10
↳ j[6]=2yz9+z11
//this is equivalent to:
bigintvec v = hilb(std(i),1);
ideal j1 = std(i,v,intvec(3,2,1)); j1;
↳ j1[1]=y2+xz-z4
↳ j1[2]=x2-xyz+yz4
↳ j1[3]=2xz5-z8
↳ j1[4]=2xyz4-yz7+z9
↳ j1[5]=z10
↳ j1[6]=yz9
size(NF(j,j1))+size(NF(j1,j)); //j and j1 define the same ideal
↳ 0

```

See also: [Section 5.1.53 \[groebner\]](#), page 191; [Section 5.1.145 \[slimgb\]](#), page 265; [Section 5.1.151 \[std\]](#), page 271; [Section 5.1.152 \[stdfglm\]](#), page 272.

### 5.1.154 subst

**Syntax:**    `subst ( poly_expression, variable, poly_expression )`  
               `subst ( poly_expression, variable, poly_expression ,... variable, poly_expression )`  
               `subst ( vector_expression, variable, poly_expression )`  
               `subst ( ideal_expression, variable, poly_expression )`  
               `subst ( module_expression, variable, poly_expression )`

**Type:**     poly, vector, ideal or module (corresponding to the first argument)

**Purpose:**    substitutes one or more ring variable(s)/parameter variable(s) by (a) polynomial(s). Note that in the case of more than one substitution pair, the substitutions will be performed sequentially and not simultaneously. The below examples illustrate this



behaviour.

Note, that the coefficients must be polynomial when substituting a parameter.

**Example:**

```
ring r=0,(x,y,z),dp;
poly f=x2+y2+z2+x+y+z;
subst(f,x,y,y,z);    // first substitute x by y, then y by z
↳ 3z2+3z
subst(f,y,z,x,y);    // first substitute y by z, then x by y
↳ y2+2z2+y+2z
```

See [Section 4.6 \[ideal\]](#), page 80; [Section 4.12 \[map\]](#), page 106; [Section 4.14 \[module\]](#), page 112; [Section 4.17 \[poly\]](#), page 120; [Section D.2.8.17 \[substitute\]](#), page 896; [Section 4.23 \[vector\]](#), page 134.

### 5.1.155 system

**Syntax:**    `system ( string_expression )`  
               `system ( string_expression, expression )`

**Type:**     depends on the desired function, may be none

**Purpose:**   interface to internal data and the operating system. The string-expression determines the command to execute. Some commands require an additional argument (second form) where the type of the argument depends on the command. See below for a list of all possible commands.

**Note:**     Not all functions work on every platform.

**Functions:**

```
system("alarm", int )
    abort the Singular process after computing for that many seconds (system+user cpu time).

system("absFact", poly )
    absolute factorization of the polynomial (from a polynomial ring over a transcendental extension) Returns a list of the ideal of the factors, intvec of multiplicities, ideal of minimal polynomials and the number of factors.

system("blackbox")
    list all blackbox data types.

system("browsers");
    returns a string about available help browsers. See Section 3.1.3 \[The online help system\], page 15.

system("bracket", poly, poly )
    returns the Lie bracket [p,q].

system("complexNearZero", number_expression )
    checks for a small value for floating point numbers

system("contributors")
    returns names of people who contributed to the SINGULAR kernel as string.

system("content",p)
    returns p/content(p) for poly/vector
```

`system("cpu")`  
 returns the number of cpus as int (for creating multiple threads/processes).  
 (see `system("--cpus")`).

`system("denom_list")`  
 returns the list of denominators (number) which occurred in the latest std  
 computation(s). Is reset to the empty list at ring changes or by this system  
 call.

`system("eigenvals", matrix )`  
 returns the list of the eigenvalues of the matrix (as ideal, intvec). (see  
`system("hessenberg")`).

`system("env", ring )`  
 returns the enveloping algebra (i.e.  $R \otimes R^{\text{opp}}$ ) See `system("opp")`.

`system("executable", string )`  
 returns the path of the command given as argument or the  
 empty string (for: not found) See `system("Singular")`. See  
`system("getenv", "PATH")`.

`system("getenv", string_expression)`  
 returns the value of the shell environment variable given as the second  
 argument. The return type is string.

`system("getPrecDigits")`  
 returns the precision for floating point numbers

`system("gmsnf", ideal, ideal, matrix, int, int )`  
 Gauss-Manin system: for `gmshpoly.lib`, `gmshsing.lib`

`system("HC")`  
 returns the degree of the "highest corner" from the last std computation  
 (or 0).

`system("hessenberg", matrix )`  
 returns the Hessenberg matrix (via QR algorithm).

`system("install", s1, s2, p3, i4 )`  
 install a new method p3 for s2 for the newstruct type s1. s2 must be a  
 reserved operator with i4 operands (i4 may be 1,2,3; use 4 for more than 3  
 or a varying number of arguments) See [Section 4.24.4 \[Commands for  
 user defined types\]](#), page 138.

`system("LLL", B )`  
 B must be a matrix or an intmat. Interface to NTLs LLL (Exact Arithmetic  
 Variant over  $\mathbb{Z}$ ). Returns the same type as the input.  
 B is an  $m \times n$  matrix, viewed as  $m$  rows of  $n$ -vectors.  $m$  may be less than,  
 equal to, or greater than  $n$ , and the rows need not be linearly independent.  
 B is transformed into an LLL-reduced basis. The first  $m - \text{rank}(B)$  rows of  
 B are zero.  
 More specifically, elementary row transformations are performed on B so  
 that the non-zero rows of new-B form an LLL-reduced basis for the lattice  
 spanned by the rows of old-B.

`system("nblocks")` or `system("nblocks", ring_name )`  
 returns the number of blocks of the given ring, or of the current basering,  
 if no second argument is given. The return type is int.

```

system("nc_hilb", ideal, int, [...] )
    internal support for ncHilb.lib, return nothing
system("neworder", ideal )
    string of the ring variables in an heurically good order for char_series
system("newstruct")
    list all newstruct data types.
system("opp", ring )
    returns the opposite ring.
system("oppose", ring R, poly p )
    returns the opposite polynomial of p from R.
system("pcvLAddL", list, list )
    system("pcvPMulL", poly, list )
    system("pcvMinDeg", poly )
    system("pcvP2CV", list, int, int )
    system("pcvCV2P", list, int, int )
    system("pcvDim", int, int )
    system("pcvBasis", int, int ) internal for mondromy.lib
system("pid")
    returns the process number as int (for creating unique names).
system("random") or system("random", int )
    returns or sets the seed of the random generator.
system("reduce_bound", poly, ideal, int )

    or system("reduce_bound", ideal, ideal, int )
    or system("reduce_bound", vector, module, int )
    or system("reduce_bound", module, module, int ) returns the normal-
    form of the first argument wrt. the second up to the given degree bound
    (wrt. total degree)
system("reserve", int )
    reserve a port and listen with the given backlog.      (see
    system("reservedLink")).
system("reservedLink")
    accept a connect at the reserved port and return a (write-only) link to it.
    (see system("reserve")).
system("rref", matrix )
    return a reduced row echelon form of the constant matrix M (see
    system("rref")).
system("semaphore", string, int )
    operations for semaphores: string may be "init", "exists", "acquire",
    "try_acquire", "release", "get_value", and int is the number of the
    semaphore. Returns -2 for wrong command, -1 for error or the result of
    the command.
system("semic", list, list )
    or system("semic", list, list, int ) computes from list of spectrum numbers
    and list of spectrum numbers the semicontinuity index (qh, if 3rd argument
    is 1).

```

`system("setenv", string_expression, string_expression)`  
 sets the shell environment variable given as the second argument to the value given as the third argument. Returns the third argument. Might not be available on all platforms.

`system("sh", string_expression )`  
 shell escape, returns the return code of the shell as int. The string is sent literally to the shell.

`system("shrinktest", poly, i2 )`  
 internal for shift algebra (with i2 variables): shrink the poly

`system("Singular")`  
 returns the absolute (path) name of the running SINGULAR as string.

`system("SingularBin")`  
 returns the absolute path name of directory of the running SINGULAR as string (ending in /)

`system("SingularLib")`  
 returns the colon separated library search path name as string.

`system("spadd", list, list )`  
 or `system("spadd", list, list, int )` computes from list of spectrum numbers and list of spectrum numbers the sum of the lists.

`system("spectrum", poly )`  
 or `system("spectrum", poly, int )`

`system("spmul", list, int )`  
 or `system("spmul", list, list, int )` computes from list of spectrum numbers the multiple of it.

`system("std_syz", module, int )`  
 compute a partial groebner base of a module, stop after the given column

`system("tensorModuleMult", int, module )`  
 internal for sheafcoh.lib (see `id_TensorModuleMult`)

`system("twostd", ideal )`  
 returns the two-sided standard basis of the two-sided ideal.

`system("uname")`  
 returns a string identifying the architecture for which SINGULAR was compiled.

`system("verifyGB", ideal_expression/module_expression )`  
 checks, if an ideal/module is a Groebner base

`system("version")`  
 returns the version number of SINGULAR as int. (Version a-b-c-d returns  $a*1000+b*100+c*10+d$ )

`system("with")`  
 without an argument: returns a string describing the current version of SINGULAR, its build options, the used path names and other configurations  
 with a string argument: test for that feature and return an int.

`system("--cpus")`  
 returns the number of available cpu cores as int (for using multiple cores). (see `system("cpu")`).

`system("--")`  
prints the values of all options.

`system("long_option_name")`  
returns the value of the (command-line) option `long_option_name`. The type of the returned value is either string or int. See [Section 3.1.6 \[Command line options\]](#), page 19, for more info.

`system("long_option_name", expression)`  
sets the value of the (command-line) option `long_option_name` to the value given by the expression. Type of the expression must be string, or int. See [Section 3.1.6 \[Command line options\]](#), page 19, for more info. Among others, this can be used for setting the seed of the random number generator, the used help browser, the minimal display time, or the timer resolution.

**Example:**

```
// a listing of the current directory:
system("sh","ls");
// execute a shell, return to SINGULAR with exit:
system("sh","sh");
string unique_name="/tmp/xx"+string(system("pid"));
unique_name;
↳ /tmp/xx4711
system("uname")
↳ ix86-Linux
system("getenv","PATH");
↳ /bin:/usr/bin:/usr/local/bin
system("Singular");
↳ /usr/local/bin/Singular
// report value of all options
system("--");
↳ // --batch           0
↳ // --execute
↳ // --sdb             0
↳ // --echo            1
↳ // --profile         0
↳ // --quiet           1
↳ // --sort            0
↳ // --random          12345678
↳ // --no-tty          1
↳ // --user-option
↳ // --allow-net       0
↳ // --browser
↳ // --cntrlc
↳ // --emacs           0
↳ // --log
↳ // --no-stdlib       0
↳ // --no-rc           1
↳ // --no-warn         0
↳ // --no-out          0
↳ // --no-shell        0
↳ // --min-time        "0.5"
↳ // --cpus            4
↳ // --threads         4
```

```

⇒ // --flint-threads    1
⇒ // --MPport
⇒ // --MPhost
⇒ // --link
⇒ // --ticks-per-sec    1
// set minimal display time to 0.02 seconds
system("--min-time", "0.02");
// set timer resolution to 0.01 seconds
system("--ticks-per-sec", 100);
// re-seed random number generator
system("--random", 12345678);
// allow your web browser to access HTML pages from the net
system("--allow-net", 1);
// and set help browser to firefox
system("--browser", "firefox");
⇒ // ** No help browser 'firefox' available.
⇒ // ** Setting help browser to 'dummy'.

```

### 5.1.156 syz

**Syntax:**    `syz ( ideal-expression )`  
              `syz ( module-expression )`  
              `syz ( ideal-expression, string-expression )`  
              `syz ( module-expression, string-expression )`

**Type:**     module

**Purpose:**    computes the first syzygy (i.e., the module of relations of the given generators) of the ideal, resp. module.  
 An optional second argument specifies the Groebner base algorithm to use. Possible values are "std"(default) and "slimgb".  
 Only for use of "std": If `option(returnSB)` is set, a standard basis is returned, otherwise a generating set.

**Example:**

```

ring R=0,(x,y),(c,dp);
ideal i=x,y;
module s=syz(i);
s;
⇒ s[1]=[y,-x]
matrix(i)*matrix(s);
⇒ _[1,1]=0
s=syz(i,"slimgb");
s;
⇒ s[1]=[y,-x]

```

See [Section 5.1.48 \[fres\]](#), page 188; [Section 5.1.58 \[hres\]](#), page 197; [Section 4.6 \[ideal\]](#), page 80; [Section 5.1.80 \[lift\]](#), page 211; [Section 5.1.81 \[liftstd\]](#), page 212; [Section 5.1.83 \[lres\]](#), page 215; [Section 4.14 \[module\]](#), page 112; [Section 5.1.98 \[mres\]](#), page 225; [Section D.4.21 \[nfmodsyz\\_lib\]](#), page 1178; [Section 5.1.106 \[nres\]](#), page 232; [Section 5.1.111 \[option\]](#), page 234; [Section 5.1.134 \[res\]](#), page 253; [Section 5.1.149 \[sres\]](#), page 269.

### 5.1.157 tensor

**Syntax:**     `tensor ( matrix_expression , matrix_expression )`  
               `tensor ( module_expression , module_expression )`  
               `tensor ( smatrix_expression , smatrix_expression )`

**Type:**       same as the first argument

**Purpose:**     computes the tensor product (Kronecker product) of A and B

**Example:**

```

ring r=32003,(x,y,z),(c,ds);
matrix A[3][3]=1,2,3,4,5,6,7,8,9;
matrix B[2][2]=x,y,0,z;
print(A);
↪ 1,2,3,
↪ 4,5,6,
↪ 7,8,9
print(B);
↪ x,y,
↪ 0,z
print(tensor(A,B));
↪ x, y, 2x,2y,3x,3y,
↪ 0, z, 0, 2z,0, 3z,
↪ 4x,4y,5x,5y,6x,6y,
↪ 0, 4z,0, 5z,0, 6z,
↪ 7x,7y,8x,8y,9x,9y,
↪ 0, 7z,0, 8z,0, 9z

```

See [Section 4.13 \[matrix\]](#), page 108; [Section 4.14 \[module\]](#), page 112.

### 5.1.158 trace

**Syntax:**     `trace ( intmat_expression )`  
               `trace ( matrix_expression )`

**Type:**       int, if the argument is an intmat, resp.  
               poly, if the argument is a matrix

**Purpose:**     returns the trace of an intmat, resp. matrix.

**Example:**

```

intmat m[2][2]=1,2,3,4;
print(m);
↪      1      2
↪      3      4
trace(m);
↪ 5

```

See [Section 4.8 \[intmat\]](#), page 90; [Section 4.13 \[matrix\]](#), page 108.

### 5.1.159 transpose

**Syntax:**     `transpose ( intmat_expression )`  
               `transpose ( matrix_expression )`  
               `transpose ( smatrix_expression )`  
               `transpose ( module_expression )`

**Type:** intmat, matrix, or module, corresponding to the argument

**Purpose:** transposes a matrix.

**Example:**

```

ring R=0,x,dp;
matrix m[2][3]=1,2,3,4,5,6;
print(m);
↪ 1,2,3,
↪ 4,5,6
print(transpose(m));
↪ 1,4,
↪ 2,5,
↪ 3,6

```

See [Section 4.8 \[intmat\], page 90](#); [Section 4.13 \[matrix\], page 108](#); [Section 4.14 \[module\], page 112](#); [Section 4.21 \[smatrix\], page 129](#).

### 5.1.160 type

**Syntax:** type name ;  
type ( name ) ;

**Type:** none

**Purpose:** prints the name, level, type and value of a variable. To display the value of an expression, it is sufficient to type the expression followed by ;.

**Example:**

```

int i=3;
i;
↪ 3
type(i);
↪ // i int 3

```

See [Chapter 4 \[Data types\], page 73](#); [Section 5.1.82 \[listvar\], page 213](#); [Section 5.1.120 \[print\], page 242](#).

### 5.1.161 typeof

**Syntax:** typeof ( expression )

**Type:** string

**Purpose:** returns the type of an expression as string.

Returns the type of the first list element if the expression is an expression list.

Possible types are: "ideal", "int", "intmat", "intvec", "list", "map", "matrix", "module", "number", "none", "poly", "proc", "qring", "resolution", "ring", "string", "vector".

For internal use only is the type "?unknown type?".

**Example:**

```

int i=9; i;
↪ 9
typeof(_);
↪ int

```



```

    print(i);
    ↪ 9
    typeof(_);
    ↪ string
    type i;
    ↪ // i int 9
    typeof(_);
    ↪ string
    string s=typeof(i);
    s;
    ↪ int
    typeof(s);
    ↪ string
    proc p() { "hello"; return();}
    p();
    ↪ hello
    typeof(_);
    ↪ ?unknown type?

```

See [Chapter 4 \[Data types\]](#), page 73; [Section 5.1.160 \[type\]](#), page 282.

### 5.1.162 univariate

**Syntax:** `univariate ( poly_expression )`

**Type:** `int`

**Purpose:** returns 0 for not univariate, -1 for a constant or the number of the variable of the univariate polynomial.

**Example:**

```

    ring r=0,(x,y,z),dp;
    univariate(x2+1);
    ↪ 1
    univariate(x2+y+1);
    ↪ 0
    univariate(1);
    ↪ -1
    univariate(var(2));
    ↪ 2
    var(univariate(z));
    ↪ z

```

See [Section 5.1.77 \[leadexp\]](#), page 210; [Section 5.1.165 \[var\]](#), page 284.

### 5.1.163 uresolve

**Syntax:** `uresolve ( ideal_expression, int_expression, int_expression, int_expression )`

**Type:** `list`

**Purpose:** computes all complex roots of a zerodimensional ideal.  
 Makes either use of the multipolynomial resultant of Macaulay (second argument = 1), which works only for homogeneous ideals, or uses the sparse resultant of Gelfand, Kapranov and Zelevinsky (second argument = 0).  
 The sparse resultant algorithm uses a mixed polyhedral subdivision of the Minkowski

sum of the Newton polytopes in order to construct the sparse resultant matrix. Its determinant is a nonzero multiple of the sparse resultant. The u-resultant of B.\ L. van der Waerden and Laguerre's algorithm are used to determine the complex roots. The third argument defines the precision of the fractional part if the ground field is the field of rational numbers, otherwise it will be ignored. The fourth argument (can be 0, 1 or 2) gives the number of extra runs of Laguerre's algorithm (with corrupted roots), leading to better results.

**Note:** If the ground field is the field of complex numbers, the elements of the list are of type number, otherwise of type string.

See [Section 5.1.74 \[laguerre\]](#), page 208; [Section 5.1.97 \[mpresmat\]](#), page 225.

### 5.1.164 vandermonde

**Syntax:** `vandermonde ( ideal_expression, ideal_expression, int_expression )`

**Type:** poly

**Purpose:** `vandermonde(p,v,d)` computes the (unique) polynomial of degree  $d$  with prescribed values  $v[1], \dots, v[N]$  at the points  $p^0, \dots, p^{N-1}$ ,  $N=(d+1)^n$ ,  $n$  the number of ring variables.

The returned polynomial is  $\sum c_{\alpha_1 \dots \alpha_n} \cdot x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n}$ , where the coefficients  $c_{\alpha_1 \dots \alpha_n}$  are the solution of the (transposed) Vandermonde system of linear equations

$$\sum_{\alpha_1 + \dots + \alpha_n \leq d} c_{\alpha_1 \dots \alpha_n} \cdot p_1^{(k-1)\alpha_1} \cdot \dots \cdot p_n^{(k-1)\alpha_n} = v[k], \quad k = 1, \dots, N.$$

**Note:** the ground field has to be the field of rational numbers. Moreover, `ncols(p)==n`, the number of variables in the basering, and all the given generators have to be numbers different from 0,1 or -1. Finally, `ncols(v)==(d+1)^n`, and all given generators have to be numbers.

**Example:**

```
ring r=0,(x,y),dp;
// determine f with deg(f)=2 and with given values v of f
// at 9 points: (2,3)^0=(1,1),..., (2,3)^8=(2^8,3^8)
// valuation point: (2,3)
ideal p=2,3;
ideal v=1,2,3,4,5,6,7,8,9;
poly ip=vandermonde(p,v,2);
ip[1..5]; // the 5 first terms of ip:
→ -1/9797760x2y2-595/85536x2y+55/396576xy2+935/384x2-1309/3240xy
// compute value of ip at the point 2^8,3^8, result must be 9
subst(subst(ip,x,2^8),y,3^8);
→ 9
```

### 5.1.165 var

**Syntax:** `var ( int_expression )`

**Type:** poly

**Purpose:** `var(n)` returns the  $n$ -th ring variable.

**Example:**

```

ring r=0,(x,y,z),dp;
var(2);
↪ y

```

See [Section 4.7 \[int\]](#), page 85; [Section 5.1.109 \[nvars\]](#), page 233; [Section 4.20 \[ring\]](#), page 127; [Section 5.1.139 \[rvar\]](#), page 258; [Section 5.1.162 \[univariate\]](#), page 283; [Section 5.1.167 \[varstr\]](#), page 285.

### 5.1.166 variables

**Syntax:**    `variables ( poly_expression )`  
              `variables ( ideal_expression )`  
              `variables ( matrix_expression )`

**Type:**     ideal

**Purpose:**    `variables(p)` returns the list of all ring variables the argument depends on.

**Example:**

```

ring r=0,(x,y,z),dp;
variables(2);
↪ _[1]=0
variables(x+y2);
↪ _[1]=x
↪ _[2]=y
variables(ideal(x+y2,x3y,z));
↪ _[1]=x
↪ _[2]=y
↪ _[3]=z
string(variables(ideal(x+y2,x3y,z)));
↪ x,y,z

```

See [Section 5.1.77 \[leadexp\]](#), page 210; [Section 5.1.109 \[nvars\]](#), page 233; [Section 5.1.162 \[univariate\]](#), page 283; [Section 5.1.165 \[var\]](#), page 284; [Section 5.1.167 \[varstr\]](#), page 285.

### 5.1.167 varstr

**Syntax:**    `varstr ( ring_name )`  
              `varstr ( int_expression )`  
              `varstr ( ring_name, int_expression )`

**Type:**     string

**Purpose:**    returns the list of the names of the ring variables as a string or the name of the n-th ring variable, where n is given by the int\_expression.  
 If the ring name is omitted, the basering is used, thus `varstr(n)` is equivalent to `varstr(basing,n)`.

**Example:**

```

ring r=0,(x,y,z),dp;
varstr(r);
↪ x,y,z
varstr(r,1);
↪ x
varstr(2);
↪ y

```

See [Section 5.1.7 \[charstr\]](#), page 162; [Section 4.7 \[int\]](#), page 85; [Section 5.1.109 \[nvars\]](#), page 233; [Section 5.1.113 \[ordstr\]](#), page 239; [Section 5.1.116 \[parstr\]](#), page 240; [Section 4.20 \[ring\]](#), page 127; [Section 5.1.165 \[var\]](#), page 284.

### 5.1.168 vdim

**Syntax:**    vdim ( ideal\_expression )  
              vdim ( module\_expression )

**Type:**       int

**Purpose:**    computes the vector space dimension of the ring, resp. free module, modulo the ideal, resp. module, generated by the initial terms of the given generators. If the generators form a standard basis, this is the same as the vector space dimension of the ring, resp. free module, modulo the ideal, resp. module.  
If the ideal, resp. module, is not zero-dimensional, -1 is returned.

**Example:**

```
ring r=0,(x,y),ds;
ideal i=x2+y2,x2-y2;
ideal j=std(i);
vdim(j);
↦ 4
```

See [Section D.6.20.1 \[codim\]](#), page 1757; [Section 5.1.20 \[degree\]](#), page 171; [Section 5.1.25 \[dim\]](#), page 174; [Section 4.6 \[ideal\]](#), page 80; [Section 5.1.69 \[kbase\]](#), page 205; [Section 5.1.101 \[mult\]](#), page 228; [Section 5.1.151 \[std\]](#), page 271.

### 5.1.169 waitall

**Syntax:**    waitall ( list\_expression )  
              waitall ( list\_expression , int\_expression )

**Type:**       int

**Purpose:**    Expects a list of open links (of mode ssi:fork, ssi:tcp) and waits until all of them are finished, i.e., are ready to be read.  
In the first case, the command waits for all links to finish (or to crash, see below) and may therefore run forever.  
In the second case, a timeout in milliseconds can be provided, forcing the command to terminate after the specified time. If the given timeout is 0, the command checks whether all links are finished or not, but does not wait for any link (polling).  
Return values are:  
-1: The read state of all links is "eof", see [Section 4.10 \[link\]](#), page 96, [Section 5.1.150 \[status\]](#), page 270. This might happen if all the links crashed.  
0: timeout (or polling): None of the links is ready.  
1: All links are ready. (Note: There might be links whose read state is "eof", but at least one link is ready.)

**Example:**

```
link l1 = "ssi:fork"; open(l1);
link l2 = "ssi:fork"; open(l2);
link l3 = "ssi:fork"; open(l3);
list l = list(l1,l2,l3);
```

```

        write(l1, quote(system("sh", "sleep 15")));
        write(l2, quote(system("sh", "sleep 10")));
        write(l3, quote(system("sh", "sleep 11")));
        waitall(1, 5000); // terminates after 5sec with result 0
    ↪ 0
        waitall(1);      // terminates after 10 more sec
    ↪ 1
        close(l1);
        close(l2);
        close(l3);

```

See [Section 5.1.170 \[waitfirst\]](#), page 287.

### 5.1.170 waitfirst

**Syntax:**    `waitfirst ( list_expression )`  
               `waitfirst ( list_expression , int_expression )`

**Type:**     `int`

**Purpose:**    Expects a list of open links (of mode `ssi:fork`, `ssi:tcp`) and waits until the first of them is finished, i.e., is ready to be read.  
               In the first case, the command waits until the first link is finished (or all of them crashed, see below) and may therefore run forever.  
               In the second case, a timeout in milliseconds can be provided, forcing the command to terminate after the specified time. If the given timeout is 0, the command checks whether one of the links is finished or not, but does not wait for any link (polling).  
               Return values are:  
               -1: The read state of all links is "eof", see [Section 4.10 \[link\]](#), page 96, [Section 5.1.150 \[status\]](#), page 270. This might happen if all the links crashed.  
               0: timeout (or polling): None of the links is ready.  
               i>1: At least the link whose list index is i is ready.

**Example:**

```

        link l1 = "ssi:fork"; open(l1);
        link l2 = "ssi:fork"; open(l2);
        link l3 = "ssi:fork"; open(l3);
        list l = list(l1,l2,l3);
        write(l1, quote(system("sh", "sleep 15")));
        write(l2, quote(system("sh", "sleep 13")));
        write(l3, quote(system("sh", "sleep 11")));
        waitfirst(1, 5000); // terminates after 5sec with result 0
    ↪ 0
        waitfirst(1);      // terminates after 6 more sec with result 3
    ↪ 3
        close(l1);
        close(l2);
        close(l3);

```

See [Section 5.1.169 \[waitall\]](#), page 286.

### 5.1.171 wedge

**Syntax:**    `wedge ( matrix_expression , int_expression )`

**Type:** matrix

**Purpose:** `wedge(M,n)` computes the  $n$ -th exterior power of the matrix  $M$ .

**Example:**

```
ring r;
matrix m[2][3]=x,y,y,z,z,x;
print(m);
↪ x,y,y,
↪ z,z,x
print(wedge(m,2));
↪ xz-yz,-x2+yz,xy-yz
```

See [Section 4.7 \[int\]](#), page 85; [Section 4.13 \[matrix\]](#), page 108; [Section 5.1.92 \[minor\]](#), page 221.

### 5.1.172 weight

**Syntax:** `weight ( ideal_expression )`  
`weight ( module_expression )`

**Type:** intvec

**Purpose:** computes an "optimal" weight vector for an ideal, resp. module, which may be used as weight vector for the variables in order to speed up the standard basis algorithm. If the input is weighted homogeneous, a weight vector for which the input is weighted homogeneous is found.

**Example:**

```
ring h1=32003,(t,x,y,z),dp;
ideal i=
9x8+y7t3z4+5x4y2t2+2xy2z3t2,
9y8+7xy6t+2x5y4t2+2x2yz3t2,
9z8+3x2y3z2t4;
intvec e=weight(i);
e;
↪ 5,7,5,7
ring r=32003,(a,b,c,d),wp(e);
map f=h1,a,b,c,d;
ideal i0=std(f(i));
```

See [Section 4.6 \[ideal\]](#), page 80; [Section 4.9 \[intvec\]](#), page 93; [Section 5.1.124 \[qhweight\]](#), page 246.

### 5.1.173 weightKB

Procedure from library `standard.lib` (see [Section D.1 \[standard.lib\]](#), page 794).

**Syntax:** `weightKB ( module_expression, int_expression , list_expression )`  
`weightKB ( ideal_expression, int_expression, list_expression )`

**Return:** the same as the input type of the first argument

**Purpose:** If  $I, d, \mathbf{wim}$  denotes the three arguments then `weightKB` computes the weighted degree- $d$  part of a vector space basis (consisting of monomials) of the quotient ring, resp. of the quotient module, modulo  $I$  w.r.t. weights given by  $\mathbf{wim}$ . The information about the weights is given as a list of two intvec:  $\mathbf{wim}[1]$  weights for all variables (positive),  $\mathbf{wim}[2]$  weights for the module generators.

**Note:** This is a generalization of the command `kbase` with the same first two arguments.

**Example:**

```
ring R=0, (x,y), wp(1,2);
weightKB(ideal(0),3,intvec(1,2));
⇨ _[1]=x3
⇨ _[2]=xy
```

See also: [Section 5.1.69 \[kbase\]](#), page 205.

### 5.1.174 write

**Syntax:** `write ( link_expression, expression_list )`  
for DBM links:  
`write ( link, string_expression, string_expression )`  
`write ( link, string_expression )`

**Type:** none

**Purpose:** writes data to a link.  
If the link is of type `ASCII`, all expressions are converted to strings (and separated by a newline character) before they are written. As a consequence, only such values which can be converted to a string can be written to an `ASCII` link.  
For `ssi` links, ring-dependent expressions are written together with a ring description. To prevent an evaluation of the expression before it is written, the `quote` command (possibly together with `eval`) can be used. A `write` blocks (i.e., does not return to the prompt), as long as a `ssi` link is not ready for writing.  
For DBM links, `write` with three arguments inserts the first string as key and the second string as value into the dbm data base.  
Called with two arguments, it deletes the entry with the key specified by the string from the data base.

**Example:**

```
// write the values of the variables f and i as strings into
// the file "outfile" (overwrite it, if it exists)
write(":w outfile",f,i);

// now append the string "that was f,i" (without the quotes)
// at the end of the file "outfile"
write(":a outfile","that was f,i");
// alternatively, links could be used:
link l=":a outfile"; l;
// type : ASCII
// mode : a
// name : outfile
// open : no
// read : not ready
// write: not ready
write(l," that was f,i");
// saving and retrieving data (ASCII format):
ring r=32003,(x,y,z),dp;
ideal i=x+y,z3+22y;
write(":w save_i",i);// this writes x+y,z3+22y to the file save_i
ring r=32003,(x,y,z),dp;
```

```

string s=read("save_i");    //creates the string x+y,z3+22y
execute("ideal k="+s+");    // this defines an ideal k which
                           // is equal to i.
// for large objects, the ssi format and ssi links are better:
write("ssi:w save_i.ssi",i);
def j=read("ssi:r save_i.ssi");

```

See [Chapter 4 \[Data types\]](#), page 73; [Section 5.1.27 \[dump\]](#), page 175; [Section 5.1.29 \[eval\]](#), page 177; [Section 4.10 \[link\]](#), page 96; [Section 5.1.120 \[print\]](#), page 242; [Section 5.1.121 \[printf\]](#), page 244; [Section 5.1.126 \[quote\]](#), page 247; [Section 5.1.130 \[read\]](#), page 250; [Section 5.3.7 \[short\]](#), page 305.

## 5.2 Control structures

A sequence of commands surrounded by curly brackets (`{` and `}`) is a so-called block. Blocks are used in SINGULAR in order to define procedures and to collect commands belonging to `if`, `else`, `for` and `while` statements and to the `example` part in libraries. Even if the sequence of statements consists of only a single command it has to be surrounded by curly brackets! Variables which are defined inside a block are not local to that block. Note that there need not be an ending semicolon at the end of the block.

**Example:**

```

if ( i>j )
{
    // This is the block
    int temp;
    temp=i;
    i=j;
    j=temp;
    kill temp;
}

```

### 5.2.1 apply

**Syntax:** `apply( expression , function );`

**Purpose:** applies the function to all elements of the first argument. The first argument must be of type `intvec`, `intmat`, or `list`. The result will be an expression list, its type and format will be set by the following assign. The function must be a kernel command or a procedure which takes one argument and returns a value.

**Example:**

```

proc p(int x) {return(x^2);}
intvec v=1,2,3;
apply(v,p);
↪ 1 4 9
intvec vv=apply(v,p);vv;
↪ 1,4,9
list ll=apply(v,p);ll;
↪ [1]:
↪ 1
↪ [2]:
↪ 4
↪ [3]:
↪ 9

```



### 5.2.2 break

**Syntax:** `break;`

**Purpose:** leaves the innermost `for` or `while` block.

**Example:**

```
while (1)
{
    ...
    if ( ... )
    {
        break; // leave the while block
    }
}
```

See [Section 5.2 \[Control structures\]](#), page 290; [Section 5.2.8 \[for\]](#), page 295; [Section 5.2.15 \[while\]](#), page 301.

### 5.2.3 breakpoint

**Syntax:** `breakpoint( proc_name );`  
`breakpoint( proc_name, line_no );`

**Purpose:** sets a breakpoint at the beginning of the specified procedure or at the given line.

**Note:** Line number 1 is the first line of a library (for procedures from libraries), resp. the line with the `{`.

A line number of -1 removes all breakpoint from that procedure.

**Example:**

```
breakpoint(groebner);
↳ breakpoint 1, at line 830 in groebner
breakpoint(groebner, 176);
↳ breakpoint 2, at line 176 in groebner
breakpoint(groebner, -1);
↳ breakpoints in groebner deleted(0x6)
```

See [Section 3.9.3 \[Source code debugger\]](#), page 69; [Section 5.2.16 \[~\]](#), page 302.

### 5.2.4 continue

**Syntax:** `continue;`

**Purpose:** skips the rest of the innermost `for` or `while` loop und jumps to the beginning of the block. This command is only valid inside a `for` or a `while` construction.

**Note:** Unlike the C-construct it **does not execute the increment statement**. The command `continue` is mainly for internal use.

**Example:**

```
for (int i = 1 ; i<=10; i=i+1)
{
    ...
    if (i==3) { i=8;continue; }
    // skip the rest if i is 3 and
    // continue with the next i: 8
}
```

```

        i;
    }
    ↦ 1
    ↦ 2
    ↦ 8
    ↦ 9
    ↦ 10

```

See [Section 5.2 \[Control structures\]](#), page 290; [Section 5.2.8 \[for\]](#), page 295; [Section 5.2.15 \[while\]](#), page 301.

### 5.2.5 else

**Syntax:** `if ( boolean_expression ) true_block else false_block`

**Purpose:** executes `false_block` if the `boolean_expression` of the `if` statement is false. This command is only valid in combination with an `if` command.

**Example:**

```

    int i=3;
    if (i > 5)
    {
        "i is bigger than 5";
    }
    else
    {
        "i is smaller than 6";
    }
    ↦ i is smaller than 6

```

See [Section 5.2 \[Control structures\]](#), page 290; [Section 4.7.5 \[boolean expressions\]](#), page 89; [Section 5.2.9 \[if\]](#), page 296.

### 5.2.6 export

**Syntax:** `export name ;`  
`export list_of_names ;`

**Purpose:** converts a local variable of a procedure to a global one, that is the identifier is not removed at the end of the procedure. However, the package the variable belongs to is not changed.

**Note:** Objects defined in a ring are not automatically exported when exporting the ring.

**Example:**

```

proc p1
{
    int i,j;
    export(i);
    intmat m;
    listvar();
    export(m);
}
p1();
↦ // m

```

[1] intmat 1 x 1

```

⇒ // j                [1]  int 0
⇒ // i                [0]  int 0
listvar();
⇒ // m                [0]  intmat 1 x 1
⇒ // i                [0]  int 0

```

See [Section 5.2.7 \[exportto\]](#), page 293; [Section 5.2.10 \[importfrom\]](#), page 296; [Section 5.2.11 \[keep-ring\]](#), page 298.

### 5.2.7 exportto

**Syntax:** `exportto( package_name , name );`  
`exportto( package_name , list_of_names );`

**Purpose:** transfers an identifier in the current package into the one specified by `package_name`. `package_name` can be `Current`, `Top` or any other identifier of type package.

**Note:** Objects defined in a ring are not automatically exported when exporting the ring.

**Warning:** The identifier is transferred to the other package. It does no longer exist in the current package. If the identifier should only be copied, [Section 5.2.10 \[importfrom\]](#), page 296 should be used instead.

**Example:**

```

proc p1
{
  int i,j;
  exportto(Current,i);
  intmat m;
  listvar(Current);
  exportto(Top,m);
}
p1();
⇒ // Top                [0]  package Top (T)
⇒ // ::m                [1]  intmat 1 x 1
⇒ // ::i                [0]  int 0
⇒ // ::j                [1]  int 0
⇒ // ::#                [1]  list, size: 0
⇒ // ::p1               [0]  proc
⇒ // ::mathicgb_prOrder [0]  proc from singmathic.so (C)
⇒ // ::mathicgb         [0]  proc from singmathic.so (C)
⇒ // ::create_ring      [0]  proc from standard.lib
⇒ // ::min              [0]  proc from standard.lib
⇒ // ::max              [0]  proc from standard.lib
⇒ // ::datetime         [0]  proc from standard.lib
⇒ // ::weightKB         [0]  proc from standard.lib
⇒ // ::fprintf          [0]  proc from standard.lib
⇒ // ::printf           [0]  proc from standard.lib
⇒ // ::sprintf          [0]  proc from standard.lib
⇒ // ::quotient4        [0]  proc from standard.lib
⇒ // ::quotient5        [0]  proc from standard.lib
⇒ // ::quotient3        [0]  proc from standard.lib
⇒ // ::quotient2        [0]  proc from standard.lib
⇒ // ::quotient1        [0]  proc from standard.lib
⇒ // ::quot             [0]  proc from standard.lib

```

```

⇒ // ::res [0] proc from standard.lib
⇒ // ::groebner [0] proc from standard.lib
⇒ // ::qslimgb [0] proc from standard.lib
⇒ // ::hilbRing [0] proc from standard.lib
⇒ // ::par2varRing [0] proc from standard.lib
⇒ // ::quotientList [0] proc from standard.lib
⇒ // ::stdhilb [0] proc from standard.lib
⇒ // ::stdfglm [0] proc from standard.lib
⇒ // ::Float [0] proc from kernel (C)
⇒ // ::crossprod [0] proc from kernel (C)
package Test1;
exportto(Test1,p1);
listvar(Top);
⇒ // Top [0] package Top (T)
⇒ // ::m [0] intmat 1 x 1
⇒ // ::i [0] int 0
⇒ // ::mathicgb_prOrder [0] proc from singmathic.so (C)
⇒ // ::mathicgb [0] proc from singmathic.so (C)
⇒ // ::create_ring [0] proc from standard.lib
⇒ // ::min [0] proc from standard.lib
⇒ // ::max [0] proc from standard.lib
⇒ // ::datetime [0] proc from standard.lib
⇒ // ::weightKB [0] proc from standard.lib
⇒ // ::fprintf [0] proc from standard.lib
⇒ // ::printf [0] proc from standard.lib
⇒ // ::sprintf [0] proc from standard.lib
⇒ // ::quotient4 [0] proc from standard.lib
⇒ // ::quotient5 [0] proc from standard.lib
⇒ // ::quotient3 [0] proc from standard.lib
⇒ // ::quotient2 [0] proc from standard.lib
⇒ // ::quotient1 [0] proc from standard.lib
⇒ // ::quot [0] proc from standard.lib
⇒ // ::res [0] proc from standard.lib
⇒ // ::groebner [0] proc from standard.lib
⇒ // ::qslimgb [0] proc from standard.lib
⇒ // ::hilbRing [0] proc from standard.lib
⇒ // ::par2varRing [0] proc from standard.lib
⇒ // ::quotientList [0] proc from standard.lib
⇒ // ::stdhilb [0] proc from standard.lib
⇒ // ::stdfglm [0] proc from standard.lib
⇒ // ::Float [0] proc from kernel (C)
⇒ // ::crossprod [0] proc from kernel (C)
listvar(Test1);
⇒ // Test1 [0] package Test1 (N)
⇒ // ::p1 [0] proc
Test1::p1();
⇒ // Test1 [0] package Test1 (N)
⇒ // ::m [1] intmat 1 x 1
⇒ // ::i [0] int 0
⇒ // ::j [1] int 0
⇒ // ::# [1] list, size: 0
⇒ // ::p1 [0] proc
⇒ // ** redefining m ( exportto(Top,m);)

```

```
listvar(Top);
⇒ // Top                                [0] package Top (T)
⇒ // ::m                                [0] intmat 1 x 1
⇒ // ::i                                [0] int 0
⇒ // ::mathicgb_prOrder                 [0] proc from singmathic.so (C)
⇒ // ::mathicgb                         [0] proc from singmathic.so (C)
⇒ // ::create_ring                      [0] proc from standard.lib
⇒ // ::min                              [0] proc from standard.lib
⇒ // ::max                              [0] proc from standard.lib
⇒ // ::datetime                         [0] proc from standard.lib
⇒ // ::weightKB                        [0] proc from standard.lib
⇒ // ::fprintf                         [0] proc from standard.lib
⇒ // ::printf                         [0] proc from standard.lib
⇒ // ::sprintf                        [0] proc from standard.lib
⇒ // ::quotient4                       [0] proc from standard.lib
⇒ // ::quotient5                       [0] proc from standard.lib
⇒ // ::quotient3                       [0] proc from standard.lib
⇒ // ::quotient2                       [0] proc from standard.lib
⇒ // ::quotient1                       [0] proc from standard.lib
⇒ // ::quot                            [0] proc from standard.lib
⇒ // ::res                             [0] proc from standard.lib
⇒ // ::groebner                        [0] proc from standard.lib
⇒ // ::qslimgb                         [0] proc from standard.lib
⇒ // ::hilbRing                       [0] proc from standard.lib
⇒ // ::par2varRing                    [0] proc from standard.lib
⇒ // ::quotientList                   [0] proc from standard.lib
⇒ // ::stdhilb                        [0] proc from standard.lib
⇒ // ::stdfglm                        [0] proc from standard.lib
⇒ // ::Float                          [0] proc from kernel (C)
⇒ // ::crossprod                      [0] proc from kernel (C)
listvar(Test1);
⇒ // Test1                             [0] package Test1 (N)
⇒ // ::i                               [0] int 0
⇒ // ::p1                              [0] proc
```

See [Section 5.2.6 \[export\]](#), page 292; [Section 5.2.10 \[importfrom\]](#), page 296; [Section 5.2.11 \[keeping\]](#), page 298.

## 5.2.8 for

**Syntax:**    `for ( init_command; boolean_expression; iterate_commands) block`

**Purpose:** repetitive, conditional execution of a command block.

The command `init_command` is executed first. Then `boolean_expression` is evaluated. If its value is `TRUE` the block is executed, otherwise the `for` statement is complete. After each execution of the block, the command `iterate_command` is executed and `boolean_expression` is evaluated. This is repeated until `boolean_expression` evaluates to `FALSE`.

The command `break;` leaves the innermost `for` construct.

**Example:**

```
// sum of 1 to 10:
int s=0;
for (int i=1; i<=10; i=i+1)
```

```

{
    s=s+i;
}
s;
↪ 55

```

See [Section 5.2 \[Control structures\]](#), page 290; [Section 4.7.5 \[boolean expressions\]](#), page 89; [Section 5.2.2 \[break\]](#), page 291; [Section 5.2.4 \[continue\]](#), page 291; [Section 5.2.9 \[if\]](#), page 296; [Section 5.2.15 \[while\]](#), page 301.

### 5.2.9 if

**Syntax:**    `if ( boolean_expression ) true_block`  
               `if ( boolean_expression ) true_block else false_block`

**Purpose:**    executes `true_block` if the boolean condition is true. If the `if` statement is followed by an `else` statement and the boolean condition is false, then `false_block` is executed.

**Example:**

```

int i = 9;
matrix m[i][i];
if ( i > 5 and typeof(m) == "matrix" )
{
    m[i][i] = i;
}

```

See [Section 5.2 \[Control structures\]](#), page 290; [Section 4.7.5 \[boolean expressions\]](#), page 89; [Section 5.2.2 \[break\]](#), page 291; [Section 5.2.5 \[else\]](#), page 292.

### 5.2.10 importfrom

**Syntax:**    `importfrom( package_name , name );`  
               `importfrom( package_name , list_of_names );`

**Purpose:**    creates a new identifier in the current package which is a copy of the one specified by `name` in the package `package_name`. `package_name` can be `Top` or any other identifier of type package.

**Note:**       Objects defined in a ring are not automatically imported when importing the ring.

**Warning:**    The identifier is copied to the current package. It does still exist (independently) in the package `package_name`. If the identifier should be erased in the package from which it originates, [Section 5.2.7 \[exportto\]](#), page 293 should be used instead.

**Example:**

<code>listvar(Top);</code>	
<code>↪ // Top</code>	<code>[0] package Top (T)</code>
<code>↪ // ::mathicgb_prOrder</code>	<code>[0] proc from singmathic.so (C)</code>
<code>↪ // ::mathicgb</code>	<code>[0] proc from singmathic.so (C)</code>
<code>↪ // ::create_ring</code>	<code>[0] proc from standard.lib</code>
<code>↪ // ::min</code>	<code>[0] proc from standard.lib</code>
<code>↪ // ::max</code>	<code>[0] proc from standard.lib</code>
<code>↪ // ::datetime</code>	<code>[0] proc from standard.lib</code>
<code>↪ // ::weightKB</code>	<code>[0] proc from standard.lib</code>
<code>↪ // ::fprintf</code>	<code>[0] proc from standard.lib</code>
<code>↪ // ::printf</code>	<code>[0] proc from standard.lib</code>

⇒ // ::sprintf	[0] proc from standard.lib
⇒ // ::quotient4	[0] proc from standard.lib
⇒ // ::quotient5	[0] proc from standard.lib
⇒ // ::quotient3	[0] proc from standard.lib
⇒ // ::quotient2	[0] proc from standard.lib
⇒ // ::quotient1	[0] proc from standard.lib
⇒ // ::quot	[0] proc from standard.lib
⇒ // ::res	[0] proc from standard.lib
⇒ // ::groebner	[0] proc from standard.lib
⇒ // ::qslimb	[0] proc from standard.lib
⇒ // ::hilbRing	[0] proc from standard.lib
⇒ // ::par2varRing	[0] proc from standard.lib
⇒ // ::quotientList	[0] proc from standard.lib
⇒ // ::stdhilb	[0] proc from standard.lib
⇒ // ::stdfglm	[0] proc from standard.lib
⇒ // ::Float	[0] proc from kernel (C)
⇒ // ::crossprod	[0] proc from kernel (C)
load("inout.lib");	
listvar(Top);	
⇒ // Top	[0] package Top (T)
⇒ // ::mathicgb_prOrder	[0] proc from singmathic.so (C)
⇒ // ::mathicgb	[0] proc from singmathic.so (C)
⇒ // ::create_ring	[0] proc from standard.lib
⇒ // ::min	[0] proc from standard.lib
⇒ // ::max	[0] proc from standard.lib
⇒ // ::datetime	[0] proc from standard.lib
⇒ // ::weightKB	[0] proc from standard.lib
⇒ // ::fprintf	[0] proc from standard.lib
⇒ // ::printf	[0] proc from standard.lib
⇒ // ::sprintf	[0] proc from standard.lib
⇒ // ::quotient4	[0] proc from standard.lib
⇒ // ::quotient5	[0] proc from standard.lib
⇒ // ::quotient3	[0] proc from standard.lib
⇒ // ::quotient2	[0] proc from standard.lib
⇒ // ::quotient1	[0] proc from standard.lib
⇒ // ::quot	[0] proc from standard.lib
⇒ // ::res	[0] proc from standard.lib
⇒ // ::groebner	[0] proc from standard.lib
⇒ // ::qslimb	[0] proc from standard.lib
⇒ // ::hilbRing	[0] proc from standard.lib
⇒ // ::par2varRing	[0] proc from standard.lib
⇒ // ::quotientList	[0] proc from standard.lib
⇒ // ::stdhilb	[0] proc from standard.lib
⇒ // ::stdfglm	[0] proc from standard.lib
⇒ // ::Float	[0] proc from kernel (C)
⇒ // ::crossprod	[0] proc from kernel (C)
importfrom(Inout,pause);	
listvar(Top);	
⇒ // Top	[0] package Top (T)
⇒ // ::pause	[0] proc from inout.lib
⇒ // ::mathicgb_prOrder	[0] proc from singmathic.so (C)
⇒ // ::mathicgb	[0] proc from singmathic.so (C)
⇒ // ::create_ring	[0] proc from standard.lib

⇒ // ::min	[0] proc from standard.lib
⇒ // ::max	[0] proc from standard.lib
⇒ // ::datetime	[0] proc from standard.lib
⇒ // ::weightKB	[0] proc from standard.lib
⇒ // ::fprintf	[0] proc from standard.lib
⇒ // ::printf	[0] proc from standard.lib
⇒ // ::sprintf	[0] proc from standard.lib
⇒ // ::quotient4	[0] proc from standard.lib
⇒ // ::quotient5	[0] proc from standard.lib
⇒ // ::quotient3	[0] proc from standard.lib
⇒ // ::quotient2	[0] proc from standard.lib
⇒ // ::quotient1	[0] proc from standard.lib
⇒ // ::quot	[0] proc from standard.lib
⇒ // ::res	[0] proc from standard.lib
⇒ // ::groebner	[0] proc from standard.lib
⇒ // ::qslimb	[0] proc from standard.lib
⇒ // ::hilbRing	[0] proc from standard.lib
⇒ // ::par2varRing	[0] proc from standard.lib
⇒ // ::quotientList	[0] proc from standard.lib
⇒ // ::stdhilb	[0] proc from standard.lib
⇒ // ::stdfglm	[0] proc from standard.lib
⇒ // ::Float	[0] proc from kernel (C)
⇒ // ::crossprod	[0] proc from kernel (C)

See [Section 5.2.6 \[export\]](#), page 292; [Section 5.2.7 \[exportto\]](#), page 293; [Section 5.2.11 \[keepring\]](#), page 298.

### 5.2.11 keepring

**Syntax:**    `keepring name ;`

**Warning:**    This command is obsolete. Instead the respective identifiers in the ring should be exported and the ring itself should subsequently be returned. The command is only included for backward compatibility and may be removed in future releases.

**Purpose:**    moves the specified ring to the next (upper) level. This command can only be used inside of procedures and it should be the last command before the `return` statement. There it provides the possibility to keep a ring which is local to the procedure (and its objects) accessible after the procedure ended without making the ring global.

**Example:**

```

proc P1
{
  ring r=0,x,dp;
  keepring r;
}
proc P2
{
  "inside P2: " + nameof(basering);
  P1();
  "inside P2, after call of P1: " + nameof(basering);
}
ring r1= 0,y,dp;
P2();

```



```

↳ inside P2: r1
↳ inside P2, after call of P1: r
"at top level: " + nameof(basering);
↳ at top level: r1

```

See [Section 4.20 \[ring\]](#), page 127.

### 5.2.12 load

**Syntax:**    `load( string_expression );`  
               `load( string_expression , string_expression );`

Type: none

**Purpose:** reads a library of procedures from a file. In contrast to the command `LIB` (see note below), the command `load` does not add the procedures of the library to the package `Top`, but only to the package corresponding to the library. If the given filename does not start with `.` or `/`, the following directories are searched for it (in the given order): the current directory, the directories given in the environment variable `SINGULARPATH`, some default directories relative to the location of the SINGULAR executable program, and finally some default absolute directories. You can view the search path which SINGULAR uses to locate its libraries, by starting up SINGULAR with the option `-v`, or by issuing the command `system("with");`.

The second string selections options for loading.

**Note:** `load(<string_expr>,"with")` is equivalent to `LIB <string_expr>`.

**Note:** `load(<string_expr>,"try")` is equivalent to  
`LIB <string_expr>` which never fails - test the package name to distinguish.

All loaded libraries are displayed by the `listvar(package);` command:

```

option(loadLib);      // show loading of libraries;
                      // standard.lib is loaded

listvar(package);
⇒ // Singmathic                [0] package Singmathic (C,singmathic.s\
o)
⇒ // Standard                  [0] package Standard (S,standard.lib)
⇒ // Top                       [0] package Top (T)
                      // the names of the procedures of inout.lib
load("inout.lib"); // are now known to Singular
⇒ // ** loaded inout.lib (4.1.2.0, Feb_2019)
listvar(package);
⇒ // Inout                     [0] package Inout (S,inout.lib)
⇒ // Singmathic                [0] package Singmathic (C,singmathic.s\
o)
⇒ // Standard                  [0] package Standard (S,standard.lib)
⇒ // Top                       [0] package Top (T)
load("blabla.lib","try");
listvar(package);
⇒ // Inout                     [0] package Inout (S,inout.lib)
⇒ // Singmathic                [0] package Singmathic (C,singmathic.s\
o)
⇒ // Standard                  [0] package Standard (S,standard.lib)
⇒ // Top                       [0] package Top (T)

```

```

option(noloadLib); // do not show loading of libraries;
load("matrix.lib","try");
listvar(package);
⇒ // Elim [0] package Elim (S,elim.lib)
⇒ // Triang [0] package Triang (S,triang.lib)
⇒ // Absfact [0] package Absfact (S,absfact.lib)
⇒ // Tasks [0] package Tasks (S,tasks.lib)
⇒ // Parallel [0] package Parallel (S,parallel.lib)
⇒ // Resources [0] package Resources (S,resources.lib\
)
⇒ // Modular [0] package Modular (S,modular.lib)
⇒ // Modfinduni [0] package Modfinduni (S,modfinduni.l\
ib)
⇒ // Primdec [0] package Primdec (S,primdec.lib)
⇒ // Ring [0] package Ring (S,ring.lib)
⇒ // Random [0] package Random (S,random.lib)
⇒ // General [0] package General (S,general.lib)
⇒ // Polylib [0] package Polylib (S,polylib.lib)
⇒ // Nctools [0] package Nctools (S,nctools.lib)
⇒ // Matrix [0] package Matrix (S,matrix.lib)
⇒ // Inout [0] package Inout (S,inout.lib)
⇒ // Singmathic [0] package Singmathic (C,singmathic.s\
o)
⇒ // Standard [0] package Standard (S,standard.lib)
⇒ // Top [0] package Top (T)

```

Each time a library ([Section 3.8 \[Libraries\]](#), page 55) / dynamic module ([Section 3.10 \[Dynamic loading\]](#), page 71) is loaded, the corresponding package is created, if it does not already exist.

The name of a package corresponding to a SINGULAR library is derived from the name of the library file. The first letter is capitalized and everything to right of the left-most dot is dropped. For a dynamic module the packagename is hard-coded in the binary file.

Only the names of the procedures in the library are loaded, the body of the procedures is read during the first call of this procedure. This minimizes memory consumption by unused procedures. When SINGULAR is started with the `-q` or `--quiet` option, no message about the loading of a library is displayed.

```

option(loadLib); // show loading of libraries; standard.lib is loaded
// the names of the procedures of inout.lib
load("inout.lib"); // are now known to Singular
⇒ // ** loaded inout.lib (4.1.2.0, Feb_2019)
listvar();

```

See [Section 3.1.6 \[Command line options\]](#), page 19; [Section A.1.9 \[Dynamic modules\]](#), page 709; [Section 5.1.79 \[LIB\]](#), page 211; [Section 2.3.3 \[Procedures and libraries\]](#), page 10; [Appendix D \[SINGULAR libraries\]](#), page 794; [Section 5.2.7 \[exportto\]](#), page 293; [Section 5.2.10 \[importfrom\]](#), page 296; [Section 4.16 \[package\]](#), page 119; [Section 4.18 \[proc\]](#), page 124; [Section D.1 \[standard.lib\]](#), page 794; [Section 4.22 \[string\]](#), page 130; [Section 5.1.155 \[system\]](#), page 275.

### 5.2.13 quit

**Syntax:**    `exit;`  
              `quit;`

**Purpose:**    quits SINGULAR; works also from inside a procedure or from an interrupt. Instead of `quit`, the synonymous command `exit` may be used.

**Example:**

```
quit;
```

**5.2.14 return**

**Syntax:**    `return ( expression_list );`  
              `return ();`

**Type:**     any

**Purpose:**   returns the result(s) of a procedure and can only be used inside a procedure. Note that the brackets are required even if no return value is given.

**Example:**

```
proc p2
{
  int i,j;
  for(i=1;i<=10;i++)
  {
    j=j+i;
  }
  return(j);
}
// can also return an expression list, i.e., more than one value
proc tworeturn ()
{ return (1,2); }
int i,j = tworeturn();
// return type may even depend on the input
proc type_return (int i)
{
  if (i > 0) {return (i);}
  else {return (list(i));}
}
// then we need def type (or list) to collect value
def t1 = type_return(1);
def t2 = type_return(-1);
```

See [Chapter 4 \[Data types\]](#), page 73; [Section 4.18 \[proc\]](#), page 124.

**5.2.15 while**

**Syntax:**    `while (boolean_expression) block`

**Purpose:**    repetitive, conditional execution of block.  
 The `boolean_expression` is evaluated and if its value is `TRUE`, the block gets executed. This is repeated until `boolean_expression` evaluates to `FALSE`. The command `break` leaves the innermost `while` construction.

**Example:**

```
int i = 9;
while (i>0)
{
  // ... // do something for i=9, 8, ..., 1
  i = i - 1;
}
```

```

while (1)
{
    // ...    // do something forever
    if (i == -5) // but leave the loop if i is -5
    {
        break;
    }
}

```

See [Section 5.2 \[Control structures\]](#), page 290; [Section 4.7.5 \[boolean expressions\]](#), page 89; [Section 5.2.2 \[break\]](#), page 291.

### 5.2.16 ~ (break point)

**Syntax:** ~;

**Purpose:** sets a break point. Whenever SINGULAR reaches the command ~; in a sequence of commands it prompts for input. The user may now input lines of SINGULAR commands. The line length cannot exceed 80 characters. SINGULAR proceeds with the execution of the command following ~; as soon as it receives an empty line. Furthermore, the debug mode will be activated: See [Section 3.9.3 \[Source code debugger\]](#), page 69.

**Example:**

```

proc t
{
    int i=2;
    ~;
    return(i+1);
}
t();
↳ -- break point in t --
↳ -- 0: called      from STDIN --
// here local variables of the procedure can be accessed
i;
↳ 2
↳ -- break point in t --

↳ 3

```

See [Section 3.9.4 \[Break points\]](#), page 70.

## 5.3 System variables

### 5.3.1 degBound

**Type:** int

**Purpose:** The standard basis computation is stopped if the total (weighted) degree exceeds degBound - used in std, slingb, system("verifyGB",...) degBound should not be used for a global ordering with inhomogeneous input, if the ordering is not dp or Dp. (Remark: elimination requires always an elimination ordering).

Reset this bound by setting `degBound` to 0.

The exact meaning of "degree" depends on the ring ordering and the command: `slimgb` uses always the total degree with weights 1, `std` does so for block orderings, only.

**Example:**

```
degBound = 7;
option();
↳ //options for 'std'-command: degBound
ideal j=std(i);
degBound;
↳ 7
degBound = 0; //resets degree bound to infinity
```

See [Section 5.1.19 \[deg\]](#), page 170; [Section 4.7 \[int\]](#), page 85; [Section 5.1.111 \[option\]](#), page 234; [Section 5.1.151 \[std\]](#), page 271; [Section 5.1.155 \[system\]](#), page 275.

### 5.3.2 echo

**Type:** int

**Purpose:** input is echoed if `echo >= voice`.  
`echo` is a local setting for a procedure and defaulted to 0.  
`echo` does not affect the output of commands.

**Example:**

```
echo = 1;
int i = echo;
↳ int i = echo;
```

See [Section 4.7 \[int\]](#), page 85; [Section 5.3.11 \[voice\]](#), page 308.

### 5.3.3 minpoly

**Type:** number

**Purpose:** describes the coefficient field of the current basering as an algebraic extension with the minimal polynomial equal to `minpoly`. Setting the `minpoly` should be the first command after defining the ring.

**Note:** The minimal polynomial has to be specified in the syntax of a polynomial. Its variable is not one of the ring variables, but the algebraic element which is being adjoined to the field. Algebraic extensions in SINGULAR are only possible over the rational numbers or over  $\mathbb{Z}/p$ ,  $p$  a prime number.

SINGULAR does not check whether the given polynomial is irreducible! It can be checked in advance with the function `factorize` (see [Section 5.1.36 \[factorize\]](#), page 180).

**Example:**

```
//(Q[i]/(i^2+1))[x,y,z]:
ring Cxyz=(0,i),(x,y,z),dp;
minpoly=i^2+1;
i2; //this is a number, not a poly
↳ -1
```

See [Section 5.1.36 \[factorize\]](#), page 180; [Section 4.20 \[ring\]](#), page 127.

### 5.3.4 multBound

**Type:** int

**Purpose:** The standard basis computation is stopped if the ideal is zero-dimensional in a ring with local ordering and its multiplicity (**mult**) is lower than **multBound**.  
Reset this bound by setting **multBound** to 0.

**Example:**

```
ring r=0,(x,y,z),ds;
ideal i,j;
i=x7+y7+z6,x6+y8+z7,x7+y5+z8,
x2y3+y2z3+x3z2,x3y2+y3z2+x2z3;
multBound=100;
j=std(i);
degree(j);
↳ // dimension (local)    = 0
↳ // multiplicity = 98
multBound=0; //disables multBound
j=std(i);
degree(j);
↳ // dimension (local)    = 0
↳ // multiplicity = 86
```

See [Section 4.7 \[int\]](#), page 85; [Section 5.1.101 \[mult\]](#), page 228; [Section 5.1.111 \[option\]](#), page 234; [Section 5.1.151 \[std\]](#), page 271.

### 5.3.5 noether

**Type:** poly

**Purpose:** The standard basis computation in local rings cuts off all monomials above (in the sense of the monomial ordering) the monomial **noether** during the computation.  
Reset **noether** by setting **noether** to 0.

**Example:**

```
ring R=32003,(x,y,z),ds;
ideal i=x2+y12,y13;
std(i);
↳ _[1]=x2+y12
↳ _[2]=y13
noether=x11;
std(i);
↳ _[1]=x2
noether=0; //disables noether
```

See [Section 4.17 \[poly\]](#), page 120; [Section 5.1.151 \[std\]](#), page 271.

### 5.3.6 printlevel

**Type:** int

**Purpose:** sets the debug level for **dbprint**. If **printlevel**  $\geq$  **voice** then **dbprint** is equivalent to **print**, otherwise nothing is printed.

**Note:** See [Section 3.8.6 \[Procedures in a library\]](#), page 58, for a small example about how this is used for the display of comments while procedures are executed.

**Example:**

```

    voice;
    ↦ 1
    printlevel=0;
    dbprint(1);
    printlevel=voice;
    dbprint(1);
    ↦ 1

```

See [Section 5.1.17 \[dbprint\]](#), page 169; [Section 4.7 \[int\]](#), page 85; [Section 5.3.11 \[voice\]](#), page 308.

### 5.3.7 short

**Type:** int

**Purpose:** the output of monomials is done in the short manner, if **short** is non-zero. A C-like notion is used, if **short** is zero. Both notations may be used as input. The default depends on the names of the ring variables (0 if there are names of variables longer than 1 character, 1 otherwise). Every change of the basering sets **short** to the previous value for that ring. In other words, the value of the variable **short** is "ring-local".

If the names are long, or the ring non-commutative, **short** can not be changed to 1.

**Example:**

```

    ring r=23,x,dp;
    int save=short;
    short=1;
    2x2,x2;
    ↦ 2x2 x2
    short=0;
    2x2,x2;
    ↦ 2*x^2 x^2
    short=save; //resets short to the previous value

```

See [Section 4.7 \[int\]](#), page 85.

### 5.3.8 timer

**Type:** int

**Purpose:**

1. the CPU time (i.e, user and system time) used for each command is printed if **timer** > 0, if this time is bigger than a (customizable) minimal time and if **printlevel+1** >= **voice** (which is by default true on the SINGULAR top level, but not true while procedures are executed).
2. yields the CPU time used since the start-up of SINGULAR in a (customizable) resolution.

The default setting of **timer** is 0, the default minimal time is 0.5 seconds, and the default timer resolution is 1 (i.e., the default unit of time is one second). The minimal time and timer resolution can be set using the command line options **--min-time** and

--ticks-per-sec and can be checked using `system("--min-time")` and `system("--ticks-per-sec")`.

How to use `timer` in order to measure the time for a sequence of commands, see example below.

**Note for Windows95/98:**

The value of the `timer` cannot be used (resp. trusted) when SINGULAR is run under Windows95/98 (this is due to the shortcomings of the Windows95/98 operating system). Use [Section 5.3.10 \[rtimer\]](#), page 308, instead.

**Example:**

```

timer=1; // The time of each command is printed
int t=timer; // initialize t by timer
ring r=0,(x,y,z),dp;
poly p=(x+2y+3z+4xy+5xz+6yz)^20;
// timer as int_expression:
t=timer-t;
t; // yields the time in ticks-per-sec (default 1)
↳ 0
    // since t was initialized by timer
int tps=system("--ticks-per-sec");
t div tps; // yields the time in seconds truncated to int
↳ 0
timer=0;
system("--ticks-per-sec",1000); // set timer resolution to ms
t=timer; // initialize t by timer
p=(x+2y+3z+4xy+5xz+6yz)^20;
timer-t; // time in ms
↳ 40

```

See [Section 3.1.6 \[Command line options\]](#), page 19; [Section 5.3.6 \[printlevel\]](#), page 304; [Section 5.3.10 \[rtimer\]](#), page 308; [Section 5.1.15 \[system\]](#), page 275; [Section 5.3.11 \[voice\]](#), page 308.

### 5.3.9 TRACE

**Type:** int

**Purpose:** sets level of debugging.

TRACE=0	No debugging messages are printed.
TRACE=1	Messages about entering and leaving of procedures are displayed.
TRACE=3	Messages about entering and leaving of procedures together with line numbers are displayed.
TRACE=4	Each line is echoed and the interpretation of commands in this line is suspended until the user presses RETURN.
TRACE=8	(debug version only:) show basering for all levels
TRACE=128	show all calls to kernel routines
TRACE=256	show all assigns



TRACE=512

show all automatic type conversions

TRACE=1024

profiling: print line numbers to smon.out

TRACE is defaulted to 0.

TRACE does not affect the output of commands.

**Example:**

```

TRACE=1;
LIB "general.lib";
↪ entering mod_init (level 1)
↪ entering mod_init (level 3)
↪ entering semaphore (level 4)
↪ leaving semaphore (level 4)
↪ leaving mod_init (level 3)
↪ leaving mod_init (level 1)
↪ entering mod_init (level 1)
↪ entering semaphore (level 2)
↪ leaving semaphore (level 2)
↪ entering semaphore (level 2)
↪ leaving semaphore (level 2)
↪ leaving mod_init (level 1)
sum(1..100);
↪ entering sum (level 0)
↪ entering lsum (level 1)
↪ entering lsum (level 2)
↪ entering lsum (level 3)
↪ entering lsum (level 4)
↪ leaving lsum (level 4)
↪ entering lsum (level 4)
↪ leaving lsum (level 4)
↪ leaving lsum (level 3)
↪ entering lsum (level 3)
↪ entering lsum (level 4)
↪ leaving lsum (level 4)
↪ entering lsum (level 4)
↪ leaving lsum (level 4)
↪ leaving lsum (level 3)
↪ leaving lsum (level 2)
↪ entering lsum (level 2)
↪ entering lsum (level 3)
↪ entering lsum (level 4)
↪ leaving lsum (level 4)
↪ entering lsum (level 4)
↪ leaving lsum (level 4)
↪ leaving lsum (level 3)
↪ entering lsum (level 3)
↪ entering lsum (level 4)
↪ leaving lsum (level 4)
↪ entering lsum (level 4)
↪ leaving lsum (level 4)

```

```

⇒ leaving      lsum (level 3)
⇒ leaving      lsum (level 2)
⇒ leaving      lsum (level 1)
⇒ entering     lsum (level 1)
⇒ entering     lsum (level 2)
⇒ entering     lsum (level 3)
⇒ entering     lsum (level 4)
⇒ leaving      lsum (level 4)
⇒ entering     lsum (level 4)
⇒ leaving      lsum (level 4)
⇒ leaving      lsum (level 3)
⇒ entering     lsum (level 3)
⇒ entering     lsum (level 4)
⇒ leaving      lsum (level 4)
⇒ entering     lsum (level 4)
⇒ leaving      lsum (level 4)
⇒ leaving      lsum (level 3)
⇒ leaving      lsum (level 2)
⇒ entering     lsum (level 2)
⇒ entering     lsum (level 3)
⇒ entering     lsum (level 4)
⇒ leaving      lsum (level 4)
⇒ entering     lsum (level 4)
⇒ leaving      lsum (level 4)
⇒ leaving      lsum (level 3)
⇒ entering     lsum (level 3)
⇒ entering     lsum (level 4)
⇒ leaving      lsum (level 4)
⇒ entering     lsum (level 4)
⇒ leaving      lsum (level 4)
⇒ leaving      lsum (level 3)
⇒ leaving      lsum (level 2)
⇒ leaving      lsum (level 1)
⇒ leaving      sum (level 0)
⇒ 5050

```

See [Section 4.7 \[int\]](#), page 85.

### 5.3.10 rtimer

**Type:** int

**Purpose:** identical to `timer` (see [Section 5.3.8 \[timer\]](#), page 305), except that real times (i.e., wall-clock) times are reported, instead of CPU times. This can be trusted on all operating systems (including Windows95/98).

### 5.3.11 voice

**Type:** int

**Purpose:** shows the nesting level of procedures.

**Note:** See [Section 3.8 \[Libraries\]](#), page 55, for a small example how this is used for the display of comments while procedures are executed.

**Example:**

```
    voice;  
    ↦ 1  
proc p  
{  
    voice;  
};  
p();  
↦ 2
```

See [Section 5.1.17 \[dbprint\]](#), page 169; [Section 5.1.82 \[listvar\]](#), page 213; [Section 5.3.6 \[printlevel\]](#), page 304.

## 6 Tricks and pitfalls

### 6.1 Limitations

SINGULAR has the following limitations:

- the characteristic of a prime field must be less than or equal to 2147483647 ( $2^{31}$ )  
(the characteristic of a prime field in the factory routines must be less than 536870912 ( $2^{29}$ ))  
(the characteristic of a prime field in the NTL routines must be less than NTL\_SP\_BOUND ( $2^{30}$ ) on 32bit machines - This is always the case since currently, only factory uses NTL.)
- if only NTL is used, extgcd has a limit on the coefficients: NTL\_MAX\_FFTPRIMES // At the current setting of 20000, on 64-bit machines with 50-bit // FFT primes, this allows for polynomials with  $20 \cdot 50 / 2 = 500\text{K-bit}$  // coefficients, while the table itself takes 160KB.
- the number of elements in  $\text{GF}(p,n)$  must be less than 65536 ( $2^{16}$ )
- the (weighted) degree of a monomial must be less than 2147483648 ( $2^{31}$ )
- the rank of any free module must be less than 2147483648 ( $2^{31}$ )
- the maximal allowed exponent of a ring variable depends on the ordering of the ring and is at least 32767 ( $2^{15}-1$ ). See also [Section B.2 \[Monomial orderings\]](#), page 769 for setting other limits.
- the precision of long floating point numbers (for ground field `real`) must be less than 32768 ( $2^{15}$ )
- integers (of type `int`) have the limited range from -2147483648 to 2147483647 ( $-2^{31}..2^{31}-1$ )
- floating point numbers (type `number` from field `real`) have a limited range which is machine dependent. A typical range is -1.0e-38 to 1.0e+38. The string representation of overflow and underflow is machine dependent, as well. For example "Inf" on Linux, or "+. +00e+00" on HP-UX.  
Their input syntax is given by `scanf`, but must start with a digit.
- floating point numbers (type `number` from field `real` with a precision `p` larger then 3) use internally `mpf_set_default_prec(3.5*p+1)`.  
Their input syntax is given by `mpf_set_str` from GMP, but must start with a digit.
- the length of an identifier is unlimited but `listvar` displays only the first 20 characters
- statements may not contain more than 10000 tokens
- tokens (i.e. strings, numbers, ...) may not be longer than 16382 characters
- All input to SINGULAR must be 7-bit clean, i.e. special characters like the German Umlaute (ä, ö, etc.), or the French accent characters may neither appear as input to SINGULAR, nor in libraries or procedure definitions.
- `vspace`, used in `system("verifyGB",I)` and `farey` can not use more than MAX\_PROCESS (64) cpus. This limit can be changed in the source file `kernel/oswrapper/vspace.h`.

### 6.2 System dependent limitations

Ports of SINGULAR to different systems do not always implement all possible parts of SINGULAR:

- dynamic modules are implemented for
  - unix systems with ELF format for executables (Linux, Solaris, FreeBSD)

## 6.3 Major differences to the C programming language

Although many constructs from SINGULAR's programming language are similar to those from the C programming language, there are some subtle differences. Most notably:

### 6.3.1 No rvalue of increments and assignments

The increment operator `++` (resp. decrement operator `--`) has no rvalue, i.e., cannot be used on the right-hand sides of assignments. So, instead of

```
j = i++; // WRONG!!!
```

(which results in an error), it must be written

```
i++; j = i;
```

Likewise, an assignment expression does not have a result. Therefore, compound assignments like `i = j = k;` are not allowed and result in an error.

### 6.3.2 Evaluation of logical expressions

**All** arguments of a logical expression are first evaluated and then the value of the logical expression is determined. For example, the logical expressions `(a || b)` is evaluated by first evaluating `a` and `b`, even though the value of `b` has no influence on the value of `(a || b)`, if `a` evaluates to true.

Note, that this evaluation is different from the left-to-right, conditional evaluation of logical expressions (as found in most programming languages). For example, in these other languages, the value of `(1 || b)` is determined without ever evaluating `b`. This causes some problems with boolean tests on variables, which might not be defined at evaluation time. For example, the following results in an error, if the variable `i` is undefined:

```
if (defined(i) && i > 0) {} // WRONG!!!
```

This must be written instead as:

```
if (defined(i))
{
    if (i > 0) {}
}
```

However, there are several short work-arounds for this problem:

1. If a variable (say, `i`) is only to be used as a boolean flag, then define (value is TRUE) and undefine (value is FALSE) `i` instead of assigning a value. Using this scheme, it is sufficient to simply write

```
if (defined(i))
```

in order to check whether `i` is TRUE. Use the command `kill` to undefine a variable, i.e. to assign it a FALSE value (see [Section 5.1.71 \[kill\]](#), page 206).

2. If a variable can have more than two values, then define it, if necessary, before it is used for the first time. For example, if the following is used within a procedure

```
if (! defined(DEBUG)) { int DEBUG = 1;}
...
if (DEBUG == 3) {...}
if (DEBUG == 2) {...}
...
```

then a user of this procedure does not need to care about the existence of the `DEBUG` variable – this remains hidden from the user. However, if `DEBUG` exists globally, then its local default value is overwritten by its global one.

### 6.3.3 No case or switch statement

SINGULAR does not offer a `case` (or `switch`) statement. However, it can be imitated in the following way:

```
while (1)
{
    if (choice == choice_1) { ...; break;}
    ...
    if (choice == choice_n) { ...; break;}
    // default case
    ...; break;
}
```

### 6.3.4 Usage of commas

In SINGULAR, a comma separates list elements and the value of a comma expression is a list. Hence, commas cannot be used to combine several expressions into a single expression. For example, instead of writing

```
for (i=1, j=5; i<5 || j<10; i++, j++) {...} // WRONG!!!!!!
```

one has to write

```
for (i,j = 1,5; i<5 || j<10; i++, j++) {...}
```

### 6.3.5 Usage of brackets

In SINGULAR, curly brackets (`{ }`) **must always** be used to enclose the statement body following such constructs like `if`, `else`, `for`, or `while`, even if this block consists of only a single statement. Similarly, in the return statement of a procedure, parentheses (`( )`) **must always** be used to enclose the return value. Even if there is no value to return, parentheses have to be used after a return statement (i.e., `return();`). For example,

```
if (i == 1) return i;    // WRONG!!!!!!
```

results in an error. Instead, it must be written as

```
if (i == 1) { return (i); }.
```

### 6.3.6 Behavior of continue

SINGULAR's `continue` construct is only valid inside the body of a `for` or `while` construct. It skips the rest of the loop-body and jumps to the beginning of the block. Unlike the C-construct SINGULAR's `continue` **does not execute the increment statement**. For example,

```
for (int i = 1 ; i<=10; i=i+1)
{
    ...
    if (i==3) { i=8;continue; }
    // skip the rest if i is 3 and
    // continue with the next i: 8
    i;
}
↪ 1
↪ 2
↪ 8
↪ 9
↪ 10
```

### 6.3.7 Return type of procedures

Although the SINGULAR language is a strongly typed programming language, the type of the return value of a procedure does not need to be specified. As a consequence, the return type of a procedure may vary, i.e., may, for example, depend on the input. However, the return value of such a procedure may then only be assigned to a variable of type `def`.

```
proc type_return (int i)
{
  if (i > 0) {return (i);}
  else {return (list(i));}
}
def t1 = type_return(1);
def t2 = type_return(-1);
typeof(t1); typeof(t2);
↪ int
↪ list
```

Furthermore, it is mandatory to assign the return value of a procedure to a variable of type `def`, if a procedure changes the current ring using the `keepiring` command (see [Section 5.2.11 \[keepiring\]](#), [page 298](#)) and returns a ring-dependent value (like a polynomial or module).

```
proc def_return
{
  ring r=0,(x,y),dp;
  poly p = x;
  keepiring r;
  return (x);
}
def p = def_return();
// poly p = def_return(); would be WRONG!!!
typeof(p);
↪ poly
```

On the other hand, more than one value can be returned by a single `return` statement. For example,

```
proc tworeturn () { return (1,2); }
int i,j = tworeturn();
```

### 6.3.8 First index is 1

Although the SINGULAR language is C like, the indices of all objects which may have an index start at 1.

```
ring r;
ideal i=1,x,z;
i[2];
↪ x
intvec v=1,2,3;
v[1];
↪ 1
poly p=x+y+z;
p[2];
↪ y
vector h=[x+y,x,z];
h[1];
```

```

↳ x+y
  h[1][1];
↳ x

```

## 6.4 Miscellaneous oddities

### 1. integer division

If two numerical constants (i.e., two sequences of digits) are divided using the `/` operator, the surrounding whitespace determines which division to use: if there is no space between the constants and the `/` operator (e.g., `"3/2"`), both numerical constants are treated as of type **number** and the current ring division is used. If there is at least one space surrounding the `/` operator (e.g., `"3 / 2"`), both numerical constants are treated as of type **int** and an integer division is performed. To avoid confusion, use the `div` operator instead of `/` for integer division and an explicit type cast to **number** for ring division. Note, that this problem does only occur for divisions of numerical constants. It also applies for large numerical constants which are of type **bigint**.

```

ring r=32002,x,dp;
↳ // ** 32002 is invalid as characteristic of the ground field. 32003 is us\
  ed.
  3/2;    // ring division
↳ -16000
  3 / 2;  // integer division
↳ // ** int division with '/': use 'div' instead in line >>  3 / 2;  // int\
  eger division<<
↳ 1
  3 div 2;
↳ 1
  number(3) / number(2);
↳ -16000
  number a=3;
  number b=2;
  a/b;
↳ -16000
  int c=3;
  int d=2;
  c / d;
↳ // ** int division with '/': use 'div' instead in line >>  c / d;<<
↳ 1

```

### 2. monomials and precedence

The formation of a monomial has precedence over all operators (a monomial is here an optional coefficient followed by any sequence of ring variables (possibly followed by an exponent) which only consist of letters, digits and (over the rationals) `/` without any whitespace):

```

ring r=0,(x,y),dp;
2xy^2 == (2*x*y)^2;
↳ 1
2xy^2 == 2x*y^2;
↳ 0
2x*y^2 == 2*x * (y^2);
↳ 1

```

During that formation no operator is involved: in the non-commutative case, we have



```

LIB "nctools.lib";
ring r = 0,(x,y),dp;
def S = superCommutative();
xy == yx;
↳ 1
x*y == y*x;
↳ 1
x*y, y*x;
↳ xy xy

```

### 3. meaning of `mult`

For an arbitrary ideal or module `i`, `mult(i)` returns the multiplicity of the ideal generated by the leading monomials of the given generators of `i`, hence depends on the monomial ordering!

A standard mistake is to interpret `degree(i)` or `mult(i)` for an inhomogeneous ideal `i` as the degree of the homogenization or as something like the 'degree of the affine part'. For the ordering `dp` (degree reverse lexicographical) the converse is true: if `i` is given by a standard basis, `mult(i)` is the degree of the homogeneous ideal obtained by homogenization of `i` and then putting the homogenizing variable to 0, hence it is the degree of the part at infinity (this can also be checked by looking at the initial ideal).

### 4. size of ideals

`size` counts the non-zero entries of an ideal or module. Use `ncols` to determine the actual number of entries in the ideal or module.

### 5. computations in `qring`

In order to speed up computations in quotient rings, SINGULAR usually does not reduce polynomials w.r.t. the quotient ideal; rather the given representative is used as long as possible during computations. If it is necessary, reduction is done during standard base computations. To reduce a polynomial `f` by hand w.r.t. the current quotient ideal use the command `reduce(f,std(0))` (see [Section 5.1.131 \[reduce\]](#), [page 251](#)).

### 6. degree of a polynomial

#### `degBound`

The exact meaning of "degree" depends on the ring ordering and the command: `slimgb` uses always the total degree with weights 1, `std` does so only for block orderings.

#### `hilb`

the degree is the total degree with weights 1 unless a weight vector is given

#### `kbase`

the degree is the total degree with weights 1 (to use another weight vector see [Section 5.1.173 \[weightKB\]](#), [page 288](#))

### 7. substring selection

To extract substrings from a `string`, square brackets are used, enclosing either two comma-separated `ints` or an `intvec`. Although two comma-separated `ints` represent an `intvec`, they mean different things in substring access. Square brackets enclosing two `ints` (e.g. `s[2,6]`) return a substring where the first integer denotes the starting position and the second integer denotes the length of the substring. The result is returned as a `string`. Square brackets enclosing an `intvec` (e.g. `s[intvec(2,6)]`) return the characters of the string at the position given by the values of the `intvec`. The result is returned as an expression list of strings.

```

string s = "one-word";
s[2,6];      // a substring starting at the second char
↳ ne-wor
size(_);

```

```

↳ 6
   intvec v = 2,6;
   s[v];      // the second and the sixth char
↳ n o
   string st = s[v]; // stick together by an assignment
   st;
↳ no
   size(_);
↳ 2
   v = 2,6,8;
   s[v];
↳ n o d

```

#### 8. packages and indexed variables

See example

```

package K;
string varok; exportto(K,varok);
string work(1); exportto(K,work(1));
int i(1..3); exportto(K,i(1..3));
// Toplevel does not contain i(1..3)
listvar();
// i(1..3) are stored in Package 'K'
listvar(K);
↳ // K                                [0] package K (N)
↳ // ::i(3)                           [0] int 0
↳ // ::i(2)                           [0] int 0
↳ // ::i(1)                           [0] int 0
↳ // ::work(1)                        [0] string
↳ // ::varok                          [0] string

```

## 6.5 Identifier resolution

In SINGULAR, an identifier (i.e., a "word") is resolved in the following way and order: It is checked for

1. a reserved name (like `ring`, `std`, ...),
2. a local variable (w.r.t. a procedure),
3. a local ring variable (w.r.t. the current basering locally set in a procedure),
4. a global ring variable (w.r.t. the current basering)
5. a global variable,
6. a monomial consisting of local ring variables written without operators,
7. a monomial consisting of global ring variables written without operators.

Consequently, it is allowed to have general variables with the same name as ring variables. However, the above identifier resolution order must be kept in mind. Otherwise, surprising results may come up.

```

ring r=0,(x,y),dp;
int x;
x*y; // resolved product int*poly, i.e., 0*y
↳ 0
xy; // "xy" is one identifier and resolved to monomial xy
↳ xy

```

For these reasons, we strongly recommend not to use variables which have the same name(s) as ring variables.

Moreover, we strongly recommend not to use ring variables whose name is fully contained in (i.e., is a substring of) another name of a ring variable. Otherwise, effects like the following might occur:

```
ring r=0,(x, x1),dp; // name x is substring of name x1 !!!!!!!!!
x;x1; // resolved polynomial x
⇒ x
⇒ x1
short=0; 2x1; // resolved to monomial 2*x^1 !!!!!
⇒ 2*x
2*x1; // resolved to product 2 times x1
⇒ 2*x1
```

## 7 Non-commutative subsystem

SINGULAR has three non-commutative subsystems, handling various classes of associative non-commutative rings: PLURAL, SCA and LETTERPLACE.

### 7.1 PLURAL

#### What is and what does PLURAL?

PLURAL is a kernel extension of SINGULAR, providing many algorithms for computations within non-commutative  $G$ - and  $GR$ - algebras (see [Section 7.4 \[Mathematical background \(plural\)\]](#), page 366 for detailed information on algebras and algorithms).

It uses the same data structures as SINGULAR, sometimes interpreting them in a different way and/or modifying them for its own purposes. In spite of such a difference, one can always transfer objects between commutative rings of SINGULAR and non-commutative rings of PLURAL.

With PLURAL, one can set up a non-commutative  $G$ -algebra, say  $A$ , with a Poincaré-Birkhoff-Witt (PBW) basis, (see [Section 7.4.1 \[G-algebras\]](#), page 366 for step-by-step building instructions and also [Section 7.5 \[PLURAL libraries\]](#), page 371 for procedures for setting many important algebras easily). Afterwards, one can proceed to the factor-algebra of  $A$  modulo a two-sided ideal (see [Section 7.3.29 \[twostd \(plural\)\]](#), page 364), thus obtaining a  $GR$ -algebra (see [Section 7.2.5 \[qring \(plural\)\]](#), page 330 type).

Functionalities of PLURAL (enlisted in [Section 7.3 \[Functions \(plural\)\]](#), page 335) are accessible as soon as the basering becomes non-commutative (see [Section 7.3.16 \[nc\\_algebra\]](#), page 349 and the library [Section 7.5.10 \[ncalg.lib\]](#), page 465 with many readily predefined algebras).

One can perform various computations with polynomials and ideals in  $A$  and with vectors and submodules of a free module  $A^n$ .

#### What PLURAL does not:

PLURAL does not perform computations in the free algebra or in its general factor algebras (instead, these computations can be possibly done by [Section 7.7 \[LETTERPLACE\]](#), page 617).

In PLURAL one can only work with  $G$ -algebras and with their factor-algebras by two-sided ideals ( $GR$ -algebras).

PLURAL requires a global monomial ordering (see [Section B.2.2 \[General definitions for orderings\]](#), page 769). However, SCA ([Section 7.6 \[Graded commutative algebras \(SCA\)\]](#), page 615) provides the possibility of computations in a tensor product of a non-commutative graded commutative algebra (equipped with a global ordering) with a commutative algebra (equipped with any ordering).

PLURAL does not handle non-commutative parameters, i.e. the elements of the coefficient field (or a ring) mutually commute with all variables. Defining parameters, one **cannot** impose non-commutative relations on them. Moreover, it is impossible to introduce parameters which do not commute with variables. However, [Section 7.5.21 \[olga.lib\]](#), page 582 offers a rich functionality for working within Ore localizations of  $G$ -algebras and [Section 7.5.25 \[ratgb.lib\]](#), page 613 provides Groebner bases for so-called rational localizations of  $G$ -algebras.

PLURAL does not yet support rings like  $\mathbb{Z}$  as coefficients.

#### PLURAL conventions

**\*-multiplication (plural)**

in the non-commutative case, the correct multiplication of  $y$  by  $x$  must be written as  $y*x$ .

Both expressions  $yx$  and  $xy$  are equal, since they are interpreted as commutative expressions. See example in [Section 7.2.4.2 \[poly expressions \(plural\)\]](#), page 329.

Note, that PLURAL output consists only of standard monomials, even when the signs  $*$  are omitted.

**ideal (plural)**

Unless stated otherwise, an expression of type **ideal** as understood by PLURAL as a list of generators of a **left** ideal. For more information see [Section 7.2.1 \[ideal \(plural\)\]](#), page 319.

For a **two-sided ideal**  $T$ , use the command [Section 7.3.29 \[twostd \(plural\)\]](#), page 364 for computing the two-sided Groebner basis of  $T$ .

For a **right ideal**  $I$ , use [Section 7.8.10 \[rightstd \(letterplace\)\]](#), page 633 from `nctools_lib` for computing the right Groebner basis of  $I$ .

**module (plural)**

Unless stated otherwise, a **module** as understood by PLURAL is **either** a finitely generated **left** submodule of a free module (of finite rank)

**or** a factor module of a free module (of finite rank) by its left submodule (see [Section 7.2.3 \[module \(plural\)\]](#), page 326 for details). The concrete interpretation left to a function.

**qring (plural)**

It is only possible to build factor-algebras modulo **two-sided** ideals (see [Section 7.2.5 \[qring \(plural\)\]](#), page 330), which have to be given via their two-sided Groebner basis (see [Section 7.3.29 \[twostd \(plural\)\]](#), page 364).

## 7.2 Data types (plural)

This chapter explains all data types of PLURAL in alphabetical order. For every type, there is a description of the declaration syntax

as well as information about how to build expressions of certain types.

The term "expression list" in PLURAL refers to any comma separated list of expressions.

For the general syntax of a declaration see [Section 3.5.1 \[General command syntax\]](#), page 41.

### 7.2.1 ideal (plural)

For PLURAL ideals are **left** ideals, unless stated otherwise.

Ideals are represented as lists of polynomials which are interpreted as left generators of the ideal.

For the operations with two-sided ideals see [Section 7.3.29 \[twostd \(plural\)\]](#), page 364.

Like polynomials, ideals can only be defined or accessed with respect to a basering.

**Note:** `size` counts only the non-zero generators of an ideal whereas `ncols` counts all generators.

#### 7.2.1.1 ideal declarations (plural)

**Syntax:** `ideal name = list_of_poly_and_ideal_expressions ;`

`ideal name = ideal_expression ;`

**Purpose:** defines a left ideal.

**Default:** 0

**Example:**

```

ring r=0,(x,y,z),dp;
def R=nc_algebra(-1,0); // an anti-commutative algebra
setring R;
poly s1 = x2;
poly s2 = y3;
poly s3 = z;
ideal i = s1, s2-s1, 0,s3*s2, s3^4;
i;
↳ i[1]=x2
↳ i[2]=y3-x2
↳ i[3]=0
↳ i[4]=-y3z
↳ i[5]=z4
size(i);
↳ 4
ncols(i);
↳ 5

```

**7.2.1.2 ideal expressions (plural)**

An ideal expression is:

1. an identifier of type ideal
2. a function returning an ideal
3. a combination of ideal expressions by the arithmetic operations + or \*
4. a power of an ideal expression (operator ^ or \*\*)

Note that the computation of the product  $i*i$  involves all products of generators of  $i$  while  $i^2$  involves only the different ones, and is therefore faster.

5. a type cast to ideal

**Example:**

```

ring r=0,(x,y,z),dp;
def R=nc_algebra(-1,0); // an anticommutative algebra
setring R;
ideal m = maxideal(1);
m;
↳ m[1]=x
↳ m[2]=y
↳ m[3]=z
poly f = x2;
poly g = y3;
ideal i = x*y*z , f-g, g*(x-y) + f^4 ,0, 2x-z2y;
ideal M = i + maxideal(10);
i = M*M;
ncols(i);
↳ 598
i = M^2;
ncols(i);
↳ 690
i[ncols(i)];
↳ x20

```

```

vector v = [x,y-z,x2,y-x,x2yz2-y];
ideal j = ideal(v);
j;
↳ j[1]=x
↳ j[2]=y-z
↳ j[3]=x2
↳ j[4]=-x+y
↳ j[5]=x2yz2-y

```

### 7.2.1.3 ideal operations (plural)

- +** addition (concatenation of the generators and simplification)
- \*** multiplication (with ideal, poly, vector, module; in case of multiplication with ideal or module, the result will be simplified)
- ^** exponentiation (by a non-negative integer)

ideal-expression [ intvec-expression ]

are polynomial generators of the ideal, index 1 gives the first generator.

**Note:** For simplification of an ideal, see also [Section 5.1.143 \[simplify\]](#), page 263.

#### Example:

```

ring r=0,(x,y,z),dp;
matrix D[3][3];
D[1,2]=-z; D[1,3]=y; D[2,3]=x;
def R=nc_algebra(1,D); // this algebra is U(so_3)
setring R;
ideal I = 0,x,0,1;
I;
↳ I[1]=0
↳ I[2]=x
↳ I[3]=0
↳ I[4]=1
I + 0; // simplification
↳ _[1]=1
I*x;
↳ _[1]=0
↳ _[2]=x2
↳ _[3]=0
↳ _[4]=x
ideal J = I,0,x,x-z;
I * J; // multiplication with simplification
↳ _[1]=1
vector V = [x,y,z];
print(I*V);
↳ 0,x2,0,x,
↳ 0,xy,0,y,
↳ 0,xz,0,z
ideal m = maxideal(1);
m^2;
↳ _[1]=x2
↳ _[2]=xy

```

```

↳ _[3]=xz
↳ _[4]=y2
↳ _[5]=yz
↳ _[6]=z2
ideal II = I[2..4];
II;
↳ II[1]=x
↳ II[2]=0
↳ II[3]=1

```

#### 7.2.1.4 ideal related functions (plural)

<b>dim</b>	Gelfand-Kirillov dimension of basering modulo the ideal of leading terms (see <a href="#">Section 7.3.3 [dim (plural)]</a> , page 337)
<b>eliminate</b>	elimination of variables (see <a href="#">Section 7.3.5 [eliminate (plural)]</a> , page 339)
<b>intersect</b>	ideal intersection (see <a href="#">Section 7.3.9 [intersect (plural)]</a> , page 343)
<b>kbase</b>	vector space basis of basering modulo the leading ideal (see <a href="#">Section 7.3.10 [kbase (plural)]</a> , page 343)
<b>lead</b>	leading terms of a set of generators (see <a href="#">Section 5.1.75 [lead]</a> , page 209)
<b>lift</b>	lift-matrix (see <a href="#">Section 7.3.11 [lift (plural)]</a> , page 344)
<b>liftstd</b>	left Groebner basis and transformation matrix computation (see <a href="#">Section 7.3.12 [liftstd (plural)]</a> , page 345)
<b>maxideal</b>	generators of a power of the maximal ideal at 0 (see <a href="#">Section 5.1.88 [maxideal]</a> , page 219)
<b>modulo</b>	represents $(h1+h2)/h1 \cong h2/(h1 \cap h2)$ (see <a href="#">Section 7.3.14 [modulo (plural)]</a> , page 347)
<b>mres</b>	minimal free resolution of an ideal and a minimal set of generators of the given ideal (see <a href="#">Section 7.3.15 [mres (plural)]</a> , page 348)
<b>ncols</b>	number of columns (see <a href="#">Section 5.1.104 [ncols]</a> , page 231)
<b>nres</b>	computes a free resolution of an ideal resp. module M which is minimized from the second free module on (see <a href="#">Section 7.3.18 [nres (plural)]</a> , page 351)
<b>oppose</b>	creates an opposite ideal of a given ideal from the given ring into a basering (see <a href="#">Section 7.3.19 [oppose]</a> , page 353)
<b>preimage</b>	preimage under a ring map (see <a href="#">Section 7.3.21 [preimage (plural)]</a> , page 355)
<b>quotient</b>	ideal quotient (see <a href="#">Section 7.3.22 [quotient (plural)]</a> , page 356)
<b>reduce</b>	left normal form with respect to a left Groebner basis (see <a href="#">Section 7.3.23 [reduce (plural)]</a> , page 357)
<b>simplify</b>	simplify a set of polynomials (see <a href="#">Section 5.1.143 [simplify]</a> , page 263)
<b>size</b>	number of non-zero generators (see <a href="#">Section 5.1.144 [size]</a> , page 264)
<b>slimgb</b>	left Groebner basis computation with slim technique (see <a href="#">Section 7.3.25 [slimgb (plural)]</a> , page 360)
<b>std</b>	left Groebner basis computation (see <a href="#">Section 7.3.26 [std (plural)]</a> , page 361)



<code>subst</code>	substitute a ring variable (see <a href="#">Section 7.3.27 [subst (plural)]</a> , page 363)
<code>syz</code>	computation of the first syzygy module (see <a href="#">Section 7.3.28 [syz (plural)]</a> , page 363)
<code>twostd</code>	two-sided Groebner basis computation (see <a href="#">Section 7.3.29 [twostd (plural)]</a> , page 364)
<code>vdim</code>	vector space dimension of basering modulo the leading ideal (see <a href="#">Section 7.3.30 [vdim (plural)]</a> , page 365)

### 7.2.2 map (plural)

Maps are ring maps from a preimage ring (source) into the basering (target), defined by specifying images for source variables in the target ring.

**Note:**

- the target of a map is **ALWAYS** the actual basering
- the preimage ring has to be stored "by its name", that means, maps can only be used in such contexts, where the name of the preimage ring can be resolved (this has to be considered in subprocedures). See also [Section 6.5 \[Identifier resolution\]](#), page 316, [Section 3.7.4 \[Names in procedures\]](#), page 54.

Maps between rings with different coefficient fields are possible and listed below.

Canonically realized are

- $Q \rightarrow Q(a, \dots)$  (  $Q$  : the rational numbers)
- $Q \rightarrow R$  (  $R$  : the real numbers)
- $Q \rightarrow C$  (  $C$  : the complex numbers)
- $Z/p \rightarrow (Z/p)(a, \dots)$  (  $Z$  : the integers)
- $Z/p \rightarrow GF(p^n)$  (  $GF$  : the Galois field)
- $Z/p \rightarrow R$
- $R \rightarrow C$

Possible are furthermore

- $Z/p \rightarrow Q$ ,  $[i]_p \mapsto i \in [-p/2, p/2] \subseteq Z$
- $Z/p \rightarrow Z/p'$ ,  $[i]_p \mapsto i \in [-p/2, p/2] \subseteq Z$ ,  $i \mapsto [i]_{p'} \in Z/p'$
- $C \rightarrow R$ , by taking the real part

Finally, in PLURAL we allow the mapping from rings with coefficient field  $Q$  to rings whose ground fields have finite characteristic:

- $Q \rightarrow Z/p$
- $Q \rightarrow (Z/p)(a, \dots)$

**Note:** In these cases the denominator and the numerator of a number are mapped separately by the usual map from  $Z$  to  $Z/p$ , and the image of the number is built again afterwards by division. It is thus not allowed to map numbers whose denominator is divisible by the characteristic of the target ground field, or objects containing such numbers. We, therefore, strongly recommend to use such maps only to map objects with integer coefficients.

Note that - in contrast to the commutative case - maps between non-commutative rings easily fail to be a morphism.

### 7.2.2.1 map declarations (plural)

**Syntax:**    `map name = preimage_ring_name , ideal_expression ;`  
               `map name = preimage_ring_name , list_of_poly_and_ideal_expressions ;`  
               `map name = map_expression ;`

**Purpose:**    defines a ring map from `preimage_ring` to `basing`.  
               Maps the variables of the `preimage_ring` to the generators of the ideal.  
               If the ideal contains less elements than the number of variables in the `preimage_ring`, the remaining variables are mapped to 0.  
               If the ideal contains more elements, extra elements are ignored.  
               The image ring is always the current basering. For the mapping of coefficients from different fields see [Section 7.2.2 \[map \(plural\)\]](#), page 323.

**Default:**    none

**Note:**        There are standard mappings for maps which are close to the identity map: `fetch (plural)` and `imap (plural)`.

The name of a map serves as the function which maps objects from the `preimage_ring` into the basering. These objects must be defined by names (no evaluation in the `preimage_ring` is possible).

**Example:**

```
// an easy example
ring r1 = 0,(a,b),dp; // a commutative ring
poly P = a^2+ab+b^3;
ring r2 = 0,(x,y),dp;
def W=nc_algebra(1,-1); // a Weyl algebra
setring W;
map M = r1, x^2, -y^3;
// note: M is just a map and not a morphism of K-algebras
M(P);
↪ -y9-x2y3+x4
// now, a more involved example
LIB "ncalg.lib";
def Usl2 = makeUsl2();
// this algebra is U(sl_2), generated by e,f,h
setring Usl2;
poly P = 4*e*f+h^2-2*h; // the central el-t of Usl2
poly Q = e^3*f-h^4;      // some polynomial
ring W1 = 0,(D,X),dp;
def W2=nc_algebra(1,-1);
setring W2; // this is the opposite Weyl algebra
map F = Usl2, -X, D*D*X, 2*D*X;
F(P); // 0, because P is in the kernel of F
↪ 0
F(Q);
↪ -16D4X4+96D3X3-D2X4-112D2X2+6DX3+16DX-6X2
```

See [Section 7.3.7 \[fetch \(plural\)\]](#), page 341; [Section 7.2.1.2 \[ideal expressions \(plural\)\]](#), page 320; [Section 7.3.8 \[imap \(plural\)\]](#), page 342; [Section 7.2.2 \[map \(plural\)\]](#), page 323; [Section 7.2.7 \[ring \(plural\)\]](#), page 333.

### 7.2.2.2 map expressions (plural)

A map expression is:

1. an identifier of type map
2. a function returning map
3. a composition of maps using parentheses, e.g.  $f(g)$

### 7.2.2.3 map (plural) operations

( ) composition of maps. If, for example,  $f$  and  $g$  are maps, then  $f(g)$  is a map expression giving the composition  $f \circ g$  of  $f$  and  $g$ , provided the target ring of  $g$  is the basering of  $f$ .

map\_expression [ int\_expressions ]  
is a map entry (the image of the corresponding variable)

**Example:**

```
LIB "ncalg.lib";
def Usl2 = makeUsl2(); // this algebra is U(sl_2)
setring Usl2;
map F = Usl2, f, e, -h; // involutive endomorphism of U(sl_2)
F;
  ↪ F[1]=f
  ↪ F[2]=e
  ↪ F[3]=-h
map G = F(F);
G;
  ↪ G[1]=e
  ↪ G[2]=f
  ↪ G[3]=h
poly p = (f+e*h)^2 + 3*h-e;
p;
  ↪ e2h2+2e2h+2efh-2ef+f2-h2-e+3h
F(p);
  ↪ f2h2-2efh-2f2h+e2-2ef+h2-f-h
G(p);
  ↪ e2h2+2e2h+2efh-2ef+f2-h2-e+3h
(G(p) == p); // G is the identity
  ↪ 1
```

### 7.2.2.4 map related functions (plural)

fetch (plural)

the identity map between rings and q rings (see [Section 7.3.7 \[fetch \(plural\)\]](#), page 341)

imap (plural)

a convenient map procedure for inclusions and projections of rings (see [Section 7.3.8 \[imap \(plural\)\]](#), page 342)

preimage (plural)

preimage under a ring map (see [Section 7.3.21 \[preimage \(plural\)\]](#), page 355)

subst

substitute a ring variable (see [Section 7.3.27 \[subst \(plural\)\]](#), page 363)

See also [Section 7.5.19 \[ncpreim.lib\]](#), page 555 for the advanced preimage algorithm.

### 7.2.3 module (plural)

Modules are **left** submodules of a free module over the basering with basis `gen(1)`, `gen(2)`, ..., `gen(n)` for some natural number `n`.

They are represented by lists of vectors, which generate the left submodule. Like vectors, they can only be defined or accessed with respect to a basering.

If  $M$  is a left submodule of  $R^n$  (where  $R$  is the basering) generated by vectors  $v_1, \dots, v_k$ , then these generators may be considered as the generators of relations of  $R^n/M$  between the canonical generators `gen(1)`, ..., `gen(n)`. Hence, any finitely generated  $R$ -module can be represented in PLURAL by its module of relations. This is the so-called Coker-representation.

The assignments `module M=v1,...,vk; matrix A=M;` create the presentation matrix of size  $n \times k$ , with the columns of  $A$  being the vectors  $v_1, \dots, v_k$  which generate  $M$ .

#### 7.2.3.1 module declarations (plural)

**Syntax:** `module name = list_of_vector_expressions` (which are interpreted as left generators of the module) ;

`module name = module_expression ;`

**Purpose:** defines a left module.

**Default:** `[0]`

**Example:**

```
ring r=0,(x,y,z),(c,dp);
matrix D[3][3];
D[1,2]=-z; D[1,3]=y; D[2,3]=x;
def R=nc_algebra(1,D); // this algebra is U(so_3)
setring R;
vector s1 = [x2,y3,z];
vector s2 = [xy,1,0];
vector s3 = [0,x2-y2,z];
poly f = -x*y;
module m = s1, s2-s1,f*(s3-s1);
m;
↪ m[1]=[x2,y3,z]
↪ m[2]=[-x2+xy,-y3+1,-z]
↪ m[3]=[x3y-2x2z-xy,xy4-x3y+xy3+2x2z+xy]
// show m in matrix format (columns generate m)
print(m);
↪ x2,-x2+xy,x3y-2x2z-xy,
↪ y3,-y3+1, xy4-x3y+xy3+2x2z+xy,
↪ z, -z, 0
```

#### 7.2.3.2 module expressions (plural)

A module expression is:

1. an identifier of type module
2. a function returning module
3. module expressions combined by the arithmetic operation `+`
4. multiplication of a module expression with an ideal or a poly expression: `*`
5. a type cast to module

### 7.2.3.3 module operations (plural)

**+** addition (concatenation of the generators and simplification) Note that “-” implicitly converts a module into a matrix; see below example.

**\*** right or left multiplication with number, ideal, or poly (but not ‘module’ \* ‘module’!)

`module_expression [ int_expression , int_expression ]`

is a module entry, where the first index indicates the row and the second the column

`module_expressions [ int_expression ]`

is a vector, where the index indicates the column (generator)

#### Example:

```
ring A=0,(x,y,z),Dp;
matrix D[3][3];
D[1,2]=-z; D[1,3]=y; D[2,3]=x; // this algebra is U(so_3)
def B=nc_algebra(1,D);
setring B;
module M = [x,y],[0,0,x*z];
module N = matrix((x+y-z)*M) - matrix(M*(x+y-z)); // no - for type module
print(N);
↦ -y-z,0,
↦ -x+z,0,
↦ 0, -x2-xy-yz-z2
```

### 7.2.3.4 module related functions (plural)

**eliminate**

elimination of variables (see [Section 7.3.5 \[eliminate \(plural\)\]](#), page 339)

**freemodule**

the free module of given rank (see [Section 5.1.47 \[freemodule\]](#), page 188)

**intersect**

module intersection (see [Section 7.3.9 \[intersect \(plural\)\]](#), page 343)

**kbase**

vector space basis of free module over the basering modulo the module of leading terms (see [Section 7.3.10 \[kbase \(plural\)\]](#), page 343)

**lead**

initial module (see [Section 5.1.75 \[lead\]](#), page 209)

**lift**

lift-matrix (see [Section 7.3.11 \[lift \(plural\)\]](#), page 344)

**liftstd**

left Groebner basis and transformation matrix computation (see [Section 7.3.12 \[liftstd \(plural\)\]](#), page 345)

**modulo**

represents  $(h1+h2)/h1 \cong h2/(h1 \cap h2)$  (see [Section 7.3.14 \[modulo \(plural\)\]](#), page 347)

**mres**

minimal free resolution of a module and a minimal set of generators of the given ideal module (see [Section 7.3.15 \[mres \(plural\)\]](#), page 348)

**ncols**

number of columns (see [Section 5.1.104 \[ncols\]](#), page 231)

**nres**

computes a free resolution of an ideal resp. module M which is minimized from the second free module on (see [Section 7.3.18 \[nres \(plural\)\]](#), page 351)

**nrows**

number of rows (see [Section 5.1.107 \[nrows\]](#), page 232)

<b>oppose</b>	creates an opposite module of a given module from the given ring into a basering (see <a href="#">Section 7.3.19 [oppose]</a> , page 353)
<b>print</b>	nice print format (see <a href="#">Section 5.1.120 [print]</a> , page 242)
<b>prune</b>	minimize the embedding into a free module (see <a href="#">Section 5.1.122 [prune]</a> , page 245)
<b>quotient</b>	module quotient (see <a href="#">Section 7.3.22 [quotient (plural)]</a> , page 356)
<b>reduce</b>	left normal form with respect to a left Groebner basis (see <a href="#">Section 7.3.23 [reduce (plural)]</a> , page 357)
<b>simplify</b>	simplify a set of vectors (see <a href="#">Section 5.1.143 [simplify]</a> , page 263)
<b>size</b>	number of non-zero generators (see <a href="#">Section 5.1.144 [size]</a> , page 264)
<b>std</b>	left Groebner basis computation (see <a href="#">Section 7.3.26 [std (plural)]</a> , page 361)
<b>subst</b>	substitute a ring variable (see <a href="#">Section 7.3.27 [subst (plural)]</a> , page 363)
<b>syz</b>	computation of the first syzygy module (see <a href="#">Section 7.3.28 [syz (plural)]</a> , page 363)
<b>vdim</b>	vector space dimension of free module over the basering modulo module of leading terms (see <a href="#">Section 7.3.30 [vdim (plural)]</a> , page 365)

## 7.2.4 poly (plural)

Polynomials and vectors are the basic data for all main algorithms in PLURAL. Polynomials consist of finitely many terms (coefficient\*monomial) which are combined by the usual polynomial operations (see [Section 7.2.4.2 \[poly expressions \(plural\)\]](#), page 329). Polynomials can only be defined or accessed with respect to a basering which determines the coefficient type, the names of the indeterminants and the monomial ordering.

**Example:**

```
ring r=32003,(x,y,z),dp;
poly f=x3+y5+z2;
```

**Remark:** Remember the conventions on polynomial multiplication we follow (\*-multiplication in [Section 7.1 \[PLURAL\]](#), page 318).

### 7.2.4.1 poly declarations (plural)

**Syntax:** poly name = poly\_expression ;

**Purpose:** defines a polynomial.

**Default:** 0

**Example:**

```
ring r = 32003,(x,y,z),dp;
def R=nc_algebra(-1,1);
setring R;
// ring of some differential-like operators
R;
↪ // coefficients: ZZ/32003
↪ // number of vars : 3
↪ //          block 1 : ordering dp
↪ //          : names  x y z
↪ //          block 2 : ordering C
↪ // noncommutative relations:
```

```

↳ //      yx=-xy+1
↳ //      zx=-xz+1
↳ //      zy=-yz+1
yx;      // not correct input
↳ xy
y*x;     // correct input
↳ -xy+1
poly s1  = x3y2+151x5y+186xy6+169y9;
poly s2  = 1*x^2*y^2*z^2+3z8;
poly s3  = 5/4x4y2+4/5*x*y^5+2x2y2z3+y7+11x10;
int a,b,c,t=37,5,4,1;
poly f=3*x^a+x*y^(b+c)+t*x^a*y^b*z^c;
f;
↳ x37y5z4+3x37+xy9
short = 0;
f;
↳ x^37*y^5*z^4+3*x^37+xy9

```

### 7.2.4.2 poly expressions (plural)

A polynomial expression is (optional parts in square brackets):

1. a monomial (there are NO spaces allowed inside a monomial)

[coefficient] ring\_variable [exponent] [ring\_variable [exponent] ...]

monomials which contain an indexed ring variable must be built from `ring_variable` and `coefficient` with the operations `*` and `^`

2. an identifier of type `poly`
3. a function returning `poly`
4. polynomial expressions combined by the arithmetic operations `+`, `-`, `*`, `/`, or `^`.
5. a type cast to `poly`

**Example:**

```

ring r=0,(x,y),dp;
def R=nc_algebra(1,1); // make it a Weyl algebra
setring R;
R;
↳ // coefficients: QQ
↳ // number of vars : 2
↳ //      block 1 : ordering dp
↳ //      : names x y
↳ //      block 2 : ordering C
↳ // noncommutative relations:
↳ //      yx=xy+1
yx;      // not correct input
↳ xy
y*x;     // correct input
↳ xy+1
poly f = 10x2*y3 + 2y2*x^2 - 2*x*y + y - x + 2;
lead(f);
↳ 10x2y3
leadmonom(f);

```

```

↳ x2y3
simplify(f,1);      // normalize leading coefficient
↳ x2y3+1/5x2y2+3/5xy-1/10x+1/10y+3/5
cleardenom(f);
↳ 10x2y3+2x2y2+6xy-x+y+6

```

### 7.2.4.3 poly operations (plural)

+            addition  
 -            negation or subtraction  
 \*            multiplication  
 /            commutative division by a monomial, non divisible terms yield 0  
 ^, \*\*        power by a positive integer  
 <, <=, >, >=, ==, <>        comparison (of leading monomials w.r.t. monomial ordering)  
 poly\_expression [ intvec\_expression ]  
               the sum of monomials at the indicated places w.r.t. the monomial ordering

### 7.2.4.4 poly related functions (plural)

bracket     computes the (iterated) Lie bracket of two polynomials (see [Section 7.3.2 \[bracket\]](#), page 336)  
 lead        leading term (see [Section 5.1.75 \[lead\]](#), page 209)  
 leadcoef    coefficient of the leading term (see [Section 5.1.76 \[leadcoef\]](#), page 209)  
 leadexp    the exponent vector of the leading monomial (see [Section 5.1.77 \[leadexp\]](#), page 210)  
 leadmonom   leading monomial (see [Section 5.1.78 \[leadmonom\]](#), page 210)  
 oppose     creates an opposite polynomial of a given polynomial from the given ring into a basering (see [Section 7.3.19 \[oppose\]](#), page 353)  
 reduce     left normal form with respect to a left Groebner basis (see [Section 7.3.23 \[reduce \(plural\)\]](#), page 357)  
 simplify    normalize a polynomial (see [Section 5.1.143 \[simplify\]](#), page 263)  
 size        number of monomials (see [Section 5.1.144 \[size\]](#), page 264)  
 subst      substitute a ring variable (see [Section 7.3.27 \[subst \(plural\)\]](#), page 363)  
 var        the indicated variable of the ring (see [Section 5.1.165 \[var\]](#), page 284)

### 7.2.5 qring (plural)

PLURAL offers the possibility to compute within factor-rings modulo two-sided ideals. The ideal has to be given as a two-sided Groebner basis (see [Section 7.3.29 \[twostd \(plural\)\]](#), page 364 command).

For a detailed description of the concept of rings and quotient rings see [Section 3.3 \[Rings and orderings\]](#), page 30.

**Note:** we highly recommend to turn on `option(redSB); option(redTail);` while computing in qrings. Otherwise results may have a difficult interpretation.



### 7.2.5.1 qring declaration (plural)

**Syntax:** `qring name = ideal_expression ;`

**Default:** `none`

**Purpose:** declares a quotient ring as the basering modulo an `ideal_expression` and sets it as current basering.

**Note:** reports error if an ideal is not a two-sided Groebner basis.

**Example:**

```
ring r=0,(z,u,v,w),dp;
def R=nc_algebra(-1,0); // an anticommutative algebra
setring R;
option(redSB);
option(redTail);
ideal i=z^2,u^2,v^2,w^2, zuv-w;
qring Q = i; // incorrect call produces error
↳ // ** i is no standard basis
↳ // ** i is no twosided standard basis
kill Q;
setring R; // go back to the ring R
qring q=twostd(i); // now it is an exterior algebra modulo <zuv-w>
q;
↳ // coefficients: QQ
↳ // number of vars : 4
↳ //          block 1 : ordering dp
↳ //          : names  z u v w
↳ //          block 2 : ordering C
↳ // noncommutative relations:
↳ //      uz=-zu
↳ //      vz=-zv
↳ //      wz=-zw
↳ //      vu=-uv
↳ //      wu=-uw
↳ //      wv=-vw
↳ // quotient ring from ideal
↳ _[1]=w2
↳ _[2]=vw
↳ _[3]=uw
↳ _[4]=zw
↳ _[5]=v2
↳ _[6]=u2
↳ _[7]=z2
↳ _[8]=zuv-w
poly k = (v-u)*(zv+u-w);
k; // the output is not yet totally reduced
↳ zuv-uv+uw-vw
poly ek=reduce(k,std(0));
ek; // the reduced form
↳ -uv+w
```

### 7.2.5.2 qring related functions (plural)

**envelope**    enveloping ring (see [Section 7.3.6 \[envelope\]](#), page 340)  
**nvars**        number of ring variables (see [Section 5.1.109 \[nvars\]](#), page 233)  
**opposite**    opposite ring (see [Section 7.3.20 \[opposite\]](#), page 354)  
**setring**     set a new basering (see [Section 5.1.141 \[setring\]](#), page 260)

## 7.2.6 resolution (plural)

The type resolution is intended as an intermediate representation which internally retains additional information obtained during computation of resolutions. It furthermore enables the use of partial results to compute, for example, Betti numbers or minimal resolutions. Like ideals and modules, a resolution can only be defined w.r.t. a basering.

**Note:** to access the elements of a resolution, it has to be assigned to a list. This assignment also completes computations and may therefore take time, (resp. an access directly with the brackets [ , ] causes implicitly a cast to a list).

### 7.2.6.1 resolution declarations (plural)

**Syntax:**     resolution name = resolution\_expression ;

**Purpose:**     defines a resolution.

**Default:**    none

**Example:**

```
ring r=0,(x,y,z),dp;
matrix D[3][3];
D[1,2]=z;
def R=nc_algebra(1,D); // it is a Heisenberg algebra
setring R;
ideal i=z2+z,x+y;
resolution re=nres(i,0);
re;
↳ 1      2      1
↳ R <--  R <--  R
↳
↳ 0      1      2
↳ resolution not minimized yet
↳
list l = re;
l;
↳ [1]:
↳   _[1]=z2+z
↳   _[2]=x+y
↳ [2]:
↳   _[1]=z2*gen(2)-x*gen(1)-y*gen(1)+z*gen(2)
↳ [3]:
↳   _[1]=0
print(matrix(l[2]));
↳ -x-y,
↳ z2+z
print(module(transpose(matrix(l[2]))*transpose(matrix(l[1])))); // check t
↳ 0
```

### 7.2.6.2 resolution expressions (plural)

A resolution expression is:

1. an identifier of type resolution
2. a function returning a resolution
3. a type cast to resolution from a list of ideals, resp. modules.

### 7.2.6.3 resolution related functions (plural)

<b>betti</b>	Betti numbers of a resolution (see <a href="#">Section 7.3.1 [betti (plural)]</a> , page 335)
<b>minres</b>	minimizes a free resolution (see <a href="#">Section 7.3.13 [minres (plural)]</a> , page 346)
<b>mres</b>	computes a minimal free resolution of an ideal resp. module and a minimal set of generators of the given ideal resp. module (see <a href="#">Section 7.3.15 [mres (plural)]</a> , page 348)
<b>nres</b>	computes a free resolution of an ideal resp. module M which is minimized from the second module on (see <a href="#">Section 7.3.18 [nres (plural)]</a> , page 351)

## 7.2.7 ring (plural)

Rings are used to describe properties of polynomials, ideals etc. Almost all computations in PLURAL require a basering. For a detailed description of the concept of rings see [Section 3.3 \[Rings and orderings\]](#), page 30.

**Note:** PLURAL usually works with global orderings (see [Section 7.1 \[PLURAL\]](#), page 318) but one can use certain local once when graded commutative rings are being used.

### 7.2.7.1 ring declarations (plural)

**Syntax:**    `ring name = ( coefficient_field ), ( names_of_ring_variables ), ( ordering );`

**Default:**    `32003, (x,y,z), (dp,C);`

**Purpose:**    declares a ring and sets it as the actual basering.

The `coefficient_field` is given by one of the following:

1. a non-negative `int_expression` less or equal 2147483647.
2. an `expression_list` of an `int_expression` and one or more names.
3. the name `real`.
4. an `expression_list` of the name `real` and an `int_expression`.
5. an `expression_list` of the name `complex`, an optional `int_expression` and a name.

'names\_of\_ring\_variables' must be a list of names or indexed names.

'ordering' is a list of block orderings where each block ordering is either

1. `lp`, `dp`, `Dp`, optionally followed by a size parameter in parentheses.
2. `wp`, `Wp`, or `a` followed by a weight vector given as an `intvec_expression` in parentheses.
3. `M` followed by an `intmat_expression` in parentheses.
4. `c` or `C`.

As long as all non-commuting variables are global, any ordering may be used. In graded commutative algebras, one may also use `ls`, `ds`, `Ds`, `ws`, and `Ws`.

If one of `coefficient_field`, `names_of_ring_variables`, and `ordering` consists of only one entry, the parentheses around this entry may be omitted.

**In order to create a non-commutative structure over a commutative ring, use [Section 7.3.16 \[nc\\_algebra\]](#), page 349.**

### 7.2.7.2 ring operations (plural)

**+** construct a tensor product  $C = A \otimes_{\mathbf{K}} B$  of two  $G$ -algebras  $A$  and  $B$  over the ground field. Let, e.g.,

$$A = k_1 \langle x_1, \dots, x_n \mid \{x_j x_i = c_{ij} \cdot x_i x_j + d_{ij}\}, 1 \leq i < j \leq n \rangle, \text{ and } B = k_2 \langle y_1, \dots, y_m \mid \{y_j y_i = q_{ij} \cdot y_i y_j + r_{ij}\}, 1 \leq i < j \leq m \rangle$$

be two  $G$ -algebras, then  $C$  is defined to be the algebra

$$C = K \langle x_1, \dots, x_n, y_1, \dots, y_m \mid \{x_j x_i = c_{ij} \cdot x_i x_j + d_{ij}, 1 \leq i < j \leq n\}, \{y_j y_i = q_{ij} \cdot y_i y_j + r_{ij}, 1 \leq i < j \leq m\}, \{y_j x_i = x_i y_j, 1 \leq j \leq m, 1 \leq i \leq n\} \rangle.$$

Concerning the ground fields  $k_1$  resp.  $k_2$  of  $A$  resp.  $B$ , take the following guidelines for  $A \otimes_{\mathbf{K}} B$  into consideration:

- Neither  $k_1$  nor  $k_2$  may be  $R$  or  $C$ .
- If the characteristic of  $k_1$  and  $k_2$  differs, then one of them must be  $Q$ .
- At most one of  $k_1$  and  $k_2$  may have parameters.
- If one of  $k_1$  and  $k_2$  is an algebraic extension of  $Z/p$  it may not be defined by a `charstr` of type  $(p^n, a)$ .

One can create a ring using `ring(list)`, see also `ringlist`.

**Example:**

```
LIB "ncalg.lib";
def a = makeUsl2();           // U(sl_2) in e,f,h presentation
ring W0 = 0,(x,d),dp;
def W = Weyl();               // 1st Weyl algebra in x,d
def S = a+W;
setring S;
S;
⇒ // coefficients: QQ
⇒ // number of vars : 5
⇒ //      block 1 : ordering dp
⇒ //      : names e f h
⇒ //      block 2 : ordering dp
⇒ //      : names x d
⇒ //      block 3 : ordering C
⇒ // noncommutative relations:
⇒ //      fe=ef-h
⇒ //      he=eh+2e
⇒ //      hf=fh-2f
⇒ //      dx=xd+1
```

### 7.2.7.3 ring related functions (plural)

`charstr` description of the coefficient field of a ring (see [Section 5.1.7 \[charstr\]](#), page 162)

<code>envelope</code>	enveloping ring (see <a href="#">Section 7.3.6 [envelope]</a> , page 340)
<code>npars</code>	number of ring parameters (see <a href="#">Section 5.1.105 [npars]</a> , page 231)
<code>nvars</code>	number of ring variables (see <a href="#">Section 5.1.109 [nvars]</a> , page 233)
<code>opposite</code>	opposite ring (see <a href="#">Section 7.3.20 [opposite]</a> , page 354)
<code>ordstr</code>	monomial ordering of a ring (see <a href="#">Section 5.1.113 [ordstr]</a> , page 239)
<code>parstr</code>	names of all ring parameters or the name of the n-th ring parameter (see <a href="#">Section 5.1.116 [parstr]</a> , page 240)
<code>qring</code>	quotient ring (see <a href="#">Section 7.2.5 [qring (plural)]</a> , page 330)
<code>ringlist</code>	decomposes a ring into a list of its components (see <a href="#">Section 7.3.24 [ringlist (plural)]</a> , page 358)
<code>setring</code>	set a new basering (see <a href="#">Section 5.1.141 [setring]</a> , page 260)
<code>varstr</code>	names of all ring variables or the name of the n-th ring variable (see <a href="#">Section 5.1.167 [varstr]</a> , page 285)

## 7.3 Functions (plural)

This chapter gives a complete reference of all functions and commands of the PLURAL kernel, i.e. all built-in commands (for the PLURAL libraries see [Section 7.5 \[PLURAL libraries\]](#), page 371).

The general syntax of a function is

[target =] function\_name (<arguments>);

Note, that both **Control structures** and **System variables** of PLURAL are the same as of SINGULAR (see [Section 5.2 \[Control structures\]](#), page 290, [Section 5.3 \[System variables\]](#), page 302).

### 7.3.1 betti (plural)

**Syntax:**    `betti ( list_expression )`  
               `betti ( resolution_expression )`  
               `betti ( list_expression , int_expression )`  
               `betti ( resolution_expression , int_expression )`

**Type:**     intmat

**Note:**     in the non-commutative case, computing Betti numbers makes sense only if the basering  $R$  has homogeneous relations. The output of the command can be pretty-printed using `print( , 'betti' )`, i.e., with "betti" as second argument; see below example.

**Purpose:**    with 1 argument: computes the graded Betti numbers of a minimal resolution of  $R^n/M$ , if  $R$  denotes the basering and  $M$  a homogeneous submodule of  $R^n$  and the argument represents a resolution of  $R^n/M$ .

The entry  $d$  of the intmat at place  $(i, j)$  is the minimal number of generators in degree  $i+j$  of the  $j$ -th syzygy module (= module of relations) of  $R^n/M$  (the 0th (resp. 1st) syzygy module of  $R^n/M$  is  $R^n$  (resp.  $M$ )). The argument is considered to be the result of a `mres` or `nres` command. This implies that a zero is only allowed (and counted) as a generator in the first module.

For the computation `betti` uses only the initial monomials. This could lead to confusing results for a non-homogeneous input.

If the optional second argument is non-zero, the Betti numbers will be minimized.

**Example:**

```

int i;int N=2;
ring r=0,(x(1..N),d(1..N),q(1..N)),Dp;
matrix D[3*N][3*N];
for (i=1;i<=N;i++)
{ D[i,N+i]=q(i)^2; }
def W=nc_algebra(1,D); setring W;
// this algebra is a kind of homogenized Weyl algebra
W;
↪ // coefficients: QQ
↪ // number of vars : 6
↪ //          block 1 : ordering Dp
↪ //          : names  x(1) x(2) d(1) d(2) q(1) q(2)
↪ //          block 2 : ordering C
↪ // noncommutative relations:
↪ //      d(1)x(1)=x(1)*d(1)+q(1)^2
↪ //      d(2)x(2)=x(2)*d(2)+q(2)^2
ideal I = x(1),x(2),d(1),d(2),q(1),q(2);
option(redSB);
option(redTail);
resolution R = mres(I,0);
// thus R will be the full length minimal resolution
print(betti(R),"betti");
↪
↪          0      1      2      3      4      5      6
↪ -----
↪    0:      1      6     15     20     15      6      1
↪ -----
↪ total:      1      6     15     20     15      6      1
↪

```

**7.3.2 bracket**

**Syntax:**    `bracket ( poly_expression, poly_expression )`  
              `bracket ( poly_expression, poly_expression, int_expression )`

**Type:**     `poly`

**Purpose:**    Computes the Lie bracket  $[p,q]=pq-qp$  of the first polynomial with the second. Uses special routines, based on the Leibniz rule. If the third argument is  $N>1$ , then the right normed bracket  $[a,[\dots[a,b]]]$  will be computed.

**Note:**      effective both with PLURAL and LETTERPLACE rings.

**Example:**

```

ring r=(0,Q),(x,y,z),Dp; // first, let us do a Plural example
minpoly=Q^2-Q+1;
matrix C[3][3]; matrix D[3][3];
C[1,2]=Q2;    C[1,3]=1/Q2;    C[2,3]=Q2;
D[1,2]=-Q*z;    D[1,3]=1/Q*y;    D[2,3]=-Q*x;
def R=nc_algebra(C,D); setring R; R;
↪ // coefficients: QQ[Q]/(Q2-Q+1)
↪ // number of vars : 3
↪ //          block 1 : ordering Dp
↪ //          : names  x y z

```

```

⇒ //          block  2 : ordering C
⇒ // noncommutative relations:
⇒ //      yx=(Q-1)*xy+(-Q)*z
⇒ //      zx=(-Q)*xz+(-Q+1)*y
⇒ //      zy=(Q-1)*yz+(-Q)*x
// this is a quantum deformation of U(so_3),
// where Q is a 6th root of unity
poly p=Q^4*x2+y2+Q^4*z2+Q*(1-Q^4)*x*y*z;
// p is the central element of the algebra
p=p^3; // any power of a central element is central
poly q=(x+Q*y+Q^2*z)^4;
// take q to be some big noncentral element
size(q); // check how many monomials are in big polynomial q
⇒ 28
bracket(p,q); // check p*q=q*p
⇒ 0
// a more common behaviour of the bracket follows:
bracket(x+Q*y+Q^2*z,z);
⇒ (Q+1)*xz+(Q+1)*yz+(Q-1)*x+(Q-1)*y
kill R; setring r; // Now consider an example for Letterplace
LIB "freegb.lib";
ring R = freeAlgebra(r,5); // F<x,y,z> with deg left lex ordering
bracket(x,y);
⇒ x*y-y*x
bracket(x,y,2);
⇒ x*x*y-2*x*y*x+y*x*x
bracket(x,y,3);
⇒ x*x*x*y-3*x*x*y*x+3*x*y*x*x-y*x*x*x
bracket(z^2,x+Q*y,2);
⇒ x*z*z*z*z+(Q)*y*z*z*z*z-2*z*z*x*z*z+(-2*Q)*z*z*y*z*z+z*z*z*z*x+(Q)*z*z*
z*y

```

### 7.3.3 dim (plural)

**Syntax:**    dim ( ideal\_expression )  
              dim ( module\_expression )

**Type:**     int

**Purpose:**    computes the Gelfand-Kirillov dimension of the ideal, resp. module, generated by the leading monomials of the given generators of the ideal, resp. module. This is also the dimension of the ideal resp. submodule, if it is represented by a left Groebner basis.

**Note:**      The dimension of a submodule of a free module is defined to be the Gelfand-Kirillov dimension of the left module with the presentation via given submodule.  
 The computed Gelfand-Kirillov dimension is taken relative to the ground field. In order to compute the complete Gelfand-Kirillov dimension, one has to add the transcendence degree of the ground field over its prime field.

**Example:**

```

ring r=0,(x,y,Dx,Dy),dp;
matrix M[4][4]; M[1,3]=1;M[2,4]=1;
def R = nc_algebra(1,M); // 2nd Weyl algebra
setring R;

```

```

dim(std(0)); // the GK dimension of the ring itself
↪ 4
ideal I=x*Dy^2-2*y*Dy^2+2*Dy, Dx^3+3*Dy^2;
dim(std(I)); // the GK dimension of the module R/I
↪ 2
module T = (x*Dx -2)*gen(1), Dx^3*gen(1), (y*Dy +3)*gen(2);
dim(std(T)); // the GK dimension of the module R^2/T
↪ 3

```

See [Section 4.6 \[ideal\]](#), page 80; [Section 4.14 \[module\]](#), page 112; [Section 5.1.151 \[std\]](#), page 271; [Section 5.1.168 \[vdim\]](#), page 286.

### 7.3.4 division (plural)

**Syntax:** `division ( ideal-expression, ideal-expression )`  
`division ( module-expression, module-expression )`  
`division ( ideal-expression, ideal-expression, int-expression )`  
`division ( module-expression, module-expression, int-expression )`  
`division ( ideal-expression, ideal-expression, int-expression, intvec-expression )`  
`division ( module-expression, module-expression, int-expression, intvec-expression )`

**Type:** list

**Purpose:** `division` computes a left division with remainder. For two left ideals resp. modules  $M$  (first argument) and  $N$  (second argument), it returns a list  $T, R, U$  where  $T$  is a matrix,  $R$  is a left ideal resp. a module, and  $U$  is a diagonal matrix of units such that  $\text{transpose}(U) * \text{transpose}(\text{matrix}(M)) = \text{transpose}(T) * \text{transpose}(\text{matrix}(N)) + \text{transpose}(\text{matrix}(R))$ . From this data one gets a left standard representation for the left normal form  $R$  of  $M$  with respect to a left Groebner basis of  $N$ . `division` uses different algorithms depending on whether  $N$  is represented by a Groebner basis. For a GR-algebra, the matrix  $U$  is the identity matrix. A matrix  $T$  as above is also computed by `lift`.

For additional arguments  $n$  (third argument) and  $w$  (fourth argument), `division` returns a list  $T, R$  as above such that  $\text{transpose}(\text{matrix}(M)) = \text{transpose}(T) * \text{transpose}(\text{matrix}(N)) + \text{transpose}(\text{matrix}(R))$  is a left standard representation for the left normal form  $R$  of  $M$  with respect to  $N$  up to weighted degree  $n$  with respect to the weight vector  $w$ . The weighted degree of  $T$  and  $R$  respect to  $w$  is at most  $n$ . If the weight vector  $w$  is not given, `division` uses the standard weight vector  $w=1, \dots, 1$ .

**Example:**

```

LIB "dmod.lib";
ring r = 0, (x,y), dp;
poly f = x^3+xy;
def S = Sannfs(f); setring S; // compute the annihilator of f^s
LD; // is not a Groebner basis yet!
↪ LD[1]=3*x^2*Dy-x*Dx+y*Dy
↪ LD[2]=x*Dx+2*y*Dy-3*s
poly f = imap(r,f);
poly P = f*Dx-s*diff(f,x);
division(P,LD); // so P is in the ideal via the cofactors in _[1]
↪ [1]:

```



```

⇒      _[1,1]=-2/3*y
⇒      _[2,1]=x^2+1/3*y
⇒ [2]:
⇒      _[1]=0
⇒ [3]:
⇒      _[1,1]=1
ideal I = LD, f; // consider a bigger ideal
list L = division(s^2, I); // the normal form is -2s-1
L;
⇒ [1]:
⇒      _[1,1]=2/3*x^2*Dy-1/3*x*Dx+2/3*s+1/3
⇒      _[2,1]=2/3*x^2*Dy-1/3*x*Dx-1/3*s-2/3
⇒      _[3,1]=-2*x*Dy^2+Dx*Dy
⇒ [2]:
⇒      _[1]=-2*s-1
⇒ [3]:
⇒      _[1,1]=1
// now we show that the formula above holds
matrix M[1][1] = s^2; matrix N = matrix(I);
matrix T = matrix(L[1]); matrix R = matrix(L[2]); matrix U = matrix(L[3])
// the formula must return zero:
transpose(U)*transpose(M) - transpose(T)*transpose(N) - transpose(R);
⇒ _[1,1]=0

```

See [Section 4.6 \[ideal\]](#), page 80; [Section 5.1.80 \[lift\]](#), page 211; [Section 4.14 \[module\]](#), page 112; [Section 4.17 \[poly\]](#), page 120; [Section 4.23 \[vector\]](#), page 134.

### 7.3.5 eliminate (plural)

**Syntax:** `eliminate ( ideal_expression, product_of_ring_variables )`  
`eliminate ( module_expression, product_of_ring_variables )`

**Type:** the same as the type of the first argument

**Purpose:** eliminates variables occurring as factors of the second argument from an ideal (resp. a submodule of a free module), by intersecting it (resp. each component of the submodule) with the subring not containing these variables.

**Note:** `eliminate` neither needs a special ordering on the basering nor a Groebner basis as input. Moreover, `eliminate` does not work in non-commutative quotients.

**Remark:** in a non-commutative algebra, not every subset of a set of variables generates a proper subalgebra. But if it is so, there may be cases, when no elimination (by means of Groebner bases) is possible; in such situations error messages will be reported. See also [Section 7.5.19 \[ncpreim\\_lib\]](#), page 555 for the advanced algorithm for elimination and preimage.

**Example:**

```

ring r=0,(e,f,h,a),Dp;
matrix d[4][4];
d[1,2]=-h; d[1,3]=2*e; d[2,3]=-2*f;
def R=nc_algebra(1,d); setring R;
// this algebra is U(sl_2), tensored with K[a] over K
option(redSB);
option(redTail);

```

```

poly p = 4*e*f+h^2-2*h - a;
// p is a central element with parameter
ideal I = e^3, f^3, h^3-4*h, p; // take this ideal
// and intersect I with the ring K[a]
ideal J = eliminate(I,e*f*h);
// if we want substitute 'a' with a value,
// it has to be a root of this polynomial
J;
⇒ J[1]=a3-32a2+192a
// now we try to eliminate h,
// that is we intersect I with the subalgebra S,
// generated by e and f.
// But S is not closed in itself, since f*e-e*f=-h !
// the next command will definitely produce an error
eliminate(I,h);
⇒ ? no elimination is possible: subalgebra is not admissible
⇒ ? error occurred in or before ./examples/eliminate_(plural).sing 1\
   ine 21: 'eliminate(I,h)';
// since a commutes with e,f,h, we can eliminate it:
eliminate(I,a);
⇒ _[1]=h3-4h
⇒ _[2]=fh2-2fh
⇒ _[3]=f3
⇒ _[4]=eh2+2eh
⇒ _[5]=2efh-h2-2h
⇒ _[6]=e3

```

See [Section 7.2.1 \[ideal \(plural\)\]](#), page 319; [Section 7.2.3 \[module \(plural\)\]](#), page 326; [Section 7.3.26 \[std \(plural\)\]](#), page 361.

### 7.3.6 envelope

**Syntax:**    `envelope ( ring_name )`

**Type:**     `ring`

**Purpose:**    creates an enveloping algebra of a given algebra, that is  $A^{env} = A \otimes_K A^{opp}$ , where  $A^{opp}$  is the opposite algebra of  $A$ .

**Remark:**    You have to activate the ring with the `setring` command. For the presentation, see explanation of opposite in [Section 7.3.20 \[opposite\]](#), page 354.

```

LIB "ncalg.lib";
def A = makeUsl2();
setring A; A;
⇒ // coefficients: QQ
⇒ // number of vars : 3
⇒ //          block 1 : ordering dp
⇒ //          : names    e f h
⇒ //          block 2 : ordering C
⇒ // noncommutative relations:
⇒ //    fe=ef-h
⇒ //    he=eh+2e
⇒ //    hf=fh-2f
def Aenv = envelope(A);

```

```

setring Aenv;
Aenv;
⇨ // coefficients: QQ
⇨ // number of vars : 6
⇨ //      block 1 : ordering dp
⇨ //      : names   e f h
⇨ //      block 2 : ordering a
⇨ //      : names   H F E
⇨ //      : weights 1 1 1
⇨ //      block 3 : ordering ls
⇨ //      : names   H F E
⇨ //      block 4 : ordering C
⇨ // noncommutative relations:
⇨ //      fe=ef-h
⇨ //      he=eh+2e
⇨ //      hf=fh-2f
⇨ //      FH=HF-2F
⇨ //      EH=HE+2E
⇨ //      EF=FE-H

```

See [Section 7.3.19 \[oppose\]](#), page 353; [Section 7.3.20 \[opposite\]](#), page 354.

### 7.3.7 fetch (plural)

**Syntax:** `fetch ( ring_name, name )`

**Type:** number, poly, vector, ideal, module, matrix or list (the same type as the second argument)

**Purpose:** maps objects between rings. `fetch` is the identity map between rings and q rings, the  $i$ -th variable of the source ring is mapped to the  $i$ -th variable of the basering. The coefficient fields must be compatible. (See [Section 7.2.2 \[map \(plural\)\]](#), page 323 for a description of possible mappings between different ground fields). `fetch` offers a convenient way to change variable names or orderings, or to map objects from a ring to a quotient ring of that ring or vice versa.

**Note:** Compared with `imap`, `fetch` uses the position of the ring variables, not their names.

**Example:**

```

LIB "ncalg.lib";
def Us12 = makeUs12(); // this algebra is U(sl_2)
setring Us12;
option(redSB);
option(redTail);
poly C = 4*e*f+h^2-2*h; // the central element of Us12
ideal I = e^3,f^3,h^3-4*h;
ideal J = twostd(I);
// print a compact presentation of J:
print(matrix(ideal(J[1..5]))); // first 5 generators
⇨ h3-4h,fh2-2fh,eh2+2eh,f2h-2f2,2efh-h2-2h
print(matrix(ideal(J[6..size(J)]))); // last generators
⇨ e2h+2e2,f3,ef2-fh,e2f-eh-2e,e3
ideal QC = twostd(C-8);
qring Q = QC;
ideal QJ = fetch(Us12,J);

```

```

QJ = std(QJ);
// thus QJ is the image of I in the factor-algebra QC
print(matrix(QJ)); // print QJ compactly
↦ h3-4h,fh2-2fh,eh2+2eh,f2h-2f2,e2h+2e2,f3,e3

```

See [Section 7.3.8 \[imap \(plural\)\]](#), page 342; [Section 7.2.2 \[map \(plural\)\]](#), page 323; [Section 7.2.5 \[qring \(plural\)\]](#), page 330; [Section 7.2.7 \[ring \(plural\)\]](#), page 333.

### 7.3.8 imap (plural)

**Syntax:**    `imap ( ring_name, name )`

**Type:**     number, poly, vector, ideal, module, matrix or list (the same type as the second argument)

**Purpose:**    identity map on common subrings. `imap` is the map between rings and qrings with compatible ground fields which is the identity on variables and parameters of the same name and 0 otherwise. (See [Section 7.2.2 \[map \(plural\)\]](#), page 323 for a description of possible mappings between different ground fields). Useful for mappings from a homogenized ring to the original ring or for mappings from/to rings with/without parameters. Compared with `fetch`, `imap` uses the names of variables and parameters. **Unlike `map` and `fetch`, `imap` can map parameters to variables.**

**Example:**

```

LIB "ncalg.lib";
ring ABP=0,(p4,p5,a,b),dp; // a commutative ring
def Usl3 = makeUsl(3);
def BIG = Usl3+ABP;
setring BIG;
poly P4 = 3*x(1)*y(1)+3*x(2)*y(2)+3*x(3)*y(3);
P4 = P4 +h(1)^2+h(1)*h(2)+h(2)^2-3*h(1)-3*h(2);
// P4 is a central element of Usl3 of degree 2
poly P5 = 4*x(1)*y(1) + h(1)^2 - 2*h(1);
// P5 is a central element of the subalgebra of U(sl_3),
// generated by x(1),y(1),h(1)
ideal J = x(1),x(2),h(1)-a,h(2)-b;
// we are interested in the module U(sl_3)/J,
// which depends on parameters a,b
ideal I = p4-P4, p5-P5;
ideal K = I, J;
ideal E = eliminate(K,x(1)*x(2)*x(3)*y(1)*y(2)*y(3)*h(1)*h(2));
E; // this is the ideal of central characters in ABP
↦ E[1]=a*b+b^2-p4+p5+a+3*b
↦ E[2]=a^2-p5+2*a
↦ E[3]=b^3+p4*a-p5*a-a^2-p4*b+3*b^2
// what are the characters on nonzero a,b?
ring abP = (0,a,b),(p4,p5),dp;
ideal abE = imap(BIG, E);
option(redSB);
option(redTail);
abE = std(abE);
// here come characters (indeed, we have only one)
// that is a maximal ideal in K[p4,p5]
abE;

```

```

↦ abE[1]=p5+(-a^2-2*a)
↦ abE[2]=p4+(-a^2-a*b-3*a-b^2-3*b)

```

See [Section 7.3.7 \[fetch \(plural\)\]](#), page 341; [Section 7.2.2 \[map \(plural\)\]](#), page 323; [Section 7.2.5 \[qring \(plural\)\]](#), page 330; [Section 7.2.7 \[ring \(plural\)\]](#), page 333.

### 7.3.9 intersect (plural)

**Syntax:**    `intersect (expression_list of ideal_expression )`  
               `intersect (expression_list of module_expression )`

**Type:**     ideal, resp. module

**Purpose:**    computes the intersection of ideals, resp. modules.

**Example:**

```

ring r=0,(x,y),dp;
def R=nc_algebra(-1,0); //anti-commutative algebra
setring R;
module M=[x,x],[y,0];
module N=[0,y^2],[y,x];
option(redSB);
module Res;
Res=intersect(M,N);
print(Res);
↦ y2, 0,
↦ -xy,xy2
kill r,R;
//-----
LIB "ncalg.lib";
ring r=0,(x,d),dp;
def RR=Weyl(); // make r into Weyl algebra
setring RR;
ideal I = x+d^2;
ideal J = d-1;
ideal H = intersect(I,J);
H;
↦ H[1]=d4+xd2-2d3-2xd+d2+x+2d-2
↦ H[2]=xd3+x2d-xd2+d3-x2+xd-2d2-x+1

```

### 7.3.10 kbase (plural)

**Syntax:**    `kbase ( ideal_expression )`  
               `kbase ( module_expression )`  
               `kbase ( ideal_expression, int_expression )`  
               `kbase ( module_expression, int_expression )`

**Type:**     the same as the input type of the first argument

**Purpose:**    with one argument: computes the vector space basis of the factor-module that equals ring (resp. free module) modulo the ideal (resp. submodule), generated by the initial terms of the given generators.

If the factor-module is not of finite dimension, -1 is returned.

If the generators form a Groebner basis, this is the same as the vector space basis of the factor-module.

when called with two arguments: computes the part of a vector space basis of the respective quotient with degree (of monomials) equal to the second argument. Here, the quotient does not need to be finite dimensional.

**Note:** in the non-commutative case, a ring modulo an ideal has a ring structure if and only if the ideal is two-sided.

Also, `kbase` respects module grading given by the `isHomog` attribute of input modules.

**Example:**

```
ring r=0,(x,y,z),dp;
matrix d[3][3];
d[1,2]=-z; d[1,3]=2x; d[2,3]=-2y;
def R=nc_algebra(1,d); // this algebra is U(sl_2)
setring R;
ideal i=x2,y2,z2-1;
i=std(i);
print(matrix(i)); // print a compact presentation of i
↳ z2-1,yz-y,xz+x,y2,2xy-z-1,x2
kbase(i);
↳ _[1]=z
↳ _[2]=y
↳ _[3]=x
↳ _[4]=1
vdim(i);
↳ 4
ideal j=x,z-1;
j=std(j);
kbase(j,3);
↳ _[1]=y3
```

See [Section 7.2.1 \[ideal \(plural\)\]](#), page 319; [Section 7.2.3 \[module \(plural\)\]](#), page 326; [Section 7.3.30 \[vdim \(plural\)\]](#), page 365.

### 7.3.11 lift (plural)

**Syntax:** `lift ( ideal_expression, subideal_expression )`  
`lift ( module_expression, submodule_expression )`

**Type:** matrix

**Purpose:** computes the (left) transformation matrix which expresses the (left) generators of a submodule in terms of the (left) generators of a module. Uses different algorithms for modules which are (resp. are not) represented by Groebner bases. More precisely, if `m` is the module, `sm` the submodule, and `T` the transformation matrix returned by `lift`, then `transpose(matrix(sm)) = transpose(T)*transpose(matrix(m))`.

If `m` and `sm` are ideals, `ideal(sm) = ideal(transpose(T)*transpose(matrix(m)))`.

**Note:** Gives a warning if `sm` is not a submodule.

**Example:**

```
ring r = (0,a),(e,f,h),(c,dp);
matrix D[3][3];
D[1,2]=-h; D[1,3]=2*e; D[2,3]=-2*f;
```

```

def R=nc_algebra(1,D); // this algebra is a parametric U(sl_2)
setring R;
ideal I = e,h-a; // consider this parametric ideal
I = std(I); // left Groebner basis
print(matrix(I)); // print a compact presentation of I
↪ h+(-a),e
poly Z = 4*e*f+h^2-2*h; // a central element in R
Z = Z - NF(Z,I); // a central character
ideal j = std(Z);
j;
↪ j[1]=4*ef+h2-2*h+(-a2-2a)
matrix T = lift(I,j);
print(T);
↪ h+(a+2),
↪ 4*f
ideal tj = ideal(transpose(T)*transpose(matrix(I)));
size(ideal(matrix(j)-matrix(tj))); // test for 0
↪ 0

```

See [Section 7.2.1 \[ideal \(plural\)\]](#), page 319; [Section 7.3.12 \[liftstd \(plural\)\]](#), page 345; [Section 7.2.3 \[module \(plural\)\]](#), page 326.

### 7.3.12 liftstd (plural)

**Syntax:** liftstd ( ideal\_expression, matrix\_name )  
 liftstd ( module\_expression, matrix\_name )  
 liftstd ( ideal\_expression, matrix\_name, module\_name )  
 liftstd ( module\_expression, matrix\_name, module\_name )

**Type:** ideal or module

**Purpose:** returns a left Groebner basis of an ideal or module and a left transformation matrix from the given ideal, resp. module, to the Groebner basis.  
 That is, if  $m$  is the ideal or module,  $sm$  is the left Groebner basis of  $m$ , returned by `liftstd`, and  $T$  is a left transformation matrix, then  $sm=module(transpose(transpose(T)*transpose(matrix(m))))$ .  
 If  $m$  is an ideal,  $sm=ideal(transpose(T)*transpose(matrix(m)))$ .  
 In an optional third argument the left syzygy module will be returned.

**Example:**

```

LIB "ncalg.lib";
def A = makeUsl2();
setring A; // this algebra is U(sl_2)
ideal i = e2,f;
option(redSB);
option(redTail);
matrix T;
ideal j = liftstd(i,T);
// the Groebner basis in a compact form:
print(matrix(j));
↪ f,2h2+2h,2eh+2e,e2
print(T); // the transformation matrix
↪ 0,f2, -f,1,
↪ 1,-e2f+4eh+8e,e2,0

```

```

ideal tj = ideal(transpose(T)*transpose(matrix(i)));
size(ideal(matrix(j)-matrix(tj))); // test for 0
⇨ 0
module S; ideal k = liftstd(i,T,S); // the third argument
S = std(S); print(S); // the syzygy module
⇨ -ef-2h+6,-f3,
⇨ e3,          e2f2-6efh-6ef+6h2+18h+12

```

See [Section 7.2.1 \[ideal \(plural\)\]](#), page 319; [Section 7.2.7 \[ring \(plural\)\]](#), page 333; [Section 7.3.26 \[std \(plural\)\]](#), page 361.

### 7.3.13 minres (plural)

**Syntax:** minres ( list\_expression )

**Type:** list

**Syntax:** minres ( resolution\_expression )

**Type:** resolution

**Purpose:** minimizes a free resolution of an ideal or module given by the list\_expression, resp. resolution\_expression.

**Example:**

```

LIB "ncalg.lib";
def A = makeUsl2();
setring A; // this algebra is U(sl_2)
ideal i=e,f,h;
i=std(i);
resolution F=nres(i,0); F;
⇨ 1      3      3      1
⇨ A <--  A <--  A <--  A
⇨
⇨ 0      1      2      3
⇨ resolution not minimized yet
⇨
list lF = F; lF;
⇨ [1]:
⇨   _[1]=h
⇨   _[2]=f
⇨   _[3]=e
⇨ [2]:
⇨   _[1]=f*gen(1)-h*gen(2)-2*gen(2)
⇨   _[2]=e*gen(1)-h*gen(3)+2*gen(3)
⇨   _[3]=e*gen(2)-f*gen(3)-gen(1)
⇨ [3]:
⇨   _[1]=e*gen(1)-f*gen(2)+h*gen(3)
print(betti(lF), "betti");
⇨          0      1      2      3
⇨ -----
⇨    0:      1      -      3      1
⇨ -----
⇨ total:      1      0      3      1
⇨
resolution MF=minres(F); MF;

```



```

↳ 1      2      2      1
↳ A <--  A <--  A <--  A
↳
↳ 0      1      2      3
↳
list LMF = F; LMF;
↳ [1]:
↳   _[1]=f
↳   _[2]=e
↳ [2]:
↳   _[1]=-ef*gen(1)+f2*gen(2)+2h*gen(1)+2*gen(1)
↳   _[2]=-e2*gen(1)+ef*gen(2)+h*gen(2)-2*gen(2)
↳ [3]:
↳   _[1]=e*gen(1)-f*gen(2)
print(betti(LMF), "betti");
↳           0      1      2      3
↳ -----
↳      0:      1      -      -      -
↳      1:      -      -      2      1
↳ -----
↳ total:      1      0      2      1
↳

```

See [Section 7.3.15 \[mres \(plural\)\]](#), page 348; [Section 7.3.18 \[nres \(plural\)\]](#), page 351.

### 7.3.14 modulo (plural)

**Syntax:**    modulo ( ideal\_expression, ideal\_expression )  
              modulo ( module\_expression, module\_expression )

**Type:**     module

**Purpose:**    modulo(h1,h2) represents  $h_1/(h_1 \cap h_2) \cong (h_1 + h_2)/h_2$ , where  $h_1$  and  $h_2$  are considered as submodules of the same free module  $R^l$  ( $l=1$  for ideals).  
 Let  $H_1$  (resp.  $H_2$ ) be the matrix of size  $l \times k$  (resp.  $l \times m$ ), having the generators of  $h_1$  (resp.  $h_2$ ) as columns.  
 Then  $h_1/(h_1 \cap h_2) \cong R^k / \ker(\overline{H_1})$ , where  $\overline{H_1} : R^k \rightarrow R^l / \text{Im}(H_2) = R^l / h_2$  is the induced map given by  $H_1$ .  
 modulo(h1,h2) returns generators of the kernel of this induced map.

**Note:** If, for at least one of  $h_1$  or  $h_2$ , the attribute isHomog is st, then modulo(h1,h2) also sets this attribute (if the weights are compatible).

**Example:**

```

LIB "ncalg.lib";
def A = makeUsl2();
setring A; // this algebra is U(sl_2)
option(redSB);
option(redTail);
ideal H2 = e2,f2,h2-1;
H2 = twostd(H2);
print(matrix(H2)); // print H2 in a compact form
↳ h2-1,fh-f,eh+e,f2,2ef-h-1,e2
ideal H1 = std(e);

```

```

ideal T = modulo(H1,H2);
T = NF(std(H2+T),H2);
T = std(T);
T;
↪ T[1]=h-1
↪ T[2]=e

```

See also [Section 7.3.28 \[syz \(plural\)\]](#), page 363.

### 7.3.15 mres (plural)

**Syntax:** `mres ( ideal_expression, int_expression )`  
`mres ( module_expression, int_expression )`

**Type:** resolution

**Purpose:** computes a minimal free resolution of an ideal or module  $M$  with the Groebner basis method. More precisely, let  $A = \text{matrix}(M)$ , then `mres` computes a free resolution of  $\text{coker}(A) = F_0/M \rightarrow 0$

$$\dots \rightarrow F_2 \xrightarrow{A_2} F_1 \xrightarrow{A_1} F_0 \rightarrow F_0/M \rightarrow 0,$$

where the columns of the matrix  $A_1$  are a (possibly) minimal set of generators of  $M$ . If the int expression  $k$  is not zero, then the computation stops after  $k$  steps and returns a resolution consisting of modules  $M_i = \text{module}(A_i)$ ,  $i = 1 \dots k$ .

`mres(M,0)` returns a resolution consisting of at most  $n+2$  modules, where  $n$  is the number of variables of the basering. Let `list L=mres(M,0)`; then `L[1]` consists of a minimal set of generators  $M$ , `L[2]` consists of a minimal set of generators for the first syzygy module of `L[1]`, etc., until `L[p+1]`, such that `L[i] ≠ 0` for  $i \leq p$ , but `L[p+1]` (the first syzygy module of `L[p]`) is 0 (if the basering is not a qring).

**Note:** Accessing single elements of a resolution may require that some partial computations have to be finished and may therefore take some time. Hence, assigning right away to a list is the recommended way to do it.

**Example:**

```

LIB "ncalg.lib";
def A = makeUs12();
setring A; // this algebra is U(sl_2)
option(redSB);
option(redTail);
ideal i = e,f,h;
i = std(i);
resolution M=mres(i,0);
M;
↪ 1      2      2      1
↪ A <--  A <--  A <--  A
↪
↪ 0      1      2      3
↪
list l = M; l;
↪ [1]:
↪   _[1]=f
↪   _[2]=e

```

```

⇒ [2]:
⇒   _[1]=ef*gen(1)-f2*gen(2)-2h*gen(1)-2*gen(1)
⇒   _[2]=e2*gen(1)-ef*gen(2)-h*gen(2)+2*gen(2)
⇒ [3]:
⇒   _[1]=e*gen(1)-f*gen(2)
// see the exactness at this point
size(ideal(transpose(matrix(1[2]))*transpose(matrix(1[1]))));
⇒ 0
print(matrix(M[3]));
⇒ e,
⇒ -f
// see the exactness at this point
size(ideal(transpose(matrix(1[3]))*transpose(matrix(1[2]))));
⇒ 0

```

See [Section 7.2.1 \[ideal \(plural\)\]](#), page 319; [Section 7.3.13 \[minres \(plural\)\]](#), page 346; [Section 7.2.3 \[module \(plural\)\]](#), page 326; [Section 7.3.18 \[nres \(plural\)\]](#), page 351.

### 7.3.16 nc\_algebra

#### Syntax:

```

nc_algebra( matrix_expression C, matrix_expression D )
nc_algebra( number_expression n, matrix_expression D )
nc_algebra( matrix_expression C, poly_expression p )
nc_algebra( number_expression n, poly_expression p )

```

**Type:** ring

**Purpose:** Executed in the basering  $\mathbf{r}$ , say, in  $k$  variables  $x_1, \dots, x_k$ , `nc_algebra` creates and returns the non-commutative extension of  $\mathbf{r}$  subject to relations  $\{x_j x_i = c_{ij} \cdot x_i x_j + d_{ij}, 1 \leq i < j \leq k\}$ , where  $c_{ij}$  and  $d_{ij}$  must be put into two strictly upper triangular matrices  $\mathbf{C}$  with entries  $c_{ij}$  from the ground field of  $\mathbf{r}$  and  $\mathbf{D}$  with (commutative) polynomial entries  $d_{ij}$  from  $\mathbf{r}$ . See all the details in [Section 7.4.1 \[G-algebras\]](#), page 366.  
 If  $\forall i < j, c_{ij} = n$ , one can input the number  $n$  instead of matrix  $\mathbf{C}$ .  
 If  $\forall i < j, d_{ij} = p$ , one can input the polynomial  $p$  instead of matrix  $\mathbf{D}$ .

**Note:** The returned ring should be activated afterwards, using the command `setring`.

**Note:** The coefficients must be a field (see [Section 7.4.1 \[G-algebras\]](#), page 366).

**Remark:** At present, PLURAL does not check the non-degeneracy conditions (see [Section 7.4.1 \[G-algebras\]](#), page 366) while setting an algebra.

#### Example:

```

LIB "nctools.lib";
// ----- first example: C, D are matrices -----
ring r1 = (0,Q),(x,y,z),Dp;
minpoly = rootofUnity(6);
matrix C[3][3];
matrix D[3][3];
C[1,2]=Q2; C[1,3]=1/Q2; C[2,3]=Q2;
D[1,2]=-Q*z; D[1,3]=1/Q*y; D[2,3]=-Q*x;
def S=nc_algebra(C,D);
// this algebra is a quantum deformation U'_q(so_3),
// where Q is a 6th root of unity
setring S;S;

```

```

↳ // coefficients: QQ[Q]/(Q2-Q+1)
↳ // number of vars : 3
↳ //          block 1 : ordering Dp
↳ //          : names  x y z
↳ //          block 2 : ordering C
↳ // noncommutative relations:
↳ //      yx=(Q-1)*xy+(-Q)*z
↳ //      zx=(-Q)*xz+(-Q+1)*y
↳ //      zy=(Q-1)*yz+(-Q)*x
kill r1,S;
// ---- second example: number n=1, D is a matrix
ring r2=0,(Xa,Xb,Xc,Ya,Yb,Yc,Ha,Hb),dp;
matrix d[8][8];
d[1,2]=-Xc; d[1,4]=-Ha; d[1,6]=Yb; d[1,7]=2*Xa;
d[1,8]=-Xa; d[2,5]=-Hb; d[2,6]=-Ya; d[2,7]=-Xb;
d[2,8]=2*Xb; d[3,4]=Xb; d[3,5]=-Xa; d[3,6]=-Ha-Hb;
d[3,7]=Xc; d[3,8]=Xc; d[4,5]=Yc; d[4,7]=-2*Ya;
d[4,8]=Ya; d[5,7]=Yb; d[5,8]=-2*Yb;
d[6,7]=-Yc; d[6,8]=-Yc;
def S=nc_algebra(1,d); // this algebra is U(sl_3)
setring S;S;
↳ // coefficients: QQ
↳ // number of vars : 8
↳ //          block 1 : ordering dp
↳ //          : names  Xa Xb Xc Ya Yb Yc Ha Hb
↳ //          block 2 : ordering C
↳ // noncommutative relations:
↳ //      XbXa=Xa*Xb-Xc
↳ //      YaXa=Xa*Ya-Ha
↳ //      YcXa=Xa*Yc+Yb
↳ //      HaXa=Xa*Ha+2*Xa
↳ //      HbXa=Xa*Hb-Xa
↳ //      YbXb=Xb*Yb-Hb
↳ //      YcXb=Xb*Yc-Ya
↳ //      HaXb=Xb*Ha-Xb
↳ //      HbXb=Xb*Hb+2*Xb
↳ //      YaXc=Xc*Ya+Xb
↳ //      YbXc=Xc*Yb-Xa
↳ //      YcXc=Xc*Yc-Ha-Hb
↳ //      HaXc=Xc*Ha+Xc
↳ //      HbXc=Xc*Hb+Xc
↳ //      YbYa=Ya*Yb+Yc
↳ //      HaYa=Ya*Ha-2*Ya
↳ //      HbYa=Ya*Hb+Ya
↳ //      HaYb=Yb*Ha+Yb
↳ //      HbYb=Yb*Hb-2*Yb
↳ //      HaYc=Yc*Ha-Yc
↳ //      HbYc=Yc*Hb-Yc
kill r2,S;
// ---- third example: C is a matrix, p=0 is a poly
ring r3=0,(a,b,c,d),lp;
matrix c[4][4];
c[1,2]=1; c[1,3]=3; c[1,4]=-2;

```

```

c[2,3]=-1; c[2,4]=-3; c[3,4]=1;
def S=nc_algebra(c,0); // it is a quasi--commutative algebra
setring S;S;
↳ // coefficients: QQ
↳ // number of vars : 4
↳ //          block 1 : ordering lp
↳ //          : names  a b c d
↳ //          block 2 : ordering C
↳ // noncommutative relations:
↳ //      ca=3ac
↳ //      da=-2ad
↳ //      cb=-bc
↳ //      db=-3bd
kill r3,S;
// -- fourth example : number n = -1, poly p = 3w
ring r4=0,(u,v,w),dp;
def S=nc_algebra(-1,3w);
setring S;S;
↳ // coefficients: QQ
↳ // number of vars : 3
↳ //          block 1 : ordering dp
↳ //          : names  u v w
↳ //          block 2 : ordering C
↳ // noncommutative relations:
↳ //      vu=-uv+3w
↳ //      wu=-uw+3w
↳ //      wv=-vw+3w
kill r4,S;

```

See also [Section 7.5.10 \[ncalg\\_lib\], page 465](#); [Section 7.5.20 \[nctools\\_lib\], page 563](#); [Section 7.5.24 \[qmatrix\\_lib\], page 610](#).

### 7.3.17 ncalgebra

#### Syntax:

```

ncalgebra( matrix_expression C, matrix_expression D )
ncalgebra( number_expression n, matrix_expression D )
ncalgebra( matrix_expression C, poly_expression p )
ncalgebra( number_expression n, poly_expression p )

```

**Type:** none

**Purpose:** Works like [Section 7.3.16 \[nc\\_algebra\], page 349](#) but changes the basering.

**Remark:** This function is **deprecated** and should be substituted by `nc_algebra`, since it violates the general SINGULAR policy: only [Section 4.20 \[ring\], page 127](#) and [Section 5.1.141 \[setring\], page 260](#) can change the basering. More concretely, replace by `def A = nc_algebra(C, D); setring A;` which will additionally introduce a new ring. Afterwards, some objects may have to be mapped into the new ring.

See also [Section 7.3.16 \[nc\\_algebra\], page 349](#); [Section 7.5.20 \[nctools\\_lib\], page 563](#).

### 7.3.18 nres (plural)

**Syntax:** `nres ( ideal_expression, int_expression )`  
`nres ( module_expression, int_expression )`

**Type:** resolution

**Purpose:** computes a free resolution of an ideal or module which is minimized from the second module on (by the Groebner basis method).

**Note:** Assigning a resolution to a list is the best choice of usage. The resolution may be minimized by using the command `minres`. Use the command `beti` to compute Betti numbers.

**Example:**

```
LIB "ncalg.lib";
def A = makeUsl2();
setring A; // this algebra is U(sl_2)
option(redSB);
option(redTail);
ideal i = e,f,h;
i = std(i);
resolution F=nres(i,0); F;
↪ 1      3      3      1
↪ A <--  A <--  A <--  A
↪
↪ 0      1      2      3
↪ resolution not minimized yet
↪
list l = F; l;
↪ [1]:
↪  _[1]=h
↪  _[2]=f
↪  _[3]=e
↪ [2]:
↪  _[1]=f*gen(1)-h*gen(2)-2*gen(2)
↪  _[2]=e*gen(1)-h*gen(3)+2*gen(3)
↪  _[3]=e*gen(2)-f*gen(3)-gen(1)
↪ [3]:
↪  _[1]=e*gen(1)-f*gen(2)+h*gen(3)
// see the exactness at this point:
size(ideal(transpose(matrix(l[2]))*transpose(matrix(l[1]))));
↪ 0
// see the exactness at this point:
size(ideal(transpose(matrix(l[3]))*transpose(matrix(l[2]))));
↪ 0
print(betti(l), "beti");
↪          0      1      2      3
↪ -----
↪  0:      1      -      3      1
↪ -----
↪ total:    1      0      3      1
↪
print(betti(minres(l)), "beti");
↪          0      1      2      3
↪ -----
↪  0:      1      -      -      -
```

```

      1:      -      -      2      1
      -----
      total:      1      0      2      1
      ↪

```

See [Section 7.2.1 \[ideal \(plural\)\], page 319](#); [Section 7.3.13 \[minres \(plural\)\], page 346](#); [Section 7.2.3 \[module \(plural\)\], page 326](#); [Section 7.3.15 \[mres \(plural\)\], page 348](#).

### 7.3.19 oppose

**Syntax:** `oppose ( ring_name, name )`

**Type:** poly, vector, ideal, module or matrix (the same type as the second argument)

**Purpose:** for a given object in the given ring, creates its opposite object in the opposite ([Section 7.3.20 \[opposite\], page 354](#)) ring (the last one is assumed to be the current ring).

**Remark:** for any object  $O$ ,  $(O^{opp})^{opp} = O$ .

```

LIB "ncalg.lib";
def R = makeUs12();
setring R;
matrix m[3][4];
poly p = (h^2-1)*f*e;
vector v = [1,e*h,0,p];
ideal i = h*e, f^2*e,h*f*e;
m = e,f,h,1,0,h^2, p,0,0,1,e^2,e*f*h+1;
module mm = module(m);
def b = opposite(R);
setring b; b;
↪ // coefficients: QQ
↪ // number of vars : 3
↪ //      block 1 : ordering a
↪ //      : names   H F E
↪ //      : weights 1 1 1
↪ //      block 2 : ordering ls
↪ //      : names   H F E
↪ //      block 3 : ordering C
↪ // noncommutative relations:
↪ //      FH=HF-2F
↪ //      EH=HE+2E
↪ //      EF=FE-H
// we will oppose these objects: p,v,i,m,mm
poly P = oppose(R,p);
vector V = oppose(R,v);
ideal I = oppose(R,i);
matrix M = oppose(R,m);
module MM = oppose(R,mm);
def c = opposite(b);
setring c; // now let's check the correctness:
// print compact presentations of objects
print(oppose(b,P)-imap(R,p));
↪ 0
print(oppose(b,V)-imap(R,v));
↪ [0]
print(matrix(oppose(b,I))-imap(R,i));

```

```

    ↦ 0,0,0
    print(matrix(oppose(b,M))-imap(R,m));
    ↦ 0,0,0,0,
    ↦ 0,0,0,0,
    ↦ 0,0,0,0
    print(matrix(oppose(b,MM))-imap(R,mm));
    ↦ 0,0,0,0,
    ↦ 0,0,0,0,
    ↦ 0,0,0,0

```

See [Section 7.3.6 \[envelope\]](#), page 340; [Section 7.3.20 \[opposite\]](#), page 354.

### 7.3.20 opposite

**Syntax:**    `opposite ( ring_name )`

**Type:**     `ring`

**Purpose:**    creates an opposite algebra of a given algebra.

**Note:**      activate the ring with the `setring` command.  
 An opposite algebra of a given algebra  $(A, \#)$  is an algebra  $(A, *)$  with the same vector space but with the opposite multiplication, i.e.  
 $\forall f, g \in A^{opp}$ , a new multiplication  $*$  on  $A^{opp}$  is defined to be  $f * g := g \# f$ .  
 This is an identity functor on commutative algebras.

**Remark:**    Starting from the variables  $x_1, \dots, x_N$  and the ordering  $<$  of the given algebra, an opposite algebra will have variables  $X_N, \dots, X_1$  (where the case and the position are reverted). Moreover, it is equipped with an opposed ordering  $<_{opp}$  (it is given by the matrix, obtained from the matrix ordering of  $<$  with the reverse order of columns).  
 Currently not implemented for non-global orderings.

```

LIB "ncalg.lib";
def B = makeQso3(3);
// this algebra is a quantum deformation of U(so_3),
// where the quantum parameter is a 6th root of unity
setring B; B;
↦ // coefficients: QQ[Q]/(Q^2-Q+1)
↦ // number of vars : 3
↦ //      block   1 : ordering dp
↦ //                  : names    x y z
↦ //      block   2 : ordering C
↦ // noncommutative relations:
↦ //      yx=(Q-1)*xy+(-Q)*z
↦ //      zx=(-Q)*xz+(-Q+1)*y
↦ //      zy=(Q-1)*yz+(-Q)*x
def Bopp = opposite(B);
setring Bopp;
Bopp;
↦ // coefficients: QQ[Q]/(Q^2-Q+1)
↦ // number of vars : 3
↦ //      block   1 : ordering a
↦ //                  : names    Z Y X
↦ //                  : weights  1 1 1
↦ //      block   2 : ordering ls

```



```

⇒ //                               : names      Z Y X
⇒ //          block      3 : ordering C
⇒ // noncommutative relations:
⇒ //      YZ=(Q-1)*ZY+(-Q)*X
⇒ //      XZ=(-Q)*ZX+(-Q+1)*Y
⇒ //      XY=(Q-1)*YX+(-Q)*Z
def Bcheck = opposite(Bopp);
setring Bcheck; Bcheck; // check that (B-opp)-opp = B
⇒ // coefficients: QQ[Q]/(Q2-Q+1)
⇒ // number of vars : 3
⇒ //          block      1 : ordering wp
⇒ //                               : names      x y z
⇒ //                               : weights    1 1 1
⇒ //          block      2 : ordering C
⇒ //          block      3 : ordering C
⇒ // noncommutative relations:
⇒ //      yx=(Q-1)*xy+(-Q)*z
⇒ //      zx=(-Q)*xz+(-Q+1)*y
⇒ //      zy=(Q-1)*yz+(-Q)*x

```

See [Section B.2.6 \[Matrix orderings\]](#), page 771; [Section 7.3.6 \[envelope\]](#), page 340; [Section 7.3.19 \[oppose\]](#), page 353.

### 7.3.21 preimage (plural)

**Syntax:**    `preimage ( ring_name, map_name, ideal_name )`  
              `preimage ( ring_name, ideal_expression, ideal_name )`

**Type:**     ideal

**Purpose:**    returns the preimage of an ideal under a given map. The second argument has to be a map from the basering to the given ring (or an ideal defining such a map), and the ideal has to be an ideal in the given ring.

**Note:**      To compute the kernel of a map, the preimage of zero has to be determined. Hence there is no special command for computing the kernel of a map in PLURAL.

**Remark:**    In the non-commutative case, the command `preimage` is implemented only for maps  $A \rightarrow B$ , where  $A$  is a commutative ring. See [Section 7.5.19 \[ncpreim\\_lib\]](#), page 555 for the most general available implementation.

**Example:**

```

LIB "ncalg.lib";
ring R = 0,a,dp;
def Usl2 = makeUsl2();
setring Usl2;
poly C = 4*e*f+h^2-2*h;
// C is a central element of U(sl2)
ideal I = e^3, f^3, h^3-4*h;
ideal Z = 0; // zero
ideal J = twostd(I); // two-sided GB
ideal K = std(I); // left GB
map Phi = R,C; // phi maps a (in R) to C (in U(sl2))
setring R;
ideal PreJ = preimage(Usl2,Phi,J);

```

```

// the central character of J
PreJ;
↪ PreJ[1]=a2-8a
factorize(PreJ[1],1);
↪ _[1]=a
↪ _[2]=a-8
// hence, there are two simple characters for J
ideal PreK = preimage(Us12,Phi,K);
// the central character of K
PreK;
↪ PreK[1]=a3-32a2+192a
factorize(PreK[1],1);
↪ _[1]=a
↪ _[2]=a-24
↪ _[3]=a-8
// hence, there are three simple characters for K
preimage(Us12, Phi, Z); // kernel of phi
↪ _[1]=0

```

See [Section 7.2.1 \[ideal \(plural\)\]](#), page 319; [Section 7.2.2 \[map \(plural\)\]](#), page 323; [Section 7.2.7 \[ring \(plural\)\]](#), page 333.

### 7.3.22 quotient (plural)

**Syntax:**    `quotient ( ideal_expression, ideal_expression )`  
              `quotient ( module_expression, module_expression )`

**Type:**     `ideal`

**Syntax:**    `quotient ( module_expression, ideal_expression )`

**Type:**     `module`

**Purpose:**    computes the ideal quotient, resp. module quotient. Let  $R$  be the basering,  $I, J$  ideals and  $M, N$  submodules in  $R^n$ . Then

$$\begin{aligned}\text{quotient}(I, J) &= \{a \in R \mid aJ \subset I\}, \\ \text{quotient}(M, J) &= \{b \in R^n \mid bJ \subset M\}.\end{aligned}$$

**Note:**      It can only be used for two-sided ideals (bimodules) in the second argument, otherwise the result may have no meaning.

**Example:**

```

//----- a very simple example -----
ring r=(0,q),(x,y),Dp;
def R=nc_algebra(q,0); // this algebra is a quantum plane
setring R;
option(returnSB);
poly f1 = x^3+2*x*y^2+2*x^2*y;
poly f2 = y;
poly f3 = x^2;
poly f4 = x+y;
ideal i = f1,f2;
ideal I = twostd(i);
ideal j = f3,f4;

```

```

ideal J = twostd(j);
quotient(I,J);
↦ _[1]=y
↦ _[2]=x2
module M = x*freemodule(3), y*freemodule(2);
quotient(M, ideal(x,y));
↦ _[1]=gen(1)
↦ _[2]=gen(2)
↦ _[3]=x*gen(3)
kill r,R;
//----- a bit more involved example
LIB "ncalg.lib";
def Usl2 = makeUsl2();
// this algebra is U(sl_2)
setring Usl2;
ideal i = e3,f3,h3-4*h;
ideal I = std(i);
poly C = 4*e*f+h^2-2*h;
ideal H = twostd(C-8);
option(returnSB);
ideal Q = quotient(I,H);
// print a compact presentation of Q:
print(matrix(Q));
↦ h,f3,ef2-4f,e2f-6e,e3

```

See [Section 7.2.1 \[ideal \(plural\)\]](#), page 319; [Section 7.2.3 \[module \(plural\)\]](#), page 326.

### 7.3.23 reduce (plural)

#### Syntax:

```

reduce ( poly_expression, ideal_expression )
reduce ( poly_expression, ideal_expression, int_expression )
reduce ( vector_expression, ideal_expression )
reduce ( vector_expression, ideal_expression, int_expression )
reduce ( vector_expression, module_expression )
reduce ( vector_expression, module_expression, int_expression )
reduce ( ideal_expression, ideal_expression )
reduce ( ideal_expression, ideal_expression, int_expression )
reduce ( module_expression, ideal_expression )
reduce ( module_expression, ideal_expression, int_expression )
reduce ( module_expression, module_expression )
reduce ( module_expression, module_expression, int_expression )

```

**Type:** the type of the first argument

**Purpose:** reduces a polynomial, vector, ideal or module to its **left** normal form with respect to an ideal or module represented by a left Groebner basis, if the second argument is a left Groebner basis.  
 returns 0 if and only if the polynomial (resp. vector, ideal, module) is an element (resp. subideal, submodule) of the ideal (resp. module).  
 Otherwise, the result may have no meaning.  
 The third (optional) argument 1 of type int forces a reduction which considers only the leading term and does no tail reduction.

**Note:** The commands `reduce` and `NF` are synonymous.

**Example:**

```

ring r=(0,a),(e,f,h),Dp;
matrix d[3][3];
d[1,2]=-h; d[1,3]=2e; d[2,3]=-2f;
def R=nc_algebra(1,d);
setring R;
// this algebra is U(sl_2) over Q(a)
ideal I = e2, f2, h2-1;
I = std(I);
// print a compact presentation of I
print(matrix(I));
↪ h2-1,fh-f,f2,eh+e,2*ef-h2-h,e2
ideal J = e, h-a;
J = std(J);
// print a compact presentation of J
print(matrix(J));
↪ h+(-a),e
poly z=4*e*f+h^2-2*h;
// z is the central element of U(sl_2)
reduce(z,I); // the central character of I:
↪ 3
reduce(z,J); // the central character of J:
↪ (a2+2a)
poly nz = z - NF(z,J); // nz will belong to J
reduce(nz,J);
↪ 0
reduce(I,J);
↪ _[1]=(a2-1)
↪ _[2]=(a-1)*f
↪ _[3]=f2
↪ _[4]=0
↪ _[5]=(-a2+a)
↪ _[6]=0

```

See also [Section 7.2.1 \[ideal \(plural\)\]](#), page 319; [Section 7.2.3 \[module \(plural\)\]](#), page 326; [Section 7.3.26 \[std \(plural\)\]](#), page 361.

### 7.3.24 ringlist (plural)

**Syntax:** `ringlist ( ring_expression )`  
`ringlist ( qring_expression )`

**Type:** list

**Purpose:** decomposes a ring/qring into a list of 6 (or 4 in the commutative case) components. The first 4 components are common both for the commutative and for the non-commutative cases, the 5th and the 6th appear only in the non-commutative case.

5. upper triangle square matrix with nonzero upper triangle, containing structural coefficients of a G-algebra (this corresponds to the matrix C from the definition of [Section 7.4.1 \[G-algebras\]](#), page 366)
6. square matrix, containing structural polynomials of a G-algebra (this corresponds to the matrix D from the definition of [Section 7.4.1 \[G-algebras\]](#), page 366)

**Note:** After modifying a list acquired with `ringlist`, one can construct a corresponding ring with `ring(list)`.

**Example:**

```
// consider the quantized Weyl algebra
ring r = (0,q),(x,d),Dp;
def RS=nc_algebra(q,1);
setring RS; RS;
⇨ // coefficients: QQ(q)
⇨ // number of vars : 2
⇨ //      block 1 : ordering Dp
⇨ //      : names x d
⇨ //      block 2 : ordering C
⇨ // noncommutative relations:
⇨ //      dx=(q)*xd+1
list l = ringlist(RS);
l;
⇨ [1]:
⇨   [1]:
⇨     0
⇨   [2]:
⇨     [1]:
⇨       q
⇨   [3]:
⇨     [1]:
⇨       [1]:
⇨         lp
⇨     [2]:
⇨       1
⇨   [4]:
⇨     _[1]=0
⇨ [2]:
⇨   [1]:
⇨     x
⇨   [2]:
⇨     d
⇨ [3]:
⇨   [1]:
⇨     [1]:
⇨       Dp
⇨   [2]:
⇨     1,1
⇨   [2]:
⇨     [1]:
⇨       C
⇨   [2]:
⇨     0
⇨ [4]:
⇨   _[1]=0
⇨ [5]:
⇨   _[1,1]=0
⇨   _[1,2]=(q)
⇨   _[2,1]=0
```

```

↳      _[2,2]=0
↳ [6]:
↳      _[1,1]=0
↳      _[1,2]=1
↳      _[2,1]=0
↳      _[2,2]=0
// now, change the relation d*x = q*x*d +1
// into the relation d*x=(q2+1)*x*d + q*d + 1
matrix S = l[5]; // matrix of coefficients
S[1,2] = q^2+1;
l[5] = S;
matrix T = l[6]; // matrix of polynomials
T[1,2] = q*d+1;
l[6] = T;
def rr = ring(l);
setring rr; rr;
↳ // coefficients: QQ(q)
↳ // number of vars : 2
↳ //      block 1 : ordering Dp
↳ //      : names x d
↳ //      block 2 : ordering C
↳ // noncommutative relations:
↳ //      dx=(q2+1)*xd+(q)*d+1

```

See also [Section 7.2.7 \[ring \(plural\)\]](#), page 333; [Section 5.1.137 \[ringlist\]](#), page 255.

### 7.3.25 slimgb (plural)

**Syntax:**    slimgb ( ideal\_expression )  
              slimgb ( module\_expression )

**Type:**     same type as argument

**Purpose:**    returns a left Groebner basis of a left ideal or module with respect to the global monomial ordering of the basering.

**Note:**      The commutative algorithm is described in the diploma thesis of Michael Brickenstein "Neue Varianten zur Berechnung von Groebnerbasen", written 2004 under supervision of G.-M. Greuel in Kaiserslautern.

It is designed to keep polynomials or vectors slim (short with small coefficients). Currently best results are examples over function fields (parameters).

The current implementation may not be optimal for weighted degree orderings.

The program only supports the options **prot**, which will give protocol output and **redSB** for returning a reduced Groebner basis. The protocol messages of **slimgb** mean the following:

**M[n,m]** means a parallel reduction of **n** elements with **m** non-zero output elements,  
**b** notices an exchange trick described in the thesis and  
**e** adds a reductor with non-minimal leading term.

**slimgb** works for grade commutative algebras but not for general GR-algebras. Please use **qslimgb** instead.

For a detailed commutative example see [Section A.2.3 \[slim Groebner bases\]](#), page 715.

**Example:**

```

LIB "nctools.lib";
LIB "ncalg.lib";
def U = makeUsl(2); setring U;
// U is the U(sl_2) algebra
ideal I = e^3, f^3, h^3-4*h;
option(redSB);
ideal J = slimgb(I);
J;
↳ J[1]=h3-4h
↳ J[2]=fh2-2fh
↳ J[3]=eh2+2eh
↳ J[4]=2efh-h2-2h
↳ J[5]=f3
↳ J[6]=e3
// compare slimgb with std:
ideal K = std(I);
print(matrix(NF(K,J)));
↳ 0,0,0,0,0,0
print(matrix(NF(J,K)));
↳ 0,0,0,0,0,0
// hence both Groebner bases are equal.
// Another example for exterior algebras
ring r;
def E = Exterior(); setring E; E;
↳ // coefficients: ZZ/32003
↳ // number of vars : 3
↳ //      block 1 : ordering dp
↳ //      : names x y z
↳ //      block 2 : ordering C
↳ // noncommutative relations:
↳ //      yx=-xy
↳ //      zx=-xz
↳ //      zy=-yz
↳ // quotient ring from ideal
↳ _[1]=z2
↳ _[2]=y2
↳ _[3]=x2
slimgb(xy+z);
↳ _[1]=yz
↳ _[2]=xz
↳ _[3]=xy+z

```

See [Section 5.1.111 \[option\]](#), page 234; [Section 7.3.26 \[std \(plural\)\]](#), page 361.

### 7.3.26 std (plural)

**Syntax:**    `std ( ideal_expression )`  
               `std ( module_expression )`  
               `std ( ideal_expression, poly_expression )`  
               `std ( module_expression, vector_expression )`

**Type:**     ideal or module

**Purpose:** returns a left Groebner basis (see [Section 7.4.2 \[Groebner bases in G-algebras\]](#), page 367 for a definition) of an ideal or module with respect to the monomial ordering of the basering.

Use an optional second argument of type poly, resp. vector, to construct the Groebner basis from an already computed one (given as the first argument) and one additional generator (the second argument).

**Note:** To view the progress of long running computations, use `option(prot)`. (see [Section 5.1.111 \[option\]](#), page 234(prot)).

**Example:**

```
LIB "ncalg.lib";
option(prot);
def R = makeUsl2();
// this algebra is U(sl_2)
setring R;
ideal I = e2, f2, h2-1;
I=std(I);
↳ 2(2)s
↳ s
↳ s
↳ 3s
↳ (3)2(2)s
↳ s
↳ (4)(3)(2)3s
↳ 2(4)(3)(2)32product criterion:6 chain criterion:3
I;
↳ I[1]=h2-1
↳ I[2]=fh-f
↳ I[3]=eh+e
↳ I[4]=f2
↳ I[5]=2ef-h-1
↳ I[6]=e2
kill R;
//-----
def RQ = makeQso3(3);
// this algebra is U'_q(so_3),
// where Q is a 6th root of unity
setring RQ;
RQ;
↳ // coefficients: QQ[Q]/(Q2-Q+1)
↳ // number of vars : 3
↳ //      block 1 : ordering dp
↳ //      : names x y z
↳ //      block 2 : ordering C
↳ // noncommutative relations:
↳ //      yx=(Q-1)*xy+(-Q)*z
↳ //      zx=(-Q)*xz+(-Q+1)*y
↳ //      zy=(Q-1)*yz+(-Q)*x
ideal J=x2, y2, z2;
J=std(J);
↳ 2(2)s
↳ s
```



```

↦ s
↦ 3s
↦ (4)s
↦ 2(3)s
↦ (5)s
↦ (6)s
↦ 1(8)s
↦ (7)(5)s
↦ (3)(2)product criterion:0 chain criterion:17
J;
↦ J[1]=z
↦ J[2]=y
↦ J[3]=x

```

See also [Section 7.2.1 \[ideal \(plural\)\]](#), page 319; [Section 7.2.7 \[ring \(plural\)\]](#), page 333.

### 7.3.27 subst (plural)

**Syntax:**    `subst ( poly_expression, ring_variable, poly_expression )`  
               `subst ( vector_expression, ring_variable, poly_expression )`  
               `subst ( ideal_expression, ring_variable, poly_expression )`  
               `subst ( module_expression, ring_variable, poly_expression )`

**Type:**     poly, vector, ideal or module (corresponding to the first argument)

**Purpose:**   substitutes a ring variable by a polynomial.

**Example:**

```

LIB "ncalg.lib";
def R = makeUs12();
// this algebra is U(sl_2)
setring R;
poly C = e*f*h;
poly C1 = subst(C,e,h^3);
C1;
↦ fh4-6fh3+12fh2-8fh
poly C2 = subst(C,f,e+f);
C2;
↦ e2h+efh

```

See also [Section 7.2.2 \[map \(plural\)\]](#), page 323.

### 7.3.28 syz (plural)

**Syntax:**    `syz ( ideal_expression )`  
               `syz ( module_expression )`

**Type:**     module

**Purpose:**   computes the first syzygy (i.e., the module of relations of the given generators) of the ideal, resp. module.

**Note:**     if  $S$  is a matrix of a left syzygy module of left submodule given by matrix  $M$ , then  $\text{transpose}(S) * \text{transpose}(M) = 0$ .

**Example:**

```

LIB "ncalg.lib";
def R = makeQso3(3); setring R;
// we wish to have completely reduced bases:
option(redSB); option(redTail);
ideal tst;
ideal J = x3+x,x*y*z;
print(syz(J));
↳ -yz,
↳ x2+1
ideal K = x+y+z,y+z,z;
module S = syz(K);
print(S);
↳ (Q-1),          (-Q+1)*z,   (-Q)*y,
↳ (Q)*z+(-Q+1), (Q-1)*z+(Q), -x+(Q)*y,
↳ y+(-Q)*z,      x+(-Q),      x+(-Q+1)
tst = ideal(transpose(S)*transpose(K));
// check the property of a syzygy module (tst==0):
size(tst);
↳ 0
// Now compute the Groebner basis of K ...
K = std(K);
// ... print a matrix presentation of K ...
print(matrix(K));
↳ z,y,x
S = syz(K); // ... and its syzygy module
print(S);
↳ y,      x,          (Q-1),
↳ (Q)*z,(Q),      x,
↳ (Q-1),(-Q+1)*z,(Q)*y
tst = ideal(transpose(S)*transpose(K));
// check the property of a syzygy module (tst==0):
size(tst);
↳ 0
// Note the "commutative" (not transposed) syzygy property does not hold
size(ideal(matrix(K)*matrix(S)));
↳ 3

```

See also [Section 7.2.1 \[ideal \(plural\)\]](#), page 319; [Section 7.3.13 \[minres \(plural\)\]](#), page 346; [Section 7.2.3 \[module \(plural\)\]](#), page 326; [Section 7.3.15 \[mres \(plural\)\]](#), page 348; [Section 7.3.18 \[nres \(plural\)\]](#), page 351.

### 7.3.29 twostd (plural)

**Syntax:** `twostd( ideal.expression);`

**Type:** `ideal`

**Purpose:** returns a two-sided Groebner basis of an input

**Note:** Treating the input as a set of

two-sided generators of a two-sided ideal  $T$ , two-sided Groebner basis is a left (and a right) Groebner basis of  $T$ . (see [Section 5.1.151 \[std\]](#), page 271).

**Remark:** There are algebras with no two-sided ideals except 0 and the whole algebra (like Weyl algebras).

**Example:**

```

LIB "ncalg.lib";
def U = makeUsl2(); // this algebra is U(sl_2)
setring U;
ideal i= e^3, f^3, h^3 - 4*h;
option(redSB);
option(redTail);
ideal I = std(i);
print(matrix(I)); // print a compact presentation of I
↪ h3-4h,fh2-2fh,eh2+2eh,2efh-h2-2h,f3,e3
ideal J = twostd(i);
// print a compact presentation of J:
print(matrix(ideal(J[1..6]))); // first 6 gen's
↪ h3-4h,fh2-2fh,eh2+2eh,f2h-2f2,2efh-h2-2h,e2h+2e2
print(matrix(ideal(J[7..size(J)]))); // the rest of gen's
↪ f3,ef2-fh,e2f-eh-2e,e3
// compute the set of elements present in J but not in I
ideal K = NF(J,I);
K = K+0; // simplify K
print(matrix(K));
↪ f2h-2f2,e2h+2e2,ef2-fh,e2f-eh-2e

```

**7.3.30 vdim (plural)**

**Syntax:**    vdim ( ideal\_expression )  
               vdim ( module\_expression )

**Type:**     int

**Purpose:**    computes the vector space dimension of the factor-module that equals ring (resp. free module) modulo the ideal (resp. submodule), generated by the leading terms of the given generators.

If the factor-module is not of finite dimension, -1 is returned.

If the generators form a left Groebner basis, this is the same as the vector space dimension of the left factor module.

**Note:**      In the non-commutative case, a ring modulo an ideal has a ring structure if and only if the ideal is two-sided.

**Example:**

```

ring R=0,(x,y,z),dp;
matrix d[3][3];
d[1,2]=-z; d[1,3]=2x; d[2,3]=-2y;
def RS=nc_algebra(1,d); //U(sl_2)
setring RS;
option(redSB); option(redTail);
ideal I=x3,y3,z3-z;
I=std(I);
I;
↪ I[1]=z3-z
↪ I[2]=y3
↪ I[3]=x3
↪ I[4]=y2z2-y2z
↪ I[5]=x2z2+x2z

```

```

↦ I[6]=x2y2z-2xyz2-2xyz+2z2+2z
vdim(I);
↦ 21

```

See also [Section 7.2.1 \[ideal \(plural\)\]](#), page 319; [Section 7.3.10 \[kbase \(plural\)\]](#), page 343; [Section 7.3.26 \[std \(plural\)\]](#), page 361.

## 7.4 Mathematical background (plural)

This section introduces some of the mathematical notions and definitions used throughout the PLURAL manual. For details, please, refer to appropriate articles or text books (see [Section 7.4.4 \[References \(plural\)\]](#), page 370). A detailed discussion of the subjects in this section can be found in the doctoral thesis [LV] of V. Levandovskyy (see [Section 7.4.4 \[References \(plural\)\]](#), page 370).

All algebras are assumed to be associative  $K$ -algebras for some field  $K$ .

### 7.4.1 G-algebras

#### Definition (PBW basis)

Let  $K$  be a field, and let a  $K$ -algebra  $A$  be generated by variables  $x_1, \dots, x_n$  subject to some relations. We call  $A$  an algebra with **PBW basis** (Poincaré-Birkhoff-Witt basis), if a  $K$ -basis of  $A$  is  $\text{Mon}(x_1, \dots, x_n) = \{x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} \mid a_i \in \mathbb{N} \cup \{0\}\}$ , where a power-product  $x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$  (in this particular order) is called a **monomial**. For example,  $x_1 x_2$  is a monomial, while  $x_2 x_1$  is, in general, not a monomial.

#### Definition (G-algebra)

Let  $K$  be a field, and let a  $K$ -algebra  $A$  be given in terms of generators subject to the following relations:

$$A = K\langle x_1, \dots, x_n \mid \{x_j x_i = c_{ij} \cdot x_i x_j + d_{ij}\}, 1 \leq i < j \leq n \rangle, \text{ where } c_{ij} \in K^*, d_{ij} \in K[x_1, \dots, x_n].$$

$A$  is called a **G-algebra**, if the following conditions hold:

- there is a monomial well-ordering  $<$  on  $K[x_1, x_2, \dots, x_n]$  such that  $\forall i < j \quad \text{LM}(d_{ij}) < x_i x_j$ ,
- **non-degeneracy conditions**:  $\forall 1 \leq i < j < k \leq n : \mathcal{NDC}_{ijk} = 0$ , where

$$\mathcal{NDC}_{ijk} = c_{ik} c_{jk} \cdot d_{ij} x_k - x_k d_{ij} + c_{jk} \cdot x_j d_{ik} - c_{ij} \cdot d_{ik} x_j + d_{jk} x_i - c_{ij} c_{ik} \cdot x_i d_{jk}.$$

**Note:** Note that non-degeneracy conditions ensure associativity of multiplication, defined by the relations. It is also proved, that they are necessary and sufficient to guarantee the PBW property of an algebra, defined via  $C_{ij}$  and  $D_{ij}$  as above.

#### Theorem (properties of G-algebras)

Let  $A$  be a  $G$ -algebra. Then

- $A$  has a PBW (Poincaré-Birkhoff-Witt) basis,
- $A$  is left and right noetherian,
- $A$  is an integral domain.

## Setting up a G-algebra

In order to set up a  $G$ -algebra one has to do the following steps:

- define a commutative ring  $R = K[x_1, \dots, x_n]$ , equipped with a monomial ordering  $<$  (see [Section 7.2.7.1 \[ring declarations \(plural\)\]](#), page 333).  
This provides us with the information on a field  $K$  (together with its parameters), variables  $\{x_i\}$  and an ordering  $<$ .  
From the sequence of variables we will build a  $G$ -algebra with the Poincaré-Birkhoff-Witt (PBW) basis  $\{x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}\}$ .
- define strictly  $n \times n$  upper triangular matrices (of type `matrix`)
  1.  $C = \{c_{ij}, i < j\}$ , with nonzero entries  $c_{ij}$  of type number ( $c_{ij}$  for  $i \geq j$  will be ignored).
  2.  $D = \{d_{ij}, i < j\}$ , with polynomial entries  $d_{ij}$  from  $R$  ( $d_{ij}$  for  $i \geq j$  will be ignored).
- Call the initialization function `nc_algebra(C,D)` (see [Section 7.3.16 \[nc\\_algebra\]](#), page 349) with the data  $C$  and  $D$ .

PLURAL does not check automatically whether the non-degeneracy conditions hold but it provides a procedure [Section 7.5.20.3 \[ndcond\]](#), page 565 from the library [Section 7.5.20 \[nctools\\_lib\]](#), page 563 to check this.

### 7.4.2 Groebner bases in G-algebras

We follow the notations, used in the SINGULAR Manual (e.g. in [Section C.1 \[Standard bases\]](#), page 775).

For a  $G$ -algebra  $A$ , we denote by  ${}_A\langle g_1, \dots, g_s \rangle$  the left submodule of a free module  $A^r$ , generated by elements  $\{g_1, \dots, g_s\} \subset A^r$ .

Let  $<$  be a fixed monomial well-ordering on the  $G$ -algebra  $A$  with the Poincaré-Birkhoff-Witt (PBW) basis  $\{x^\alpha = x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}\}$ . For a given free module  $A^r$  with the basis  $\{e_1, \dots, e_r\}$ ,  $<$  denotes also a fixed module ordering on the set of monomials  $\{x^\alpha e_i \mid \alpha \in \mathbb{N}^n, 1 \leq i \leq r\}$ .

#### Definition

For a set  $S \subset A^r$ , define  $L(S)$  to be the  $K$ -vector space, spanned on the leading monomials of elements of  $S$ ,  $L(S) = \oplus \{Kx^\alpha e_i \mid \exists s \in S, \text{LM}(s) = x^\alpha e_i\}$ .

We call  $L(S)$  the **span of leading monomials of  $S$** .

Let  $I \subset A^r$  be a left  $A$ -submodule. A finite set  $G \subset I$  is called a **left Groebner basis** of  $I$  if and only if  $L(G) = L(I)$ , that is for any  $f \in I \setminus \{0\}$  there exists a  $g \in G$  satisfying  $\text{LM}(g) \mid \text{LM}(f)$ , i.e., if  $\text{LM}(f) = x^\alpha e_i$ , then  $\text{LM}(f) = x^\beta e_i$  with  $\beta_j \leq \alpha_j$ ,  $1 \leq j \leq n$ .

**Remark:** In general non-commutative algorithms are working with global well-orderings only (see [Section 7.1 \[PLURAL\]](#), page 318, [Section B.2 \[Monomial orderings\]](#), page 769 and [Section 3.3.3 \[Term orderings\]](#), page 34), unless we deal with graded commutative algebras via [Section 7.6 \[Graded commutative algebras \(SCA\)\]](#), page 615.

A Groebner basis  $G \subset A^r$  is called **minimal** (or **reduced**) if  $0 \notin G$  and if  $\text{LM}(g) \notin L(G \setminus \{g\})$  for all  $g \in G$ . Note, that any Groebner basis can be made minimal by deleting successively those  $g$  with  $\text{LM}(h) \mid \text{LM}(g)$  for some  $h \in G \setminus \{g\}$ .

For  $f \in A^r$  and  $G \subset A^r$  we say that  $f$  is **completely reduced with respect to  $G$**  if no monomial of  $f$  is contained in  $L(G)$ .

## Left Normal Form

A map  $\text{NF} : A^r \times \{G \mid G \text{ a (left) Groebner basis}\} \rightarrow A^r, (f|G) \mapsto \text{NF}(f|G)$ , is called a **(left) normal form** on  $A^r$  if for any  $f \in A^r$  and any left Groebner basis  $G$  the following holds:

- (i)  $\text{NF}(0|G) = 0$ ,
- (ii) if  $\text{NF}(f|G) \neq 0$  then  $\text{LM}(g)$  does not divide  $\text{LM}(\text{NF}(f|G))$  for all  $g \in G$ ,
- (iii)  $f - \text{NF}(f|G) \in {}_A\langle G \rangle$ .

$\text{NF}(f|G)$  is called a **left normal form of  $f$  with respect to  $G$**  (note that such a map is not unique).

**Remark:** As we have already mentioned in the definitions `ideal` and `module` (see [Section 7.1 \[PLURAL\]](#), page 318), by `NF` (or `reduce`) `PLURAL` understands a left normal form. Note, that `rightNF` from [Section 7.5.20 \[nctools\\_lib\]](#), page 563 allows to compute a right normal form.

## Left ideal membership (plural)

For a left Groebner basis  $G$  of  $I$  the following holds:  $f \in I$  if and only if the left normal form  $\text{NF}(f|G) = 0$ .

For computing a left Groebner basis  $G$  of  $I$ , use [Section 7.3.26 \[std \(plural\)\]](#), page 361.

For computing a left normal form of  $f$  with respect to  $G$ , use [Section 7.3.23 \[reduce \(plural\)\]](#), page 357.

## Right ideal membership (plural)

The right ideal membership is analogous to the left one:

for computing a right Groebner basis  $G$  of  $I$ , use [Section 7.8.10 \[rightstd \(letterplace\)\]](#), page 633 from [Section 7.5.20 \[nctools\\_lib\]](#), page 563,

for computing a right normal form of  $f$  with respect to  $G$ , use [Section 7.5.20.11 \[rightNF\]](#), page 572 from [Section 7.5.20 \[nctools\\_lib\]](#), page 563.

## Two-sided ideal membership (plural)

Let  $J$  be a two-sided ideal and  $T$  be a two-sided Groebner basis of  $J$ .

Then  $f \in J$  if and only if the left normal form  $\text{NF}(f|T) = 0$ .

For computing a two-sided Groebner basis  $T$  of  $J$ , use [Section 7.3.29 \[twostd \(plural\)\]](#), page 364,

for computing a normal form of  $f$  with respect to  $T$ , use [Section 7.3.23 \[reduce \(plural\)\]](#), page 357.

## 7.4.3 Syzygies and resolutions (plural)

### Syzygies

Let  $A$  be a GR-algebra. A **left** (resp. **right**) **syzygy** between  $k$  elements  $\{f_1, \dots, f_k\} \subset A^r$  is a  $k$ -tuple  $(g_1, \dots, g_k) \in A^k$  satisfying

$$\sum_{i=1}^k g_i f_i = 0 \quad \text{resp.} \quad \sum_{i=1}^k f_i g_i = 0.$$

The set of all left (resp. right) syzygies between  $\{f_1, \dots, f_k\}$  is a left (resp. right) submodule  $S$  of  $A^k$ .

**Remark:** With respect to the definitions of `ideal` and `module` (see [Section 7.1 \[PLURAL\]](#), [page 318](#)), by `syz PLURAL` understands an inquiry to compute the left syzygy module.

Note, that `rightModulo(M, std(0))` from [Section 7.5.20 \[nctools.lib\]](#), [page 563](#) computes the right syzygy module of `M`.

If `S` is a matrix of a left syzygy module of left submodule given by matrix `M`, then `transpose(S)*transpose(M) = 0` (but, in general,  $M \cdot S \neq 0$ ).

Note, that the syzygy modules of  $I$  depend on a choice of generators  $\{g_1, \dots, g_s\}$ , but one can show that they depend on  $I$  uniquely up to direct summands.

## Free resolutions

Let  $I = {}_A\langle g_1, \dots, g_s \rangle \subseteq A^r$  and  $M = A^r/I$ . A **free resolution of  $M$**  is a long exact sequence

$$\dots \longrightarrow F_2 \xrightarrow{B_2} F_1 \xrightarrow{B_1} F_0 \longrightarrow M \longrightarrow 0,$$

with  $\text{transpose}(B_{i+1}) \cdot \text{transpose}(B_i) = 0$

and where the columns of the matrix  $B_1$  generate  $I$ . Note, that resolutions over factor-algebras need not to be of finite length.

## Generalized Hilbert Syzygy Theorem

For a  $G$ -algebra  $A$ , generated by  $n$  variables, there exists a free resolution of length smaller or equal than  $n$ .

**Example:**

```
ring R=0,(x,y,z),dp;
matrix d[3][3];
d[1,2]=-z; d[1,3]=2x; d[2,3]=-2y;
def U=nc_algebra(1,d); // this algebra is U(sl_2)
setring U;
option(redSB); option(redTail);
ideal I=x3,y3,z3-z;
I=std(I);
I;
↪ I[1]=z3-z
↪ I[2]=y3
↪ I[3]=x3
↪ I[4]=y2z2-y2z
↪ I[5]=x2z2+x2z
↪ I[6]=x2y2z-2xyz2-2xyz+2z2+2z
resolution resI = mres(I,0);
resI;
↪ 1      5      7      3
↪ U <--  U <--  U <--  U
↪
↪ 0      1      2      3
↪
list l = resI;
// The matrix A_1 is given by
print(matrix(l[1]));
```

```

⇒ z3-z,y3,x3,y2z2-y2z,x2z2+x2z
// We see that the columns of A_1 generate I.
// The matrix A_2 is given by
print(matrix(l[2]));
⇒ 0,      0,      y2,  x2,  6yz,      -36xy+18z+24,-6xz,
⇒ z2+11z+30,0,      0,  0,  2x2z+12x2,  2x3,      0,
⇒ 0,      z2-11z+30,0,  0,  0,      -2y3,      2y2z-12y2,
⇒ -y,      0,      -z-5,0,  x2y-6xz-30x,9x2,      x3,
⇒ 0,      -x,      0,  -z+5,-y3,      -9y2,      -xy2-4yz+28y
ideal tst; // now let us show that the resolution is exact
matrix TST;
TST = transpose(matrix(l[3]))*transpose(matrix(l[2])); // 2nd term
size(ideal(TST));
⇒ 0
TST = transpose(matrix(l[2]))*transpose(matrix(l[1])); // 1st term
size(ideal(TST));
⇒ 0

```

#### 7.4.4 References (plural)

The Centre for Computer Algebra Kaiserslautern publishes a series of preprints which are electronically available at <https://www.mathematik.uni-kl.de/organisation/zca/reports-on-ca/>. Other sources to check are the following books and articles:

##### Text books

- [DK] Y. Drozd and V. Kirichenko. Finite dimensional algebras. With an appendix by Vlastimil Dlab. Springer, 1994
- [GPS] Greuel, G.-M. and Pfister, G. with contributions by Bachmann, O. ; Lossen, C. and Schönemann, H. A SINGULAR Introduction to Commutative Algebra. Springer, 2002
- [BGV] Bueso, J.; Gomez Torrecillas, J.; Verschoren, A. Algorithmic methods in non-commutative algebra. Applications to quantum groups. Kluwer Academic Publishers, 2003
- [Kr] Kredel, H. Solvable polynomial rings. Shaker, 1993 [http://krum.rz.uni-mannheim.de/kredel/kredel\\_solvable\\_polynomial\\_rings.pdf](http://krum.rz.uni-mannheim.de/kredel/kredel_solvable_polynomial_rings.pdf)
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- [L2] Levandovskyy, V. On preimages of ideals in certain non-commutative algebras. In Pfister G., Cojocaru S. and Ufnarovski, V. (editors), Computational Commutative and Non-Commutative Algebraic Geometry, IOS Press, 2005.
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## 7.5 PLURAL libraries

The content of libraries, created for PLURAL is described in the following subsections.

Use the LIB command for loading of single libraries.

**Note:** For any computation in PLURAL, the monomial ordering must be a global ordering.

See also [Section D.11.3 \[jacobson.lib\]](#), [page 1978](#) for the diagonalization of matrices over Ore Euclidean domains.

### 7.5.1 bimodules.lib

**Library:** bimodules.lib

**Purpose:** Tools for handling bimodules

**Authors:** Ann Christina Foldenauer, Christina.Foldenauer@rwth-aachen.de  
Viktor Levandovskyy, levandov@math.rwth-aachen.de

**Overview:**

The main purpose of this library is the handling of bimodules which will help e.g. to determine weak normal forms of representation matrices and total divisors within non-commutative, non-simple G-algebras. We will use modules homomorphisms between a G-algebra and its enveloping algebra in order to work left Groebner basis theory on bimodules. Assume we have defined a (non-commutative) G-algebra  $A$  over the field  $K$ , and an  $(A,A)$ -bimodule  $M$ . Instead of working with  $M$  over  $A$ , we define the enveloping algebra  $A^{\text{env}} = A \otimes_K A^{\text{opp}}$  (this can be done with command `envelope(A)`) and embed  $M$  into  $A^{\text{env}}$  via `imap()`. Thus we obtain the left  $A^{\text{env}}$ -module  $M \otimes 1$  in  $A^{\text{env}}$ . This has a lot of advantages, because left module theory has much more commands that are already implemented in SINGULAR:PLURAL. Two important procedures that we can use are `std()` which computes the left Groebner basis, and `NF()` which computes the left normal form. With the help of this method we are also able to determine the set of bisyzygies of a bimodule.

A built-in command `twostd` in PLURAL computes the two-sided Groebner basis of an ideal by using the right completion algorithm of [2]. `bistd` from this library uses very different approach, which is often superior to the right completion.

**References:**

- The procedure `bistd()` is the implementation of an algorithm M. del Socorro Garcia Roman presented in [1](page 66-78).
- [1] Maria del Socorro Garcia Roman, Effective methods in Algebras with PBW bases: G-algebras and Yang-Baxter Algebras, Ph.D. thesis, Universidad de La Laguna, 2005.
  - [2] Viktor Levandovskyy, Non-commutative Computer Algebra for polynomial Algebras: Groebner Bases, Applications and Implementations, Ph.D. thesis, Kaiserslautern, 2005.
  - [3] N. Jacobson, The theory of rings, AMS, 1943.
  - [4] P. M. Cohn, Free Rings and their Relations, Academic Press Inc. (London) Ltd., 1971.

**Procedures:** See also: [Section 7.5.10 \[ncalg\\_lib\]](#), page 465; [Section 7.5.20 \[nctools\\_lib\]](#), page 563.

**7.5.1.1 bistd**

Procedure from library `bimodules.lib` (see [Section 7.5.1 \[bimodules\\_lib\]](#), page 371).

**Usage:** `bistd(M)`;  $M$  is (two-sided) ideal/module

**Return:** ideal or module (same type as the argument)

**Purpose:** Computes the two-sided Groebner basis of an ideal/module with the help the enveloping algebra of the basering, alternative to `twostd()` for ideals.

**Example:**

```

LIB "bimodules.lib";
ring w = 0,(x,s),Dp;
def W=nc_algebra(1,s); // 1st shift algebra
setring W;
matrix m[3][3]=[s^2,s+1,0],[s+1,0,s^3-x^2*s],[2*s+1,s^3+s^2,s^2];
print(m);
  ↪ s2,    s+1,    s+1,
  ↪ 0,     -x2s+s3,2s+1,
  ↪ s3+s2,s2,     0
module L = m; module M2 = bistd(L);
print(M2);
  ↪ 1,1,s+1,
  ↪ 0,1,0,
  ↪ 0,0,s2

```

### 7.5.1.2 bitrinity

Procedure from library `bimodules.lib` (see [Section 7.5.1 \[bimodules.lib\]](#), page 371).

**Usage:** `bitrinity(M)`;  $M$  is (two-sided) ideal/module

**Return:** ring, the enveloping algebra of the basering, with objects in it. additionally it exports a list  $L = \text{Coeff}, \text{BiSyz}$ .

**Theory:** Let  $\text{psi}_s$  be the epimorphism of left  $R$  ( $X$ )  $R^{\{\text{opp}\}}$  modules:  
 $\text{psi}_s(s(X)_K t) = \text{smt} := (s_1 m_{t_1}, \dots, s_s m_{t_s}) = (\text{psi}(s_1(X) t_1), \dots, \text{psi}(s_s(X) t_s))$  in  $R^s$ .  
 Then  $\text{psi}_s(A) := (\text{psi}_s(a_{ij}))$  for every matrix  $A$  in  $\text{Mat}(n \times m, R)$ .  
 For a two-sided ideal  $I = \langle f_1, \dots, f_j \rangle$  with Groebner basis  $G = \{g_1, \dots, g_k\}$  in  $R$ ,  $\text{Coeff}$  is the Coefficient-Matrix and  $\text{BiSyz}$  a bisyzygy matrix.  
 Let  $C$  be the submatrix of  $\text{Coeff}$ , where  $C$  is  $\text{Coeff}$  without the first row. Then  $(g_1, \dots, g_k) = \text{psi}_s(C^T * (f_1 \dots f_j)^T)$  and  $(0, \dots, 0) = \text{psi}_s(\text{BiSyz}^T * (f_1 \dots f_j)^T)$ .  
 The first row of  $\text{Coeff}$  ( $G_1 \dots G_n$ ) corresponds to the image of the Groebner basis of  $I$ :  $\text{psi}_s((G_1 \dots G_n)) = G = \{g_1 \dots g_k\}$ .  
 For a  $(R,R)$ -bimodule  $M$  with Groebner basis  $G = \{g_1, \dots, g_k\}$  in  $R^r$ ,  $\text{Coeff}$  is the coefficient matrix and  $\text{BiSyz}$  a bisyzygy matrix.  
 Let  $C$  be the submatrix of  $\text{Coeff}$ , where  $C$  is  $\text{Coeff}$  without the first  $r$  rows. Then  $(g_1 \dots g_k) = \text{psi}_s(C^T * (f_1 \dots f_j)^T)$  and  $(0 \dots 0) = \text{psi}_s(\text{BiSyz}^T * (f_1 \dots f_j)^T)$ .  
 The first  $r$  rows of  $\text{Coeff} = (G_1 \dots G_n)$  (Here  $G_i$  denotes to the  $i$ -th column of the first  $r$  rows) corresponds to the image of the Groebner basis of  $M$ :  $\text{psi}_s((G_1 \dots G_n)) = G = \{g_1 \dots g_k\}$ .

**Purpose:** This procedure returns a coefficient matrix in the enveloping algebra of the basering  $R$ , that gives implicitly the two-sided Groebner basis of a  $(R,R)$ -bimodule  $M$  and the coefficients that produce the Groebner basis with the help of the originally used generators of  $M$ . Additionally it calculates the bisyzygies of  $M$  as left-module of the enveloping algebra of  $R$ .

**Auxiliary procedures:**

**Note:** To get list  $L = \text{Coeff}, \text{BiSyz}$ , we set: `def G = bitrinity(); setring G; L;` or `$L[1]; L[2];`.

**Example:**

```

LIB "bimodules.lib";
ring r = 0,(x,s),dp;

```

```

def R = nc_algebra(1,s); setring R; // 1st shift algebra
poly f = x*s + s^2; // only one generator
ideal I = f; // note, two sided Groebner basis of I is xs, s^2
def G = bitrinity(I);
setring G;
print(L[1]); // Coeff
  ↦ S2, SX,
  ↦ s-S, -s+S+1
//the first row shows the Groebnerbasis of I consists of
// psi_s(SX) = xs , phi(S^2) = s^2:
// remember phi(a (X) b - c (X) d) = psi_s(a (X) b) - phi(c (X) d) := ab - cd in R.
// psi_s((-s+S+1)*(x*s + s^2)) = psi_s(-xs2-s3+xsS+xs+s2S)
// = -xs^2-s^3+xs^2+xs+s^3 = xs
// psi_s((s-S)*(x*s + s^2)) = psi_s(xs2+s3-xsS-s2S+s2) = s^2
print(L[2]); //Biszygies
  ↦ sX+sS-2s-SX-S2,x+s-X-S+1,s2-2sS+S2
// e.g. psi_s((x2-2sS+s-X2+2S2+2X+S-1)(x*s + s^2))
// = psi_s(x3s+x2s2-2xs2S+xs2-2s3S+s3-xsX2+2xsS2+2xsX+xsS-xs-s2X2+2s2S2+2s2X-s2S)
// = x^3s+x^2s^2-2xs^3+xs^2-2s^4+s^3-xsx^2+2xs^3+2xsx+xs^2-xs-s^2x^2+2s^4+2s^2x-s^3
// = 0 in R

```

### 7.5.1.3 liftenvelope

Procedure from library `bimodules.lib` (see [Section 7.5.1 \[bimodules.lib\]](#), page 371).

**Usage:** `liftenvelope(M,g);` M ideal/module, g poly

**Return:** ring, the enveloping algebra of the basering R.

Given a two-sided ideal M in R and a polynomial g in R this procedure returns the enveloping algebra of R. Additionally it exports a list  $l = C, B$ ; where B is the left Groebner basis of the left-syzygies of  $M \otimes 1$  and C is a vector of coefficients in the enveloping algebra of R such that  $\text{psi}_s(C^T * (f_1 \dots f_n)) = g$ .

$\text{psi}_s$  is an epimorphism of left  $R(X) R^{\text{opp}}$  modules:

$\text{psi}_s(s(X)_K t) = \text{smt} := (s_1 m t_1, \dots, s_s m t_s) = (\psi(s_1(X) t_1), \dots, \psi(s_s(X) t_s))$  in  $R^s$ .

Then  $\text{psi}_s(A) := (\text{psi}_s(a_{ij}))$  for every matrix A in  $\text{Mat}(n \times m, R)$ .

**Assume:** The second component has to be an element of the first component.

**Purpose:** This procedure is used for computing total divisors. Let  $\{f_1, \dots, f_n\}$  be the generators of the first component and let the second component be called g. Then the returned list  $l = C, B = (b_1, \dots, b_n)$ ; defines an affine set  $A = C + \sum_i a_i b_i$  with  $(a_1, \dots, a_n)$  in the enveloping algebra of the basering R such that  $\text{psi}_s(a^T * (f_1 \dots f_n)) = g$  for all a in A. For certain rings R, we can find pure tensors within this set A, and if we do, `liftenvelope()` helps us to decide whether f is a total divisor of g.

**Note:** To get list  $l = C, B$ . we set: `def G = liftenvelope(); setring G; l; or l[1]; l[2];`.

**Example:**

```

LIB "bimodules.lib";
ring r = 0,(x,s),dp;
def R = nc_algebra(1,s); setring R;
ideal I = x*s;
poly p = s*x*s*x; // = (s (x) x) * x*s = (sX) * x*s
p;

```

```

↪ x2s2+3xs2+2s2
def J = liftenvelope(I,p);
setring J;
print(l[1]);
↪ 0
//2s+SX = (2s (x) 1) + (1 (x) sx)
print(l[2]);
↪ sX-2s-SX,x-X+1,s2-2sS+S2
// Groebnerbasis of BiSyz(I) as LeftSyz in R^{env}
// We get : 2s+SX + ( sX - 2s -SX) = sX - a pure tensor!!!!

```

### 7.5.1.4 CompDecomp

Procedure from library `bimodules.lib` (see [Section 7.5.1 \[bimodules.lib\]](#), page 371).

**Usage:** `CompDecomp(p)`;  $p$  poly

**Note:** This procedure only works if the basering is an enveloping algebra  $A^{\{env\}}$  of a (non-commutative) ring  $A$ . Thus also the polynomial in the argument has to be in  $A^{\{env\}}$ .

**Return:** Returns an ideal  $I$  in  $A^{\{env\}}$ , where the sum of all terms of the argument with the same right side (of the tensor summands) are stored as a generator of  $I$ .  
Let  $b \neq c$ , then for  $p = (a(X) b) + (c(X) b) + (a(X) c)$  the ideal  $I := \text{CompDecomp}(p)$  is given by:  $I[1] = (a(X) b) + (c(X) b)$ ;  $I[2] = a(X) c$ .

**Purpose:** By decomposing the polynomial we can easily check whether the given polynomial is a pure tensor.

**Example:**

```

LIB "bimodules.lib";
ring r = 0,(x,s),dp;
def R = nc_algebra(1,s); setring R; //1st shift algebra
def Re = envelope(R); setring Re; //basing is now R^{env} = R (X) R^{opp}
poly f = X*S*x^2+5*x*S*X+S*X; f;
↪ x2SX+x2S+5xSX+SX
ideal I = CompDecomp(f);
print(matrix(I)); // what means that f = (x2+5x+1)*SX + x2*S
↪ x2SX+5xSX+SX,x2S
poly p = x*S+X^2*S+2*s+x*X^2*s+5*x*s; p;
↪ xsX2+5xs+xS+2s+SX2+2SX+S
ideal Q = CompDecomp(p);
print(matrix(Q));
↪ xsX2,5xs+2s,xS+S,SX2,2SX

```

### 7.5.1.5 isPureTensor

Procedure from library `bimodules.lib` (see [Section 7.5.1 \[bimodules.lib\]](#), page 371).

**Usage:** `isPureTensor(g)`;  $g$  poly

**Note:** This procedure only works if the basering is an enveloping algebra  $A^{\{env\}}$  of a (non-commutative) ring  $A$ . Thus also the polynomial in the argument has to be in  $A^{\{env\}}$ .

**Return:** Returns 0 if  $g$  is not a pure tensor and if  $g$  is a pure tensor then `isPureTensor()` returns a vector  $v$  with  $v = a*\text{gen}(1)+b*\text{gen}(2) = (a,b)^T$  with  $a(X) b = g$ .

**Purpose:** Checks whether a given polynomial in  $\mathbb{A}^{\text{env}}$  is a pure tensor. This is also an auxiliary procedure for checking total divisibility.

**Example:**

```
LIB "bimodules.lib";
ring r = 0,(x,s),dp;
def R = nc_algebra(1,s); setring R; //1st shift algebra
def Re = envelope(R); setring Re; //basing is now  $R^{\text{env}} = R(X) R^{\text{opp}}$ 
poly p = x*(x*s)*x + s^2*x; p;
 $\mapsto x^3s+x^2s+xs^2+2s^2$ 
// p is of the form  $q(X)1$ , a pure tensor indeed:
isPureTensor(p);
 $\mapsto x^3s*\text{gen}(1)+x^2s*\text{gen}(1)+xs^2*\text{gen}(1)+2s^2*\text{gen}(1)+\text{gen}(2)$ 
// v = transpose(  $x^3s+x^2s+xs^2+2s^2$  1 ) i.e.  $p = x^3s+x^2s+xs^2+2s^2(X) 1$ 
poly g = S*X+ x*s*X+ S^2*x;
g;
 $\mapsto xsX+xS^2+SX$ 
isPureTensor(g); // indeed g is not a pure tensor
 $\mapsto 0$ 
poly d = x*X+s*X+x*S*X+s*S*X;d;
 $\mapsto xSX+xX+sSX+sX$ 
isPureTensor(d); // d is a pure tensor indeed
 $\mapsto x*\text{gen}(1)+s*\text{gen}(1)+SX*\text{gen}(2)+X*\text{gen}(2)$ 
// v = transpose(  $x+s S*X+X$  ) i.e.  $d = x+s(X) s*x+x$ 
// remember that * denotes to the opposite multiplication  $s*x = xs$  in R.
```

### 7.5.1.6 isTwoSidedGB

Procedure from library `bimodules.lib` (see [Section 7.5.1 \[bimodules.lib\]](#), page 371).

**Usage:** `isTwoSidedGB(I);` I ideal

**Return:** Returns 0 if the generators of a given ideal are not two-sided, 1 if they are.

**Note:** This procedure should only be used for non-commutative rings, as every element is two-sided in a commutative ring.

**Purpose:** Auxiliary procedure for diagonal forms. Let  $R$  be a non-commutative ring (e.g.  $G$ -algebra), and  $p$  in  $R$ , this program checks whether  $p$  is two-sided i.e.  $Rp = pR$ .

**Example:**

```
LIB "bimodules.lib";
ring r = 0,(x,s),dp;
def R = nc_algebra(1,s); setring R; //1st shift algebra
ideal I = s^2, x*s, s^2 + 3*x*s;
isTwoSidedGB(I); // I is two-sided
 $\mapsto 1$ 
ideal J = s^2+x;
isTwoSidedGB(J); // J is not two-sided; twostd(J) = s,x;
 $\mapsto 0$ 
```

### 7.5.2 bfun\_lib

**Library:** `bfun.lib`

**Purpose:** Algorithms for b-functions and Bernstein-Sato polynomial

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**Overview:** Given a polynomial ring  $R = K[x_1, \dots, x_n]$  and a polynomial  $F$  in  $R$ , one is interested in the global b-function (also known as Bernstein-Sato polynomial)  $b(s)$  in  $K[s]$ , defined to be the non-zero monic polynomial of minimal degree, satisfying a functional identity  $L * F^{s+1} = b(s) F^s$ , for some operator  $L$  in  $D[s]$  (\* stands for the action of differential operator)

By  $D$  one denotes the  $n$ -th Weyl algebra

$K\langle x_1, \dots, x_n, d_1, \dots, d_n \mid d_j x_i = x_i d_j + \delta_{ij} \rangle$ .

One is interested in the following data:

- Bernstein-Sato polynomial  $b(s)$  in  $K[s]$ ,
- the list of its roots, which are known to be rational
- the multiplicities of the roots.

There is a constructive definition of a b-function of a holonomic ideal  $I$  in  $D$  (that is, an ideal  $I$  in a Weyl algebra  $D$ , such that  $D/I$  is holonomic module) with respect to the given weight vector  $w$ : For a polynomial  $p$  in  $D$ , its initial form w.r.t.  $(-w, w)$  is defined as the sum of all terms of  $p$  which have maximal weighted total degree where the weight of  $x_i$  is  $-w_i$  and the weight of  $d_i$  is  $w_i$ . Let  $J$  be the initial ideal of  $I$  w.r.t.  $(-w, w)$ , i.e. the  $K$ -vector space generated by all initial forms w.r.t.  $(-w, w)$  of elements of  $I$ . Put  $s = w_1 x_1 d_1 + \dots + w_n x_n d_n$ . Then the monic generator  $b_w(s)$  of the intersection of  $J$  with the PID  $K[s]$  is called the b-function of  $I$  w.r.t.  $w$ . Unlike Bernstein-Sato polynomial, general b-function with respect to arbitrary weights need not have rational roots at all. However, b-function of a holonomic ideal is known to be non-zero as well.

**References:**

[SST] Saito, Sturmfels, Takayama: Groebner Deformations of Hypergeometric Differential Equations (2000),  
 Noro: An Efficient Modular Algorithm for Computing the Global b-function, (2002).

**Procedures:** See also: [Section 7.5.4 \[dmod.lib\], page 401](#); [Section 7.5.5 \[dmodapp.lib\], page 421](#); [Section 7.5.7 \[dmodvar.lib\], page 454](#); [Section D.6.13 \[gmssing.lib\], page 1703](#).

### 7.5.2.1 bfct

Procedure from library `bfun.lib` (see [Section 7.5.2 \[bfun.lib\], page 376](#)).

**Usage:** `bfct(f [,s,t,v]);`  $f$  a poly,  $s, t$  optional ints,  $v$  an optional intvec

**Return:** list of ideal and intvec

**Purpose:** computes the roots of the Bernstein-Sato polynomial  $b(s)$  for the hypersurface defined by  $f$ .

**Assume:** The basering is commutative and of characteristic 0.

**Background:**

In this proc, the initial Malgrange ideal is computed according to the algorithm by Masayuki Noro and then a system of linear equations is solved by linear reductions.

**Note:** In the output list, the ideal contains all the roots and the intvec their multiplicities.



If  $s < 0$ , `std` is used for GB computations, otherwise, and by default, `slimgb` is used.  
 If  $t < 0$ , a matrix ordering is used for Groebner basis computations, otherwise, and by default, a block ordering is used.  
 If  $v$  is a positive weight vector,  $v$  is used for homogenization computations, otherwise and by default, no weights are used.

**Display:** If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "bfun.lib";
ring r = 0,(x,y),dp;
poly f = x^2+y^3+x*y^2;
bfct(f);
⇒ [1]:
⇒   _[1]=-5/6
⇒   _[2]=-1
⇒   _[3]=-7/6
⇒ [2]:
⇒   1,1,1
intvec v = 3,2;
bfct(f,1,0,v);
⇒ [1]:
⇒   _[1]=-5/6
⇒   _[2]=-1
⇒   _[3]=-7/6
⇒ [2]:
⇒   1,1,1
```

### 7.5.2.2 bfctSyz

Procedure from library `bfun.lib` (see [Section 7.5.2 \[bfun.lib\]](#), page 376).

**Usage:** `bfctSyz(f [r,s,t,u,v]);`  $f$  poly,  $r,s,t,u$  optional ints,  $v$  opt. intvec

**Return:** list of ideal and intvec

**Purpose:** computes the roots of the Bernstein-Sato polynomial  $b(s)$  for the hypersurface defined by  $f$

**Assume:** The basering is commutative and of characteristic 0.

**Background:**

In this proc, the initial Malgrange ideal is computed according to the algorithm by Masayuki Noro and then a system of linear equations is solved by computing syzygies.

**Note:** In the output list, the ideal contains all the roots and the intvec their multiplicities.

If  $r < 0$ , `std` is used for GB computations in characteristic 0, otherwise, and by default, `slimgb` is used.

If  $s < 0$ , a matrix ordering is used for GB computations, otherwise, and by default, a block ordering is used.

If  $t < 0$ , the computation of the intersection is solely performed over characteristic 0, otherwise and by default, a modular method is used.



If  $u < 0$  and by default, `std` is used for GB computations in characteristic  $> 0$ , otherwise, `slimgb` is used.

If  $v$  is a positive weight vector,  $v$  is used for homogenization computations, otherwise and by default, no weights are used.

**Display:** If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "bfun.lib";
ring r = 0, (x,y), dp;
poly f = x^2+y^3+x*y^2;
bfctSyz(f);
⇒ [1]:
⇒   _[1]=-5/6
⇒   _[2]=-1
⇒   _[3]=-7/6
⇒ [2]:
⇒   1,1,1
intvec v = 3,2;
bfctSyz(f,0,1,1,0,v);
⇒ [1]:
⇒   _[1]=-5/6
⇒   _[2]=-1
⇒   _[3]=-7/6
⇒ [2]:
⇒   1,1,1
```

### 7.5.2.3 bfctAnn

Procedure from library `bfun.lib` (see [Section 7.5.2 \[bfun.lib\]](#), page 376).

**Usage:** `bfctAnn(f [a,b,c]);`  $f$  a poly,  $a$ ,  $b$ ,  $c$  optional ints

**Return:** list of ideal and intvec

**Purpose:** computes the roots of the Bernstein-Sato polynomial  $b(s)$  for the hypersurface defined by  $f$ .

**Assume:** The basering is commutative and of characteristic 0.

**Background:**

In this proc,  $\text{Ann}(f^s)$  is computed and then a system of linear equations is solved by linear reductions.

**Note:** In the output list, the ideal contains all the roots and the intvec their multiplicities.

If  $a < 0$ , only  $f$  is appended to  $\text{Ann}(f^s)$ , otherwise, and by default,  $f$  and all its partial derivatives are appended.

If  $b < 0$ , `std` is used for GB computations, otherwise, and by default, `slimgb` is used.

If  $c < 0$ , `std` is used for Groebner basis computations of ideals  $\langle I+J \rangle$  when  $I$  is already a Groebner basis of  $\langle I \rangle$ .

Otherwise, and by default the engine determined by the switch  $b$  is used.

Note that in the case  $c < 0$ , the choice for  $b$  will be overwritten only for the types of ideals mentioned above.

This means that if  $b < 0$ , specifying  $c$  has no effect.

**Display:** If `printlevel=1`, progress debug messages will be printed,  
if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "bfun.lib";
ring r = 0,(x,y),dp;
poly f = x^2+y^3+x*y^2;
bfctAnn(f);
⇒ [1]:
⇒ _[1]=-5/6
⇒ _[2]=-1
⇒ _[3]=-7/6
⇒ [2]:
⇒ 1,1,1
def R = reiffen(4,5); setring R;
RC; // the Reiffen curve in 4,5
⇒ xy4+y5+x4
bfctAnn(RC,0,1);
⇒ [1]:
⇒ _[1]=-9/20
⇒ _[2]=-11/20
⇒ _[3]=-13/20
⇒ _[4]=-7/10
⇒ _[5]=-17/20
⇒ _[6]=-9/10
⇒ _[7]=-19/20
⇒ _[8]=-1
⇒ _[9]=-21/20
⇒ _[10]=-11/10
⇒ _[11]=-23/20
⇒ _[12]=-13/10
⇒ _[13]=-27/20
⇒ [2]:
⇒ 1,1,1,1,1,1,1,1,1,1,1,1,1
```

### 7.5.2.4 bfctOneGB

Procedure from library `bfun.lib` (see [Section 7.5.2 \[bfun.lib\]](#), page 376).

**Usage:** `bfctOneGB(f [,s,t]);` `f` a poly, `s,t` optional ints

**Return:** list of ideal and intvec

**Purpose:** computes the roots of the Bernstein-Sato polynomial  $b(s)$  for the hypersurface defined by `f`, using only one GB computation

**Assume:** The basering is commutative and of characteristic 0.

**Background:**

In this proc, the initial Malgrange ideal is computed based on the algorithm by Masayuki Noro and combined with an elimination ordering.

**Note:** In the output list, the ideal contains all the roots and the intvec their multiplicities.

If `s<>0`, `std` is used for the GB computation, otherwise, and by default, `slingb` is used.

If  $t < 0$ , a matrix ordering is used for GB computations, otherwise, and by default, a block ordering is used.

**Display:** If `printlevel=1`, progress debug messages will be printed, if `printlevel ≥ 2`, all the debug messages will be printed.

**Example:**

```
LIB "bfun.lib";
ring r = 0, (x,y), dp;
poly f = x^2+y^3+x*y^2;
bfctOneGB(f);
⇒ [1]:
⇒   _[1]=-5/6
⇒   _[2]=-1
⇒   _[3]=-7/6
⇒ [2]:
⇒   1,1,1
bfctOneGB(f,1,1);
⇒ [1]:
⇒   _[1]=-5/6
⇒   _[2]=-1
⇒   _[3]=-7/6
⇒ [2]:
⇒   1,1,1
```

### 7.5.2.5 bfctIdeal

Procedure from library `bfun.lib` (see [Section 7.5.2 \[bfun.lib\]](#), page 376).

**Usage:** `bfctIdeal(I,w[s,t]);` I an ideal, w an intvec, s,t optional ints

**Return:** list of ideal and intvec

**Purpose:** computes the roots of the global b-function of I w.r.t. the weight  $(-w,w)$ .

**Assume:** The basering is the  $n$ -th Weyl algebra in characteristic 0 and for all  $1 ≤ i ≤ n$  the identity  $\text{var}(i+n) \cdot \text{var}(i) = \text{var}(i) \cdot \text{var}(i+1) + 1$  holds, i.e. the sequence of variables is given by  $x(1), \dots, x(n), D(1), \dots, D(n)$ , where  $D(i)$  is the differential operator belonging to  $x(i)$ . Further we assume that I is holonomic.

**Background:**

In this proc, the initial ideal of I is computed according to the algorithm by Masayuki Noro and then a system of linear equations is solved by linear reductions.

**Note:** In the output list, say L,

- L[1] of type ideal contains all the rational roots of a b-function,
- L[2] of type intvec contains the multiplicities of above roots,
- optional L[3] of type string is the part of b-function without rational roots.

Note, that a b-function of degree 0 is encoded via  $L[1][1]=0$ ,  $L[2]=0$  and  $L[3]$  is 1 (for nonzero constant) or 0 (for zero b-function).

If  $s < 0$ , `std` is used for GB computations in characteristic 0, otherwise, and by default, `slimgb` is used.

If  $t < 0$ , a matrix ordering is used for GB computations, otherwise, and by default, a block ordering is used.

**Display:** If `printlevel=1`, progress debug messages will be printed,  
if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "bfun.lib";
ring @D = 0, (x,y,Dx,Dy), dp;
def D = Weyl();
setring D;
ideal I = 3*x^2*Dy+2*y*Dx, 2*x*Dx+3*y*Dy+6; I = std(I);
intvec w1 = 0,1;
intvec w2 = 2,3;
bfctIdeal(I,w1);
⇒ [1]:
⇒   _[1]=0
⇒   _[2]=-2/3
⇒   _[3]=-4/3
⇒ [2]:
⇒   1,1,1
bfctIdeal(I,w2,0,1);
⇒ [1]:
⇒   _[1]=-6
⇒ [2]:
⇒   1
ideal J = I[size(I)]; // J is not holonomic by construction
bfctIdeal(J,w1); // b-function of D/J w.r.t. w1 is non-zero
⇒ WARNING: given ideal is not holonomic
⇒ ... setting bound for degree of b-function to 10 and proceeding
⇒ [1]:
⇒   _[1]=0
⇒   _[2]=-2/3
⇒   _[3]=-4/3
⇒ [2]:
⇒   1,1,1
bfctIdeal(J,w2); // b-function of D/J w.r.t. w2 is zero
⇒ WARNING: given ideal is not holonomic
⇒ ... setting bound for degree of b-function to 10 and proceeding
⇒ // Intersection is zero
⇒ [1]:
⇒   _[1]=0
⇒ [2]:
⇒   0
⇒ [3]:
⇒   0
```

### 7.5.2.6 pIntersect

Procedure from library `bfun.lib` (see [Section 7.5.2 \[bfun.lib\]](#), page 376).

**Usage:** `pIntersect(f, I [,s]);` `f` a poly, `I` an ideal, `s` an optional int

**Return:** vector, coefficient vector of the monic polynomial

**Purpose:** compute the intersection of ideal `I` with the subalgebra  $K[f]$

**Assume:** `I` is given as Groebner basis, basering is not a qring.

**Note:** If the intersection is zero, this proc might not terminate.  
If  $s > 0$  is given, it is searched for the generator of the intersection only up to degree  $s$ . Otherwise (and by default), no bound is assumed.

**Display:** If `printlevel=1`, progress debug messages will be printed,  
if `printlevel ≥ 2`, all the debug messages will be printed.

**Example:**

```
LIB "bfun.lib";
ring r = 0,(x,y),dp;
poly f = x^2+y^3+x*y^2;
def D = initialMalgrange(f);
setring D;
inF;
⇒ inF[1]=x*Dt
⇒ inF[2]=2*x*y*Dx+3*y^2*Dx-y^2*Dy-2*x*Dy
⇒ inF[3]=2*x^2*Dx+x*y*Dx+x*y*Dy+18*t*Dt+9*x*Dx-x*Dy+6*y*Dy+4*x+18
⇒ inF[4]=18*t*Dt^2+6*y*Dt*Dy-y*Dt+27*Dt
⇒ inF[5]=y^2*Dt
⇒ inF[6]=2*t*y*Dt+2*x*y*Dx+2*y^2*Dx-6*t*Dt-3*x*Dx-x*Dy-2*y*Dy+2*y-6
⇒ inF[7]=x*y^2+y^3+x^2
⇒ inF[8]=2*y^3*Dx-2*y^3*Dy-3*y^2*Dx-2*x*y*Dy+y^2*Dy-4*y^2+36*t*Dt+18*x*Dx+1\
2*y*Dy+36
pIntersect(t*Dt,inF);
⇒ gen(4)-1/36*gen(2)
pIntersect(t*Dt,inF,1);
⇒ // Try a bound of at least 2
⇒ 0
```

### 7.5.2.7 pIntersectSyz

Procedure from library `bfun.lib` (see [Section 7.5.2 \[bfun.lib\]](#), page 376).

**Usage:** `pIntersectSyz(f, I [,p,s,t]);`  $f$  poly,  $I$  ideal,  $p, t$  optional ints,  $p$  prime

**Return:** vector, coefficient vector of the monic polynomial

**Purpose:** compute the intersection of an ideal  $I$  with the subalgebra  $K[f]$

**Assume:**  $I$  is given as Groebner basis.

**Note:** If the intersection is zero, this procedure might not terminate.  
If  $p > 0$  is given, this proc computes the generator of the intersection in char  $p$  first and then only searches for a generator of the obtained degree in the basering. Otherwise, it searches for all degrees by computing syzygies.  
If  $s < 0$ , `std` is used for Groebner basis computations in char 0, otherwise, and by default, `slimgb` is used.  
If  $t < 0$  and by default, `std` is used for Groebner basis computations in char  $> 0$ , otherwise, `slimgb` is used.

**Display:** If `printlevel=1`, progress debug messages will be printed,  
if `printlevel ≥ 2`, all the debug messages will be printed.

**Example:**

```

LIB "bfun.lib";
ring r = 0,(x,y),dp;
poly f = x^2+y^3+x*y^2;
def D = initialMalgrange(f);
setring D;
inF;
↳ inF[1]=x*Dt
↳ inF[2]=2*x*y*Dx+3*y^2*Dx-y^2*Dy-2*x*Dy
↳ inF[3]=2*x^2*Dx+x*y*Dx+x*y*Dy+18*t*Dt+9*x*Dx-x*Dy+6*y*Dy+4*x+18
↳ inF[4]=18*t*Dt^2+6*y*Dt*Dy-y*Dt+27*Dt
↳ inF[5]=y^2*Dt
↳ inF[6]=2*t*y*Dt+2*x*y*Dx+2*y^2*Dx-6*t*Dt-3*x*Dx-x*Dy-2*y*Dy+2*y-6
↳ inF[7]=x*y^2+y^3+x^2
↳ inF[8]=2*y^3*Dx-2*y^3*Dy-3*y^2*Dx-2*x*y*Dy+y^2*Dy-4*y^2+36*t*Dt+18*x*Dx+1\
2*y*Dy+36
poly s = t*Dt;
pIntersectSyz(s,inF);
↳ gen(4)-1/36*gen(2)
int p = prime(20000);
pIntersectSyz(s,inF,p,0,0);
↳ gen(4)-1/36*gen(2)

```

### 7.5.2.8 linReduce

Procedure from library `bfun.lib` (see [Section 7.5.2 \[bfun.lib\]](#), page 376).

**Usage:** `linReduce(f, I [,s,t,u])`; `f` a poly, `I` an ideal, `s,t,u` optional ints

**Return:** poly or list, linear reductum (over field) of `f` by elements from `I`

**Purpose:** reduce a polynomial only by linear reductions (no monomial multiplications)

**Note:** If `s<>0`, a list consisting of the reduced polynomial and the coefficient vector of the used reductions is returned, otherwise (and by default) only reduced polynomial is returned.  
 If `t<>0` (and by default) all monomials are reduced (if possible), otherwise, only leading monomials are reduced.  
 If `u<>0` (and by default), the ideal is linearly pre-reduced, i.e. instead of the given ideal, the output of `linReduceIdeal` is used.  
 If `u` is set to 0 and the given ideal does not equal the output of `linReduceIdeal`, the result might not be as expected.

**Display:** If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```

LIB "bfun.lib";
ring r = 0,(x,y),dp;
ideal I = 1,y,xy;
poly f = 5xy+7y+3;
poly g = 7x+5y+3;
linReduce(g,I);      // reduces tails
↳ 7x
linReduce(g,I,0,0); // no reductions of tails
↳ 7x+5y+3

```

```

linReduce(f,I,1); // reduces tails and shows reductions used
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ -5*gen(3)-7*gen(2)-3*gen(1)
f = x3+y2+x2+y+x;
I = x3-y3, y3-x2,x3-y2,x2-y,y2-x;
list l = linReduce(f,I,1);
l;
⇒ [1]:
⇒ 5y
⇒ [2]:
⇒ gen(5)-4*gen(4)+2*gen(3)-3*gen(2)-3*gen(1)
module M = I;
f - (l[1]-(M*l[2])[1,1]);
⇒ 0

```

### 7.5.2.9 linReduceIdeal

Procedure from library `bfun.lib` (see [Section 7.5.2 \[bfun.lib\]](#), page 376).

**Usage:** `linReduceIdeal(I [,s,t,u]);` I an ideal, s,t,u optional ints

**Return:** ideal or list, linear reductum (over field) of I by its elements

**Purpose:** reduces a list of polys only by linear reductions (no monomial multiplications)

**Note:** If `s<>0`, a list consisting of the reduced ideal and the coefficient vectors of the used reductions given as module is returned.  
 Otherwise (and by default), only the reduced ideal is returned.  
 If `t<>0` (and by default) all monomials are reduced (if possible), otherwise, only leading monomials are reduced.  
 If `u<>0` (and by default), the ideal is first sorted in increasing order.  
 If `u` is set to 0 and the given ideal is not sorted in the way described, the result might not be as expected.

**Display:** If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```

LIB "bfun.lib";
ring r = 0,(x,y),dp;
ideal I = 3,x+9,y4+5x,2y4+7x+2;
linReduceIdeal(I); // reduces tails
⇒ _[1]=0
⇒ _[2]=3
⇒ _[3]=x
⇒ _[4]=y4
linReduceIdeal(I,0,0); // no reductions of tails
⇒ _[1]=0
⇒ _[2]=3
⇒ _[3]=x+9
⇒ _[4]=y4+5x
list l = linReduceIdeal(I,1); // reduces tails and shows reductions used

```

```

1;
⇒ [1]:
⇒ _[1]=0
⇒ _[2]=3
⇒ _[3]=x
⇒ _[4]=y4
⇒ [2]:
⇒ _[1]=gen(4)-2*gen(3)+3*gen(2)-29/3*gen(1)
⇒ _[2]=gen(1)
⇒ _[3]=gen(2)-3*gen(1)
⇒ _[4]=gen(3)-5*gen(2)+15*gen(1)
module M = I;
matrix(l[1]) - matrix(M)*matrix(l[2]);
⇒ _[1,1]=0
⇒ _[1,2]=0
⇒ _[1,3]=0
⇒ _[1,4]=0

```

### 7.5.2.10 linSyzSolve

Procedure from library `bfun.lib` (see [Section 7.5.2 \[bfun.lib\]](#), page 376).

- Usage:** `linSyzSolve(I[,s]);` I an ideal, s an optional int
- Return:** vector, coefficient vector of linear combination of 0 in elements of I
- Purpose:** compute a linear dependency between the elements of an ideal if such one exists
- Note:** If `s<>0`, `std` is used for Groebner basis computations, otherwise, `slimgb` is used.  
By default, `slimgb` is used in char 0 and `std` in char >0.
- Display:** If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```

LIB "bfun.lib";
ring r = 0,x,dp;
ideal I = x,2x;
linSyzSolve(I);
⇒ gen(2)-2*gen(1)
ideal J = x,x2;
linSyzSolve(J);
⇒ 0

```

### 7.5.2.11 allPositive

Procedure from library `bfun.lib` (see [Section 7.5.2 \[bfun.lib\]](#), page 376).

- Usage:** `allPositive(v);` v an intvec
- Return:** int, 1 if all components of v are positive, or 0 otherwise
- Purpose:** check whether all components of an intvec are positive
- Example:**



```

LIB "bfun.lib";
intvec v = 1,2,3;
allPositive(v);
 $\mapsto$  1
intvec w = 1,-2,3;
allPositive(w);
 $\mapsto$  0

```

### 7.5.2.12 scalarProd

Procedure from library `bfun.lib` (see [Section 7.5.2 \[bfun.lib\]](#), page 376).

**Usage:** `scalarProd(v,w)`;  $v, w$  intvecs  
**Return:** int, the standard scalar product of  $v$  and  $w$   
**Purpose:** computes the scalar product of two intvecs  
**Assume:** the arguments are of the same size

**Example:**

```

LIB "bfun.lib";
intvec v = 1,2,3;
intvec w = 4,5,6;
scalarProd(v,w);
 $\mapsto$  32

```

### 7.5.2.13 vec2poly

Procedure from library `bfun.lib` (see [Section 7.5.2 \[bfun.lib\]](#), page 376).

**Usage:** `vec2poly(v [,i])`;  $v$  a vector or an intvec,  $i$  an optional int  
**Return:** poly, an univariate polynomial in  $i$ -th variable with coefficients given by  $v$   
**Purpose:** constructs an univariate polynomial in  $K[\text{var}(i)]$  with given coefficients, such that the coefficient at  $\text{var}(i)^{j-1}$  is  $v[j]$ .  
**Note:** The optional argument  $i$  must be positive, by default  $i$  is 1.

**Example:**

```

LIB "bfun.lib";
ring r = 0,(x,y),dp;
vector v = gen(1) + 3*gen(3) + 22/9*gen(4);
intvec iv = 3,2,1;
vec2poly(v,2);
 $\mapsto$  22/9y3+3y2+1
vec2poly(iv);
 $\mapsto$  x2+2x+3

```

## 7.5.3 central.lib

**Library:** `central.lib`  
**Purpose:** Computation of central elements of GR-algebras  
**Author:** Oleksandr Motsak, U@D, where  $U=\{\text{motsak}\}$ ,  $D=\{\text{mathematik.uni-kl.de}\}$   
**Overview:** A library for computing elements of the center and centralizers of sets of elements in GR-algebras.  
**Procedures:**

### 7.5.3.1 centralizeSet

Procedure from library `central.lib` (see [Section 7.5.3 \[central.lib\]](#), page 387).

**Usage:** `centralizeSet( F, V );` F, V ideals

**Input:** F, V finite sets of elements of the base algebra

**Return:** ideal, generated by computed elements

**Purpose:** computes a vector space basis of the centralizer of the set F in the vector space generated by V over the ground field

**Example:**

```
LIB "central.lib";
ring A = 0,(e(1..4)),dp;
matrix D[4][4]=0;
D[2,4] = -e(1);
D[3,4] = -e(2);
// This is A_4_1 - the first real Lie algebra of dimension 4.
def A_4_1 = nc_algebra(1,D); setring A_4_1;
ideal F = variablesSorted(); F;
⇨ F[1]=e(1)
⇨ F[2]=e(4)
⇨ F[3]=e(3)
⇨ F[4]=e(2)
// the center of A_4_1 is generated by
// e(1) and -1/2*e(2)^2+e(1)*e(3)
// therefore one may consider computing it in the following way:
// 1. Compute a PBW basis consisting of
//    monomials with exponent <= (1,2,1,0)
ideal V = PBW_maxMonom( e(1) * e(2)^2 * e(3) );
// 2. Compute the centralizer of F within the vector space
//    spanned by these monomials:
ideal C = centralizeSet( F, V ); C;
⇨ C[1]=e(1)
⇨ C[2]=e(2)^2-2*e(1)*e(3)
inCenter(C); // check the result
⇨ 1
```

See also: [Section 7.5.3.7 \[centralizer\]](#), page 392; [Section 7.5.3.2 \[centralizerVS\]](#), page 388; [Section 7.5.3.11 \[inCentralizer\]](#), page 394.

### 7.5.3.2 centralizerVS

Procedure from library `central.lib` (see [Section 7.5.3 \[central.lib\]](#), page 387).

**Usage:** `centralizerVS( F, D );` F ideal, D int

**Return:** ideal, generated by computed elements

**Purpose:** computes a vector space basis of `centralizer(F)` up to degree D

**Note:** D must be non-negative

**Example:**

```
LIB "central.lib";
ring AA = 0,(x,y,z),dp;
```

```

matrix D[3][3]=0;
D[1,2]=-z; D[1,3]=2*x; D[2,3]=-2*y;
def A = nc_algebra(1,D); setring A; // this algebra is U(sl_2)
ideal F = x, y;
// find generators of the vector space of elements
// of degree <= 4 commuting with x and y:
ideal C = centralizerVS(F, 4);
C;
⇒ C[1]=4xy+z2-2z
⇒ C[2]=16x2y2+8xyz2+z4-32xyz-4z3-4z2+16z
inCentralizer(C, F); // check the result
⇒ 1

```

See also: [Section 7.5.3.4 \[centerVS\], page 389](#); [Section 7.5.3.7 \[centralizer\], page 392](#); [Section 7.5.3.11 \[inCentralizer\], page 394](#).

### 7.5.3.3 centralizerRed

Procedure from library `central.lib` (see [Section 7.5.3 \[central.lib\], page 387](#)).

**Usage:** `centralizerRed( F, D[, N] );` F ideal, D int, N optional int

**Return:** ideal, generated by computed elements

**Purpose:** computes subalgebra generators of `centralizer(F)` up to degree D.

**Note:** In general, one cannot compute the whole `centralizer(F)`.  
Hence, one has to specify a termination condition via arguments D and/or N.  
If D is positive, only centralizing elements up to degree D are computed.  
If D is negative, the termination is determined by N only.  
If N is given, the computation stops if at least N elements have been found.  
Warning: if N is given and bigger than the actual number of generators,  
the procedure may not terminate.  
Current ordering must be a degree compatible well-ordering.

**Example:**

```

LIB "central.lib";
ring AA = 0,(x,y,z),dp;
matrix D[3][3]=0;
D[1,2]=-z; D[1,3]=2*x; D[2,3]=-2*y;
def A = nc_algebra(1,D); setring A; // this algebra is U(sl_2)
ideal F = x, y;
// find subalgebra generators of degree <= 4 of the subalgebra of
// all elements commuting with x and y:
ideal C = centralizerRed(F, 4);
C;
⇒ C[1]=4xy+z2-2z
inCentralizer(C, F); // check the result
⇒ 1

```

See also: [Section 7.5.3.5 \[centerRed\], page 390](#); [Section 7.5.3.7 \[centralizer\], page 392](#); [Section 7.5.3.2 \[centralizerVS\], page 388](#); [Section 7.5.3.11 \[inCentralizer\], page 394](#).

### 7.5.3.4 centerVS

Procedure from library `central.lib` (see [Section 7.5.3 \[central.lib\], page 387](#)).

**Usage:**      `centerVS( D );`  $D$  int  
**Return:**     ideal, generated by computed elements  
**Purpose:**     computes a vector space basis of the center up to degree  $D$   
**Note:**         $D$  must be non-negative

**Example:**

```
LIB "central.lib";
ring AA = 0,(x,y,z),dp;
matrix D[3][3]=0;
D[1,2]=-z; D[1,3]=2*x; D[2,3]=-2*y;
def A = nc_algebra(1,D); setring A; // this algebra is U(sl_2)
// find a basis of the vector space of all
// central elements of degree <= 4:
ideal Z = centerVS(4);
Z;
↪ Z[1]=4xy+z2-2z
↪ Z[2]=16x2y2+8xyz2+z4-32xyz-4z3-4z2+16z
// note that the second element is the square of the first
// plus a multiple of the first:
Z[2] - Z[1]^2 + 8*Z[1];
↪ 0
inCenter(Z); // check the result
↪ 1
```

See also: [Section 7.5.3.6 \[center\]](#), page 391; [Section 7.5.3.2 \[centralizerVS\]](#), page 388; [Section 7.5.3.10 \[inCenter\]](#), page 394.

**7.5.3.5 centerRed**

Procedure from library `central.lib` (see [Section 7.5.3 \[central.lib\]](#), page 387).

**Usage:**      `centerRed( D[, N] );`  $D$  int,  $N$  optional int  
**Return:**     ideal, generated by computed elements  
**Purpose:**     computes subalgebra generators of the center up to degree  $D$   
**Note:**        In general, one cannot compute the whole center.  
                 Hence, one has to specify a termination condition via arguments  $D$  and/or  $N$ .  
                 If  $D$  is positive, only central elements up to degree  $D$  will be found.  
                 If  $D$  is negative, the termination is determined by  $N$  only.  
                 If  $N$  is given, the computation stops if at least  $N$  elements have been found.  
                 Warning: if  $N$  is given and bigger than the actual number of generators,  
                 the procedure may not terminate.  
                 Current ordering must be a degree compatible well-ordering.

**Example:**

```
LIB "central.lib";
ring AA = 0,(x,y,z),dp;
matrix D[3][3]=0;
D[1,2]=z;
def A = nc_algebra(1,D); setring A; // it is a Heisenberg algebra
// find a basis of the vector space of
// central elements of degree <= 3:
```

```

ideal VSZ = centerVS(3);
// There should be 3 degrees of z.
VSZ;
⇨ VSZ[1]=z
⇨ VSZ[2]=z2
⇨ VSZ[3]=z3
inCenter(VSZ); // check the result
⇨ 1
// find "minimal" central elements of degree <= 3
ideal SAZ = centerRed(3);
// Only 'z' must be computed
SAZ;
⇨ SAZ[1]=z
inCenter(SAZ); // check the result
⇨ 1

```

See also: [Section 7.5.3.6 \[center\], page 391](#); [Section 7.5.3.4 \[centerVS\], page 389](#); [Section 7.5.3.3 \[centralizerRed\], page 389](#); [Section 7.5.3.10 \[inCenter\], page 394](#).

### 7.5.3.6 center

Procedure from library `central.lib` (see [Section 7.5.3 \[central.lib\], page 387](#)).

**Usage:** `center(D[, N]);` D int, N optional int

**Return:** ideal, generated by computed elements

**Purpose:** computes subalgebra generators of the center up to degree D

**Note:** In general, one cannot compute the whole center.  
Hence, one has to specify a termination condition via arguments D and/or N.  
If D is positive, only central elements up to degree D will be found.  
If D is negative, the termination is determined by N only.  
If N is given, the computation stops if at least N elements have been found.  
Warning: if N is given and bigger than the actual number of generators,  
the procedure may not terminate.  
Current ordering must be a degree compatible well-ordering.

**Example:**

```

LIB "central.lib";
ring AA = 0,(x,y,z,t),dp;
matrix D[4][4]=0;
D[1,2]=-z; D[1,3]=2*x; D[2,3]=-2*y;
def A = nc_algebra(1,D); setring A; // this algebra is U(sl_2) tensored with K[t]
// find generators of the center of degree <= 3:
ideal Z = center(3);
Z;
⇨ Z[1]=t
⇨ Z[2]=4xy+z2-2z
inCenter(Z); // check the result
⇨ 1
// find at least one generator of the center:
ideal Z2 = center(-1, 1);
Z2;
⇨ Z2[1]=t
inCenter(Z2); // check the result

```

↪ 1

See also: [Section 7.5.3.7 \[centralizer\]](#), page 392; [Section 7.5.3.10 \[inCenter\]](#), page 394.

### 7.5.3.7 centralizer

Procedure from library `central.lib` (see [Section 7.5.3 \[central.lib\]](#), page 387).

**Usage:** `centralizer(F, D[, N]);` F poly/ideal, D int, N optional int

**Return:** ideal, generated by computed elements

**Purpose:** computes subalgebra generators of `centralizer(F)` up to degree D

**Note:** In general, one cannot compute the whole `centralizer(F)`.  
Hence, one has to specify a termination condition via arguments D and/or N.  
If D is positive, only centralizing elements up to degree D will be found.  
If D is negative, the termination is determined by N only.  
If N is given, the computation stops if at least N elements have been found.  
Warning: if N is given and bigger than the actual number of generators, the procedure may not terminate.  
Current ordering must be a degree compatible well-ordering.

**Example:**

```
LIB "central.lib";
ring AA = 0,(x,y,z),dp;
matrix D[3][3]=0;
D[1,2]=-z; D[1,3]=2*x; D[2,3]=-2*y;
def A = nc_algebra(1,D); setring A; // this algebra is U(sl_2)
poly f = 4*x*y+z^2-2*z; // a central polynomial
f;
↪ 4xy+z2-2z
// find generators of the centralizer of f of degree <= 2:
ideal c = centralizer(f, 2);
c; // since f is central, the answer consists of generators of A
↪ c[1]=z
↪ c[2]=y
↪ c[3]=x
inCentralizer(c, f); // check the result
↪ 1
// find at least two generators of the centralizer of f:
ideal cc = centralizer(f,-1,2);
cc;
↪ cc[1]=z
↪ cc[2]=y
↪ cc[3]=x
inCentralizer(cc, f); // check the result
↪ 1
poly g = z^2-2*z; // some non-central polynomial
// find generators of the centralizer of g of degree <= 2:
c = centralizer(g, 2);
c;
↪ c[1]=z
↪ c[2]=xy
inCentralizer(c, g); // check the result
↪ 1
```

```

// find at least one generator of the centralizer of g:
centralizer(g,-1,1);
⇒ _[1]=z
// find at least two generators of the centralizer of g:
cc = centralizer(g,-1,2);
cc;
⇒ cc[1]=z
⇒ cc[2]=xy
inCentralizer(cc, g); // check the result
⇒ 1

```

See also: [Section 7.5.3.6 \[center\]](#), page 391; [Section 7.5.3.11 \[inCentralizer\]](#), page 394.

### 7.5.3.8 sa\_reduce

Procedure from library `central.lib` (see [Section 7.5.3 \[central.lib\]](#), page 387).

**Usage:** `sa_reduce(V);`  $V$  ideal

**Return:** ideal, generated by computed elements

**Purpose:** compute a subalgebra basis of an algebra generated by the elements of  $V$

**Note:** At the moment the usage of this procedure is limited to  $G$ -algebras

**Example:**

```

LIB "central.lib";
ring AA = 0,(x,y,z),dp;
matrix D[3][3]=0;
D[1,2]=-z; D[1,3]=2*x; D[2,3]=-2*y;
def A = nc_algebra(1,D); setring A; // this algebra is U(sl_2)
poly f = 4*x*y+z^2-2*z; // a central polynomial
ideal I = f, f*f, f*f*f - 10*f*f, f+3*z^3; I;
⇒ I[1]=4xy+z2-2z
⇒ I[2]=16x2y2+8xyz2+z4-32xyz-4z3+32xy+4z2
⇒ I[3]=64x3y3+48x2y2z2+12xyz4+z6-288x2y2z-96xyz3-6z5+352x2y2+224xyz2+2z4-12\
8xyz+32z3-64xy-40z2
⇒ I[4]=3z3+4xy+z2-2z
sa_reduce(I); // should be just f and z^3
⇒ _[1]=4xy+z2-2z
⇒ _[2]=z3

```

See also: [Section 7.5.3.9 \[sa\\_poly\\_reduce\]](#), page 393.

### 7.5.3.9 sa\_poly\_reduce

Procedure from library `central.lib` (see [Section 7.5.3 \[central.lib\]](#), page 387).

**Usage:** `sa_poly_reduce(p, V);`  $p$  poly,  $V$  ideal

**Return:** polynomial, a reduction of  $p$  w.r.t.  $V$

**Purpose:** computes a reduction of the polynomial  $p$  w.r.t. the subalgebra generated by elements of  $V$

**Note:** At the moment the usage of this procedure is limited to  $G$ -algebras

**Example:**

```

LIB "central.lib";
ring AA = 0,(x,y,z),dp;
matrix D[3][3]=0;
D[1,2]=-z; D[1,3]=2*x; D[2,3]=-2*y;
def A = nc_algebra(1,D); setring A; // this algebra is U(sl_2)
poly f = 4*x*y+z^2-2*z; // a central polynomial
sa_poly_reduce(f + 3*f*f + x, ideal(f) ); // should be just 'x'
↦ x

```

See also: [Section 7.5.3.8 \[sa\\_reduce\]](#), page 393.

### 7.5.3.10 inCenter

Procedure from library `central.lib` (see [Section 7.5.3 \[central.lib\]](#), page 387).

**Usage:** `inCenter(E)`; E poly/list/ideal

**Return:** integer, 1 if E is in the center, 0 otherwise

**Purpose:** check whether the elements of E are central

**Example:**

```

LIB "central.lib";
ring R=0,(x,y,z),dp;
matrix D[3][3]=0;
D[1,2]=-z;
D[1,3]=2*x;
D[2,3]=-2*y;
def r = nc_algebra(1,D); setring r; // this is U(sl_2)
poly p=4*x*y+z^2-2*z;
inCenter(p);
↦ 1
poly f=4*x*y;
inCenter(f);
↦ 0
list l= list( 1, p, p^2, p^3);
inCenter(l);
↦ 1
ideal I= p, f;
inCenter(I);
↦ 0

```

### 7.5.3.11 inCentralizer

Procedure from library `central.lib` (see [Section 7.5.3 \[central.lib\]](#), page 387).

**Usage:** `inCentralizer(E, S)`; E poly/list/ideal, S poly/ideal

**Return:** integer, 1 if E is in the centralizer(S), 0 otherwise

**Purpose:** check whether the elements of E are in the centralizer(S)

**Example:**

```

LIB "central.lib";
ring R = 0,(x,y,z),dp;
matrix D[3][3]=0;
D[1,2]=-z;

```



```

def r = nc_algebra(1,D); setring r; // the Heisenberg algebra
poly f = x^2;
poly a = z; // 'z' is central => it lies in every centralizer!
poly b = y^2;
inCentralizer(a, f);
↳ 1
inCentralizer(b, f);
↳ 0
list l = list(1, a);
inCentralizer(l, f);
↳ 1
ideal I = a, b;
inCentralizer(I, f);
↳ 0
printlevel = 2;
inCentralizer(a, f); // yes
↳ 1
inCentralizer(b, f); // no
↳ [1]:
↳ POLY: y2 is NOT in the centralizer of polynomial {x2}
↳ 0

```

### 7.5.3.12 isCartan

Procedure from library `central.lib` (see [Section 7.5.3 \[central.lib\]](#), page 387).

**Usage:** `isCartan(f);`  $f$  poly

**Purpose:** check whether  $f$  is a Cartan element.

**Return:** integer, 1 if  $f$  is a Cartan element and 0 otherwise.

**Note:**  $f$  is a Cartan element of the algebra  $A$   
 if and only if for all  $g$  in  $A$  there exists  $C$  in  $K$  such that  $[f, g] = C * g$   
 if and only if for all variables  $v_i$  there exist  $C$  in  $K$  such that  $[f, v_i] = C * v_i$ .

**Example:**

```

LIB "central.lib";
ring R=0,(x,y,z),dp;
matrix D[3][3]=0;
D[1,2]=-z;
D[1,3]=2*x;
D[2,3]=-2*y;
def r = nc_algebra(1,D); setring r; // this is U(sl_2) with cartan - z
isCartan(z); // yes!
↳ 1
poly p=4*x*y+z^2-2*z;
isCartan(p); // central elements are Cartan elements!
↳ 1
poly f=4*x*y;
isCartan(f); // no way!
↳ 0
isCartan( 10 + p + z ); // scalar + central + cartan
↳ 1

```

### 7.5.3.13 applyAdF

Procedure from library `central.lib` (see [Section 7.5.3 \[central.lib\]](#), page 387).

**Usage:**      `applyAdF(B, f);` B ideal, f poly

**Purpose:**      Apply `Ad.f` to every element of B

**Return:**      ideal, generated by `Ad.f(B[i])`,  $1 \leq i \leq \text{size}(B)$

**Note:**         $\text{Ad.f}(v) := [f, v] = f*v - v*f$

**Example:**

```
LIB "central.lib";
ring AA = 0,(e,f,h),dp;
matrix D[3][3]=0;
D[1,2]=-h; D[1,3]=2*e; D[2,3]=-2*f;
def A = nc_algebra(1,D); setring A; // this algebra is U(sl_2)
// Let us consider the linear map Ad_{e} from A_2 into A.
// Compute the PBW basis of A_2:
ideal Basis = PBW_maxDeg( 2 ); Basis;
⇨ Basis[1]=e
⇨ Basis[2]=f
⇨ Basis[3]=h
⇨ Basis[4]=h^2
⇨ Basis[5]=fh
⇨ Basis[6]=f^2
⇨ Basis[7]=eh
⇨ Basis[8]=ef
⇨ Basis[9]=e^2
// Compute images of basis elements under the linear map Ad_e:
ideal Image = applyAdF( Basis, e ); Image;
⇨ Image[1]=0
⇨ Image[2]=h
⇨ Image[3]=-2e
⇨ Image[4]=-4eh-4e
⇨ Image[5]=-2ef+h^2+2h
⇨ Image[6]=2fh-2f
⇨ Image[7]=-2e^2
⇨ Image[8]=eh
⇨ Image[9]=0
// Now we have a linear map given by: Basis_i --> Image_i
// Let's compute its kernel K:
// 1. compute syzygy module C:
module C = linearMapKernel( Image ); C;
⇨ C[1]=gen(1)
⇨ C[2]=gen(8)+1/4*gen(4)-1/2*gen(3)
⇨ C[3]=gen(9)
// 2. compute corresponding combinations of basis vectors:
ideal K = linearCombinations(Basis, C); K;
⇨ K[1]=e
⇨ K[2]=ef+1/4h^2-1/2h
⇨ K[3]=e^2
// Let's check that Ad_e(K) is zero:
applyAdF( K, e );
⇨ _[1]=0
```

$\mapsto \_ [2]=0$   
 $\mapsto \_ [3]=0$

See also: [Section 7.5.3.14 \[linearMapKernel\]](#), page 397.

### 7.5.3.14 linearMapKernel

Procedure from library `central.lib` (see [Section 7.5.3 \[central.lib\]](#), page 387).

**Usage:** `linearMapKernel( Images );` Images ideal

**Purpose:** Computes the syzygy module of the linear map given by Images.

**Return:** syzygy module, or `int(0)` if all images are zeroes

**Example:**

```

LIB "central.lib";
ring AA = 0,(e,f,h),dp;
matrix D[3][3]=0;
D[1,2]=-h; D[1,3]=2*e; D[2,3]=-2*f;
def A = nc_algebra(1,D); // this algebra is U(sl_2)
setring A;
// Let us consider the linear map Ad_{e} from A_2 into A.
// Compute the PBW basis of A_2:
ideal Basis = PBW_maxDeg( 2 ); Basis;
 $\mapsto$  Basis[1]=e
 $\mapsto$  Basis[2]=f
 $\mapsto$  Basis[3]=h
 $\mapsto$  Basis[4]=h2
 $\mapsto$  Basis[5]=fh
 $\mapsto$  Basis[6]=f2
 $\mapsto$  Basis[7]=eh
 $\mapsto$  Basis[8]=ef
 $\mapsto$  Basis[9]=e2
// Compute images of basis elements under the linear map Ad_e:
ideal Image = applyAdF( Basis, e ); Image;
 $\mapsto$  Image[1]=0
 $\mapsto$  Image[2]=h
 $\mapsto$  Image[3]=-2e
 $\mapsto$  Image[4]=-4eh-4e
 $\mapsto$  Image[5]=-2ef+h2+2h
 $\mapsto$  Image[6]=2fh-2f
 $\mapsto$  Image[7]=-2e2
 $\mapsto$  Image[8]=eh
 $\mapsto$  Image[9]=0
// Now we have a linear map given by: Basis_i --> Image_i
// Let's compute its kernel K:
// 1. compute syzygy module C:
module C = linearMapKernel( Image ); C;
 $\mapsto$  C[1]=gen(1)
 $\mapsto$  C[2]=gen(8)+1/4*gen(4)-1/2*gen(3)
 $\mapsto$  C[3]=gen(9)
// 2. compute corresponding combinations of basis vectors:
ideal K = linearCombinations(Basis, C); K;
 $\mapsto$  K[1]=e
 $\mapsto$  K[2]=ef+1/4h2-1/2h

```

```

↳ K[3]=e2
// Let's check that Ad_e(K) is zero:
ideal Z = applyAdF( K, e ); Z;
↳ Z[1]=0
↳ Z[2]=0
↳ Z[3]=0
// Now linearMapKernel will return a single integer 0:
def CC = linearMapKernel(Z); typeof(CC); CC;
↳ int
↳ 0

```

See also: [Section 7.5.3.13 \[applyAdF\], page 396](#); [Section 7.5.3.14 \[linearMapKernel\], page 397](#).

### 7.5.3.15 linearCombinations

Procedure from library `central.lib` (see [Section 7.5.3 \[central.lib\], page 387](#)).

**Usage:** `linearCombinations( Basis, C );` Basis ideal, C module

**Purpose:** forms linear combinations of elements from Basis by replacing `gen(i)` by `Basis[i]` in C

**Return:** ideal generated by computed linear combinations

**Example:**

```

LIB "central.lib";
ring AA = 0,(e,f,h),dp;
matrix D[3][3]=0;
D[1,2]=-h; D[1,3]=2*e; D[2,3]=-2*f;
def A = nc_algebra(1,D); setring A; // this algebra is U(sl_2)
// Let us consider the linear map Ad_{e} from A_2 into A.
// Compute the PBW basis of A_2:
ideal Basis = PBW_maxDeg( 2 ); Basis;
↳ Basis[1]=e
↳ Basis[2]=f
↳ Basis[3]=h
↳ Basis[4]=h2
↳ Basis[5]=fh
↳ Basis[6]=f2
↳ Basis[7]=eh
↳ Basis[8]=ef
↳ Basis[9]=e2
// Compute images of basis elements under the linear map Ad_e:
ideal Image = applyAdF( Basis, e ); Image;
↳ Image[1]=0
↳ Image[2]=h
↳ Image[3]=-2e
↳ Image[4]=-4eh-4e
↳ Image[5]=-2ef+h2+2h
↳ Image[6]=2fh-2f
↳ Image[7]=-2e2
↳ Image[8]=eh
↳ Image[9]=0
// Now we have a linear map given by: Basis_i --> Image_i
// Let's compute its kernel K:
// 1. compute syzygy module C:
module C = linearMapKernel( Image ); C;

```

```

    ↪ C[1]=gen(1)
    ↪ C[2]=gen(8)+1/4*gen(4)-1/2*gen(3)
    ↪ C[3]=gen(9)
    // 2. compute corresponding combinations of basis vectors:
    ideal K = linearCombinations(Basis, C); K;
    ↪ K[1]=e
    ↪ K[2]=ef+1/4h2-1/2h
    ↪ K[3]=e2
    // Let's check that Ad_e(K) is zero:
    applyAdF( K, e );
    ↪ _[1]=0
    ↪ _[2]=0
    ↪ _[3]=0

```

See also: [Section 7.5.3.13 \[applyAdF\], page 396](#); [Section 7.5.3.14 \[linearMapKernel\], page 397](#).

### 7.5.3.16 variablesStandard

Procedure from library `central.lib` (see [Section 7.5.3 \[central.lib\], page 387](#)).

**Usage:**      `variablesStandard();`

**Return:**     ideal, generated by algebra variables

**Purpose:**     computes the set of algebra variables taken in their natural order

**Example:**

```

LIB "central.lib";
ring AA = 0,(x,y,z),dp;
matrix D[3][3]=0;
D[1,2]=-z; D[1,3]=2*x; D[2,3]=-2*y;
def A = nc_algebra(1,D); setring A; // this algebra is U(sl_2)
// Variables in their natural order:
variablesStandard();
↪ _[1]=x
↪ _[2]=y
↪ _[3]=z

```

See also: [Section 7.5.3.17 \[variablesSorted\], page 399](#).

### 7.5.3.17 variablesSorted

Procedure from library `central.lib` (see [Section 7.5.3 \[central.lib\], page 387](#)).

**Usage:**      `variablesSorted();`

**Return:**     ideal, generated by sorted algebra variables

**Purpose:**     computes the set of algebra variables sorted so that  
Cartan variables go first

**Note:**       This is a heuristics for the computation of the center:  
it is better to compute centralizers of Cartan variables first since in this  
case we can omit solving the system of equations.

**Example:**

```

LIB "central.lib";
ring AA = 0,(x,y,z),dp;
matrix D[3][3]=0;
D[1,2]=-z; D[1,3]=2*x; D[2,3]=-2*y;
def A = nc_algebra(1,D); setring A; // this algebra is U(sl_2)
// There is only one Cartan variable - z in U(sl_2),
// it must go 1st:
variablesSorted();
⇒ _[1]=z
⇒ _[2]=y
⇒ _[3]=x

```

See also: [Section 7.5.3.16 \[variablesStandard\]](#), page 399.

### 7.5.3.18 PBW\_eqDeg

Procedure from library `central.lib` (see [Section 7.5.3 \[central.lib\]](#), page 387).

**Usage:** PBW\_eqDeg(Deg); Deg int

**Purpose:** Compute PBW elements of a given degree.

**Return:** ideal consisting of found elements.

**Note:** Unit is omitted. Weights are ignored!

**Example:**

```

LIB "central.lib";
ring AA = 0,(e,f,h),dp;
matrix D[3][3]=0;
D[1,2]=-h; D[1,3]=2*e; D[2,3]=-2*f;
def A = nc_algebra(1,D); setring A; // this algebra is U(sl_2)
// PBW Basis of A_2 \ A_1 - monomials of degree == 2:
PBW_eqDeg( 2 );
⇒ _[1]=h2
⇒ _[2]=fh
⇒ _[3]=f2
⇒ _[4]=eh
⇒ _[5]=ef
⇒ _[6]=e2

```

### 7.5.3.19 PBW\_maxDeg

Procedure from library `central.lib` (see [Section 7.5.3 \[central.lib\]](#), page 387).

**Usage:** PBW\_maxDeg(MaxDeg); MaxDeg int

**Purpose:** Compute PBW elements up to a given maximal degree.

**Return:** ideal consisting of found elements.

**Note:** unit is omitted. Weights are ignored!

**Example:**

```

LIB "central.lib";
ring AA = 0,(e,f,h),dp;
matrix D[3][3]=0;
D[1,2]=-h; D[1,3]=2*e; D[2,3]=-2*f;

```

```

def A = nc_algebra(1,D); // this algebra is U(sl_2)
setring A;
// PBW Basis of A_2 - monomials of degree <= 2, without unit:
PBW_maxDeg( 2 );
↪ _[1]=e
↪ _[2]=f
↪ _[3]=h
↪ _[4]=h2
↪ _[5]=fh
↪ _[6]=f2
↪ _[7]=eh
↪ _[8]=ef
↪ _[9]=e2

```

### 7.5.3.20 PBW\_maxMonom

Procedure from library `central.lib` (see [Section 7.5.3 \[central.lib\]](#), page 387).

**Usage:** PBW\_maxMonom(m); m poly

**Purpose:** Compute PBW elements up to a given maximal one.

**Return:** ideal consisting of found elements.

**Note:** Unit is omitted. Weights are ignored!

**Example:**

```

LIB "central.lib";
ring AA = 0,(e,f,h),dp;
matrix D[3][3]=0;
D[1,2]=-h; D[1,3]=2*e; D[2,3]=-2*f;
def A = nc_algebra(1,D); // this algebra is U(sl_2)
setring A;
// At most 1st degree in e, h and at most 2nd degree in f, unit is omitted:
PBW_maxMonom( e*(f^2)* h );
↪ _[1]=e
↪ _[2]=f
↪ _[3]=ef
↪ _[4]=f2
↪ _[5]=ef2
↪ _[6]=h
↪ _[7]=eh
↪ _[8]=fh
↪ _[9]=efh
↪ _[10]=f2h
↪ _[11]=ef2h

```

## 7.5.4 dmod\_lib

**Library:** dmod.lib

**Purpose:** Algorithms for algebraic D-modules

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**Overview:** Let  $K$  be a field of characteristic 0. Given a polynomial ring  $R = K[x_1, \dots, x_n]$  and a polynomial  $F$  in  $R$ , one is interested in the  $R[1/F]$ -module of rank one, generated by the symbol  $F^s$  for a symbolic discrete variable  $s$ . In fact, the module  $R[1/F]^*F^s$  has a structure of a  $D(R)[s]$ -module, where  $D(R)$  is an  $n$ -th Weyl algebra  $K\langle x_1, \dots, x_n, d_1, \dots, d_n \mid d_j x_j = x_j d_j + 1 \rangle$  and  $D(R)[s] = D(R)$  tensored with  $K[s]$  over  $K$ . Constructively, one needs to find a left ideal  $I = I(F^s)$  in  $D(R)$ , such that  $K[x_1, \dots, x_n, 1/F]^*F^s$  is isomorphic to  $D(R)/I$  as a  $D(R)$ -module. We often write just  $D$  for  $D(R)$  and  $D[s]$  for  $D(R)[s]$ . One is interested in the following data:

- $\text{Ann } F^s = I = I(F^s)$  in  $D(R)[s]$ , denoted by  $LD$  in the output
- global Bernstein polynomial in  $K[s]$ , denoted by  $bs$ ,
- its minimal integer root  $s_0$ , the list of all roots of  $bs$ , which are known to be rational, with their multiplicities, which is denoted by  $BS$
- $\text{Ann } F^{s_0} = I(F^{s_0})$  in  $D(R)$ , denoted by  $LD_0$  in the output ( $LD_0$  is a holonomic ideal in  $D(R)$ )
- $\text{Ann}^{(1)} F^s$  in  $D(R)[s]$ , denoted by  $LD_1$  (logarithmic derivations)
- an operator in  $D(R)[s]$ , denoted by  $PS$ , such that the functional equality  $PS^*F^{(s+1)} = bs^*F^s$  holds in  $K[x_1, \dots, x_n, 1/F]^*F^s$ .

#### References:

We provide the following implementations of algorithms:

- (OT) the classical  $\text{Ann } F^s$  algorithm from Oaku and Takayama (Journal of Pure and Applied Math., 1999),
- (LOT) Levandovskyy's modification of the Oaku-Takayama algorithm (ISSAC 2007)
- (BM) the  $\text{Ann } F^s$  algorithm by Briancon and Maisonobe (Remarques sur l'ideal de Bernstein associe a des polynomes, preprint, 2002)
- (LM08) V. Levandovskyy and J. Martin-Morales, ISSAC 2008
- (C) Countinho, A Primer of Algebraic D-Modules,
- (SST) Saito, Sturmfels, Takayama 'Groebner Deformations of Hypergeometric Differential Equations', Springer, 2000

Guide:

- $\text{Ann } F^s = I(F^s) = LD$  in  $D(R)[s]$  can be computed by `Sannfs` [BM, OT, LOT]
- $\text{Ann}^{(1)} F^s$  in  $D(R)[s]$  can be computed by `Sannfslog`
- global Bernstein polynomial  $bs$  in  $K[s]$  can be computed by `bernsteinBM`
- $\text{Ann } F^{s_0} = I(F^{s_0}) = LD_0$  in  $D(R)$  can be computed by `annfs0`, `annfs`, `annfsBM`, `annfsOT`, `annfsLOT`, `annfs2`, `annfsRB` etc.
- all the relevant data to  $F^s$  ( $LD$ ,  $LD_0$ ,  $bs$ ,  $PS$ ) are computed by `operatorBM`
- operator  $PS$  can be computed via `operatorModulo` or `operatorBM`
- annihilator of  $F^{\{s_1\}}$  for a number  $s_1$  is computed with `annfspecial`
- annihilator of  $F_1^{s_1} * \dots * F_p^{s_p}$  is computed with `annfsBMI`
- computing the multiplicity of a rational number  $r$  in the Bernstein poly of a given ideal goes with `checkRoot`
- check, whether a given univariate polynomial divides the Bernstein poly goes with `checkFactor`

**Procedures:** See also: [Section 7.5.2 \[bfun\\_lib\], page 376](#); [Section 7.5.5 \[dmodapp\\_lib\], page 421](#); [Section 7.5.7 \[dmodvar\\_lib\], page 454](#); [Section D.6.13 \[gmssing\\_lib\], page 1703](#).



### 7.5.4.1 annfs

Procedure from library `dmod.lib` (see [Section 7.5.4 \[dmod.lib\], page 401](#)).

**Usage:** `annfs(f [,S,eng]);`  $f$  a poly,  $S$  a string, `eng` an optional int

**Return:** ring

**Purpose:** compute the D-module structure of `basing[1/f]*f^s` with the algorithm given in  $S$  and with the Groebner basis engine given in "eng"

**Note:** activate the output ring with the `setring` command.

String  $S$ ;  $S$  can be one of the following:

'bm' (default) - for the algorithm of Briancon and Maisonobe,

'ot' - for the algorithm of Oaku and Takayama,

'lot' - for the Levandovskyy's modification of the algorithm of OT.

If `eng <> 0`, `std` is used for Groebner basis computations, otherwise and by default `slimgb` is used.

In the output ring:

- the ideal `LD` (which is a Groebner basis) is the needed D-module structure,
- the list `BS` contains roots and multiplicities of a BS-polynomial of  $f$ .

**Display:** If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "dmod.lib";
ring r = 0,(x,y,z),Dp;
poly F = z*x^2+y^3;
def A = annfs(F); // here, the default BM algorithm will be used
setring A; // the Weyl algebra in (x,y,z,Dx,Dy,Dz)
LD; //the annihilator of F^{-1} over A
⇨ LD[1]=y*Dy+3*z*Dz+3
⇨ LD[2]=x*Dx-2*z*Dz
⇨ LD[3]=x^2*Dy-3*y^2*Dz
⇨ LD[4]=3*y^2*Dx-2*x*z*Dy
⇨ LD[5]=y^3*Dz+x^2*z*Dz+x^2
⇨ LD[6]=2*x*z*Dy^2+9*y*z*Dx*Dz+3*y*Dx
⇨ LD[7]=9*y*z*Dx^2*Dz+4*z^2*Dy^2*Dz+3*y*Dx^2+2*z*Dy^2
⇨ LD[8]=4*z^2*Dy^3*Dz-27*z^2*Dx^2*Dz^2+2*z*Dy^3-54*z*Dx^2*Dz-6*Dx^2
BS; // roots with multiplicities of BS polynomial
⇨ [1]:
⇨ _[1]=-1
⇨ _[2]=-4/3
⇨ _[3]=-5/3
⇨ _[4]=-5/6
⇨ _[5]=-7/6
⇨ [2]:
⇨ 1,1,1,1,1
```

### 7.5.4.2 annfspecial

Procedure from library `dmod.lib` (see [Section 7.5.4 \[dmod.lib\], page 401](#)).

**Usage:** `annfspecial(F,n);`  $F$  a poly, number  $n$

**Return:** ring

- Purpose:** compute the annihilator ideal of  $F^n$  in the Weyl Algebra for the given rational number  $n$
- Assume:** basering is commutative, the number  $n$  is rational.
- Note:** Activate the output ring by `setring` command. In the ring the ideal called `annfalpha` is exported.  
We compute the real annihilator for any rational value of  $n$  (both generic and exceptional). The implementation fixes a bug in the Algorithm 5.3.15 from Saito-Sturmfels-Takayama.
- Display:** If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "dmod.lib";
ring r = 0,(x,y),dp;
poly F = x3-y2;
bernsteinBM(F); // the roots of Bernstein-Sato poly: -7/6, -1, -5/6
⇒ [1]:
⇒   _[1]=-1
⇒   _[2]=-5/6
⇒   _[3]=-7/6
⇒ [2]:
⇒   1,1,1
// *** first example: generic root
def A = annfspecial(F,-5/6);
setring A; print(annfalpha); kill A; setring r;
⇒ 2*x*Dx+3*y*Dy+5,
⇒ 3*x^2*Dy+2*y*Dx,
⇒ 9*x*y*Dy^2-4*y*Dx^2+12*x*Dy,
⇒ 27*y^2*Dy^3+8*y*Dx^3+117*y*Dy^2+72*Dy
// *** second example: exceptional root since its distance to -1 is integer 2
def A = annfspecial(F,1);
setring A; print(annfalpha); kill A; setring r;
⇒ Dx*Dy,
⇒ 2*x*Dx+3*y*Dy-6,
⇒ Dy^3,
⇒ y*Dy^2-Dy,
⇒ 3*x*Dy^2+Dx^2,
⇒ 3*x^2*Dy+2*y*Dx,
⇒ Dx^3+3*Dy^2,
⇒ y*Dx^2+3*x*Dy
// *** third example: exceptional root since its distance to -5/6 is integer 1
def A = annfspecial(F,1/6);
setring A; print(annfalpha); kill A;
⇒ 2*x*Dx+3*y*Dy-1,
⇒ 3*x^2*Dy+2*y*Dx,
⇒ 27*y*Dy^3+8*Dx^3+9*Dy^2,
⇒ 9*x*y*Dy^2-4*y*Dx^2-6*x*Dy
```

**7.5.4.3 annfspecialOld**

Procedure from library `dmod.lib` (see [Section 7.5.4 \[dmod.lib\]](#), page 401).

- Usage:** `annfspecialOld(I,F,mir,n)`; I an ideal, F a poly, int mir, number n
- Return:** ideal
- Purpose:** compute the annihilator ideal of  $F^n$  in the Weyl Algebra for the given rational number n
- Assume:** The basering is  $D[s]$  and contains 's' explicitly as a variable, the ideal I is the  $\text{Ann } F^n$  in  $D[s]$  (obtained with e.g. `SannfsBM`), the integer 'mir' is the minimal integer root of the BS polynomial of F, and the number n is rational.
- Note:** We compute the real annihilator for any rational value of n (both generic and exceptional). The implementation goes along the lines of the Algorithm 5.3.15 from Saito-Sturmfels-Takayama, which has a bug. This procedure is correct for integer values of n.
- Display:** If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "dmod.lib";
ring r = 0,(x,y),dp;
poly F = x3-y2;
def B = annfs(F); setring B;
minIntRoot(BS[1],0);
⇨ -1
// So, the minimal integer root is -1
setring r;
def A = SannfsBM(F);
setring A;
poly F = x3-y2;
annfspecialOld(LD,F,-1,3/4); // generic root
⇨ _[1]=4*x*Dx+6*y*Dy-9
⇨ _[2]=3*x^2*Dy+2*y*Dx
⇨ _[3]=18*x*y*Dy^2-8*y*Dx^2-33*x*Dy
⇨ _[4]=54*y^2*Dy^3+16*y*Dx^3+66*x*Dx*Dy-9*y*Dy^2+66*Dy
annfspecialOld(LD,F,-1,-2); // integer but still generic root
⇨ _[1]=2*x*Dx+3*y*Dy+12
⇨ _[2]=3*x^2*Dy+2*y*Dx
⇨ _[3]=9*x*y*Dy^2-4*y*Dx^2+33*x*Dy
⇨ _[4]=27*y^2*Dy^3+8*y*Dx^3-66*x*Dx*Dy+144*y*Dy^2-66*Dy
annfspecialOld(LD,F,-1,1); // exceptional integer root
⇨ _[1]=Dx*Dy
⇨ _[2]=2*x*Dx+3*y*Dy-6
⇨ _[3]=Dy^3
⇨ _[4]=y*Dy^2-Dy
⇨ _[5]=3*x*Dy^2+Dx^2
⇨ _[6]=3*x^2*Dy+2*y*Dx
⇨ _[7]=Dx^3+3*Dy^2
⇨ _[8]=y*Dx^2+3*x*Dy
```

**7.5.4.4 Sannfs**

Procedure from library `dmod.lib` (see [Section 7.5.4 \[dmod.lib\]](#), page 401).

- Usage:** `Sannfs(f [,S,eng]);`  $f$  a poly,  $S$  a string,  $eng$  an optional int
- Return:** ring
- Purpose:** compute the D-module structure of `basing[f^s]` with the algorithm given in  $S$  and with the Groebner basis engine given in  $eng$
- Note:** activate the output ring with the `setring` command.  
 The value of a string  $S$  can be  
 'bm' (default) - for the algorithm of Briancon and Maisonobe,  
 'lot' - for the Levandovskyy's modification of the algorithm of OT,  
 'ot' - for the algorithm of Oaku and Takayama.  
 If  $eng < 0$ , `std` is used for Groebner basis computations,  
 otherwise, and by default `slingb` is used.  
 In the output ring:  
 - the ideal  $LD$  is the needed D-module structure.
- Display:** If `printlevel=1`, progress debug messages will be printed,  
 if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "dmod.lib";
ring r = 0,(x,y,z),Dp;
poly F = x^3+y^3+z^3;
printlevel = 0;
def A = Sannfs(F); // here, the default BM algorithm will be used
setring A;
LD;
⇨ LD[1]=z^2*Dy-y^2*Dz
⇨ LD[2]=x*Dx+y*Dy+z*Dz-3*s
⇨ LD[3]=z^2*Dx-x^2*Dz
⇨ LD[4]=y^2*Dx-x^2*Dy
```

**7.5.4.5 Sannfslog**

Procedure from library `dmod.lib` (see [Section 7.5.4 \[dmod.lib\]](#), page 401).

- Usage:** `Sannfslog(f [,eng]);`  $f$  a poly,  $eng$  an optional int
- Return:** ring
- Purpose:** compute the D-module structure of `basing[1/f]*f^s`
- Note:** activate the output ring with the `setring` command.  
 In the output ring  $D[s]$ , the ideal  $LD1$  is generated by the elements in  $\text{Ann } F^s$  in  $D[s]$ , coming from logarithmic derivations.  
 If  $eng < 0$ , `std` is used for Groebner basis computations,  
 otherwise, and by default `slingb` is used.
- Display:** If `printlevel=1`, progress debug messages will be printed,  
 if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "dmod.lib";
ring r = 0,(x,y),Dp;
poly F = x4+y5+x*y4;
printlevel = 0;
```

```

def A = Sannfslog(F);
setring A;
LD1;
⇒ LD1[1]=4*x^2*Dx+5*x*y*Dx+3*x*y*Dy+4*y^2*Dy-16*x*s-20*y*s
⇒ LD1[2]=16*x*y^2*Dx+4*y^3*Dx+12*y^3*Dy-125*x*y*Dx-4*x^2*Dy+5*x*y*Dy-100*y^2*Dy-64*y^2*s+500*y*s

```

#### 7.5.4.6 bernsteinBM

Procedure from library `dmod.lib` (see [Section 7.5.4 \[dmod.lib\], page 401](#)).

- Usage:** `bernsteinBM(f [,eng]);` `f` a poly, `eng` an optional int
- Return:** list (of roots of the Bernstein polynomial `b` and their multiplicities)
- Purpose:** compute the global Bernstein-Sato polynomial for a hypersurface, defined by `f`, according to the algorithm by Briancon and Maisonobe
- Note:** If `eng <> 0`, `std` is used for Groebner basis computations, otherwise, and by default `slimgb` is used.
- Display:** If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```

LIB "dmod.lib";
ring r = 0, (x,y,z,w), Dp;
poly F = x^3+y^3+z^2*w;
printlevel = 0;
bernsteinBM(F);
⇒ [1]:
⇒ _[1]=-1
⇒ _[2]=-2
⇒ _[3]=-3/2
⇒ _[4]=-5/3
⇒ _[5]=-7/3
⇒ _[6]=-7/6
⇒ _[7]=-11/6
⇒ [2]:
⇒ 1,1,1,1,1,1,1

```

#### 7.5.4.7 bernsteinLift

Procedure from library `dmod.lib` (see [Section 7.5.4 \[dmod.lib\], page 401](#)).

- Usage:** `bernsteinLift(I, F [,eng]);` `I` an ideal, `F` a poly, `eng` an optional int
- Return:** list
- Purpose:** compute the (multiple of) Bernstein-Sato polynomial with lift-like method, based on the output of `Sannfs`-like procedure
- Note:** the output list contains the roots with multiplicities of the candidate for being Bernstein-Sato polynomial of `f`.  
If `eng <> 0`, `std` is used for Groebner basis computations, otherwise and by default `slimgb` is used.  
If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```

LIB "dmod.lib";
ring r = 0, (x,y,z), Dp;
poly F = x^3+y^3+z^3;
printlevel = 0;
def A = Sannfs(F);   setring A;
LD;
↳ LD[1]=z^2*Dy-y^2*Dz
↳ LD[2]=x*Dx+y*Dy+z*Dz-3*s
↳ LD[3]=z^2*Dx-x^2*Dz
↳ LD[4]=y^2*Dx-x^2*Dy
poly F = imap(r,F);
list L = bernsteinLift(LD,F); L;
↳ [1]:
↳   _[1]=-2
↳   _[2]=-4/3
↳   _[3]=-5/3
↳   _[4]=-1
↳ [2]:
↳   1,1,1,2
poly bs = fl2poly(L,"s"); bs; // the candidate for Bernstein-Sato polynomial
↳ s^5+7*s^4+173/9*s^3+233/9*s^2+154/9*s+40/9

```

**7.5.4.8 operatorBM**

Procedure from library `dmod.lib` (see [Section 7.5.4 \[dmod.lib\]](#), page 401).

**Usage:**     `operatorBM(f [,eng]);` `f` a poly, `eng` an optional int

**Return:**    ring

**Purpose:**     compute the B-operator and other relevant data for  $\text{Ann } F^s$ ,  
using e.g. algorithm by Briancon and Maisonobe for  $\text{Ann } F^s$  and BS.

**Note:**       activate the output ring with the `setring` command. In the output ring  $D[s]$

- the polynomial `F` is the same as the input,
- the ideal `LD` is the annihilator of  $f^s$  in  $D_n[s]$ ,
- the ideal `LD0` is the needed D-mod structure, where  $LD0 = LD|_{s=s_0}$ ,
- the polynomial `bs` is the global Bernstein polynomial of `f` in the variable `s`,
- the list `BS` contains all the roots with multiplicities of the global Bernstein polynomial of `f`,
- the polynomial `PS` is an operator in  $D_n[s]$  such that  $PS*f^{(s+1)} = bs*f^s$ .

If `eng <> 0`, `std` is used for Groebner basis computations,  
otherwise and by default `slimgb` is used.

**Display:**   If `printlevel=1`, progress debug messages will be printed,  
if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```

LIB "dmod.lib";
ring r = 0, (x,y,z), Dp;
poly F = x^3+y^3+z^3;
printlevel = 0;
def A = operatorBM(F);
setring A;

```

```

F; // the original polynomial itself
⇨ x^3+y^3+z^3
LD; // generic annihilator
⇨ LD[1]=x*Dx+y*Dy+z*Dz-3*s
⇨ LD[2]=z^2*Dy-y^2*Dz
⇨ LD[3]=z^2*Dx-x^2*Dz
⇨ LD[4]=y^2*Dx-x^2*Dy
⇨ LD[5]=x^3*Dz+y^3*Dz+z^3*Dz-3*z^2*s
⇨ LD[6]=x^3*Dy+y^3*Dy+y^2*z*Dz-3*y^2*s
LD0; // annihilator
⇨ LD0[1]=x*Dx+y*Dy+z*Dz+6
⇨ LD0[2]=z^2*Dy-y^2*Dz
⇨ LD0[3]=z^2*Dx-x^2*Dz
⇨ LD0[4]=y^2*Dx-x^2*Dy
⇨ LD0[5]=x^3*Dz+y^3*Dz+z^3*Dz+6*z^2
⇨ LD0[6]=x^3*Dy+y^3*Dy+y^2*z*Dz+6*y^2
bs; // normalized Bernstein poly
⇨ s^5+7*s^4+173/9*s^3+233/9*s^2+154/9*s+40/9
BS; // roots and multiplicities of the Bernstein poly
⇨ [1]:
⇨   _[1]=-2
⇨   _[2]=-4/3
⇨   _[3]=-5/3
⇨   _[4]=-1
⇨ [2]:
⇨   1,1,1,2
PS; // the operator, s.t. PS*F^{s+1} = bs*F^s mod LD
⇨ 2/81*y*z*Dx^3*Dy*Dz-2/81*y*z*Dy^4*Dz-4/81*y^2*Dy^2*Dz^3-2/81*y*z*Dy*Dz^4+\
  2/81*y*Dx^3*Dy*s-2/81*y*Dy^4*s+2/81*z*Dx^3*Dz*s+2/27*z*Dy^3*Dz*s+2/27*y*D\
  y*Dz^3*s-2/81*z*Dz^4*s+2/27*y*Dx^3*Dy-2/27*y*Dy^4+2/27*z*Dx^3*Dz+2/27*z*D\
  y^3*Dz-10/81*y*Dy*Dz^3-2/27*z*Dz^4+1/27*Dx^3*s^2+1/9*Dy^3*s^2+1/9*Dz^3*s^2+\
  2+5/27*Dx^3*s+11/27*Dy^3*s+11/27*Dz^3*s+20/81*Dx^3+8/27*Dy^3+16/81*Dz^3
reduce(PS*F-bs,LD); // check the property of PS
⇨ 0

```

### 7.5.4.9 operatorModulo

Procedure from library `dmod.lib` (see [Section 7.5.4 \[dmod.lib\]](#), page 401).

**Usage:** `operatorModulo(f,I,b)`;  $f$  a poly,  $I$  an ideal,  $b$  a poly

**Return:** poly

**Purpose:** compute the B-operator from the polynomial  $f$ , ideal  $I = \text{Ann } f^s$  and Bernstein-Sato polynomial  $b$  using modulo i.e. kernel of module homomorphism

**Note:** The computations take place in the ring, similar to the one returned by `Sannfs` procedure.  
 Note, that operator is not completely reduced wrt  $\text{Ann } f^{s+1}$ .  
 If `printlevel=1`, progress debug messages will be printed,  
 if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "dmod.lib";
```

```
// LIB "dmod.lib"; option(prot); option(mem);
ring r = 0,(x,y),Dp;
poly F = x^3+y^3+x*y^3;
def A = Sannfs(F); // here we get LD = ann f^s
setring A;
poly F = imap(r,F);
def B = annfs0(LD,F); // to obtain BS polynomial
list BS = imap(B,BS); poly bs = fl2poly(BS,"s");
poly PS = operatorModulo(F,LD,bs);
LD = groebner(LD);
PS = NF(PS,subst(LD,s,s+1)); // reduction modulo Ann s^{s+1}
⇒ // ** _ is no standard basis
size(PS);
⇒ 56
lead(PS);
⇒ -2/243*y^3*Dx*Dy^3
reduce(PS*F-bs,LD); // check the defining property of PS
⇒ 0
```

#### 7.5.4.10 annfsParamBM

Procedure from library `dmod.lib` (see [Section 7.5.4 \[dmod.lib\]](#), page 401).

**Usage:** `annfsParamBM(f [,eng]);` `f` a poly, `eng` an optional int

**Return:** ring

**Purpose:** compute the generic  $\text{Ann } F^s$  and exceptional parametric constellations of a polynomial with parametric coefficients with the BM algorithm

**Note:** activate the output ring with the `setring` command. In this ring,  
 - the ideal `LD` is the D-module structure of  $\text{Ann } F^s$   
 - the ideal `Param` contains special parameters as entries  
 If `eng < 0`, `std` is used for Groebner basis computations, otherwise, and by default `slimgb` is used.

**Display:** If `printlevel=1`, progress debug messages will be printed, if `printlevel ≥ 2`, all the debug messages will be printed.

**Example:**

```
LIB "dmod.lib";
ring r = (0,a,b),(x,y),Dp;
poly F = x^2 - (y-a)*(y-b);
printlevel = 0;
def A = annfsParamBM(F); setring A;
LD;
⇒ LD[1]=2*y*Dx+2*x*Dy+(-a-b)*Dx
⇒ LD[2]=x^2*Dy-y^2*Dy+(a+b)*y*Dy+2*y*s+(-a*b)*Dy+(-a-b)*s
⇒ LD[3]=4*x^2*Dx+4*x*y*Dy+(-2*a-2*b)*x*Dy-8*x*s+(a^2-2*a*b+b^2)*Dx
Param;
⇒ Param[1]=(a-b)
setring r;
poly G = x^2-(y-a)^2; // try the exceptional value b=a of parameters
def B = annfsParamBM(G); setring B;
LD;
```



```

↳ LD[1]=y*Dx+x*Dy+(-a)*Dx
↳ LD[2]=x*Dx+y*Dy+(-a)*Dy-2*s
↳ LD[3]=x^2*Dy-y^2*Dy+(2*a)*y*Dy+2*y*s+(-a^2)*Dy+(-2*a)*s
Param;
↳ Param[1]=0

```

### 7.5.4.11 annfsBMI

Procedure from library `dmod.lib` (see [Section 7.5.4 \[dmod.lib\]](#), page 401).

**Usage:** `annfsBMI(F [,eng,met,us]);`  $F$  an ideal, `eng`, `met`, `us` optional ints

**Return:** ring

**Purpose:** compute two kinds of Bernstein-Sato ideals, associated to  $f = F[1]*..*F[P]$ , with the multivariate algorithm by Briancon and Maisonobe.

**Note:** activate the output ring with the `setring` command. In this ring,  
- the ideal `LD` is the annihilator of  $F[1]^{s_1}*..*F[P]^{s_p}$ ,  
- the list or ideal `BS` is a Bernstein-Sato ideal of a polynomial  $f = F[1]*..*F[P]$ .  
If `eng <> 0`, `std` is used for Groebner basis computations,  
otherwise, and by default `slimgb` is used.  
If `met < 0`, the B-Sigma ideal (cf. Castro and Ucha, 'On the computation of Bernstein-Sato ideals', 2005) is computed.  
If  $0 < \text{met} < P$ , then the ideal `B_P` (cf. the paper) is computed.  
Otherwise, and by default, the ideal `B` (cf. the paper) is computed.  
If `us <> 0`, then syzygies-driven method is used.  
If the output ideal happens to be principal, the list of factors with their multiplicities is returned instead of the ideal.  
If `printlevel=1`, progress debug messages will be printed,  
if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```

LIB "dmod.lib";
ring r = 0,(x,y),Dp;
ideal F = x,y,x+y;
printlevel = 0;
// *1* let us compute the B ideal
def A = annfsBMI(F);   setring A;
LD; // annihilator
↳ LD[1]=x*Dx+y*Dy-s(1)-s(2)-s(3)
↳ LD[2]=x*y*Dy+y^2*Dy-x*s(2)-y*s(2)-y*s(3)
↳ LD[3]=y^2*Dx*Dy-y^2*Dy^2+y*Dy*s(1)-y*Dx*s(2)+2*y*Dy*s(2)-y*Dx*s(3)+y*Dy*s\
(3)-s(1)*s(2)-s(2)^2-s(2)*s(3)-s(2)
BS; // Bernstein-Sato ideal
↳ [1]:
↳   _[1]=s(1)+1
↳   _[2]=s(2)+1
↳   _[3]=s(3)+1
↳   _[4]=s(1)+s(2)+s(3)+2
↳   _[5]=s(1)+s(2)+s(3)+3
↳   _[6]=s(1)+s(2)+s(3)+4
↳ [2]:
↳   1,1,1,1,1,1
// *2* now, let us compute B-Sigma ideal

```

```

setring r;
def Sigma = annfsBMI(F,0,-1); setring Sigma;
print(matrix(lead(LD))); // compact form of leading
↳ x*Dx,x*y*Dy,y^2*Dx*Dy
// monomials from the annihilator
BS; // Bernstein-Sato B-Sigma ideal: it is principal,
↳ [1]:
↳ _[1]=s(1)+s(2)+s(3)+2
↳ [2]:
↳ 1
// so factors and multiplicities are returned
// *3* and now, let us compute B-i ideal
setring r;
def Bi = annfsBMI(F,0,3); // that is F[3]=x+y is taken
setring Bi;
print(matrix(lead(LD))); // compact form of leading
↳ x*Dx,x*y*Dy,y^2*Dx*Dy
// monomials from the annihilator
BS; // the B_3 ideal: it is principal, so factors
↳ [1]:
↳ _[1]=s(3)+1
↳ _[2]=s(1)+s(2)+s(3)+2
↳ [2]:
↳ 1,1
// and multiplicities are returned

```

#### 7.5.4.12 checkRoot

Procedure from library `dmod.lib` (see [Section 7.5.4 \[dmod.lib\]](#), page 401).

**Usage:** `checkRoot(f,alpha [,S,eng]);` poly f, number alpha, string S, int eng

**Return:** int

**Assume:** Basing is a commutative ring, alpha is a positive rational number.

**Purpose:** check whether a negative rational number  $-\alpha$  is a root of the global Bernstein-Sato polynomial of f and compute its multiplicity, with the algorithm given by S and with the Groebner basis engine given by eng.

**Note:** The annihilator of  $f^s$  in  $D[s]$  is needed and hence it is computed with the algorithm by Briancon and Maisonobe. The value of a string S can be 'alg1' (default) - for the algorithm 1 of [LM08] 'alg2' - for the algorithm 2 of [LM08]  
Depending on the value of S, the output of type int is:  
'alg1': 1 only if  $-\alpha$  is a root of the global Bernstein-Sato polynomial  
'alg2': the multiplicity of  $-\alpha$  as a root of the global Bernstein-Sato polynomial of f. If  $-\alpha$  is not a root, the output is 0.  
If `eng <> 0`, `std` is used for Groebner basis computations, otherwise (and by default) `slimgb` is used.

**Display:** If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```

LIB "dmod.lib";
printlevel=0;
ring r = 0,(x,y),Dp;
poly F = x^4+y^5+x*y^4;
checkRoot(F,11/20);    // -11/20 is a root of bf
↪ 1
poly G = x*y;
checkRoot(G,1,"alg2"); // -1 is a root of bg with multiplicity 2
↪ 2

```

### 7.5.4.13 SannfsBFCT

Procedure from library `dmod.lib` (see [Section 7.5.4 \[dmod.lib\]](#), page 401).

**Usage:** SannfsBFCT(f [,a,b,c]); f a poly, a,b,c optional ints

**Return:** ring

**Purpose:** compute a Groebner basis either of  $\text{Ann}(f^s) + \langle f \rangle$  or of  $\text{Ann}(f^s) + \langle f_1, \dots, f_n \rangle$  in  $D[s]$

**Note:** Activate the output ring with the `setring` command.  
This procedure, unlike `SannfsBM`, returns the ring  $D[s]$  with an anti-elimination ordering for  $s$ .

The output ring contains an ideal `LD`, being a Groebner basis either of  $\text{Ann}(f^s) + \langle f \rangle$ , if  $a=0$  (and by default), or of  $\text{Ann}(f^s) + \langle f_1, \dots, f_n \rangle$ , otherwise.

Here,  $f_i$  stands for the  $i$ -th partial derivative of  $f$ .

If  $b < 0$ , `std` is used for Groebner basis computations, otherwise, and by default `slimgb` is used.

If  $c < 0$ , `std` is used for Groebner basis computations of ideals  $\langle I+J \rangle$  when  $I$  is already a Groebner basis of  $\langle I \rangle$ .

Otherwise, and by default the engine determined by the switch  $b$  is used. Note that in the case  $c < 0$ , the choice for  $b$  will be overwritten only for the types of ideals mentioned above.

This means that if  $b < 0$ , specifying  $c$  has no effect.

**Display:** If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```

LIB "dmod.lib";
ring r = 0,(x,y,z,w),Dp;
poly F = x^3+y^3+z^3*w;
// compute Ann(F^s)+<F> using slimgb only
def A = SannfsBFCT(F);
setring A; A;
↪ // coefficients: QQ
↪ // number of vars : 9
↪ //      block  1 : ordering dp
↪ //              : names      s
↪ //      block  2 : ordering dp
↪ //              : names      x y z w Dx Dy Dz Dw
↪ //      block  3 : ordering C
↪ // noncommutative relations:

```

```

⇒ //      Dxx=x*Dx+1
⇒ //      Dyy=y*Dy+1
⇒ //      Dzz=z*Dz+1
⇒ //      Dww=w*Dw+1
LD;
⇒ LD[1]=z*Dz-3*w*Dw
⇒ LD[2]=3*s-x*Dx-y*Dy-3*w*Dw
⇒ LD[3]=y^2*Dx-x^2*Dy
⇒ LD[4]=z^2*w*Dy-y^2*Dz
⇒ LD[5]=z^3*Dy-3*y^2*Dw
⇒ LD[6]=z^2*w*Dx-x^2*Dz
⇒ LD[7]=z^3*Dx-3*x^2*Dw
⇒ LD[8]=z^3*w+x^3+y^3
⇒ LD[9]=x^3*Dy+y^3*Dy+3*y^2*w*Dw+3*y^2
⇒ LD[10]=x^3*Dx+x^2*y*Dy+3*x^2*w*Dw+3*x^2
⇒ LD[11]=3*z*w^2*Dy*Dw-y^2*Dz^2+2*z*w*Dy
⇒ LD[12]=3*z*w^2*Dx*Dw-x^2*Dz^2+2*z*w*Dx
⇒ LD[13]=3*z^2*w^2*Dw+x^3*Dz+y^3*Dz+3*z^2*w
⇒ LD[14]=9*w^3*Dy*Dw^2-y^2*Dz^3+18*w^2*Dy*Dw+2*w*Dy
⇒ LD[15]=9*w^3*Dx*Dw^2-x^2*Dz^3+18*w^2*Dx*Dw+2*w*Dx
⇒ LD[16]=9*z*w^3*Dw^2+x^3*Dz^2+y^3*Dz^2+24*z*w^2*Dw+6*z*w
⇒ LD[17]=27*w^4*Dw^3+x^3*Dz^3+y^3*Dz^3+135*w^3*Dw^2+114*w^2*Dw+6*w
// the Bernstein-Sato poly of F:
vec2poly(pIntersect(s,LD));
⇒ s^6+28/3*s^5+320/9*s^4+1910/27*s^3+2093/27*s^2+1198/27*s+280/27
// a fancier example:
def R = reiffen(4,5); setring R;
RC; // the Reiffen curve in 4,5
⇒ xy4+y5+x4
// compute Ann(RC^s)+<RC,diff(RC,x),diff(RC,y)>
// using std for GB computations of ideals <I+J>
// where I is already a GB of <I>
// and slimgb for other ideals
def B = SannfsBFCT(RC,1,0,1);
setring B;
// the Bernstein-Sato poly of RC:
(s-1)*vec2poly(pIntersect(s,LD));
⇒ s^13+10*s^12+44*s^11+44099/400*s^10+13355001/80000*s^9+22138611/160000*s^8\
8+1747493/160000*s^7-7874303503/64000000*s^6-4244944536107/25600000000*s^5\
5-3066298289417/25600000000*s^4-2787777479229/51200000000*s^3-19980507461\
787/1280000000000*s^2-663659243177931/25600000000000*s-48839201079669/25\
600000000000

```

#### 7.5.4.14 annfs0

Procedure from library `dmod.lib` (see [Section 7.5.4 \[dmod.lib\]](#), page 401).

**Usage:** `annfs0(I, F [,eng]);` I an ideal, F a poly, eng an optional int

**Return:** ring

**Purpose:** compute the annihilator ideal of  $f^s$  in the Weyl Algebra, based on the output of Sannfs-like procedure

**Note:** activate the output ring with the `setring` command. In this ring,  
- the ideal LD (which is a Groebner basis) is the annihilator of  $f^s$ ,

- the list BS contains the roots with multiplicities of BS polynomial of f.
- If eng <>0, **std** is used for Groebner basis computations, otherwise and by default **slimgb** is used.
- If printlevel=1, progress debug messages will be printed, if printlevel>=2, all the debug messages will be printed.

**Example:**

```
LIB "dmod.lib";
ring r = 0,(x,y,z),Dp;
poly F = x^3+y^3+z^3;
printlevel = 0;
def A = SannfsBM(F);  setring A;
// alternatively, one can use SannfsOT or SannfsLOT
LD;
⇨ LD[1]=z^2*Dy-y^2*Dz
⇨ LD[2]=x*Dx+y*Dy+z*Dz-3*s
⇨ LD[3]=z^2*Dx-x^2*Dz
⇨ LD[4]=y^2*Dx-x^2*Dy
poly F = imap(r,F);
def B = annfs0(LD,F);  setring B;
LD;
⇨ LD[1]=x*Dx+y*Dy+z*Dz+6
⇨ LD[2]=z^2*Dy-y^2*Dz
⇨ LD[3]=z^2*Dx-x^2*Dz
⇨ LD[4]=y^2*Dx-x^2*Dy
⇨ LD[5]=x^3*Dz+y^3*Dz+z^3*Dz+6*z^2
⇨ LD[6]=x^3*Dy+y^3*Dy+y^2*z*Dz+6*y^2
BS;
⇨ [1]:
⇨   _[1]=-2
⇨   _[2]=-4/3
⇨   _[3]=-5/3
⇨   _[4]=-1
⇨ [2]:
⇨   1,1,1,2
```

**7.5.4.15 annfs2**

Procedure from library **dmod.lib** (see [Section 7.5.4 \[dmod.lib\]](#), page 401).

**Usage:** annfs2(I, F [,eng]); I an ideal, F a poly, eng an optional int

**Return:** ring

**Purpose:** compute the annihilator ideal of  $f^s$  in the Weyl Algebra, based on the output of Sannfs-like procedure  
annfs2 uses shorter expressions in the variable s (the idea of Noro).

**Note:** activate the output ring with the **setring** command. In this ring,  
- the ideal LD (which is a Groebner basis) is the annihilator of  $f^s$ ,  
- the list BS contains the roots with multiplicities of the BS polynomial.  
If eng <>0, **std** is used for Groebner basis computations, otherwise and by default **slimgb** is used.

**Display:** If printlevel=1, progress debug messages will be printed, if printlevel>=2, all the debug messages will be printed.

**Example:**

```

LIB "dmod.lib";
ring r = 0,(x,y,z),Dp;
poly F = x^3+y^3+z^3;
printlevel = 0;
def A = SannfsBM(F);
setring A;
LD;
↳ LD[1]=z^2*Dy-y^2*Dz
↳ LD[2]=x*Dx+y*Dy+z*Dz-3*s
↳ LD[3]=z^2*Dx-x^2*Dz
↳ LD[4]=y^2*Dx-x^2*Dy
poly F = imap(r,F);
def B = annfs2(LD,F);
setring B;
LD;
↳ LD[1]=x*Dx+y*Dy+z*Dz+6
↳ LD[2]=z^2*Dy-y^2*Dz
↳ LD[3]=z^2*Dx-x^2*Dz
↳ LD[4]=y^2*Dx-x^2*Dy
↳ LD[5]=x^3*Dz+y^3*Dz+z^3*Dz+6*z^2
↳ LD[6]=x^3*Dy+y^3*Dy+y^2*z*Dz+6*y^2
BS;
↳ [1]:
↳   _[1]=-2
↳   _[2]=-5/3
↳   _[3]=-4/3
↳   _[4]=-1
↳ [2]:
↳   1,1,1,2

```

**7.5.4.16 annfsRB**

Procedure from library `dmod.lib` (see [Section 7.5.4 \[dmod.lib\]](#), page 401).

**Usage:** `annfsRB(I, F [,eng]);` I an ideal, F a poly, eng an optional int

**Return:** ring

**Purpose:** compute the annihilator ideal of  $f^s$  in the Weyl Algebra, based on the output of `Sannfs` like procedure

**Note:** activate the output ring with the `setring` command. In this ring,  
 - the ideal LD (which is a Groebner basis) is the annihilator of  $f^s$ ,  
 - the list BS contains the roots with multiplicities of a Bernstein polynomial of  $f$ .  
 If `eng <> 0`, `std` is used for Groebner basis computations, otherwise and by default `slimgb` is used.  
 This procedure uses in addition to F its Jacobian ideal.

**Display:** If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```

LIB "dmod.lib";
ring r = 0,(x,y,z),Dp;

```

```

poly F = x^3+y^3+z^3;
printlevel = 0;
def A = SannfsBM(F); setring A;
LD; // s-parametric annihilator
↳ LD[1]=z^2*Dy-y^2*Dz
↳ LD[2]=x*Dx+y*Dy+z*Dz-3*s
↳ LD[3]=z^2*Dx-x^2*Dz
↳ LD[4]=y^2*Dx-x^2*Dy
poly F = imap(r,F);
def B = annfsRB(LD,F); setring B;
LD;
↳ LD[1]=x*Dx+y*Dy+z*Dz+6
↳ LD[2]=z^2*Dy-y^2*Dz
↳ LD[3]=z^2*Dx-x^2*Dz
↳ LD[4]=y^2*Dx-x^2*Dy
↳ LD[5]=x^3*Dz+y^3*Dz+z^3*Dz+6*z^2
↳ LD[6]=x^3*Dy+y^3*Dy+y^2*z*Dz+6*y^2
BS;
↳ [1]:
↳   _[1]=-2
↳   _[2]=-5/3
↳   _[3]=-4/3
↳   _[4]=-1
↳ [2]:
↳   1,1,1,2

```

#### 7.5.4.17 checkFactor

Procedure from library `dmod.lib` (see [Section 7.5.4 \[dmod.lib\]](#), page 401).

- Usage:** `checkFactor(I,f,qs [,eng]);` I an ideal, f a poly, qs a poly, eng an optional int
- Assume:** `checkFactor` is called from the basering, created by Sannfs-like proc, that is, from the Weyl algebra in  $x_1, \dots, x_N, d_1, \dots, d_N$  tensored with  $K[s]$ . The ideal I is the annihilator of  $f^s$  in  $D[s]$ , that is the ideal, computed by Sannfs-like procedure (usually called LD there). Moreover, f is a polynomial in  $K[x_1, \dots, x_N]$  and qs is a polynomial in  $K[s]$ .
- Return:** int, 1 if qs is a factor of the global Bernstein polynomial of f and 0 otherwise
- Purpose:** check whether a univariate polynomial qs is a factor of the Bernstein-Sato polynomial of f without explicit knowledge of the latter.
- Note:** If `eng <> 0`, `std` is used for Groebner basis computations, otherwise (and by default) `slimgb` is used.
- Display:** If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```

LIB "dmod.lib";
ring r = 0,(x,y),Dp;
poly F = x^4+y^5+x*y^4;
printlevel = 0;
def A = Sannfs(F);
setring A;

```

```

poly F = imap(r,F);
checkFactor(LD,F,20*s+31);      // -31/20 is not a root of bs
↳ 0
checkFactor(LD,F,20*s+11);      // -11/20 is a root of bs
↳ 1
checkFactor(LD,F,(20*s+11)^2); // the multiplicity of -11/20 is 1
↳ 0

```

#### 7.5.4.18 arrange

Procedure from library `dmod.lib` (see [Section 7.5.4 \[dmod.lib\], page 401](#)).

**Usage:**     `arrange(p); int p`

**Return:**    `ring`

**Purpose:**    set up the polynomial, describing a hyperplane arrangement

**Note:**      must be executed in a commutative ring

**Assume:**    `basering` is present and it is commutative

**Example:**

```

LIB "dmod.lib";
ring X = 0,(x,y,z,t),dp;
poly q = arrange(3);
factorize(q,1);
↳ _[1]=x
↳ _[2]=y
↳ _[3]=x+y
↳ _[4]=z
↳ _[5]=x+z
↳ _[6]=y+z
↳ _[7]=x+y+z

```

#### 7.5.4.19 reiffen

Procedure from library `dmod.lib` (see [Section 7.5.4 \[dmod.lib\], page 401](#)).

**Usage:**     `reiffen(p, q); int p, int q`

**Return:**    `ring`

**Purpose:**    set up the polynomial, describing a Reiffen curve

**Note:**      activate the output ring with the `setring` command and  
               find the curve as a polynomial RC.  
               A Reiffen curve is defined as  $RC = x^p + y^q + xy^{\{q-1\}}$ ,  $q \geq p+1 \geq 5$

**Example:**

```

LIB "dmod.lib";
def r = reiffen(4,5);
setring r;
RC;
↳ xy4+y5+x4

```



#### 7.5.4.20 isHolonomic

Procedure from library `dmod.lib` (see [Section 7.5.4 \[dmod.lib\], page 401](#)).

**Usage:** `isHolonomic(M)`;  $M$  an ideal/module/matrix  
**Return:** int, 1 if  $M$  is holonomic over the base ring, and 0 otherwise  
**Assume:** `basering` is a Weyl algebra in characteristic 0  
**Purpose:** check whether  $M$  is holonomic over the base ring  
**Note:**  $M$  is holonomic if  $2 \cdot \dim(M) = \dim(R)$ , where  $R$  is the base ring; `dim` stands for Gelfand-Kirillov dimension

**Example:**

```
LIB "dmod.lib";
ring R = 0,(x,y),dp;
poly F = x*y*(x+y);
def A = annfsBM(F,0);
setring A;
LD;
↳ LD[1]=x*Dx+y*Dy+3
↳ LD[2]=x*y*Dy+y^2*Dy+x+2*y
↳ LD[3]=y^2*Dx*Dy-y^2*Dy^2+2*y*Dx-4*y*Dy-2
isHolonomic(LD);
↳ 1
ideal I = std(LD[1]);
I;
↳ I[1]=x*Dx+y*Dy+3
isHolonomic(I);
↳ 0
```

#### 7.5.4.21 convloc

Procedure from library `dmod.lib` (see [Section 7.5.4 \[dmod.lib\], page 401](#)).

**Usage:** `convloc(L)`;  $L$  a list  
**Return:** list  
**Purpose:** convert a ringlist  $L$  into another ringlist, where all the 'p' orderings are replaced with the 's' orderings, e.g. `dp` by `ds`.  
**Assume:**  $L$  is a result of a ringlist command

**Example:**

```
LIB "dmod.lib";
ring r = 0,(x,y,z),(Dp(2),dp(1));
list L = ringlist(r);
list N = convloc(L);
def rs = ring(N);
setring rs;
rs;
↳ // coefficients: QQ
↳ // number of vars : 3
↳ //          block 1 : ordering Ds
↳ //          : names x y
```

```

⇒ //      block   2 : ordering ds
⇒ //      : names   z
⇒ //      block   3 : ordering C

```

#### 7.5.4.22 minIntRoot

Procedure from library `dmod.lib` (see [Section 7.5.4 \[dmod.lib\]](#), page 401).

**Usage:** `minIntRoot(P, fact)`; P an ideal, fact an int

**Return:** int

**Purpose:** minimal integer root of a maximal ideal P

**Note:** if `fact==1`, P is the result of some 'factorize' call,  
 else P is treated as the result of `bernstein::gmssing.lib`  
 in both cases without constants and multiplicities

**Example:**

```

LIB "dmod.lib";
ring r = 0,(x,y),ds;
poly f1 = x*y*(x+y);
ideal I1 = bernstein(f1)[1]; // a local Bernstein poly
I1;
⇒ I1[1]==-4/3
⇒ I1[2]==-1
⇒ I1[3]==-2/3
minIntRoot(I1,0);
⇒ -1
poly f2 = x2-y3;
ideal I2 = bernstein(f2)[1];
I2;
⇒ I2[1]==-7/6
⇒ I2[2]==-1
⇒ I2[3]==-5/6
minIntRoot(I2,0);
⇒ -1
// now we illustrate the behaviour of factorize
// together with a global ordering
ring r2 = 0,x,dp;
poly f3 = 9*(x+2/3)*(x+1)*(x+4/3); //global b-polynomial of f1=x*y*(x+y)
ideal I3 = factorize(f3,1);
I3;
⇒ I3[1]=x+1
⇒ I3[2]=3x+2
⇒ I3[3]=3x+4
minIntRoot(I3,1);
⇒ -1
// and a more interesting situation
ring s = 0,(x,y,z),ds;
poly f = x3 + y3 + z3;
ideal I = bernstein(f)[1];
I;
⇒ I[1]==-2
⇒ I[2]==-5/3

```

```

 $\mapsto$  I[3]=-4/3
 $\mapsto$  I[4]=-1
minIntRoot(I,0);
 $\mapsto$  -2

```

### 7.5.4.23 isRational

Procedure from library `dmod.lib` (see [Section 7.5.4 \[dmod.lib\]](#), page 401).

**Usage:** `isRational(n);` n number

**Return:** int

**Purpose:** determine whether n is a rational number, that is it does not contain parameters.

**Assume:** ground field is of characteristic 0

**Example:**

```

LIB "dmod.lib";
ring r = (0,a),(x,y),dp;
number n1 = 11/73;
isRational(n1);
 $\mapsto$  1
number n2 = (11*a+3)/72;
isRational(n2);
 $\mapsto$  0

```

## 7.5.5 dmodapp.lib

**Library:** `dmodapp.lib`

**Purpose:** Applications of algebraic D-modules

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Support: DFG Graduiertenkolleg 1632 'Experimentelle und konstruktive Algebra'

**Overview:** Let  $K$  be a field of characteristic 0,  $R = K[x_1, \dots, x_N]$  and  $D$  be the Weyl algebra in variables  $x_1, \dots, x_N, d_1, \dots, d_N$ . In this library there are the following procedures for algebraic D-modules:

- given a cyclic representation  $D/I$  of a holonomic module and a polynomial  $F$  in  $R$ , it is proved that the localization of  $D/I$  with respect to the mult. closed set of all powers of  $F$  is a holonomic D-module. Thus we aim to compute its cyclic representation  $D/L$  for an ideal  $L$  in  $D$ . The procedures for the localization are `DLoc`, `SDLoc` and `DLoc0`.
- annihilator in  $D$  of a given polynomial  $F$  from  $R$  as well as of a given rational function  $G/F$  from  $\text{Quot}(R)$ . These can be computed via procedures `annPoly` resp. `annRat`.
- Groebner bases with respect to weights (according to (SST)), given an arbitrary integer vector containing weights for variables, one computes the homogenization of a given ideal relative to this vector, then one computes a Groebner basis and returns the dehomogenization of the result), initial forms and initial ideals in Weyl algebras with respect to a given weight vector can be computed with `GBWeight`, `inForm`, `initialMalgrange` and `initialIdealW`.
- restriction and integration of a holonomic module  $D/I$ . Suppose  $I$  annihilates a function  $F(x_1, \dots, x_N)$ . Our aim is to compute an ideal  $J$  directly from  $I$ , which annihilates

- $F(0, \dots, 0, x_k, \dots, x_n)$  in case of restriction or
- the integral of  $F$  with respect to  $x_1, \dots, x_m$  in case of integration. The corresponding procedures are `restrictionModule`, `restrictionIdeal`, `integralModule` and `integralIdeal`.
- characteristic varieties defined by ideals in Weyl algebras can be computed with `charVariety` and `charInfo`.
- `appelF1`, `appelF2` and `appelF4` return ideals in parametric Weyl algebras, which annihilate corresponding Appel hypergeometric functions.

**References:**

- (SST) Saito, Sturmfels, Takayama 'Groebner Deformations of Hypergeometric Differential Equations', Springer, 2000  
 (OTW) Oaku, Takayama, Walther 'A Localization Algorithm for D-modules', Journal of Symbolic Computation, 2000  
 (OT) Oaku, Takayama 'Algorithms for D-modules', Journal of Pure and Applied Algebra, 1998

**Procedures:** D-module See also: [Section 7.5.2 \[bfun\\_lib\]](#), page 376; [Section 7.5.4 \[dmod\\_lib\]](#), page 401; [Section 7.5.7 \[dmodvar\\_lib\]](#), page 454; [Section D.6.13 \[gmssing\\_lib\]](#), page 1703.

**7.5.5.1 annPoly**

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp\\_lib\]](#), page 421).

**Usage:** `annPoly(f)`;  $f$  a poly

**Return:** ring (a Weyl algebra) containing an ideal 'LD'

**Purpose:** compute the complete annihilator ideal of  $f$  in the corresponding Weyl algebra

**Assume:** basering is commutative and over a field of characteristic 0

**Note:** Activate the output ring with the `setring` command.  
 In the output ring, the ideal 'LD' (in Groebner basis) is the annihilator.

**Display:** If `printlevel=1`, progress debug messages will be printed,  
 if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "dmodapp.lib";
ring r = 0,(x,y,z),dp;
poly f = x^2*z - y^3;
def A = annPoly(f);
setring A;      // A is the 3rd Weyl algebra in 6 variables
LD;             // the Groebner basis of annihilator
⇨ LD[1]=Dz^2
⇨ LD[2]=Dy*Dz
⇨ LD[3]=Dx*Dy
⇨ LD[4]=y*Dy+3*z*Dz-3
⇨ LD[5]=x*Dx-2*z*Dz
⇨ LD[6]=z*Dx*Dz-Dx
⇨ LD[7]=Dy^3+3*Dx^2*Dz
⇨ LD[8]=x*Dy^2+3*y*Dx*Dz
⇨ LD[9]=x^2*Dy+3*y^2*Dz
⇨ LD[10]=Dx^3
```

```

↳ LD[11]=3*y*Dx^2+z*Dy^2
↳ LD[12]=3*y^2*Dx+2*x*z*Dy
↳ LD[13]=y^3*Dz-x^2*z*Dz+x^2
gkdim(LD); // must be 3 = 6/2, since A/LD is holonomic module
↳ 3
NF(Dy^4, LD); // must be 0 since Dy^4 clearly annihilates f
↳ 0
poly f = imap(r,f);
NF(LD*f,std(ideal(Dx,Dy,Dz))); // must be zero if LD indeed annihilates f
↳ _[1]=0
↳ _[2]=0
↳ _[3]=0
↳ _[4]=0
↳ _[5]=0
↳ _[6]=0
↳ _[7]=0
↳ _[8]=0
↳ _[9]=0
↳ _[10]=0
↳ _[11]=0
↳ _[12]=0
↳ _[13]=0

```

See also: [Section 7.5.5.2 \[annRat\]](#), page 423.

### 7.5.5.2 annRat

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

**Usage:** `annRat(g,f)`;  $f, g$  polynomials

**Return:** ring (a Weyl algebra) containing an ideal 'LD'

**Purpose:** compute the annihilator of the rational function  $g/f$  in the corresponding Weyl algebra

**Assume:** basering is commutative and over a field of characteristic 0

**Note:** Activate the output ring with the `setring` command.  
 In the output ring, the ideal 'LD' (in Groebner basis) is the annihilator of  $g/f$ .  
 The algorithm uses the computation of  $\text{Ann}(f^{-1})$  via D-modules, see (SST).

**Display:** If `printlevel=1`, progress debug messages will be printed,  
 if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```

LIB "dmodapp.lib";
ring r = 0,(x,y),dp;
poly g = 2*x*y; poly f = x^2 - y^3;
def B = annRat(g,f);
setring B;
LD;
↳ LD[1]=3*y^2*Dx^2*Dy+2*x*Dx*Dy^2+9*y*Dx^2+4*Dy^2
↳ LD[2]=3*y^3*Dx^2-10*x*y*Dx*Dy-8*y^2*Dy^2+10*x*Dx
↳ LD[3]=y^3*Dy^2-x^2*Dy^2-6*x*y*Dx+2*y^2*Dy+4*y

```

```

⇒ LD[4]=3*x*Dx+2*y*Dy+1
⇒ LD[5]=y^4*Dy-x^2*y*Dy+2*y^3+x^2
// Now, compare with the output of Macaulay2:
ideal tst = 3*x*Dx + 2*y*Dy + 1, y^3*Dy^2 - x^2*Dy^2 + 6*y^2*Dy + 6*y,
9*y^2*Dx^2*Dy-4*y*Dy^3+27*y*Dx^2+2*Dy^2, 9*y^3*Dx^2-4*y^2*Dy^2+10*y*Dy -10;
option(redSB); option(redTail);
LD = groebner(LD);
tst = groebner(tst);
print(matrix(NF(LD,tst))); print(matrix(NF(tst,LD)));
⇒ 0,0,0,0,0
⇒ 0,0,0,0,0
// So, these two answers are the same

```

See also: [Section 7.5.5.1 \[annPoly\]](#), page 422.

### 7.5.5.3 DLoc

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

**Usage:** DLoc(I, f); I an ideal, f a poly

**Return:** list of ideal and list

**Assume:** the basering is a Weyl algebra

**Purpose:** compute the presentation of the localization of  $D/I$  w.r.t.  $f$ 's

**Note:** In the output list L,  
- L[1] is an ideal (given as Groebner basis), the presentation of the localization,  
- L[2] is a list containing roots with multiplicities of Bernstein polynomial of  $(D/I)_f$ .

**Display:** If `printlevel=1`, progress debug messages will be printed,  
if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```

LIB "dmodapp.lib";
ring r = 0,(x,y,Dx,Dy),dp;
def R = Weyl(); setring R; // Weyl algebra in variables x,y,Dx,Dy
poly F = x2-y3;
ideal I = (y^3 - x^2)*Dx - 2*x, (y^3 - x^2)*Dy + 3*y^2; // I = Dx*F, Dy*F;
// I is not holonomic, since its dimension is not 4/2=2
gkdim(I);
⇒ 3
list L = DLoc(I, x2-y3);
L[1]; // localized module (R/I)_f is isomorphic to R/LD0
⇒ _[1]=3*x*Dx+2*y*Dy+12
⇒ _[2]=3*y^2*Dx+2*x*Dy
⇒ _[3]=y^3*Dy-x^2*Dy+6*y^2
L[2]; // description of b-function for localization
⇒ [1]:
⇒ _[1]=0
⇒ _[2]=1/6
⇒ _[3]=-1/6
⇒ [2]:
⇒ 1,1,1

```

### 7.5.5.4 SLoc

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

- Usage:** `SLoc(I, f)`;  $I$  an ideal,  $f$  a poly
- Return:** ring (basing extended by a new variable) containing an ideal 'LD'
- Purpose:** compute a generic presentation of the localization of  $D/I$  w.r.t.  $f^s$
- Assume:** the basing  $D$  is a Weyl algebra over a field of characteristic 0
- Note:** Activate this ring with the `setring` command. In this ring, the ideal LD (given as Groebner basis) is the presentation of the localization.
- Display:** If `printlevel = 1`, progress debug messages will be printed, if `printlevel ≥ 2`, all the debug messages will be printed.

**Example:**

```
LIB "dmodapp.lib";
ring r = 0, (x,y,Dx,Dy), dp;
def R = Weyl(); // Weyl algebra on the variables x,y,Dx,Dy
setring R;
poly F = x^2-y^3;
ideal I = Dx*F, Dy*F;
// note, that I is not holonomic, since it's dimension is not 2
gkdim(I); // 3, while dim R = 4
⇒ 3
def W = SLoc(I,F);
setring W; // = R[s], where s is a new variable
LD; // Groebner basis of s-parametric presentation
⇒ LD[1]=3*x*Dx*s+2*y*Dy*s-6*s^2+6*s
⇒ LD[2]=3*y^2*Dx*s+2*x*Dy*s
⇒ LD[3]=y^3*Dy-x^2*Dy-3*y^2*s+3*y^2
⇒ LD[4]=y^3*Dx-x^2*Dx+2*x*s-2*x
```

### 7.5.5.5 DLoc0

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

- Usage:** `DLoc0(I, f)`;  $I$  an ideal,  $f$  a poly
- Return:** ring (a Weyl algebra) containing an ideal 'LD0' and a list 'BS'
- Purpose:** compute the presentation of the localization of  $D/I$  w.r.t.  $f^s$ , where  $D$  is a Weyl Algebra, based on the output of procedure `SLoc`
- Assume:** the basing is similar to the output ring of `SLoc` procedure
- Note:** activate the output ring with the `setring` command. In this ring,  
- the ideal LD0 (given as Groebner basis) is the presentation of the localization,  
- the list BS contains roots and multiplicities of Bernstein polynomial of  $(D/I)_f$ .
- Display:** If `printlevel = 1`, progress debug messages will be printed, if `printlevel ≥ 2`, all the debug messages will be printed.

**Example:**

```

LIB "dmodapp.lib";
ring r = 0,(x,y,Dx,Dy),dp;
def R = Weyl();    setring R; // Weyl algebra in variables x,y,Dx,Dy
poly F = x2-y3;
ideal I = (y^3 - x^2)*Dx - 2*x, (y^3 - x^2)*Dy + 3*y^2; // I = Dx*F, Dy*F;
// moreover I is not holonomic, since its dimension is not 2 = 4/2
gkdim(I); // 3
↳ 3
def W = SDLoc(I,F); setring W; // creates ideal LD in W = R[s]
def U = DLoc0(LD, x2-y3); setring U; // compute in R
LD0; // Groebner basis of the presentation of localization
↳ LD0[1]=3*x*Dx+2*y*Dy+12
↳ LD0[2]=3*y^2*Dx+2*x*Dy
↳ LD0[3]=y^3*Dy-x^2*Dy+6*y^2
BS; // description of b-function for localization
↳ [1]:
↳   _[1]=0
↳   _[2]=1/6
↳   _[3]=-1/6
↳ [2]:
↳   1,1,1

```

**7.5.5.6 GBWeight**

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

- Usage:** `GBWeight(I,u,v [s,t,w]);`  
 $I$  ideal,  $u,v$  intvecs,  $s,t$  optional ints,  $w$  an optional intvec
- Return:** ideal, Groebner basis of  $I$  w.r.t. the weights  $u$  and  $v$
- Assume:** The basering is the  $n$ -th Weyl algebra over a field of characteristic 0 and for all  $1 \leq i \leq n$  the identity  $\text{var}(i+n) \cdot \text{var}(i) = \text{var}(i) \cdot \text{var}(i+1) + 1$  holds, i.e. the sequence of variables is given by  $x(1), \dots, x(n), D(1), \dots, D(n)$ , where  $D(i)$  is the differential operator belonging to  $x(i)$ .
- Purpose:** computes a Groebner basis with respect to given weights
- Note:** The weights  $u$  and  $v$  are understood as weight vectors for  $x(i)$  and  $D(i)$ , respectively. According to (SST), one computes the homogenization of a given ideal relative to  $(u,v)$ , then one computes a Groebner basis and returns the dehomogenization of the result.  
 If  $s < 0$ , `std` is used for Groebner basis computations, otherwise, and by default, `slimgb` is used.  
 If  $t < 0$ , a matrix ordering is used for Groebner basis computations, otherwise, and by default, a block ordering is used.  
 If  $w$  is given and consists of exactly  $2 \cdot n$  strictly positive entries,  $w$  is used for constructing the weighted homogenized Weyl algebra, see Noro (2002). Otherwise, and by default, the homogenization weight  $(1, \dots, 1)$  is used.
- Display:** If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.



**Example:**

```

LIB "dmodapp.lib";
ring r = 0,(x,y,Dx,Dy),dp;
def D2 = Weyl();
setring D2;
ideal I = 3*x^2*Dy+2*y*Dx,2*x*Dx+3*y*Dy+6;
intvec u = -2,-3;
intvec v = -u;
GBWeight(I,u,v);
↪ _[1]=2*x*Dx+3*y*Dy+6
↪ _[2]=3*x^2*Dy+2*y*Dx
↪ _[3]=9*x*y*Dy^2-4*y*Dx^2+15*x*Dy
↪ _[4]=27*y^2*Dy^3+8*y*Dx^3+135*y*Dy^2+105*Dy
ideal J = std(I);
GBWeight(J,u,v); // same as above
↪ _[1]=2*x*Dx+3*y*Dy+6
↪ _[2]=3*x^2*Dy+2*y*Dx
↪ _[3]=9*x*y*Dy^2-4*y*Dx^2+15*x*Dy
↪ _[4]=27*y^2*Dy^3+8*y*Dx^3+135*y*Dy^2+105*Dy
u = 0,1;
GBWeight(I,u,v);
↪ _[1]=2*x*Dx+3*y*Dy+6
↪ _[2]=3*x^2*Dy+2*y*Dx
↪ _[3]=-x^3*Dx+y^2*Dx-3*x^2

```

**7.5.5.7 initialMalgrange**

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

- Usage:** `initialMalgrange(f,[a,b,v]);` `f` poly, `a,b` optional ints, `v` opt. intvec
- Return:** ring, Weyl algebra induced by basering, extended by two new vars `t,Dt`
- Purpose:** computes the initial Malgrange ideal of a given polynomial w.r.t. the weight vector  $(-1,0,\dots,0,1,0,\dots,0)$  such that the weight of `t` is -1 and the weight of `Dt` is 1.
- Assume:** The basering is commutative and over a field of characteristic 0.
- Note:** Activate the output ring with the `setring` command.  
 The returned ring contains the ideal 'inF', being the initial ideal of the Malgrange ideal of `f`.  
 Varnames of the basering should not include `t` and `Dt`.  
 If `a<>0`, `std` is used for Groebner basis computations, otherwise, and by default, `slimgb` is used.  
 If `b<>0`, a matrix ordering is used for Groebner basis computations, otherwise, and by default, a block ordering is used.  
 If a positive weight vector `v` is given, the weight  $(d,v[1],\dots,v[n],1,d+1-v[1],\dots,d+1-v[n])$  is used for homogenization computations, where `d` denotes the weighted degree of `f`.  
 Otherwise and by default, `v` is set to  $(1,\dots,1)$ . See Noro (2002).
- Display:** If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```

LIB "dmodapp.lib";
ring r = 0,(x,y),dp;
poly f = x^2+y^3+x*y^2;
def D = initialMalgrange(f);
setring D;
inF;
⇨ inF[1]=x*Dt
⇨ inF[2]=2*x*y*Dx+3*y^2*Dx-y^2*Dy-2*x*Dy
⇨ inF[3]=2*x^2*Dx+x*y*Dx+x*y*Dy+18*t*Dt+9*x*Dx-x*Dy+6*y*Dy+4*x+18
⇨ inF[4]=18*t*Dt^2+6*y*Dt*Dy-y*Dt+27*Dt
⇨ inF[5]=y^2*Dt
⇨ inF[6]=2*t*y*Dt+2*x*y*Dx+2*y^2*Dx-6*t*Dt-3*x*Dx-x*Dy-2*y*Dy+2*y-6
⇨ inF[7]=x*y^2+y^3+x^2
⇨ inF[8]=2*y^3*Dx-2*y^3*Dy-3*y^2*Dx-2*x*y*Dy+y^2*Dy-4*y^2+36*t*Dt+18*x*Dx+1\
    2*y*Dy+36
setring r;
intvec v = 3,2;
def D2 = initialMalgrange(f,1,1,v);
setring D2;
inF;
⇨ inF[1]=x*Dt
⇨ inF[2]=2*x*y*Dx+3*y^2*Dx-y^2*Dy-2*x*Dy
⇨ inF[3]=4*x^2*Dx-3*y^2*Dx+2*x*y*Dy+y^2*Dy+36*t*Dt+18*x*Dx+12*y*Dy+8*x+36
⇨ inF[4]=18*t*Dt^2+6*y*Dt*Dy-y*Dt+27*Dt
⇨ inF[5]=y^2*Dt
⇨ inF[6]=2*t*y*Dt-y^2*Dx+y^2*Dy-6*t*Dt-3*x*Dx+x*Dy-2*y*Dy+2*y-6
⇨ inF[7]=x*y^2+y^3+x^2
⇨ inF[8]=2*y^3*Dx-2*y^3*Dy-3*y^2*Dx-2*x*y*Dy+y^2*Dy-4*y^2+36*t*Dt+18*x*Dx+1\
    2*y*Dy+36

```

### 7.5.5.8 initialIdealW

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

- Usage:** `initialIdealW(I,u,v [,s,t,w]);`  
 $I$  ideal,  $u,v$  intvecs,  $s,t$  optional ints,  $w$  an optional intvec
- Return:** ideal, GB of initial ideal of the input ideal wrt the weights  $u$  and  $v$
- Assume:** The basering is the  $n$ -th Weyl algebra in characteristic 0 and for all  $1 \leq i \leq n$  the identity  $\text{var}(i+n) \cdot \text{var}(i) = \text{var}(i) \cdot \text{var}(i+1) + 1$  holds, i.e. the sequence of variables is given by  $x(1), \dots, x(n), D(1), \dots, D(n)$ , where  $D(i)$  is the differential operator belonging to  $x(i)$ .
- Purpose:** computes the initial ideal with respect to given weights.
- Note:**  $u$  and  $v$  are understood as weight vectors for  $x(1..n)$  and  $D(1..n)$  respectively.  
 If  $s < 0$ , `std` is used for Groebner basis computations, otherwise, and by default, `slimgb` is used.  
 If  $t < 0$ , a matrix ordering is used for Groebner basis computations, otherwise, and by default, a block ordering is used.  
 If  $w$  is given and consists of exactly  $2 \cdot n$  strictly positive entries,  $w$  is used as homogenization weight.  
 Otherwise, and by default, the homogenization weight  $(1, \dots, 1)$  is used.

**Display:** If `printlevel=1`, progress debug messages will be printed,  
if `printlevel≥2`, all the debug messages will be printed.

**Example:**

```
LIB "dmodapp.lib";
ring r = 0, (x,y,Dx,Dy), dp;
def D2 = Weyl();
setring D2;
ideal I = 3*x^2*Dy+2*y*Dx, 2*x*Dx+3*y*Dy+6;
intvec u = -2,-3;
intvec v = -u;
initialIdealW(I,u,v);
↪ _[1]=2*x*Dx+3*y*Dy+6
↪ _[2]=3*x^2*Dy+2*y*Dx
↪ _[3]=9*x*y*Dy^2-4*y*Dx^2+15*x*Dy
↪ _[4]=27*y^2*Dy^3+8*y*Dx^3+135*y*Dy^2+105*Dy
ideal J = std(I);
initialIdealW(J,u,v); // same as above
↪ _[1]=2*x*Dx+3*y*Dy+6
↪ _[2]=3*x^2*Dy+2*y*Dx
↪ _[3]=9*x*y*Dy^2-4*y*Dx^2+15*x*Dy
↪ _[4]=27*y^2*Dy^3+8*y*Dx^3+135*y*Dy^2+105*Dy
u = 0,1;
initialIdealW(I,u,v);
↪ _[1]=Dx
↪ _[2]=Dy
```

### 7.5.5.9 inForm

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

**Usage:** `inForm(I,w)`; `I` ideal or poly, `w` intvec

**Return:** ideal, generated by initial forms of generators of `I` w.r.t. `w`, or  
poly, initial form of input poly w.r.t. `w`

**Purpose:** computes the initial form of an ideal or a poly w.r.t. the weight `w`

**Note:** The size of the weight vector must be equal to the number of variables  
of the basering.

**Example:**

```
LIB "dmodapp.lib";
ring r = 0, (x,y,Dx,Dy), dp;
def D = Weyl(); setring D;
poly F = 3*x^2*Dy+2*y*Dx;
poly G = 2*x*Dx+3*y*Dy+6;
ideal I = F,G;
intvec w1 = -1,-1,1,1;
intvec w2 = -1,-2,1,2;
intvec w3 = -2,-3,2,3;
inForm(I,w1);
↪ _[1]=2*y*Dx
↪ _[2]=2*x*Dx+3*y*Dy+6
inForm(I,w2);
```

```

↳ _[1]=3*x^2*Dy
↳ _[2]=2*x*Dx+3*y*Dy+6
inForm(I,w3);
↳ _[1]=3*x^2*Dy+2*y*Dx
↳ _[2]=2*x*Dx+3*y*Dy+6
inForm(F,w1);
↳ 2*y*Dx

```

### 7.5.5.10 restrictionIdeal

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

- Usage:** `restrictionIdeal(I,w,[eng,m,G]);`  
 $I$  ideal,  $w$  intvec,  $eng$  and  $m$  optional ints,  $G$  optional ideal
- Return:** ring (a Weyl algebra) containing an ideal 'resIdeal'
- Assume:** The basering is the  $n$ -th Weyl algebra over a field of characteristic 0 and for all  $1 \leq i \leq n$  the identity  $\text{var}(i+n) \cdot \text{var}(i) = \text{var}(i) \cdot \text{var}(i+1) + 1$  holds, i.e. the sequence of variables is given by  $x(1), \dots, x(n), D(1), \dots, D(n)$ , where  $D(i)$  is the differential operator belonging to  $x(i)$ .  
Further, assume that  $I$  is holonomic and that  $w$  is  $n$ -dimensional with non-negative entries.
- Purpose:** computes the restriction ideal of a holonomic ideal to the subspace defined by the variables corresponding to the non-zero entries of the given intvec
- Note:** The output ring is the Weyl algebra defined by the zero entries of  $w$ . It contains an ideal 'resIdeal' being the restriction ideal of  $I$  wrt  $w$ .  
If there are no zero entries, the input ring is returned.  
If  $eng < 0$ , `std` is used for Groebner basis computations, otherwise, and by default, `slingb` is used.  
The minimal integer root of the  $b$ -function of  $I$  wrt the weight  $(-w, w)$  can be specified via the optional argument  $m$ .  
The optional argument  $G$  is used for specifying a Groebner basis of  $I$  wrt the weight  $(-w, w)$ , that is, the initial form of  $G$  generates the initial ideal of  $I$  wrt the weight  $(-w, w)$ .  
Further note, that the assumptions on  $m$  and  $G$  (if given) are not checked.
- Display:** If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```

LIB "dmodapp.lib";
ring r = 0, (a,x,b,Da,Dx,Db), dp;
def D3 = Weyl();
setring D3;
ideal I = a*Db-Dx+2*Da,
x*Db-Da,
x*Da+a*Da+b*Db+1,
x*Dx-2*x*Da-a*Da,
b*Db^2+Dx*Da-Da^2+Db,

```

```

a*Dx*Da+2*x*Da^2+a*Da^2+b*Dx*Db+Dx+2*Da;
intvec w = 1,0,0;
def D2 = restrictionIdeal(I,w);
setring D2; D2;
⇒ // coefficients: QQ
⇒ // number of vars : 4
⇒ //          block 1 : ordering C
⇒ //          block 2 : ordering dp
⇒ //          : names      x b Dx Db
⇒ // noncommutative relations:
⇒ //      Dxx=x*Dx+1
⇒ //      Dbb=b*Db+1
resIdeal;
⇒ resIdeal[1]=2*x*Db-Dx
⇒ resIdeal[2]=x*Dx+2*b*Db+2
⇒ resIdeal[3]=4*b*Db^2+Dx^2+6*Db

```

### 7.5.5.11 restrictionModule

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

- Usage:** `restrictionModule(I,w,[eng,m,G]);`  
 $I$  ideal,  $w$  intvec,  $eng$  and  $m$  optional ints,  $G$  optional ideal
- Return:** ring (a Weyl algebra) containing a module 'resMod'
- Assume:** The basering is the  $n$ -th Weyl algebra over a field of characteristic 0 and for all  $1 \leq i \leq n$  the identity  $\text{var}(i+n) \cdot \text{var}(i) = \text{var}(i) \cdot \text{var}(i+1) + 1$  holds, i.e. the sequence of variables is given by  $x(1), \dots, x(n), D(1), \dots, D(n)$ , where  $D(i)$  is the differential operator belonging to  $x(i)$ .  
Further, assume that  $I$  is holonomic and that  $w$  is  $n$ -dimensional with non-negative entries.
- Purpose:** computes the restriction module of a holonomic ideal to the subspace defined by the variables corresponding to the non-zero entries of the given intvec
- Note:** The output ring is the Weyl algebra defined by the zero entries of  $w$ . It contains a module 'resMod' being the restriction module of  $I$  wrt  $w$ . If there are no zero entries, the input ring is returned.  
If  $eng < 0$ , `std` is used for Groebner basis computations, otherwise, and by default, `slingb` is used.  
The minimal integer root of the  $b$ -function of  $I$  wrt the weight  $(-w, w)$  can be specified via the optional argument  $m$ .  
The optional argument  $G$  is used for specifying a Groebner Basis of  $I$  wrt the weight  $(-w, w)$ , that is, the initial form of  $G$  generates the initial ideal of  $I$  wrt the weight  $(-w, w)$ .  
Further note, that the assumptions on  $m$  and  $G$  (if given) are not checked.
- Display:** If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```

LIB "dmodapp.lib";
ring r = 0,(a,x,b,Da,Dx,Db),dp;
def D3 = Weyl();
setring D3;
ideal I = a*Db-Dx+2*Da, x*Db-Da, x*Da+a*Da+b*Db+1,
x*Dx-2*x*Da-a*Da, b*Db^2+Dx*Da-Da^2+Db,
a*Dx*Da+2*x*Da^2+a*Da^2+b*Dx*Db+Dx+2*Da;
intvec w = 1,0,0;
def rm = restrictionModule(I,w);
setring rm; rm;
⇒ // coefficients: QQ
⇒ // number of vars : 4
⇒ //          block 1 : ordering C
⇒ //          block 2 : ordering dp
⇒ //          : names      x b Dx Db
⇒ // noncommutative relations:
⇒ //      Dxx=x*Dx+1
⇒ //      Dbb=b*Db+1
print(resMod);
⇒ 2*x*Db-Dx,x*Dx+2*b*Db+2,4*b*Db^2+Dx^2+6*Db

```

### 7.5.5.12 integralIdeal

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

- Usage:** `integralIdeal(I,w,[eng,m,G]);`  
 $I$  ideal,  $w$  intvec,  $eng$  and  $m$  optional ints,  $G$  optional ideal
- Return:** ring (a Weyl algebra) containing an ideal 'intIdeal'
- Assume:** The basering is the  $n$ -th Weyl algebra over a field of characteristic 0 and for all  $1 \leq i \leq n$  the identity  $\text{var}(i+n) * \text{var}(i) = \text{var}(i) * \text{var}(i+1) + 1$  holds, i.e. the sequence of variables is given by  $x(1), \dots, x(n), D(1), \dots, D(n)$ , where  $D(i)$  is the differential operator belonging to  $x(i)$ .  
Further, assume that  $I$  is holonomic and that  $w$  is  $n$ -dimensional with non-negative entries.
- Purpose:** computes the integral ideal of a holonomic ideal w.r.t. the subspace defined by the variables corresponding to the non-zero entries of the given intvec.
- Note:** The output ring is the Weyl algebra defined by the zero entries of  $w$ . It contains ideal 'intIdeal' being the integral ideal of  $I$  w.r.t.  $w$ .  
If there are no zero entries, the input ring is returned.  
If  $eng \neq 0$ , `std` is used for Groebner basis computations, otherwise, and by default, `slimgb` is used.  
The minimal integer root of the  $b$ -function of  $I$  wrt the weight  $(-w, w)$  can be specified via the optional argument  $m$ .  
The optional argument  $G$  is used for specifying a Groebner basis of  $I$  wrt the weight  $(-w, w)$ , that is, the initial form of  $G$  generates the initial ideal of  $I$  wrt the weight  $(-w, w)$ .  
Further note, that the assumptions on  $m$  and  $G$  (if given) are not checked.

**Display:** If `printlevel=1`, progress debug messages will be printed,  
if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "dmodapp.lib";
ring r = 0,(x,b,Dx,Db),dp;
def D2 = Weyl();
setring D2;
ideal I = x*Dx+2*b*Db+2, x^2*Dx+b*Dx+2*x;
intvec w = 1,0;
def D1 = integralIdeal(I,w);
setring D1; D1;
↪ // coefficients: QQ
↪ // number of vars : 2
↪ //          block 1 : ordering C
↪ //          block 2 : ordering dp
↪ //          : names      b Db
↪ // noncommutative relations:
↪ //      Dbb=b*Db+1
intIdeal;
↪ intIdeal[1]=2*b*Db+1
```

### 7.5.5.13 integralModule

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

**Usage:** `integralModule(I,w,[eng,m,G]);`  
I ideal, w intvec, eng and m optional ints, G optional ideal

**Return:** ring (a Weyl algebra) containing a module 'intMod'

**Assume:** The basering is the n-th Weyl algebra over a field of characteristic 0 and for all  $1 \leq i \leq n$  the identity  $\text{var}(i+n) \cdot \text{var}(i) = \text{var}(i) \cdot \text{var}(i+1) + 1$  holds, i.e. the sequence of variables is given by  $x(1), \dots, x(n), D(1), \dots, D(n)$ , where  $D(i)$  is the differential operator belonging to  $x(i)$ .  
Further, assume that I is holonomic and that w is n-dimensional with non-negative entries.

**Purpose:** computes the integral module of a holonomic ideal w.r.t. the subspace defined by the variables corresponding to the non-zero entries of the given intvec

**Note:** The output ring is the Weyl algebra defined by the zero entries of w. It contains a module 'intMod' being the integral module of I wrt w. If there are no zero entries, the input ring is returned. If `eng <> 0`, `std` is used for Groebner basis computations, otherwise, and by default, `slingb` is used. Let  $F(I)$  denote the Fourier transform of I w.r.t. w. The minimal integer root of the b-function of  $F(I)$  w.r.t. the weight  $(-w, w)$  can be specified via the optional argument m. The optional argument G is used for specifying a Groebner Basis of  $F(I)$  wrt the weight  $(-w, w)$ , that is, the initial form of G generates the initial ideal of  $F(I)$  w.r.t. the weight  $(-w, w)$ .

Further note, that the assumptions on  $m$  and  $G$  (if given) are not checked.

**Display:** If `printlevel=1`, progress debug messages will be printed,  
if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "dmodapp.lib";
ring r = 0,(x,b,Dx,Db),dp;
def D2 = Weyl();
setring D2;
ideal I = x*Dx+2*b*Db+2, x^2*Dx+b*Dx+2*x;
intvec w = 1,0;
def im = integralModule(I,w);
setring im; im;
⇒ // coefficients: QQ
⇒ // number of vars : 2
⇒ //          block 1 : ordering C
⇒ //          block 2 : ordering dp
⇒ //          : names      b Db
⇒ // noncommutative relations:
⇒ //      Dbb=b*Db+1
print(intMod);
⇒ 2*b*Db+1,0,
⇒ 0,          b*Db
```

#### 7.5.5.14 deRhamCohom

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

**Usage:** `deRhamCohom(f[,w,eng,m]);`  $f$  poly,  $w$  optional intvec,  
 $eng$  and  $m$  optional ints

**Return:** ring (a Weyl Algebra) containing a list 'DR' of ideal and int

**Assume:** Basing is commutative and over a field of characteristic 0.

**Purpose:** computes a basis of the  $n$ -th de Rham cohomology group of the complement of the hypersurface defined by  $f$ , where  $n$  denotes the number of variables of the basing

**Note:** The output ring is the  $n$ -th Weyl algebra. It contains a list 'DR' with two entries (ideal  $J$  and int  $m$ ) such that  $\{f^m \cdot J[i] : i=1..size(I)\}$  is a basis of the  $n$ -th de Rham cohomology group of the complement of the hypersurface defined by  $f$ .

If  $w$  is an intvec with exactly  $n$  strictly positive entries,  $w$  is used in the computation. Otherwise, and by default,  $w$  is set to  $(1,...,1)$ .

If  $eng < 0$ , `std` is used for Groebner basis computations, otherwise, and by default, `slingb` is used.

If  $m$  is given, it is assumed to be less than or equal to the minimal integer root of the Bernstein-Sato polynomial of  $f$ . This assumption is not checked. If not specified,  $m$  is set to the minimal integer root of the Bernstein-Sato polynomial of  $f$ .

**Theory:** (SST) pp. 232-235



**Display:** If `printlevel=1`, progress debug messages will be printed,  
if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "dmodapp.lib";
ring r = 0,(x,y,z),dp;
poly f = x^3+y^3+z^3;
def A = deRhamCohom(f); // we see that the K-dim is 2
setring A;
DR;
⇒ [1]:
⇒   _[1]=-x^3*Dx*Dy*Dz
⇒   _[2]=-x*y*z*Dx*Dy*Dz
⇒ [2]:
⇒   -2
```

See also: [Section 7.5.5.15 \[deRhamCohomIdeal\], page 435](#).

### 7.5.5.15 deRhamCohomIdeal

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\], page 421](#)).

**Usage:** `deRhamCohomIdeal (I[w,eng,k,G]);`  
I ideal, w optional intvec, eng and k optional ints, G optional ideal

**Return:** ideal

**Assume:** The basering is the  $n$ -th Weyl algebra  $D$  over a field of characteristic zero and for all  $1 \leq i \leq n$  the identity  $\text{var}(i+n) \cdot \text{var}(i) = \text{var}(i) \cdot \text{var}(i+1) + 1$  holds, i.e. the sequence of variables is given by  $x(1), \dots, x(n), D(1), \dots, D(n)$ , where  $D(i)$  is the differential operator belonging to  $x(i)$ .  
Further, assume that  $I$  is of special kind, namely let  $f$  in  $K[x]$  and consider the module  $K[x, 1/f]f^m$ , where  $m$  is smaller than or equal to the minimal integer root of the Bernstein-Sato polynomial of  $f$ .  
Since this module is known to be a holonomic  $D$ -module, it has a cyclic presentation  $D/I$ .

**Purpose:** computes a basis of the  $n$ -th de Rham cohomology group of the complement of the hypersurface defined by  $f$

**Note:** The elements of the basis are of the form  $f^m \cdot p$ , where  $p$  runs over the entries of the returned ideal.

If  $I$  does not satisfy the assumptions described above, the result might have no meaning. Note that  $I$  can be computed with `annfs`.

If  $w$  is an intvec with exactly  $n$  strictly positive entries,  $w$  is used in the computation. Otherwise, and by default,  $w$  is set to  $(1, \dots, 1)$ .

If  $\text{eng} < 0$ , `std` is used for Groebner basis computations, otherwise, and by default, `slingb` is used.

Let  $F(I)$  denote the Fourier transform of  $I$  wrt  $w$ .

An integer smaller than or equal to the minimal integer root of the  $b$ -function of  $F(I)$  wrt the weight  $(-w, w)$  can be specified via the optional argument  $k$ .

The optional argument  $G$  is used for specifying a Groebner Basis of  $F(I)$  wrt the weight  $(-w, w)$ , that is, the initial form of  $G$  generates the

initial ideal of  $F(I)$  wrt the weight  $(-w, w)$ .

Further note, that the assumptions on  $I$ ,  $k$  and  $G$  (if given) are not checked.

**Theory:** (SST) pp. 232-235

**Display:** If `printlevel=1`, progress debug messages will be printed,  
if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "dmodapp.lib";
ring r = 0, (x,y,z), dp;
poly F = x^3+y^3+z^3;
bfctAnn(F);           // Bernstein-Sato poly of F has minimal integer root -2
⇨ [1]:
⇨   _[1]=-1
⇨   _[2]=-4/3
⇨   _[3]=-5/3
⇨   _[4]=-2
⇨ [2]:
⇨   2,1,1,1
def W = annRat(1,F^2); // so we compute the annihilator of 1/F^2
setring W; W;         // Weyl algebra, contains LD = Ann(1/F^2)
⇨ // coefficients: QQ
⇨ // number of vars : 6
⇨ //           block 1 : ordering dp
⇨ //           : names  x y z Dx Dy Dz
⇨ //           block 2 : ordering C
⇨ // noncommutative relations:
⇨ //   Dxx=x*Dx+1
⇨ //   Dyy=y*Dy+1
⇨ //   Dzz=z*Dz+1
LD;                   // K[x,y,z,1/F]F^(-2) is isomorphic to W/LD as W-module
⇨ LD[1]=x*Dx+y*Dy+z*Dz+6
⇨ LD[2]=z^2*Dy-y^2*Dz
⇨ LD[3]=z^2*Dx-x^2*Dz
⇨ LD[4]=y^2*Dx-x^2*Dy
⇨ LD[5]=x^3*Dz+y^3*Dz+z^3*Dz+6*z^2
⇨ LD[6]=x^3*Dy+y^3*Dy+y^2*z*Dz+6*y^2
deRhamCohomIdeal(LD); // we see that the K-dim is 2
⇨ _[1]=-x^3*Dx*Dy*Dz
⇨ _[2]=-x*y*z*Dx*Dy*Dz
```

See also: [Section 7.5.5.14 \[deRhamCohom\]](#), page 434.

### 7.5.5.16 charVariety

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

**Usage:** `charVariety(I [,eng])`;  $I$  an ideal, `eng` an optional int

**Return:** ring (commutative) containing an ideal 'charVar'

**Purpose:** computes an ideal whose zero set is the characteristic variety of  $I$  in the sense of D-module theory

**Assume:** The basering is the  $n$ -th Weyl algebra over a field of characteristic 0 and for all  $1 \leq i \leq n$  the identity  $\text{var}(i+n) \cdot \text{var}(i) = \text{var}(i) \cdot \text{var}(i+1) + 1$  holds, i.e. the sequence of variables is given by  $x(1), \dots, x(n), D(1), \dots, D(n)$ , where  $D(i)$  is the differential operator belonging to  $x(i)$ .

**Note:** The output ring is commutative. It contains an ideal 'charVar'.  
If `eng<>0`, `std` is used for Groebner basis computations, otherwise, and by default, `slingb` is used.

**Display:** If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "dmodapp.lib";
ring r = 0, (x,y), Dp;
poly F = x3-y2;
printlevel = 0;
def A = annfs(F);
setring A;          // Weyl algebra
LD;                // the annihilator of F
--> LD[1]=2*x*Dx+3*y*Dy+6
--> LD[2]=3*x^2*Dy+2*y*Dx
--> LD[3]=9*x*y*Dy^2-4*y*Dx^2+15*x*Dy
--> LD[4]=27*y^2*Dy^3+8*y*Dx^3+135*y*Dy^2+105*Dy
def CA = charVariety(LD);
setring CA; CA; // commutative ring
--> // coefficients: QQ
--> // number of vars : 4
--> //          block 1 : ordering dp
--> //          : names  x y Dx Dy
--> //          block 2 : ordering C
charVar;
--> charVar[1]=2*x*Dx+3*y*Dy
--> charVar[2]=3*x^2*Dy+2*y*Dx
--> charVar[3]=9*x*y*Dy^2-4*y*Dx^2
--> charVar[4]=27*y^2*Dy^3+8*y*Dx^3
dim(std(charVar)); // hence I is holonomic
--> 2
```

See also: [Section 7.5.5.17 \[charInfo\]](#), page 437.

### 7.5.5.17 charInfo

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

**Usage:** `charInfo(I)`;  $I$  an ideal

**Return:** ring (commut.) containing ideals 'charVar', 'singLoc' and list 'primDec'

**Purpose:** computes characteristic variety of  $I$  (in the sense of D-module theory), its singular locus and primary decomposition

**Assume:** The basering is the  $n$ -th Weyl algebra over a field of characteristic 0 and for all  $1 \leq i \leq n$  the identity  $\text{var}(i+n) \cdot \text{var}(i) = \text{var}(i) \cdot \text{var}(i+1) + 1$  holds, i.e. the sequence of variables is given by

$x(1), \dots, x(n), D(1), \dots, D(n)$ , where  $D(i)$  is the differential operator belonging to  $x(i)$ .

**Note:** In the output ring, which is commutative:

- the ideal 'charVar' is the characteristic variety  $\text{char}(I)$ ,
- the ideal 'SingLoc' is the singular locus of  $\text{char}(I)$ ,
- the list 'primDec' is the primary decomposition of  $\text{char}(I)$ .

**Display:** If `printlevel=1`, progress debug messages will be printed,  
if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "dmodapp.lib";
ring r = 0, (x, y), Dp;
poly F = x^3 - y^2;
printlevel = 0;
def A = annfs(F);
setring A;          // Weyl algebra
LD;                // the annihilator of F
⇨ LD[1] = 2*x*Dx + 3*y*Dy + 6
⇨ LD[2] = 3*x^2*Dy + 2*y*Dx
⇨ LD[3] = 9*x*y*Dy^2 - 4*y*Dx^2 + 15*x*Dy
⇨ LD[4] = 27*y^2*Dy^3 + 8*y*Dx^3 + 135*y*Dy^2 + 105*Dy
def CA = charInfo(LD);
setring CA; CA;    // commutative ring
⇨ // coefficients: QQ
⇨ // number of vars : 4
⇨ //          block 1 : ordering dp
⇨ //          : names  x y Dx Dy
⇨ //          block 2 : ordering C
charVar;           // characteristic variety
⇨ charVar[1] = 2*x*Dx + 3*y*Dy
⇨ charVar[2] = 3*x^2*Dy + 2*y*Dx
⇨ charVar[3] = 9*x*y*Dy^2 - 4*y*Dx^2
⇨ charVar[4] = 27*y^2*Dy^3 + 8*y*Dx^3
singLoc;           // singular locus
⇨ singLoc[1] = y*Dy
⇨ singLoc[2] = y*Dx
⇨ singLoc[3] = 2*x*Dx - 3*y*Dy
⇨ singLoc[4] = 9*x*Dy^2 - 2*Dx^2
⇨ singLoc[5] = 3*x^2*Dy - y*Dx
⇨ singLoc[6] = Dx^3
⇨ singLoc[7] = x^3 - y^2
primDec;           // primary decomposition
⇨ [1]:
⇨   [1]:
⇨     _[1] = Dy
⇨     _[2] = Dx
⇨   [2]:
⇨     _[1] = Dy
⇨     _[2] = Dx
⇨ [2]:
⇨   [1]:
⇨     _[1] = 27*y*Dy^3 + 8*Dx^3
```

```

⇒      _[2]=9*x*Dy^2-4*Dx^2
⇒      _[3]=2*x*Dx+3*y*Dy
⇒      _[4]=3*x^2*Dy+2*y*Dx
⇒      _[5]=x^3-y^2
⇒      [2]:
⇒      _[1]=27*y*Dy^3+8*Dx^3
⇒      _[2]=9*x*Dy^2-4*Dx^2
⇒      _[3]=2*x*Dx+3*y*Dy
⇒      _[4]=3*x^2*Dy+2*y*Dx
⇒      _[5]=x^3-y^2
⇒      [3]:
⇒      [1]:
⇒      _[1]=y
⇒      _[2]=x
⇒      [2]:
⇒      _[1]=y
⇒      _[2]=x

```

#### 7.5.5.18 isFsat

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

**Usage:** `isFsat(I, F)`; I an ideal, F a poly

**Return:** int, 1 if I is F-saturated and 0 otherwise

**Purpose:** checks whether the ideal I is F-saturated

**Note:** We check indeed that  $\text{Ker}(D \rightarrow F \rightarrow D/I)$  is 0, where D is the basering.

**Example:**

```

LIB "dmodapp.lib";
ring r = 0,(x,y),dp;
poly G = x*(x-y)*y;
def A = annfs(G);
setring A;
poly F = x3-y2;
isFsat(LD,F);
⇒ 1
ideal J = LD*F;
isFsat(J,F);
⇒ 0

```

#### 7.5.5.19 appelF1

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

**Usage:** `appelF1()`;

**Return:** ring (a parametric Weyl algebra) containing an ideal 'IAappel1'

**Purpose:** defines the ideal in a parametric Weyl algebra,  
which annihilates Appel F1 hypergeometric function

**Note:** The output ring is a parametric Weyl algebra. It contains an ideal 'IAappel1' annihilating Appel F1 hypergeometric function.  
See (SST) p. 48.

**Example:**

```

LIB "dmodapp.lib";
def A = appelF1();
setring A;
IAappel1;
⇒ IAappel1[1]=-x^3*Dx^2+x^2*Dx^2-x^2*y*Dx*Dy+x*y*Dx*Dy+(-a-b-1)*x^2*Dx+(c)*x\
  *Dx+(-b)*x*y*Dy+(-a*b)*x
⇒ IAappel1[2]=-x*y^2*Dx*Dy+x*y*Dx*Dy-y^3*Dy^2+y^2*Dy^2+(-d)*x*y*Dx+(-a-d-1)*\
  y^2*Dy+(c)*y*Dy+(-a*d)*y
⇒ IAappel1[3]=x*Dx*Dy-y*Dx*Dy+(-d)*Dx+(b)*Dy

```

**7.5.5.20 appelF2**

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

**Usage:** `appelF2();`

**Return:** ring (a parametric Weyl algebra) containing an ideal 'IAappel2'

**Purpose:** defines the ideal in a parametric Weyl algebra,  
which annihilates Appel F2 hypergeometric function

**Note:** The output ring is a parametric Weyl algebra. It contains an ideal  
'IAappel2' annihilating Appel F2 hypergeometric function.  
See (SST) p. 85.

**Example:**

```

LIB "dmodapp.lib";
def A = appelF2();
setring A;
IAappel2;
⇒ IAappel2[1]=-x^3*Dx^2+x^2*Dx^2-x^2*y*Dx*Dy+(-a-b-1)*x^2*Dx+x*Dx+(-b)*x*y*D\
  y+(-a*b)*x
⇒ IAappel2[2]=-x*y^2*Dx*Dy-y^3*Dy^2+y^2*Dy^2+(-c)*x*y*Dx+(-a-c-1)*y^2*Dy+y*D\
  y+(-a*c)*y

```

**7.5.5.21 appelF4**

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

**Usage:** `appelF4();`

**Return:** ring (a parametric Weyl algebra) containing an ideal 'IAappel4'

**Purpose:** defines the ideal in a parametric Weyl algebra,  
which annihilates Appel F4 hypergeometric function

**Note:** The output ring is a parametric Weyl algebra. It contains an ideal  
'IAappel4' annihilating Appel F4 hypergeometric function.  
See (SST) p. 39.

**Example:**

```

LIB "dmodapp.lib";
def A = appelF4();
setring A;
IAappel4;
⇒ IAappel4[1]=-x^2*Dx^2+x*Dx^2-2*x*y*Dx*Dy-y^2*Dy^2+(-a-b-1)*x*Dx+(c)*Dx+(-a\

```

```

-b-1)*y*Dy+(-a*b)
↦ IAppel4[2]=-x^2*Dx^2-2*x*y*Dx*Dy-y^2*Dy^2+y*Dy^2+(-a-b-1)*x*Dx+(-a-b-1)*y\
*Dy+(d)*Dy+(-a*b)

```

### 7.5.5.22 fourier

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

**Usage:** `fourier(I[,v]);` I an ideal, v an optional intvec

**Return:** ideal

**Purpose:** computes the Fourier transform of an ideal in a Weyl algebra

**Assume:** The basering is the n-th Weyl algebra over a field of characteristic 0 and for all  $1 \leq i \leq n$  the identity  $\text{var}(i+n) \cdot \text{var}(i) = \text{var}(i) \cdot \text{var}(i+1) + 1$  holds, i.e. the sequence of variables is given by  $x(1), \dots, x(n), D(1), \dots, D(n)$ , where  $D(i)$  is the differential operator belonging to  $x(i)$ .

**Note:** The Fourier automorphism is defined by mapping  $x(i)$  to  $-D(i)$  and  $D(i)$  to  $x(i)$ .  
If v is an intvec with entries ranging from 1 to n, the Fourier transform of I restricted to the variables given by v is computed.

**Example:**

```

LIB "dmodapp.lib";
ring r = 0, (x,y,Dx,Dy), dp;
def D2 = Weyl();
setring D2;
ideal I = x*Dx+2*y*Dy+2, x^2*Dx+y*Dx+2*x;
intvec v = 2;
fourier(I,v);
↦ _[1]=x*Dx-2*y*Dy
↦ _[2]=x^2*Dx-Dx*Dy+2*x
fourier(I);
↦ _[1]=-x*Dx-2*y*Dy-1
↦ _[2]=x*Dx^2-x*Dy

```

See also: [Section 7.5.5.23 \[inverseFourier\]](#), page 441.

### 7.5.5.23 inverseFourier

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

**Usage:** `inverseFourier(I[,v]);` I an ideal, v an optional intvec

**Return:** ideal

**Purpose:** computes the inverse Fourier transform of an ideal in a Weyl algebra

**Assume:** The basering is the n-th Weyl algebra over a field of characteristic 0 and for all  $1 \leq i \leq n$  the identity  $\text{var}(i+n) \cdot \text{var}(i) = \text{var}(i) \cdot \text{var}(i+1) + 1$  holds, i.e. the sequence of variables is given by  $x(1), \dots, x(n), D(1), \dots, D(n)$ , where  $D(i)$  is the differential operator belonging to  $x(i)$ .

**Note:** The Fourier automorphism is defined by mapping  $x(i)$  to  $-D(i)$  and  $D(i)$  to  $x(i)$ .  
If  $v$  is an intvec with entries ranging from 1 to  $n$ , the inverse Fourier transform of  $I$  restricted to the variables given by  $v$  is computed.

**Example:**

```
LIB "dmodapp.lib";
ring r = 0,(x,y,Dx,Dy),dp;
def D2 = Weyl();
setring D2;
ideal I = x*Dx+2*y*Dy+2, x^2*Dx+y*Dx+2*x;
intvec v = 2;
ideal FI = fourier(I);
inverseFourier(FI);
↪ _[1]=x*Dx+2*y*Dy+2
↪ _[2]=x^2*Dx+y*Dx+2*x
```

See also: [Section 7.5.5.22 \[fourier\]](#), page 441.

### 7.5.5.24 bFactor

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

**Usage:** `bFactor(f); f poly`

**Return:** list of ideal and intvec and possibly a string

**Purpose:** tries to compute the roots of a univariate poly  $f$

**Note:** The output list consists of two or three entries:  
roots of  $f$  as an ideal, their multiplicities as intvec, and,  
if present, a third one being the product of all irreducible factors  
of degree greater than one, given as string.  
If  $f$  is the zero polynomial or if  $f$  has no roots in the ground field,  
this is encoded as root 0 with multiplicity 0.

**Display:** If `printlevel=1`, progress debug messages will be printed,  
if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "dmodapp.lib";
ring r = 0,(x,y),dp;
bFactor((x^2-1)^2);
↪ [1]:
↪   _[1]=1
↪   _[2]=-1
↪ [2]:
↪   2,2
bFactor((x^2+1)^2);
↪ [1]:
↪   _[1]=0
↪ [2]:
↪   0
↪ [3]:
↪   x4+2x2+1
bFactor((y^2+1/2)*(y+9)*(y-7));
```



```

⇒ [1]:
⇒   _[1]=7
⇒   _[2]=-9
⇒ [2]:
⇒   1,1
⇒ [3]:
⇒   2y2+1
bFactor(1);
⇒ [1]:
⇒   _[1]=0
⇒ [2]:
⇒   0
⇒ [3]:
⇒   1
bFactor(0);
⇒ [1]:
⇒   _[1]=0
⇒ [2]:
⇒   0
⇒ [3]:
⇒   0

```

#### 7.5.5.25 intRoots

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

**Usage:**     `isInt(L)`; L a list

**Return:**    list

**Purpose:**    extracts integer roots from a list given in `bFactor` format

**Assume:**   The input list must be given in the format of `bFactor`.

**Note:**      Parameters are treated as integers.

**Example:**

```

LIB "dmodapp.lib";
ring r = 0,x,dp;
list L = bFactor((x-4/3)*(x+3)^2*(x-5)^4); L;
⇒ [1]:
⇒   _[1]=5
⇒   _[2]=4/3
⇒   _[3]=-3
⇒ [2]:
⇒   4,1,2
intRoots(L);
⇒ [1]:
⇒   _[1]=5
⇒   _[2]=-3
⇒ [2]:
⇒   4,2

```

See also: [Section 7.5.5.24 \[bFactor\]](#), page 442.

### 7.5.5.26 poly2list

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

**Usage:** `poly2list(f)`;  $f$  a poly

**Return:** list of exponents and corresponding terms of  $f$

**Purpose:** converts a poly to a list of pairs consisting of intvecs (1st entry) and polys (2nd entry), where the  $i$ -th pair contains the exponent of the  $i$ -th term of  $f$  and the  $i$ -th term (with coefficient) itself.

**Example:**

```
LIB "dmodapp.lib";
ring r = 0,x,dp;
poly F = x;
poly2list(F);
⇒ [1]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ x
ring r2 = 0,(x,y),dp;
poly F = x2y+5xy2;
poly2list(F);
⇒ [1]:
⇒ [1]:
⇒ 2,1
⇒ [2]:
⇒ x2y
⇒ [2]:
⇒ [1]:
⇒ 1,2
⇒ [2]:
⇒ 5xy2
poly2list(0);
⇒ [1]:
⇒ [1]:
⇒ 0,0
⇒ [2]:
⇒ 0
```

### 7.5.5.27 fl2poly

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

**Usage:** `fl2poly(L,s)`;  $L$  a list,  $s$  a string

**Return:** poly

**Purpose:** reconstruct a monic polynomial in one variable from its factorization

**Assume:**  $s$  is a string with the name of some variable and

$L$  is supposed to consist of two entries:

- $L[1]$  of the type ideal with the roots of a polynomial
- $L[2]$  of the type intvec with the multiplicities of corr. roots

**Example:**

```

LIB "dmodapp.lib";
ring r = 0,(x,y,z,s),Dp;
ideal I = -1,-4/3,0,-5/3,-2;
intvec mI = 2,1,2,1,1;
list BS = I,mI;
poly p = fl2poly(BS,"s");
p;
⇒ s7+7s6+173/9s5+233/9s4+154/9s3+40/9s2
factorize(p,2);
⇒ [1]:
⇒   _[1]=s+2
⇒   _[2]=3s+4
⇒   _[3]=3s+5
⇒   _[4]=s
⇒   _[5]=s+1
⇒ [2]:
⇒   1,1,1,2,2

```

**7.5.5.28 insertGenerator**

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

**Usage:**     `insertGenerator(id,p[,k]);`  
               `id` an ideal/module, `p` a poly/vector, `k` an optional int

**Return:**    of the same type as `id`

**Purpose:**    inserts `p` into `id` at `k`-th position and returns the enlarged object

**Note:**      If `k` is given, `p` is inserted at position `k`, otherwise (and by default),  
               `p` is inserted at the beginning (`k=1`).

**Example:**

```

LIB "dmodapp.lib";
ring r = 0,(x,y,z),dp;
ideal I = x^2,z^4;
insertGenerator(I,y^3);
⇒ _[1]=y3
⇒ _[2]=x2
⇒ _[3]=z4
insertGenerator(I,y^3,2);
⇒ _[1]=x2
⇒ _[2]=y3
⇒ _[3]=z4
module M = I*gen(3);
insertGenerator(M,[x^3,y^2,z],2);
⇒ _[1]=x2*gen(3)
⇒ _[2]=x3*gen(1)+y2*gen(2)+z*gen(3)
⇒ _[3]=z4*gen(3)
insertGenerator(M,x+y+z,4);
⇒ _[1]=x2*gen(3)
⇒ _[2]=z4*gen(3)
⇒ _[3]=0
⇒ _[4]=x*gen(1)+y*gen(1)+z*gen(1)

```

See also: [Section 7.5.5.29 \[deleteGenerator\]](#), page 446.

### 7.5.5.29 deleteGenerator

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

**Usage:** `deleteGenerator(id,k)`; `id` an ideal/module, `k` an int

**Return:** of the same type as `id`

**Purpose:** deletes the `k`-th generator from the first argument and returns the altered object

**Example:**

```
LIB "dmodapp.lib";
ring r = 0,(x,y,z),dp;
ideal I = x^2,y^3,z^4;
deleteGenerator(I,2);
↪ _[1]=x2
↪ _[2]=z4
module M = [x,y,z],[x2,y2,z2],[x3,y3,z3];
print(deleteGenerator(M,2));
↪ x,x3,
↪ y,y3,
↪ z,z3
M = M[1];
deleteGenerator(M,1);
↪ _[1]=0
```

See also: [Section 7.5.5.28 \[insertGenerator\]](#), page 445.

### 7.5.5.30 isInt

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

**Usage:** `isInt(n)`; `n` a number

**Return:** int, 1 if `n` is an integer or 0 otherwise

**Purpose:** check whether given object of type number is actually an int

**Note:** Parameters are treated as integers.

**Example:**

```
LIB "dmodapp.lib";
ring r = 0,x,dp;
number n = 4/3;
isInt(n);
↪ 0
n = 11;
isInt(n);
↪ 1
```

### 7.5.5.31 sortIntvec

Procedure from library `dmodapp.lib` (see [Section 7.5.5 \[dmodapp.lib\]](#), page 421).

**Usage:** `sortIntvec(v)`; `v` an intvec

**Return:** list of two intvecs

**Purpose:** sorts an intvec

**Note:** In the output list  $L$ , the first entry consists of the entries of  $v$  satisfying  $L[1][i] \geq L[1][i+1]$ . The second entry is a permutation such that  $v[L[2]] = L[1]$ .  
Unlike in the procedure `sort`, zeros are not dismissed.

**Example:**

```
LIB "dmodapp.lib";
ring r = 0,x,dp;
intvec v = -1,0,1,-2,0,2;
list L = sortIntvec(v); L;
⇒ [1]:
⇒ 2,1,0,0,-1,-2
⇒ [2]:
⇒ 6,3,2,5,1,4
v[L[2]];
⇒ 2 1 0 0 -1 -2
v = -3,0;
sortIntvec(v);
⇒ [1]:
⇒ 0,-3
⇒ [2]:
⇒ 2,1
v = 0,-3;
sortIntvec(v);
⇒ [1]:
⇒ 0,-3
⇒ [2]:
⇒ 1,2
```

See also: [Section D.2.3.15 \[sort\]](#), page 813.

### 7.5.6 dmodideal.lib

**Library:** dmodideal.lib

**Purpose:** Algorithms for Bernstein-Sato ideals of morphisms

**Authors:** Robert Loew, robert.loew at rwth-aachen.de  
Viktor Levandovskyy, levandov at math.rwth-aachen.de Jorge Martin Morales, jorge at unizar.es

**Overview:** Let  $K$  be a field of characteristic 0. Given a polynomial ring  $R = K[x_1, \dots, x_n]$  and a map, given by polynomials  $F_1, \dots, F_r$  from  $R$ , one is interested in the  $R[1/(F_1 \cdots F_r)]$ -module of rank one, generated by the symbol  $F^\mathbf{s} = F_1^{s_1} \cdots F_r^{s_r}$  for symbolic discrete variables  $s_1, \dots, s_r$ . This module  $R[1/(F_1 \cdots F_r)] F^\mathbf{s}$  has a structure of a  $D(R)[s_1, \dots, s_r]$ -module, where  $D(R)$  is an  $n$ -th Weyl algebra  $K\langle x_1, \dots, x_n, d_1, \dots, d_n \mid d_j x_j = x_j d_j + 1 \rangle$  and  $D(R)[s] := D(R)$  tensored with  $K[s] := K[s_1, \dots, s_r]$  over  $K$ . We often write just  $D$  for  $D(R)$  and  $D[s]$  for  $D(R)[s]$ .

One is interested in the computation of the following data:

- $\text{Ann}\{D[s]\} F^\mathbf{s}$ , the annihilator of  $F^\mathbf{s}$  in  $D[s]$ ; see `annihilatorMultiFs`
- $\text{Ann}^{\{1\}}\{D[s]\} F^\mathbf{s}$ , the logarithmic annihilator of  $F^\mathbf{s}$  in  $D[s]$ ; see `annfsLogIdeal`
- several kinds of global Bernstein-Sato ideals in  $K[s]$ , cf. (CU) and (Bud12); see `BernsteinSatoIdeal` and `BSidealFromAnn`

- `Ann_{D} F^alpha` for  $\alpha$  from  $K^r$ , the annihilator of  $F^\alpha$  in  $D$ ; see `annfalphaI`
- sub- and over-ideals, bounding the Bernstein-Sato ideal; see `BFBoundsBudur`

**References:**

- (BM) the `Ann F^s` algorithm by Briancon and Maisonobe (Remarques sur l'ideal de Bernstein associe a des polynomes, preprint, 2002)
- (LM08) V. Levandovskyy and J. Martin-Morales, ISSAC 2008
- (CU) Castro and Ucha, On the computation of Bernstein-Sato ideals, JSC 2005
- (SST) Saito, Sturmfels, Takayama 'Groebner Deformations of Hypergeometric Differential Equations', Springer, 2000
- (Bud12) N. Budur, Bernstein-Sato ideals and local systems, Annales de l'Institut Fourier, Volume 65 (2015) no. 2
- (OT99) T. Oaku and N. Takayama, An algorithm for de Rham cohomology groups of the complement of an affine variety via D-module computation, Journal of Pure and Applied Algebra, 1999

**Procedures:** See also: [Section 7.5.2 \[bfun\\_lib\], page 376](#); [Section 7.5.4 \[dmod\\_lib\], page 401](#); [Section 7.5.5 \[dmodapp\\_lib\], page 421](#); [Section 7.5.14 \[dmodloc\\_lib\], page 523](#); [Section D.6.13 \[gmssing\\_lib\], page 1703](#).

**7.5.6.1 annfsLogIdeal**

Procedure from library `dmodideal.lib` (see [Section 7.5.6 \[dmodideal\\_lib\], page 447](#)).

**Usage:** `annfsLogIdeal(F)`;  $F$  an ideal

**Return:** ring

**Purpose:** compute the logarithmic annihilator of  $F[1]^s(1)*\dots*F[P]^s(P)$

**Assume:** basering is a commutative polynomial ring in characteristic 0

**Note:** activate the output ring with the `setring` command. In this ring, `annfsLog` is the logarithmic annihilator of  $F^s$  (no Groebner basis). If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "dmodideal.lib";
ring R = 0,(x,y),dp;
ideal F = x^3+y^4+x*y^3, x;
def S1 = annfsLogIdeal(F);
setring S1;
annfsLog;
↪ annfsLog[1]=-9*s(1)*x-12*s(1)*y-3*s(2)*x-4*s(2)*y+3*x^2*Dx+4*x*y*Dx+2*x*y\
  *Dy+3*y^2*Dy
↪ annfsLog[2]=-36*s(1)*y^2-192*s(1)*y-9*s(2)*y^2-64*s(2)*y+9*x*y^2*Dx+9*y^3\
  *Dy+64*x*y*Dx+3*x^2*Dy-4*x*y*Dy+48*y^2*Dy
setring R; // now compare with the full annihilator
def S = annihilatorMultiFs(F);
setring S;
annFs;
↪ annFs[1]=-3*x^2*Dx-4*x*y*Dx-2*x*y*Dy-3*y^2*Dy+9*x*s(1)+12*y*s(1)+3*x*s(2)\
  +4*y*s(2)
↪ annFs[2]=-9*x*y^2*Dx-9*y^3*Dy-64*x*y*Dx-3*x^2*Dy+4*x*y*Dy-48*y^2*Dy+36*y^3\
  *s(1)+9*y^2*s(2)+192*y*s(1)+64*y*s(2)
↪ annFs[3]=-3*x^2*y^2*Dx-4*x*y^3*Dx+x*y^3*Dy+3*x^3*Dy+3*x*y^2*s(2)+4*y^3*s(\
```

```

2)
⇒ annFs[4]=-144*x^2*y*Dx^2-84*x*y^2*Dx^2-141*x*y^2*Dx*Dy-81*y^3*Dy^2-768*x^2\
2*Dx^2-256*x*y*Dx^2+84*x^2*Dx*Dy-1008*x*y*Dx*Dy-192*y^2*Dx*Dy-51*x^2*Dy^2\
+64*x*y*Dy^2-336*y^2*Dy^2+36*x*y*Dx*s(2)+84*y^2*Dx*s(2)-27*y^2*Dy*s(2)-70\
2*x*y*Dx-84*y^2*Dx-555*y^2*Dy+768*y*Dx*s(1)-384*y*Dy*s(1)+1296*y*s(1)^2+2\
56*y*Dx*s(2)-128*y*Dy*s(2)+756*y*s(1)*s(2)+108*y*s(2)^2-3712*x*Dx-256*y*D\
x+244*x*Dy-2528*y*Dy+1980*y*s(1)+6912*s(1)^2+558*y*s(2)+4608*s(1)*s(2)+76\
8*s(2)^2+8832*s(1)+2944*s(2)
lead(groebner(imap(S1,annfsLog)));
⇒ _[1]=3*x^2*Dx
⇒ _[2]=9*x*y^2*Dx
⇒ _[3]=x*y^3*Dy
⇒ _[4]=27*y^4*Dx*Dy
lead(groebner(annFs)); // and we see the difference
⇒ _[1]=3*x^2*Dx
⇒ _[2]=9*x*y^2*Dx
⇒ _[3]=3*y^3*Dx*Dy
⇒ _[4]=x*y^3*Dy

```

### 7.5.6.2 annihilatorMultiFs

Procedure from library `dmodideal.lib` (see [Section 7.5.6 \[dmodideal.lib\]](#), page 447).

**Usage:** `annihilatorMultiFs(F [,eng,us,ord]);`  $F$  an ideal, `eng`, `us`, `ord` optional ints

**Return:** ring

**Purpose:** compute  $\text{Ann}(F[1]^{s_1} \dots F[P]^{s_P})$   
with the multivariate algorithm by Briancon and Maisonobe.

**Assume:** basering is a commutative polynomial ring in characteristic 0

**Note:** activate the output ring with the `setring` command. In this ring, the ideal `annFs` is the annihilator of  $F[1]^{s_1} \dots F[P]^{s_P}$ . If `eng <> 0`, `std` is used for Groebner basis computations, otherwise, and by default `slingb` is used. If `us <> 0`, then syzygies-driven method is used additionally. If specified, `ord` describes the desired order from the following choices: 0 - 'dp'

1 - elimination order for  $x$ , 'dp' in the parts

2 - elimination order for  $s$ , 'dp' in the parts

3 - elimination order for  $x$  and  $s$ , 'dp' in the parts

4 - elimination order for  $x$  and  $D$ , 'dp' in the parts

(used for the further work in the Bernstein-Sato ideal) If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```

LIB "dmodideal.lib";
ring R = 0,(x,y),dp;
ideal F = x^2-y,y;
def S = annihilatorMultiFs(F,0,0,0);
setring S;
annFs;
⇒ annFs[1]=-2*x*y*Dy-y*Dx+2*x*s(2)
⇒ annFs[2]=-x*Dx-2*y*Dy+2*s(1)+2*s(2)
groebner(annFs);
⇒ _[1]=x*Dx+2*y*Dy-2*s(1)-2*s(2)

```

```

↪ _[2]=2*x*y*Dy+y*Dx-2*x*s(2)
↪ _[3]=4*y^2*Dy^2-y*Dx^2-4*y*Dy*s(1)-8*y*Dy*s(2)+2*y*Dy+4*s(1)*s(2)+4*s(2)^2+2*s(2)

```

### 7.5.6.3 BSidealFromAnn

Procedure from library `dmodideal.lib` (see [Section 7.5.6 \[dmodideal.lib\]](#), page 447).

**Usage:** BSidealFromAnn(F, @R [,eng,met]); F an ideal, @R a ring, eng, met optional ints

**Return:** ring

**Purpose:** compute several kinds of Bernstein-Sato ideals, associated to  $f = F[1] \dots F[P]$ , with the multivariate algorithm by Briancon and Maisonobe from `ann(F^s)` as input.

**Assume:** basering is a commutative polynomial ring in characteristic 0 @R is a ring as returned from `annihilatorMultiFs`.

**Note:** activate the output ring with the `setring` command. In this ring, the ideal BS is a Bernstein-Sato ideal of a polynomial  $f = F[1] \dots F[P]$ . If `eng > 0`, `std` is used for Groebner basis computations, otherwise, and by default `slimgb` is used. If `met` is of type int:

if `met < 0`, the B-Sigma ideal (cf. (CU)) is computed.

If  $0 < \text{met} < P$ , then the ideal  $B_{\text{met}}$  (cf. (CU)) is computed. If `met` is an intvec or a list of intvecs, Budurs generalized Bernstein-Sato ideal associated to `met` is computed.

Otherwise, and by default, the ideal  $B$  (cf. (CU)) is computed. If `met` is of type intvec: Budurs generalized Bernstein-Sato ideal  $B^{\text{met}}.F$  is computed. If `printlevel=1`, progress debug messages will be printed, if `printlevel ≥ 2`, all the debug messages will be printed.

**Example:**

```

LIB "dmodideal.lib";
ring R = 0,(x,y),dp;
ideal F = x+y,x-y,x;
def @R = annihilatorMultiFs(F, 0, 0, 4);
// first we compute the ideal B
def @R2 = BSidealFromAnn(F, @R, 0, 0);
setring @R2;
BS;
↪ BS[1]=s(1)^4*s(2)*s(3)+s(1)^4*s(2)+s(1)^4*s(3)+s(1)^4+3*s(1)^3*s(2)^2*s(3)\
)+3*s(1)^3*s(2)^2+3*s(1)^3*s(2)*s(3)^2+16*s(1)^3*s(2)*s(3)+13*s(1)^3*s(2)\
+3*s(1)^3*s(3)^2+13*s(1)^3*s(3)+10*s(1)^3+3*s(1)^2*s(2)^3*s(3)+3*s(1)^2*s\
(2)^3+6*s(1)^2*s(2)^2*s(3)^2+30*s(1)^2*s(2)^2*s(3)+24*s(1)^2*s(2)^2+3*s(1)\
)^2*s(2)*s(3)^3+30*s(1)^2*s(2)*s(3)^2+83*s(1)^2*s(2)*s(3)+56*s(1)^2*s(2)+\
3*s(1)^2*s(3)^3+24*s(1)^2*s(3)^2+56*s(1)^2*s(3)+35*s(1)^2+s(1)*s(2)^4*s(3)\
)+s(1)*s(2)^4+3*s(1)*s(2)^3*s(3)^2+16*s(1)*s(2)^3*s(3)+13*s(1)*s(2)^3+3*s\
(1)*s(2)^2*s(3)^3+30*s(1)*s(2)^2*s(3)^2+83*s(1)*s(2)^2*s(3)+56*s(1)*s(2)^2\
+2*s(1)*s(2)*s(3)^4+16*s(1)*s(2)*s(3)^3+83*s(1)*s(2)*s(3)^2+162*s(1)*s(2)*\
s(3)+94*s(1)*s(2)+s(1)*s(3)^4+13*s(1)*s(3)^3+56*s(1)*s(3)^2+94*s(1)*s(3)+\
50*s(1)+s(2)^4*s(3)+s(2)^4+3*s(2)^3*s(3)^2+13*s(2)^3*s(3)+10*s(2)^3+3*s(2)\
)^2*s(3)^3+24*s(2)^2*s(3)^2+56*s(2)^2*s(3)+35*s(2)^2+s(2)*s(3)^4+13*s(2)*\
s(3)^3+56*s(2)*s(3)^2+94*s(2)*s(3)+50*s(2)+s(3)^4+10*s(3)^3+35*s(3)^2+50*\
s(3)+24
setring R;
// secondly we compute the ideal B_1

```



```

@R2 = BSidealFromAnn(F, @R, 0, 1);
setring @R2;
BS;
↪ BS[1]=s(1)^2+s(1)*s(2)+s(1)*s(3)+3*s(1)+s(2)+s(3)+2

```

#### 7.5.6.4 BernsteinSatoIdeal

Procedure from library `dmodideal.lib` (see [Section 7.5.6 \[dmodideal.lib\]](#), page 447).

**Usage:** `BernsteinSatoIdeal(F [,eng,met,us]);`  $F$  an ideal, `eng`, `us` optional ints, `met` optional int or intvec

**Return:** ring

**Purpose:** compute two kinds of Bernstein-Sato ideals, associated to  $f = F[1]*..*F[P]$ , with the multivariate algorithm by Briancon and Maisonobe.

**Assume:** `basering` is a commutative polynomial ring in characteristic 0

**Note:** activate the output ring with the `setring` command. In this ring,  
 - the ideal `LD` is the annihilator of  $F[1]^{s_1}*..*F[P]^{s_p}$ ,  
 - the list or ideal `BS` is a Bernstein-Sato ideal of a polynomial  $f = F[1]*..*F[P]$ . If `eng`  $\neq 0$ , `std` is used for Groebner basis computations, otherwise, and by default `slimgb` is used. If `met`  $< 0$ , the B-Sigma ideal (cf. Castro and Ucha, 'On the computation of Bernstein-Sato ideals', 2005) is computed. If  $0 < \text{met} < P$ , then the ideal `B_P` (cf. the paper) is computed. If `met` is an intvec, Budurs generalized Bernstein-Sato ideal associated to `met` is computed. Otherwise, and by default, the ideal `B` (cf. the paper) is computed. If `us`  $\neq 0$ , then syzygies-driven method is used.  
 If `printlevel=1`, progress debug messages will be printed, if `printlevel`  $\geq 2$ , all the debug messages will be printed.

**Example:**

```

LIB "dmodideal.lib";
ring R = 0,(x,y),dp;
ideal F = x^2-y,y;
// first we compute the ideal B:
def S = BernsteinSatoIdeal(F);
setring S;
BS;
↪ BS[1]=4*s(1)^3*s(2)+4*s(1)^3+8*s(1)^2*s(2)^2+28*s(1)^2*s(2)+20*s(1)^2+4*s\
(1)*s(2)^3+28*s(1)*s(2)^2+55*s(1)*s(2)+31*s(1)+4*s(2)^3+20*s(2)^2+31*s(2)\
+15
// secondly we compute the ideal B_1:
setring R;
def S = BernsteinSatoIdeal(F,0,1);
↪ // ** redefining S (def S = BernsteinSatoIdeal(F,0,1);) ./examples/Bernst\
einSatoIdeal.sing:10
setring S;
BS;
↪ BS[1]=2*s(1)^2+2*s(1)*s(2)+5*s(1)+2*s(2)+3
// thirdly we compute the ideal B_sigma:
setring R;
def S = BernsteinSatoIdeal(F,0,-1);
↪ // ** redefining S (def S = BernsteinSatoIdeal(F,0,-1);) ./examples/Berns\

```

```

      teinSatoIdeal.sing:15
setring S;
BS;
↪ BS[1]=2*s(1)*s(2)+2*s(1)+2*s(2)^2+5*s(2)+3
↪ BS[2]=2*s(1)^2+3*s(1)-2*s(2)^2-3*s(2)

```

### 7.5.6.5 BFBoundsBudur

Procedure from library `dmodideal.lib` (see [Section 7.5.6 \[dmodideal.lib\], page 447](#)).

**Usage:** BFBoundsBudur(F,m); F an ideal, m an intvec

**Return:** ring

**Assume:** basering is a commutative polynomial ring in characteristic 0

**Purpose:** determine upper and lower bounds of the Bernstein-Sato ideal associated to m with the method of (Bud12)

**Note:** The returned ring contains lists Bj, containing the Bernstein-Sato ideals associated to  $e_j$ , shiftedIdeals, containing the shifted ideals from (Bud12) 4.7, and ideals upperBound, lowerBound which give upper bound and lower bound for the Bernstein-Sato-Ideal associated to m respectively.

**Example:**

```

LIB "dmodideal.lib";
ring r = 0,(x,y,z),dp;
setring r;
ideal F = x*z,2*x^2*y^2*z+x^4+y^4;
def A = BFBoundsBudur(F,intvec(1,1));
setring A;
lead(upperBound);
↪ _[1]=2*s(1)^8*s(2)^2
↪ _[2]=s(1)^9*s(2)
lead(lowerBound);
↪ _[1]=s(1)^11*s(2)
↪ _[2]=2*s(1)^10*s(2)^2
↪ _[3]=2*s(1)^10*s(2)^2
↪ _[4]=4*s(1)^9*s(2)^3

```

### 7.5.6.6 annfalphaI

Procedure from library `dmodideal.lib` (see [Section 7.5.6 \[dmodideal.lib\], page 447](#)).

**Usage:** annfalphaI(f,alpha); f,alpha ideals

**Return:** ring

**Assume:** basering is a commutative polynomial ring in characteristic 0

**Purpose:** determine annihilator of  $f^\alpha$  with the method of (OT99)

**Note:** The returned ring contains the annihilator of  $f^\alpha$  over D as annfalpha. alpha should contain the desired rational exponents.  
The procedure may also be applied to the univariate case, i.e. for  $r=1$ .

**Example:**

```

LIB "dmodideal.lib";
ring R = 0,(x,y,z),dp;
ideal f = x,y,z;
ideal alpha = 1/4, 2/3, 1;
def A = annfalphaI(f,alpha);
setring A;
annfalpha;
↪ annfalpha[1]=Dz^2
↪ annfalpha[2]=z*Dz-1
↪ annfalpha[3]=3*y*Dy-2
↪ annfalpha[4]=4*x*Dx-1

```

### 7.5.6.7 extractS

Procedure from library `dmodideal.lib` (see [Section 7.5.6 \[dmodideal.lib\]](#), page 447).

**Usage:** `extractS(I,r);` I ideal, r int

**Return:** ring

**Assume:** I is an ideal in the first r variables of the basering and these r variables generate a commutative subring

**Purpose:** give the ideal generated by I in the commutative subring generated by the first r variables, ordering dp

**Note:** The returned ring contains I.

**Example:**

```

LIB "dmodideal.lib";
ring R = 0,(x,y),dp;
ideal f = x^2-y^2,y;
def S = BernsteinSatoIdeal(f);
setring S;
BS;
↪ BS[1]=8*s(1)^4*s(2)+8*s(1)^4+12*s(1)^3*s(2)^2+56*s(1)^3*s(2)+44*s(1)^3+6*\
s(1)^2*s(2)^3+54*s(1)^2*s(2)^2+136*s(1)^2*s(2)+88*s(1)^2+s(1)*s(2)^4+16*s(\
(1)*s(2)^3+77*s(1)*s(2)^2+138*s(1)*s(2)+76*s(1)+s(2)^4+10*s(2)^3+35*s(2)^2\
+50*s(2)+24
def T = extractS(BS,2);
setring T;
I;
↪ I[1]=8*s(1)^4*s(2)+12*s(1)^3*s(2)^2+6*s(1)^2*s(2)^3+s(1)*s(2)^4+8*s(1)^4+\
56*s(1)^3*s(2)+54*s(1)^2*s(2)^2+16*s(1)*s(2)^3+s(2)^4+44*s(1)^3+136*s(1)^2\
2*s(2)+77*s(1)*s(2)^2+10*s(2)^3+88*s(1)^2+138*s(1)*s(2)+35*s(2)^2+76*s(1)\
+50*s(2)+24
factorize(I[1]);
↪ [1]:
↪ _[1]=1
↪ _[2]=s(1)+1
↪ _[3]=s(2)+1
↪ _[4]=2*s(1)+s(2)+2
↪ _[5]=2*s(1)+s(2)+3
↪ _[6]=2*s(1)+s(2)+4
↪ [2]:
↪ 1,1,1,1,1,1

```

### 7.5.7 dmodvar.lib

**Library:** dmodvar.lib

**Purpose:** Algebraic D-modules for varieties

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Support: DFG Graduiertenkolleg 1632 'Experimentelle und konstruktive Algebra'

**Overview:** Let  $K$  be a field of characteristic 0. Given a polynomial ring  $R = K[x_1, \dots, x_n]$  and polynomials  $f_1, \dots, f_r$  in  $R$ , define  $F = f_1 \cdots f_r$  and  $F^s = f_1^{s_1} \cdots f_r^{s_r}$  for symbolic discrete (that is shiftable) variables  $s_1, \dots, s_r$ . The module  $R[1/F]^s$  has the structure of a  $D\langle S \rangle$ -module, where  $D\langle S \rangle = D(R)$  tensored with  $S$  over  $K$ , where

- $D(R)$  is the  $n$ -th Weyl algebra  $K\langle x_1, \dots, x_n, d_1, \dots, d_n \mid d_j x_j = x_j d_j + 1 \rangle$
- $S$  is the universal enveloping algebra of  $gl_r$ , generated by  $s_i = s_{\{i\}}$ .

One is interested in the following data:

- the left ideal  $\text{Ann } F^s$  in  $D\langle S \rangle$ , usually denoted by  $LD$  in the output
- global Bernstein polynomial in one variable  $s = s_1 + \dots + s_r$ , denoted by  $bs$ ,
- its minimal integer root  $s_0$ , the list of all roots of  $bs$ , which are known to be negative rational numbers, with their multiplicities, which is denoted by  $BS$
- an  $r$ -tuple of operators in  $D\langle S \rangle$ , denoted by  $PS$ , such that the functional equality  $\sum_{k=1}^r P_k f_k F^s = bs F^s$  holds in  $R[1/F]^s$ .

**References:**

- (BMS06) Budur, Mustata, Saito: Bernstein-Sato polynomials of arbitrary varieties (2006).  
 (ALM09) Andres, Levandovskyy, Martin-Morales: Principal Intersection and Bernstein-Sato Polynomial of an Affine Variety (2009).

**Procedures:** See also: [Section 7.5.2 \[bfun.lib\]](#), page 376; [Section 7.5.4 \[dmod.lib\]](#), page 401; [Section 7.5.5 \[dmodapp.lib\]](#), page 421; [Section D.6.13 \[gmssing.lib\]](#), page 1703.

#### 7.5.7.1 bfctVarIn

Procedure from library `dmodvar.lib` (see [Section 7.5.7 \[dmodvar.lib\]](#), page 454).

**Usage:** bfctVarIn( $I$  [ $a, b, c$ ]);  $I$  an ideal,  $a, b, c$  optional ints

**Return:** list of ideal and intvec

**Purpose:** computes the roots of the Bernstein-Sato polynomial and their multiplicities for an affine algebraic variety defined by  $I$ .

**Assume:** The basering is commutative and over a field of characteristic 0.  
 Varnames of the basering do not include  $t(1), \dots, t(r)$  and  $Dt(1), \dots, Dt(r)$ , where  $r$  is the number of entries of the input ideal.

**Note:** In the output list, say  $L$ ,

- $L[1]$  of type ideal contains all the rational roots of a b-function,
- $L[2]$  of type intvec contains the multiplicities of above roots,
- optional  $L[3]$  of type string is the part of b-function without rational roots.

Note, that a b-function of degree 0 is encoded via  $L[1][1]=0$ ,  $L[2]=0$  and  $L[3]$  is 1 (for nonzero constant) or 0 (for zero b-function).  
 If  $a > 0$ , the ideal is used as given. Otherwise, and by default, a heuristically better

suited generating set is used to reduce computation time.

If  $b < 0$ , `std` is used for GB computations in characteristic 0, otherwise, and by default, `slimgb` is used.

If  $c < 0$ , a matrix ordering is used for GB computations, otherwise, and by default, a block ordering is used.

Further note, that in this proc, the initial ideal of the multivariate Malgrange ideal defined by  $I$  is computed and then a system of linear equations is solved by linear reductions following the ideas by Noro.

The result is shifted by  $1 - \text{codim}(\text{Var}(F))$  following (BMS06).

**Display:** If `printlevel=1`, progress debug messages will be printed,  
if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "dmodvar.lib";
ring R = 0,(x,y,z),dp;
ideal F = x^2+y^3, z;
list L = bfctVarIn(F);
L;
⇒ [1]:
⇒   _[1]==-5/6
⇒   _[2]==-1
⇒   _[3]==-7/6
⇒ [2]:
⇒   1,1,1
```

### 7.5.7.2 bfctVarAnn

Procedure from library `dmodvar.lib` (see [Section 7.5.7 \[dmodvar.lib\], page 454](#)).

**Usage:** `bfctVarAnn(F[,gid,eng])`;  $F$  an ideal, `gid`, `eng` optional ints

**Return:** list of an ideal and an intvec

**Purpose:** computes the roots of the Bernstein-Sato polynomial and their multiplicities for an affine algebraic variety defined by  $F = F[1], \dots, F[r]$ .

**Assume:** The basering is commutative and over a field in char 0.

**Note:** In the output list, the ideal contains all the roots and the intvec their multiplicities.  
If  $gid < 0$ , the ideal is used as given. Otherwise, and by default, a heuristically better suited generating set is used.

If  $eng < 0$ , `std` is used for GB computations, otherwise, and by default, `slimgb` is used.

Computational remark: The time of computation can be very different depending on the chosen generators of  $F$ , although the result is always the same.

Further note that in this proc, the annihilator of  $f^s$  in  $D[s]$  is computed and then a system of linear equations is solved by linear reductions in order to find the minimal polynomial of  $S = s(1)(1) + \dots + s(P)(P)$ . The result is shifted by  $1 - \text{codim}(\text{Var}(F))$  following (BMS06).

**Display:** If `printlevel=1`, progress debug messages will be printed,  
if `printlevel=2`, all the debug messages will be printed.

**Example:**

```
LIB "dmodvar.lib";
ring R = 0,(x,y,z),Dp;
```

```

ideal F = x^2+y^3, z;
bfctVarAnn(F);
↳ [1]:
↳   _[1]=-5/6
↳   _[2]=-1
↳   _[3]=-7/6
↳ [2]:
↳   1,1,1

```

### 7.5.7.3 SannfsVar

Procedure from library `dmodvar.lib` (see [Section 7.5.7 \[dmodvar.lib\]](#), page 454).

**Usage:** `SannfsVar(F [,ORD,eng]);`  $F$  an ideal,  $ORD$  an optional string,  $eng$  an optional int

**Return:** ring (Weyl algebra tensored with  $U(\mathfrak{gl}_P)$ ), containing an ideal  $LD$

**Purpose:** compute the  $D\langle S \rangle$ -module structure of  $D\langle S \rangle * f^s$  where  $f = F[1] * \dots * F[P]$  and  $D\langle S \rangle$  is the Weyl algebra  $D$  tensored with  $K\langle S \rangle = U(\mathfrak{gl}_P)$ , according to the generalized algorithm by Briancon and Maisonobe for affine varieties

**Assume:** The basering is commutative and over a field of characteristic 0.

**Note:** Activate the output ring  $D\langle S \rangle$  with the `setring` command. In the ring  $D\langle S \rangle$ , the ideal  $LD$  is the needed  $D\langle S \rangle$ -module structure.

The value of  $ORD$  must be an elimination ordering in  $D\langle Dt, S \rangle$  for  $Dt$  written in the string form, otherwise the result may have no meaning. By default  $ORD = '(a(1..(P)..1), a(1..(P+P^2)..1), dp)'$ .

If  $eng < 0$ , `std` is used for Groebner basis computations, otherwise, and by default `slimgb` is used.

**Display:** If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```

LIB "dmodvar.lib";
ring R = 0, (x,y), Dp;
ideal F = x^3, y^5;
//ORD = "(a(1,1),a(1,1,1,1,1,1),dp)";
//eng = 0;
def A = SannfsVar(F);
setring A;
A;
↳ // coefficients: QQ
↳ // number of vars : 8
↳ //      block 1 : ordering a
↳ //      : names      s(1)(1) s(1)(2) s(2)(1) s(2)(2)
↳ //      : weights      1      1      1      1
↳ //      block 2 : ordering dp
↳ //      : names      s(1)(1) s(1)(2) s(2)(1) s(2)(2) x y Dx Dy
↳ //      block 3 : ordering C
↳ // noncommutative relations:
↳ //      s(1)(2)s(1)(1)=s(1)(1)*s(1)(2)-s(1)(2)
↳ //      s(2)(1)s(1)(1)=s(1)(1)*s(2)(1)+s(2)(1)
↳ //      s(2)(1)s(1)(2)=s(1)(2)*s(2)(1)-s(1)(1)+s(2)(2)
↳ //      s(2)(2)s(1)(2)=s(1)(2)*s(2)(2)-s(1)(2)

```

```

⇒ //      s(2)(2)s(2)(1)=s(2)(1)*s(2)(2)+s(2)(1)
⇒ //      Dxx=x*Dx+1
⇒ //      Dyy=y*Dy+1
LD;
⇒ LD[1]=5*s(2)(2)-y*Dy
⇒ LD[2]=3*s(1)(1)-x*Dx
⇒ LD[3]=15*s(1)(2)*s(2)(1)-x*y*Dx*Dy-5*x*Dx
⇒ LD[4]=5*s(2)(1)*y^4-x^3*Dy
⇒ LD[5]=3*s(1)(2)*x^2-y^5*Dx

```

#### 7.5.7.4 makeMalgrange

Procedure from library `dmodvar.lib` (see [Section 7.5.7 \[dmodvar.lib\], page 454](#)).

**Usage:** `makeMalgrange(F [,ORD]);` F an ideal, ORD an optional string

**Return:** ring (Weyl algebra) containing an ideal IF

**Purpose:** create the ideal by Malgrange associated with  $F = F[1], \dots, F[P]$ .

**Note:** Activate the output ring with the `setring` command. In this ring, the ideal IF is the ideal by Malgrange corresponding to F.  
The value of ORD must be an arbitrary ordering in  $K\langle t, x, Dt, Dx \rangle$  written in the string form. By default `ORD = 'dp'`.

**Display:** If `printlevel=1`, progress debug messages will be printed,  
if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```

LIB "dmodvar.lib";
ring R = 0,(x,y,z),Dp;
ideal I = x^2+y^3, z;
def W = makeMalgrange(I);
setring W;
W;
⇒ // coefficients: QQ
⇒ // number of vars : 10
⇒ //      block 1 : ordering dp
⇒ //      : names  t(1) t(2) x y z Dt(1) Dt(2) Dx Dy Dz
⇒ //      block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      Dt(1)t(1)=t(1)*Dt(1)+1
⇒ //      Dt(2)t(2)=t(2)*Dt(2)+1
⇒ //      Dxx=x*Dx+1
⇒ //      Dyy=y*Dy+1
⇒ //      Dzz=z*Dz+1
IF;
⇒ IF[1]=-y^3-x^2+t(1)
⇒ IF[2]=t(2)-z
⇒ IF[3]=2*x*Dt(1)+Dx
⇒ IF[4]=3*y^2*Dt(1)+Dy
⇒ IF[5]=Dt(2)+Dz

```

#### 7.5.8 involut.lib

**Library:** `involut.lib`

**Purpose:** Computations and operations with involutions

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**Overview:** Involution is an anti-automorphism of a non-commutative K-algebra with the property that applied an involution twice, one gets an identity. Involution is linear with respect to the ground field. In this library we compute linear involutions, distinguishing the case of a diagonal matrix (such involutions are called homothetic) and a general one. Also, linear automorphisms of different order can be computed.

**Support:** Forschungsschwerpunkt 'Mathematik und Praxis' (Project of Dr. E. Zerz and V. Levandovskyy), Uni Kaiserslautern

**Remark:** This library provides algebraic tools for computations and operations with algebraic involutions and linear automorphisms of non-commutative algebras

**Procedures:**

### 7.5.8.1 findInvo

Procedure from library `involut.lib` (see [Section 7.5.8 \[involut.lib\]](#), page 457).

**Usage:** `findInvo();`

**Return:** a ring containing a list L of pairs, where  
 $L[i][1]$  = ideal; a Groebner Basis of an i-th associated prime,  
 $L[i][2]$  = matrix, defining a linear map, with entries, reduced with respect to  $L[i][1]$

**Purpose:** computed the ideal of linear involutions of the basering

**Assume:** the relations on the algebra are of the form  $YX = XY + D$ , that is the current ring is a G-algebra of Lie type.

**Note:** for convenience, the full ideal of relations `idJ` and the initial matrix with indeterminates `matD` are exported in the output ring

**Example:**

```
LIB "involut.lib";
def a = makeWeyl(1);
setring a; // this algebra is a first Weyl algebra
a;
⇒ // coefficients: QQ
⇒ // number of vars : 2
⇒ //      block 1 : ordering dp
⇒ //      : names x D
⇒ //      block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      Dx=xD+1
def X = findInvo();
setring X; // ring with new variables, corr. to unknown coefficients
X;
⇒ // coefficients: QQ
⇒ // number of vars : 4
⇒ //      block 1 : ordering dp
⇒ //      : names a11 a12 a21 a22
⇒ //      block 2 : ordering C
```



```

L;
⇒ [1]:
⇒ [1]:
⇒ [1]=a11+a22
⇒ [2]=a12*a21+a22^2-1
⇒ [2]:
⇒ [1,1]=-a22
⇒ [1,2]=a12
⇒ [2,1]=a21
⇒ [2,2]=a22
// look at the matrix in the new variables, defining the linear involution
print(L[1][2]);
⇒ -a22,a12,
⇒ a21, a22
L[1][1]; // where new variables obey these relations
⇒ [1]=a11+a22
⇒ [2]=a12*a21+a22^2-1
idJ;
⇒ idJ[1]=-a12*a21+a11*a22+1
⇒ idJ[2]=a11^2+a12*a21-1
⇒ idJ[3]=a11*a12+a12*a22
⇒ idJ[4]=a11*a21+a21*a22
⇒ idJ[5]=a12*a21+a22^2-1

```

See also: [Section 7.5.8.2 \[findInvoDiag\]](#), page 459; [Section 7.5.8.5 \[involution\]](#), page 462.

### 7.5.8.2 findInvoDiag

Procedure from library `involut.lib` (see [Section 7.5.8 \[involut.lib\]](#), page 457).

**Usage:** `findInvoDiag();`

**Return:** a ring together with a list of pairs `L`, where  
`L[i][1]` = ideal; a Groebner Basis of an  $i$ -th associated prime,  
`L[i][2]` = matrix, defining a linear map, with entries, reduced with respect to `L[i][1]`

**Purpose:** compute homothetic (diagonal) involutions of the basering

**Assume:** the relations on the algebra are of the form  $YX = XY + D$ , that is the current ring is a  $G$ -algebra of Lie type.

**Note:** for convenience, the full ideal of relations `idJ` and the initial matrix with indeterminates `matD` are exported in the output ring

**Example:**

```

LIB "involut.lib";
def a = makeWeyl(1);
setring a; // this algebra is a first Weyl algebra
a;
⇒ // coefficients: QQ
⇒ // number of vars : 2
⇒ //      block 1 : ordering dp
⇒ //      : names x D
⇒ //      block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      Dx=xD+1

```

```

def X = findInvoDiag();
setring X; // ring with new variables, corresponding to unknown coefficients
X;
⇒ // coefficients: QQ
⇒ // number of vars : 2
⇒ //      block 1 : ordering dp
⇒ //      : names  a11 a22
⇒ //      block 2 : ordering C
// print matrices, defining linear involutions
print(L[1][2]); // a first matrix: we see it is constant
⇒ -1,0,
⇒ 0, 1
print(L[2][2]); // and a second possible matrix; it is constant too
⇒ 1,0,
⇒ 0,-1
L; // let us take a look on the whole list
⇒ [1]:
⇒   [1]:
⇒   _[1]=a22-1
⇒   _[2]=a11+1
⇒   [2]:
⇒   _[1,1]=-1
⇒   _[1,2]=0
⇒   _[2,1]=0
⇒   _[2,2]=1
⇒ [2]:
⇒   [1]:
⇒   _[1]=a22+1
⇒   _[2]=a11-1
⇒   [2]:
⇒   _[1,1]=1
⇒   _[1,2]=0
⇒   _[2,1]=0
⇒   _[2,2]=-1
idJ;
⇒ idJ[1]=a11*a22+1
⇒ idJ[2]=a11^2-1
⇒ idJ[3]=a22^2-1

```

See also: [Section 7.5.8.1 \[findInvo\]](#), page 458; [Section 7.5.8.5 \[involution\]](#), page 462.

### 7.5.8.3 findAuto

Procedure from library `involut.lib` (see [Section 7.5.8 \[involut.lib\]](#), page 457).

**Usage:** findAuto(n); n an integer

**Return:** a ring together with a list of pairs L, where  
 $L[i][1]$  = ideal; a Groebner Basis of an i-th associated prime,  
 $L[i][2]$  = matrix, defining a linear map, with entries, reduced with respect to  $L[i][1]$

**Purpose:** compute the ideal of linear automorphisms of the basering,  
given by a matrix, n-th power of which gives identity (i.e. unipotent matrix)

**Assume:** the relations on the algebra are of the form  $YX = XY + D$ , that is the current ring is a G-algebra of Lie type.

**Note:** if  $n=0$ , a matrix, defining an automorphism is not assumed to be unipotent but just non-degenerate. A nonzero parameter  $@p$  is introduced as the value of the determinant of the matrix above.  
 For convenience, the full ideal of relations  $idJ$  and the initial matrix with indeterminates  $matD$  are mutually exported in the output ring

**Example:**

```
LIB "involut.lib";
def a = makeWeyl(1);
setring a; // this algebra is a first Weyl algebra
a;
⇒ // coefficients: QQ
⇒ // number of vars : 2
⇒ //          block 1 : ordering dp
⇒ //          : names    x D
⇒ //          block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      Dx=xD+1
def X = findAuto(2); // in contrast to findInvo look for automorphisms
setring X; // ring with new variables - unknown coefficients
X;
⇒ // coefficients: QQ
⇒ // number of vars : 4
⇒ //          block 1 : ordering dp
⇒ //          : names    a11 a12 a21 a22
⇒ //          block 2 : ordering C
size(L); // we have (size(L)) families in the answer
⇒ 2
// look at matrices, defining linear automorphisms:
print(L[1][2]); // a first one: we see it is the identity
⇒ 1,0,
⇒ 0,1
print(L[2][2]); // and a second possible matrix; it is diagonal
⇒ -1,0,
⇒ 0, -1
// L; // we can take a look on the whole list, too
idJ;
⇒ idJ[1]=-a12*a21+a11*a22-1
⇒ idJ[2]=a11^2+a12*a21-1
⇒ idJ[3]=a11*a12+a12*a22
⇒ idJ[4]=a11*a21+a21*a22
⇒ idJ[5]=a12*a21+a22^2-1
kill X; kill a;
//----- find all the linear automorphisms -----
//----- use the call findAuto(0) -----
ring R = 0,(x,s),dp;
def r = nc_algebra(1,s); setring r; // the shift algebra
s*x; // the only relation in the algebra is:
⇒ xs+s
def Y = findAuto(0);
setring Y;
size(L); // here, we have 1 parametrized family
```

```

↳ 1
print(L[1][2]); // here, @p is a nonzero parameter
↳ 1,a12,
↳ 0,(@p)
det(L[1][2]-@p); // check whether determinante is zero
↳ 0

```

See also: [Section 7.5.8.1 \[findInvo\]](#), page 458.

#### 7.5.8.4 ncdetection

Procedure from library `involut.lib` (see [Section 7.5.8 \[involut.lib\]](#), page 457).

**Usage:** `ncdetection()`;

**Return:** ideal, representing an involution map

**Purpose:** compute classical involutions (i.e. acting rather on operators than on variables) for some particular noncommutative algebras

**Assume:** the procedure is aimed at non-commutative algebras with differential, shift or advance operators arising in Control Theory. It has to be executed in a ring.

**Example:**

```

LIB "involut.lib";
ring R = 0,(x,y,z,D(1..3)),dp;
matrix D[6][6];
D[1,4]=1; D[2,5]=1; D[3,6]=1;
def r = nc_algebra(1,D); setring r;
ncdetection();
↳ _[1]=x
↳ _[2]=y
↳ _[3]=z
↳ _[4]=-D(1)
↳ _[5]=-D(2)
↳ _[6]=-D(3)
kill r, R;
//-----
ring R=0,(x,S),dp;
def r = nc_algebra(1,-S); setring r;
ncdetection();
↳ _[1]=-x
↳ _[2]=S
kill r, R;
//-----
ring R=0,(x,D(1),S),dp;
matrix D[3][3];
D[1,2]=1; D[1,3]=-S;
def r = nc_algebra(1,D); setring r;
ncdetection();
↳ _[1]=-x
↳ _[2]=D(1)
↳ _[3]=S

```

#### 7.5.8.5 involution

Procedure from library `involut.lib` (see [Section 7.5.8 \[involut.lib\]](#), page 457).

- Usage:** `involution(m, theta)`; `m` is a poly/vector/ideal/matrix/module, `theta` is a map
- Return:** object of the same type as `m`
- Purpose:** applies the involution, presented by `theta` to the object `m`
- Theory:** for an involution `theta` and two polynomials `a, b` from the algebra,  
 $\text{theta}(ab) = \text{theta}(b) \text{theta}(a)$ ; `theta` is linear with respect to the ground field
- Note:** This is generalized "`theta(m)`" for data types unsupported by "`map`".

**Example:**

```
LIB "involut.lib";
ring R = 0,(x,d),dp;
def r = nc_algebra(1,1); setring r; // Weyl-Algebra
map F = r,x,-d;
F(F); // should be maxideal(1) for an involution
⇨ _[1]=x
⇨ _[2]=d
poly f = x*d^2+d;
poly If = involution(f,F);
f-If;
⇨ 0
poly g = x^2*d+2*x*d+3*x+7*d;
poly tg = -d*x^2-2*d*x+3*x-7*d;
poly Ig = involution(g,F);
tg-Ig;
⇨ 0
ideal I = f,g;
ideal II = involution(I,F);
II;
⇨ II[1]=xd2+d
⇨ II[2]=-x2d-2xd+x-7d-2
matrix(I) - involution(II,F);
⇨ _[1,1]=0
⇨ _[1,2]=0
module M = [f,g,0],[g,0,x^2*d];
module IM = involution(M,F);
print(IM);
⇨ xd2+d, -x2d-2xd+x-7d-2,
⇨ -x2d-2xd+x-7d-2,0,
⇨ 0, -x2d-2x
print(matrix(M) - involution(IM,F));
⇨ 0,0,
⇨ 0,0,
⇨ 0,0
```

**7.5.8.6 isInvolution**

Procedure from library `involut.lib` (see [Section 7.5.8 \[involut.lib\]](#), page 457).

- Usage:** `isInvolution(F)`; `F` is a map from current ring to itself
- Return:** integer, 1 if `F` determines an involution and 0 otherwise
- Theory:** involution is an antiautomorphism of order 2

**Assume:** F is a map from current ring to itself

**Example:**

```
LIB "involut.lib";
def A = makeUs1(2); setring A;
map I = A,-e,-f,-h; //correct antiauto involution
isInvolution(I);
 $\mapsto$  1
map J = A,3*e,1/3*f,-h; // antiauto but not involution
isInvolution(J);
 $\mapsto$  0
map K = A,f,e,-h; // not antiauto
isInvolution(K);
 $\mapsto$  0
```

See also: [Section 7.5.8.1 \[findInvo\]](#), page 458; [Section 7.5.8.5 \[involution\]](#), page 462; [Section 7.5.8.7 \[isAntiEndo\]](#), page 464.

### 7.5.8.7 isAntiEndo

Procedure from library `involut.lib` (see [Section 7.5.8 \[involut.lib\]](#), page 457).

**Usage:** isAntiEndo(F); F is a map from current ring to itself

**Return:** integer, 1 if F determines an antiendomorphism of current ring and 0 otherwise

**Assume:** F is a map from current ring to itself

**Example:**

```
LIB "involut.lib";
def A = makeUs1(2); setring A;
map I = A,-e,-f,-h; //correct antiauto involution
isAntiEndo(I);
 $\mapsto$  1
map J = A,3*e,1/3*f,-h; // antiauto but not involution
isAntiEndo(J);
 $\mapsto$  1
map K = A,f,e,-h; // not antiendo
isAntiEndo(K);
 $\mapsto$  0
```

See also: [Section 7.5.8.1 \[findInvo\]](#), page 458; [Section 7.5.8.5 \[involution\]](#), page 462; [Section 7.5.8.6 \[isInvolution\]](#), page 463.

### 7.5.9 gkdim\_lib

**Library:** gkdim.lib

**Purpose:** Procedures for calculating the Gelfand-Kirillov dimension

**Authors:** Lobillo, F.J., jlobillo@ugr.es  
Rabelo, C., crabelo@ugr.es

Support: 'Metodos algebraicos y efectivos en grupos cuanticos', BFM2001-3141, MCYT, Jose Gomez-Torrecillas (Main researcher).

**Note:** The built-in command `dim`, executed for a module in `@plural`, computes the Gelfand-Kirillov dimension.

**Procedures:**

### 7.5.9.1 GKdim

Procedure from library `gkdim.lib` (see [Section 7.5.9 \[gkdim.lib\]](#), page 464).

**Usage:** `GKdim(L)`;  $L$  is a left ideal/module/matrix

**Return:** `int`

**Purpose:** compute the Gelfand-Kirillov dimension of the factor-module, whose presentation is given by  $L$ , e.g.  $R^r/L$

**Note:** if the factor-module is zero, -1 is returned

**Example:**

```
LIB "gkdim.lib";
ring R = 0,(x,y,z),Dp;
matrix C[3][3]=0,1,1,0,0,-1,0,0,0;
matrix D[3][3]=0,0,0,0,0,x;
def r = nc_algebra(C,D); setring r;
r;
↪ // coefficients: QQ
↪ // number of vars : 3
↪ //          block 1 : ordering Dp
↪ //          : names  x y z
↪ //          block 2 : ordering C
↪ // noncommutative relations:
↪ //      zy=-yz+x
ideal I=x;
GKdim(I);
↪ 2
ideal J=x^2,y;
GKdim(J);
↪ 1
module M=[x^2,y,1],[x,y^2,0];
GKdim(M);
↪ 3
ideal A = x,y,z;
GKdim(A);
↪ 0
ideal B = 1;
GKdim(B);
↪ -1
GKdim(ideal(0)) == nvars(basering); // should be true, i.e., evaluated to 1
↪ 1
```

### 7.5.10 ncalg\_lib

**Library:** `ncalg.lib`

**Purpose:** Definitions of important G- and GR-algebras

**Authors:** Viktor Levandovskyy, levandov@mathematik.uni-kl.de,  
Oleksandr Motsak, U@D, where  $U=\{\text{motsak}\}$ ,  $D=\{\text{mathematik.uni-kl.de}\}$

**Conventions:**

This library provides pre-defined important noncommutative algebras.

For universal enveloping algebras of finite dimensional Lie algebras `sl_n`, `gl_n`, `g_2` etc.

there are functions `makeUsl`, `makeUgl`, `makeUg2` etc.

For quantized enveloping algebras  $U_q(\mathfrak{sl}_2)$  and  $U_q(\mathfrak{sl}_3)$ , there are functions `makeQsl2`, `makeQsl3`) and for non-standard quantum deformation of  $\mathfrak{so}_3$ , there is the function `makeQso3`.

For bigger algebras we suppress the output of the (lengthy) list of non-commutative relations and provide only the number of these relations instead.

#### Procedures:

##### 7.5.10.1 `makeUsl2`

Procedure from library `ncalg.lib` (see [Section 7.5.10 \[ncalg-lib\]](#), page 465).

**Usage:** `makeUsl2([p])`,  $p$  an optional integer (field characteristic)

**Return:** ring

**Purpose:** set up the  $U(\mathfrak{sl}_2)$  in the variables  $e, f, h$  over the field of char  $p$

**Note:** activate this ring with the `setring` command

#### Example:

```
LIB "ncalg.lib";
def a=makeUsl2();
setring a;
a;
⇒ // coefficients: QQ
⇒ // number of vars : 3
⇒ //          block 1 : ordering dp
⇒ //          : names      e f h
⇒ //          block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      fe=ef-h
⇒ //      he=eh+2e
⇒ //      hf=fh-2f
```

See also: [Section 7.5.10.17 \[makeUg2\]](#), page 476; [Section 7.5.10.3 \[makeUgl\]](#), page 467; [Section 7.5.10.2 \[makeUsl\]](#), page 466.

##### 7.5.10.2 `makeUsl`

Procedure from library `ncalg.lib` (see [Section 7.5.10 \[ncalg-lib\]](#), page 465).

**Usage:** `makeUsl(n,[p])`;  $n$  an integer,  $n > 1$ ;  $p$  an optional integer (field characteristic)

**Return:** ring

**Purpose:** set up the  $U(\mathfrak{sl}_n)$  in the variables  $(x(i), y(i), h(i) \mid i=1..n+1)$  over the field of char  $p$

**Note:** activate this ring with the `setring` command

This presentation of  $U(\mathfrak{sl}_n)$  is the standard one, i.e. positive resp. negative roots are denoted by  $x(i)$  resp.  $y(i)$  and the Cartan elements are denoted by  $h(i)$ .

The variables are ordered as  $x(1), \dots, x(n), y(1), \dots, y(n), h(1), \dots, h(n)$ .

#### Example:

```
LIB "ncalg.lib";
def a=makeUsl(3);
setring a;
```



```

a;
⇒ // coefficients: QQ
⇒ // number of vars : 8
⇒ //          block 1 : ordering dp
⇒ //          : names  x(1) x(2) x(3) y(1) y(2) y(3) h(1) h(2)
⇒ //          block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      x(2)x(1)=x(1)*x(2)+x(3)
⇒ //      y(1)x(1)=x(1)*y(1)-h(1)
⇒ //      y(3)x(1)=x(1)*y(3)-y(2)
⇒ //      h(1)x(1)=x(1)*h(1)+2*x(1)
⇒ //      h(2)x(1)=x(1)*h(2)-x(1)
⇒ //      y(2)x(2)=x(2)*y(2)-h(2)
⇒ //      y(3)x(2)=x(2)*y(3)+y(1)
⇒ //      h(1)x(2)=x(2)*h(1)-x(2)
⇒ //      h(2)x(2)=x(2)*h(2)+2*x(2)
⇒ //      y(1)x(3)=x(3)*y(1)-x(2)
⇒ //      y(2)x(3)=x(3)*y(2)+x(1)
⇒ //      y(3)x(3)=x(3)*y(3)-h(1)-h(2)
⇒ //      h(1)x(3)=x(3)*h(1)+x(3)
⇒ //      h(2)x(3)=x(3)*h(2)+x(3)
⇒ //      y(2)y(1)=y(1)*y(2)-y(3)
⇒ //      h(1)y(1)=y(1)*h(1)-2*y(1)
⇒ //      h(2)y(1)=y(1)*h(2)+y(1)
⇒ //      h(1)y(2)=y(2)*h(1)+y(2)
⇒ //      h(2)y(2)=y(2)*h(2)-2*y(2)
⇒ //      h(1)y(3)=y(3)*h(1)-y(3)
⇒ //      h(2)y(3)=y(3)*h(2)-y(3)

```

See also: [Section 7.5.10.24 \[makeQsl3\], page 480](#); [Section 7.5.10.22 \[makeQso3\], page 479](#); [Section 7.5.10.17 \[makeUg2\], page 476](#); [Section 7.5.10.3 \[makeUgl\], page 467](#); [Section 7.5.10.1 \[makeUsl2\], page 466](#).

### 7.5.10.3 makeUgl

Procedure from library `ncalg.lib` (see [Section 7.5.10 \[ncalg-lib\], page 465](#)).

**Usage:**     `makeUgl(n,[p]);`  $n$  an int,  $n > 1$ ;  $p$  an optional int (field characteristic)

**Return:**    ring

**Purpose:**    set up the  $U(\mathfrak{gl}_n)$  in the  $(e_{ij} \ (1 \leq i, j \leq n))$  presentation (where  $e_{ij}$  corresponds to a matrix with 1 at  $i, j$  only) over the field of char  $p$

**Note:**      activate this ring with the `setring` command  
               the variables are ordered as  $e_{12}, e_{13}, \dots, e_{1n}, e_{21}, \dots, e_{nn}$ .

**Example:**

```

LIB "ncalg.lib";
def a=makeUgl(3);
setring a; a;
⇒ // coefficients: QQ
⇒ // number of vars : 9
⇒ //          block 1 : ordering dp
⇒ //          : names  e_1_1 e_1_2 e_1_3 e_2_1 e_2_2 e_2_3 e_3_1 \
e_3_2 e_3_3

```

```

⇒ //      block  2 : ordering C
⇒ // noncommutative relations:
⇒ //      e_1_2e_1_1=e_1_1*e_1_2-e_1_2
⇒ //      e_1_3e_1_1=e_1_1*e_1_3-e_1_3
⇒ //      e_2_1e_1_1=e_1_1*e_2_1+e_2_1
⇒ //      e_3_1e_1_1=e_1_1*e_3_1+e_3_1
⇒ //      e_2_1e_1_2=e_1_2*e_2_1-e_1_1+e_2_2
⇒ //      e_2_2e_1_2=e_1_2*e_2_2-e_1_2
⇒ //      e_2_3e_1_2=e_1_2*e_2_3-e_1_3
⇒ //      e_3_1e_1_2=e_1_2*e_3_1+e_3_2
⇒ //      e_2_1e_1_3=e_1_3*e_2_1+e_2_3
⇒ //      e_3_1e_1_3=e_1_3*e_3_1-e_1_1+e_3_3
⇒ //      e_3_2e_1_3=e_1_3*e_3_2-e_1_2
⇒ //      e_3_3e_1_3=e_1_3*e_3_3-e_1_3
⇒ //      e_2_2e_2_1=e_2_1*e_2_2+e_2_1
⇒ //      e_3_2e_2_1=e_2_1*e_3_2+e_3_1
⇒ //      e_2_3e_2_2=e_2_2*e_2_3-e_2_3
⇒ //      e_3_2e_2_2=e_2_2*e_3_2+e_3_2
⇒ //      e_3_1e_2_3=e_2_3*e_3_1-e_2_1
⇒ //      e_3_2e_2_3=e_2_3*e_3_2-e_2_2+e_3_3
⇒ //      e_3_3e_2_3=e_2_3*e_3_3-e_2_3
⇒ //      e_3_3e_3_1=e_3_1*e_3_3+e_3_1
⇒ //      e_3_3e_3_2=e_3_2*e_3_3+e_3_2

```

See also: [Section 7.5.10.17 \[makeUg2\]](#), page 476; [Section 7.5.10.2 \[makeUsl\]](#), page 466.

#### 7.5.10.4 makeUso5

Procedure from library `ncalg.lib` (see [Section 7.5.10 \[ncalg.lib\]](#), page 465).

**Usage:**     `makeUso5([p]);` `p` an optional integer (field characteristic)

**Return:**    a ring, describing  $U(\mathfrak{so}_5)$

**Note:**     You have to activate this ring with the 'setring' command. The presentation of  $U(\mathfrak{so}_5)$  is derived from the Chevalley representation of  $\mathfrak{so}_5$ , positive resp. negative roots are denoted by  $x(i)$  resp.  $y(i)$ ; Cartan elements are denoted by  $h(i)$ .

**Example:**

```

LIB "ncalg.lib";
def ncAlgebra = makeUso5();
ncAlgebra;
⇒ // coefficients: QQ
⇒ // number of vars : 10
⇒ //      block  1 : ordering dp
⇒ //      : names  X(1) X(2) X(3) X(4) Y(1) Y(2) Y(3) Y(4) H(\
1) H(2)
⇒ //      block  2 : ordering C
⇒ // noncommutative relations: ...
setring ncAlgebra;
// ... 28 noncommutative relations

```

See also: [Section 7.5.10.19 \[makeUe6\]](#), page 477; [Section 7.5.10.20 \[makeUe7\]](#), page 478; [Section 7.5.10.21 \[makeUe8\]](#), page 478; [Section 7.5.10.18 \[makeUf4\]](#), page 476; [Section 7.5.10.17 \[makeUg2\]](#), page 476; [Section 7.5.10.2 \[makeUsl\]](#), page 466; [Section 7.5.10.12 \[makeUsp1\]](#), page 473.

### 7.5.10.5 makeUso6

Procedure from library `ncalg.lib` (see [Section 7.5.10 \[ncalg.lib\]](#), page 465).

**Usage:**     `makeUso6([p]);` `p` an optional integer (field characteristic)

**Return:**    a ring, describing  $U(\mathfrak{so}_6)$

**Note:**     You have to activate this ring with the 'setring' command. The presentation of  $U(\mathfrak{so}_6)$  is derived from the Chevalley representation of  $\mathfrak{so}_6$ , positive resp. negative roots are denoted by  $x(i)$  resp.  $y(i)$ ; Cartan elements are denoted by  $h(i)$ .

**Example:**

```
LIB "ncalg.lib";
def ncAlgebra = makeUso6();
ncAlgebra;
⇒ // coefficients: QQ
⇒ // number of vars : 15
⇒ //          block 1 : ordering dp
⇒ //          : names  X(1) X(2) X(3) X(4) X(5) X(6) Y(1) Y(2) Y(\
3) Y(4) Y(5) Y(6) H(1) H(2) H(3)
⇒ //          block 2 : ordering C
⇒ // noncommutative relations: ...
setring ncAlgebra;
// ... 60 noncommutative relations
```

See also: [Section 7.5.10.19 \[makeUe6\]](#), page 477; [Section 7.5.10.20 \[makeUe7\]](#), page 478; [Section 7.5.10.21 \[makeUe8\]](#), page 478; [Section 7.5.10.18 \[makeUf4\]](#), page 476; [Section 7.5.10.17 \[makeUg2\]](#), page 476; [Section 7.5.10.2 \[makeUsl\]](#), page 466; [Section 7.5.10.4 \[makeUso5\]](#), page 468; [Section 7.5.10.12 \[makeUsp1\]](#), page 473.

### 7.5.10.6 makeUso7

Procedure from library `ncalg.lib` (see [Section 7.5.10 \[ncalg.lib\]](#), page 465).

**Usage:**     `makeUso7([p]);` `p` an optional integer (field characteristic)

**Return:**    a ring, describing  $U(\mathfrak{so}_7)$

**Note:**     You have to activate this ring with the 'setring' command. The presentation of  $U(\mathfrak{so}_7)$  is derived from the Chevalley representation of  $\mathfrak{so}_7$ , positive resp. negative roots are denoted by  $x(i)$  resp.  $y(i)$ ; Cartan elements are denoted by  $h(i)$ .

**Example:**

```
LIB "ncalg.lib";
def ncAlgebra = makeUso7();
ncAlgebra;
⇒ // coefficients: QQ
⇒ // number of vars : 21
⇒ //          block 1 : ordering dp
⇒ //          : names  X(1) X(2) X(3) X(4) X(5) X(6) X(7) X(8) X(\
9) Y(1) Y(2) Y(3) Y(4) Y(5) Y(6) Y(7) Y(8) Y(9) H(1) H(2) H(3)
⇒ //          block 2 : ordering C
⇒ // noncommutative relations: ...
setring ncAlgebra;
// ... 107 noncommutative relations
```

See also: [Section 7.5.10.19 \[makeUe6\], page 477](#); [Section 7.5.10.20 \[makeUe7\], page 478](#); [Section 7.5.10.21 \[makeUe8\], page 478](#); [Section 7.5.10.18 \[makeUf4\], page 476](#); [Section 7.5.10.17 \[makeUg2\], page 476](#); [Section 7.5.10.2 \[makeUsl\], page 466](#); [Section 7.5.10.4 \[makeUso5\], page 468](#); [Section 7.5.10.12 \[makeUsp1\], page 473](#).

### 7.5.10.7 makeUso8

Procedure from library `ncalg.lib` (see [Section 7.5.10 \[ncalg.lib\], page 465](#)).

**Usage:** `makeUso8([p]);` `p` an optional integer (field characteristic)

**Return:** a ring, describing  $U(\mathfrak{so}_8)$

**Note:** You have to activate this ring with the 'setring' command. The presentation of  $U(\mathfrak{so}_8)$  is derived from the Chevalley representation of  $\mathfrak{so}_8$ , positive resp. negative roots are denoted by  $x(i)$  resp.  $y(i)$ ; Cartan elements are denoted by  $h(i)$ .

**Example:**

```
LIB "ncalg.lib";
def ncAlgebra = makeUso8();
ncAlgebra;
⇒ // coefficients: QQ
⇒ // number of vars : 28
⇒ //          block 1 : ordering dp
⇒ //          : names  X(1) X(2) X(3) X(4) X(5) X(6) X(7) X(8) X(\
9) X(10) X(11) X(12) Y(1) Y(2) Y(3) Y(4) Y(5) Y(6) Y(7) Y(8) Y(9) Y(10) Y\
(11) Y(12) H(1) H(2) H(3) H(4)
⇒ //          block 2 : ordering C
⇒ // noncommutative relations: ...
setring ncAlgebra;
// ... 180 noncommutative relations
```

See also: [Section 7.5.10.19 \[makeUe6\], page 477](#); [Section 7.5.10.20 \[makeUe7\], page 478](#); [Section 7.5.10.21 \[makeUe8\], page 478](#); [Section 7.5.10.18 \[makeUf4\], page 476](#); [Section 7.5.10.17 \[makeUg2\], page 476](#); [Section 7.5.10.2 \[makeUsl\], page 466](#); [Section 7.5.10.4 \[makeUso5\], page 468](#); [Section 7.5.10.12 \[makeUsp1\], page 473](#).

### 7.5.10.8 makeUso9

Procedure from library `ncalg.lib` (see [Section 7.5.10 \[ncalg.lib\], page 465](#)).

**Usage:** `makeUso9([p]);` `p` an optional integer (field characteristic)

**Return:** a ring, describing  $U(\mathfrak{so}_9)$

**Note:** You have to activate this ring with the 'setring' command. The presentation of  $U(\mathfrak{so}_9)$  is derived from the Chevalley representation of  $\mathfrak{so}_9$ , positive resp. negative roots are denoted by  $x(i)$  resp.  $y(i)$ ; Cartan elements are denoted by  $h(i)$ .

**Example:**

```
LIB "ncalg.lib";
def ncAlgebra = makeUso9();
ncAlgebra;
⇒ // coefficients: QQ
⇒ // number of vars : 36
⇒ //          block 1 : ordering dp
⇒ //          : names  X(1) X(2) X(3) X(4) X(5) X(6) X(7) X(8) X(\
```

```

9) X(10) X(11) X(12) X(13) X(14) X(15) X(16) Y(1) Y(2) Y(3) Y(4) Y(5) Y(6)
) Y(7) Y(8) Y(9) Y(10) Y(11) Y(12) Y(13) Y(14) Y(15) Y(16) H(1) H(2) H(3)
H(4)
⇒ //          block 2 : ordering C
⇒ // noncommutative relations: ...
setring ncAlgebra;
// ... 264 noncommutative relations

```

See also: [Section 7.5.10.19 \[makeUe6\]](#), page 477; [Section 7.5.10.20 \[makeUe7\]](#), page 478; [Section 7.5.10.21 \[makeUe8\]](#), page 478; [Section 7.5.10.18 \[makeUf4\]](#), page 476; [Section 7.5.10.17 \[makeUg2\]](#), page 476; [Section 7.5.10.2 \[makeUsl\]](#), page 466; [Section 7.5.10.4 \[makeUso5\]](#), page 468; [Section 7.5.10.12 \[makeUsp1\]](#), page 473.

### 7.5.10.9 makeUso10

Procedure from library `ncalg.lib` (see [Section 7.5.10 \[ncalg\\_lib\]](#), page 465).

**Usage:** `makeUso10([p]);`  $p$  an optional integer (field characteristic)

**Return:** a ring, describing  $U(\mathfrak{so}_{10})$

**Note:** You have to activate this ring with the 'setring' command. The presentation of  $U(\mathfrak{so}_{10})$  is derived from the Chevalley representation of  $\mathfrak{so}_{10}$ , positive resp. negative roots are denoted by  $x(i)$  resp.  $y(i)$ ; Cartan elements are denoted by  $h(i)$ .

**Example:**

```

LIB "ncalg.lib";
def ncAlgebra = makeUso10();
ncAlgebra;
⇒ // coefficients: QQ
⇒ // number of vars : 45
⇒ //          block 1 : ordering dp
⇒ //          : names  X(1) X(2) X(3) X(4) X(5) X(6) X(7) X(8) X(
9) X(10) X(11) X(12) X(13) X(14) X(15) X(16) X(17) X(18) X(19) X(20) Y(1)\
Y(2) Y(3) Y(4) Y(5) Y(6) Y(7) Y(8) Y(9) Y(10) Y(11) Y(12) Y(13) Y(14) Y(
15) Y(16) Y(17) Y(18) Y(19) Y(20) H(1) H(2) H(3) H(4) H(5)
⇒ //          block 2 : ordering C
⇒ // noncommutative relations: ...
setring ncAlgebra;
// ... 390 noncommutative relations

```

See also: [Section 7.5.10.19 \[makeUe6\]](#), page 477; [Section 7.5.10.20 \[makeUe7\]](#), page 478; [Section 7.5.10.21 \[makeUe8\]](#), page 478; [Section 7.5.10.18 \[makeUf4\]](#), page 476; [Section 7.5.10.17 \[makeUg2\]](#), page 476; [Section 7.5.10.2 \[makeUsl\]](#), page 466; [Section 7.5.10.4 \[makeUso5\]](#), page 468; [Section 7.5.10.12 \[makeUsp1\]](#), page 473.

### 7.5.10.10 makeUso11

Procedure from library `ncalg.lib` (see [Section 7.5.10 \[ncalg\\_lib\]](#), page 465).

**Usage:** `makeUso11([p]);`  $p$  an optional integer (field characteristic)

**Return:** a ring, describing  $U(\mathfrak{so}_{11})$

**Note:** You have to activate this ring with the 'setring' command. The presentation of  $U(\mathfrak{so}_{11})$  is derived from the Chevalley representation of  $\mathfrak{so}_{11}$ , positive resp. negative roots are denoted by  $x(i)$  resp.  $y(i)$ ; Cartan elements are denoted by  $h(i)$ .

**Example:**

```

LIB "ncalg.lib";
def ncAlgebra = makeUso11();
ncAlgebra;
⇒ // coefficients: QQ
⇒ // number of vars : 55
⇒ //          block 1 : ordering dp
⇒ //          : names  X(1) X(2) X(3) X(4) X(5) X(6) X(7) X(8) X(\
9) X(10) X(11) X(12) X(13) X(14) X(15) X(16) X(17) X(18) X(19) X(20) X(21\
) X(22) X(23) X(24) X(25) Y(1) Y(2) Y(3) Y(4) Y(5) Y(6) Y(7) Y(8) Y(9) Y(\
10) Y(11) Y(12) Y(13) Y(14) Y(15) Y(16) Y(17) Y(18) Y(19) Y(20) Y(21) Y(2\
2) Y(23) Y(24) Y(25) H(1) H(2) H(3) H(4) H(5)
⇒ //          block 2 : ordering C
⇒ // noncommutative relations: ...
setring ncAlgebra;
// ... 523 noncommutative relations

```

See also: [Section 7.5.10.19 \[makeUe6\], page 477](#); [Section 7.5.10.20 \[makeUe7\], page 478](#); [Section 7.5.10.21 \[makeUe8\], page 478](#); [Section 7.5.10.18 \[makeUf4\], page 476](#); [Section 7.5.10.17 \[makeUg2\], page 476](#); [Section 7.5.10.2 \[makeUsl\], page 466](#); [Section 7.5.10.4 \[makeUso5\], page 468](#); [Section 7.5.10.12 \[makeUsp1\], page 473](#).

**7.5.10.11 makeUso12**

Procedure from library `ncalg.lib` (see [Section 7.5.10 \[ncalg.lib\], page 465](#)).

**Usage:** `makeUso12([p]);` `p` an optional integer (field characteristic)

**Return:** a ring, describing  $U(\mathfrak{so}_{\{12\}})$

**Note:** You have to activate this ring with the 'setring' command. The presentation of  $U(\mathfrak{so}_{\{12\}})$  is derived from the Chevalley representation of  $\mathfrak{so}_{\{12\}}$ , positive resp. negative roots are denoted by  $x(i)$  resp.  $y(i)$ ; Cartan elements are denoted by  $h(i)$ .

**Example:**

```

LIB "ncalg.lib";
def ncAlgebra = makeUso12();
ncAlgebra;
⇒ // coefficients: QQ
⇒ // number of vars : 66
⇒ //          block 1 : ordering dp
⇒ //          : names  X(1) X(2) X(3) X(4) X(5) X(6) X(7) X(8) X(\
9) X(10) X(11) X(12) X(13) X(14) X(15) X(16) X(17) X(18) X(19) X(20) X(21\
) X(22) X(23) X(24) X(25) X(26) X(27) X(28) X(29) X(30) Y(1) Y(2) Y(3) Y(\
4) Y(5) Y(6) Y(7) Y(8) Y(9) Y(10) Y(11) Y(12) Y(13) Y(14) Y(15) Y(16) Y(1\
7) Y(18) Y(19) Y(20) Y(21) Y(22) Y(23) Y(24) Y(25) Y(26) Y(27) Y(28) Y(29\
) Y(30) H(1) H(2) H(3) H(4) H(5) H(6)
⇒ //          block 2 : ordering C
⇒ // noncommutative relations: ...
setring ncAlgebra;
// ... 714 noncommutative relations

```

See also: [Section 7.5.10.19 \[makeUe6\], page 477](#); [Section 7.5.10.20 \[makeUe7\], page 478](#); [Section 7.5.10.21 \[makeUe8\], page 478](#); [Section 7.5.10.18 \[makeUf4\], page 476](#); [Section 7.5.10.17 \[makeUg2\], page 476](#); [Section 7.5.10.2 \[makeUsl\], page 466](#); [Section 7.5.10.4 \[makeUso5\], page 468](#); [Section 7.5.10.12 \[makeUsp1\], page 473](#).

### 7.5.10.12 makeUsp1

Procedure from library `ncalg.lib` (see [Section 7.5.10 \[ncalg-lib\]](#), page 465).

**Usage:** `makeUsp1([p]);`  $p$  an optional integer (field characteristic)

**Return:** a ring, describing  $U(\mathfrak{sp}_1)$

**Note:** You have to activate this ring with the 'setring' command. The presentation of  $U(\mathfrak{sp}_1)$  is derived from the Chevalley representation of  $\mathfrak{sp}_1$ , positive resp. negative roots are denoted by  $x(i)$  resp.  $y(i)$ ; Cartan elements are denoted by  $h(i)$ .

**Example:**

```
LIB "ncalg.lib";
def ncAlgebra = makeUsp1();
setring ncAlgebra;
ncAlgebra;
⇒ // coefficients: QQ
⇒ // number of vars : 3
⇒ //      block 1 : ordering dp
⇒ //      : names X(1) Y(1) H(1)
⇒ //      block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      Y(1)X(1)=X(1)*Y(1)-H(1)
⇒ //      H(1)X(1)=X(1)*H(1)+2*X(1)
⇒ //      H(1)Y(1)=Y(1)*H(1)-2*Y(1)
```

See also: [Section 7.5.10.19 \[makeUe6\]](#), page 477; [Section 7.5.10.20 \[makeUe7\]](#), page 478; [Section 7.5.10.21 \[makeUe8\]](#), page 478; [Section 7.5.10.18 \[makeUf4\]](#), page 476; [Section 7.5.10.17 \[makeUg2\]](#), page 476; [Section 7.5.10.2 \[makeUsl\]](#), page 466; [Section 7.5.10.4 \[makeUso5\]](#), page 468.

### 7.5.10.13 makeUsp2

Procedure from library `ncalg.lib` (see [Section 7.5.10 \[ncalg-lib\]](#), page 465).

**Usage:** `makeUsp2([p]);`  $p$  an optional integer (field characteristic)

**Return:** a ring, describing  $U(\mathfrak{sp}_2)$

**Note:** You have to activate this ring with the 'setring' command. The presentation of  $U(\mathfrak{sp}_2)$  is derived from the Chevalley representation of  $\mathfrak{sp}_2$ , positive resp. negative roots are denoted by  $x(i)$  resp.  $y(i)$ ; Cartan elements are denoted by  $h(i)$ .

**Example:**

```
LIB "ncalg.lib";
def ncAlgebra = makeUsp2();
setring ncAlgebra;
ncAlgebra;
⇒ // coefficients: QQ
⇒ // number of vars : 10
⇒ //      block 1 : ordering dp
⇒ //      : names X(1) X(2) X(3) X(4) Y(1) Y(2) Y(3) Y(4) H(\
1) H(2)
⇒ //      block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      X(2)X(1)=X(1)*X(2)+X(3)
⇒ //      X(3)X(1)=X(1)*X(3)+2*X(4)
```

```

⇒ //      Y(1)X(1)=X(1)*Y(1)-H(1)
⇒ //      Y(3)X(1)=X(1)*Y(3)-2*Y(2)
⇒ //      Y(4)X(1)=X(1)*Y(4)-Y(3)
⇒ //      H(1)X(1)=X(1)*H(1)+2*X(1)
⇒ //      H(2)X(1)=X(1)*H(2)-X(1)
⇒ //      Y(2)X(2)=X(2)*Y(2)-H(2)
⇒ //      Y(3)X(2)=X(2)*Y(3)+Y(1)
⇒ //      H(1)X(2)=X(2)*H(1)-2*X(2)
⇒ //      H(2)X(2)=X(2)*H(2)+2*X(2)
⇒ //      Y(1)X(3)=X(3)*Y(1)-2*X(2)
⇒ //      Y(2)X(3)=X(3)*Y(2)+X(1)
⇒ //      Y(3)X(3)=X(3)*Y(3)-H(1)-2*H(2)
⇒ //      Y(4)X(3)=X(3)*Y(4)+Y(1)
⇒ //      H(2)X(3)=X(3)*H(2)+X(3)
⇒ //      Y(1)X(4)=X(4)*Y(1)-X(3)
⇒ //      Y(3)X(4)=X(4)*Y(3)+X(1)
⇒ //      Y(4)X(4)=X(4)*Y(4)-H(1)-H(2)
⇒ //      H(1)X(4)=X(4)*H(1)+2*X(4)
⇒ //      Y(2)Y(1)=Y(1)*Y(2)-Y(3)
⇒ //      Y(3)Y(1)=Y(1)*Y(3)-2*Y(4)
⇒ //      H(1)Y(1)=Y(1)*H(1)-2*Y(1)
⇒ //      H(2)Y(1)=Y(1)*H(2)+Y(1)
⇒ //      H(1)Y(2)=Y(2)*H(1)+2*Y(2)
⇒ //      H(2)Y(2)=Y(2)*H(2)-2*Y(2)
⇒ //      H(2)Y(3)=Y(3)*H(2)-Y(3)
⇒ //      H(1)Y(4)=Y(4)*H(1)-2*Y(4)

```

See also: [Section 7.5.10.19 \[makeUe6\]](#), page 477; [Section 7.5.10.20 \[makeUe7\]](#), page 478; [Section 7.5.10.21 \[makeUe8\]](#), page 478; [Section 7.5.10.18 \[makeUf4\]](#), page 476; [Section 7.5.10.17 \[makeUg2\]](#), page 476; [Section 7.5.10.2 \[makeUs1\]](#), page 466; [Section 7.5.10.4 \[makeUso5\]](#), page 468; [Section 7.5.10.12 \[makeUsp1\]](#), page 473.

### 7.5.10.14 makeUsp3

Procedure from library `ncalg.lib` (see [Section 7.5.10 \[ncalg\\_lib\]](#), page 465).

**Usage:**     `makeUsp3([p]);` `p` an optional integer (field characteristic)

**Return:**    a ring, describing  $U(\mathfrak{sp}_3)$

**Note:**     You have to activate this ring with the 'setring' command. The presentation of  $U(\mathfrak{sp}_3)$  is derived from the Chevalley representation of  $\mathfrak{sp}_3$ , positive resp. negative roots are denoted by  $x(i)$  resp.  $y(i)$ ; Cartan elements are denoted by  $h(i)$ .

**Example:**

```

LIB "ncalg.lib";
def ncAlgebra = makeUsp3();
ncAlgebra;
⇒ // coefficients: QQ
⇒ // number of vars : 21
⇒ //      block   1 : ordering dp
⇒ //      : names   X(1) X(2) X(3) X(4) X(5) X(6) X(7) X(8) X(\
    9) Y(1) Y(2) Y(3) Y(4) Y(5) Y(6) Y(7) Y(8) Y(9) H(1) H(2) H(3)
⇒ //      block   2 : ordering C
⇒ // noncommutative relations: ...

```



```

setring ncAlgebra;
// ... 107 noncommutative relations

```

See also: [Section 7.5.10.19 \[makeUe6\]](#), page 477; [Section 7.5.10.20 \[makeUe7\]](#), page 478; [Section 7.5.10.21 \[makeUe8\]](#), page 478; [Section 7.5.10.18 \[makeUf4\]](#), page 476; [Section 7.5.10.17 \[makeUg2\]](#), page 476; [Section 7.5.10.2 \[makeUsl\]](#), page 466; [Section 7.5.10.4 \[makeUso5\]](#), page 468; [Section 7.5.10.12 \[makeUsp1\]](#), page 473.

### 7.5.10.15 makeUsp4

Procedure from library `ncalg.lib` (see [Section 7.5.10 \[ncalg-lib\]](#), page 465).

**Usage:** `makeUsp4([p]);`  $p$  an optional integer (field characteristic)

**Return:** a ring, describing  $U(\mathfrak{sp}_4)$

**Note:** You have to activate this ring with the 'setring' command. The presentation of  $U(\mathfrak{sp}_4)$  is derived from the Chevalley representation of  $\mathfrak{sp}_4$ , positive resp. negative roots are denoted by  $x(i)$  resp.  $y(i)$ ; Cartan elements are denoted by  $h(i)$ .

**Example:**

```

LIB "ncalg.lib";
def ncAlgebra = makeUsp4();
ncAlgebra;
⇒ // coefficients: QQ
⇒ // number of vars : 36
⇒ //      block 1 : ordering dp
⇒ //      : names  X(1) X(2) X(3) X(4) X(5) X(6) X(7) X(8) X(\
9) X(10) X(11) X(12) X(13) X(14) X(15) X(16) Y(1) Y(2) Y(3) Y(4) Y(5) Y(6\
) Y(7) Y(8) Y(9) Y(10) Y(11) Y(12) Y(13) Y(14) Y(15) Y(16) H(1) H(2) H(3)\
H(4)
⇒ //      block 2 : ordering C
⇒ // noncommutative relations: ...
setring ncAlgebra;
// ... 264 noncommutative relations

```

See also: [Section 7.5.10.19 \[makeUe6\]](#), page 477; [Section 7.5.10.20 \[makeUe7\]](#), page 478; [Section 7.5.10.21 \[makeUe8\]](#), page 478; [Section 7.5.10.18 \[makeUf4\]](#), page 476; [Section 7.5.10.17 \[makeUg2\]](#), page 476; [Section 7.5.10.2 \[makeUsl\]](#), page 466; [Section 7.5.10.4 \[makeUso5\]](#), page 468; [Section 7.5.10.12 \[makeUsp1\]](#), page 473.

### 7.5.10.16 makeUsp5

Procedure from library `ncalg.lib` (see [Section 7.5.10 \[ncalg-lib\]](#), page 465).

**Usage:** `makeUsp5([p]);`  $p$  an optional integer (field characteristic)

**Return:** a ring, describing  $U(\mathfrak{sp}_5)$

**Note:** You have to activate this ring with the 'setring' command. The presentation of  $U(\mathfrak{sp}_5)$  is derived from the Chevalley representation of  $\mathfrak{sp}_5$ , positive resp. negative roots are denoted by  $x(i)$  resp.  $y(i)$ ; Cartan elements are denoted by  $h(i)$ .

**Example:**

```

LIB "ncalg.lib";
def ncAlgebra = makeUsp5();
ncAlgebra;

```

```

⇒ // coefficients: QQ
⇒ // number of vars : 55
⇒ //          block 1 : ordering dp
⇒ //          : names  X(1) X(2) X(3) X(4) X(5) X(6) X(7) X(8) X(\
9) X(10) X(11) X(12) X(13) X(14) X(15) X(16) X(17) X(18) X(19) X(20) X(21\
) X(22) X(23) X(24) X(25) Y(1) Y(2) Y(3) Y(4) Y(5) Y(6) Y(7) Y(8) Y(9) Y(\
10) Y(11) Y(12) Y(13) Y(14) Y(15) Y(16) Y(17) Y(18) Y(19) Y(20) Y(21) Y(2\
2) Y(23) Y(24) Y(25) H(1) H(2) H(3) H(4) H(5)
⇒ //          block 2 : ordering C
⇒ // noncommutative relations: ...
setring ncAlgebra;
// ... 523 noncommutative relations

```

See also: [Section 7.5.10.19 \[makeUe6\], page 477](#); [Section 7.5.10.20 \[makeUe7\], page 478](#); [Section 7.5.10.21 \[makeUe8\], page 478](#); [Section 7.5.10.18 \[makeUf4\], page 476](#); [Section 7.5.10.17 \[makeUg2\], page 476](#); [Section 7.5.10.2 \[makeUsl\], page 466](#); [Section 7.5.10.4 \[makeUso5\], page 468](#); [Section 7.5.10.12 \[makeUsp1\], page 473](#).

### 7.5.10.17 makeUg2

Procedure from library `ncalg.lib` (see [Section 7.5.10 \[ncalg.lib\], page 465](#)).

**Usage:** `makeUg2([p])`, `p` an optional int (field characteristic)

**Return:** ring

**Purpose:** set up the  $U(\mathfrak{g}_2)$  in variables  $(x(i), y(i), Ha, Hb)$  for  $i=1..6$  over the field of char `p`

**Note:** activate this ring with the `setring` command  
the variables are ordered as  $x(1), \dots, x(6), y(1), \dots, y(6), Ha, Hb$ .

**Example:**

```

LIB "ncalg.lib";
def a = makeUg2();
a;
⇒ // coefficients: QQ
⇒ // number of vars : 14
⇒ //          block 1 : ordering dp
⇒ //          : names  x(1) x(2) x(3) x(4) x(5) x(6) y(1) y(2) y(\
3) y(4) y(5) y(6) Ha Hb
⇒ //          block 2 : ordering C
⇒ // noncommutative relations: ...
setring a;
// ... 56 noncommutative relations

```

See also: [Section 7.5.10.3 \[makeUgl\], page 467](#); [Section 7.5.10.2 \[makeUsl\], page 466](#).

### 7.5.10.18 makeUf4

Procedure from library `ncalg.lib` (see [Section 7.5.10 \[ncalg.lib\], page 465](#)).

**Usage:** `makeUf4([p])`; `p` an optional integer (field characteristic)

**Return:** a ring, describing  $U(\mathfrak{f}_4)$

**Note:** You have to activate this ring with the 'setring' command. The presentation of  $U(\mathfrak{f}_4)$  is derived from the Chevalley representation of  $\mathfrak{f}_4$ , positive resp. negative roots are denoted by  $x(i)$  resp.  $y(i)$ ; Cartan elements are denoted by  $h(i)$ .

**Example:**

```

LIB "ncalg.lib";
def ncAlgebra = makeUf4();
ncAlgebra;
⇒ // coefficients: QQ
⇒ // number of vars : 52
⇒ //      block 1 : ordering dp
⇒ //      : names  X(1) X(2) X(3) X(4) X(5) X(6) X(7) X(8) X(\
9) X(10) X(11) X(12) X(13) X(14) X(15) X(16) X(17) X(18) X(19) X(20) X(21\
) X(22) X(23) X(24) Y(1) Y(2) Y(3) Y(4) Y(5) Y(6) Y(7) Y(8) Y(9) Y(10) Y(\
11) Y(12) Y(13) Y(14) Y(15) Y(16) Y(17) Y(18) Y(19) Y(20) Y(21) Y(22) Y(2\
3) Y(24) H(1) H(2) H(3) H(4)
⇒ //      block 2 : ordering C
⇒ // noncommutative relations: ...
setring ncAlgebra;
// ... 552 noncommutative relations

```

See also: [Section 7.5.10.19 \[makeUe6\], page 477](#); [Section 7.5.10.20 \[makeUe7\], page 478](#); [Section 7.5.10.21 \[makeUe8\], page 478](#); [Section 7.5.10.17 \[makeUg2\], page 476](#); [Section 7.5.10.2 \[makeUsl\], page 466](#); [Section 7.5.10.4 \[makeUso5\], page 468](#); [Section 7.5.10.12 \[makeUsp1\], page 473](#).

**7.5.10.19 makeUe6**

Procedure from library `ncalg.lib` (see [Section 7.5.10 \[ncalg.lib\], page 465](#)).

**Usage:** `makeUe6([p]);` `p` an optional integer (field characteristic)

**Return:** a ring, describing  $U(e_6)$

**Note:** You have to activate this ring with the 'setring' command. The presentation of  $U(e_6)$  is derived from the Chevalley representation of  $e_6$ , positive resp. negative roots are denoted by  $x(i)$  resp.  $y(i)$ ; Cartan elements are denoted by  $h(i)$ .

**Example:**

```

LIB "ncalg.lib";
def ncAlgebra = makeUe6();
ncAlgebra;
⇒ // coefficients: QQ
⇒ // number of vars : 78
⇒ //      block 1: ordering dp
⇒ //      : names  X(1) X(2) X(3) X(4) X(5) X(6) X(7) X(8) X(\
9) X(10) X(11) X(12) X(13) X(14) X(15) X(16) X(17) X(18) X(19) X(20) X(21\
) X(22) X(23) X(24) X(25) X(26) X(27) X(28) X(29) X(30) X(31) X(32) X(33)\
X(34) X(35) X(36) Y(1) Y(2) Y(3) Y(4) Y(5) Y(6) Y(7) Y(8) Y(9) Y(10) Y(1\
1) Y(12) Y(13) Y(14) Y(15) Y(16) Y(17) Y(18) Y(19) Y(20) Y(21) Y(22) Y(23\
) Y(24) Y(25) Y(26) Y(27) Y(28) Y(29) Y(30) Y(31) Y(32) Y(33) Y(34) Y(35)\
Y(36) H(1) H(2) H(3) H(4) H(5) H(6)
⇒ //      block 2 : ordering C
⇒ // noncommutative relations: ...
setring ncAlgebra;
// ... 1008 noncommutative relations

```

See also: [Section 7.5.10.19 \[makeUe6\], page 477](#); [Section 7.5.10.20 \[makeUe7\], page 478](#); [Section 7.5.10.21 \[makeUe8\], page 478](#); [Section 7.5.10.18 \[makeUf4\], page 476](#); [Section 7.5.10.17 \[makeUg2\], page 476](#); [Section 7.5.10.2 \[makeUsl\], page 466](#); [Section 7.5.10.4 \[makeUso5\], page 468](#); [Section 7.5.10.12 \[makeUsp1\], page 473](#).

### 7.5.10.20 makeUe7

Procedure from library `ncalg.lib` (see [Section 7.5.10 \[ncalg\\_lib\]](#), page 465).

**Usage:** `makeUe7([p]);` `p` an optional integer (field characteristic)

**Return:** a ring, describing  $U(e_7)$

**Note:** You have to activate this ring with the 'setring' command. The presentation of  $U(e_7)$  is derived from the Chevalley representation of  $e_7$ , positive resp. negative roots are denoted by  $x(i)$  resp.  $y(i)$ ; Cartan elements are denoted by  $h(i)$ .

**Example:**

```
LIB "ncalg.lib";
def ncAlgebra = makeUe7();
ncAlgebra;
⇒ // coefficients: QQ
⇒ // number of vars : 133
⇒ //          block 1 : ordering dp
⇒ //          : names  X(1) X(2) X(3) X(4) X(5) X(6) X(7) X(8) X(\
9) X(10) X(11) X(12) X(13) X(14) X(15) X(16) X(17) X(18) X(19) X(20) X(21)\
) X(22) X(23) X(24) X(25) X(26) X(27) X(28) X(29) X(30) X(31) X(32) X(33)\
X(34) X(35) X(36) X(37) X(38) X(39) X(40) X(41) X(42) X(43) X(44) X(45) \
X(46) X(47) X(48) X(49) X(50) X(51) X(52) X(53) X(54) X(55) X(56) X(57) X\
(58) X(59) X(60) X(61) X(62) X(63) Y(1) Y(2) Y(3) Y(4) Y(5) Y(6) Y(7) Y(8)\
) Y(9) Y(10) Y(11) Y(12) Y(13) Y(14) Y(15) Y(16) Y(17) Y(18) Y(19) Y(20) \
Y(21) Y(22) Y(23) Y(24) Y(25) Y(26) Y(27) Y(28) Y(29) Y(30) Y(31) Y(32) Y\
(33) Y(34) Y(35) Y(36) Y(37) Y(38) Y(39) Y(40) Y(41) Y(42) Y(43) Y(44) Y(\
45) Y(46) Y(47) Y(48) Y(49) Y(50) Y(51) Y(52) Y(53) Y(54) Y(55) Y(56) Y(5\
7) Y(58) Y(59) Y(60) Y(61) Y(62) Y(63) H(1) H(2) H(3) H(4) H(5) H(6) H(7)
⇒ //          block 2 : ordering C
⇒ // noncommutative relations: ...
setring ncAlgebra;
// ... 2541 noncommutative relations
```

See also: [Section 7.5.10.19 \[makeUe6\]](#), page 477; [Section 7.5.10.20 \[makeUe7\]](#), page 478; [Section 7.5.10.21 \[makeUe8\]](#), page 478; [Section 7.5.10.18 \[makeUf4\]](#), page 476; [Section 7.5.10.17 \[makeUg2\]](#), page 476; [Section 7.5.10.2 \[makeUsl\]](#), page 466; [Section 7.5.10.4 \[makeUso5\]](#), page 468; [Section 7.5.10.12 \[makeUsp1\]](#), page 473.

### 7.5.10.21 makeUe8

Procedure from library `ncalg.lib` (see [Section 7.5.10 \[ncalg\\_lib\]](#), page 465).

**Usage:** `makeUe8([p]);` `p` an optional integer (field characteristic)

**Return:** a ring, describing  $U(e_8)$

**Note:** You have to activate this ring with the 'setring' command. The presentation of  $U(e_8)$  is derived from the Chevalley representation of  $e_8$ , positive resp. negative roots are denoted by  $x(i)$  resp.  $y(i)$ ; Cartan elements are denoted by  $h(i)$ .

**Example:**

```
LIB "ncalg.lib";
def ncAlgebra = makeUe8();
ncAlgebra;
⇒ // coefficients: QQ
```

```

⇒ // number of vars : 248
⇒ //      block 1 : ordering dp
⇒ //      : names  X(1) X(2) X(3) X(4) X(5) X(6) X(7) X(8) X(\
9) X(10) X(11) X(12) X(13) X(14) X(15) X(16) X(17) X(18) X(19) X(20) X(21\
) X(22) X(23) X(24) X(25) X(26) X(27) X(28) X(29) X(30) X(31) X(32) X(33)\
X(34) X(35) X(36) X(37) X(38) X(39) X(40) X(41) X(42) X(43) X(44) X(45) \
X(46) X(47) X(48) X(49) X(50) X(51) X(52) X(53) X(54) X(55) X(56) X(57) X\
(58) X(59) X(60) X(61) X(62) X(63) X(64) X(65) X(66) X(67) X(68) X(69) X(\
70) X(71) X(72) X(73) X(74) X(75) X(76) X(77) X(78) X(79) X(80) X(81) X(8\
2) X(83) X(84) X(85) X(86) X(87) X(88) X(89) X(90) X(91) X(92) X(93) X(94\
) X(95) X(96) X(97) X(98) X(99) X(100) X(101) X(102) X(103) X(104) X(105)\
X(106) X(107) X(108) X(109) X(110) X(111) X(112) X(113) X(114) X(115) X(\
116) X(117) X(118) X(119) X(120) Y(1) Y(2) Y(3) Y(4) Y(5) Y(6) Y(7) Y(8) \
Y(9) Y(10) Y(11) Y(12) Y(13) Y(14) Y(15) Y(16) Y(17) Y(18) Y(19) Y(20) Y(\
21) Y(22) Y(23) Y(24) Y(25) Y(26) Y(27) Y(28) Y(29) Y(30) Y(31) Y(32) Y(3\
3) Y(34) Y(35) Y(36) Y(37) Y(38) Y(39) Y(40) Y(41) Y(42) Y(43) Y(44) Y(45\
) Y(46) Y(47) Y(48) Y(49) Y(50) Y(51) Y(52) Y(53) Y(54) Y(55) Y(56) Y(57)\
Y(58) Y(59) Y(60) Y(61) Y(62) Y(63) Y(64) Y(65) Y(66) Y(67) Y(68) Y(69) \
Y(70) Y(71) Y(72) Y(73) Y(74) Y(75) Y(76) Y(77) Y(78) Y(79) Y(80) Y(81) Y\
(82) Y(83) Y(84) Y(85) Y(86) Y(87) Y(88) Y(89) Y(90) Y(91) Y(92) Y(93) Y(\
94) Y(95) Y(96) Y(97) Y(98) Y(99) Y(100) Y(101) Y(102) Y(103) Y(104) Y(10\
5) Y(106) Y(107) Y(108) Y(109) Y(110) Y(111) Y(112) Y(113) Y(114) Y(115) \
Y(116) Y(117) Y(118) Y(119) Y(120) H(1) H(2) H(3) H(4) H(5) H(6) H(7) H(8\
)
⇒ //      block 2 : ordering C
⇒ // noncommutative relations: ...
setring ncAlgebra;
// ... 7752 noncommutative relations

```

See also: [Section 7.5.10.19 \[makeUe6\]](#), page 477; [Section 7.5.10.20 \[makeUe7\]](#), page 478; [Section 7.5.10.21 \[makeUe8\]](#), page 478; [Section 7.5.10.18 \[makeUf4\]](#), page 476; [Section 7.5.10.17 \[makeUg2\]](#), page 476; [Section 7.5.10.2 \[makeUs1\]](#), page 466; [Section 7.5.10.4 \[makeUso5\]](#), page 468; [Section 7.5.10.12 \[makeUsp1\]](#), page 473.

### 7.5.10.22 makeQso3

Procedure from library `ncalg.lib` (see [Section 7.5.10 \[ncalg-lib\]](#), page 465).

**Usage:** `makeQso3([n])`, `n` an optional int

**Purpose:** set up the  $U_q(\mathfrak{so}_3)$  in the presentation of Klimyk; if `n` is specified, the quantum parameter  $Q$  will be specialized at the  $(2n)$ -th root of unity

**Return:** ring

**Note:** activate this ring with the `setring` command

**Example:**

```

LIB "ncalg.lib";
def K = makeQso3(3);
setring K;
K;
⇒ // coefficients: QQ[Q]/(Q^2-Q+1)
⇒ // number of vars : 3
⇒ //      block 1 : ordering dp
⇒ //      : names  x y z

```

```

⇒ //      block  2 : ordering C
⇒ // noncommutative relations:
⇒ //      yx=(Q-1)*xy+(-Q)*z
⇒ //      zx=(-Q)*xz+(-Q+1)*y
⇒ //      zy=(Q-1)*yz+(-Q)*x

```

See also: [Section 7.5.10.25 \[Qso3Casimir\]](#), page 482; [Section 7.5.10.23 \[makeQsl2\]](#), page 480; [Section 7.5.10.24 \[makeQsl3\]](#), page 480; [Section 7.5.10.17 \[makeUg2\]](#), page 476; [Section 7.5.10.3 \[makeUgl\]](#), page 467; [Section 7.5.10.2 \[makeUsl\]](#), page 466.

### 7.5.10.23 makeQsl2

Procedure from library `ncalg.lib` (see [Section 7.5.10 \[ncalg-lib\]](#), page 465).

**Usage:**     `makeQsl2([n])`, `n` an optional int

**Return:**    ring

**Purpose:**    define the  $U_q(\mathfrak{sl}_2)$  as a factor-ring of a ring  $V_q(\mathfrak{sl}_2)$  modulo the ideal `Qideal`

**Note:**     the output consists of a ring, presenting  $V_q(\mathfrak{sl}_2)$  together with the ideal called `Qideal` in this ring  
               activate this ring with the `setring` command  
               in order to create the  $U_q(\mathfrak{sl}_2)$  from the output, execute the command like `qring Usl2q = Qideal;`  
               If `n` is specified, the quantum parameter `q` will be specialized at the `n`-th root of unity

**Example:**

```

LIB "ncalg.lib";
def A = makeQsl2(3);
setring A;
Qideal;
⇒ Qideal[1]=Ke*Kf-1
qring Usl2q = Qideal;
Usl2q;
⇒ // coefficients: QQ[q]/(q^2+q+1)
⇒ // number of vars : 4
⇒ //      block  1 : ordering dp
⇒ //      : names      E F Ke Kf
⇒ //      block  2 : ordering C
⇒ // noncommutative relations:
⇒ //      FE=E*F+(2/3*q+1/3)*Ke+(-2/3*q-1/3)*Kf
⇒ //      KeE=(-q-1)*E*Ke
⇒ //      KfE=(q)*E*Kf
⇒ //      KeF=(q)*F*Ke
⇒ //      KfF=(-q-1)*F*Kf
⇒ // quotient ring from ideal
⇒ _[1]=Ke*Kf-1

```

See also: [Section 7.5.10.24 \[makeQsl3\]](#), page 480; [Section 7.5.10.22 \[makeQso3\]](#), page 479; [Section 7.5.10.2 \[makeUsl\]](#), page 466.

### 7.5.10.24 makeQsl3

Procedure from library `ncalg.lib` (see [Section 7.5.10 \[ncalg-lib\]](#), page 465).

**Usage:**     `makeQsl3([n])`, `n` an optional int

**Return:** ring

**Purpose:** define the  $U_q(\mathfrak{sl}_3)$  as a factor-ring of a ring  $V_q(\mathfrak{sl}_3)$  modulo the ideal `Qideal`

**Note:** the output consists of a ring, presenting  $V_q(\mathfrak{sl}_3)$  together with the ideal called `Qideal` in this ring  
activate this ring with the `setring` command  
in order to create the  $U_q(\mathfrak{sl}_3)$  from the output, execute the command like `qring Usl3q = Qideal;`  
If  $n$  is specified, the quantum parameter  $q$  will be specialized at the  $n$ -th root of unity

**Example:**

```
LIB "ncalg.lib";
def B = makeQsl3(5);
setring B;
qring Usl3q = Qideal;
Usl3q;
⇒ // coefficients: QQ[q]/(q^4+q^3+q^2+q+1)
⇒ // number of vars : 10
⇒ //      block 1 : ordering wp
⇒ //      : names      f12 f13 f23 k1 k2 l1 l2 e12 e13 e23
⇒ //      : weights    2   3   2   1   1   1   1   2   3   2
⇒ //      block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      f13f12=(q^3)*f12*f13
⇒ //      f23f12=(q^2)*f12*f23+(-q)*f13
⇒ //      k1f12=(q^3)*f12*k1
⇒ //      k2f12=(q)*f12*k2
⇒ //      l1f12=(q^2)*f12*l1
⇒ //      l2f12=(-q^3-q^2-q-1)*f12*l2
⇒ //      e12f12=f12*e12+(1/5*q^3-3/5*q^2-2/5*q-1/5)*k1^2+(-1/5*q^3+3/5*q^2+2/5*q+1/5)*l1^2
⇒ //      e13f12=f12*e13+(q^3+q^2+q+1)*l1^2*e23
⇒ //      f23f13=(q^3)*f13*f23
⇒ //      k1f13=(-q^3-q^2-q-1)*f13*k1
⇒ //      k2f13=(-q^3-q^2-q-1)*f13*k2
⇒ //      l1f13=(q)*f13*l1
⇒ //      l2f13=(q)*f13*l2
⇒ //      e12f13=f13*e12+(q)*f23*k1^2
⇒ //      e13f13=f13*e13+(-1/5*q^3+3/5*q^2+2/5*q+1/5)*k1^2*k2^2+(1/5*q^3-3/5*q^2-2/5*q-1/5)*l1^2*l2^2
⇒ //      e23f13=f13*e23+(q^3+q^2+q+1)*f12*l2^2
⇒ //      k1f23=(q)*f23*k1
⇒ //      k2f23=(q^3)*f23*k2
⇒ //      l1f23=(-q^3-q^2-q-1)*f23*l1
⇒ //      l2f23=(q^2)*f23*l2
⇒ //      e13f23=f23*e13+(q)*k2^2*e12
⇒ //      e23f23=f23*e23+(1/5*q^3-3/5*q^2-2/5*q-1/5)*k2^2+(-1/5*q^3+3/5*q^2+2/5*q+1/5)*l2^2
⇒ //      e12k1=(q^3)*k1*e12
⇒ //      e13k1=(-q^3-q^2-q-1)*k1*e13
⇒ //      e23k1=(q)*k1*e23
⇒ //      e12k2=(q)*k2*e12
⇒ //      e13k2=(-q^3-q^2-q-1)*k2*e13
```

```

⇒ //      e23k2=(q^3)*k2*e23
⇒ //      e12l1=(q^2)*l1*e12
⇒ //      e13l1=(q)*l1*e13
⇒ //      e23l1=(-q^3-q^2-q-1)*l1*e23
⇒ //      e12l2=(-q^3-q^2-q-1)*l2*e12
⇒ //      e13l2=(q)*l2*e13
⇒ //      e23l2=(q^2)*l2*e23
⇒ //      e13e12=(q^3)*e12*e13
⇒ //      e23e12=(q^2)*e12*e23+(-q)*e13
⇒ //      e23e13=(q^3)*e13*e23
⇒ // quotient ring from ideal
⇒ _[1]=k2*l2-1
⇒ _[2]=k1*l1-1

```

See also: [Section 7.5.10.23 \[makeQsl2\]](#), page 480; [Section 7.5.10.22 \[makeQso3\]](#), page 479; [Section 7.5.10.2 \[makeUsl\]](#), page 466.

### 7.5.10.25 Qso3Casimir

Procedure from library `ncalg.lib` (see [Section 7.5.10 \[ncalg.lib\]](#), page 465).

**Usage:** `Qso3Casimir(n [,m])`,  $n$  an integer,  $m$  an optional integer

**Return:** list (of polynomials)

**Purpose:** compute the Casimir (central) elements of  $U_q(\mathfrak{so}_3)$  for the quantum parameter specialized at the  $n$ -th root of unity; if  $m \neq 0$  is given, polynomials will be normalized

**Assume:** the basering must be  $U_q(\mathfrak{so}_3)$

**Example:**

```

LIB "ncalg.lib";
def R = makeQso3(5);
setring R;
list C = Qso3Casimir(5);
C;
⇒ [1]:
⇒ 1/5*x5+(1/5Q3-1/5Q2+2/5)*x3+(1/5Q3-1/5Q2+1/5)*x
⇒ [2]:
⇒ 1/5*y5+(1/5Q3-1/5Q2+2/5)*y3+(1/5Q3-1/5Q2+1/5)*y
⇒ [3]:
⇒ 1/5*z5+(1/5Q3-1/5Q2+2/5)*z3+(1/5Q3-1/5Q2+1/5)*z
list Cnorm = Qso3Casimir(5,1);
Cnorm;
⇒ [1]:
⇒ x5+(Q3-Q2+2)*x3+(Q3-Q2+1)*x
⇒ [2]:
⇒ y5+(Q3-Q2+2)*y3+(Q3-Q2+1)*y
⇒ [3]:
⇒ z5+(Q3-Q2+2)*z3+(Q3-Q2+1)*z

```

See also: [Section 7.5.10.22 \[makeQso3\]](#), page 479.

### 7.5.10.26 GKZsystem

Procedure from library `ncalg.lib` (see [Section 7.5.10 \[ncalg.lib\]](#), page 465).



- Usage:** `GKZsystem(A, sord, alg, [v]);` A intmat, sord, alg string, v intvec
- Return:** ring
- Purpose:** define a ring (Weyl algebra) and create a Gelfand-Kapranov-Zelevinsky (GKZ) system of equations in a ring from the following data:  
 A is an intmat, defining the system,  
 sord is a string with desired term ordering,  
 alg is a string, saying which algorithm to use (exactly like in `toric.lib`),  
 v is an optional intvec.  
 In addition, the ideal called `GKZid` containing actual equations is calculated and exported to the ring.
- Note:** activate the output ring with the `setring` command. This procedure is elaborated by Oleksandr Iena
- Assume:** This procedure uses `toric.lib` and therefore inherits its input requirements:  
 possible values for input variable `alg` are: "ect", "pt", "blr", "hs", "du".  
 As for the term ordering, it should be a string `sord` in Singular format like "lp", "dp", etc.  
 Please consult the `toric.lib` for allowed orderings and more details.

**Example:**

```
LIB "ncalg.lib";
// example 3.1.4 from the [SST] without the vector w
intmat A[2][4]=3,2,1,0,0,1,2,3;
print(A);
⇒      3      2      1      0
⇒      0      1      2      3
def D1 = GKZsystem(A,"lp","ect");
setring D1;
D1;
⇒ // coefficients: QQ(b(1), b(2))
⇒ // number of vars : 8
⇒ //      block   1 : ordering a
⇒ //                : names      x(1) x(2) x(3) x(4)
⇒ //                : weights    0    0    0    0
⇒ //      block   2 : ordering lp
⇒ //                : names      x(1) x(2) x(3) x(4) d(1) d(2) d(3) d(4)
⇒ //      block   3 : ordering C
⇒ // noncommutative relations:
⇒ //      d(1)x(1)=x(1)*d(1)+1
⇒ //      d(2)x(2)=x(2)*d(2)+1
⇒ //      d(3)x(3)=x(3)*d(3)+1
⇒ //      d(4)x(4)=x(4)*d(4)+1
print(GKZid);
⇒ 3*x(1)*d(1)+2*x(2)*d(2)+x(3)*d(3)+(-b(1)),
⇒ x(2)*d(2)+2*x(3)*d(3)+3*x(4)*d(4)+(-b(2)),
⇒ d(2)*d(4)-d(3)^2,
⇒ d(1)*d(4)-d(2)*d(3),
⇒ d(1)*d(3)-d(2)^2
// now, consider A with the vector w=1,1,1,1
intvec v=1,1,1,1;
def D2 = GKZsystem(A,"lp","blr",v);
setring D2;
```

```

print(GKZid);
 $\mapsto 3*x(1)*d(1)+2*x(2)*d(2)+x(3)*d(3)+(-b(1)),$ 
 $\mapsto x(2)*d(2)+2*x(3)*d(3)+3*x(4)*d(4)+(-b(2)),$ 
 $\mapsto d(2)*d(4)-d(3)^2,$ 
 $\mapsto d(1)*d(4)-d(2)*d(3),$ 
 $\mapsto d(1)*d(3)-d(2)^2$ 

```

See also: [Section D.4.35 \[toric\\_lib\]](#), page 1388.

### 7.5.11 ncdecomp\_lib

**Library:** ncdecomp.lib

**Purpose:** Decomposition of a module into its central characters

**Authors:** Viktor Levandovskyy, levandov@mathematik.uni-kl.de.

**Overview:**

This library presents algorithms for the central character decomposition of a module, i.e. a decomposition into generalized weight modules with respect to the center. Based on ideas of O. Khomenko and V. Levandovskyy (see the article [L2] in the References for details).

**Procedures:**

#### 7.5.11.1 CentralQuot

Procedure from library ncdecomp.lib (see [Section 7.5.11 \[ncdecomp\\_lib\]](#), page 484).

**Usage:** CentralQuot(M, G), M a module, G an ideal

**Assume:** G is an ideal in the center of the base ring

**Return:** module

**Purpose:** compute the central quotient M:G

**Theory:** for an ideal G of the center of an algebra and a submodule M of  $A^n$ , the central quotient of M by G is defined to be  $M:G := \{ v \text{ in } A^n \mid z*v \text{ in } M, \text{ for all } z \text{ in } G \}$ .

**Note:** the output module is not necessarily given in a Groebner basis

**Example:**

```

LIB "ncdecomp.lib";
option(returnSB);
def a = makeUs12();
setring a;
ideal I = e3,f3,h3-4*h;
I = std(I);
poly C=4*e*f+h^2-2*h; // C in Z(U(sl2)), the central element
ideal G = (C-8)*(C-24); // G normal factor in Z(U(sl2)) as an ideal in the center
ideal R = CentralQuot(I,G); // same as I:G
R;
 $\mapsto R[1]=h$ 
 $\mapsto R[2]=f$ 
 $\mapsto R[3]=e$ 

```

See also: [Section 7.5.11.3 \[CenCharDec\]](#), page 485; [Section 7.5.11.2 \[CentralSaturation\]](#), page 485.

### 7.5.11.2 CentralSaturation

Procedure from library `ncdecomp.lib` (see [Section 7.5.11 \[ncdecomp.lib\]](#), page 484).

**Usage:** `CentralSaturation(M, T)`, for a module  $M$  and an ideal  $T$

**Assume:**  $T$  is an ideal in the center of the base ring

**Return:** module

**Purpose:** compute the central saturation of  $M$  by  $T$ , that is  $M:T^{\infty}$ , by repetitive application of `CentralQuot`

**Note:** the output module is not necessarily a Groebner basis

**Example:**

```
LIB "ncdecomp.lib";
option(returnSB);
def a = makeUs12();
setring a;
ideal I = e3,f3,h3-4*h;
I = std(I);
poly C=4*e*f+h^2-2*h;
ideal G = C*(C-8);
ideal R = CentralSaturation(I,G);
R=std(R);
vdim(R);
↳ 5
R;
↳ R[1]=h
↳ R[2]=ef-6
↳ R[3]=f3
↳ R[4]=e3
```

See also: [Section 7.5.11.3 \[CenCharDec\]](#), page 485; [Section 7.5.11.1 \[CentralQuot\]](#), page 484.

### 7.5.11.3 CenCharDec

Procedure from library `ncdecomp.lib` (see [Section 7.5.11 \[ncdecomp.lib\]](#), page 484).

**Usage:** `CenCharDec(I, C)`;  $I$  a module,  $C$  an ideal

**Assume:**  $C$  consists of generators of the center of the base ring

**Return:** a list  $L$ , where each entry consists of three records (if a finite decomposition exists)  
 $L[*][1]$  ('ideal' type), the central character as a maximal ideal in the center,  
 $L[*][2]$  ('module' type), the Groebner basis of the weight module, corresponding to the character in  $L[*][1]$ ,  
 $L[*][3]$  ('int' type) is the vector space dimension of the weight module (-1 in case of infinite dimension);

**Purpose:** compute a finite decomposition of  $C$  into central characters or determine that there is no finite decomposition

**Note:** actual decomposition is the sum of  $L[i][2]$  above;  
 some modules have no finite decomposition (in such case one gets warning message)  
 The function `central` in `central.lib` may be used to obtain  $C$ , when needed.

**Example:**

```

LIB "ncdecomp.lib";
printlevel=0;
option(returnSB);
def a = makeUs12(); // U(sl_2) in characteristic 0
setring a;
ideal I = e3,f3,h3-4*h;
I = twostd(I);          // two-sided ideal generated by I
vdim(I);               // it is finite-dimensional
⇒ 10
ideal Cn = 4*e*f+h^2-2*h; // the only central element
list T = CenCharDec(I,Cn);
T;
⇒ [1]:
⇒   [1]:
⇒   _[1]=4ef+h2-2h-8
⇒   [2]:
⇒   _[1]=h
⇒   _[2]=f
⇒   _[3]=e
⇒   [3]:
⇒   1
⇒ [2]:
⇒   [1]:
⇒   _[1]=4ef+h2-2h
⇒   [2]:
⇒   _[1]=4ef+h2-2h-8
⇒   _[2]=h3-4h
⇒   _[3]=fh2-2fh
⇒   _[4]=eh2+2eh
⇒   _[5]=f2h-2f2
⇒   _[6]=e2h+2e2
⇒   _[7]=f3
⇒   _[8]=e3
⇒   [3]:
⇒   9
// consider another example
ideal J = e*f*h;
CenCharDec(J,Cn);
⇒ There is no finite decomposition
⇒ 0

```

See also: [Section 7.5.11.1 \[CentralQuot\]](#), page 484; [Section 7.5.11.2 \[CentralSaturation\]](#), page 485.

#### 7.5.11.4 IntersectWithSub

Procedure from library `ncdecomp.lib` (see [Section 7.5.11 \[ncdecomp.lib\]](#), page 484).

**Usage:** `IntersectWithSub(M,Z)`, `M` an ideal, `Z` an ideal

**Assume:** `Z` consists of pairwise commutative elements

**Return:** ideal of two-sided generators, not a Groebner basis

**Purpose:** computes the intersection of `M` with the subalgebra, generated by `Z`

**Note:** usually `Z` consists of generators of the center

The function `central` from `central.lib` may be used to obtain the center `Z`, if needed.

**Example:**

```

LIB "ncdecomp.lib";
ring R=(0,a),(e,f,h),Dp;
matrix @d[3][3];
@d[1,2]=-h; @d[1,3]=2e; @d[2,3]=-2f;
def r = nc_algebra(1,@d); setring r; // parametric U(sl_2)
ideal I = e,h-a;
ideal C;
C[1] = h^2-2*h+4*e*f; // the center of U(sl_2)
ideal X = IntersectWithSub(I,C);
X;
↪ X[1]=4*e*f+h^2-2*h+(-a^2-2a)
ideal G = e*f, h; // the biggest comm. subalgebra of U(sl_2)
ideal Y = IntersectWithSub(I,G);
Y;
↪ Y[1]=h+(-a)
↪ Y[2]=e*f+(-a)

```

**7.5.12 ncfactor.lib****Library:** ncfactor.lib**Purpose:** Tools for factorization in some noncommutative algebras

**Authors:** Albert Heinle, aheinle at uwaterloo.ca  
 Viktor Levandovskyy, levandov at math.rwth-aachen.de

**Overview:** In this library, new methods for factorization on polynomials are implemented for several types of algebras, namely

- finitely presented (and also free) associative algebras (Letterplace subsystem)
- G-algebras (Plural subsystem), including (q)-Weyl and (q)-shift algebras in 2n variables

The determination of the best algorithm available for users input is done automatically in the procedure `ncfactor()`.

More detailed description of the algorithms and related publications can be found at [@url{https://cs.uwaterloo.ca/~aheinle/}](https://cs.uwaterloo.ca/~aheinle/).

**Procedures:****7.5.12.1 ncfactor**

Procedure from library `ncfactor.lib` (see [Section 7.5.12 \[ncfactor.lib\]](#), page 487).

**Usage:** `ncfactor(h)`;  $h$  is a polynomial in a non-commutative polynomial algebra over a field  $k$ .**Return:** `list(list)`**Purpose:** Compute all factorizations of  $h$ .**Theory:** Implements an ansatz-driven factorization method as outlined by Bell, Heinle and Levandovskyy in "On Noncommutative Finite Factorization Domains".

**Assume:**

- $k$  is a ring, such that `factorize` can factor any univariate and multivariate commutative polynomial over  $k$ .
- There exists at least one variable in the ring.

**Note:**

- works for both PLURAL and LETTERPLACE subsystems
- Every entry of the output list is a list with factors for one possible factorization. The first factor is always a constant (1, if no nontrivial constant could be excluded).

**Example:**

```
LIB "ncfactor.lib";
// first, an example with PLURAL
def R = makeUs12();
setring(R);
poly p = e^3*f+e^2*f^2-e^3+e^2*f+2*e*f^2-3*e^2*h-2*e*f*h-8*e^2
+e*f+f^2-4*e*h-2*f*h-7*e+f-h;
ncfactor(p);
⇒ [1]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ e+1
⇒ [3]:
⇒ ef-e+f-2h-3
⇒ [4]:
⇒ e+f
⇒ [2]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ e2f+ef2-e2+f2-2eh-3e-f-2h
⇒ [3]:
⇒ e+1
kill R;
// an example with LETTERPLACE
LIB "freegb.lib";
⇒ // ** redefining tstfreegb (LIB "freegb.lib"); ./examples/ncfactor.sing:1\
0
⇒ // ** redefining tstfreegb (LIB "freegb.lib"); ./examples/ncfactor.sing:1\
0
⇒ // ** redefining setLetterplaceAttributes (LIB "freegb.lib"); ./examples/\
ncfactor.sing:10
⇒ // ** redefining setLetterplaceAttributes (LIB "freegb.lib"); ./examples/\
ncfactor.sing:10
⇒ // ** redefining lst2str (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining mod2str (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining vct2str (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining isVar (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining letplaceGBasis (LIB "freegb.lib"); ./examples/ncfactor.s\
ing:10
⇒ // ** redefining letplaceGBasis (LIB "freegb.lib"); ./examples/ncfactor.s\
ing:10
⇒ // ** redefining lieBracket (LIB "freegb.lib"); ./examples/ncfactor.sing:\
10
⇒ // ** redefining lieBracket (LIB "freegb.lib"); ./examples/ncfactor.sing:\
10
⇒ // ** redefining lpPrint (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining lpPrint (LIB "freegb.lib"); ./examples/ncfactor.sing:10
```

```

⇒ // ** redefining freeGBasis (LIB "freegb.lib");) ./examples/ncfactor.sing:\
  10
⇒ // ** redefining freeGBasis (LIB "freegb.lib");) ./examples/ncfactor.sing:\
  10
⇒ // ** redefining crs (LIB "freegb.lib");) ./examples/ncfactor.sing:10
⇒ // ** redefining polylen (LIB "freegb.lib");) ./examples/ncfactor.sing:10
⇒ // ** redefining lpDegBound (LIB "freegb.lib");) ./examples/ncfactor.sing:\
  10
⇒ // ** redefining lpDegBound (LIB "freegb.lib");) ./examples/ncfactor.sing:\
  10
⇒ // ** redefining lpVarBlockSize (LIB "freegb.lib");) ./examples/ncfactor.s\
  ing:10
⇒ // ** redefining lpVarBlockSize (LIB "freegb.lib");) ./examples/ncfactor.s\
  ing:10
⇒ // ** redefining isFreeAlgebra (LIB "freegb.lib");) ./examples/ncfactor.si\
  ng:10
⇒ // ** redefining isFreeAlgebra (LIB "freegb.lib");) ./examples/ncfactor.si\
  ng:10
⇒ // ** redefining lpNcgenCount (LIB "freegb.lib");) ./examples/ncfactor.sin\
  g:10
⇒ // ** redefining lpNcgenCount (LIB "freegb.lib");) ./examples/ncfactor.sin\
  g:10
⇒ // ** redefining makeLetterplaceRing (LIB "freegb.lib");) ./examples/ncfac\
  tor.sing:10
⇒ // ** redefining makeLetterplaceRing (LIB "freegb.lib");) ./examples/ncfac\
  tor.sing:10
⇒ // ** redefining makeLetterplaceRing1 (LIB "freegb.lib");) ./examples/ncfa\
  ctor.sing:10
⇒ // ** redefining makeLetterplaceRing2 (LIB "freegb.lib");) ./examples/ncfa\
  ctor.sing:10
⇒ // ** redefining makeLetterplaceRing4 (LIB "freegb.lib");) ./examples/ncfa\
  ctor.sing:10
⇒ // ** redefining makeLetterplaceRing3 (LIB "freegb.lib");) ./examples/ncfa\
  ctor.sing:10
⇒ // ** redefining freegbold (LIB "freegb.lib");) ./examples/ncfactor.sing:1\
  0
⇒ // ** redefining stringpoly2lplace (LIB "freegb.lib");) ./examples/ncfacto\
  r.sing:10
⇒ // ** redefining addplaces (LIB "freegb.lib");) ./examples/ncfactor.sing:1\
  0
⇒ // ** redefining sent2lplace (LIB "freegb.lib");) ./examples/ncfactor.sing\
  :10
⇒ // ** redefining testnumber (LIB "freegb.lib");) ./examples/ncfactor.sing:\
  10
⇒ // ** redefining str2lplace (LIB "freegb.lib");) ./examples/ncfactor.sing:\
  10
⇒ // ** redefining strpower2rep (LIB "freegb.lib");) ./examples/ncfactor.sin\
  g:10
⇒ // ** redefining shiftPoly (LIB "freegb.lib");) ./examples/ncfactor.sing:1\
  0
⇒ // ** redefining lastBlock (LIB "freegb.lib");) ./examples/ncfactor.sing:1\
  0
⇒ // ** redefining test_shift (LIB "freegb.lib");) ./examples/ncfactor.sing:\

```

```

10
⇒ // ** redefining lp2lstr (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining lp2lstr (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining strList2poly (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining file2lplace (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining lpPower (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining lpNF (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining lpNF (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining lpDivision (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining lpDivision (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining lpGBPres2Poly (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining lpGBPres2Poly (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining getExpVecs (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining delSupZero (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining delSupZeroList (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining makeDVec (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining makeDVecL (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining makeDVecI (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining dShiftDiv (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining lpNormalForm1 (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining lpNormalForm2 (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining isOrderingShiftInvariant (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining isOrderingShiftInvariant (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining lpMonomialsWithHoles (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining getlpCoeffs (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining lpReduce (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining entryViolation (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining checkAssumptionsLPIV (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining checkAssumptions (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining checkLPRing (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining checkAssumptionIdeal (LIB "freegb.lib"); ./examples/ncfactor.sing:10

```



```

    ctor.sing:10
⇒ // ** redefining checkAssumptionPoly (LIB "freegb.lib"); ./examples/ncfactor\
    tor.sing:10
⇒ // ** redefining isContainedInVp (LIB "freegb.lib"); ./examples/ncfactor.\
    sing:10
⇒ // ** redefining extractLinearPart (LIB "freegb.lib"); ./examples/ncfactor\
    r.sing:10
⇒ // ** redefining isLinearVector (LIB "freegb.lib"); ./examples/ncfactor.s\
    ing:10
⇒ // ** redefining lpAssumeViolation (LIB "freegb.lib"); ./examples/ncfactor\
    r.sing:10
⇒ // ** redefining skip0 (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining ivL2lpI (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining ivL2lpI (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining iv2lp (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining iv2lp (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining iv2lpList (LIB "freegb.lib"); ./examples/ncfactor.sing:1\
    0
⇒ // ** redefining iv2lpList (LIB "freegb.lib"); ./examples/ncfactor.sing:1\
    0
⇒ // ** redefining iv2lpMat (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining iv2lpMat (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining lpId2ivLi (LIB "freegb.lib"); ./examples/ncfactor.sing:1\
    0
⇒ // ** redefining lpId2ivLi (LIB "freegb.lib"); ./examples/ncfactor.sing:1\
    0
⇒ // ** redefining lp2iv (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining lp2iv (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining lp2ivId (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining lp2ivId (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining testLift (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining testLift (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining testSyz (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining testSyz (LIB "freegb.lib"); ./examples/ncfactor.sing:10
⇒ // ** redefining mod_init (LIB "freegb.lib"); ./examples/ncfactor.sing:10
ring r = 0,(x,y),Dp;
def R = freeAlgebra(r,5); setring(R);
poly p = x*y*x - x;
ncfactor(p);
⇒ [1]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ x*y-1
⇒ [3]:
⇒ x
⇒ [2]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ x
⇒ [3]:
⇒ y*x-1

```

See also: [Section 7.5.12.5 \[facSubWeyl\]](#), page 494; [Section 7.5.12.2 \[facWeyl\]](#), page 492; [Section 7.5.12.4 \[testNCfac\]](#), page 493.

### 7.5.12.2 facWeyl

Procedure from library `ncfactor.lib` (see [Section 7.5.12 \[ncfactor.lib\]](#), page 487).

**Usage:** `facWeyl(h)`;  $h$  a polynomial in the  $n$ th Weyl algebra

**Return:** list

**Purpose:** compute all factorizations of a polynomial in the first Weyl algebra

**Theory:** Implements the new algorithm by A. Heinle and V. Levandovskyy, see the thesis of A. Heinle

**Assume:** `basing` is the  $n$ th Weyl algebra, where  $n$  in  $\mathbb{N}$ .

**Note:** Every entry of the output list is a list with factors for one possible factorization. The first factor is always a constant (1, if no nontrivial constant could be excluded).

**Example:**

```
LIB "ncfactor.lib";
ring R = 0,(x1,x2,d1,d2),dp;
matrix C[4][4] = 1,1,1,1,
1,1,1,1,
1,1,1,1,
1,1,1,1;
matrix D[4][4] = 0,0,1,0,
0,0,0,1,
-1,0,0,0,
0,-1,0,0;
def r = nc_algebra(C,D);
setring(r);
poly h = (d1+1)^2*(d1 + x1*d2);
facWeyl(h);
⇒ [1]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ d1+1
⇒ [3]:
⇒ d1+1
⇒ [4]:
⇒ x1*d2+d1
⇒ [2]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ x1*d1*d2+d1^2+x1*d2+d1+2*d2
⇒ [3]:
⇒ d1+1
```

See also: [Section 7.5.12.7 \[facFirstShift\]](#), page 496; [Section 7.5.12.3 \[facFirstWeyl\]](#), page 493; [Section 7.5.12.5 \[facSubWeyl\]](#), page 494; [Section 7.5.12.4 \[testNCfac\]](#), page 493.

### 7.5.12.3 facFirstWeyl

Procedure from library `ncfactor.lib` (see [Section 7.5.12 \[ncfactor.lib\]](#), page 487).

**Usage:** `facFirstWeyl(h)`;  $h$  a polynomial in the first Weyl algebra

**Return:** list

**Purpose:** compute all factorizations of a polynomial in the first Weyl algebra

**Theory:** This function is a wrapper for `facWeyl`. It exists to make this library downward-compatible with older versions.

**Assume:** `basing` is the first Weyl algebra

**Note:** Every entry of the output list is a list with factors for one possible factorization. The first factor is always a constant (1, if no nontrivial constant could be excluded).

**Example:**

```
LIB "ncfactor.lib";
ring R = 0,(x,y),dp;
def r = nc_algebra(1,1);
setring(r);
poly h = (x^2*y^2+x)*(x+1);
facFirstWeyl(h);
⇒ [1]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ x
⇒ [3]:
⇒ xy2+1
⇒ [4]:
⇒ x+1
```

See also: [Section 7.5.12.6 \[facShift\]](#), page 495; [Section 7.5.12.5 \[facSubWeyl\]](#), page 494; [Section 7.5.12.4 \[testNCfac\]](#), page 493.

### 7.5.12.4 testNCfac

Procedure from library `ncfactor.lib` (see [Section 7.5.12 \[ncfactor.lib\]](#), page 487).

**Usage:** `testNCfac(l[,p,b])`;  $l$  is a list,  $p$  is an optional poly,  $b$  is 1 or 0

**Return:** Case 1: No optional argument. In this case the output is 1, if the entries in the given list represent the same polynomial or 0 otherwise.

Case 2: One optional argument  $p$  is given. In this case it returns 1, if all the entries in  $l$  are factorizations of  $p$ , otherwise 0. Case 3: Second optional  $b$  is given. In this case a list is returned containing the difference between the product of each entry in  $l$  and  $p$ .

**Assume:** `basing` is the first Weyl algebra, the entries of  $l$  are polynomials

**Purpose:** Checks whether a list of factorizations contains factorizations of the same element in the first Weyl algebra

**Theory:** `testNCfac` multiplies out each factorization and checks whether each factorization was a factorization of the same element.

- if there is only a list given, the output will be 0, if it does not contain factorizations of the same element. Otherwise the output will be 1.

- if there is a polynomial in the second argument, then the procedure checks whether the given list contains factorizations of this polynomial. If it does, then the output depends on the third argument. If it is not given, the procedure will check whether the factorizations in the list  $l$  are associated to this polynomial and return either 1 or 0, respectively. If the third argument is given, the output will be a list with the length of the given one and in each entry is the product of one entry in  $l$  subtracted by the polynomial.

**Example:**

```
LIB "ncfactor.lib";
ring r = 0,(x,y),dp;
def R = nc_algebra(1,1);
setring R;
poly h = (x^2*y^2+1)*(x^2);
def t1 = facFirstWeyl(h);
//first a correct list
testNCfac(t1);
⇒ 1
//now a correct list with the factorized polynomial
testNCfac(t1,h);
⇒ 1
//now we put in an incorrect list without a polynomial
t1[3][3] = y;
testNCfac(t1);
⇒ 0
// take h as additional input
testNCfac(t1,h);
⇒ 0
// take h as additional input and output list of differences
testNCfac(t1,h,1);
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ -x4y2+x3y3-4x3y+3x2y2-3x2+xy+1
```

See also: [Section 7.5.12.7 \[facFirstShift\]](#), page 496; [Section 7.5.12.3 \[facFirstWeyl\]](#), page 493; [Section 7.5.12.5 \[facSubWeyl\]](#), page 494.

**7.5.12.5 facSubWeyl**

Procedure from library `ncfactor.lib` (see [Section 7.5.12 \[ncfactor.lib\]](#), page 487).

**Usage:** `facSubWeyl(h,[x_1,...,x_n,d_1,...,d_n])`;  $h$  is a polynomial,  $x_i$ ,  $d_i$  are variables in the basering for  $i$  in  $\{1,...,n\}$

**Return:** `list(list)`

**Assume:**  $x_i$ ,  $d_i$  are variables of a basering with  $d_i x_i = x_i d_{i+1}$  for  $i$  in  $\{1,...,n\}$ .

That is, they generate the copy of the first Weyl algebra in a basering.

Moreover,  $h$  is a polynomial in the  $x_i, d_i$  only.

If the list of variables is omitted, this function will try to figure out itself if  $h$  is in a subalgebra that resembles the Weyl algebra.

This function produces an error if the conditions on the variables do not line up or if the

variables contained in  $h$  do not belong to a subalgebra of the basering that resembles the Weyl algebra.

**Purpose:** compute factorizations of the polynomial, depending on  $x_i$  and  $d_i$ .

**Example:**

```
LIB "ncfactor.lib";
ring r = 0,(x,y,z),dp;
matrix D[3][3]; D[1,3]=-1;
def R = nc_algebra(1,D); // x,z generate Weyl subalgebra
setring R;
poly h = (x^2*z^2+x)*x;
list fact1 = facSubWeyl(h,x,z);
// compare with facFirstWeyl:
ring s = 0,(z,x),dp;
def S = nc_algebra(1,1); setring S;
poly h = (x^2*z^2+x)*x;
list fact2 = facFirstWeyl(h);
map F = R,x,0,z;
list fact1 = F(fact1); // it is identical to list fact2
testNCfac(fact1); // check the correctness again
↳ 1
```

See also: [Section 7.5.12.7 \[facFirstShift\], page 496](#); [Section 7.5.12.3 \[facFirstWeyl\], page 493](#); [Section 7.5.12.4 \[testNCfac\], page 493](#).

### 7.5.12.6 facShift

Procedure from library `ncfactor.lib` (see [Section 7.5.12 \[ncfactor.lib\], page 487](#)).

**Usage:** `facShift(h)`;  $h$  a polynomial in the  $n$ 'th shift algebra

**Return:** list

**Purpose:** compute all factorizations of a polynomial in the  $n$ th shift algebra

**Theory:** Currently, we do not have a specialized algorithm for the shift algebra in this library that takes advantage of the graded structure, hence this function is mapping to the general factorization algorithm for  $G$ -Algebras

**Note:** Every entry of the output list is a list with factors for one possible factorization.

**Example:**

```
LIB "ncfactor.lib";
ring R = 0,(x1,x2,s1,s2),dp;
matrix C[4][4] = 1,1,1,1,
1,1,1,1,
1,1,1,1,
1,1,1,1;
matrix D[4][4] = 0,0,s1,0,
0,0,0,s2,
-s1,0,0,0,
0,-s2,0,0;
def r = nc_algebra(C,D);
setring(r);
poly h = x1*(x1+1)*s1^2-2*x1*(x1+100)*s1+(x1+99)*(x1+100);
facShift(h);
```

```

⇒ [1]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ x1*s1-x1+s1-100
⇒ [3]:
⇒ x1*s1-x1-s1-99
⇒ [2]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ x1*s1-x1-100
⇒ [3]:
⇒ x1*s1-x1-99
⇒ [3]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ x1*s1-x1-99
⇒ [3]:
⇒ x1*s1-x1-100

```

See also: [Section 7.5.12.3 \[facFirstWeyl\]](#), page 493; [Section 7.5.12.5 \[facSubWeyl\]](#), page 494; [Section 7.5.12.4 \[testNCfac\]](#), page 493.

### 7.5.12.7 facFirstShift

Procedure from library `ncfactor.lib` (see [Section 7.5.12 \[ncfactor.lib\]](#), page 487).

**Usage:** `facFirstShift(h)`;  $h$  a polynomial in the first shift algebra

**Return:** list

**Purpose:** compute all factorizations of a polynomial in the first shift algebra

**Theory:** This function is a wrapper for `facShift`. It exists to make this library downward-compatible with older versions.

**Assume:** `basing` is the first shift algebra

**Note:** Every entry of the output list is a list with factors for one possible factorization.

**Example:**

```

LIB "ncfactor.lib";
ring R = 0,(x,s),dp;
def r = nc_algebra(1,s);
setring(r);
poly h = (s^2*x+x)*s;
facFirstShift(h);
⇒ [1]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ s
⇒ [3]:
⇒ s2+1
⇒ [4]:

```

```

⇒      x-1
⇒ [2] :
⇒      [1] :
⇒ 1
⇒      [2] :
⇒      s2+1
⇒      [3] :
⇒      s
⇒      [4] :
⇒      x-1
⇒ [3] :
⇒      [1] :
⇒ 1
⇒      [2] :
⇒      s2+1
⇒      [3] :
⇒      x
⇒      [4] :
⇒      s

```

See also: [Section 7.5.12.3 \[facFirstWeyl\]](#), page 493; [Section 7.5.12.5 \[facSubWeyl\]](#), page 494; [Section 7.5.12.4 \[testNCfac\]](#), page 493.

### 7.5.12.8 homogfacNthWeyl

Procedure from library `ncfactor.lib` (see [Section 7.5.12 \[ncfactor.lib\]](#), page 487).

**Usage:** `homogfacNthWeyl(h)`;  $h$  is a homogeneous polynomial in the  $n$ th Weyl algebra with respect to the  $-1,1$ -grading

**Return:** list

**Purpose:** Computes a factorization of a homogeneous polynomial  $h$  with respect to the  $ZZ$ -grading on the  $n$ -th Weyl algebra.

**Theory:** `homogfacFirstWeyl` returns a list with a factorization of the given,  $[-1,1]$ -homogeneous polynomial. For every  $i$  in  $1..n$ : If the degree of the polynomial in  $[d_i, x_i]$  is  $k$  with  $k$  positive, the last  $k$  entries in the output list are the second variable. If  $k$  is positive, the last  $k$  entries will be  $x_i$ . The other entries will be irreducible polynomials of degree zero or 1 resp.  $-1$ . resp. other variables

**General assumptions:**

- The basering is the  $n$ th Weyl algebra and has the form, that the first  $n$  variables represent  $x_1, \dots, x_n$ , and the second  $n$  variables do represent the  $d_1, \dots, d_n$ .

### 7.5.12.9 homogfacNthQWeyl

Procedure from library `ncfactor.lib` (see [Section 7.5.12 \[ncfactor.lib\]](#), page 487).

**Usage:** `homogfacNthQWeyl(h)`;  $h$  is a homogeneous polynomial in the  $n$ 'th  $q$ -Weyl algebra with respect to the weight vector

$@ [-1, \dots, -1, 1, \dots, 1]$ .

$@ \sqrt[n]{q} \dots \sqrt[n]{q} @ \sqrt[n]{q} \sqrt[n]{q} @ n/2 \ n/2$

**Return:** list

**Purpose:** Computes a factorization of a homogeneous polynomial  $h$  in the  $n$ 'th  $q$ -Weyl algebra

**Theory:** `homogfacNthQWeyl` returns a list with a factorization of the given,  $[-1,1]$ -homogeneous polynomial. For every  $i$  in  $1..n$ : If the degree of the polynomial in  $[d_i, x_i]$  is  $k$  with  $k$  positive, the last entries in the output list are the second variable. If  $k$  is positive, the last  $k$  entries will be  $x_i$ . The other entries will be irreducible polynomials of degree zero or 1 resp. -1. resp. other variables

**General assumptions:**

- The basering is the  $n$ th Weyl algebra and has the form, that the first  $n$  variables represent  $x_1, \dots, x_n$ , and the second  $n$  variables do represent the  $d_1, \dots, d_n$ .
- We have  $n$  parameters  $q_1, \dots, q_n$  given.

**Example:**

```
LIB "ncfactor.lib";
ring R = (0,q1,q2,q3),(x1,x2,x3,d1,d2,d3),dp;
matrix C[6][6] = 1,1,1,q1,1,1,
1,1,1,1,q2,1,
1,1,1,1,1,q3,
1,1,1,1,1,1,
1,1,1,1,1,1,
1,1,1,1,1,1;
matrix D[6][6] = 0,0,0,1,0,0,
0,0,0,0,1,0,
0,0,0,0,0,1,
-1,0,0,0,0,0,
0,-1,0,0,0,0,
0,0,-1,0,0,0;
def r = nc_algebra(C,D);
setring(r);
poly h = x1*x2^2*x3^3*d1*d2^2+x2*x3^3*d2;
homogfacNthQWeyl(h);
⇒ [1]:
⇒ 1/(q2)
⇒ [2]:
⇒ x2
⇒ [3]:
⇒ d2
⇒ [4]:
⇒ x1*x2*d1*d2-x1*d1+(q2)
⇒ [5]:
⇒ x3
⇒ [6]:
⇒ x3
⇒ [7]:
⇒ x3
```

See also: [Section 7.5.12.10 \[homogfacFirstQWeyl\], page 498](#); [Section 7.5.12.12 \[homogfacFirstQWeyl\\_all\], page 517](#); [Section 7.5.12.11 \[homogfacNthQWeyl\\_all\], page 499](#).

### 7.5.12.10 homogfacFirstQWeyl

Procedure from library `ncfactor.lib` (see [Section 7.5.12 \[ncfactor.lib\], page 487](#)).

**Usage:** `homogfacFirstQWeyl(h)`;  $h$  is a homogeneous polynomial in the first  $q$ -Weyl algebra with respect to the weight vector  $[-1,1]$



**Return:** list

**Purpose:** Computes a factorization of a homogeneous polynomial  $h$  with respect to the weight vector  $[-1,1]$  in the first  $q$ -Weyl algebra

**Theory:** This function is a wrapper for `homogfacNthQWeyl`. It exists to make this library downward-compatible with older versions.

**Example:**

```
LIB "ncfactor.lib";
ring R = (0,q),(x,d),dp;
def r = nc_algebra (q,1);
setring(r);
poly h = q^25*x^10*d^10+q^16*(q^4+q^3+q^2+q+1)^2*x^9*d^9+
q^9*(q^13+3*q^12+7*q^11+13*q^10+20*q^9+26*q^8+30*q^7+
31*q^6+26*q^5+20*q^4+13*q^3+7*q^2+3*q+1)*x^8*d^8+
q^4*(q^9+2*q^8+4*q^7+6*q^6+7*q^5+8*q^4+6*q^3+
4*q^2+2*q+1)*(q^4+q^3+q^2+q+1)*(q^2+q+1)*x^7*d^7+
q*(q^2+q+1)*(q^5+2*q^4+2*q^3+3*q^2+2*q+1)*(q^4+q^3+q^2+q+1)*(q^2+1)*(q+1)*x^6*d^6+
(q^10+5*q^9+12*q^8+21*q^7+29*q^6+33*q^5+31*q^4+24*q^3+15*q^2+7*q+12)*x^5*d^5+
6*x^3*d^3+24;
homogfacFirstQWeyl(h);
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ x5d5+6
⇒ [3]:
⇒ x5d5+x3d3+4
```

See also: [Section 7.5.12.12 \[homogfacFirstQWeyl\\_all\]](#), page 517.

### 7.5.12.11 homogfacNthQWeyl\_all

Procedure from library `ncfactor.lib` (see [Section 7.5.12 \[ncfactor.lib\]](#), page 487).

**Usage:** `homogfacNthQWeyl_all(h)`;  $h$  is a homogeneous polynomial in the  $n$ 'th  $q$ -Weyl algebra with respect to the weight vector  
 $@ [-1, \dots, -1, 1, \dots, 1]$ .  
 $@ \setminus \dots \setminus / \setminus \dots \setminus / @ \setminus / \setminus @ n/2 \ n/2$

**Return:** list

**Purpose:** Computes all factorizations of a homogeneous polynomial  $h$  in the  $n$ 'th  $q$ -Weyl algebra

**Theory:** `homogfacNthQWeyl` returns a list with lists representing each a factorization of the given,  
 $[-1, \dots, -1, 1, \dots, 1]$ -homogeneous polynomial.

**General assumptions:**

- The basering is the  $n$ th Weyl algebra and has the form, that the first  $n$  variables represent  $x_1, \dots, x_n$ , and the second  $n$  variables do represent the  $d_1, \dots, d_n$ .
- We have  $n$  parameters  $q_1, \dots, q_n$  given.

**Example:**

```
LIB "ncfactor.lib";
ring R = (0,q1,q2,q3),(x1,x2,x3,d1,d2,d3),dp;
matrix C[6][6] = 1,1,1,q1,1,1,
```

```

1,1,1,1,q2,1,
1,1,1,1,1,q3,
1,1,1,1,1,1,
1,1,1,1,1,1,
1,1,1,1,1,1;
matrix D[6][6] = 0,0,0,1,0,0,
0,0,0,0,1,0,
0,0,0,0,0,1,
-1,0,0,0,0,0,
0,-1,0,0,0,0,
0,0,-1,0,0,0;
def r = nc_algebra(C,D);
setring(r);
poly h =x1*x2^2*x3^3*d1*d2^2+x2*x3^3*d2;
homogfacNthQWeyl_all(h);
↳ [1]:
↳ [1]:
↳ 1
↳ [2]:
↳ x2
↳ [3]:
↳ x1*x2*d1*d2+1
↳ [4]:
↳ d2
↳ [5]:
↳ x3
↳ [6]:
↳ x3
↳ [7]:
↳ x3
↳ [2]:
↳ [1]:
↳ 1
↳ [2]:
↳ x2
↳ [3]:
↳ x1*x2*d1*d2+1
↳ [4]:
↳ x3
↳ [5]:
↳ d2
↳ [6]:
↳ x3
↳ [7]:
↳ x3
↳ [3]:
↳ [1]:
↳ 1
↳ [2]:
↳ x2
↳ [3]:
↳ x1*x2*d1*d2+1
↳ [4]:

```

```

↳      x3
↳      [5]:
↳      x3
↳      [6]:
↳      d2
↳      [7]:
↳      x3
↳ [4]:
↳      [1]:
↳ 1
↳      [2]:
↳      x2
↳      [3]:
↳      x1*x2*d1*d2+1
↳      [4]:
↳      x3
↳      [5]:
↳      x3
↳      [6]:
↳      x3
↳      [7]:
↳      d2
↳ [5]:
↳      [1]:
↳ 1
↳      [2]:
↳      x2
↳      [3]:
↳      x3
↳      [4]:
↳      x1*x2*d1*d2+1
↳      [5]:
↳      d2
↳      [6]:
↳      x3
↳      [7]:
↳      x3
↳ [6]:
↳      [1]:
↳ 1
↳      [2]:
↳      x2
↳      [3]:
↳      x3
↳      [4]:
↳      x1*x2*d1*d2+1
↳      [5]:
↳      x3
↳      [6]:
↳      d2
↳      [7]:
↳      x3
↳ [7]:

```

```

⇨ [1]:
⇨ 1
⇨ [2]:
⇨ x2
⇨ [3]:
⇨ x3
⇨ [4]:
⇨ x1*x2*d1*d2+1
⇨ [5]:
⇨ x3
⇨ [6]:
⇨ x3
⇨ [7]:
⇨ d2
⇨ [8]:
⇨ [1]:
⇨ 1
⇨ [2]:
⇨ x2
⇨ [3]:
⇨ x3
⇨ [4]:
⇨ x3
⇨ [5]:
⇨ x1*x2*d1*d2+1
⇨ [6]:
⇨ d2
⇨ [7]:
⇨ x3
⇨ [9]:
⇨ [1]:
⇨ 1
⇨ [2]:
⇨ x2
⇨ [3]:
⇨ x3
⇨ [4]:
⇨ x3
⇨ [5]:
⇨ x1*x2*d1*d2+1
⇨ [6]:
⇨ x3
⇨ [7]:
⇨ d2
⇨ [10]:
⇨ [1]:
⇨ 1
⇨ [2]:
⇨ x2
⇨ [3]:
⇨ x3
⇨ [4]:
⇨ x3

```

```

⇒      [5]:
⇒      x3
⇒      [6]:
⇒      x1*x2*d1*d2+1
⇒      [7]:
⇒      d2
⇒ [11]:
⇒      [1]:
⇒      1
⇒      [2]:
⇒      x3
⇒      [3]:
⇒      x2
⇒      [4]:
⇒      x1*x2*d1*d2+1
⇒      [5]:
⇒      d2
⇒      [6]:
⇒      x3
⇒      [7]:
⇒      x3
⇒ [12]:
⇒      [1]:
⇒      1
⇒      [2]:
⇒      x3
⇒      [3]:
⇒      x2
⇒      [4]:
⇒      x1*x2*d1*d2+1
⇒      [5]:
⇒      x3
⇒      [6]:
⇒      d2
⇒      [7]:
⇒      x3
⇒ [13]:
⇒      [1]:
⇒      1
⇒      [2]:
⇒      x3
⇒      [3]:
⇒      x2
⇒      [4]:
⇒      x1*x2*d1*d2+1
⇒      [5]:
⇒      x3
⇒      [6]:
⇒      x3
⇒      [7]:
⇒      d2
⇒ [14]:
⇒      [1]:

```

```

↳ 1
↳ [2]:
↳ x3
↳ [3]:
↳ x2
↳ [4]:
↳ x3
↳ [5]:
↳ x1*x2*d1*d2+1
↳ [6]:
↳ d2
↳ [7]:
↳ x3
↳ [15]:
↳ [1]:
↳ 1
↳ [2]:
↳ x3
↳ [3]:
↳ x2
↳ [4]:
↳ x3
↳ [5]:
↳ x1*x2*d1*d2+1
↳ [6]:
↳ x3
↳ [7]:
↳ d2
↳ [16]:
↳ [1]:
↳ 1
↳ [2]:
↳ x3
↳ [3]:
↳ x2
↳ [4]:
↳ x3
↳ [5]:
↳ x3
↳ [6]:
↳ x1*x2*d1*d2+1
↳ [7]:
↳ d2
↳ [17]:
↳ [1]:
↳ 1
↳ [2]:
↳ x3
↳ [3]:
↳ x3
↳ [4]:
↳ x2
↳ [5]:

```

```

↳      x1*x2*d1*d2+1
↳      [6]:
↳      d2
↳      [7]:
↳      x3
↳ [18]:
↳      [1]:
↳      1
↳      [2]:
↳      x3
↳      [3]:
↳      x3
↳      [4]:
↳      x2
↳      [5]:
↳      x1*x2*d1*d2+1
↳      [6]:
↳      x3
↳      [7]:
↳      d2
↳ [19]:
↳      [1]:
↳      1
↳      [2]:
↳      x3
↳      [3]:
↳      x3
↳      [4]:
↳      x2
↳      [5]:
↳      x3
↳      [6]:
↳      x1*x2*d1*d2+1
↳      [7]:
↳      d2
↳ [20]:
↳      [1]:
↳      1
↳      [2]:
↳      x3
↳      [3]:
↳      x3
↳      [4]:
↳      x3
↳      [5]:
↳      x2
↳      [6]:
↳      x1*x2*d1*d2+1
↳      [7]:
↳      d2
↳ [21]:
↳      [1]:
↳      1/(q2)

```

```

↳ [2]:
↳ x1*x2*d1*d2-x1*d1+(q2)
↳ [3]:
↳ x2
↳ [4]:
↳ d2
↳ [5]:
↳ x3
↳ [6]:
↳ x3
↳ [7]:
↳ x3
↳ [22]:
↳ [1]:
↳ 1/(q2)
↳ [2]:
↳ x1*x2*d1*d2-x1*d1+(q2)
↳ [3]:
↳ x2
↳ [4]:
↳ x3
↳ [5]:
↳ d2
↳ [6]:
↳ x3
↳ [7]:
↳ x3
↳ [23]:
↳ [1]:
↳ 1/(q2)
↳ [2]:
↳ x1*x2*d1*d2-x1*d1+(q2)
↳ [3]:
↳ x2
↳ [4]:
↳ x3
↳ [5]:
↳ x3
↳ [6]:
↳ d2
↳ [7]:
↳ x3
↳ [24]:
↳ [1]:
↳ 1/(q2)
↳ [2]:
↳ x1*x2*d1*d2-x1*d1+(q2)
↳ [3]:
↳ x2
↳ [4]:
↳ x3
↳ [5]:
↳ x3

```



```

↳ [6]:
↳ x3
↳ [7]:
↳ d2
↳ [25]:
↳ [1]:
↳ 1/(q2)
↳ [2]:
↳ x1*x2*d1*d2-x1*d1+(q2)
↳ [3]:
↳ x3
↳ [4]:
↳ x2
↳ [5]:
↳ d2
↳ [6]:
↳ x3
↳ [7]:
↳ x3
↳ [26]:
↳ [1]:
↳ 1/(q2)
↳ [2]:
↳ x1*x2*d1*d2-x1*d1+(q2)
↳ [3]:
↳ x3
↳ [4]:
↳ x2
↳ [5]:
↳ x3
↳ [6]:
↳ d2
↳ [7]:
↳ x3
↳ [27]:
↳ [1]:
↳ 1/(q2)
↳ [2]:
↳ x1*x2*d1*d2-x1*d1+(q2)
↳ [3]:
↳ x3
↳ [4]:
↳ x2
↳ [5]:
↳ x3
↳ [6]:
↳ x3
↳ [7]:
↳ d2
↳ [28]:
↳ [1]:
↳ 1/(q2)
↳ [2]:

```

```

↳      x1*x2*d1*d2-x1*d1+(q2)
↳ [3]:
↳      x3
↳ [4]:
↳      x3
↳ [5]:
↳      x2
↳ [6]:
↳      d2
↳ [7]:
↳      x3
↳ [29]:
↳ [1]:
↳ 1/(q2)
↳ [2]:
↳      x1*x2*d1*d2-x1*d1+(q2)
↳ [3]:
↳      x3
↳ [4]:
↳      x3
↳ [5]:
↳      x2
↳ [6]:
↳      x3
↳ [7]:
↳      d2
↳ [30]:
↳ [1]:
↳ 1/(q2)
↳ [2]:
↳      x1*x2*d1*d2-x1*d1+(q2)
↳ [3]:
↳      x3
↳ [4]:
↳      x3
↳ [5]:
↳      x3
↳ [6]:
↳      x2
↳ [7]:
↳      d2
↳ [31]:
↳ [1]:
↳ 1/(q2)
↳ [2]:
↳      x2
↳ [3]:
↳      d2
↳ [4]:
↳      x1*x2*d1*d2-x1*d1+(q2)
↳ [5]:
↳      x3
↳ [6]:

```

```

⊢      x3
⊢      [7]:
⊢      x3
⊢ [32]:
⊢      [1]:
⊢ 1/(q2)
⊢      [2]:
⊢      x2
⊢      [3]:
⊢      d2
⊢      [4]:
⊢      x3
⊢      [5]:
⊢      x1*x2*d1*d2-x1*d1+(q2)
⊢      [6]:
⊢      x3
⊢      [7]:
⊢      x3
⊢ [33]:
⊢      [1]:
⊢ 1/(q2)
⊢      [2]:
⊢      x2
⊢      [3]:
⊢      d2
⊢      [4]:
⊢      x3
⊢      [5]:
⊢      x3
⊢      [6]:
⊢      x1*x2*d1*d2-x1*d1+(q2)
⊢      [7]:
⊢      x3
⊢ [34]:
⊢      [1]:
⊢ 1/(q2)
⊢      [2]:
⊢      x2
⊢      [3]:
⊢      d2
⊢      [4]:
⊢      x3
⊢      [5]:
⊢      x3
⊢      [6]:
⊢      x3
⊢      [7]:
⊢      x1*x2*d1*d2-x1*d1+(q2)
⊢ [35]:
⊢      [1]:
⊢ 1/(q2)
⊢      [2]:
⊢      x2

```

```

⇒ [3]:
⇒ x3
⇒ [4]:
⇒ d2
⇒ [5]:
⇒ x1*x2*d1*d2-x1*d1+(q2)
⇒ [6]:
⇒ x3
⇒ [7]:
⇒ x3
⇒ [36]:
⇒ [1]:
⇒ 1/(q2)
⇒ [2]:
⇒ x2
⇒ [3]:
⇒ x3
⇒ [4]:
⇒ d2
⇒ [5]:
⇒ x3
⇒ [6]:
⇒ x1*x2*d1*d2-x1*d1+(q2)
⇒ [7]:
⇒ x3
⇒ [37]:
⇒ [1]:
⇒ 1/(q2)
⇒ [2]:
⇒ x2
⇒ [3]:
⇒ x3
⇒ [4]:
⇒ d2
⇒ [5]:
⇒ x3
⇒ [6]:
⇒ x3
⇒ [7]:
⇒ x1*x2*d1*d2-x1*d1+(q2)
⇒ [38]:
⇒ [1]:
⇒ 1/(q2)
⇒ [2]:
⇒ x2
⇒ [3]:
⇒ x3
⇒ [4]:
⇒ x3
⇒ [5]:
⇒ d2
⇒ [6]:
⇒ x1*x2*d1*d2-x1*d1+(q2)

```

```

⇒      [7]:
⇒      x3
⇒ [39]:
⇒      [1]:
⇒ 1/(q2)
⇒      [2]:
⇒      x2
⇒      [3]:
⇒      x3
⇒      [4]:
⇒      x3
⇒      [5]:
⇒      d2
⇒      [6]:
⇒      x3
⇒      [7]:
⇒      x1*x2*d1*d2-x1*d1+(q2)
⇒ [40]:
⇒      [1]:
⇒ 1/(q2)
⇒      [2]:
⇒      x2
⇒      [3]:
⇒      x3
⇒      [4]:
⇒      x3
⇒      [5]:
⇒      x3
⇒      [6]:
⇒      d2
⇒      [7]:
⇒      x1*x2*d1*d2-x1*d1+(q2)
⇒ [41]:
⇒      [1]:
⇒ 1/(q2)
⇒      [2]:
⇒      x3
⇒      [3]:
⇒      x1*x2*d1*d2-x1*d1+(q2)
⇒      [4]:
⇒      x2
⇒      [5]:
⇒      d2
⇒      [6]:
⇒      x3
⇒      [7]:
⇒      x3
⇒ [42]:
⇒      [1]:
⇒ 1/(q2)
⇒      [2]:
⇒      x3
⇒      [3]:

```

```

↳      x1*x2*d1*d2-x1*d1+(q2)
↳      [4]:
↳      x2
↳      [5]:
↳      x3
↳      [6]:
↳      d2
↳      [7]:
↳      x3
↳ [43]:
↳      [1]:
↳ 1/(q2)
↳      [2]:
↳      x3
↳      [3]:
↳      x1*x2*d1*d2-x1*d1+(q2)
↳      [4]:
↳      x2
↳      [5]:
↳      x3
↳      [6]:
↳      x3
↳      [7]:
↳      d2
↳ [44]:
↳      [1]:
↳ 1/(q2)
↳      [2]:
↳      x3
↳      [3]:
↳      x1*x2*d1*d2-x1*d1+(q2)
↳      [4]:
↳      x3
↳      [5]:
↳      x2
↳      [6]:
↳      d2
↳      [7]:
↳      x3
↳ [45]:
↳      [1]:
↳ 1/(q2)
↳      [2]:
↳      x3
↳      [3]:
↳      x1*x2*d1*d2-x1*d1+(q2)
↳      [4]:
↳      x3
↳      [5]:
↳      x2
↳      [6]:
↳      x3
↳      [7]:

```

```

↳      d2
↳ [46] :
↳      [1] :
↳ 1/(q2)
↳      [2] :
↳      x3
↳      [3] :
↳      x1*x2*d1*d2-x1*d1+(q2)
↳      [4] :
↳      x3
↳      [5] :
↳      x3
↳      [6] :
↳      x2
↳      [7] :
↳      d2
↳ [47] :
↳      [1] :
↳ 1/(q2)
↳      [2] :
↳      x3
↳      [3] :
↳      x2
↳      [4] :
↳      d2
↳      [5] :
↳      x1*x2*d1*d2-x1*d1+(q2)
↳      [6] :
↳      x3
↳      [7] :
↳      x3
↳ [48] :
↳      [1] :
↳ 1/(q2)
↳      [2] :
↳      x3
↳      [3] :
↳      x2
↳      [4] :
↳      d2
↳      [5] :
↳      x3
↳      [6] :
↳      x1*x2*d1*d2-x1*d1+(q2)
↳      [7] :
↳      x3
↳ [49] :
↳      [1] :
↳ 1/(q2)
↳      [2] :
↳      x3
↳      [3] :
↳      x2

```

```

⇒ [4]:
⇒ d2
⇒ [5]:
⇒ x3
⇒ [6]:
⇒ x3
⇒ [7]:
⇒ x1*x2*d1*d2-x1*d1+(q2)
⇒ [50]:
⇒ [1]:
⇒ 1/(q2)
⇒ [2]:
⇒ x3
⇒ [3]:
⇒ x2
⇒ [4]:
⇒ x3
⇒ [5]:
⇒ d2
⇒ [6]:
⇒ x1*x2*d1*d2-x1*d1+(q2)
⇒ [7]:
⇒ x3
⇒ [51]:
⇒ [1]:
⇒ 1/(q2)
⇒ [2]:
⇒ x3
⇒ [3]:
⇒ x2
⇒ [4]:
⇒ x3
⇒ [5]:
⇒ d2
⇒ [6]:
⇒ x3
⇒ [7]:
⇒ x1*x2*d1*d2-x1*d1+(q2)
⇒ [52]:
⇒ [1]:
⇒ 1/(q2)
⇒ [2]:
⇒ x3
⇒ [3]:
⇒ x2
⇒ [4]:
⇒ x3
⇒ [5]:
⇒ x3
⇒ [6]:
⇒ d2
⇒ [7]:
⇒ x1*x2*d1*d2-x1*d1+(q2)

```



```

⇒ [53]:
⇒ [1]:
⇒ 1/(q2)
⇒ [2]:
⇒ x3
⇒ [3]:
⇒ x3
⇒ [4]:
⇒ x1*x2*d1*d2-x1*d1+(q2)
⇒ [5]:
⇒ x2
⇒ [6]:
⇒ d2
⇒ [7]:
⇒ x3
⇒ [54]:
⇒ [1]:
⇒ 1/(q2)
⇒ [2]:
⇒ x3
⇒ [3]:
⇒ x3
⇒ [4]:
⇒ x1*x2*d1*d2-x1*d1+(q2)
⇒ [5]:
⇒ x2
⇒ [6]:
⇒ x3
⇒ [7]:
⇒ d2
⇒ [55]:
⇒ [1]:
⇒ 1/(q2)
⇒ [2]:
⇒ x3
⇒ [3]:
⇒ x3
⇒ [4]:
⇒ x1*x2*d1*d2-x1*d1+(q2)
⇒ [5]:
⇒ x3
⇒ [6]:
⇒ x2
⇒ [7]:
⇒ d2
⇒ [56]:
⇒ [1]:
⇒ 1/(q2)
⇒ [2]:
⇒ x3
⇒ [3]:
⇒ x3
⇒ [4]:

```

```

↳      x2
↳      [5]:
↳      d2
↳      [6]:
↳      x1*x2*d1*d2-x1*d1+(q2)
↳      [7]:
↳      x3
↳ [57]:
↳      [1]:
↳ 1/(q2)
↳      [2]:
↳      x3
↳      [3]:
↳      x3
↳      [4]:
↳      x2
↳      [5]:
↳      d2
↳      [6]:
↳      x3
↳      [7]:
↳      x1*x2*d1*d2-x1*d1+(q2)
↳ [58]:
↳      [1]:
↳ 1/(q2)
↳      [2]:
↳      x3
↳      [3]:
↳      x3
↳      [4]:
↳      x2
↳      [5]:
↳      x3
↳      [6]:
↳      d2
↳      [7]:
↳      x1*x2*d1*d2-x1*d1+(q2)
↳ [59]:
↳      [1]:
↳ 1/(q2)
↳      [2]:
↳      x3
↳      [3]:
↳      x3
↳      [4]:
↳      x3
↳      [5]:
↳      x1*x2*d1*d2-x1*d1+(q2)
↳      [6]:
↳      x2
↳      [7]:
↳      d2
↳ [60]:

```

```

⇒ [1]:
⇒ 1/(q2)
⇒ [2]:
⇒ x3
⇒ [3]:
⇒ x3
⇒ [4]:
⇒ x3
⇒ [5]:
⇒ x2
⇒ [6]:
⇒ d2
⇒ [7]:
⇒ x1*x2*d1*d2-x1*d1+(q2)

```

See also: [Section 7.5.12.10 \[homogfacFirstQWeyl\]](#), page 498; [Section 7.5.12.12 \[homogfacFirstQWeyl\\_all\]](#), page 517; [Section 7.5.12.8 \[homogfacNthWeyl\]](#), page 497.

### 7.5.12.12 homogfacFirstQWeyl\_all

Procedure from library `ncfactor.lib` (see [Section 7.5.12 \[ncfactor.lib\]](#), page 487).

**Usage:** `homogfacFirstQWeyl_all(h)`;  $h$  is a homogeneous polynomial in the first  $q$ -Weyl algebra with respect to the weight vector  $[-1,1]$

**Return:** list

**Purpose:** Computes all factorizations of a homogeneous polynomial  $h$  with respect to the weight vector  $[-1,1]$  in the first  $q$ -Weyl algebra

**Theory:** This function is a wrapper for `homogFacNthQWeyl_all`. It exists to make this library downward-compatible with older versions.

**Example:**

```

LIB "ncfactor.lib";
ring R = (0,q),(x,d),dp;
def r = nc_algebra (q,1);
setring(r);
poly h = q^25*x^10*d^10+q^16*(q^4+q^3+q^2+q+1)^2*x^9*d^9+
q^9*(q^13+3*q^12+7*q^11+13*q^10+20*q^9+26*q^8+30*q^7+
31*q^6+26*q^5+20*q^4+13*q^3+7*q^2+3*q+1)*x^8*d^8+
q^4*(q^9+2*q^8+4*q^7+6*q^6+7*q^5+8*q^4+6*q^3+
4*q^2+2*q+1)*(q^4+q^3+q^2+q+1)*(q^2+q+1)*x^7*d^7+
q*(q^2+q+1)*(q^5+2*q^4+2*q^3+3*q^2+2*q+1)*(q^4+q^3+q^2+q+1)*(q^2+1)*(q+1)*x^6*d^6+
(q^10+5*q^9+12*q^8+21*q^7+29*q^6+33*q^5+31*q^4+24*q^3+15*q^2+7*q+12)*x^5*d^5+
6*x^3*d^3+24;
homogfacFirstQWeyl_all(h);
⇒ [1]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ x5d5+6
⇒ [3]:
⇒ x5d5+x3d3+4
⇒ [2]:
⇒ [1]:

```

```

⇒ 1
⇒ [2]:
⇒ x5d5+x3d3+4
⇒ [3]:
⇒ x5d5+6

```

See also: [Section 7.5.12.10 \[homogfacFirstQWeyl\]](#), page 498.

### 7.5.13 nchilbert.lib

**Library:** nchilbert.lib

**Purpose:** Hilbert series, polynomial and multiplicity for G-Algebras (Plural)

**Authors:** Andre Ranft, andre.ranft at rwth-aachen.de  
 Viktor Levandovskyy, levandov at rwth-aachen.de

**Overview:** The theory is found in the book by Bueso, Gomez-Torrecillas, and Verschoren Algorithmic Methods in Non-Commutative Algebra. Applications to Quantum Groups. and in the bachelor thesis by Andre Ranft, Hilbert polynomials of modules over noncommutative G-algebras, RWTH Aachen, 2014.

**Procedures:** See also: [Section 5.1.56 \[hilb\]](#), page 195; [Section D.15.12 \[multigrading.lib\]](#), page 2459; [Section D.7.3 \[rinvar.lib\]](#), page 1807.

#### 7.5.13.1 ncHilb

Procedure from library nchilbert.lib (see [Section 7.5.13 \[nchilbert.lib\]](#), page 518).

**Usage:** ncHilb(M,j); M a module, j an int

**Return:** intvec

**Assume:** - M is given via a Groebner basis; j is 1 or 2;  
 - the weights of all the ring variables are positive

**Note:** - computes the first (if j=1) or second (j=2) Hilbert series of I as intvec - the procedure works analogously to the commutative procedure hilb - If the returned vector has the form  $v=(v_0, v_1, \dots, v_d, 0)$ , then the Hilbert series is  $v_0 + v_1*t + \dots + v_d*t^d$

**Example:**

```

LIB "nchilbert.lib";
def A = makeUs12(); setring A;
ideal I = e,h-1; I = std(I);
ncHilb(I,1); // first Hilbert series of A/I
⇒ Warning: the input generators are not a Groebner basis
⇒ The result might have no meaning
⇒ 1,-2,1,0
ncHilb(I,2); // second Hilbert series of A/I
⇒ Warning: the input generators are not a Groebner basis
⇒ The result might have no meaning
⇒ 1,0
ideal J = I, f^2; J = std(J);
ncHilb(J,2);
⇒ Warning: the input generators are not a Groebner basis
⇒ The result might have no meaning

```

```

⇒ 1,1,0
// now with weights 1,2,3
ring r = 0,(e,f,h),wp(1,2,3);
matrix D[3][3]; D[1,2]=-h; D[1,3]=2*e;D[2,3]=-2*f;
def R = nc_algebra(1,D); setring R;
ideal I = imap(A,I); I = std(I);
ncHilb(I,1); // first weighted Hilbert series of R/I
⇒ Warning: the input generators are not a Groebner basis
⇒ The result might have no meaning
⇒ 1,-1,0,-1,1,0
ncHilb(I,2); // second weighted Hilbert series of R/I
⇒ Warning: the input generators are not a Groebner basis
⇒ The result might have no meaning
⇒ 1,1,1,0
matrix M[2][5] =
e,h-1,f^2, 0,0,
0,0,0, e,h+1;
module G = std(M);
print(G);
⇒ e,0,h-1,0, f2,
⇒ 0,e,0, h+1,0
ncHilb(G,1); // first weighted Hilbert series of R^2/G
⇒ 2,-2,0,-2,1,1,0,1,-1,0
ncHilb(G,2); // second weighted Hilbert series of R^2/G
⇒ 2,2,2,0,-1,-1,-1,0

```

See also: [Section 5.1.56 \[hilb\]](#), page 195.

### 7.5.13.2 ncHilbertSeries

Procedure from library `nchilbert.lib` (see [Section 7.5.13 \[nchilbert.lib\]](#), page 518).

**Usage:** `ncHilbertSeries(M,j)`;  $M$  is a module,  $j$  is an int

**Return:** ring

**Purpose:** computes the first (if  $j=1$ ) and second ( $j=2$ ) Hilbert Series of  $A^m/M$

**Assume:** -  $M$  is given via a Groebner basis;  $j$  is 1 or 2;  
- the weights of all the ring variables are positive

**Note:** - the procedure returns an univariate ring and a polynomial called `ncHS` in it.

**Example:**

```

LIB "nchilbert.lib";
def A = makeUs12(); setring A;
ideal I = e,h-1; I = std(I);
def r = ncHilbertSeries(I,1); setring r;
⇒ Warning: the input generators are not a Groebner basis
⇒ The result might have no meaning
ncHS;
⇒ t2-2t+1
setring A; kill r;
def s= ncHilbertSeries(I,2); setring s;
⇒ Warning: the input generators are not a Groebner basis
⇒ The result might have no meaning

```

```

ncHS;
↳ 1
// now consider admissible weights 1,2,3
ring r = 0,(e,f,h),wp(1,2,3);
matrix D[3][3]; D[1,2]=-h; D[1,3]=2*e;D[2,3]=-2*f;
def R = nc_algebra(1,D); setring R;
matrix M[2][5] =
e,h-1,f^2, 0,0,
0,0,0, e,h+1;
module G = std(M);
print(G);
↳ e,0,h-1,0, f2,
↳ 0,e,0, h+1,0
def r= ncHilbertSeries(G,1); setring r;
↳ // ** redefining r (def r= ncHilbertSeries(G,1); setring r;) ./examples/\
ncHilbertSeries.sing:18
ncHS; // first weighted Hilbert series of R^2/G
↳ -t8+t7+t5+t4-2t3-2t+2
setring R; kill r;
def s=ncHilbertSeries(G,2); setring s;
↳ // ** redefining s (def s=ncHilbertSeries(G,2); setring s;) ./examples/nc\
HilbertSeries.sing:21
ncHS;// second weighted Hilbert series of R^2/G
↳ -t6-t5-t4+2t2+2t+2

```

### 7.5.13.3 ncHilbertPolynomial

Procedure from library `nchilbert.lib` (see [Section 7.5.13 \[nchilbert.lib\]](#), page 518).

**Usage:** `ncHilbertPolynomial(M)`;  $M$  is a module

**Return:** ring

**Purpose:** computes the Hilbert polynomial of  $A^m/M$

**Assume:** -  $M$  is given via a Groebner basis  
- the weights of all the ring variables are positive

**Note:** - the procedure returns an univariate ring and a polynomial called `ncHP` in it.

**Example:**

```

LIB "nchilbert.lib";
def A = makeUs12(); setring A;
ideal I = h^4,e*f*h^3,e^2*f^2*h^2+2*e*f*h^2; I = std(I);
dim(I); // 2
↳ 2
def r = ncHilbertPolynomial(I); setring r;
ncHP; // 2t+7
↳ 2t+7
kill r;
// now consider admissible weights 1,2,3
ring r = 0,(e,f,h),wp(1,2,3);
matrix D[3][3]; D[1,2]=-h; D[1,3]=2*e;D[2,3]=-2*f;
def R = nc_algebra(1,D); setring R;
ideal I = imap(A,I);
I = std(I);

```

```

dim(I); // 2
↳ 2
def r = ncHilbertPolynomial(I); setring r;
↳ // ** redefining r (def r = ncHilbertPolynomial(I); setring r;) ./example\
  s/ncHilbertPolynomial.sing:15
ncHP; // 6t+18
↳ 6t+18

```

See also: [Section D.2.8.20 \[hilbPoly\]](#), page 898.

### 7.5.13.4 ncHilbertMultiplicity

Procedure from library `nchilbert.lib` (see [Section 7.5.13 \[nchilbert.lib\]](#), page 518).

**Usage:** `ncHilbertMultiplicity(M)`;  $M$  is a module

**Return:** `int`

**Purpose:** compute the (Hilbert) multiplicity of the module `coker(M)`

**Assume:** -  $M$  is given via a Groebner basis  
 - the weights of all the ring variables are positive

**Note:** the multiplicity depends on the selected weights of variables

**Example:**

```

LIB "nchilbert.lib";
def A = makeUs12(); setring A;
ideal I = e,h-1; I = std(I);
ncHilbertMultiplicity(I); // multiplicity of A/I
↳ Warning: the input generators are not a Groebner basis
↳ The result might have no meaning
↳ 1
ideal J = I, f^2; J = std(J);
ncHilbertMultiplicity(J);
↳ Warning: the input generators are not a Groebner basis
↳ The result might have no meaning
↳ 2
// now the same algebra with weights 1,2,3
ring r = 0,(e,f,h),wp(1,2,3);
matrix D[3][3]; D[1,2]=-h; D[1,3]=2*e;D[2,3]=-2*f;
def R = nc_algebra(1,D); setring R;
ideal I = imap(A,I); I = std(I);
ncHilbertMultiplicity(I);
↳ Warning: the input generators are not a Groebner basis
↳ The result might have no meaning
↳ 3
matrix M[2][5] =
e,h-1,f^2, 0,0,
0,0,0, e,h+1;
module G = std(M);
print(G);
↳ e,0,h-1,0, f2,
↳ 0,e,0, h+1,0
ncHilbertMultiplicity(G);
↳ 3

```

### 7.5.13.5 GKExp

Procedure from library `nchilbert.lib` (see [Section 7.5.13 \[nchilbert.lib\]](#), page 518).

**Usage:** GKExp(M); M a module

**Return:** int

**Purpose:** computes the Gelfand-Kirillov-Dimension of `coker(M)` via `Exp(M)`

**Assume:** basering is G-Algebra

**Note:** for zero module -1 is returned

**Example:**

```
LIB "nchilbert.lib";
def A = makeUs12(); setring A;
ideal I = e,h-1; I = std(I);
GKExp(I); // computes GKdim(A/I), should be 1
↳ Warning: the input generators are not a Groebner basis
↳ Proceed with Groebner basis computation
↳ 1
ideal J = I, f^2; J = std(J);
GKExp(J); // should be 0
↳ Warning: the input generators are not a Groebner basis
↳ Proceed with Groebner basis computation
↳ 0
matrix M[2][4] =
e,h-1,0,0,
0,0,e,h+1;
module G = std(M);
print(G);
↳ h-1,0, e,0,
↳ 0, h+1,0,e
GKExp(G);
↳ 1
```

See also: [Section 7.5.9.1 \[GKdim\]](#), page 465; [Section 7.3.3 \[dim \(plural\)\]](#), page 337.

### 7.5.13.6 mondim

Procedure from library `nchilbert.lib` (see [Section 7.5.13 \[nchilbert.lib\]](#), page 518).

**Usage:** mondim(B,i); B is list of elements of  $N_0^i$ ,

**Return:** int

**Purpose:** computes the dimension of the monoid ideal generated by B

**Example:**

```
LIB "nchilbert.lib";
ring A = 0,(x,y,z),dp;
setring A;
list I = [1,0,1],[0,1,1]; // belongs to the ideal <xz,yz>
mondim(I,3);
↳ 1
mondim(I,5); // treat generators of I as extended in  $N_0^5$ 
↳ 3
```



### 7.5.14 dmodloc.lib

Status: experimental

**Library:** dmodloc.lib

**Purpose:** Localization of algebraic D-modules and applications

**Author:** Daniel Andres, daniel.andres@math.rwth-aachen.de

Support: DFG Graduiertenkolleg 1632 ‘Experimentelle und konstruktive Algebra’

**Overview:** Let  $I$  be a left ideal in the  $n$ -th polynomial Weyl algebra  $D = K[x]\langle d \rangle$  and let  $f$  be a polynomial in  $K[x]$ .

If  $D/I$  is a holonomic module over  $D$ , it is known that the localization of  $D/I$  at  $f$  is also holonomic. The procedure `Dlocalization` computes an ideal  $J$  in  $D$  such that this localization is isomorphic to  $D/J$  as  $D$ -modules.

If one regards  $I$  as an ideal in the rational Weyl algebra as above,  $K(x)\langle d \rangle^* I$ , and intersects with  $K[x]\langle d \rangle$ , the result is called the Weyl closure of  $I$ . The procedures `WeylClosure` (if  $I$  has finite holonomic rank) and `WeylClosure1` (if  $I$  is in the first Weyl algebra) can be used for computations.

As an application of the Weyl closure, the procedure `annRatSyz` computes a holonomic part of the annihilator of a rational function by computing certain syzygies. The full annihilator can be obtained by taking the Weyl closure of the result.

If one regards the left ideal  $I$  as system of linear PDEs, one can find its polynomial solutions with `polSol` (if  $I$  is holonomic) or `polSolFiniteRank` (if  $I$  is of finite holonomic rank). Rational solutions can be obtained with `ratSol`.

The procedure `bfctBound` computes a possible multiple of the  $b$ -function for  $f^s \cdot u$  at a generic root of  $f$ . Here,  $u$  stands for  $[1]$  in  $D/I$ .

This library also offers the procedures `holonomicRank` and `DsingularLocus` to compute the holonomic rank and the singular locus of the  $D$ -module  $D/I$ .

**References:**

(OT) T. Oaku, N. Takayama: ‘Algorithms for D-modules’, Journal of Pure and Applied Algebra, 1998.

(OTT) T. Oaku, N. Takayama, H. Tsai: ‘Polynomial and rational solutions of holonomic systems’, Journal of Pure and Applied Algebra, 2001.

(OTW) T. Oaku, N. Takayama, U. Walther: ‘A Localization Algorithm for D-modules’, Journal of Symbolic Computation, 2000.

(Tsa) H. Tsai: ‘Algorithms for algebraic analysis’, PhD thesis, 2000.

**Procedures:** See also: [Section 7.5.2 \[bfun.lib\], page 376](#); [Section 7.5.4 \[dmod.lib\], page 401](#); [Section 7.5.5 \[dmodapp.lib\], page 421](#); [Section 7.5.7 \[dmodvar.lib\], page 454](#); [Section D.6.13 \[gmssing.lib\], page 1703](#).

#### 7.5.14.1 Dlocalization

Procedure from library `dmodloc.lib` (see [Section 7.5.14 \[dmodloc.lib\], page 523](#)).

**Usage:** `Dlocalization(I,f[,k,e]);`  $I$  ideal,  $f$  poly,  $k,e$  optional ints

**Assume:** The basering is the  $n$ -th Weyl algebra over a field of characteristic 0 and for all  $1 \leq i \leq n$  the identity

$\text{var}(i+n) \cdot \text{var}(i) = \text{var}(i) \cdot \text{var}(i+1) + 1$  holds, i.e. the sequence of variables is given by  $x(1), \dots, x(n), D(1), \dots, D(n)$ , where  $D(i)$  is the differential operator belonging to  $x(i)$ .

Further, assume that  $f$  does not contain any  $D(i)$  and that  $I$  is holonomic on  $K^n \setminus V(f)$ .

- Return:** ideal or list, computes an ideal  $J$  such that  $D/J$  is isomorphic to  $D/I$  localized at  $f$  as  $D$ -modules.  
If  $k > 0$ , a list consisting of  $J$  and an integer  $m$  is returned, such that  $f^m$  represents the natural map from  $D/I$  to  $D/J$ . Otherwise (and by default), only the ideal  $J$  is returned.
- Remarks:** It is known that a localization at  $f$  of a holonomic  $D$ -module is again a holonomic  $D$ -module.  
Reference: (OTW)
- Note:** If  $e > 0$ , `std` is used for Groebner basis computations, otherwise (and by default) `slimgb` is used.  
If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "dmodloc.lib";
// (OTW), Example 8
ring r = 0, (x,y,z,Dx,Dy,Dz), dp;
def W = Weyl();
setring W;
poly f = x^3-y^2*z^2;
ideal I = f^2*Dx+3*x^2, f^2*Dy-2*y*z^2, f^2*Dz-2*y^2*z;
// I annihilates exp(1/f)
ideal J = Dlocalization(I,f);
J;
⇒ J[1]=y*Dy-z*Dz
⇒ J[2]=2*y*z^2*Dx+3*x^2*Dy
⇒ J[3]=2*y^2*z*Dx+3*x^2*Dz
⇒ J[4]=2*z^3*Dx*Dz+3*x^2*Dy^2+2*z^2*Dx
⇒ J[5]=3*y^2*z^3*Dz-2*x^4*Dx-6*x^3*z*Dz+12*y^2*z^2-12*x^3-6
⇒ J[6]=4*x^4*Dx^2+12*x^3*z*Dx*Dz+9*x^2*z^2*Dz^2+40*x^3*Dx+63*x^2*z*Dz+72*x^2+12*Dx
⇒ J[7]=3*y*z^4*Dz^2-2*x^4*Dx*Dy-6*x^3*z*Dy*Dz+21*y*z^3*Dz-12*x^3*Dy+24*y*z^2-6*Dy
⇒ J[8]=3*z^5*Dz^3-2*x^4*Dx*Dy^2-6*x^3*z*Dy^2*Dz+30*z^4*Dz^2-12*x^3*Dy^2+66*z^3*Dz+24*z^2-6*Dy^2
Dlocalization(I,f,1); // The natural map D/I -> D/J is given by 1/f^2
⇒ [1]:
⇒ _[1]=y*Dy-z*Dz
⇒ _[2]=2*y*z^2*Dx+3*x^2*Dy
⇒ _[3]=2*y^2*z*Dx+3*x^2*Dz
⇒ _[4]=2*z^3*Dx*Dz+3*x^2*Dy^2+2*z^2*Dx
⇒ _[5]=3*y^2*z^3*Dz-2*x^4*Dx-6*x^3*z*Dz+12*y^2*z^2-12*x^3-6
⇒ _[6]=4*x^4*Dx^2+12*x^3*z*Dx*Dz+9*x^2*z^2*Dz^2+40*x^3*Dx+63*x^2*z*Dz+72*x^2+12*Dx
⇒ _[7]=3*y*z^4*Dz^2-2*x^4*Dx*Dy-6*x^3*z*Dy*Dz+21*y*z^3*Dz-12*x^3*Dy+24*y*z^2-6*Dy
⇒ _[8]=3*z^5*Dz^3-2*x^4*Dx*Dy^2-6*x^3*z*Dy^2*Dz+30*z^4*Dz^2-12*x^3*Dy^2+66*z^3*Dz+24*z^2-6*Dy^2
⇒ [2]:
⇒ 2
```

See also: [Section 7.5.5.3 \[DLoc\], page 424](#); [Section 7.5.5.5 \[DLoc0\], page 425](#); [Section 7.5.5.4 \[SDLoc\], page 425](#).

### 7.5.14.2 WeylClosure

Procedure from library `dmodloc.lib` (see [Section 7.5.14 \[dmodloc.lib\]](#), page 523).

**Usage:** `WeylClosure(I)`;  $I$  an ideal

**Assume:** The basering is the  $n$ -th Weyl algebra  $W$  over a field of characteristic 0 and for all  $1 \leq i \leq n$  the identity  $\text{var}(i+n) * \text{var}(i) = \text{var}(i) * \text{var}(i+1) + 1$  holds, i.e. the sequence of variables is given by  $x(1), \dots, x(n), D(1), \dots, D(n)$ , where  $D(i)$  is the differential operator belonging to  $x(i)$ . Moreover, assume that the holonomic rank of  $W/I$  is finite.

**Return:** ideal, the Weyl closure of  $I$

**Remarks:** The Weyl closure of a left ideal  $I$  in the Weyl algebra  $W$  is defined to be the intersection of  $I$  regarded as left ideal in the rational Weyl algebra  $K(x(1..n)) \langle D(1..n) \rangle$  with the polynomial Weyl algebra  $W$ .  
Reference: (Tsa), Algorithm 2.2.4

**Note:** If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "dmodloc.lib";
// (OTW), Example 8
ring r = 0, (x,y,z,Dx,Dy,Dz), dp;
def D3 = Weyl();
setring D3;
poly f = x^3-y^2*z^2;
ideal I = f^2*Dx + 3*x^2, f^2*Dy-2*y*z^2, f^2*Dz-2*y^2*z;
// I annihilates exp(1/f)
WeylClosure(I);
→ _[1]=y*Dy-z*Dz
→ _[2]=2*y*z^2*Dx+3*x^2*Dy
→ _[3]=2*y^2*z*Dx+3*x^2*Dz
→ _[4]=2*z^3*Dx*Dz+3*x^2*Dy^2+2*z^2*Dx
→ _[5]=4*x^4*Dx^2+12*x^3*z*Dx*Dz+9*x^2*z^2*Dz^2+16*x^3*Dx+27*x^2*z*Dz+12*Dx
→ _[6]=3*y*z^4*Dz^2-2*x^4*Dx*Dy-6*x^3*z*Dy*Dz+9*y*z^3*Dz-6*Dy
→ _[7]=3*y^2*z^3*Dz-2*x^4*Dx-6*x^3*z*Dz-6
→ _[8]=3*z^5*Dz^3-2*x^4*Dx*Dy^2-6*x^3*z*Dy^2*Dz+18*z^4*Dz^2+18*z^3*Dz-6*Dy^2
```

See also: [Section 7.5.14.3 \[WeylClosure1\]](#), page 525.

### 7.5.14.3 WeylClosure1

Procedure from library `dmodloc.lib` (see [Section 7.5.14 \[dmodloc.lib\]](#), page 523).

**Usage:** `WeylClosure1(L)`;  $L$  a poly

**Assume:** The basering is the first Weyl algebra  $D = K\langle x, d \mid dx = xd + 1 \rangle$  over a field  $K$  of characteristic 0.

**Return:** ideal, the Weyl closure of the principal left ideal generated by  $L$

**Remarks:** The Weyl closure of a left ideal  $I$  in the Weyl algebra  $D$  is defined to be the intersection of  $I$  regarded as left ideal in the rational Weyl algebra  $K(x) \langle d \rangle$  with the polynomial Weyl algebra  $D$ .  
Reference: (Tsa), Algorithm 1.2.4

**Note:** If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "dmodloc.lib";
ring r = 0,(x,Dx),dp;
def W = Weyl();
setring W;
poly L = (x^3+2)*Dx-3*x^2;
WeylClosure1(L);
⇨ _[1]=x^3*Dx-3*x^2+2*Dx
⇨ _[2]=x^2*Dx^2-2*x*Dx
⇨ _[3]=x^2*Dx+Dx^2-3*x
⇨ _[4]=x*Dx^2-2*Dx
L = (x^4-4*x^3+3*x^2)*Dx^2+(-6*x^3+20*x^2-12*x)*Dx+(12*x^2-32*x+12);
WeylClosure1(L);
⇨ _[1]=x^4*Dx^2-4*x^3*Dx^2-6*x^3*Dx+3*x^2*Dx^2+20*x^2*Dx+12*x^2-12*x*Dx-32*\
x+12
⇨ _[2]=x^2*Dx^3-21/10*x^2*Dx^2+3/10*x*Dx^3-6/5*x*Dx^2+63/5*x*Dx-3/5*Dx^2-12\
/5*Dx-126/5
⇨ _[3]=x^3*Dx^2-43/10*x^2*Dx^2+9/10*x*Dx^3-6*x^2*Dx+12/5*x*Dx^2+109/5*x*Dx-\
9/5*Dx^2+12*x-36/5*Dx-178/5
⇨ _[4]=x^3*Dx^3-48/5*x^2*Dx^2+9/5*x*Dx^3+24/5*x*Dx^2+198/5*x*Dx-18/5*Dx^2-7\
2/5*Dx-336/5
⇨ _[5]=x^3*Dx^4-4*x^2*Dx^4+2*x^2*Dx^3+3*x*Dx^4-69/10*x^2*Dx^2+67/10*x*Dx^3-\
24/5*x*Dx^2-3*Dx^3+207/5*x*Dx-27/5*Dx^2-18/5*Dx-414/5
⇨ _[6]=x^3*Dx^6+8/3*x^3*Dx^5-4*x^2*Dx^6-2/3*x^2*Dx^5+3*x*Dx^6+16*x^2*Dx^4-2\
0*x*Dx^5-92/3*x*Dx^4+12*Dx^5+126/5*x^2*Dx^2-258/5*x*Dx^3-168/5*x*Dx^2+92/\
3*Dx^3-756/5*x*Dx+356/5*Dx^2+504/5*Dx+1512/5
```

See also: [Section 7.5.14.2 \[WeylClosure\]](#), page 525.

#### 7.5.14.4 holonomicRank

Procedure from library `dmodloc.lib` (see [Section 7.5.14 \[dmodloc.lib\]](#), page 523).

**Usage:** `holonomicRank(I[,e]);` I ideal, e optional int

**Assume:** The basering is the n-th Weyl algebra over a field of characteristic 0 and for all  $1 \leq i \leq n$  the identity  $\text{var}(i+n) \cdot \text{var}(i) = \text{var}(i) \cdot \text{var}(i+1) + 1$  holds, i.e. the sequence of variables is given by  $x(1), \dots, x(n), D(1), \dots, D(n)$ , where  $D(i)$  is the differential operator belonging to  $x(i)$ .

**Return:** int, the holonomic rank of I

**Remarks:** The holonomic rank of I is defined to be the  $K(x(1..n))$ -dimension of the module  $W/WI$ , where W is the rational Weyl algebra  $K(x(1..n))\langle D(1..n) \rangle$ . If this dimension is infinite, -1 is returned.

**Note:** If  $e < 0$ , `std` is used for Groebner basis computations, otherwise (and by default) `slingb` is used.

If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```

LIB "dmodloc.lib";
// (OTW), Example 8
ring r3 = 0, (x,y,z,Dx,Dy,Dz), dp;
def D3 = Weyl();
setring D3;
poly f = x^3-y^2*z^2;
ideal I = f^2*Dx+3*x^2, f^2*Dy-2*y*z^2, f^2*Dz-2*y^2*z;
// I annihilates exp(1/f)
holonomicRank(I);
↦ 1

```

### 7.5.14.5 DsingularLocus

Procedure from library `dmodloc.lib` (see [Section 7.5.14 \[dmodloc.lib\]](#), page 523).

**Usage:** DsingularLocus(I); I ideal

**Assume:** The basering is the  $n$ -th Weyl algebra over a field of characteristic 0 and for all  $1 \leq i \leq n$  the identity  $\text{var}(i+n) \cdot \text{var}(i) = \text{var}(i) \cdot \text{var}(i+1) + 1$  holds, i.e. the sequence of variables is given by  $x(1), \dots, x(n), D(1), \dots, D(n)$ , where  $D(i)$  is the differential operator belonging to  $x(i)$ .

**Return:** ideal, describing the singular locus of the  $D$ -module  $D/I$

**Note:** If `printlevel`  $\geq 1$ , progress debug messages will be printed, if `printlevel`  $\geq 2$ , all the debug messages will be printed

**Example:**

```

LIB "dmodloc.lib";
// (OTW), Example 8
ring @D3 = 0, (x,y,z,Dx,Dy,Dz), dp;
def D3 = Weyl();
setring D3;
poly f = x^3-y^2*z^2;
ideal I = f^2*Dx + 3*x^2, f^2*Dy-2*y*z^2, f^2*Dz-2*y^2*z;
// I annihilates exp(1/f)
DsingularLocus(I);
↦ _[1]=y^2*z^2-x^3

```

### 7.5.14.6 polSol

Procedure from library `dmodloc.lib` (see [Section 7.5.14 \[dmodloc.lib\]](#), page 523).

**Usage:** polSol(I[,w,m]); I ideal, w optional intvec, m optional int

**Assume:** The basering is the  $n$ -th Weyl algebra  $W$  over a field of characteristic 0 and for all  $1 \leq i \leq n$  the identity  $\text{var}(i+n) \cdot \text{var}(i) = \text{var}(i) \cdot \text{var}(i+1) + 1$  holds, i.e. the sequence of variables is given by  $x(1), \dots, x(n), D(1), \dots, D(n)$ , where  $D(i)$  is the differential operator belonging to  $x(i)$ . Moreover, assume that  $I$  is holonomic.

**Return:** ideal, a basis of the polynomial solutions to the given system of linear PDEs with polynomial coefficients, encoded via  $I$

**Remarks:** If  $w$  is given,  $w$  should consist of  $n$  strictly negative entries. Otherwise and by default,  $w$  is set to  $-1:n$ . In this case,  $w$  is used as weight vector for the computation of a  $b$ -function.

If  $m$  is given,  $m$  is assumed to be the minimal integer root of the  $b$ -function of  $I$  w.r.t.  $w$ . Note that this assumption is not checked.

Reference: (OTT), Algorithm 2.4

**Note:** If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "dmodloc.lib";
ring r = 0,(x,y,Dx,Dy),dp;
def W = Weyl();
setring W;
poly tx,ty = x*Dx, y*Dy;
ideal I = // Appel F1 with parameters (2,-3,-2,5)
tx*(tx+ty+4)-x*(tx+ty+2)*(tx-3),
ty*(tx+ty+4)-y*(tx+ty+2)*(ty-2),
(x-y)*Dx*Dy+2*Dx-3*Dy;
intvec w = -1,-1;
polSol(I,w);
↦ _[1]=10*x^3*y^2-30*x^3*y-45*x^2*y^2+24*x^3+144*x^2*y+72*x*y^2-126*x^2-252\
*x*y-42*y^2+252*x+168*y-210
```

See also: [Section 7.5.14.7 \[polSolFiniteRank\]](#), page 528; [Section 7.5.14.8 \[ratSol\]](#), page 529.

### 7.5.14.7 polSolFiniteRank

Procedure from library `dmodloc.lib` (see [Section 7.5.14 \[dmodloc.lib\]](#), page 523).

**Usage:** `polSolFiniteRank(I,w)`;  $I$  ideal,  $w$  optional intvec

**Assume:** The basering is the  $n$ -th Weyl algebra  $W$  over a field of characteristic 0 and for all  $1 \leq i \leq n$  the identity  $\text{var}(i+n) \cdot \text{var}(i) = \text{var}(i) \cdot \text{var}(i+1) + 1$  holds, i.e. the sequence of variables is given by  $x(1), \dots, x(n), D(1), \dots, D(n)$ , where  $D(i)$  is the differential operator belonging to  $x(i)$ . Moreover, assume that  $I$  is of finite holonomic rank.

**Return:** ideal, a basis of the polynomial solutions to the given system of linear PDEs with polynomial coefficients, encoded via  $I$

**Remarks:** If  $w$  is given,  $w$  should consist of  $n$  strictly negative entries. Otherwise and by default,  $w$  is set to  $-1:n$ .

In this case,  $w$  is used as weight vector for the computation of a  $b$ -function.

Reference: (OTT), Algorithm 2.6

**Note:** If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "dmodloc.lib";
ring r = 0,(x,y,Dx,Dy),dp;
def W = Weyl();
setring W;
poly tx,ty = x*Dx, y*Dy;
ideal I = // Appel F1 with parameters (2,-3,-2,5)
tx*(tx+ty+4)-x*(tx+ty+2)*(tx-3),
ty*(tx+ty+4)-y*(tx+ty+2)*(ty-2),
(x-y)*Dx*Dy+2*Dx-3*Dy;
```

```

intvec w = -1,-1;
polSolFiniteRank(I,w);
↦ _[1]=10*x^3*y^2-30*x^3*y-45*x^2*y^2+24*x^3+144*x^2*y+72*x*y^2-126*x^2-252\
*x*y-42*y^2+252*x+168*y-210

```

See also: [Section 7.5.14.6 \[polSol\]](#), page 527; [Section 7.5.14.8 \[ratSol\]](#), page 529.

### 7.5.14.8 ratSol

Procedure from library `dmodloc.lib` (see [Section 7.5.14 \[dmodloc.lib\]](#), page 523).

**Usage:** `ratSol(I)`;  $I$  ideal

**Assume:** The basering is the  $n$ -th Weyl algebra  $W$  over a field of characteristic 0 and for all  $1 \leq i \leq n$  the identity  $\text{var}(i+n) \cdot \text{var}(i) = \text{var}(i) \cdot \text{var}(i+1) + 1$  holds, i.e. the sequence of variables is given by  $x(1), \dots, x(n), D(1), \dots, D(n)$ , where  $D(i)$  is the differential operator belonging to  $x(i)$ . Moreover, assume that  $I$  is holonomic.

**Return:** module, a basis of the rational solutions to the given system of linear PDEs with polynomial coefficients, encoded via  $I$ . Note that each entry has two components, the first one standing for the numerator, the second one for the denominator.

**Remarks:** Reference: (OTT), Algorithm 3.10

**Note:** If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```

LIB "dmodloc.lib";
ring r = 0, (x,y,Dx,Dy), dp;
def W = Weyl();
setring W;
poly tx,ty = x*Dx, y*Dy;
ideal I = // Appel F1 with parameters (3,-1,1,1) is a solution
tx*(tx+ty)-x*(tx+ty+3)*(tx-1),
ty*(tx+ty)-y*(tx+ty+3)*(ty+1);
module M = ratSol(I);
// We obtain a basis of the rational solutions to I represented by a
// module / matrix with two rows.
// Each column of the matrix represents a rational function, where
// the first row correspond to the numerator and the second row to
// the denominator.
print(M);
↦ x-y, x,
↦ y^4-3*y^3+3*y^2-y,y

```

See also: [Section 7.5.14.6 \[polSol\]](#), page 527; [Section 7.5.14.7 \[polSolFiniteRank\]](#), page 528.

### 7.5.14.9 bfctBound

Procedure from library `dmodloc.lib` (see [Section 7.5.14 \[dmodloc.lib\]](#), page 523).

**Usage:** `bfctBound (I,f[,primdec])`;  $I$  ideal,  $f$  poly, `primdec` optional string

**Assume:** The basering is the  $n$ -th Weyl algebra  $W$  over a field of characteristic 0 and for all  $1 \leq i \leq n$  the identity  $\text{var}(i+n) \cdot \text{var}(i) = \text{var}(i) \cdot \text{var}(i+1) + 1$  holds, i.e. the sequence of variables is given by

$x(1), \dots, x(n), D(1), \dots, D(n)$ , where  $D(i)$  is the differential operator belonging to  $x(i)$ . Moreover, assume that  $I$  is holonomic.

**Return:** list of roots (of type ideal) and multiplicities (of type intvec) of a multiple of the b-function for  $f^s \cdot u$  at a generic root of  $f$ . Here,  $u$  stands for  $[1]$  in  $D/I$ .

**Remarks:** Reference: (OTT), Algorithm 3.4

**Note:** This procedure requires to compute a primary decomposition in a commutative ring. The optional string `primdec` can be used to specify the algorithm to do so. It may either be ‘GTZ’ (Gianni, Trager, Zacharias) or ‘SY’ (Shimoyama, Yokoyama). By default, ‘GTZ’ is used.

If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "dmodloc.lib";
ring r = 0, (x,y,Dx,Dy), dp;
def W = Weyl();
setring W;
poly tx,ty = x*Dx, y*Dy;
ideal I = // Appel F1 with parameters (2,-3,-2,5)
tx*(tx+ty+4)-x*(tx+ty+2)*(tx-3),
ty*(tx+ty+4)-y*(tx+ty+2)*(ty-2),
(x-y)*Dx*Dy+2*Dx-3*Dy;
kill tx,ty;
poly f = x-1;
bfctBound(I,f);
⇒ [1]:
⇒ _[1]==-1
⇒ _[2]==-7
⇒ [2]:
⇒ 1,1
```

See also: [Section D.6.13.4 \[bernstein\]](#), page 1706; [Section 7.5.2.1 \[bfct\]](#), page 377; [Section 7.5.2.3 \[bfctAnn\]](#), page 379.

#### 7.5.14.10 annRatSyz

Procedure from library `dmodloc.lib` (see [Section 7.5.14 \[dmodloc.lib\]](#), page 523).

**Usage:** `annRatSyz(f,g[,db,eng]);`  $f, g$  polynomials,  $db, eng$  optional integers

**Assume:** The basering is commutative and over a field of characteristic 0.

**Return:** ring (a Weyl algebra) containing an ideal ‘LD’, which is (part of) the annihilator of the rational function  $g/f$  in the corresponding Weyl algebra

**Remarks:** This procedure uses the computation of certain syzygies. One can obtain the full annihilator by computing the Weyl closure of the ideal LD.

**Note:** Activate the output ring with the `setring` command. In the output ring, the ideal ‘LD’ (in Groebner basis) is (part of) the annihilator of  $g/f$ .

If  $db > 0$  is given, operators of order up to  $db$  are considered, otherwise, and by default, a minimal holonomic solution is computed.

If  $eng < 0$ , `std` is used for Groebner basis computations, otherwise, and by default, `slimgb` is used.



If `printlevel = 1`, progress debug messages will be printed, if `printlevel >= 2`, all the debug messages will be printed.

**Example:**

```
LIB "dmodloc.lib";
// printlevel = 3;
ring r = 0,(x,y),dp;
poly f = 2*x*y; poly g = x^2 - y^3;
def A = annRatSyz(f,g); // compute a holonomic solution
setring A; A;
⇒ // coefficients: QQ
⇒ // number of vars : 4
⇒ //          block 1 : ordering dp
⇒ //          : names  x y Dx Dy
⇒ //          block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      Dxx=x*Dx+1
⇒ //      Dyy=y*Dy+1
LD;
⇒ LD[1]=3*x*Dx+2*y*Dy+1
⇒ LD[2]=y^4*Dy-x^2*y*Dy+2*y^3+x^2
setring r;
def B = annRatSyz(f,g,5); // compute a solution up to degree 5
setring B;
LD; // this is the full annihilator as we will check below
⇒ LD[1]=3597*y^2*Dx^2*Dy-866*x*Dx*Dy^2-2176*y*Dy^3+10791*y*Dx^2-644*Dy^2
⇒ LD[2]=111*y^3*Dx^2+2760*x^2*Dx^2+1470*x*y*Dx*Dy-296*y^2*Dy^2+4050*x*Dx
⇒ LD[3]=3*x*Dx+2*y*Dy+1
⇒ LD[4]=y^3*Dy^2-x^2*Dy^2+6*y^2*Dy+6*y
⇒ LD[5]=y^4*Dy-x^2*y*Dy+2*y^3+x^2
setring r;
def C = annRat(f,g); setring C;
LD; // the full annihilator
⇒ LD[1]=3*y^2*Dx^2*Dy+2*x*Dx*Dy^2+9*y*Dx^2+4*Dy^2
⇒ LD[2]=3*y^3*Dx^2-10*x*y*Dx*Dy-8*y^2*Dy^2+10*x*Dx
⇒ LD[3]=y^3*Dy^2-x^2*Dy^2-6*x*y*Dx+2*y^2*Dy+4*y
⇒ LD[4]=3*x*Dx+2*y*Dy+1
⇒ LD[5]=y^4*Dy-x^2*y*Dy+2*y^3+x^2
ideal BLD = imap(B,LD);
NF(LD,std(BLD));
⇒ _[1]=0
⇒ _[2]=0
⇒ _[3]=0
⇒ _[4]=0
⇒ _[5]=0
```

See also: [Section 7.5.5.1 \[annPoly\]](#), page 422; [Section 7.5.5.2 \[annRat\]](#), page 423.

#### 7.5.14.11 dmodGeneralAssumptionCheck

Procedure from library `dmodloc.lib` (see [Section 7.5.14 \[dmodloc.lib\]](#), page 523).

**Usage:** `dmodGeneralAssumptionCheck();`

**Return:** nothing, but checks general assumptions on the basering

**Note:** This procedure checks the following conditions on the basering  $R$  and prints an error message if any of them is violated:

- $R$  is the  $n$ -th Weyl algebra over a field of characteristic 0,
- $R$  is not a qring,
- for all  $1 \leq i \leq n$  the identity  $\text{var}(i+n) * \text{var}(i) = \text{var}(i) * \text{var}(i+1) + 1$  holds, i.e. the sequence of variables is given by  $x(1), \dots, x(n), D(1), \dots, D(n)$ , where  $D(i)$  is the differential operator belonging to  $x(i)$ .

**Example:**

```
LIB "dmodloc.lib";
ring r = 0, (x, D), dp;
dmodGeneralAssumptionCheck(); // prints error message
⇒ ? Basing is not a Weyl algebra
⇒ ? leaving dmodloc.lib::dmodGeneralAssumptionCheck (0)
def W = Weyl();
setring W;
dmodGeneralAssumptionCheck(); // returns nothing
```

#### 7.5.14.12 extendWeyl

Procedure from library `dmodloc.lib` (see [Section 7.5.14 \[dmodloc.lib\]](#), page 523).

**Usage:** `extendWeyl(S)`;  $S$  string or list of strings

**Assume:** The basering is the  $n$ -th Weyl algebra over a field of characteristic 0 and for all  $1 \leq i \leq n$  the identity  $\text{var}(i+n) * \text{var}(i) = \text{var}(i) * \text{var}(i+1) + 1$  holds, i.e. the sequence of variables is given by  $x(1), \dots, x(n), D(1), \dots, D(n)$ , where  $D(i)$  is the differential operator belonging to  $x(i)$ .

**Return:** ring, Weyl algebra extended by vars given by  $S$

**Example:**

```
LIB "dmodloc.lib";
ring @D2 = 0, (x, y, Dx, Dy), dp;
def D2 = Weyl();
setring D2;
def D3 = extendWeyl("t");
setring D3; D3;
⇒ // coefficients: QQ
⇒ // number of vars : 6
⇒ //          block 1 : ordering dp
⇒ //          : names   t x y Dt Dx Dy
⇒ //          block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      Dtt=t*Dt+1
⇒ //      Dxx=x*Dx+1
⇒ //      Dyy=y*Dy+1
list L = "u", "v";
def D5 = extendWeyl(L);
setring D5;
D5;
⇒ // coefficients: QQ
⇒ // number of vars : 10
⇒ //          block 1 : ordering dp
⇒ //          : names   u v t x y Du Dv Dt Dx Dy
```

```

⇒ //      block  2 : ordering C
⇒ // noncommutative relations:
⇒ //      Duu=u*Du+1
⇒ //      Dvv=v*Dv+1
⇒ //      Dtt=t*Dt+1
⇒ //      Dxx=x*Dx+1
⇒ //      Dyy=y*Dy+1

```

#### 7.5.14.13 polyVars

Procedure from library `dmodloc.lib` (see [Section 7.5.14 \[dmodloc.lib\]](#), page 523).

**Usage:** `polyVars(f,v);`  $f$  poly,  $v$  intvec

**Return:** int, 1 if  $f$  contains only variables indexed by  $v$ , 0 otherwise

**Example:**

```

LIB "dmodloc.lib";
ring r = 0,(x,y,z),dp;
poly f = y^2+zy;
intvec v = 1,2;
polyVars(f,v); // does f depend only on x,y?
⇒ 0
v = 2,3;
polyVars(f,v); // does f depend only on y,z?
⇒ 1

```

#### 7.5.14.14 monomialInIdeal

Procedure from library `dmodloc.lib` (see [Section 7.5.14 \[dmodloc.lib\]](#), page 523).

**Usage:** `monomialInIdeal(I);`  $I$  ideal

**Return:** ideal consisting of all monomials appearing in generators of ideal

**Example:** `example monomialInIdeal;` shows examples

**Example:**

```

LIB "dmodloc.lib";
ring r = 0,(x,y),dp;
ideal I = x2+5x3y7, x-x2-6xy;
monomialInIdeal(I);
⇒ _[1]=x3y7
⇒ _[2]=x2
⇒ _[3]=xy
⇒ _[4]=x

```

#### 7.5.14.15 vars2pars

Procedure from library `dmodloc.lib` (see [Section 7.5.14 \[dmodloc.lib\]](#), page 523).

**Usage:** `vars2pars(v);`  $v$  intvec

**Assume:** The basering is commutative.

**Return:** ring with variables specified by  $v$  converted into parameters

**Example:**

```

LIB "dmodloc.lib";
ring r = 0,(x,y,z,a,b,c),dp;
intvec v = 4,5,6;
def R = vars2pars(v);
setring R;
R;
↪ // coefficients: QQ(a, b, c)
↪ // number of vars : 3
↪ //          block 1 : ordering dp
↪ //          : names  x y z
↪ //          block 2 : ordering C
v = 1,2;
def RR = vars2pars(v);
setring RR;
RR;
↪ // coefficients: QQ(a, b, c, x, y)
↪ // number of vars : 1
↪ //          block 1 : ordering dp
↪ //          : names  z
↪ //          block 2 : ordering C

```

#### 7.5.14.16 minIntRoot2

Procedure from library `dmodloc.lib` (see [Section 7.5.14 \[dmodloc.lib\]](#), page 523).

**Usage:**     `minIntRoot2(L)`; L list

**Assume:**   L is the output of `bFactor`.

**Return:**    int, the minimal integer root in a list of roots

**Example:**

```

LIB "dmodloc.lib";
ring r = 0,x,dp;
poly f = x*(x+1)*(x-2)*(x-5/2)*(x+5/2);
list L = bFactor(f);
minIntRoot2(L);
↪ -1

```

See also: [Section 7.5.5.24 \[bFactor\]](#), page 442; [Section 7.5.14.17 \[maxIntRoot\]](#), page 534; [Section 7.5.4.22 \[minIntRoot\]](#), page 420.

#### 7.5.14.17 maxIntRoot

Procedure from library `dmodloc.lib` (see [Section 7.5.14 \[dmodloc.lib\]](#), page 523).

**Usage:**     `maxIntRoot(L)`; L list

**Assume:**   L is the output of `bFactor`.

**Return:**    int, the maximal integer root in a list of roots

**Example:**

```

LIB "dmodloc.lib";
ring r = 0,x,dp;
poly f = x*(x+1)*(x-2)*(x-5/2)*(x+5/2);
list L = bFactor(f);

```

```
maxIntRoot(L);
↳ 2
```

See also: [Section 7.5.5.24 \[bFactor\]](#), page 442; [Section 7.5.14.16 \[minIntRoot2\]](#), page 534.

### 7.5.14.18 dmodAction

Procedure from library `dmodloc.lib` (see [Section 7.5.14 \[dmodloc.lib\]](#), page 523).

**Usage:** `dmodAction(id,f[,v]);` id ideal or poly, f poly, v optional intvec

**Assume:** If v is not given, the basering is the n-th Weyl algebra W over a field of characteristic 0 and for all  $1 \leq i \leq n$  the identity  $\text{var}(i+n) * \text{var}(i) = \text{var}(i) * \text{var}(i+1) + 1$  holds, i.e. the sequence of variables is given by  $x(1), \dots, x(n), D(1), \dots, D(n)$ , where  $D(i)$  is the differential operator belonging to  $x(i)$ .

Otherwise, v is assumed to specify positions of variables, which form a Weyl algebra as a subalgebra of the basering:

If  $\text{size}(v)$  equals  $2*n$ , then  $\text{bracket}(\text{var}(v[i]), \text{var}(v[j]))$  must equal 1 if and only if j equals  $i+n$ , and 0 otherwise, for all  $1 \leq i, j \leq n$ .

Further, assume that f does not contain any  $D(i)$ .

**Return:** same type as id, the result of the natural D-module action of id on f

**Note:** The assumptions made are not checked.

**Example:**

```
LIB "dmodloc.lib";
ring r = 0, (x,y,z), dp;
poly f = x^2*z - y^3;
def A = annPoly(f);
setring A;
poly f = imap(r,f);
dmodAction(LD,f);
↳ _[1]=0
↳ _[2]=0
↳ _[3]=0
↳ _[4]=0
↳ _[5]=0
↳ _[6]=0
↳ _[7]=0
↳ _[8]=0
↳ _[9]=0
↳ _[10]=0
↳ _[11]=0
↳ _[12]=0
↳ _[13]=0
poly P = y*Dy+3*z*Dz-3;
dmodAction(P,f);
↳ 0
dmodAction(P[1],f);
↳ -3*y^3
```

### 7.5.14.19 dmodActionRat

Procedure from library `dmodloc.lib` (see [Section 7.5.14 \[dmodloc.lib\]](#), page 523).

- Usage:** `dmodActionRat(id,w)`; `id` ideal or poly, `f` vector
- Assume:** The basering is the  $n$ -th Weyl algebra  $W$  over a field of characteristic 0 and for all  $1 \leq i \leq n$  the identity  $\text{var}(i+n) * \text{var}(i) = \text{var}(i) * \text{var}(i+1) + 1$  holds, i.e. the sequence of variables is given by  $x(1), \dots, x(n), D(1), \dots, D(n)$ , where  $D(i)$  is the differential operator belonging to  $x(i)$ . Further, assume that  $w$  has exactly two components, second one not 0, and that  $w$  does not contain any  $D(i)$ .
- Return:** same type as `id`, the result of the natural  $D$ -module action of `id` on the rational function  $w[1]/w[2]$

**Example:**

```
LIB "dmodloc.lib";
ring r = 0,(x,y),dp;
poly f = 2*x; poly g = y;
def A = annRat(f,g); setring A;
poly f = imap(r,f); poly g = imap(r,g);
vector v = [f,g]; // represents f/g
// x and y act by multiplication
dmodActionRat(x,v);
  => _[1]=2*x^2*gen(1)+y*gen(2)
dmodActionRat(y,v);
  => _[1]=2*x*gen(1)+gen(2)
// Dx and Dy act by partial derivation
dmodActionRat(Dx,v);
  => _[1]=y*gen(2)+2*gen(1)
dmodActionRat(Dy,v);
  => _[1]=y^2*gen(2)-2*x*gen(1)
dmodActionRat(x*Dx+y*Dy,v);
  => _[1]=gen(2)
setring r;
f = 2*x*y; g = x^2 - y^3;
def B = annRat(f,g); setring B;
poly f = imap(r,f); poly g = imap(r,g);
vector v = [f,g];
dmodActionRat(LD,v); // hence LD is indeed the annihilator of f/g
  => _[1]=gen(2)
  => _[2]=gen(2)
  => _[3]=gen(2)
  => _[4]=gen(2)
  => _[5]=gen(2)
```

**7.5.14.20 simplifyRat**

Procedure from library `dmodloc.lib` (see [Section 7.5.14 \[dmodloc.lib\]](#), page 523).

- Usage:** `simplifyRat(v)`; `v` vector
- Assume:** Assume that  $v$  has exactly two components, second one not 0.
- Return:** vector, representing simplified rational function  $v[1]/v[2]$
- Note:** Possibly present non-commutative relations of the basering are ignored.
- Example:**

```

LIB "dmodloc.lib";
ring r = 0,(x,y),dp;
vector v = [x2-1,x+1];
simplifyRat(v);
 $\mapsto x*\text{gen}(1)+\text{gen}(2)-\text{gen}(1)$ 
simplifyRat(v) - [x-1,1];
 $\mapsto 0$ 

```

### 7.5.14.21 addRat

Procedure from library `dmodloc.lib` (see [Section 7.5.14 \[dmodloc.lib\]](#), page 523).

**Usage:** `addRat(v,w)`;  $v,w$  vectors

**Assume:** Assume that  $v,w$  have exactly two components, second ones not 0.

**Return:** vector, representing rational function  $(v[1]/v[2])+(w[1]/w[2])$

**Note:** Possibly present non-commutative relations of the basering are ignored.

**Example:**

```

LIB "dmodloc.lib";
ring r = 0,(x,y),dp;
vector v = [x,y];
vector w = [y,x];
addRat(v,w);
 $\mapsto x2*\text{gen}(1)+xy*\text{gen}(2)+y2*\text{gen}(1)$ 
addRat(v,w) - [x2+y2,xy];
 $\mapsto 0$ 

```

### 7.5.14.22 multRat

Procedure from library `dmodloc.lib` (see [Section 7.5.14 \[dmodloc.lib\]](#), page 523).

**Usage:** `multRat(v,w)`;  $v,w$  vectors

**Assume:** Assume that  $v,w$  have exactly two components, second ones not 0.

**Return:** vector, representing rational function  $(v[1]/v[2])*(w[1]/w[2])$

**Note:** Possibly present non-commutative relations of the basering are ignored.

**Example:**

```

LIB "dmodloc.lib";
ring r = 0,(x,y),dp;
vector v = [x,y];
vector w = [y,x];
multRat(v,w);
 $\mapsto \text{gen}(2)+\text{gen}(1)$ 
multRat(v,w) - [1,1];
 $\mapsto 0$ 

```

### 7.5.14.23 diffRat

Procedure from library `dmodloc.lib` (see [Section 7.5.14 \[dmodloc.lib\]](#), page 523).

**Usage:** `diffRat(v,j)`;  $v$  vector,  $j$  int

**Assume:** Assume that  $v$  has exactly two components, second one not 0.

**Return:** vector, representing rational function derivative of rational function  $(v[1]/v[2])$  w.r.t.  $\text{var}(j)$

**Note:** Possibly present non-commutative relations of the basering are ignored.

**Example:**

```
LIB "dmodloc.lib";
ring r = 0,(x,y),dp;
vector v = [x,y];
diffRat(v,1);
  ↪ y*gen(2)+gen(1)
diffRat(v,1) - [1,y];
  ↪ 0
diffRat(v,2);
  ↪ y2*gen(2)-x*gen(1)
diffRat(v,2) - [-x,y2];
  ↪ 0
```

#### 7.5.14.24 commRing

Procedure from library `dmodloc.lib` (see [Section 7.5.14 \[dmodloc.lib\]](#), page 523).

**Usage:** `commRing();`

**Return:** ring, basering without non-commutative relations

**Example:**

```
LIB "dmodloc.lib";
def W = makeWeyl(3);
setring W; W;
  ↪ // coefficients: QQ
  ↪ // number of vars : 6
  ↪ //          block 1 : ordering dp
  ↪ //          : names   x(1) x(2) x(3) D(1) D(2) D(3)
  ↪ //          block 2 : ordering C
  ↪ // noncommutative relations:
  ↪ //    D(1)x(1)=x(1)*D(1)+1
  ↪ //    D(2)x(2)=x(2)*D(2)+1
  ↪ //    D(3)x(3)=x(3)*D(3)+1
def W2 = commRing();
setring W2; W2;
  ↪ // coefficients: QQ
  ↪ // number of vars : 6
  ↪ //          block 1 : ordering dp
  ↪ //          : names   x(1) x(2) x(3) D(1) D(2) D(3)
  ↪ //          block 2 : ordering C
ring r = 0,(x,y),dp;
def r2 = commRing(); // same as r
setring r2; r2;
  ↪ // coefficients: QQ
  ↪ // number of vars : 2
  ↪ //          block 1 : ordering dp
  ↪ //          : names   x y
  ↪ //          block 2 : ordering C
```



### 7.5.14.25 rightNFWeyl

Procedure from library `dmodloc.lib` (see [Section 7.5.14 \[dmodloc.lib\]](#), page 523).

**Usage:** `rightNFWeyl(id,k);` id ideal or poly, k int

**Assume:** The basering is the  $n$ -th Weyl algebra over a field of characteristic 0 and for all  $1 \leq i \leq n$  the identity  $\text{var}(i+n) \cdot \text{var}(i) = \text{var}(i) \cdot \text{var}(i+1) + 1$  holds, i.e. the sequence of variables is given by  $x(1), \dots, x(n), D(1), \dots, D(n)$ , where  $D(i)$  is the differential operator belonging to  $x(i)$ .

**Return:** same type as id, the right normal form of id with respect to the principal right ideal generated by the  $k$ -th variable

**Note:** No Groebner basis computation is used.

**Example:**

```
LIB "dmodloc.lib";
ring r = 0, (x,y,Dx,Dy), dp;
def W = Weyl();
setring W;
ideal I = x^3*Dx^3, y^2*Dy^2, x*Dy, y*Dx;
rightNFWeyl(I,1); // right NF wrt principal right ideal x*W
  => _[1]=0
  => _[2]=y^2*Dy^2
  => _[3]=0
  => _[4]=y*Dx
rightNFWeyl(I,3); // right NF wrt principal right ideal Dx*W
  => _[1]=-6
  => _[2]=y^2*Dy^2
  => _[3]=x*Dy
  => _[4]=0
rightNFWeyl(I,2); // right NF wrt principal right ideal y*W
  => _[1]=x^3*Dx^3
  => _[2]=0
  => _[3]=x*Dy
  => _[4]=0
rightNFWeyl(I,4); // right NF wrt principal right ideal Dy*W
  => _[1]=x^3*Dx^3
  => _[2]=2
  => _[3]=0
  => _[4]=y*Dx
poly p = x*Dx+1;
rightNFWeyl(p,1); // right NF wrt principal right ideal x*W
  => 1
```

### 7.5.15 ncfrac.lib

**Library:** `ncfrac.lib`

**Purpose:** object-oriented interface for `olga.lib`

**Author:** Johannes Hoffmann, email: johannes.hoffmann at math.rwth-aachen.de

**Overview:** This library introduces a new type: `ncfrac`.  
This type wraps the data defining a (non-commutative) fraction in an Ore localization

of a G-algebra as in `olga.lib`.

An element of type `ncfrac` has five members:

- `polys lnum, lden, rnum, rden`
- `ncloc loc`

**Operations:**

`string(ncfrac);`  
 give a string representation of the data describing the fraction  
`print(ncfrac);`  
 prints the string representation of the fraction  
`status(ncfrac);`  
 report on the status/validity of the fraction  
`test(ncfrac);`  
 check if the fraction is valid

**Infix operations:**

`ncfrac == ncfrac;`  
 compare two fractions  
`ncfrac != ncfrac;`  
 compare two fractions  
`ncfrac + ncfrac;`  
 add two fractions  
`ncfrac - ncfrac`  
 subtract two fractions  
`ncfrac * ncfrac`  
 multiply two fractions  
`ncfrac / ncfrac`  
 divide two fractions  
`ncfrac = int/number/poly`  
 create a fraction with:  
 - left and right denominator equal to 1  
 - left and right numerator determined by the input - localization data describing the trivial monoidal localization at 1  
`ncfrac = vector`  
 create a fraction from a vector `v` with unspecified localization such that  
`lden,lnum,rnum,rden = v[1],v[2],v[3],v[4]`  
 (note: without specifying a localization afterwards this results is an invalid fraction)  
`ncfrac = list`  
 create a fraction from a list `L` as follows:  
 - try to create a fraction from `L[1]` as above  
 - if `L[2]` is of type `ncloc` set the localization of the fraction to `L[2]`

**Procedures:**

### 7.5.15.1 hasLeftDenom

Procedure from library `ncfrac.lib` (see [Section 7.5.15 \[ncfrac.lib\]](#), page 539).

**Usage:** `hasLeftDenom(frac), ncfrac frac`

**Purpose:** checks if `frac` has a left representation

**Return:** `int`, 1 if `frac` has a left representation, 0 otherwise

**Example:**

```
LIB "ncfrac.lib";
↪ // ** redefining testNcfrac (LIB "ncfrac.lib"); ./examples/hasLeftDenom.s\
```

```

ing:1
⇒ // ** redefining testNcloc (    LIB "ncloc.lib");) ncfrac.lib::mod_init:11\
3
ring R = 0,(x,y,Dx,Dy),dp;
def S = Weyl();
setring S;
ncloc loc = ideal(x-3,y+7);
ncfrac noLeft = list([0,0,3*y*Dx,x+2], loc);
hasLeftDenom(noLeft);
⇒ 0
ncfrac left = list([1,Dx,Dx,1], loc);
hasLeftDenom(left);
⇒ 1

```

### 7.5.15.2 hasRightDenom

Procedure from library `ncfrac.lib` (see [Section 7.5.15 \[ncfrac.lib\]](#), page 539).

**Usage:**     `hasRightDenom(frac), ncfrac frac`

**Purpose:**    checks if `frac` has a right representation

**Return:**    `int`, 1 if `frac` has a right representation, 0 otherwise

**Example:**

```

LIB "ncfrac.lib";
⇒ // ** redefining testNcfrac (LIB "ncfrac.lib");) ./examples/hasRightDenom.\
sing:1
⇒ // ** redefining testNcloc (    LIB "ncloc.lib");) ncfrac.lib::mod_init:11\
3
ring R = 0,(x,y,Dx,Dy),dp;
def S = Weyl();
setring S;
ncloc loc = ideal(x-3,y+7);
ncfrac noRight = list([x+2,3*y*Dx,0,0], loc);
hasRightDenom(noRight);
⇒ 0
ncfrac right = list([1,Dx,Dx,1], loc);
hasRightDenom(right);
⇒ 1

```

### 7.5.15.3 isZeroNcfrac

Procedure from library `ncfrac.lib` (see [Section 7.5.15 \[ncfrac.lib\]](#), page 539).

**Usage:**     `isZeroNcfrac(frac), ncfrac frac`

**Purpose:**    checks if `frac` is zero

**Return:**    `int`, 1 if `frac` is zero, 0 otherwise

**Example:**

```

LIB "ncfrac.lib";
⇒ // ** redefining testNcfrac (LIB "ncfrac.lib");) ./examples/isZeroNcfrac.s\
ing:1
⇒ // ** redefining testNcloc (    LIB "ncloc.lib");) ncfrac.lib::mod_init:11\
3

```

```

ring Q = (0,q),(x,y,Qx,Qy),dp;
matrix C[4][4] = UpOneMatrix(4);
C[1,3] = q;
C[2,4] = q;
def ncQ = nc_algebra(C,0);
setring ncQ;
ncloc loc = intvec(2);
ncfrac frac = list([y^2+7*y+1,0,0,0], loc);
isZeroNcfrac(frac);
↪ 1
frac.lnum = 42*y*Qy+7*Qx+3*x+7;
isZeroNcfrac(frac);
↪ 0

```

#### 7.5.15.4 isOneNcfrac

Procedure from library `ncfrac.lib` (see [Section 7.5.15 \[ncfrac.lib\]](#), page 539).

**Usage:** `isOneNcfrac(frac), ncfrac frac`

**Purpose:** checks if `frac` is one

**Return:** int, 1 if `frac` is one, 0 otherwise

**Example:**

```

LIB "ncfrac.lib";
↪ // ** redefining testNcfrac (LIB "ncfrac.lib";) ./examples/isOneNcfrac.si\
ng:1
↪ // ** redefining testNcloc ( LIB "ncloc.lib";) ncfrac.lib::mod_init:11\
3
ring Q = (0,q),(x,y,Qx,Qy),dp;
matrix C[4][4] = UpOneMatrix(4);
C[1,3] = q;
C[2,4] = q;
def ncQ = nc_algebra(C,0);
setring ncQ;
ncloc loc = intvec(2);
ncfrac frac = list([y^2+7*y+1,y^2+7*y+1,0,0], loc);
isOneNcfrac(frac);
↪ 1
frac.lnum = 42*y*Qy+7*Qx+3*x+7;
isOneNcfrac(frac);
↪ 0

```

#### 7.5.15.5 zeroNcfrac

Procedure from library `ncfrac.lib` (see [Section 7.5.15 \[ncfrac.lib\]](#), page 539).

**Usage:** `zeroNcfrac(loc), ncloc loc`

**Purpose:** returns the zero fraction in the localization `loc`

**Return:** `ncfrac`

**Example:**

```

LIB "ncfrac.lib";
⇨ // ** redefining testNcfrac (LIB "ncfrac.lib");) ./examples/zeroNcfrac.sin\
  g:1
⇨ // ** redefining testNcloc (    LIB "ncloc.lib");) ncfrac.lib::mod_init:11\
  3
ring R = 0,(x,y,Dx,Dy),dp;
def S = Weyl();
setring S;
ncloc loc = ideal(x-53,y-7);
zeroNcfrac(loc);
⇨ left repr.: (1,0)
⇨ right repr.: (0,1)
⇨

```

### 7.5.15.6 oneNcfrac

Procedure from library `ncfrac.lib` (see [Section 7.5.15 \[ncfrac.lib\]](#), page 539).

**Usage:**      `oneNcfrac(loc), ncloc loc`

**Purpose:**     returns the one fraction in the localization `loc`

**Return:**    `ncfrac`

**Example:**

```

LIB "ncfrac.lib";
⇨ // ** redefining testNcfrac (LIB "ncfrac.lib");) ./examples/oneNcfrac.sing\
  :1
⇨ // ** redefining testNcloc (    LIB "ncloc.lib");) ncfrac.lib::mod_init:11\
  3
ring R = 0,(x,y,Dx,Dy),dp;
def S = Weyl();
setring S;
ncloc loc = ideal(x-42,y-17);
oneNcfrac(loc);
⇨ left repr.: (1,1)
⇨ right repr.: (1,1)
⇨

```

### 7.5.15.7 ensureLeftNcfrac

Procedure from library `ncfrac.lib` (see [Section 7.5.15 \[ncfrac.lib\]](#), page 539).

**Usage:**      `ensureLeftNcfrac(frac), ncfrac frac`

**Purpose:**     ensures that `frac` has a left representation (by computing it if not already known)

**Return:**    `ncfrac`, a representation of `frac` which has a left representation

**Example:**

```

LIB "ncfrac.lib";
⇨ // ** redefining testNcfrac (LIB "ncfrac.lib");) ./examples/ensureLeftNcfr\
  ac.sing:1
⇨ // ** redefining testNcloc (    LIB "ncloc.lib");) ncfrac.lib::mod_init:11\
  3
ring R = 0,(x,y,Dx,Dy),dp;
def S = Weyl();

```

```

setring S; S;
⇨ // coefficients: QQ
⇨ // number of vars : 4
⇨ //          block 1 : ordering dp
⇨ //          : names  x y Dx Dy
⇨ //          block 2 : ordering C
⇨ // noncommutative relations:
⇨ //      Dxx=x*Dx+1
⇨ //      Dyy=y*Dy+1
// monoidal localization
poly g1 = x+3;
poly g2 = x*y;
list L = g1,g2;
ncloc loc0 = L;
poly g = g1^2*g2;
poly f = Dx;
ncfrac frac0 = [0,0,f,g];
frac0.loc = loc0;
ncfrac rm = ensureLeftNcfrac(frac0);
print(rm);
⇨ left repr.: (x^8*y^4+12*x^7*y^4+54*x^6*y^4+108*x^5*y^4+81*x^4*y^4,x^5*y^3\
    *Dx+6*x^4*y^3*Dx-3*x^4*y^3+9*x^3*y^3*Dx-12*x^3*y^3-9*x^2*y^3)
⇨ right repr.: (Dx,x^3*y+6*x^2*y+9*x*y)
rm.lnum*g-rm.liden*f;
⇨ 0
// geometric localization
ncloc loc1 = ideal(x-1,y-3);
f = Dx;
g = x^2+y;
ncfrac frac1 = [0,0,f,g];
frac1.loc = loc1;
ncfrac rg = ensureLeftNcfrac(frac1);
print(rg);
⇨ left repr.: (x^4+2*x^2*y+y^2,x^2*Dx+y*Dx-2*x)
⇨ right repr.: (Dx,x^2+y)
rg.lnum*g-rg.liden*f;
⇨ 0
// rational localization
intvec rat = 1;
ncloc loc2 = rat;
f = Dx+Dy;
g = x;
ncfrac frac2 = [0,0,f,g];
frac2.loc = loc2;
ncfrac rr = ensureLeftNcfrac(frac2);
print(rr);
⇨ left repr.: (x^2,x*Dx+x*Dy-1)
⇨ right repr.: (Dx+Dy,x)
rr.lnum*g-rr.liden*f;
⇨ 0

```

### 7.5.15.8 ensureRightNcfrac

Procedure from library `ncfrac.lib` (see [Section 7.5.15 \[ncfrac.lib\]](#), page 539).

**Usage:**      `ensureLeftNcfrac(frac), ncfrac frac`

**Purpose:**      ensures that `frac` has a right representation (by computing it if not already known)

**Return:**      `ncfrac`, a representation of `frac` which has a right representation

**Example:**

```
LIB "ncfrac.lib";
⇨ // ** redefining testNcfrac (LIB "ncfrac.lib";) ./examples/ensureRightNcf\
  rac.sing:1
⇨ // ** redefining testNcloc (    LIB "ncloc.lib";) ncfrac.lib::mod_init:11\
  3
ring R = 0,(x,y,Dx,Dy),dp;
def S = Weyl();
setring S; S;
⇨ // coefficients: QQ
⇨ // number of vars : 4
⇨ //          block  1 : ordering dp
⇨ //                  : names    x y Dx Dy
⇨ //          block  2 : ordering C
⇨ // noncommutative relations:
⇨ //      Dxx=x*Dx+1
⇨ //      Dyy=y*Dy+1
// monoidal localization
poly g = x;
poly f = Dx;
ncloc loc0 = g;
ncfrac frac0 = [g,f,0,0];
frac0.loc = loc0;
ncfrac rm = ensureRightNcfrac(frac0);
print(rm);
⇨ left repr.: (x,Dx)
⇨ right repr.: (x*Dx+2,x^2)
f*rm.rden-g*rm.rnum;
⇨ 0
// geometric localization
g = x+y;
f = Dx+Dy;
ncloc loc1 = ideal(x-1,y-3);
ncfrac frac1 = [g,f,0,0];
frac1.loc = loc1;
ncfrac rg = ensureRightNcfrac(frac1);
print(rg);
⇨ left repr.: (x+y,Dx+Dy)
⇨ right repr.: (x*Dx+y*Dx+x*Dy+y*Dy+4,x^2+2*x*y+y^2)
f*rg.rden-g*rg.rnum;
⇨ 0
// rational localization
intvec rat = 1;
f = Dx+Dy;
g = x;
```

```

ncloc loc2 = rat;
ncfrac frac2 = [g,f,0,0];
frac2.loc = loc2;
ncfrac rr = ensureRightNcfrac(frac2);
print(rr);
 $\mapsto$  left repr.:  $(x, Dx+Dy)$ 
 $\mapsto$  right repr.:  $(x*Dx+x*Dy+2, x^2)$ 
f*rr.rden-g*rr.rnum;
 $\mapsto$  0

```

### 7.5.15.9 negateNcfrac

Procedure from library `ncfrac.lib` (see [Section 7.5.15 \[ncfrac.lib\]](#), page 539).

**Usage:** `negateNcfrac(frac), ncfrac frac`

**Purpose:** compute the negative (i.e. additive inverse) of `frac`

**Return:** `ncfrac`

**Note:** returns  $(-1)*\text{frac}$

**Example:**

```

LIB "ncfrac.lib";
 $\mapsto$  // ** redefining testNcfrac (LIB "ncfrac.lib"); ./examples/negateNcfrac.s\
    ing:1
 $\mapsto$  // ** redefining testNcloc (    LIB "ncloc.lib"); ncfrac.lib::mod_init:11\
    3
ring R = 0,(x,y,Dx,Dy),dp;
def S = Weyl();
setring S;
poly g = x*y^2+4*x+7*y-98;
ncloc loc = g;
ncfrac frac = list([g, 13*x^2], loc);
frac;
 $\mapsto$  left repr.:  $(x*y^2+4*x+7*y-98, 13*x^2)$ 
 $\mapsto$  right repr.:  $(0,0)$ 
 $\mapsto$ 
ncfrac negFrac = negateNcfrac(frac);
negFrac;
 $\mapsto$  left repr.:  $(x*y^2+4*x+7*y-98, -13*x^2)$ 
 $\mapsto$  right repr.:  $(0,0)$ 
 $\mapsto$ 
frac + negFrac;
 $\mapsto$  left repr.:  $(x*y^2+4*x+7*y-98, 0)$ 
 $\mapsto$  right repr.:  $(0,0)$ 
 $\mapsto$ 

```

### 7.5.15.10 isInvertibleNcfrac

Procedure from library `ncfrac.lib` (see [Section 7.5.15 \[ncfrac.lib\]](#), page 539).

**Usage:** `isInvertibleNcfrac(frac), ncfrac frac`

**Purpose:** checks if `frac` is invertible

**Return:** `int`, 1 if `frac` is invertible, 0 otherwise



**Example:**

```

LIB "ncfrac.lib";
⇒ // ** redefining testNcfrac (LIB "ncfrac.lib";) ./examples/isInvertibleNc\
   frac.sing:1
⇒ // ** redefining testNcloc (    LIB "ncloc.lib";) ncfrac.lib::mod_init:11\
   3
ring R = 0,(x,y,Dx,Dy),dp;
def S = Weyl();
setring S;
ncloc loc = intvec(2);
ncfrac frac = list([y,y+1,0,0], loc);
isInvertibleNcfrac(frac);
⇒ 1
frac = list([y,x+1,0,0], loc);
isInvertibleNcfrac(frac);
⇒ 0

```

**7.5.15.11 invertNcfrac**

Procedure from library `ncfrac.lib` (see [Section 7.5.15 \[ncfrac.lib\]](#), page 539).

**Usage:**      `invertNcfrac(frac), ncfrac frac`

**Purpose:**     compute the inverse of `frac`

**Return:**    `ncfrac`

**Note:**       returns the zero fraction if `frac` is not invertible

**Example:**

```

LIB "ncfrac.lib";
⇒ // ** redefining testNcfrac (LIB "ncfrac.lib";) ./examples/invertNcfrac.s\
   ing:1
⇒ // ** redefining testNcloc (    LIB "ncloc.lib";) ncfrac.lib::mod_init:11\
   3
ring R = 0,(x,y,Dx,Dy),dp;
def S = Weyl();
setring S;
ncloc loc = intvec(2);
ncfrac frac1 = list([y,y+1,0,0], loc);
// frac1 is invertible
ncfrac inv = invertNcfrac(frac1);
inv;
⇒ left repr.: (y+1,y)
⇒ right repr.: (0,0)
⇒
ncfrac frac2 = list([y,x+1,0,0], loc);
// frac2 is not invertible
inv = invertNcfrac(frac2);
inv;
⇒ left repr.: (1,0)
⇒ right repr.: (0,1)
⇒

```

### 7.5.15.12 testNcfrac

Procedure from library `ncfrac.lib` (see [Section 7.5.15 \[ncfrac.lib\]](#), page 539).

**Usage:** `testNcfrac()`

**Purpose:** execute a series of internal testing procedures

**Return:** nothing

**Note:**

### 7.5.15.13 testNcfracExamples

Procedure from library `ncfrac.lib` (see [Section 7.5.15 \[ncfrac.lib\]](#), page 539).

**Usage:** `testNcfracExamples()`

**Purpose:** execute the examples of all procedures in this library

**Return:** nothing

**Note:**

## 7.5.16 nchomolog.lib

Status: experimental

**Library:** `nchomolog.lib`

**Purpose:** Procedures for Noncommutative Homological Algebra

**Authors:** Viktor Levandovskyy [levandov@math.rwth-aachen.de](mailto:levandov@math.rwth-aachen.de),  
Christian Schilli, [christian.schilli@rwth-aachen.de](mailto:christian.schilli@rwth-aachen.de),  
Gerhard Pfister, [pfister@mathematik.uni-kl.de](mailto:pfister@mathematik.uni-kl.de)

**Overview:** In this library we present tools of homological algebra for finitely presented modules over GR-algebras.

**Procedures:**

### 7.5.16.1 ncExt\_R

Procedure from library `nchomolog.lib` (see [Section 7.5.16 \[nchomolog.lib\]](#), page 548).

**Usage:** `ncExt_R(i, M); i int, M module`

**Compute:** a presentation of  $\text{Ext}^i(M', R)$ ; for  $M' = \text{coker}(M)$ .

**Return:** right module `Ext`, a presentation of  $\text{Ext}^i(M', R)$

**Example:**

```
LIB "nchomolog.lib";
ring R      = 0, (x, y), dp;
poly F      = x^2 - y^2;
def A = annfs(F); setring A; // A is the 2nd Weyl algebra
matrix M[1][size(LD)] = LD; // ideal
print(M);
⇨ y*Dx + x*Dy, x*Dx + y*Dy + 2, x^2*Dy - y^2*Dy - 2*y
print(ncExt_R(1, M)); // hence the Ext^1 is zero
⇨ 1, 0,
```

```

↳ 0,1
module E = ncExt_R(2,M); // define the right module E
print(E); // E is in the opposite algebra
↳ 1, -x,      -y,
↳ Dx,y*Dy+1,x*Dy
def Aop = opposite(A); setring Aop;
module Eop = oppose(A,E);
module T1 = ncExt_R(2,Eop);
setring A;
module T1 = oppose(Aop,T1);
print(T1); // this is a left module Ext^2(Ext^2(M,A),A)
↳ y*Dx+x*Dy,x*Dx+y*Dy+2,x^2*Dy-y^2*Dy-2*y
print(M); // it is known that M holonomic implies Ext^2(Ext^2(M,A),A) iso to M
↳ y*Dx+x*Dy,x*Dx+y*Dy+2,x^2*Dy-y^2*Dy-2*y

```

### 7.5.16.2 ncHom

Procedure from library `nchomolog.lib` (see [Section 7.5.16 \[nchomolog.lib\]](#), page 548).

**Usage:** `ncHom(M,N)`;  $M, N$  modules

**Compute:** A presentation of  $\text{Hom}(M', N')$ ,  $M' = \text{coker}(M)$ ,  $N' = \text{coker}(N)$

**Assume:**  $M'$  is a left module,  $N'$  is a centralizing bimodule

**Note:** `ncHom(M,N)` is a right module, hence a right presentation matrix is returned

**Example:**

```

LIB "nchomolog.lib";
ring A=0,(x,y,z),dp;
matrix M[3][3]=1,2,3,
4,5,6,
7,8,9;
matrix N[2][2]=x,y,
z,0;
module H = ncHom(M,N);
print(H);
↳ 0,0,0,0,y,x,
↳ 0,0,0,0,0,z,
↳ 1,0,0,0,0,0,
↳ 0,1,0,0,0,0,
↳ 0,0,1,0,0,0,
↳ 0,0,0,1,0,0

```

### 7.5.16.3 coHom

Procedure from library `nchomolog.lib` (see [Section 7.5.16 \[nchomolog.lib\]](#), page 548).

**Usage:** `coHom(A,k)`;  $A$  matrix,  $k$  int

**Purpose:** compute the matrix of a homomorphism  $\text{Hom}(R^k, A)$ , where  $R$  is the basering. Let  $A$  be a matrix defining a map  $F_1 \rightarrow F_2$  of free  $R$ -modules, then the matrix of  $\text{Hom}(R^k, F_1) \rightarrow \text{Hom}(R^k, F_2)$  is computed.

**Note:** Both  $A$  and  $\text{Hom}(A, R^k)$  are matrices for either left or right  $R$ -module homomorphisms

**Example:**

```

LIB "nchomolog.lib";
ring A=0,(x,y,z),dp;
matrix M[3][3]=1,2,3,
4,5,6,
7,8,9;
module cM = coHom(M,2);
print(cM);
↪ 1,0,2,0,3,0,
↪ 0,1,0,2,0,3,
↪ 4,0,5,0,6,0,
↪ 0,4,0,5,0,6,
↪ 7,0,8,0,9,0,
↪ 0,7,0,8,0,9

```

#### 7.5.16.4 contraHom

Procedure from library `nchomolog.lib` (see [Section 7.5.16 \[nchomolog.lib\]](#), page 548).

**Usage:** `contraHom(A,k)`; A matrix, k int

**Return:** matrix

**Purpose:** compute the matrix of a homomorphism  $\text{Hom}(A, R^k)$ , where  $R$  is the basering. Let  $A$  be a matrix defining a map  $F1 \rightarrow F2$  of free  $R$ -modules, then the matrix of  $\text{Hom}(F2, R^k) \rightarrow \text{Hom}(F1, R^k)$  is computed.

**Note:** if  $A$  is matrix of a left (resp. right)  $R$ -module homomorphism, then  $\text{Hom}(A, R^k)$  is a right (resp. left)  $R$ -module homomorphism

**Example:**

```

LIB "nchomolog.lib";
ring A=0,(x,y,z),dp;
matrix M[3][3]=1,2,3,
4,5,6,
7,8,9;
module cM = contraHom(M,2);
print(cM);
↪ 1,4,7,0,0,0,
↪ 2,5,8,0,0,0,
↪ 3,6,9,0,0,0,
↪ 0,0,0,1,4,7,
↪ 0,0,0,2,5,8,
↪ 0,0,0,3,6,9

```

#### 7.5.16.5 dmodoublext

Procedure from library `nchomolog.lib` (see [Section 7.5.16 \[nchomolog.lib\]](#), page 548).

**Usage:** `dmodoublext(M [,i])`; M module, i optional int

**Compute:** a presentation of  $\text{Ext}^i(\text{Ext}^i(M, D), D)$  for basering  $D$

**Return:** left module

**Note:** by default,  $i$  is set to the integer part of the half of number of variables of  $D$   
for holonomic modules over Weyl algebra, the double ext is known to be holonomic left module

**Example:**

```

LIB "nchomolog.lib";
ring R = 0, (x,y), dp;
poly F = x^3-y^2;
def A = annfs(F);
setring A;
dmodoublext(LD);
↳ _[1]=2*x*Dx+3*y*Dy+6
↳ _[2]=3*x^2*Dy+2*y*Dx
↳ _[3]=9*x*y*Dy^2-4*y*Dx^2+15*x*Dy
↳ _[4]=27*y^2*Dy^3+8*y*Dx^3+135*y*Dy^2+105*Dy
LD;
↳ LD[1]=2*x*Dx+3*y*Dy+6
↳ LD[2]=3*x^2*Dy+2*y*Dx
↳ LD[3]=9*x*y*Dy^2-4*y*Dx^2+15*x*Dy
↳ LD[4]=27*y^2*Dy^3+8*y*Dx^3+135*y*Dy^2+105*Dy
// fancier example:
setring A;
ideal I = Dx*(x^2-y^3), Dy*(x^2-y^3);
I = groebner(I);
print(dmodoublext(I,1));
↳ y^3-x^2
print(dmodoublext(I,2));
↳ Dy,
↳ Dx

```

**7.5.16.6 is\_cenBimodule**

Procedure from library `nchomolog.lib` (see [Section 7.5.16 \[nchomolog.lib\]](#), page 548).

**Usage:** `is_cenBimodule(M)`; M module

**Compute:** 1, if a module, presented by M can be centralizing in the sense of Artin and 0 otherwise

**Note:** only one condition for centralizing factor module can be checked algorithmically

**Example:**

```

LIB "nchomolog.lib";
def A = makeUs12(); setring A;
poly p = 4*e*f + h^2-2*h; // generator of the center
matrix M[2][2] = p, p^2-7, 0, p*(p+1);
is_cenBimodule(M); // M is centralizing
↳ 1
matrix N[2][2] = p, e*f, h, p*(p+1);
is_cenBimodule(N); // N is not centralizing
↳ 0

```

**7.5.16.7 is\_cenSubbimodule**

Procedure from library `nchomolog.lib` (see [Section 7.5.16 \[nchomolog.lib\]](#), page 548).

**Usage:** `is_cenSubbimodule(M)`; M module

**Compute:** 1, if a subbimodule, generated by the columns of M is centralizing in the sense of Artin and 0 otherwise

**Example:**

```

LIB "nchomolog.lib";
def A = makeUs12(); setring A;
poly p = 4*e*f + h^2-2*h; // generator of the center
matrix M[2][2] = p, p^2-7,0,p*(p+1);
is_cenSubbimodule(M); // M is centralizing subbimodule
  ↦ 1
matrix N[2][2] = p, e*f,h,p*(p+1);
is_cenSubbimodule(N); // N is not centralizing subbimodule
  ↦ 0

```

**7.5.17 ncloc\_lib****Library:** ncloc.lib**Purpose:** Ore-localization in G-Algebras**Author:** Johannes Hoffmann, email: johannes.hoffmann at math.rwth-aachen.de

**Overview:** This library introduces a new type: ncloc.  
 This type wraps the localization data defined as in olga.lib. An element of type ncloc has two members:

- int locType
- def locData

**Operations:**

```

string(ncloc);
  give a string representation of the data describing the localization
print(ncloc);
  prints the string representation of the localization status
test(ncloc);
  report on the status/validity of the localization
valid(ncloc);
  check if the localization is valid

```

**Infix operations:**

```

ncloc == ncloc;
  compare two localizations
ncloc != ncloc;
  compare two localizations
ncloc = list/poly
  create a monoidal localization from the given data
ncloc = ideal
  create a geometric localization from the given data
ncloc = intvec
  create a rational localization from the given data

```

**Procedures:****7.5.17.1 isDenom**

Procedure from library `ncloc.lib` (see [Section 7.5.17 \[ncloc\\_lib\]](#), page 552).

**Usage:** isDenom(p, loc), poly a, ncloc loc**Purpose:** check if p is a valid denominator in the localization loc**Return:** int**Note:** returns 1 or 0, depending whether p is a valid denominator**Example:**

```

LIB "ncloc.lib";
⇨ // ** redefining testNcloc (LIB "ncloc.lib";) ./examples/isDenom.sing:1
ring R = 0,(x,y,Dx,Dy),dp;
def S = Weyl();
setring S; S;
⇨ // coefficients: QQ
⇨ // number of vars : 4
⇨ //          block 1 : ordering dp
⇨ //          : names  x y Dx Dy
⇨ //          block 2 : ordering C
⇨ // noncommutative relations:
⇨ //      Dxx=x*Dx+1
⇨ //      Dyy=y*Dy+1
// monoidal localization
ncloc loc;
poly g1 = x^2*y+x+2;
poly g2 = y^3+x*y;
list L = g1,g2;
loc = L;
poly g = g1^2*g2;
poly f = g - 1;
isDenom(g, loc);
⇨ 1
isDenom(f, loc);
⇨ 0
// geometrical localization
loc = ideal(x-1,y-3);
g = x^2+y-3;
f = (x-1)*g;
isDenom(g, loc);
⇨ 1
isDenom(f, loc);
⇨ 0
// rational localization
intvec v = 2;
loc = v;
g = y^5+17*y^2-4;
f = x*y;
isDenom(g, loc);
⇨ 1
isDenom(f, loc);
⇨ 0

```

### 7.5.17.2 testNcloc

Procedure from library `ncloc.lib` (see [Section 7.5.17 \[ncloc.lib\]](#), page 552).

**Usage:**      `testNcloc()`

**Purpose:**    execute a series of internal testing procedures

**Return:**    nothing

**Note:**

### 7.5.17.3 testNclocExamples

Procedure from library `ncloc.lib` (see [Section 7.5.17 \[ncloc.lib\]](#), page 552).

**Usage:** `testNclocExamples()`

**Purpose:** execute the examples of all procedures in this library

**Return:** nothing

**Note:**

### 7.5.18 ncModslimgb\_lib

**Library:** `ncModslimgb.lib`

**Purpose:** A library for computing Groebner bases over G-algebras defined over the rationals using modular techniques.

**Authors:** Wolfram Decker, Christian Eder, Viktor Levandovskyy, and Sharwan K. Tiwari  
shrawant@gmail.com

**References:**

Wolfram Decker, Christian Eder, Viktor Levandovskyy, and Sharwan K. Tiwari, Modular Techniques For Noncommutative Groebner Bases, <https://link.springer.com/article/10.1007/s11786-019-00412-9> and <https://arxiv.org/abs/1704.02852>.

E. A. Arnold, Modular algorithms for computing Groebner bases. *Journal of Symbolic Computation* 35, 403-419 (2003).

N. Idrees, G. Pfister, S. Steidel, Parallelization of Modular Algorithms, *Journal of Symbolic Computation* 46, 672-684 (2011).

**Procedures:**

#### 7.5.18.1 ncmodslimgb

Procedure from library `ncModslimgb.lib` (see [Section 7.5.18 \[ncModslimgb.lib\]](#), page 554).

**Usage:** `ncmodslimgb(I[, exactness, ncores]);` I ideal, optional integers exactness and n(umber of )cores

**Return:** ideal

**Purpose:** compute a left Groebner basis of I by modular approach

**Assume:** basering is a G-algebra; base field is prime field Q of rationals.

**Note:**

- If the given algebra and ideal are graded (it is not checked by this command), then the computed Groebner basis will be exact. Otherwise, the result will be correct with a very high probability.
- The optional parameter 'exactness' justifies, whether the final (expensive) verification step will be performed or not (exactness=0, default value is 1).
- The optional parameter 'ncores' (default value is 1) provides an integer to use the number of cores (this must not exceed the number of available cores in the computing machine).

**Example:**



### 7.5.19 ncpreim\_lib

**Overview:** In G-algebras, elimination of variables is more involved than in the commutative case. One, not every subset of variables generates an algebra, which is again a G-algebra.

Two, even if the subset of variables in question generates an admissible subalgebra, there might be no admissible elimination ordering, i.e. an elimination ordering which also satisfies the ordering condition for G-algebras.

The difference between the procedure `eliminateNC` provided in this library and the procedure `eliminate (plural)` from the kernel is that `eliminateNC` will always find an admissible elimination if such one exists. Moreover, the use of `slingb` for performing Groebner basis computations is possible.

As an application of the theory of elimination, the procedure `preimageNC` is provided, which computes the preimage of an ideal under a homomorphism  $f: A \rightarrow B$  between G-algebras A and B. In contrast to the kernel procedure `preimage (plural)`, the assumption that A is commutative is not required.

#### References:

- (BGL) J.L. Bueso, J. Gomez-Torrecillas, F.J. Lobillo: ‘Re-filtering and exactness of the Gelfand-Kirillov dimension’, Bull. Sci. math. 125, 8, 689-715, 2001.  
 (GML) J.I. Garcia Garcia, J. Garcia Miranda, F.J. Lobillo: ‘Elimination orderings and localization in PBW algebras’, Linear Algebra and its Applications 430(8-9), 2133-2148, 2009.  
 (Lev) V. Levandovskyy: ‘Intersection of ideals with non-commutative subalgebras’, ISSAC’06, 212-219, ACM, 2006.

**Procedures:** See also: [Section D.4.7 \[elim.lib\]](#), page 1059; [Section 7.3.21 \[preimage \(plural\)\]](#), page 355.

### 7.5.19.1 eliminateNC

Procedure from library `ncpreim.lib` (see [Section 7.5.19 \[ncpreim.lib\]](#), page 555).

- Usage:** `eliminateNC(I,v,eng)`; I ideal, v intvec, eng optional int
- Return:** ideal, I intersected with the subring defined by the variables not index by the entries of v
- Assume:** The entries of v are in the range `1..nvars(basering)` and the corresponding variables generate an admissible subalgebra.
- Remarks:** In order to determine the required elimination ordering, a linear programming problem is solved with the simplex algorithm.  
 Reference: (GML)  
 Unlike `eliminate`, this procedure will always find an elimination ordering, if such exists.
- Note:** If `eng<>0`, `std` is used for Groebner basis computations, otherwise (and by default) `slingb` is used.  
 If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

#### Example:

```
LIB "ncpreim.lib";
// (Lev): Example 2
ring r = 0,(a,b,x,d),Dp;
matrix D[4][4];
D[1,2] = 3*a; D[1,4] = 3*x^2;
D[2,3] = -x; D[2,4] = d; D[3,4] = 1;
def A = nc_algebra(1,D);
setring A; A;
```

```

⇒ // coefficients: QQ
⇒ // number of vars : 4
⇒ //      block 1 : ordering Dp
⇒ //      : names  a b x d
⇒ //      block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      ba=ab+3a
⇒ //      da=ad+3x2
⇒ //      xb=bx-x
⇒ //      db=bd+d
⇒ //      dx=xd+1
ideal I = a,x;
// Since d*a-a*d = 3*x^2, any admissible ordering has to satisfy
// x^2 < a*d, while any elimination ordering for {x,d} additionally
// has to fulfil a << x and a << d.
// Hence, the weight (0,0,1,1) is not an elimination weight for
// (x,d) and the call eliminate(I,x*d); will produce an error.
eliminateNC(I,3..4);
⇒ _[1]=a
// This call uses the elimination weight (0,0,1,2), which works.

```

See also: [Section 7.3.5 \[eliminate \(plural\)\]](#), page 339.

### 7.5.19.2 preimageNC

Procedure from library `ncpreim.lib` (see [Section 7.5.19 \[ncpreim.lib\]](#), page 555).

**Usage:** `preimageNC(A,f,J[,P,eng]);` A ring, f map or ideal, J ideal, P optional string, eng optional int

**Assume:** f defines a map from A to the basering.

**Return:** nothing, instead exports an object 'preim' of type ideal to ring A, being the preimage of J under f.

**Note:** If P is given and not equal to the empty string, the preimage is exported to A under the name specified by P.

Otherwise (and by default), P is set to 'preim'.

If `eng<0`, `std` is used for Groebner basis computations, otherwise (and by default) `slimgb` is used.

If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Remark:** Reference: (Lev)

**Example:**

```

LIB "ncpreim.lib";
def A = makeUgl(3); setring A; A; // universal enveloping algebra of gl_3
⇒ // coefficients: QQ
⇒ // number of vars : 9
⇒ //      block 1 : ordering dp
⇒ //      : names  e_1_1 e_1_2 e_1_3 e_2_1 e_2_2 e_2_3 e_3_1 \
    e_3_2 e_3_3
⇒ //      block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      e_1_2e_1_1=e_1_1*e_1_2-e_1_2

```

```

⇒ //      e_1_3e_1_1=e_1_1*e_1_3-e_1_3
⇒ //      e_2_1e_1_1=e_1_1*e_2_1+e_2_1
⇒ //      e_3_1e_1_1=e_1_1*e_3_1+e_3_1
⇒ //      e_2_1e_1_2=e_1_2*e_2_1-e_1_1+e_2_2
⇒ //      e_2_2e_1_2=e_1_2*e_2_2-e_1_2
⇒ //      e_2_3e_1_2=e_1_2*e_2_3-e_1_3
⇒ //      e_3_1e_1_2=e_1_2*e_3_1+e_3_2
⇒ //      e_2_1e_1_3=e_1_3*e_2_1+e_2_3
⇒ //      e_3_1e_1_3=e_1_3*e_3_1-e_1_1+e_3_3
⇒ //      e_3_2e_1_3=e_1_3*e_3_2-e_1_2
⇒ //      e_3_3e_1_3=e_1_3*e_3_3-e_1_3
⇒ //      e_2_2e_2_1=e_2_1*e_2_2+e_2_1
⇒ //      e_3_2e_2_1=e_2_1*e_3_2+e_3_1
⇒ //      e_2_3e_2_2=e_2_2*e_2_3-e_2_3
⇒ //      e_3_2e_2_2=e_2_2*e_3_2+e_3_2
⇒ //      e_3_1e_2_3=e_2_3*e_3_1-e_2_1
⇒ //      e_3_2e_2_3=e_2_3*e_3_2-e_2_2+e_3_3
⇒ //      e_3_3e_2_3=e_2_3*e_3_3-e_2_3
⇒ //      e_3_3e_3_1=e_3_1*e_3_3+e_3_1
⇒ //      e_3_3e_3_2=e_3_2*e_3_3+e_3_2
ring r3 = 0,(x,y,z,Dx,Dy,Dz),dp;
def B = Weyl(); setring B; B;      // third Weyl algebra
⇒ // coefficients: QQ
⇒ // number of vars : 6
⇒ //      block 1 : ordering dp
⇒ //      : names      x y z Dx Dy Dz
⇒ //      block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      Dxx=x*Dx+1
⇒ //      Dyy=y*Dy+1
⇒ //      Dzz=z*Dz+1
ideal ff = x*Dx,x*Dy,x*Dz,y*Dx,y*Dy,y*Dz,z*Dx,z*Dy,z*Dz;
map f = A,ff;      // f: A -> B, e(i,j) |-> x(i)D(j)
ideal J = 0;
preimageNC(A,f,J,"K");      // compute K := ker(f)
setring A;
K;
⇒ K[1]=e_2_3*e_3_2-e_2_2*e_3_3-e_2_2
⇒ K[2]=e_1_3*e_3_2-e_1_2*e_3_3-e_1_2
⇒ K[3]=e_2_3*e_3_1-e_2_1*e_3_3-e_2_1
⇒ K[4]=e_2_2*e_3_1-e_2_1*e_3_2
⇒ K[5]=e_1_3*e_3_1-e_1_1*e_3_3-e_1_1
⇒ K[6]=e_1_2*e_3_1-e_1_1*e_3_2
⇒ K[7]=e_1_3*e_2_2-e_1_2*e_2_3+e_1_3
⇒ K[8]=e_1_3*e_2_1-e_1_1*e_2_3
⇒ K[9]=e_1_2*e_2_1-e_1_1*e_2_2-e_1_1

```

See also: [Section 7.3.21 \[preimage \(plural\)\]](#), page 355.

### 7.5.19.3 admissibleSub

Procedure from library `ncpreim.lib` (see [Section 7.5.19 \[ncpreim.lib\]](#), page 555).

**Usage:**      `admissibleSub(v); v intvec`

**Assume:** The entries of  $v$  are in the range  $1..nvars(basering)$ .

**Return:** int, 1 if the variables indexed by the entries of  $v$  form an admissible subalgebra, 0 otherwise

**Example:**

```
LIB "ncpreim.lib";
ring r = 0,(e,f,h),dp;
matrix d[3][3];
d[1,2] = -h; d[1,3] = 2*e; d[2,3] = -2*f;
def A = nc_algebra(1,d);
setring A; A; // A is U(sl_2)
↪ // coefficients: QQ
↪ // number of vars : 3
↪ //          block 1 : ordering dp
↪ //          : names   e f h
↪ //          block 2 : ordering C
↪ // noncommutative relations:
↪ //    fe=ef-h
↪ //    he=eh+2e
↪ //    hf=fh-2f
// the subalgebra generated by e,f is not admissible since [e,f]=h
admissibleSub(1..2);
↪ 0
// but the subalgebra generated by f,h is admissible since [f,h]=2f
admissibleSub(2..3);
↪ 1
```

#### 7.5.19.4 isUpperTriangular

Procedure from library `ncpreim.lib` (see [Section 7.5.19 \[ncpreim.lib\]](#), page 555).

**Usage:** `isUpperTriangular(M[,k]);`  $M$  a matrix,  $k$  an optional int

**Return:** int, 1 if the given matrix is upper triangular,  
0 otherwise.

**Note:** If  $k > 0$  is given, it is checked whether  $M$  is strictly upper triangular.

**Example:**

```
LIB "ncpreim.lib";
ring r = 0,x,dp;
matrix M[2][3] =
0,1,2,
0,0,3;
isUpperTriangular(M);
↪ 1
isUpperTriangular(M,1);
↪ 1
M[2,2] = 4;
isUpperTriangular(M);
↪ 1
isUpperTriangular(M,1);
↪ 0
```

### 7.5.19.5 appendWeight2Ord

Procedure from library `ncpreim.lib` (see [Section 7.5.19 \[ncpreim.lib\]](#), page 555).

**Usage:**      `appendWeight2Ord(w);` w an intvec

**Return:**    ring, the basering equipped with the ordering  $(a(w), <)$ , where  $<$  is the ordering of the basering.

**Example:**

```
LIB "ncpreim.lib";
ring r = 0,(a,b,x,d),Dp;
intvec w = 1,2,3,4;
def r2 = appendWeight2Ord(w); // for a commutative ring
r2;
⇒ // coefficients: QQ
⇒ // number of vars : 4
⇒ //      block 1 : ordering a
⇒ //      : names      a b x d
⇒ //      : weights    1 2 3 4
⇒ //      block 2 : ordering Dp
⇒ //      : names      a b x d
⇒ //      block 3 : ordering C
matrix D[4][4];
D[1,2] = 3*a; D[1,4] = 3*x^2; D[2,3] = -x;
D[2,4] = d; D[3,4] = 1;
def A = nc_algebra(1,D);
setring A; A;
⇒ // coefficients: QQ
⇒ // number of vars : 4
⇒ //      block 1 : ordering Dp
⇒ //      : names      a b x d
⇒ //      block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      ba=ab+3a
⇒ //      da=ad+3x2
⇒ //      xb=bx-x
⇒ //      db=bd+d
⇒ //      dx=xd+1
w = 2,1,1,1;
def B = appendWeight2Ord(w); // for a non-commutative ring
setring B; B;
⇒ // coefficients: QQ
⇒ // number of vars : 4
⇒ //      block 1 : ordering a
⇒ //      : names      a b x d
⇒ //      : weights    2 1 1 1
⇒ //      block 2 : ordering Dp
⇒ //      : names      a b x d
⇒ //      block 3 : ordering C
⇒ // noncommutative relations:
⇒ //      ba=ab+3a
⇒ //      da=ad+3x2
⇒ //      xb=bx-x
```

```

⇒ //      db=bd+d
⇒ //      dx=xd+1

```

### 7.5.19.6 elimWeight

Procedure from library `ncpreim.lib` (see [Section 7.5.19 \[ncpreim.lib\]](#), page 555).

**Usage:** `elimWeight(v)`;  $v$  an intvec

**Assume:** The basering is a G-algebra.  
The entries of  $v$  are in the range  $1..nvars(basering)$  and the corresponding variables generate an admissible subalgebra.

**Return:** intvec, say  $w$ , such that the ordering  $(a(w), <)$ , where  $<$  is any admissible global ordering, is an elimination ordering for the subalgebra generated by the variables indexed by the entries of the given intvec.

**Note:** If no such ordering exists, the zero intvec is returned.

**Remark:** Reference: (BGL), (GML)

**Example:**

```

LIB "ncpreim.lib";
// (Lev): Example 2
ring r = 0,(a,b,x,d),Dp;
matrix D[4][4];
D[1,2] = 3*a; D[1,4] = 3*x^2; D[2,3] = -x;
D[2,4] = d; D[3,4] = 1;
def A = nc_algebra(1,D);
setring A; A;
⇒ // coefficients: QQ
⇒ // number of vars : 4
⇒ //          block 1 : ordering Dp
⇒ //          : names  a b x d
⇒ //          block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      ba=ab+3a
⇒ //      da=ad+3x2
⇒ //      xb=bx-x
⇒ //      db=bd+d
⇒ //      dx=xd+1
// Since d*a-a*d = 3*x^2, any admissible ordering has to satisfy
// x^2 < a*d, while any elimination ordering for {x,d} additionally
// has to fulfil a << x and a << d.
// Hence neither a block ordering with weights
// (1,1,1,1) nor a weighted ordering with weight (0,0,1,1) will do.
intvec v = 3,4;
elimWeight(v);
⇒ 0,0,1,2

```

### 7.5.19.7 extendedTensor

Procedure from library `ncpreim.lib` (see [Section 7.5.19 \[ncpreim.lib\]](#), page 555).

**Usage:** `extendedTensor(A,I)`;  $A$  ring,  $I$  ideal

**Return:** ring,  $A+B$  (where  $B$  denotes the basering) extended with non-commutative relations between the vars of  $A$  and  $B$ , which arise from the homomorphism  $A \rightarrow B$  induced by  $I$  in the usual sense, i.e. if the vars of  $A$  are named  $x(i)$  and the vars of  $B$   $y(j)$ , then putting  $q(i)(j) = \text{leadcoef}(y(j)*I[i])/\text{leadcoef}(I[i]*y(j))$  and  $r(i)(j) = y(j)*I[i] - q(i)(j)*I[i]*y(j)$  yields the relation  $y(j)*x(i) = q(i)(j)*x(i)*y(j)+r(i)(j)$ .

**Remark:** Reference: (Lev)

**Example:**

```
LIB "ncpreim.lib";
def A = makeWeyl(2);
setring A; A;
⇨ // coefficients: QQ
⇨ // number of vars : 4
⇨ //          block 1 : ordering dp
⇨ //          : names  x(1) x(2) D(1) D(2)
⇨ //          block 2 : ordering C
⇨ // noncommutative relations:
⇨ //      D(1)x(1)=x(1)*D(1)+1
⇨ //      D(2)x(2)=x(2)*D(2)+1
def B = makeUgl(2);
setring B; B;
⇨ // coefficients: QQ
⇨ // number of vars : 4
⇨ //          block 1 : ordering dp
⇨ //          : names  e_1_1 e_1_2 e_2_1 e_2_2
⇨ //          block 2 : ordering C
⇨ // noncommutative relations:
⇨ //      e_1_2e_1_1=e_1_1*e_1_2-e_1_2
⇨ //      e_2_1e_1_1=e_1_1*e_2_1+e_2_1
⇨ //      e_2_1e_1_2=e_1_2*e_2_1-e_1_1+e_2_2
⇨ //      e_2_2e_1_2=e_1_2*e_2_2-e_1_2
⇨ //      e_2_2e_2_1=e_2_1*e_2_2+e_2_1
ideal I = var(1)*var(3), var(1)*var(4), var(2)*var(3), var(2)*var(4);
I;
⇨ I[1]=e_1_1*e_2_1
⇨ I[2]=e_1_1*e_2_2
⇨ I[3]=e_1_2*e_2_1
⇨ I[4]=e_1_2*e_2_2
def C = extendedTensor(A,I);
setring C; C;
⇨ // coefficients: QQ
⇨ // number of vars : 8
⇨ //          block 1 : ordering dp
⇨ //          : names  x(1) x(2) D(1) D(2)
⇨ //          block 2 : ordering dp
⇨ //          : names  e_1_1 e_1_2 e_2_1 e_2_2
⇨ //          block 3 : ordering C
⇨ // noncommutative relations:
⇨ //      D(1)x(1)=x(1)*D(1)+1
⇨ //      e_1_1x(1)=x(1)*e_1_1-e_1_1*e_2_1
⇨ //      e_1_2x(1)=x(1)*e_1_2+e_1_1^2-e_1_2*e_2_1-e_1_1*e_2_2
⇨ //      e_2_1x(1)=x(1)*e_2_1+e_2_1^2
⇨ //      e_2_2x(1)=x(1)*e_2_2+e_1_1*e_2_1
```



```

⇒ //      D(2)x(2)=x(2)*D(2)+1
⇒ //      e_1_2x(2)=x(2)*e_1_2+e_1_1*e_1_2-e_1_2*e_2_2
⇒ //      e_2_1x(2)=x(2)*e_2_1-e_1_1*e_2_1+e_2_1*e_2_2
⇒ //      e_1_2D(1)=D(1)*e_1_2+e_1_1*e_1_2-e_1_2*e_2_2-e_1_2
⇒ //      e_2_1D(1)=D(1)*e_2_1-e_1_1*e_2_1+e_2_1*e_2_2+e_2_1
⇒ //      e_1_1D(2)=D(2)*e_1_1+e_1_2*e_2_2
⇒ //      e_1_2D(2)=D(2)*e_1_2+e_1_2^2
⇒ //      e_2_1D(2)=D(2)*e_2_1-e_1_2*e_2_1-e_1_1*e_2_2+e_2_2^2
⇒ //      e_2_2D(2)=D(2)*e_2_2-e_1_2*e_2_2
⇒ //      e_1_2e_1_1=e_1_1*e_1_2-e_1_2
⇒ //      e_2_1e_1_1=e_1_1*e_2_1+e_2_1
⇒ //      e_2_1e_1_2=e_1_2*e_2_1-e_1_1+e_2_2
⇒ //      e_2_2e_1_2=e_1_2*e_2_2-e_1_2
⇒ //      e_2_2e_2_1=e_2_1*e_2_2+e_2_1

```

### 7.5.20 nctools\_lib

**Library:** nctools.lib

**Purpose:** General tools for noncommutative algebras

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**Overview:** Support: DFG (Deutsche Forschungsgesellschaft) and Metodos algebraicos y efectivos en grupos cuanticos, BFM2001-3141, MCYT, Jose Gomez-Torrecillas (Main researcher).

**Procedures:**

#### 7.5.20.1 Gweights

Procedure from library `nctools.lib` (see [Section 7.5.20 \[nctools\\_lib\]](#), page 563).

**Usage:** Gweights(r); r a ring or a square matrix

**Return:** intvec

**Purpose:** compute an appropriate weight int vector for a G-algebra, i.e., such that  $\forall i, j, k, l, m, n, \dots, \text{lm}_w(d_{ij}) <_w x_i x_j$ .  
 the polynomials  $d_{ij}$  are taken from r itself, if it is of the type ring or defined by the given square polynomial matrix

**Theory:** Gweights returns an integer vector, whose weighting should be used to redefine the G-algebra in order to get the same non-commutative structure w.r.t. a weighted ordering. If the input is a matrix and the output is the zero vector then there is not a G-algebra structure associated to these relations with respect to the given variables. Another possibility is to use `weightedRing` to obtain directly a G-algebra with the new appropriate (weighted) ordering.

**Example:**

```

LIB "nctools.lib";
ring r = (0,q),(a,b,c,d),lp;
matrix C[4][4];

```

```

C[1,2]=q; C[1,3]=q; C[1,4]=1; C[2,3]=1; C[2,4]=q; C[3,4]=q;
matrix D[4][4];
D[1,4]=(q-1/q)*b*c;
def S = nc_algebra(C,D); setring S; S;
⇒ // coefficients: QQ(q)
⇒ // number of vars : 4
⇒ //          block 1 : ordering lp
⇒ //          : names  a b c d
⇒ //          block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      ba=(q)*ab
⇒ //      ca=(q)*ac
⇒ //      da=ad+(q2-1)/(q)*bc
⇒ //      db=(q)*bd
⇒ //      dc=(q)*cd
Gweights(S);
⇒ 2,1,1,1
def D=fetch(r,D);
Gweights(D);
⇒ 2,1,1,1

```

See also: [Section 7.5.20.2 \[weightedRing\]](#), page 564.

### 7.5.20.2 weightedRing

Procedure from library `nctools.lib` (see [Section 7.5.20 \[nctools.lib\]](#), page 563).

**Usage:** `weightedRing(r)`; `r` a ring

**Return:** ring

**Purpose:** equip the variables of the given ring with weights such that the relations of new ring (with weighted variables) satisfies the ordering condition for G-algebras: e.g.  $\forall i < j, \text{lm}_w(d_{ij}) <_w x_i x_j$ .

**Note:** activate this ring with the "setring" command

**Example:**

```

LIB "nctools.lib";
ring r = (0,q),(a,b,c,d),lp;
matrix C[4][4];
C[1,2]=q; C[1,3]=q; C[1,4]=1; C[2,3]=1; C[2,4]=q; C[3,4]=q;
matrix D[4][4];
D[1,4]=(q-1/q)*b*c;
def S = nc_algebra(C,D); setring S; S;
⇒ // coefficients: QQ(q)
⇒ // number of vars : 4
⇒ //          block 1 : ordering lp
⇒ //          : names  a b c d
⇒ //          block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      ba=(q)*ab
⇒ //      ca=(q)*ac
⇒ //      da=ad+(q2-1)/(q)*bc
⇒ //      db=(q)*bd
⇒ //      dc=(q)*cd

```

```

def t=weightedRing(S);
setring t; t;
⇨ // coefficients: QQ(q)
⇨ // number of vars : 4
⇨ //          block 1 : ordering M
⇨ //          : names   a b c d
⇨ //          : weights 2 1 1 1
⇨ //          : weights 0 0 0 1
⇨ //          : weights 0 0 1 0
⇨ //          : weights 0 1 0 0
⇨ //          block 2 : ordering C
⇨ // noncommutative relations:
⇨ //      ba=(q)*ab
⇨ //      ca=(q)*ac
⇨ //      da=ad+(q2-1)/(q)*bc
⇨ //      db=(q)*bd
⇨ //      dc=(q)*cd

```

See also: [Section 7.5.20.1 \[Gweights\]](#), page 563.

### 7.5.20.3 ndcond

Procedure from library `nctools.lib` (see [Section 7.5.20 \[nctools.lib\]](#), page 563).

**Usage:** `ndcond();`

**Return:** ideal

**Purpose:** compute the non-degeneracy conditions of the basering

**Note:** if `printlevel > 0`, the procedure displays intermediate information (by default, `printlevel=0`)

**Example:**

```

LIB "nctools.lib";
ring r = (0,q1,q2),(x,y,z),dp;
matrix C[3][3];
C[1,2]=q2; C[1,3]=q1; C[2,3]=1;
matrix D[3][3];
D[1,2]=x; D[1,3]=z;
def S = nc_algebra(C,D); setring S;
S;
⇨ // coefficients: QQ(q1, q2)
⇨ // number of vars : 3
⇨ //          block 1 : ordering dp
⇨ //          : names   x y z
⇨ //          block 2 : ordering C
⇨ // noncommutative relations:
⇨ //      yx=(q2)*x*y+x
⇨ //      zx=(q1)*x*z+z
ideal j=ndcond(); // the silent version
j;
⇨ j[1]=(-q2+1)*y*z-z
printlevel=1;
ideal i=ndcond(); // the verbose version
⇨ Processing degree : 1

```

```

⇒ 1 . 2 . 3 .
⇒ failed: (-q2+1)*y*z-z
⇒ done
i;
⇒ i[1]=(-q2+1)*y*z-z

```

### 7.5.20.4 Weyl

Procedure from library `nctools.lib` (see [Section 7.5.20 \[nctools.lib\]](#), page 563).

**Usage:**      `Weyl()`

**Return:**     ring

**Purpose:**     create a Weyl algebra structure on the basering

**Note:**       Activate this ring using the command `setring`.  
                 Assume the number of variables of a basering is  $2k$ . (if the number of variables is odd,  
                 an error message will be returned)  
                 by default, the procedure treats first  $k$  variables as coordinates `x.i` and the last  $k$  as  
                 differentials `d.i`  
                 if a non-zero optional argument is given, the procedure treats  $2k$  variables of a basering  
                 as  $k$  pairs  $(x_i, d_i)$ , i.e. variables with odd numbers are treated as coordinates and with  
                 even numbers as differentials

**Example:**

```

LIB "nctools.lib";
ring A1=0,(x(1..2),d(1..2)),dp;
def S=Weyl();
setring S; S;
⇒ // coefficients: QQ
⇒ // number of vars : 4
⇒ //          block 1 : ordering dp
⇒ //          : names  x(1) x(2) d(1) d(2)
⇒ //          block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      d(1)x(1)=x(1)*d(1)+1
⇒ //      d(2)x(2)=x(2)*d(2)+1
kill A1,S;
ring B1=0,(x1,d1,x2,d2),dp;
def S=Weyl(1);
setring S; S;
⇒ // coefficients: QQ
⇒ // number of vars : 4
⇒ //          block 1 : ordering dp
⇒ //          : names  x1 d1 x2 d2
⇒ //          block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      d1x1=x1*d1+1
⇒ //      d2x2=x2*d2+1

```

See also: [Section 7.5.20.5 \[makeWeyl\]](#), page 566.

### 7.5.20.5 makeWeyl

Procedure from library `nctools.lib` (see [Section 7.5.20 \[nctools.lib\]](#), page 563).

**Usage:** `makeWeyl(n,[p]);`  $n$  an integer,  $n > 0$ ;  $p$  an optional integer (field characteristic)

**Return:** ring

**Purpose:** create the  $n$ -th Weyl algebra over the rationals  $\mathbb{Q}$  or  $\mathbb{F}_p$

**Note:** activate this ring with the "setring" command.  
 The presentation of an  $n$ -th Weyl algebra is classical:  $D(i)x(i)=x(i)D(i)+1$ , where  $x(i)$  correspond to coordinates and  $D(i)$  to partial differentiations,  $i=1,\dots,n$ .  
 If  $p$  is not prime, the next larger prime number will be used.

**Example:**

```
LIB "nctools.lib";
def a = makeWeyl(3);
setring a;
a;
⇒ // coefficients: QQ
⇒ // number of vars : 6
⇒ //          block 1 : ordering dp
⇒ //          : names  x(1) x(2) x(3) D(1) D(2) D(3)
⇒ //          block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      D(1)x(1)=x(1)*D(1)+1
⇒ //      D(2)x(2)=x(2)*D(2)+1
⇒ //      D(3)x(3)=x(3)*D(3)+1
```

See also: [Section 7.5.20.4 \[Weyl\]](#), page 566.

### 7.5.20.6 makeHeisenberg

Procedure from library `nctools.lib` (see [Section 7.5.20 \[nctools.lib\]](#), page 563).

**Usage:** `makeHeisenberg(n, [p,d]);` int  $n$  (setting  $2n+1$  variables), optional int  $p$  (field characteristic), optional int  $d$  (power of  $h$  in the commutator)

**Return:** ring

**Purpose:** create the  $n$ -th Heisenberg algebra in the variables  $x(1), y(1), \dots, x(n), y(n), h$  over the rationals  $\mathbb{Q}$  or  $\mathbb{F}_p$  with the relations  $\forall i \in \{1, 2, \dots, n\}; y(j)x(i) = x(i)y(j) + h^d$ .

**Note:** activate this ring with the `setring` command  
 If  $p$  is not prime, the next larger prime number will be used.

**Example:**

```
LIB "nctools.lib";
def a = makeHeisenberg(2);
setring a; a;
⇒ // coefficients: QQ
⇒ // number of vars : 5
⇒ //          block 1 : ordering lp
⇒ //          : names  x(1) x(2) y(1) y(2) h
⇒ //          block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      y(1)x(1)=x(1)*y(1)+h
⇒ //      y(2)x(2)=x(2)*y(2)+h
def H3 = makeHeisenberg(3, 7, 2);
```

```

setring H3; H3;
⇨ // coefficients: ZZ/7
⇨ // number of vars : 7
⇨ //          block 1 : ordering lp
⇨ //          : names  x(1) x(2) x(3) y(1) y(2) y(3) h
⇨ //          block 2 : ordering C
⇨ // noncommutative relations:
⇨ //      y(1)x(1)=x(1)*y(1)+h^2
⇨ //      y(2)x(2)=x(2)*y(2)+h^2
⇨ //      y(3)x(3)=x(3)*y(3)+h^2

```

See also: [Section 7.5.20.5 \[makeWeyl\]](#), page 566.

### 7.5.20.7 Exterior

Procedure from library `nctools.lib` (see [Section 7.5.20 \[nctools.lib\]](#), page 563).

**Usage:**     `Exterior();`

**Return:**    `qring`

**Purpose:**    create the exterior algebra of a basering

**Note:**      activate this `qring` with the "setring" command

**Theory:**    given a basering, this procedure introduces the anticommutative relations  $x(j)x(i) = -x(i)x(j)$  for all  $j > i$ ,  
 moreover, creates a factor algebra modulo the two-sided ideal, generated by  $x(i)^2$  for all  $i$

**Example:**

```

LIB "nctools.lib";
ring R = 0,(x(1..3)),dp;
def ER = Exterior();
setring ER;
ER;
⇨ // coefficients: QQ
⇨ // number of vars : 3
⇨ //          block 1 : ordering dp
⇨ //          : names  x(1) x(2) x(3)
⇨ //          block 2 : ordering C
⇨ // noncommutative relations:
⇨ //      x(2)x(1)=-x(1)*x(2)
⇨ //      x(3)x(1)=-x(1)*x(3)
⇨ //      x(3)x(2)=-x(2)*x(3)
⇨ // quotient ring from ideal
⇨ _[1]=x(3)^2
⇨ _[2]=x(2)^2
⇨ _[3]=x(1)^2

```

### 7.5.20.8 findimAlgebra

Procedure from library `nctools.lib` (see [Section 7.5.20 \[nctools.lib\]](#), page 563).

**Usage:**     `findimAlgebra(M,[r]);`  $M$  a matrix,  $r$  an optional ring

**Return:**    `ring`

**Purpose:** define a finite dimensional algebra structure on a ring

**Note:** the matrix  $M$  is used to define the relations  $x(i)*x(j) = M[i,j]$  in the basering (by default) or in the optional ring  $r$ .  
The procedure equips the ring with the noncommutative structure.  
The procedure exports the ideal (not a two-sided Groebner basis!), called `fdQuot`, for further qring definition.

**Theory:** finite dimensional algebra can be represented as a factor algebra of a G-algebra modulo certain two-sided ideal. The relations of a f.d. algebra are thus naturally divided into two groups: firstly, the relations on the variables of the ring, making it into G-algebra and the rest of them, which constitute the ideal which will be factored out.

**Example:**

```
LIB "nctools.lib";
ring r=(0,a,b),(x(1..3)),dp;
matrix S[3][3];
S[2,3]=a*x(1); S[3,2]=-b*x(1);
def A=findimAlgebra(S); setring A;
fdQuot = twostd(fdQuot);
qring Qr = fdQuot;
Qr;
⇨ // coefficients: QQ(a, b)
⇨ // number of vars : 3
⇨ //          block 1 : ordering dp
⇨ //          : names  x(1) x(2) x(3)
⇨ //          block 2 : ordering C
⇨ // noncommutative relations:
⇨ //      x(3)x(2)=(-b)/(a)*x(2)*x(3)
⇨ // quotient ring from ideal
⇨ _[1]=x(3)^2
⇨ _[2]=x(2)*x(3)+(-a)*x(1)
⇨ _[3]=x(1)*x(3)
⇨ _[4]=x(2)^2
⇨ _[5]=x(1)*x(2)
⇨ _[6]=x(1)^2
```

### 7.5.20.9 superCommutative

Procedure from library `nctools.lib` (see [Section 7.5.20 \[nctools.lib\]](#), page 563).

**Usage:** `superCommutative([b,[e,[Q]]])`;

**Return:** qring

**Purpose:** create a super-commutative algebra (as a GR-algebra) over a basering,

**Note:** activate this qring with the "setring" command.

**Note:** if  $b=e$  then the resulting ring is commutative.  
By default,  $b=1$ ,  $e=nvars(basering)$ ,  $Q=0$ .

**Theory:** given a basering, this procedure introduces the anti-commutative relations  $\text{var}(j)\text{var}(i)=-\text{var}(i)\text{var}(j)$  for all  $e \geq j > i \geq b$  and creates the quotient of the anti-commutative algebra modulo the two-sided ideal, generated by  $x(b)^2, \dots, x(e)^2 + Q$

**Display:** If `printlevel > 1`, warning debug messages will be printed

**Example:**

```
LIB "nctools.lib";
ring R = 0,(x(1..4)),dp; // global!
def ER = superCommutative(); // the same as Exterior (b = 1, e = N)
setring ER; ER;
⇨ // coefficients: QQ
⇨ // number of vars : 4
⇨ //      block 1 : ordering dp
⇨ //      : names  x(1) x(2) x(3) x(4)
⇨ //      block 2 : ordering C
⇨ // noncommutative relations:
⇨ //      x(2)x(1)=-x(1)*x(2)
⇨ //      x(3)x(1)=-x(1)*x(3)
⇨ //      x(4)x(1)=-x(1)*x(4)
⇨ //      x(3)x(2)=-x(2)*x(3)
⇨ //      x(4)x(2)=-x(2)*x(4)
⇨ //      x(4)x(3)=-x(3)*x(4)
⇨ // quotient ring from ideal
⇨ _[1]=x(4)^2
⇨ _[2]=x(3)^2
⇨ _[3]=x(2)^2
⇨ _[4]=x(1)^2
"Alternating variables: [", AltVarStart(), ",", AltVarEnd(), "].";
⇨ Alternating variables: [ 1 , 4 ].
kill R; kill ER;
ring R = 0,(x(1..4)),(lp(1), dp(3)); // global!
def ER = superCommutative(2); // b = 2, e = N
setring ER; ER;
⇨ // coefficients: QQ
⇨ // number of vars : 4
⇨ //      block 1 : ordering lp
⇨ //      : names  x(1)
⇨ //      block 2 : ordering dp
⇨ //      : names  x(2) x(3) x(4)
⇨ //      block 3 : ordering C
⇨ // noncommutative relations:
⇨ //      x(3)x(2)=-x(2)*x(3)
⇨ //      x(4)x(2)=-x(2)*x(4)
⇨ //      x(4)x(3)=-x(3)*x(4)
⇨ // quotient ring from ideal
⇨ _[1]=x(4)^2
⇨ _[2]=x(3)^2
⇨ _[3]=x(2)^2
"Alternating variables: [", AltVarStart(), ",", AltVarEnd(), "].";
⇨ Alternating variables: [ 2 , 4 ].
kill R; kill ER;
ring R = 0,(x, y, z),(ds(1), dp(2)); // mixed!
def ER = superCommutative(2,3); // b = 2, e = 3
setring ER; ER;
⇨ // coefficients: QQ
⇨ // number of vars : 3
```



```

⇒ //      block  1 : ordering ds
⇒ //      : names  x
⇒ //      block  2 : ordering dp
⇒ //      : names  y z
⇒ //      block  3 : ordering C
⇒ // noncommutative relations:
⇒ //      zy=-yz
⇒ // quotient ring from ideal
⇒ _[1]=y2
⇒ _[2]=z2
"Alternating variables: [", AltVarStart(), ",", AltVarEnd(), "].";
⇒ Alternating variables: [ 2 , 3 ].
x + 1 + z + y; // ordering on variables: y > z > 1 > x
⇒ y+z+1+x
std(x - x*x*x);
⇒ _[1]=x
std(ideal(x - x*x*x, x*x*z + y, z + y*x*x));
⇒ _[1]=y+x2z
⇒ _[2]=z+x2y
⇒ _[3]=x
kill R; kill ER;
ring R = 0,(x, y, z),(ds(1), dp(2)); // mixed!
def ER = superCommutative(2, 3, ideal(x - x*x, x*x*z + y, z + y*x*x )); // b = 2, e =
setring ER; ER;
⇒ // coefficients: QQ
⇒ // number of vars : 3
⇒ //      block  1 : ordering ds
⇒ //      : names  x
⇒ //      block  2 : ordering dp
⇒ //      : names  y z
⇒ //      block  3 : ordering C
⇒ // noncommutative relations:
⇒ //      zy=-yz
⇒ // quotient ring from ideal
⇒ _[1]=y+x2z
⇒ _[2]=z+x2y
⇒ _[3]=x
⇒ _[4]=y2
⇒ _[5]=z2
"Alternating variables: [", AltVarStart(), ",", AltVarEnd(), "].";
⇒ Alternating variables: [ 2 , 3 ].

```

### 7.5.20.10 rightStd

Procedure from library `nctools.lib` (see [Section 7.5.20 \[nctools.lib\]](#), page 563).

**Purpose:** compute a right Groebner basis of I

**Return:** the same type as input

**Example:**

```

LIB "nctools.lib";
LIB "ncalg.lib";
def A = makeUs1(2);

```

```

setring A;
ideal I = e2,f;
option(redSB);
option(redTail);
ideal LI = std(I);
LI;
⇒ LI[1]=f
⇒ LI[2]=h2+h
⇒ LI[3]=eh+e
⇒ LI[4]=e2
ideal RI = rightStd(I);
RI;
⇒ RI[1]=f
⇒ RI[2]=h2-h
⇒ RI[3]=eh+e
⇒ RI[4]=e2

```

### 7.5.20.11 rightNF

Procedure from library `nctools.lib` (see [Section 7.5.20 \[nctools.lib\]](#), page 563).

**Usage:** `rightNF(I)`;  $v$  a poly/vector,  $M$  an ideal/module

**Purpose:** compute a right normal form of  $v$  w.r.t.  $M$

**Return:** poly/vector (as of the 1st argument)

**Example:**

```

LIB "nctools.lib";
LIB "ncalg.lib";
ring r = 0,(x,d),dp;
def S = nc_algebra(1,1); setring S; // Weyl algebra
ideal I = x; I = std(I);
poly p = x*d+1;
NF(p,I); // left normal form
⇒ 0
rightNF(p,I); // right normal form
⇒ 1

```

### 7.5.20.12 rightModulo

Procedure from library `nctools.lib` (see [Section 7.5.20 \[nctools.lib\]](#), page 563).

**Usage:** `rightModulo(M,N)`;  $M,N$  are ideals/modules

**Purpose:** compute a right representation of the module  $(M+N)/N$

**Return:** module

**Assume:**  $M,N$  are presentation matrices for right modules

**Example:**

```

LIB "nctools.lib";
LIB "ncalg.lib";
def A = makeUsl(2);
setring A;
option(redSB);

```

```

option(redTail);
ideal I = e2,f2,h2-1;
I = twostd(I);
print(matrix(I));
↪ h2-1,fh-f,eh+e,f2,2ef-h-1,e2
ideal E = std(e);
ideal TL = e,h-1; // the result of left modulo
TL;
↪ TL[1]=e
↪ TL[2]=h-1
ideal T = rightModulo(E,I);
T = rightStd(T+I);
T = rightStd(rightNF(T,I)); // make the output canonic
T;
↪ T[1]=h+1
↪ T[2]=e

```

### 7.5.20.13 moduloSlim

Procedure from library `nctools.lib` (see [Section 7.5.20 \[nctools.lib\]](#), page 563).

**Usage:**      `moduloSlim(A,B)`; A,B module/matrix/ideal

**Return:**     module

**Purpose:**     compute modulo with `slimgb` as engine

**Example:**

```

LIB "nctools.lib";
LIB "ncalg.lib";
ring r; // first classical example for modulo
ideal h1=x,y,z;    ideal h2=x;
module m=moduloSlim(h1,h2);
print(m);
↪ 1,0,0, 0,
↪ 0,0,z, x,
↪ 0,x,-y,0
// now, a noncommutative example
def A = makeUsl2(); setring A; // this algebra is U(sl_2)
ideal H2 = e2,f2,h2-1; H2 = twostd(H2);
print(matrix(H2)); // print H2 in a compact form
↪ h2-1,fh-f,eh+e,f2,2ef-h-1,e2
ideal H1 = std(e);
ideal T = moduloSlim(H1,H2);
T = std( NF(std(H2+T),H2) );
T;
↪ T[1]=h-1
↪ T[2]=e
// now, a matrix example:
ring r2 = 0,(x,d), (dp);
def R = nc_algebra(1,1); setring R;
matrix M[2][2] = d, 0, 0, d*(x*d);
matrix P[2][1] = (8x+7)*d+9x, (x2+1)*d + 5*x;
module X = moduloSlim(P,M);
print(X);

```

$\mapsto 5x^2d^2-2xd^3-5xd-6d^2+5, xd^5+5xd^3+5d^4+5d^2$

### 7.5.20.14 ncRelations

Procedure from library `nctools.lib` (see [Section 7.5.20 \[nctools.lib\]](#), page 563).

**Usage:** `ncRelations(r)`; `r` a ring

**Return:** list `L` with two elements, both elements are of type matrix:  
`L[1]` = matrix of coefficients `C`,  
`L[2]` = matrix of polynomials `D`

**Purpose:** recover the noncommutative relations via matrices `C` and `D` from a noncommutative ring

**Example:**

```
LIB "nctools.lib";
ring r = 0,(x,y,z),dp;
matrix C[3][3]=0,1,2,0,0,-1,0,0,0;
print(C);
 $\mapsto 0,1,2,$ 
 $\mapsto 0,0,-1,$ 
 $\mapsto 0,0,0$ 
matrix D[3][3]=0,1,2y,0,0,-2x+y+1;
print(D);
 $\mapsto 0,1,2y,$ 
 $\mapsto 0,0,-2x+y+1,$ 
 $\mapsto 0,0,0$ 
def S=nc_algebra(C,D);setring S; S;
 $\mapsto$  // coefficients: QQ
 $\mapsto$  // number of vars : 3
 $\mapsto$  //          block 1 : ordering dp
 $\mapsto$  //                      : names    x y z
 $\mapsto$  //          block 2 : ordering C
 $\mapsto$  // noncommutative relations:
 $\mapsto$  //      yx=xy+1
 $\mapsto$  //      zx=2xz+2y
 $\mapsto$  //      zy=-yz-2x+y+1
def l=ncRelations(S);
print (l[1]);
 $\mapsto 0,1,2,$ 
 $\mapsto 0,0,-1,$ 
 $\mapsto 0,0,0$ 
print (l[2]);
 $\mapsto 0,1,2y,$ 
 $\mapsto 0,0,-2x+y+1,$ 
 $\mapsto 0,0,0$ 
```

See also: [Section 7.4.1 \[G-algebras\]](#), page 366; [Section 5.1.137 \[ringlist\]](#), page 255.

### 7.5.20.15 isCentral

Procedure from library `nctools.lib` (see [Section 7.5.20 \[nctools.lib\]](#), page 563).

**Usage:** `isCentral(p)`; `p` poly

**Return:** int, 1 if `p` commutes with all variables and 0 otherwise

**Purpose:** check whether  $p$  is central in a basering (that is, commutes with every generator of the ring)

**Note:** if `printlevel > 0`, the procedure displays intermediate information (by default, `printlevel=0`)

**Example:**

```
LIB "nctools.lib";
ring r=0,(x,y,z),dp;
matrix D[3][3]=0;
D[1,2]=-z;
D[1,3]=2*x;
D[2,3]=-2*y;
def S = nc_algebra(1,D); setring S;
S; // this is U(sl_2)
↳ // coefficients: QQ
↳ // number of vars : 3
↳ //          block 1 : ordering dp
↳ //          : names  x y z
↳ //          block 2 : ordering C
↳ // noncommutative relations:
↳ //      yx=xy-z
↳ //      zx=xz+2x
↳ //      zy=yz-2y
poly c = 4*x*y+z^2-2*z;
printlevel = 0;
isCentral(c);
↳ 1
poly h = x*c;
printlevel = 1;
isCentral(h);
↳ Non-central at: y
↳ Non-central at: z
↳ 0
```

### 7.5.20.16 isNC

Procedure from library `nctools.lib` (see [Section 7.5.20 \[nctools.lib\]](#), page 563).

**Usage:** `isNC()`;

**Purpose:** check whether a basering is commutative or not

**Return:** int, 1 if basering is noncommutative and 0 otherwise

**Example:**

```
LIB "nctools.lib";
def a = makeWeyl(2);
setring a;
isNC();
↳ 1
kill a;
ring r = 17,(x(1..7)),dp;
isNC();
↳ 0
kill r;
```

### 7.5.20.17 isCommutative

Procedure from library `nctools.lib` (see [Section 7.5.20 \[nctools.lib\]](#), page 563).

**Usage:** `isCommutative();`

**Return:** `int`, 1 if basering is commutative, or 0 otherwise

**Purpose:** check whether basering is commutative

**Example:**

```
LIB "nctools.lib";
ring r = 0,(x,y),dp;
isCommutative();
↪ 1
def D = Weyl(); setring D;
isCommutative();
↪ 0
setring r;
def R = nc_algebra(1,0); setring R;
isCommutative();
↪ 1
```

### 7.5.20.18 isWeyl

Procedure from library `nctools.lib` (see [Section 7.5.20 \[nctools.lib\]](#), page 563).

**Usage:** `isWeyl();`

**Return:** `int`, 1 if basering is a Weyl algebra, or 0 otherwise

**Purpose:** check whether basering is a Weyl algebra

**Example:**

```
LIB "nctools.lib";
ring r = 0,(a,b,c,d),dp;
isWeyl();
↪ 0
def D = Weyl(1); setring D; //make from r a Weyl algebra
b*a;
↪ ab+1
isWeyl();
↪ 1
ring t = 0,(Dx,x,y,Dy),dp;
matrix M[4][4]; M[1,2]=-1; M[3,4]=1;
def T = nc_algebra(1,M); setring T;
isWeyl();
↪ 1
```

### 7.5.20.19 UpOneMatrix

Procedure from library `nctools.lib` (see [Section 7.5.20 \[nctools.lib\]](#), page 563).

**Usage:** `UpOneMatrix(n);` `n` an integer

**Return:** `intmat`

**Purpose:** compute an `n x n` matrix with 1's in the whole upper triangle

**Note:** helpful for setting noncommutative algebras with complicated coefficient matrices

**Example:**

```
LIB "nctools.lib";
ring r = (0,q),(x,y,z),dp;
matrix C = UpOneMatrix(3);
C[1,3] = q;
print(C);
⇒ 0,1,(q),
⇒ 0,0,1,
⇒ 0,0,0
def S = nc_algebra(C,0); setring S;
S;
⇒ // coefficients: QQ(q)
⇒ // number of vars : 3
⇒ //          block 1 : ordering dp
⇒ //          : names  x y z
⇒ //          block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      zx=(q)*xz
```

### 7.5.20.20 AltVarStart

Procedure from library `nctools.lib` (see [Section 7.5.20 \[nctools.lib\]](#), page 563).

**Usage:** `AltVarStart();`

**Return:** `int`

**Purpose:** returns the number of the first alternating variable of basering

**Note:** basering should be a super-commutative algebra constructed by the procedure `superCommutative`, emits an error otherwise

**Example:**

```
LIB "nctools.lib";
ring R = 0,(x(1..4)),dp; // global!
def ER = superCommutative(2); // (b = 2, e = N)
setring ER; ER;
⇒ // coefficients: QQ
⇒ // number of vars : 4
⇒ //          block 1 : ordering dp
⇒ //          : names  x(1) x(2) x(3) x(4)
⇒ //          block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      x(3)x(2)=-x(2)*x(3)
⇒ //      x(4)x(2)=-x(2)*x(4)
⇒ //      x(4)x(3)=-x(3)*x(4)
⇒ // quotient ring from ideal
⇒ _[1]=x(4)^2
⇒ _[2]=x(3)^2
⇒ _[3]=x(2)^2
"Alternating variables: [" , AltVarStart(), ",", AltVarEnd(), "]" .";
⇒ Alternating variables: [ 2 , 4 ].
setring R;
"Alternating variables: [" , AltVarStart(), ",", AltVarEnd(), "]" .";
```

```

⇒      ? SCA rings are factors by (at least) squares!
⇒      ? leaving nctools.lib::AltVarStart (1133)
kill R, ER;
////////////////////////////////////////////////////////////////
ring R = 2,(x(1..4)),dp; // the same in char. = 2!
def ER = superCommutative(2); // (b = 2, e = N)
setring ER; ER;
⇒ // coefficients: ZZ/2
⇒ // number of vars : 4
⇒ //          block 1 : ordering dp
⇒ //          : names  x(1) x(2) x(3) x(4)
⇒ //          block 2 : ordering C
⇒ // quotient ring from ideal
⇒ _[1]=x(4)^2
⇒ _[2]=x(3)^2
⇒ _[3]=x(2)^2
"Alternating variables: [", AltVarStart(), ",", AltVarEnd(), "].";
⇒ Alternating variables: [ 4 , 4 ].
setring R;
"Alternating variables: [", AltVarStart(), ",", AltVarEnd(), "].";
⇒      ? SCA rings are factors by (at least) squares!
⇒      ? leaving nctools.lib::AltVarStart (1133)

```

### 7.5.20.21 AltVarEnd

Procedure from library `nctools.lib` (see [Section 7.5.20 \[nctools.lib\]](#), page 563).

**Usage:**     `AltVarStart();`

**Return:**    `int`

**Purpose:**    returns the number of the last alternating variable of basering

**Note:**      basing should be a super-commutative algebra constructed by  
the procedure `superCommutative`, emits an error otherwise

**Example:**

```

LIB "nctools.lib";
ring R = 0,(x(1..4)),dp; // global!
def ER = superCommutative(2); // (b = 2, e = N)
setring ER; ER;
⇒ // coefficients: QQ
⇒ // number of vars : 4
⇒ //          block 1 : ordering dp
⇒ //          : names  x(1) x(2) x(3) x(4)
⇒ //          block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      x(3)x(2)=-x(2)*x(3)
⇒ //      x(4)x(2)=-x(2)*x(4)
⇒ //      x(4)x(3)=-x(3)*x(4)
⇒ // quotient ring from ideal
⇒ _[1]=x(4)^2
⇒ _[2]=x(3)^2
⇒ _[3]=x(2)^2
"Alternating variables: [", AltVarStart(), ",", AltVarEnd(), "].";

```



```

⇒ Alternating variables: [ 2 , 4 ].
setring R;
"Alternating variables: [" , AltVarStart(), "," , AltVarEnd(), "]" .";
⇒ ? SCA rings are factors by (at least) squares!
⇒ ? leaving nctools.lib::AltVarStart (1133)
kill R, ER;
////////////////////////////////////
ring R = 2,(x(1..4)),dp; // the same in char. = 2!
def ER = superCommutative(2); // (b = 2, e = N)
setring ER; ER;
⇒ // coefficients: ZZ/2
⇒ // number of vars : 4
⇒ //          block 1 : ordering dp
⇒ //          : names  x(1) x(2) x(3) x(4)
⇒ //          block 2 : ordering C
⇒ // quotient ring from ideal
⇒ _[1]=x(4)^2
⇒ _[2]=x(3)^2
⇒ _[3]=x(2)^2
"Alternating variables: [" , AltVarStart(), "," , AltVarEnd(), "]" .";
⇒ Alternating variables: [ 4 , 4 ].
setring R;
"Alternating variables: [" , AltVarStart(), "," , AltVarEnd(), "]" .";
⇒ ? SCA rings are factors by (at least) squares!
⇒ ? leaving nctools.lib::AltVarStart (1133)

```

### 7.5.20.22 IsSCA

Procedure from library `nctools.lib` (see [Section 7.5.20 \[nctools.lib\]](#), page 563).

**Usage:** `IsSCA();`

**Return:** `int`

**Purpose:** returns 1 if basering is a super-commutative algebra and 0 otherwise

**Example:**

```

LIB "nctools.lib";
////////////////////////////////////
ring R = 0,(x(1..4)),dp; // commutative
if(IsSCA())
{ "Alternating variables: [" , AltVarStart(), "," , AltVarEnd(), "]" ."; }
else
{ "Not a super-commutative algebra!!!"; }
⇒ Not a super-commutative algebra!!!
kill R;
////////////////////////////////////
ring R = 0,(x(1..4)),dp;
def S = nc_algebra(1, 0); setring S; S; // still commutative!
⇒ // coefficients: QQ
⇒ // number of vars : 4
⇒ //          block 1 : ordering dp
⇒ //          : names  x(1) x(2) x(3) x(4)
⇒ //          block 2 : ordering C
⇒ // noncommutative relations:

```

```

if(IsSCA())
{ "Alternating variables: [" , AltVarStart(), ",", AltVarEnd(), "]."; }
else
{ "Not a super-commutative algebra!!!"; }
⇒ Not a super-commutative algebra!!!
kill R, S;
////////////////////////////////////////////////////////////////
ring R = 0,(x(1..4)),dp;
list CurrRing = ringlist(R);
def ER = ring(CurrRing);
setring ER; // R;
matrix E = UpOneMatrix(nvars(R));
int i, j; int b = 2; int e = 3;
for ( i = b; i < e; i++ )
{
for ( j = i+1; j <= e; j++ )
{
E[i, j] = -1;
}
}
def S = nc_algebra(E,0); setring S; S;
⇒ // coefficients: QQ
⇒ // number of vars : 4
⇒ //          block 1 : ordering dp
⇒ //          : names  x(1) x(2) x(3) x(4)
⇒ //          block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      x(3)x(2)=-x(2)*x(3)
if(IsSCA())
{ "Alternating variables: [" , AltVarStart(), ",", AltVarEnd(), "]."; }
else
{ "Not a super-commutative algebra!!!"; }
⇒ Not a super-commutative algebra!!!
kill R, ER, S;
////////////////////////////////////////////////////////////////
ring R = 0,(x(1..4)),dp;
def ER = superCommutative(2); // (b = 2, e = N)
setring ER; ER;
⇒ // coefficients: QQ
⇒ // number of vars : 4
⇒ //          block 1 : ordering dp
⇒ //          : names  x(1) x(2) x(3) x(4)
⇒ //          block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      x(3)x(2)=-x(2)*x(3)
⇒ //      x(4)x(2)=-x(2)*x(4)
⇒ //      x(4)x(3)=-x(3)*x(4)
⇒ // quotient ring from ideal
⇒ _[1]=x(4)^2
⇒ _[2]=x(3)^2
⇒ _[3]=x(2)^2
if(IsSCA())
{ "This is a SCA! Alternating variables: [" , AltVarStart(), ",", AltVarEnd(), "]."; }

```

```

    ↪ This is a SCA! Alternating variables: [ 2 , 4 ].
    else
    { "Not a super-commutative algebra!!!"; }
    kill R, ER;

```

### 7.5.20.23 makeModElimRing

Procedure from library `nctools.lib` (see [Section 7.5.20 \[nctools.lib\]](#), page 563).

**Usage:**     `makeModElimRing(L)`; L a list

**Return:**    ring

**Purpose:**    create a copy of a given ring equipped with the elimination ordering for module components  $(c, <)$

**Note:**      usually the list argument contains a ring to work with

**Example:**

```

LIB "nctools.lib";
ring r1 = 0,(x,y,z),(C,Dp);
def r2 = makeModElimRing(r1); setring r2; r2;   kill r2;
↪ // coefficients: QQ
↪ // number of vars : 3
↪ //           block 1 : ordering c
↪ //           block 2 : ordering Dp
↪ //           : names   x y z
ring r3 = 0,(z,t),(wp(2,3),c);
def r2 = makeModElimRing(r3); setring r2; r2; kill r2;
↪ // coefficients: QQ
↪ // number of vars : 2
↪ //           block 1 : ordering c
↪ //           block 2 : ordering wp
↪ //           : names   z t
↪ //           : weights 2 3
ring r4 = 0,(z,t,u,w),(a(1,2),C,wp(2,3,4,5));
def r2 = makeModElimRing(r4); setring r2; r2;
↪ // coefficients: QQ
↪ // number of vars : 4
↪ //           block 1 : ordering c
↪ //           block 2 : ordering a
↪ //           : names   z t
↪ //           : weights 1 2
↪ //           block 3 : ordering wp
↪ //           : names   z t u w
↪ //           : weights 2 3 4 5

```

### 7.5.20.24 embedMat

Procedure from library `nctools.lib` (see [Section 7.5.20 \[nctools.lib\]](#), page 563).

**Usage:**     `embedMat(A,m,n)`; A,B matrix/module

**Return:**    matrix

**Purpose:**    embed A in the left upper corner of  $m \times n$  matrix

**Example:**

```

LIB "nctools.lib";
ring r = 0,(a,b,c,d),dp;
matrix M[2][3]; M[1,1]=a; M[1,2]=b;M[2,2]=d;M[1,3]=c;
print(M);
  ↦ a,b,c,
  ↦ 0,d,0
print(embedMat(M,3,4));
  ↦ a,b,c,0,
  ↦ 0,d,0,0,
  ↦ 0,0,0,0
matrix N = M; N[2,2]=0;
print(embedMat(N,3,4));
  ↦ a,b,c,0,
  ↦ 0,0,0,0,
  ↦ 0,0,0,0

```

**7.5.21 olga\_lib****Library:** olga.lib**Purpose:** Ore-localization in G-Algebras**Author:** Johannes Hoffmann, email: johannes.hoffmann at math.rwth-aachen.de**Overview:** Let  $A$  be a  $G$ -algebra.

Current localization types:

Type 0: monoidal

- represented by a list of polys  $g_1, \dots, g_k$  that have to be contained in a commutative polynomial subring of  $A$  generated by a subset of the variables of  $A$

Type 1: geometric

- only for algebras with an even number of variables where the first half induces a commutative polynomial subring  $B$  of  $A$

- represented by an ideal  $p$ , which has to be a prime ideal in  $B$

- represented by an intvec  $v = [i_1, \dots, i_k]$  in the range  $1..nvars(basering)$

Localization data is an int specifying the type and a def with the corresponding information.

A fraction is represented as a vector with four entries:  $[s, r, p, t]$  Here,  $s^{-1}r$  is the left fraction representation,  $pt^{-1}$  is the right one. If  $s$  or  $t$  is zero, it means that the corresponding representation is not set. If both are zero, the fraction is not valid.

A detailed description along with further examples can be found in our paper: Johannes Hoffmann, Viktor Levandovskyy:

Constructive Arithmetics in Ore Localizations of Domains  
<https://arxiv.org/abs/1712.01773>

**Procedures:****7.5.21.1 locStatus**

Procedure from library `olga.lib` (see [Section 7.5.21 \[olga.lib\]](#), page 582).

**Usage:** `locStatus(locType, locData)`, int locType, list/vector/intvec locData**Purpose:** determine the status of a set of localization data

**Assume:**

**Return:** list

**Note:** - the first entry is 0 or 1, depending whether the input represents a valid localization  
 - the second entry is a string with a status/error message

**Example:**

```
LIB "olga.lib";
locStatus(42, list(1));
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ invalid localization: type is 42, valid types are:
⇒ 0 for a monoidal localization
⇒ 1 for a geometric localization
⇒ 2 for a rational localization
def undef;
locStatus(0, undef);
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ uninitialized or invalid localization: locData has to be defined
string s;
locStatus(0, s);
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ for a monoidal localization, locData has to be of type list, but is of\
type string
list L;
locStatus(0, L);
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ for a monoidal localization, locData has to be a non-empty list
L = s;
print(L);
⇒ [1]:
⇒
locStatus(0, L);
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ for a monoidal localization, locData has to be a list of polys, ints o\
r numbers, but entry 1 is , which is of type string
ring w = 0,(x,Dx,y,Dy),dp;
def W = Weyl(1);
setring W;
W;
⇒ // coefficients: QQ
⇒ // number of vars : 4
⇒ //          block 1 : ordering dp
⇒ //          : names    x Dx y Dy
```

```

⇒ //      block  2 : ordering C
⇒ // noncommutative relations:
⇒ //      Dxx=x*Dx+1
⇒ //      Dyy=y*Dy+1
locStatus(0, list(x, Dx));
⇒ [1]:
⇒      0
⇒ [2]:
⇒      for a monoidal localization, the variables occurring in the polys in l\
      ocData have to induce a commutative polynomial subring of basering
ring R;
setring R;
R;
⇒ // coefficients: ZZ/32003
⇒ // number of vars : 3
⇒ //      block  1 : ordering dp
⇒ //      : names      x y z
⇒ //      block  2 : ordering C
locStatus(1, s);
⇒ [1]:
⇒      0
⇒ [2]:
⇒      for a geometric localization, basering has to have an even number of v\
      ariables
setring W;
locStatus(1, s);
⇒ [1]:
⇒      0
⇒ [2]:
⇒      for a geometric localization, the first half of the variables of baser\
      ing has to induce a commutative polynomial subring of basering
ring t = 0,(x,y,Dx,Dy),dp;
def T = Weyl();
setring T;
T;
⇒ // coefficients: QQ
⇒ // number of vars : 4
⇒ //      block  1: ordering dp
⇒ //      : names      x y Dx Dy
⇒ //      block  2 : ordering C
⇒ // noncommutative relations:
⇒ //      Dxx=x*Dx+1
⇒ //      Dyy=y*Dy+1
locStatus(1, s);
⇒ [1]:
⇒      0
⇒ [2]:
⇒      for a geometric localization, locData has to be of type ideal, but is \
      of type string
locStatus(1, ideal(Dx));
⇒ [1]:
⇒      0
⇒ [2]:

```

```

⇒    for a geometric localization, locData has to be an ideal generated by \
      polynomials containing only variables from the first half of the variable\
      s
locStatus(2, s);
⇒ [1]:
⇒    0
⇒ [2]:
⇒    for a rational localization, locData has to be of type intvec, but is \
      of type string
intvec v;
locStatus(2, v);
⇒ [1]:
⇒    0
⇒ [2]:
⇒    for a rational localization, locData has to be a non-zero intvec
locStatus(2, intvec(1,2));
⇒ [1]:
⇒    1
⇒ [2]:
⇒    valid localization

```

### 7.5.21.2 testLocData

Procedure from library `olga.lib` (see [Section 7.5.21 \[olga.lib\]](#), page 582).

**Usage:** `testLocData(locType, locData)`, `int locType`,  
`list/vector/intvec locData`

**Purpose:** test if the given data specifies a denominator set wrt. the checks from `locStatus`

**Assume:**

**Return:** nothing

**Note:** throws error if checks were not successful

**Example:**

```

LIB "olga.lib";
ring R; setring R;
testLocData(0, list(1)); // correct localization, no error
testLocData(42, list(1)); // incorrect localization, results in error
⇒    ? invalid localization: type is 42, valid types are:
⇒    0 for a monoidal localization
⇒    1 for a geometric localization
⇒    2 for a rational localization
⇒    ? leaving olga.lib::testLocData (0)

```

### 7.5.21.3 isInS

Procedure from library `olga.lib` (see [Section 7.5.21 \[olga.lib\]](#), page 582).

**Usage:** `isInS(p, locType, locData(, override))`, `poly p`, `int locType`, `list/vector/intvec locData(`,  
`int override)`

**Purpose:** determine if a polynomial is in a denominator set

**Assume:**

**Return:** int

**Note:** - returns 0 or 1, depending whether  $p$  is in the denominator set specified by `locType` and `locData`  
 - if `override` is set, will not normalize `locData` (use with care)

**Example:**

```
LIB "olga.lib";
ring R = 0,(x,y,Dx,Dy),dp;
def S = Weyl();
setring S; S;
⇨ // coefficients: QQ
⇨ // number of vars : 4
⇨ //          block 1 : ordering dp
⇨ //          : names  x y Dx Dy
⇨ //          block 2 : ordering C
⇨ // noncommutative relations:
⇨ //      Dxx=x*Dx+1
⇨ //      Dyy=y*Dy+1
// monoidal localization
poly g1 = x^2*y+x+2;
poly g2 = y^3+x*y;
list L = g1,g2;
poly g = g1^2*g2;
poly f = g-1;
isInS(g, 0, L); // g is in the denominator set
⇨ 1
isInS(f, 0, L); // f is NOT in the denominator set
⇨ 0
// geometric localization
ideal p = x-1, y-3;
g = x^2+y-3;
f = (x-1)*g;
isInS(g, 1, p); // g is in the denominator set
⇨ 1
isInS(f, 1, p); // f is NOT in the denominator set
⇨ 0
// rational localization
intvec v = 2;
g = y^5+17*y^2-4;
f = x*y;
isInS(g, 2, v); // g is in the denominator set
⇨ 1
isInS(f, 2, v); // f is NOT in the denominator set
⇨ 0
```

#### 7.5.21.4 fracStatus

Procedure from library `olga.lib` (see [Section 7.5.21 \[olga.lib\]](#), page 582).

**Usage:** `fracStatus(frac, locType, locData)`, vector `frac`, int `locType`, list/intvec/vector `locData`

**Purpose:** determine if the given vector is a representation of a fraction in the specified localization

**Assume:**



**Return:** list

**Note:** - the first entry is 0 or 1, depending whether the input is valid - the second entry is a string with a status message

**Example:**

```
LIB "olga.lib";
ring r = QQ[x,y,Dx,Dy];
def R = Weyl();
setring R;
fracStatus([1,0,0,0], 42, list(1));
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ invalid localization in fraction: gen(1)
⇒ invalid localization: type is 42, valid types are:
⇒ 0 for a monoidal localization
⇒ 1 for a geometric localization
⇒ 2 for a rational localization
list L = x;
fracStatus([0,7,x,0], 0, L);
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ vector is not a valid fraction: no denominator specified in x*gen(3)+7\
*gen(2)
fracStatus([Dx,Dy,0,0], 0, L);
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ the left denominator Dx of fraction Dx*gen(1)+Dy*gen(2) is not in the \
denominator set of type 0 given by x
fracStatus([0,0,Dx,Dy], 0, L);
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ the right denominator Dy of fraction Dx*gen(3)+Dy*gen(4) is not in the \
denominator set of type 0 given by x
fracStatus([x,Dx,Dy,x], 0, L);
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ left and right representation are not equal in: x*gen(4)+x*gen(1)+Dx*ge\
n(2)+Dy*gen(3)
fracStatus([x,Dx,x*Dx+2,x^2], 0, L);
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ valid fraction
```

### 7.5.21.5 testFraction

Procedure from library `olga.lib` (see [Section 7.5.21 \[olga.lib\]](#), page 582).

**Usage:**     testFraction(frac, locType, locData), vector frac, int locType, list/intvec/vector locData

**Purpose:**     test if the given vector is a representation of a fraction in the specified localization wrt. the checks from fracStatus

**Assume:**

**Return:**     nothing

**Note:**       throws error if checks were not successful

**Example:**

```
LIB "olga.lib";
ring r = QQ[x,y,Dx,Dy];
def R = Weyl();
setring R;
list L = x;
vector frac = [x,Dx,x*Dx+2,x^2];
testFraction(frac, 0, L); // correct localization, no error
frac = [x,Dx,x*Dx,x^2];
testFraction(frac, 0, L); // incorrect localization, results in error
↳ ? left and right representation are not equal in: x^2*gen(4)+x*Dx*gen(3\
   )+x*gen(1)+Dx*gen(2)
↳ ? leaving olga.lib::testFraction (0)
```

### 7.5.21.6 leftOre

Procedure from library `olga.lib` (see [Section 7.5.21 \[olga.lib\]](#), page 582).

**Usage:**     leftOre(s, r, locType, locData), poly s, r, int locType, list/vector/intvec locData

**Purpose:**     compute left Ore data for a given tuple (s,r)

**Assume:**    s is in the denominator set determined via locType and locData

**Return:**     list

**Note:**       - the first entry of the list is a vector [ts,tr] such that  $ts*r=tr*s$  - the second entry of the list is a description of all choices for ts

**Example:**

```
LIB "olga.lib";
ring R = 0,(x,y,Dx,Dy),dp;
def S = Weyl();
setring S; S;
↳ // coefficients: QQ
↳ // number of vars : 4
↳ //          block 1 : ordering dp
↳ //          : names  x y Dx Dy
↳ //          block 2 : ordering C
↳ // noncommutative relations:
↳ //      Dxx=x*Dx+1
↳ //      Dyy=y*Dy+1
// left Ore
// monoidal localization
poly g1 = x+3;
poly g2 = x*y;
```

```

list L = g1,g2;
poly g = g1^2*g2;
poly f = Dx;
list rm = leftOre(g, f, 0, L);
print(rm[1]);
↪ [x^8*y^4+12*x^7*y^4+54*x^6*y^4+108*x^5*y^4+81*x^4*y^4,x^5*y^3*Dx+6*x^4*y^3\
   3*Dx-3*x^4*y^3+9*x^3*y^3*Dx-12*x^3*y^3-9*x^2*y^3]
rm[2];
↪ _[1]=x^8*y^4+12*x^7*y^4+54*x^6*y^4+108*x^5*y^4+81*x^4*y^4
rm[1][2]*g-rm[1][1]*f;
↪ 0
// geometric localization
ideal p = x-1, y-3;
f = Dx;
g = x^2+y;
list rg = leftOre(g, f, 1, p);
print(rg[1]);
↪ [x^4+2*x^2*y+y^2,x^2*Dx+y*Dx-2*x]
rg[2];
↪ _[1]=x^4+2*x^2*y+y^2
rg[1][2]*g-rg[1][1]*f;
↪ 0
// rational localization
intvec rat = 1;
f = Dx+Dy;
g = x;
list rr = leftOre(g, f, 2, rat);
print(rr[1]);
↪ [x^2,x*Dx+x*Dy-1]
rr[2];
↪ _[1]=x^2
rr[1][2]*g-rr[1][1]*f;
↪ 0

```

### 7.5.21.7 rightOre

Procedure from library `olga.lib` (see [Section 7.5.21 \[olga.lib\]](#), page 582).

**Usage:** `rightOre(s, r, locType, locData)`, poly `s`, `r`, int `locType`, list/vector/intvec `locData`

**Purpose:** compute right Ore data for a given tuple `(s,r)`

**Assume:** `s` is in the denominator set determined via `locType` and `locData`

**Return:** list

**Note:** - the first entry of the list is a vector `[ts,tr]` such that  $r*ts=s*tr$  - the second entry of the list is a description of all choices for `ts`

**Example:**

```

LIB "olga.lib";
ring R = 0,(x,y,Dx,Dy),dp;
def S = Weyl();
setring S; S;
↪ // coefficients: QQ
↪ // number of vars : 4

```

```

⇒ //      block  1 : ordering dp
⇒ //      : names  x y Dx Dy
⇒ //      block  2 : ordering C
⇒ // noncommutative relations:
⇒ //      Dxx=x*Dx+1
⇒ //      Dyy=y*Dy+1
// monoidal localization
poly g1 = x+3;
poly g2 = x*y;
list L = g1,g2;
poly g = g1^2*g2;
poly f = Dx;
list rm = rightOre(g, f, 0, L);
print(rm[1]);
⇒ [x^8*y^4+12*x^7*y^4+54*x^6*y^4+108*x^5*y^4+81*x^4*y^4,x^5*y^3*Dx+6*x^4*y^3\
   3*Dx+8*x^4*y^3+9*x^3*y^3*Dx+36*x^3*y^3+36*x^2*y^3]
rm[2];
⇒ _[1]=x^8*y^4+12*x^7*y^4+54*x^6*y^4+108*x^5*y^4+81*x^4*y^4
g*rm[1][2]-f*rm[1][1];
⇒ 0
// geometric localization
ideal p = x-1, y-3;
f = Dx;
g = x^2+y;
list rg = rightOre(g, f, 1, p);
print(rg[1]);
⇒ [x^4+2*x^2*y+y^2,x^2*Dx+y*Dx+4*x]
rg[2];
⇒ _[1]=x^4+2*x^2*y+y^2
g*rg[1][2]-f*rg[1][1];
⇒ 0
// rational localization
intvec rat = 1;
f = Dx+Dy;
g = x;
list rr = rightOre(g, f, 2, rat);
print(rr[1]);
⇒ [x^2,x*Dx+x*Dy+2]
rr[2];
⇒ _[1]=x^2
g*rr[1][2]-f*rr[1][1];
⇒ 0

```

### 7.5.21.8 convertRightToLeftFraction

Procedure from library `olga.lib` (see [Section 7.5.21 \[olga.lib\]](#), page 582).

**Usage:**      `convertRightToLeftFraction(frac, locType, locData),`  
                  vector frac, int locType, list/vector/intvec locData

**Purpose:**     determine a left fraction representation of a given fraction

**Assume:**

**Return:**    vector

**Note:** - the returned vector contains a repr. of frac as a left fraction - if the left representation of frac is already specified, frac will be returned.

**Example:**

```
LIB "olga.lib";
ring R = 0,(x,y,Dx,Dy),dp;
def S = Weyl();
setring S; S;
⇨ // coefficients: QQ
⇨ // number of vars : 4
⇨ //          block 1 : ordering dp
⇨ //          : names  x y Dx Dy
⇨ //          block 2 : ordering C
⇨ // noncommutative relations:
⇨ //      Dxx=x*Dx+1
⇨ //      Dyy=y*Dy+1
// monoidal localization
poly g1 = x+3;
poly g2 = x*y;
list L = g1,g2;
poly g = g1^2*g2;
poly f = Dx;
vector fracm = [0,0,f,g];
vector rm = convertRightToLeftFraction(fracm, 0, L);
print(rm);
⇨ [x^8*y^4+12*x^7*y^4+54*x^6*y^4+108*x^5*y^4+81*x^4*y^4,x^5*y^3*Dx+6*x^4*y^3\
    3*Dx-3*x^4*y^3+9*x^3*y^3*Dx-12*x^3*y^3-9*x^2*y^3,Dx,x^3*y+6*x^2*y+9*x*y]
rm[2]*g-rm[1]*f;
⇨ 0
// geometric localization
ideal p = x-1, y-3;
f = Dx;
g = x^2+y;
vector fracg = [0,0,f,g];
vector rg = convertRightToLeftFraction(fracg, 1, p);
print(rg);
⇨ [x^4+2*x^2*y+y^2,x^2*Dx+y*Dx-2*x,Dx,x^2+y]
rg[2]*g-rg[1]*f;
⇨ 0
// rational localization
intvec rat = 1;
f = Dx+Dy;
g = x;
vector fracr = [0,0,f,g];
vector rr = convertRightToLeftFraction(fracr, 2, rat);
print(rr);
⇨ [x^2,x*Dx+x*Dy-1,Dx+Dy,x]
rr[2]*g-rr[1]*f;
⇨ 0
```

### 7.5.21.9 convertLeftToRightFraction

Procedure from library `olga.lib` (see [Section 7.5.21 \[olga.lib\]](#), page 582).

**Usage:** `convertLeftToRightFraction(frac, locType, locData), vector frac, int locType, list/vector/intvec locData`

**Purpose:** determine a right fraction representation of a given fraction

**Assume:**

**Return:** vector

**Note:** - the returned vector contains a repr. of frac as a right fraction, - if the right representation of frac is already specified, frac will be returned.

**Example:**

```
LIB "olga.lib";
ring R = 0,(x,y,Dx,Dy),dp;
def S = Weyl();
setring S; S;
⇨ // coefficients: QQ
⇨ // number of vars : 4
⇨ //          block 1 : ordering dp
⇨ //          : names  x y Dx Dy
⇨ //          block 2 : ordering C
⇨ // noncommutative relations:
⇨ //      Dxx=x*Dx+1
⇨ //      Dyy=y*Dy+1
// monoidal localization
poly g = x;
poly f = Dx;
vector fracm = [g,f,0,0];
list L = g;
vector rm = convertLeftToRightFraction(fracm, 0, L);
print(rm);
⇨ [x,Dx,x*Dx+2,x^2]
f*rm[4]-g*rm[3];
⇨ 0
// geometric localization
g = x+y;
f = Dx+Dy;
vector fracg = [g,f,0,0];
ideal p = x-1, y-3;
vector rg = convertLeftToRightFraction(fracg, 1, p);
print(rg);
⇨ [x+y,Dx+Dy,x*Dx+y*Dx+x*Dy+y*Dy+4,x^2+2*x*y+y^2]
f*rg[4]-g*rg[3];
⇨ 0
// rational localization
intvec rat = 1;
f = Dx+Dy;
g = x;
vector fracr = [g,f,0,0];
vector rr = convertLeftToRightFraction(fracr, 2, rat);
print(rr);
⇨ [x,Dx+Dy,x*Dx+x*Dy+2,x^2]
f*rr[4]-g*rr[3];
⇨ 0
```

### 7.5.21.10 addLeftFractions

Procedure from library `olga.lib` (see [Section 7.5.21 \[olga.lib\]](#), page 582).

**Usage:**      `addLeftFractions(a, b, locType, locData(, override)),`  
                  `vector a, b, int locType, list/vector/intvec locData(, int override)`

**Purpose:**     add two left fractions in the specified localization

**Assume:**

**Return:**     vector

**Note:**       the returned vector is the sum of `a` and `b` as fractions in the localization specified by `locType` and `locData`.

**Example:**

```
LIB "olga.lib";
ring R = 0,(x,y,Dx,Dy),dp;
def S = Weyl();
setring S; S;
⇨ // coefficients: QQ
⇨ // number of vars : 4
⇨ //          block 1 : ordering dp
⇨ //          : names  x y Dx Dy
⇨ //          block 2 : ordering C
⇨ // noncommutative relations:
⇨ //      Dxx=x*Dx+1
⇨ //      Dyy=y*Dy+1
// monoidal localization
poly g1 = x+3;
poly g2 = x*y+y;
list L = g1,g2;
poly s1 = g1;
poly s2 = g2;
poly r1 = Dx;
poly r2 = Dy;
vector frac1 = [s1,r1,0,0];
vector frac2 = [s2,r2,0,0];
vector rm = addLeftFractions(frac1, frac2, 0, L);
print(rm);
⇨ [x^2*y+4*x*y+3*y,x*y*Dx+y*Dx+x*Dy+3*Dy]
// geometric localization
ideal p = x-1, y-3;
vector rg = addLeftFractions(frac1, frac2, 1, p);
print(rg);
⇨ [x^2*y+4*x*y+3*y,x*y*Dx+y*Dx+x*Dy+3*Dy]
// rational localization
intvec v = 2;
s1 = y^2+y+1;
s2 = y-2;
r1 = Dx;
r2 = Dy;
frac1 = [s1,r1,0,0];
frac2 = [s2,r2,0,0];
vector rr = addLeftFractions(frac1, frac2, 2, v);
```

```
print(rr);
↪ [y^3-y^2-y-2,y^2*Dy+y*Dx+y*Dy-2*Dx+Dy]
```

### 7.5.21.11 multiplyLeftFractions

Procedure from library `olga.lib` (see [Section 7.5.21 \[olga.lib\]](#), page 582).

**Usage:** `multiplyLeftFractions(a, b, locType, locData(, override))`, vector `a`, `b`, int `locType`, list/vector/intvec `locData`, int `override`

**Purpose:** multiply two left fractions in the specified localization

**Assume:**

**Return:** vector

**Note:** the returned vector is the product of `a` and `b` as fractions in the localization specified by `locType` and `locData`.

**Example:**

```
LIB "olga.lib";
ring R = 0,(x,y,Dx,Dy),dp;
def S = Weyl();
setring S; S;
↪ // coefficients: QQ
↪ // number of vars : 4
↪ //          block 1 : ordering dp
↪ //          : names  x y Dx Dy
↪ //          block 2 : ordering C
↪ // noncommutative relations:
↪ //      Dxx=x*Dx+1
↪ //      Dyy=y*Dy+1
// monoidal localization
poly g1 = x+3;
poly g2 = x*y+y;
list L = g1,g2;
poly s1 = g1;
poly s2 = g2;
poly r1 = Dx;
poly r2 = Dy;
vector frac1 = [s1,r1,0,0];
vector frac2 = [s2,r2,0,0];
vector rm = multiplyLeftFractions(frac1, frac2, 0, L);
print(rm);
↪ [x^3*y^2+5*x^2*y^2+7*x*y^2+3*y^2,x*y*Dx*Dy+y*Dx*Dy-y*Dy]
// geometric localization
ideal p = x-1, y-3;
vector rg = multiplyLeftFractions(frac1, frac2, 1, p);
print(rg);
↪ [x^3*y+5*x^2*y+7*x*y+3*y,x*Dx*Dy+Dx*Dy-Dy]
// rational localization
intvec v = 2;
s1 = y^2+y+1;
s2 = y-2;
r1 = Dx;
r2 = Dy;
```



```

frac1 = [s1,r1,0,0];
frac2 = [s2,r2,0,0];
vector rr1 = multiplyLeftFractions(frac1, frac2, 2, v);
print(rr1);
 $\mapsto [y^3-y^2-y-2, Dx*Dy]$ 
vector rr2 = multiplyLeftFractions(frac2, frac1, 2, v);
print(rr2);
 $\mapsto [y^5-y^3-4*y^2-3*y-2, y^2*Dx*Dy+y*Dx*Dy-2*y*Dx+Dx*Dy-Dx]$ 
areEqualLeftFractions(rr1, rr2, 2, v);
 $\mapsto 0$ 

```

### 7.5.21.12 areEqualLeftFractions

Procedure from library `olga.lib` (see [Section 7.5.21 \[olga.lib\]](#), page 582).

**Usage:** `areEqualLeftFractions(a, b, locType, locData), vector a, b, int locType, list/vector/intvec locData`

**Purpose:** check if two given fractions are equal

**Assume:**

**Return:** `int`

**Note:** returns 1 or 0, depending whether  $a=b$  as fractions in the localization specified by `locType` and `locData`

**Example:**

```

LIB "olga.lib";
ring R = 0,(x,y,Dx,Dy),dp;
def S = Weyl();
setring S; S;
 $\mapsto$  // coefficients: QQ
 $\mapsto$  // number of vars : 4
 $\mapsto$  //          block 1 : ordering dp
 $\mapsto$  //          : names   x y Dx Dy
 $\mapsto$  //          block 2 : ordering C
 $\mapsto$  // noncommutative relations:
 $\mapsto$  //      Dxx=x*Dx+1
 $\mapsto$  //      Dyy=y*Dy+1
// monoidal
poly g1 = x*y+3;
poly g2 = y^3;
list L = g1,g2;
poly s1 = g1;
poly s2 = s1*g2;
poly s3 = s2;
poly r1 = Dx;
poly r2 = g2*r1;
poly r3 = s1*r1+3;
vector fracm1 = [s1,r1,0,0];
vector fracm2 = [s2,r2,0,0];
vector fracm3 = [s3,r3,0,0];
areEqualLeftFractions(fracm1, fracm2, 0, L);
 $\mapsto 1$ 
areEqualLeftFractions(fracm1, fracm3, 0, L);

```

```

⇒ 0
areEqualLeftFractions(fracm2, fracm3, 0, L);
⇒ 0

```

### 7.5.21.13 isInvertibleLeftFraction

Procedure from library `olga.lib` (see [Section 7.5.21 \[olga.lib\], page 582](#)).

**Usage:** `isInvertibleLeftFraction(frac, locType, locData), vector frac, int locType, list/vector/intvec locData`

**Purpose:** check if a fraction is invertible in the specified localization

**Assume:**

**Return:** int

**Note:** - returns 1, if the numerator of frac is in the denominator set, - returns 0, otherwise (NOTE: this does NOT mean that the fraction is not invertible, it just means it could not be determined by the method above).

**Example:**

```

LIB "olga.lib";
ring R = 0,(x,y,Dx,Dy),dp;
def S = Weyl();
setring S; S;
⇒ // coefficients: QQ
⇒ // number of vars : 4
⇒ //          block 1 : ordering dp
⇒ //          : names  x y Dx Dy
⇒ //          block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      Dxx=x*Dx+1
⇒ //      Dyy=y*Dy+1
poly g1 = x+3;
poly g2 = x*y;
list L = g1,g2;
vector frac = [g1*g2, 17, 0, 0];
isInvertibleLeftFraction(frac, 0, L);
⇒ 1
ideal p = x-1, y;
frac = [g1, x, 0, 0];
isInvertibleLeftFraction(frac, 1, p);
⇒ 1
intvec rat = 1,2;
frac = [g1*g2, Dx, 0, 0];
isInvertibleLeftFraction(frac, 2, rat);
⇒ 0

```

### 7.5.21.14 invertLeftFraction

Procedure from library `olga.lib` (see [Section 7.5.21 \[olga.lib\], page 582](#)).

**Usage:** `invertLeftFraction(frac, locType, locData), vector frac, int locType, list/vector/intvec locData`

**Purpose:** invert a fraction in the specified localization

**Assume:** frac is invertible in the loc. specified by locType and locData

**Return:** vector

**Note:** - returns the multiplicative inverse of frac in the localization specified by locType and locData,  
 - throws error if frac is not invertible (NOTE: this does NOT mean that the fraction is not invertible, it just means it could not be determined by the method listed above).

**Example:**

```
LIB "olga.lib";
ring R = 0,(x,y,Dx,Dy),dp;
def S = Weyl();
setring S; S;
⇨ // coefficients: QQ
⇨ // number of vars : 4
⇨ //      block 1 : ordering dp
⇨ //      : names  x y Dx Dy
⇨ //      block 2 : ordering C
⇨ // noncommutative relations:
⇨ //      Dxx=x*Dx+1
⇨ //      Dyy=y*Dy+1
poly g1 = x+3;
poly g2 = x*y;
list L = g1,g2;
vector frac = [g1*g2, 17, 0, 0];
print(invertLeftFraction(frac, 0, L));
⇨ [17,x^2*y+3*x*y]
ideal p = x-1, y;
frac = [g1, x, 0, 0];
print(invertLeftFraction(frac, 1, p));
⇨ [x,x+3]
intvec rat = 1,2;
frac = [g1*g2, y, 0, 0];
print(invertLeftFraction(frac, 2, rat));
⇨ [y,x^2*y+3*x*y]
```

### 7.5.21.15 isZeroFraction

Procedure from library `olga.lib` (see [Section 7.5.21 \[olga.lib\]](#), page 582).

**Usage:** isZeroFraction(frac), vector frac

**Purpose:** determine if the vector frac represents zero

**Assume:** frac is a valid fraction

**Return:** int

**Note:** returns 1, if frac == 0; 0 otherwise

**Example:**

```
LIB "olga.lib";
ring R = 0,(x,y,Dx,Dy),dp;
def S = Weyl();
setring S; S;
⇨ // coefficients: QQ
```

```

⇒ // number of vars : 4
⇒ //          block 1 : ordering dp
⇒ //          : names  x y Dx Dy
⇒ //          block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      Dxx=x*Dx+1
⇒ //      Dyy=y*Dy+1
isZeroFraction([42,0,0,0]);
⇒ 1
isZeroFraction([0,0,Dx,3]);
⇒ 0
isZeroFraction([1,1,1,1]);
⇒ 0

```

### 7.5.21.16 isOneFraction

Procedure from library `olga.lib` (see [Section 7.5.21 \[olga.lib\]](#), page 582).

**Usage:**     `isOneFraction(frac)`, vector `frac`

**Purpose:**    determine if the vector `frac` represents one

**Assume:**   `frac` is a valid fraction

**Return:**    `int`

**Note:**      1, if `frac == 1`; 0 otherwise

**Example:**

```

LIB "olga.lib";
ring R = 0,(x,y,Dx,Dy),dp;
def S = Weyl();
setring S; S;
⇒ // coefficients: QQ
⇒ // number of vars : 4
⇒ //          block 1 : ordering dp
⇒ //          : names  x y Dx Dy
⇒ //          block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      Dxx=x*Dx+1
⇒ //      Dyy=y*Dy+1
isOneFraction([42,42,0,0]);
⇒ 1
isOneFraction([0,0,Dx,3]);
⇒ 0
isOneFraction([1,0,0,1]);
⇒ 0

```

### 7.5.21.17 normalizeMonoidal

Procedure from library `olga.lib` (see [Section 7.5.21 \[olga.lib\]](#), page 582).

**Usage:**     `normalizeMonoidal(L)`, list `L`

**Purpose:**    compute a normal form of monoidal localization data

**Return:**    list

**Note:** given a list of polys, returns a list of all unique factors appearing in the given polys

**Example:**

```
LIB "olga.lib";
ring R = 0,(x,y,Dx,Dy),dp;
def S = Weyl(); setring S;
list L = x^2*y^3, (x+1)*(x*y-3*y^2+1);
L = normalizeMonoidal(L);
print(L);
↦ [1]:
↦      x*y-3*y^2+1
↦ [2]:
↦      x+1
↦ [3]:
↦      x
↦ [4]:
↦      y
```

### 7.5.21.18 normalizeRational

Procedure from library `olga.lib` (see [Section 7.5.21 \[olga.lib\]](#), page 582).

**Usage:** `normalizeRational(v)`, intvec  $v$

**Purpose:** compute a normal form of rational localization data

**Return:** intvec

**Note:** purges double entries and sorts ascendingly

**Example:**

```
LIB "olga.lib";
ring R; setring R;
intvec v = 9,5,9,3,1,5;
v = normalizeRational(v);
v;
↦ 1,3,5,9
```

### 7.5.21.19 testOlga

Procedure from library `olga.lib` (see [Section 7.5.21 \[olga.lib\]](#), page 582).

**Usage:** `testOlga()`

**Purpose:** execute a series of internal testing procedures

**Return:** nothing

**Note:**

### 7.5.21.20 testOlgaExamples

Procedure from library `olga.lib` (see [Section 7.5.21 \[olga.lib\]](#), page 582).

**Usage:** `testOlgaExamples()`

**Purpose:** execute the examples of all procedures in this library

**Return:** nothing

**Note:**

## 7.5.22 perron.lib

**Library:** perron.lib

**Purpose:** computation of algebraic dependences

**Author:** Oleksandr Motsak U@D, where U={motsak}, D={mathematik.uni-kl.de}

**Procedures:**

### 7.5.22.1 perron

Procedure from library `perron.lib` (see [Section 7.5.22 \[perron.lib\]](#), page 600).

**Usage:** perron( L [, D] )

**Return:** commutative ring with ideal 'Relations'

**Purpose:** computes polynomial relations ('Relations') between pairwise commuting polynomials of L [, up to a given degree bound D]

**Note:** the implementation was partially inspired by the Perron's theorem.

**Example:**

```
LIB "perron.lib";
int p = 3;
ring AA = p,(x,y,z),dp;
matrix D[3][3]=0;
D[1,2]=-z; D[1,3]=2*x; D[2,3]=-2*y;
def A = nc_algebra(1,D); setring A; // this algebra is U(sl_2)
ideal I = x^p, y^p, z^p-z, 4*x*y+z^2-2*z; // the center
def RA = perron( I, p );
setring RA;
RA;
↳ // coefficients: ZZ/3
↳ // number of vars : 4
↳ //      block 1 : ordering dp
↳ //      : names      F(1) F(2) F(3) F(4)
↳ //      block 2 : ordering C
Relations; // it was exported from perron to be in the returned ring.
↳ Relations[1]=F(4)^3-F(1)*F(2)-F(3)^2+F(4)^2
// perron can be also used in a commutative case, for example:
ring B = 0,(x,y,z),dp;
ideal J = xy+z^2, z^2+y^2, x^2y^2-2xy^3+y^4;
def RB = perron(J);
setring RB;
Relations;
↳ Relations[1]=F(1)^2-2*F(1)*F(2)+F(2)^2-F(3)
// one more test:
setring A;
map T=RA,I;
T(Relations); // should be zero
↳ _[1]=0
```

### 7.5.23 purityfiltration.lib

Status: experimental

**Library:** purityfiltration.lib

**Purpose:** Algorithms for computing a purity filtration of a given module

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**Overview:** Purity is a notion with several meanings. In our context it is equidimensionality of a module (that is all  $M$  is pure iff any nonzero submodule of  $N$  has the same dimension as  $N$ ).  
Notably, one should define purity with respect to a given dimension function. In the context of this library the corresponding function is the homological grade number  $j_A(M)$  of a module  $M$  over an  $K$ -algebra  $A$ .  $j_A(M)$  is the minimal integer  $k$ , such that  $\text{Ext}^k_A(M, A) \neq 0$ .

**References:**

- [AQ] Alban Quadrat: Grade filtration of linear functional systems, INRIA Report 7769 (2010), to appear in Acta Applicanda Mathematica.
- [B93] Jan-Erik Bjoerk: Analytic D-modules and applications, Kluwer Acad. Publ., 1993.
- [MB10] Mohamed Barakat: Purity Filtration and the Fine Structure of Autonomy. Proc. MTNS, 2010.

**Procedures:**

#### 7.5.23.1 projectiveDimension

Procedure from library `purityfiltration.lib` (see [Section 7.5.23 \[purityfiltration.lib\]](#), [page 601](#)).

**Usage:** `projectiveDimension(R,i,j)`,  $R$  matrix representing the Modul  $M = \text{coker}(R)$   
int  $i$ , with  $i=0$  or  $i=1$ ,  $j$  a natural number

**Return:** list  $T$ , a projective resolution of  $M$  and its projective dimension

**Purpose:** if  $i=0$  (and by default),  $T[1]$  gives a shortest left resolution of  $M = D^p / D^q(R^t)$  and  $T[2]$  the left projective dimension of  $M$   
if  $i=1$ ,  $T[1]$  gives a shortest right resolution of  $M = D^p / R D^q$  and  $T[2]$  the right projective dimension of  $M$   
in both cases  $T[1][j]$  is the  $(j-1)$ -th syzygy module of  $M$

**Note:** The algorithm is due to A. Quadrat, D. Robertz, Computation of bases of free modules over the Weyl algebras, J.Symb.Comp. 42, 2007.

**Example:**

```
LIB "purityfiltration.lib";
// commutative example
ring D = 0,(x,y,z),dp;
matrix R[6][4]=
0,-2*x,z-2*y-x,-1,
0,z-2*x,2*y-3*x,1,
z,-6*x,-2*y-5*x,-1,
0,y-x,y-x,0,
```

```

y,-x,-y-x,0,
x,-x,-2*x,0;
// compute a left resolution of  $M=D^4/D^6 \cdot R$ 
list T=projectiveDimension(transpose(R),0);
// so we have the left projective dimension
T[2];
 $\mapsto 3$ 
//we could also compute a right resolution of  $M=D^6/RD^4$ 
list T1=projectiveDimension(R,1);
// and we have right projective dimension
T1[2];
 $\mapsto 1$ 
// check, that a syzygy matrix of R has left inverse:
print(leftInverse(syz(R)));
 $\mapsto 0,0,1,0$ 
// so lpd(M) must be 1.
// Non-commutative example
ring D1 = 0,(x1,x2,x3,d1,d2,d3),dp;
def S=Weyl(); setring S;
matrix R[3][3]=
1/2*x2*d1, x2*d2+1, x2*d3+1/2*d1,
-1/2*x2*d2-3/2,0,1/2*d2,
-d1-1/2*x2*d3,-d2,-1/2*d3;
list T=projectiveDimension(R,0);
// left projective dimension of coker(R) is
T[2];
 $\mapsto 1$ 
list T1=projectiveDimension(R,1);
// both modules have the same projective dimension, but different resolutions, because
print(T[1][1]);
 $\mapsto 1/2 \cdot x_2 \cdot d_1, \quad -1/2 \cdot x_2 \cdot d_2 - 3/2, \quad -1/2 \cdot x_2 \cdot d_3 - d_1,$ 
 $\mapsto x_2 \cdot d_2 + 1, \quad 0, \quad -d_2,$ 
 $\mapsto x_2 \cdot d_3 + 1/2 \cdot d_1, \quad 1/2 \cdot d_2, \quad -1/2 \cdot d_3$ 
// not the same as
print(transpose(T1[1][1]));
 $\mapsto 1/2 \cdot x_2 \cdot d_1, \quad -1/2 \cdot x_2 \cdot d_2 - 3/2, \quad -1/2 \cdot x_2 \cdot d_3 - d_1, \quad -1/2 \cdot x_2,$ 
 $\mapsto x_2 \cdot d_2 + 1, \quad 0, \quad -d_2, \quad 0,$ 
 $\mapsto x_2 \cdot d_3 + 1/2 \cdot d_1, \quad 1/2 \cdot d_2, \quad -1/2 \cdot d_3, \quad 1/2$ 

```

### 7.5.23.2 purityFiltration

Procedure from library `purityfiltration.lib` (see [Section 7.5.23 \[purityfiltration.lib\]](#), page 601).

**Usage:** `purityFiltration(S)`, S matrix with entries of an Auslander regular ring D

**Return:** a list T of two lists, purity filtration of the module  $M=D^q/D^p(S^t)$

**Purpose:** the first list  $T[1]$  gives a filtration  $\{M_i\}$  of M,  
 where the i-th entry of  $T[1]$  gives the representation matrix of  $M_{-(i-1)}$ .  
 the second list  $T[2]$  gives representations of the factor Modules,  
 i.e.  $T[2][i]$  gives the repr. matrix for  $M_{-(i-1)}/M_i$

**Example:**

```

LIB "purityfiltration.lib";
ring D = 0,(x1,x2,d1,d2),dp;

```



```

def S=Weyl();
setring S;
int i;
matrix R[3][3]=0,d2-d1,d2-d1,d2,-d1,-d1-d2,d1,-d1,-2*d1;
print(R);
↳ 0, -d1+d2,-d1+d2,
↳ d2,-d1, -d1-d2,
↳ d1,-d1, -2*d1
list T=purityFiltration(transpose(R));
// the purity filtration of coker(M)
print(T[1][1]);
↳ 0, -d1+d2,-d1+d2,
↳ d2,-d1, -d1-d2,
↳ d1,-d1, -2*d1
print(T[1][2]);
↳ d2, d2,
↳ d1-d2,0,
↳ d2, d1
print(T[1][3]);
↳ 1,0,
↳ 0,d2,
↳ 0,d1
// factor modules of the filtration
print(T[2][1]);
↳ 0, 1,1,
↳ -1,0,1
print(T[2][2]);
↳ 1, 1,
↳ d1-d2,0
print(T[2][3]);
↳ 1,0,
↳ 0,d2,
↳ 0,d1

```

### 7.5.23.3 purityTriang

Procedure from library `purityfiltration.lib` (see [Section 7.5.23 \[purityfiltration.lib\]](#), page 601).

**Usage:** `purityTriang(S)`, `S` matrix with entries of an Auslander regular ring `D`

**Return:** a matrix `T`

**Purpose:** compute a triangular block matrix `T`, such that  $M=D^p/D^q(S^t)$  is isomorphic to  $M'=D^{p'}/D^{q'}(T^t)$

**Example:**

```

LIB "purityfiltration.lib";
ring D = 0,(x1,x2,d1,d2),dp;
def S=Weyl();
setring S;
int i;
matrix R[3][3]=0,d2-d1,d2-d1,d2,-d1,-d1-d2,d1,-d1,-2*d1;
print(R);
↳ 0, -d1+d2,-d1+d2,
↳ d2,-d1, -d1-d2,

```

```

⇒ d1,-d1,    -2*d1
matrix T=purityTriang(transpose(R));
// a triangular blockmatrix representing the module coker(R)
print(T);
⇒ 0, 1,1,-1, 0,  0, 0,
⇒ -1,0,1,0,  -1, 0, 0,
⇒ 0, 0,0,-d1,-d2,-1,0,
⇒ 0, 0,0,-1, -1, 0, -1,
⇒ 0, 0,0,0,  0,  1, -d2,
⇒ 0, 0,0,0,  0,  1,  0,
⇒ 0, 0,0,0,  0,  0, d1

```

### 7.5.23.4 gradeNumber

Procedure from library `purityfiltration.lib` (see [Section 7.5.23 \[purityfiltration.lib\]](#), page 601).

**Usage:**     `gradeNumber(R)`,  $R$  matrix, representing  $M=D^p/D^q(R^t)$  over a ring  $D$

**Return:**    `int`, grade number of  $M$

**Purpose:**    computes the grade number of  $M$ , i.e. the first  $i$ , with  $\text{ext}^i(M,D) \neq 0$   
               returns -1 if  $M=0$

**Example:**

```

LIB "purityfiltration.lib";
// trivial example
ring D=0,(x,y,z),dp;
matrix R[2][1]=1,x;
gradeNumber(R);
⇒ 0
// R has left inverse, so M=D/D^2R=0
gradeNumber(transpose(R));
⇒ -1
print(ncExt_R(0,R));
⇒ 0
// so, ext^0(coker(R),D) != 0
//
// a little bit more complex
matrix R1[3][1]=x,-y,z;
gradeNumber(transpose(R1));
⇒ 3
print(ncExt_R(0,transpose(R1)));
⇒ 1
print(ncExt_R(1,transpose(R1)));
⇒ 1
print(ncExt_R(2,transpose(R1)));
⇒ 1,0,0,
⇒ 0,1,0,
⇒ 0,0,1
// ext^i are zero for i=0,1,2
matrix ext3=ncExt_R(3,transpose(R1));
print(ext3);
⇒ z,y,x
// not zero
is_zero(ext3);

```

$\mapsto 0$

### 7.5.23.5 showgrades

Procedure from library `purityfiltration.lib` (see [Section 7.5.23 \[purityfiltration.lib\]](#), page 601).

**Usage:** `showgrades(T)`, `T` list, which includes representation matrices of modules

**Return:** list, gradenumbers of the entries in `T`

**Purpose:** computes a list `L` with  $L[i] = \text{gradenumber}(M)$ ,  $M = D^p / D^q T[i]$

**Example:**

```
LIB "purityfiltration.lib";
ring D = 0,(x,y,z),dp;
matrix R[6][4]=
0,-2*x,z-2*y-x,-1,
0,z-2*x,2*y-3*x,1,
z,-6*x,-2*y-5*x,-1,
0,y-x,y-x,0,
y,-x,-y-x,0,
x,-x,-2*x,0;
list T=purityFiltration(transpose(R))[2];
showgrades(T);
 $\mapsto$  [1]:
 $\mapsto$  0
 $\mapsto$  [2]:
 $\mapsto$  -1
 $\mapsto$  [3]:
 $\mapsto$  2
 $\mapsto$  [4]:
 $\mapsto$  3
// T[i] are i-1 pure (i=1,3,4) or zero (i=2)
```

### 7.5.23.6 allExtOfLeft

Procedure from library `purityfiltration.lib` (see [Section 7.5.23 \[purityfiltration.lib\]](#), page 601).

**Usage:** `allExtOfLeft(M)`,

**Return:** list, entries are ext-modules

**Assume:** `M` presents a left module of finite left projective dimension `n`

**Purpose:** For a left module presented by `M` over the basering `D`, compute a list `T`, whose entry `T[i+1]` is a matrix, presenting the right module  $\text{Ext}^i_D(M, D)$  for  $i=0..n$

**Example:**

```
LIB "purityfiltration.lib";
ring D = 0,(x,y,z),dp;
matrix R[6][4]=
0,-2*x,z-2*y-x,-1,
0,z-2*x,2*y-3*x,1,
z,-6*x,-2*y-5*x,-1,
0,y-x,y-x,0,
y,-x,-y-x,0,
```

```

x,-x,-2*x,0;
// coker(R) consider the left module M=D^6/D^4R
list T=allExtOfLeft(transpose(R));
print(T[1]);
↳ 0
print(T[2]);
↳ 1,0,0,
↳ 0,1,0,
↳ 0,0,1
print(T[3]);
↳ 0,0, z,4y-z,4x,
↳ 0,-2,1,0, 1,
↳ 1,0, 0,0, 0,
↳ 0,1, 0,0, 0
print(T[4]);
↳ z,y,x
// right modules coker(T[i].)!!

```

### 7.5.23.7 allExtOfRight

Procedure from library `purityfiltration.lib` (see [Section 7.5.23 \[purityfiltration.lib\]](#), page 601).

**Usage:** `allExtOfRight(R)`,  $R$  matrix representing the right Module  $M=D^q/RD^p$  over a ring  $D$

$M$  module with finite right projective dimension  $n$

**Return:** list, entries are ext-modules

**Purpose:** computes a list  $T$ , which entries are representations of the left modules  $\text{ext}^i(M,D)$   
 $T[i]$  gives the repr. matrix of  $\text{ext}^{(i-1)}(M,D)$ ,  $i=1,\dots,n+1$

**Example:**

```

LIB "purityfiltration.lib";
ring D = 0,(x,y,z),dp;
matrix R[6][4]=
0,-2*x,z-2*y-x,-1,
0,z-2*x,2*y-3*x,1,
z,-6*x,-2*y-5*x,-1,
0,y-x,y-x,0,
y,-x,-y-x,0,
x,-x,-2*x,0;
// coker(R) considered as right module
projectiveDimension(R,1)[2];
↳ 1
list T=allExtOfRight(R);
print(T[1]);
↳ 4x,
↳ -4y,
↳ -z,
↳ z
print(T[2]);
↳ 1,0,0, 0,0, 0,
↳ 0,0,4y-z,0,4x-z,0,
↳ 0,z,0, y,0, x
// left modules coker(.T[i])!!

```

### 7.5.23.8 doubleExt

Procedure from library `purityfiltration.lib` (see [Section 7.5.23 \[purityfiltration.lib\]](#), page 601).

**Usage:** `doubleExt(R,i)`,  $R$  matrix representing the left Module  $M=D^p/D^q(R^t)$  over a ring  $D$   
 $i$ , less or equal the left projective dimension of  $M$

**Return:** matrix  $P$ , representing the double ext module

**Purpose:** computes a matrix  $P$ , which represents the left module  $\text{ext}^i(\text{ext}^i(M,D))$

**Example:**

```
LIB "purityfiltration.lib";
ring D = 0,(x,y,z),dp;
matrix R[7][3]=
0 ,0,1,
1 ,-4*x+z,-z,
-1,8*x-2*z,z,
1 ,0 ,0,
0 ,x-y,0,
0 ,x-y,y,
0 ,0 ,x;
// coker(R) is 2-pure, so all doubleExt are zero
print(doubleExt(transpose(R),0));
↪ 1
print(doubleExt(transpose(R),1));
↪ 1,0,0,
↪ 0,1,0,
↪ 0,0,1
print(doubleExt(transpose(R),3));
↪ 1
// except of the second
print(doubleExt(transpose(R),2));
↪ 4y-z,4x-z
```

### 7.5.23.9 allDoubleExt

Procedure from library `purityfiltration.lib` (see [Section 7.5.23 \[purityfiltration.lib\]](#), page 601).

**Usage:** `allDoubleExt(R)`,  $R$  matrix representing the left Module  $M=D^p/D^q(R^t)$  over a ring  $D$

**Return:** list  $T$ , double indexed, which include all double-ext modules

**Purpose:** computes all double ext-modules  
 $T[i][j]$  gives a representation matrix of  $\text{ext}^{(j-1)}(\text{ext}^{(i-1)}(M,D))$

**Example:**

```
LIB "purityfiltration.lib";
ring D = 0,(x1,x2,x3,d1,d2,d3),dp;
def S=Weyl();
setring S;
matrix R[6][4]=
0,-2*d1,d3-2*d2-d1,-1,
0,d3-2*d1,2*d2-3*d1,1,
```

```

d3,-6*d1,-2*d2-5*d1,-1,
0,d2-d1,d2-d1,0,
d2,-d1,-d2-d1,0,
d1,-d1,-2*d1,0;
list T=allDoubleExt(transpose(R));
// left projective dimension of M=coker(R) is 3
// ext^i(ext^0(M,D)), i=0,1,2,3
print(T[1][1]);
⇒ 0,
⇒ d1,
⇒ d3,
⇒ -d2
print(T[1][2]);
⇒ d3,d3,d2,d1
print(T[1][3]);
⇒ 1
print(T[1][4]);
⇒ 1
// ext^i(ext^1(M,D)), i=0,1,2,3
print(T[2][1]);
⇒ 1
print(T[2][2]);
⇒ 1,0,0,
⇒ 0,1,0,
⇒ 0,0,1
print(T[2][3]);
⇒ 0,0,0,4*d2-d3,4*d1-d3,
⇒ 1,0,0,0,      0,
⇒ 0,1,0,0,      0,
⇒ 0,0,1,0,      0
print(T[2][4]);
⇒ d3,d2,d1
// ext^i(ext^2(M,D)), i=0,1,2,3 (all zero)
print(T[3][1]);
⇒ 1
print(T[3][2]);
⇒ 1
print(T[3][3]);
⇒ 1
print(T[3][4]);
⇒ 1
// ext^i(ext^3(M,D)), i=0,1,2,3 (all zero)
print(T[4][1]);
⇒ 1
print(T[4][2]);
⇒ 1
print(T[4][3]);
⇒ 1
print(T[4][4]);
⇒ 1

```

### 7.5.23.10 is\_pure

Procedure from library `purityfiltration.lib` (see [Section 7.5.23 \[purityfiltration.lib\], page 601](#)).

**Usage:** `is_pure(R)`,  $R$  representing the module  $M=D^p/D^q(R^t)$

**Return:** `int`, 0 or 1

**Purpose:** checks pureness of  $M$ .  
 returns 1, if  $M$  is pure, or 0, if it's not  
 remark: if  $M$  is zero, `is_pure` returns 1

**Example:**

```
LIB "purityfiltration.lib";
ring D = 0,(x,y,z),dp;
matrix R[3][2]=y,-z,x,0,0,x;
list T=purityFiltration(transpose(R));
print(transpose(std(transpose(T[2][2]))));
  ↪ y,-z,
  ↪ x,0,
  ↪ 0,x
// so the purity filtration of coker(R) is trivial,
// i.e. coker(R) is already pure
is_pure(transpose(R));
  ↪ 1
// we can also have non-pure modules:
matrix R2[6][4]=
0,-2*x,z-2*y-x,-1,
0,z-2*x,2*y-3*x,1,
z,-6*x,-2*y-5*x,-1,
0,y-x,y-x,0,
y,-x,-y-x,0,
x,-x,-2*x,0;
is_pure(transpose(R2));
  ↪ 0
```

### 7.5.23.11 purelist

Procedure from library `purityfiltration.lib` (see [Section 7.5.23 \[purityfiltration.lib\], page 601](#)).

**Usage:** `purelist(T)`,  $T$  list, in which the  $i$ -th entry  $R=T[i]$  represents  $M=D^p/D^q(R^t)$

**Return:** list  $M$ , entries of  $M$  are 0 or 1

**Purpose:** if  $T[i]$  is pure,  $M[i]$  is 1, else  $M[i]$  is 0

**Example:**

```
LIB "purityfiltration.lib";
ring D = 0,(x,y,z),dp;
matrix R[6][4]=
0,-2*x,z-2*y-x,-1,
0,z-2*x,2*y-3*x,1,
z,-6*x,-2*y-5*x,-1,
0,y-x,y-x,0,
y,-x,-y-x,0,
x,-x,-2*x,0;
is_pure(transpose(R));
```

```

⇒ 0
// R is not pure, so we do the purity filtration
list T=purityFiltration(transpose(R));
// all Elements of T[2] are either zero or pure
purelist(T[2]);
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ 1
⇒ [4]:
⇒ 1

```

### 7.5.24 qmatrix\_lib

**Library:** qmatrix.lib

**Purpose:** Quantum matrices, quantum minors and symmetric groups

**Authors:** Lobillo, F.J., jlobillo@ugr.es  
 Rabelo, C., crabelo@ugr.es

**Support:** 'Metodos algebraicos y efectivos en grupos cuanticos', BFM2001-3141, MCYT, Jose Gomez-Torrecillas (Main researcher).

**Procedures:**

#### 7.5.24.1 quantMat

Procedure from library `qmatrix.lib` (see [Section 7.5.24 \[qmatrix\\_lib\]](#), page 610).

**Usage:** quantMat(n [, p]); n integer (n>1), p an optional integer

**Return:** ring (of quantum matrices). If p is specified, the quantum parameter q will be specialized at the p-th root of unity

**Purpose:** compute the quantum matrix ring of order n

**Note:** activate this ring with the "setring" command.  
 The usual representation of the variables in this quantum algebra is not used because double indexes are not allowed in the variables. Instead the variables are listed by reading the rows of the usual matrix representation, that is, there will be n\*n variables (one for each entry an n\*N generic matrix), listed row-wise

**Example:**

```

LIB "qmatrix.lib";
def r = quantMat(2); // generate O_q(M_2) at q generic
setring r; r;
⇒ // coefficients: QQ(q)
⇒ // number of vars : 4
⇒ //      block 1 : ordering Dp
⇒ //      : names y(1) y(2) y(3) y(4)
⇒ //      block 2 : ordering C
⇒ // noncommutative relations:

```



```

⇒ //      y(2)y(1)=1/(q)*y(1)*y(2)
⇒ //      y(3)y(1)=1/(q)*y(1)*y(3)
⇒ //      y(4)y(1)=y(1)*y(4)+(-q^2+1)/(q)*y(2)*y(3)
⇒ //      y(4)y(2)=1/(q)*y(2)*y(4)
⇒ //      y(4)y(3)=1/(q)*y(3)*y(4)
kill r;
def r = quantMat(2,5); // generate 0_q(M_2) at q^5=1
setring r;  r;
⇒ // coefficients: QQ[q]/(q^4+q^3+q^2+q+1)
⇒ // number of vars : 4
⇒ //          block 1 : ordering Dp
⇒ //          : names  y(1) y(2) y(3) y(4)
⇒ //          block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      y(2)y(1)=(-q^3-q^2-q-1)*y(1)*y(2)
⇒ //      y(3)y(1)=(-q^3-q^2-q-1)*y(1)*y(3)
⇒ //      y(4)y(1)=y(1)*y(4)+(-q^3-q^2-2*q-1)*y(2)*y(3)
⇒ //      y(4)y(2)=(-q^3-q^2-q-1)*y(2)*y(4)
⇒ //      y(4)y(3)=(-q^3-q^2-q-1)*y(3)*y(4)

```

See also: [Section 7.5.24.2 \[qminor\]](#), page 611.

### 7.5.24.2 qminor

Procedure from library `qmatrix.lib` (see [Section 7.5.24 \[qmatrix.lib\]](#), page 610).

**Usage:** `qminor(I,J,n)`; I,J intvec, n int

**Return:** poly, the quantum minor of a generic  $n \times n$  quantum matrix

**Assume:** I is the ordered list of the rows to consider in the minor,  
 J is the ordered list of the columns to consider in the minor,  
 I and J must have the same number of elements,  
 n is the order of the quantum matrix algebra you are working with (`quantMat(n)`).  
 The base ring should be constructed using `quantMat`.

**Example:**

```

LIB "qmatrix.lib";
def r = quantMat(3); // let r be a quantum matrix of order 3
setring r;
intvec u = 1,2;
intvec v = 2,3;
intvec w = 1,2,3;
qminor(w,w,3);
⇒ y(1)*y(5)*y(9)+(-q)*y(1)*y(6)*y(8)+(-q)*y(2)*y(4)*y(9)+(q^2)*y(2)*y(6)*y(\
  7)+(q^2)*y(3)*y(4)*y(8)+(-q^3)*y(3)*y(5)*y(7)
qminor(u,v,3);
⇒ y(2)*y(6)+(-q)*y(3)*y(5)
qminor(v,u,3);
⇒ y(4)*y(8)+(-q)*y(5)*y(7)
qminor(u,u,3);
⇒ y(1)*y(5)+(-q)*y(2)*y(4)

```

See also: [Section 7.5.24.1 \[quantMat\]](#), page 610.

### 7.5.24.3 SymGroup

Procedure from library `qmatrix.lib` (see [Section 7.5.24 \[qmatrix.lib\]](#), page 610).

**Usage:**      `SymGroup(n)`;  $n$  an integer (positive)

**Return:**     `intmat`

**Purpose:**     represent the symmetric group  $S(n)$  via integer vectors (permutations)

**Note:**       each row of the output integer matrix is an element of  $S(n)$

**Example:**

```
LIB "qmatrix.lib";
// "S(3)={(1,2,3),(1,3,2),(3,1,2),(2,1,3),(2,3,1),(3,2,1)}";
SymGroup(3);
↪ 1,2,3,
↪ 1,3,2,
↪ 3,1,2,
↪ 2,1,3,
↪ 2,3,1,
↪ 3,2,1
```

See also: [Section 7.5.24.5 \[LengthSym\]](#), page 612; [Section 7.5.24.4 \[LengthSymElement\]](#), page 612.

### 7.5.24.4 LengthSymElement

Procedure from library `qmatrix.lib` (see [Section 7.5.24 \[qmatrix.lib\]](#), page 610).

**Usage:**      `LengthSymElement(v)`;  $v$  `intvec`

**Return:**     `int`

**Purpose:**     determine the length of the permutation given by  $v$  in some  $S(n)$

**Assume:**     $v$  represents an element of  $S(n)$ ; otherwise the output may have no sense

**Example:**

```
LIB "qmatrix.lib";
intvec v=1,3,4,2,8,9,6,5,7,10;
LengthSymElement(v);
↪ 9
```

See also: [Section 7.5.24.5 \[LengthSym\]](#), page 612; [Section 7.5.24.3 \[SymGroup\]](#), page 612.

### 7.5.24.5 LengthSym

Procedure from library `qmatrix.lib` (see [Section 7.5.24 \[qmatrix.lib\]](#), page 610).

**Usage:**      `LengthSym(M)`;  $M$  an `intmat`

**Return:**     `intvec`

**Purpose:**     determine a vector, where the  $i$ -th element is the length of the permutation of  $S(n)$  given by the  $i$ -th row of  $M$

**Assume:**     $M$  represents a subset of  $S(n)$  (each row must be an element of  $S(n)$ ); otherwise, the output may have no sense

**Example:**

```

LIB "qmatrix.lib";
def M = SymGroup(3); M;
⇨ 1,2,3,
⇨ 1,3,2,
⇨ 3,1,2,
⇨ 2,1,3,
⇨ 2,3,1,
⇨ 3,2,1
LengthSym(M);
⇨ 0,1,2,1,2,3

```

See also: [Section 7.5.24.4 \[LengthSymElement\]](#), page 612; [Section 7.5.24.3 \[SymGroup\]](#), page 612.

## 7.5.25 ratgb\_lib

Status: experimental

**Library:** ratgb.lib

**Purpose:** Groebner bases in Ore localizations of noncommutative G-algebras

**Author:** Viktor Levandovskyy, levandov@risc.uni-linz.ac.at

**Overview:** Theory: Let  $A$  be an operator algebra with  $R = K[x_1, \dots, x_N]$  as subring. The operators are usually denoted by  $d_1, \dots, d_M$ .

Assume, that  $A$  is a  $G$ -algebra, then the set  $S=R-0$  is multiplicatively closed Ore set in  $A$ . That is, for any  $s$  in  $S$  and  $a$  in  $A$ , there exist  $t$  in  $S$  and  $b$  in  $A$ , such that  $sa=bt$ . In other words, one can transform any left fraction into a right fraction. The algebra  $A_S$  is called an Ore localization of  $A$  with respect to  $S$ .

This library provides Groebner basis procedure for  $A_S$ , performing polynomial (that is fraction-free) computations only. Note, that there is ongoing development of the subsystem called Singular:Locapal, which will provide yet another approach to Groebner bases over such Ore localizations.

Assumptions: in order to treat such localizations constructively, some care need to be taken. We will assume that the variables  $x_1, \dots, x_N$  from above (which will become invertible in the localization) come as the first block among the variables of the basering. Moreover, the ordering on the basering must be an antiblock ordering, that is its matrix form has the left upper  $N \times N$  block zero. Here is a recipe to create such an ordering easily: use 'a(w)' definitions of the ordering  $N$  times with intvecs  $w_i$  of the following form:  $w_i$  has first  $N$  components zero. The rest entries need to be positive and such, that  $w_1, \dots, w_N$  are linearly independent (see an example below).

Guide: with this library, it is possible

- to compute a Groebner basis of an ideal or a submodule in the 'rational' Ore localization  $D = A_S$
- to compute a dimension of associated graded submodule (called  $D$ -dimension) - to compute a vector space dimension over  $\text{Quot}(R)$  of a submodule of  $D$ -dimension 0 (so called  $D$ -finite submodule)
- to compute a basis over  $\text{Quot}(R)$  of a  $D$ -finite submodule

**Procedures:** See also: [Section D.11.3 \[jacobson.lib\]](#), page 1978; [Section 7.5.21 \[olga.lib\]](#), page 582.

### 7.5.25.1 ratstd

Procedure from library `ratgb.lib` (see [Section 7.5.25 \[ratgb.lib\]](#), page 613).

- Usage:** `ratstd(I, n [,eng]);` I an ideal/module, n an integer, eng an optional integer
- Return:** ring
- Purpose:** compute the Groebner basis of I in the Ore localization of the basering with respect to the subalgebra, generated by first n variables
- Assume:** the variables of basering are organized in two blocks and - the first block of length n contains the elements with respect to which one localizes, - the basering is equipped with anti-block ordering, giving block dominance for the variables in the second block
- Note:** the output ring C is commutative. The ideal `rGBid` in C represents the rational form of the output ideal `pGBid` in the basering. - During the computation, the D-dimension of I, `Ratgb::Ddim` and the corresponding dimension as  $K(x)$ -vector space of I (`Ratgb::KXdim`, if `Ratgb::Ddim=0`) are computed and exported. - Setting optional integer `eng` to 1, `std` is taken as Groebner engine; default is `slimgb`.
- Display:** In order to see the steps of the computation, set `printlevel` to  $\geq 2$

**Example:**

```
LIB "ratgb.lib";
ring r = (0,c),(x,y,Dx,Dy),(a(0,0,1,1),a(0,0,1,0),dp);
// this ordering is an antiblock ordering, as it must be
def S = Weyl(); setring S;
// the ideal I below annihilates parametric Appel F4 function
// where we set parameters to a=-2, b=-1 and d=0
ideal I =
x*Dx*(x*Dx+c-1) - x*(x*Dx+y*Dy-2)*(x*Dx+y*Dy-1),
y*Dy*(y*Dy-1) - y*(x*Dx+y*Dy-2)*(x*Dx+y*Dy-1);
int is = 2; // hence 1st and 2nd variables, that is x and y
// will become invertible in the localization
def A = ratstd(I,2); // main call
pGBid; // polynomial form of the basis in the localized ring
⇒ pGBid[1]=2*x*y*Dx*Dy+x*y*Dy^2+y^2*Dy^2-y*Dy^2+(-c-2)*x*Dx-2*y*Dy+2
⇒ pGBid[2]=x*Dx^2-y*Dy^2+(c)*Dx
⇒ pGBid[3]=2*x^2*y*Dy^3-4*x*y^2*Dy^3+2*y^3*Dy^3-4*x*y*Dy^3-4*y^2*Dy^3+2*y*D\
y^3+(-2*c)*x^2*Dx*Dy+(2*c)*x*Dx*Dy+2*x^2*Dy^2+(c-2)*x*y*Dy^2+(-3*c)*y^2*D\
y^2-4*x*Dy^2+(3*c-2)*y*Dy^2+2*Dy^2+(-c^2+2*c)*x*Dx+(6*c)*y*Dy+(-6*c)
setring A; // A is a commutative ring used for presentation
rGBid; // "rational" or "localized" form of the basis
⇒ rGBid[1]=(2*x*y)*Dx*Dy+(x*y+y^2-y)*Dy^2+(-c*x-2*x)*Dx+(-2*y)*Dy+2
⇒ rGBid[2]=(x)*Dx^2+(-y)*Dy^2+(c)*Dx
⇒ rGBid[3]=(2*x^2*y-4*x*y^2-4*x*y+2*y^3-4*y^2+2*y)*Dy^3+(-2*c*x^2+2*c*x)*Dx\
*Dy+(c*x*y-3*c*y^2+3*c*y+2*x^2-2*x*y-4*x-2*y+2)*Dy^2+(-c^2*x+2*c*x)*Dx+(6\
*c*y)*Dy+(-6*c)
Ratgb::Ddim; // the Krull-like dimension of A/I
⇒ 0
Ratgb::KXdim; // the dimension of A/I as a left K(x,y)-vector space
⇒ 4
//--- Now, let us compute a K(x,y) basis explicitly
print(matrix(kbase(rGBid)));
⇒ // ** rGBid is no standard basis
⇒ Dy^2,Dy,Dx,1
```

## 7.6 Graded commutative algebras (SCA)

This section describes basic mathematical notions, definition, and a little bit the implementation of the experimental non-commutative kernel extension SCA of SINGULAR which improves performance of many algorithms in graded commutative algebras.

In order to improve performance of SINGULAR in specific non-commutative algebras one can extend the internal implementation for them in a virtual-method-overloading-like manner. At the moment graded commutative algebras (SCA) and in particular exterior algebras are implemented this way. Note that graded commutative algebras require no special user actions apart from defining an appropriate non-commutative GR-algebra in SINGULAR. Upon doing that, the super-commutative structure will be automatically detected and special multiplication will be used. Moreover, in most SCA-aware (e.g. `std`) algorithms special internal improvements will be used (otherwise standard generic non-commutative implementations will be used).

All considered algebras are assumed to be associative  $K$ -algebras for some ground field  $K$ .

Definition

**Polynomial graded commutative algebras** are factors of tensor products of commutative algebras with an exterior algebra over a ground field  $K$ .

These algebras can be naturally endowed with a  $\mathbb{Z}/2\mathbb{Z}$ -grading, where anti-commutative algebra generators have degree 1 and commutative algebra generators (and naturally scalars) have degree 0. In this particular case they may be considered as super-commutative algebras.

GR-algebra representation

A graded commutative algebra with  $n$  commutative and  $m$  anti-commutative algebra generators can be represented as factors of the following GR-algebra by some two-sided ideal:

$$K \langle x_1, \dots, x_n; y_1, \dots, y_m \mid y_j * y_i = -y_i y_j, i < j \rangle / \langle y_1^2, \dots, y_m^2 \rangle.$$

Distinctive features

Graded commutative algebras are Noetherian.

Graded commutative algebras have zero divisors if and only if  $m > 0 : y_i * y_i = 0$ .

Unlike other non-commutative algebras one may use any monomial ordering where only the non-commutative variables are required to be global. In particular, commutative variables are allowed to be local. This means that one can work in tensor products of any commutative ring with an exterior algebra.

Example of defining graded commutative algebras in SINGULAR:SCA and computing with them

Given a commutative polynomial ring  $r$ , super-commutative structure on it can be introduced as follows:

```
LIB "nctools.lib";
ring r = 0,(a, b, x,y,z, Q, W),(lp(2), dp(3), Dp(2));
// Let us make variables x = var(3), ..., z = var(5) to be anti-commutative
// and add additionally a quotient ideal:
def S = superCommutative(3, 5, ideal(a*W + b*Q*x + z) ); setring S; S;
--> // coefficients: QQ
--> // number of vars : 7
--> //      block   1 : ordering lp
--> //                : names    a b
--> //      block   2 : ordering dp
--> //                : names    x y z
--> //      block   3 : ordering Dp
```

```

⇒ //                               : names    Q W
⇒ //          block    4 : ordering C
⇒ // noncommutative relations:
⇒ //      yx=-xy
⇒ //      zx=-xz
⇒ //      zy=-yz
⇒ // quotient ring from ideal
⇒ _[1]=xz
⇒ _[2]=bxyQ-yz
⇒ _[3]=aW+bxQ+z
⇒ _[4]=z2
⇒ _[5]=y2
⇒ _[6]=x2
ideal I = a*x*y + z*Q + b, y*Q + a; I;
⇒ I[1]=axy+b+zQ
⇒ I[2]=a+yQ
std(I); // Groebner basis is used here since > is global
⇒ _[1]=yQW-z
⇒ _[2]=yz
⇒ _[3]=b+zQ
⇒ _[4]=a+yQ
kill r, S;
// Let's do the same but this time with some local commutative variables:
ring r = 0,(a, b, x,y,z, Q, W),(dp(1), ds(1), lp(3), ds(2));
def S = superCommutative(3, 5, ideal(a*W + b*Q*x + z) ); setring S; S;
⇒ // coefficients: QQ
⇒ // number of vars : 7
⇒ //          block    1 : ordering dp
⇒ //                               : names    a
⇒ //          block    2 : ordering ds
⇒ //                               : names    b
⇒ //          block    3 : ordering lp
⇒ //                               : names    x y z
⇒ //          block    4 : ordering ds
⇒ //                               : names    Q W
⇒ //          block    5 : ordering C
⇒ // noncommutative relations:
⇒ //      yx=-xy
⇒ //      zx=-xz
⇒ //      zy=-yz
⇒ // quotient ring from ideal
⇒ _[1]=xz
⇒ _[2]=yz-bxyQ
⇒ _[3]=aW+z+bxQ
⇒ _[4]=x2
⇒ _[5]=y2
⇒ _[6]=z2
ideal I = a*x*y + z*Q + b, y*Q + a; I;
⇒ I[1]=axy+zQ+b
⇒ I[2]=a+yQ
std(I);
⇒ _[1]=yQW-z-bxQ
⇒ _[2]=zQ+b

```

```

↳ _[3]=bx
↳ _[4]=by
↳ _[5]=bz
↳ _[6]=b2
↳ _[7]=a+yQ

```

See example of [Section 7.5.20.9 \[superCommutative\]](#), page 569 from the library `nctools.lib`.

Reference: Ph.D thesis by Oleksandr Motsak (2010), <https://nbn-resolving.org/urn:nbn:de:hbz:386-kluedo-26479>.

## 7.7 LETTERPLACE

This section describes mathematical notions and definitions used in the LETTERPLACE subsystem of SINGULAR.

All algebras are assumed to be associative  $R$ -algebras for  $R$  being a field  $K$  or a ring  $Z$ .

### What is and what does LETTERPLACE?

What is LETTERPLACE? It is a subsystem of SINGULAR, providing the manipulations and computations within free associative algebras over rings  $R$

$\langle x_1, \dots, x_n \rangle$ , where the coefficient domain  $R$  is either a ring  $Z$  or a field, supported by SINGULAR.

LETTERPLACE can perform computations also in the factor-algebras of the above (via data type `qring`) by two-sided ideals.

Free algebras are internally represented in SINGULAR as so-called Letterplace rings.

Each such ring is constructed from a commutative ring  $R[x_1, \dots, x_n]$  and a **degree (length) bound**  $d$ .

This encodes a sub- $K$ -vector space (also called a filtered part) of  $K$

$\langle x_1, \dots, x_n \rangle$ , spanned by all monomials of **length** at most  $d$ . Analogously for free  $R$ -subbimodules of a free  $R$ -bimodule of a fixed rank.

Within such a construction we offer the computations of Groebner (also known as Groebner-Shirshov) bases, normal forms, syzygies and many more.

We address both two-sided ideals and subbimodules of the free bimodule of the fixed rank.

A variety of monomial and module orderings is supported, including **elimination** orderings for both variables and bimodule components. A monomial ordering has to be a well-ordering.

LETTERPLACE works with every field, supported by SINGULAR, and with the coefficient ring  $Z$ .

Note, that the elements of the coefficient field (or a ring) mutually commute with all variables.

### 7.7.1 Examples of use of LETTERPLACE

First, define a commutative ring  $K[X]$  in SINGULAR, equipped with a monomial well-ordering and call it, say, `r`.

Then, decide what should be the degree (length) bound  $d$ , that is how long may the words (monomials in the free algebra) become and run the procedure `freeAlgebra(r, d)`.

In the case you wish to work with subbimodules of the free bimodule of rank  $k$ , use `freeAlgebra(r, d, k)` instead of the previous.

The `freeAlgebra.` procedure creates free algebra  $K < X >$  resp. the free bimodule of rank  $k$  over  $K < X >$  subject to a monomial (module) ordering, corresponding to the one in the original commutative ring  $K[X]$ , see [Section 7.9.2 \[Monomial orderings on free algebras\]](#), page 637.

Polynomial (vector) arithmetics in this  $K$ -algebra is the usual one:  $+, -, *, ^$  while of course,  $x*y$  and  $y*x$  are different monomials while  $x*7=7*x$ .

Let us define an ideal  $I$  as a list of polynomials in the free algebra and run, for example, `twostd` (see [Section 7.8.14 \[twostd \(letterplace\)\]](#), page 635). The answer is a two-sided Groebner basis  $J$  of the two-sided ideal  $I$

up to the length bound  $d$ .

Then, we want to compute the following: 1. The two-sided normal form of `xyzy` with respect to  $J$  using the function `reduce` (see [Section 7.8.9 \[reduce \(letterplace\)\]](#), page 632). 2. By introducing a factor algebra  $K < x, y, z > / J$  of type `qring`, and demonstrate the functions `reduce` and `rightstd` (for right Groebner bases) over the factor algebra. 3. By creating the free R-bimodule of rank 8, we demonstrate how `embeddins` works with `imap` and also, how to express a subbimodule (or a single element) in terms of bimodule generators with `lift`. In other words, we compute and compare presentations of a polynomials with respect to the original generating set of ideal and with respect to a Groebner basis. 4. In the same free R-bimodule we will compute the module of bisyzygies of  $J$  and do some syzygy tests. 5. We demonstrate the bimodule membership problem: a boolean answer via NF and the certified version (with a Groebner presentation) via `lift`. 6. We show how elimination of module components works for bimodules.

We illustrate the approach with the following example:

```
//***** Part 1 *****/
LIB "freegb.lib";
ring r = 0,(x,y,z),dp; // the ordering on the free algebra will be degree right lex
ring R = freeAlgebra(r, 5); // 5 the is degree (length) bound;
ideal I = x*y + y*z, x*x + x*y - z; // define an ideal via the set of polynomials
ideal J = twostd(I);
J; // as we see, with respect to the current ordering this Groebner basis
⇨ J[1]=x*y+y*z
⇨ J[2]=x*x-y*z-z
⇨ J[3]=y*z*y-y*z*z+z*y
⇨ J[4]=y*z*x+y*z*z+z*x-x*z
⇨ J[5]=y*z*z*y-y*z*z*z-x*z*y
⇨ J[6]=y*z*z*x+y*z*z*z-x*z*x+y*z*z+z*z
⇨ J[7]=y*z*z*z*y-y*z*z*z*z+y*z*z*z+x*z*y+z*z*y
⇨ J[8]=y*z*z*z*x+y*z*z*z*z+x*z*x+z*z*x-x*z*z-y*z*z-z*z
// tends to be infinite. Increasing the bound and recomputing helps to check it.
poly p = reduce(x*y*z*y,J);
p; // since p!=0, x*y*z*y is not contained in J up to length 5
⇨ -y*z*z*z-x*z*y
// however this does not imply a definite answer on whether p is in J
poly q = x*(y+1)*z*y-x*y*z^2;
reduce(q, J); // 0, thus q is in J
⇨ 0
//***** Part 2 *****/
qring Q = J; // J is a Groebner basis, computed above
poly p = reduce(x*x, twostd(0)); // the canonical representative of x*x in Q
p;
```



```

↳ y*z+z
rightstd(ideal(p)); // right Groebner basis of the right ideal, generated by p in Q
↳ _[1]=z*z
↳ _[2]=y*z+z
↳ _[3]=x*z
//***** Part 3 *****//
setring r;
ring R8 = freeAlgebra(r, 5, 8); // 5 the is length bound; 8 is the rank of the free
ideal J = imap(R, J); // we map J identically from R (of rank 1)
J = twostd(J);
poly q = imap(R, q);
NF(q, J); // NF is an alias to reduce, we have rechecked that q is in J
↳ 0
matrix L = lift(J, q); // creates the presentation for q in terms of J
// since J is a Groebner basis, this is a Groebner presentation of q
print(transpose(matrix(L))); // J has 8 generators and these are the needed coefficients
↳ ncgen(1)*z*y-ncgen(1)*z*z,0,0,0,-ncgen(5),0,0,0
// here, the generators of the free bimodule are ncgen(1)*gen(1), ... , ncgen(8)*gen(1)
// the output means, that substituting ncgen(i) by the i-th generator of J, we get q
J[1]*z*y - J[1]*z*z - J[5] - q; // 0, so this is the sought expression of q
↳ 0
testLift(J,L); // recovers q from the lift matrix
↳ _[1]=x*y*z*y-x*y*z*z+x*z*y
// Let us compare now this nice Groebner presentation with the one
// obtained from the original set of generators
ideal I = imap(R,I); // note: I is not a Groebner basis of itself
matrix M = lift(I, q); // creates the presentation for q in terms of I
M; // presentation is longer and more complicated than the one in L
↳ M[1,1]=-ncgen(1)*x*y+x*ncgen(1)*y-ncgen(1)*y*z-x*ncgen(1)*z+y*z*ncgen(1)+\
z*ncgen(1)
↳ M[2,1]=ncgen(2)*x*y-x*ncgen(2)*y+ncgen(2)*y*z
testLift(I,M); // a routine test to ensure that indeed we recover q
↳ _[1]=x*y*z*y-x*y*z*z+x*z*y
//***** Part 4 *****//
// Let us compute the module of biszygies of J and analyze it
module S = syz(J); size(S); // 18
↳ 18
S[6]; // consider, for example, this element
↳ ncgen(1)*z*y*gen(1)-ncgen(1)*z*z*gen(1)+y*z*ncgen(1)*gen(1)-ncgen(4)*y*ge\
n(4)-ncgen(3)*z*gen(3)+z*ncgen(1)*gen(1)-x*ncgen(3)*gen(3)
// plugging the i-th generator of J instead of ncgen(i), we obtain a biszygy:
J[1]*z*y - J[1]*z*z - x*J[3] - J[5]; //0
↳ 0
module S2 = S[6..8]; // pick just three generators
testSyz(J,S2); // tests the biszygy property for the generators
↳ _[1]=0
↳ _[2]=0
↳ _[3]=0
//***** Part 5 *****//
option(redSB); option(redTail); // to compute minimal and tail-reduced bases
module GS = twostd(S); size(GS); // 30
↳ 30
// let us construct a vector, belonging to GS:

```

```

vector v = GS[11]*y - x*GS[7] + z*GS[3]*z;
print(v);
↪ [-x*ncgen(1)*x*y-x*ncgen(1)*y*z+x*x*y*ncgen(1)-x*z*ncgen(1),y*z*z*ncgen(2\
   )*y+z*y*z*ncgen(2)*z-x*z*ncgen(2)*y+z*z*ncgen(2)*z,z*ncgen(3)*z*z+x*ncgen\
   (3)*z,-z*ncgen(4)*x*z+x*ncgen(4)*y,ncgen(5)*z*y-x*ncgen(5),-ncgen(6)*x*y+\
   z*ncgen(6)*z+ncgen(6)*y,0,ncgen(8)*y]
NF(v, GS); // 0, by the construction
↪ 0
// now we wish to compute the expression of v via GS
ring r3 = 0,(x,y,z),(c,dp);
ring R30 = freeAlgebra(r3,5,30);
module GS = imap(R8,GS);
vector v = imap(R8,v);
matrix L = lift(GS,v); // via printing we see only three components involved:
L[3,1]; // =z*ncgen(3)*z, as well as
↪ z*ncgen(3)*z
L[7,1]; // =-x*ncgen(7) and
↪ -x*ncgen(7)
L[11,1]; // =ncgen(11)*y
↪ ncgen(11)*y
//***** Part 6 *****/
// Notice, that the module ordering is (c,dp): it is a position-over-term ordering
// which eliminates module components in an descending way.
GS = GS[1..5]; // consider just first five syzygies, for a smaller example
GS = twostd(GS); // a nice finite Groebner basis
print(matrix(GS)); // shows the structure
↪ 0,      0,      0,      0,      0,      _[1,6],  _[1,7],  _[1,8],
↪ 0,      _[2,2],  _[2,3], _[2,4], _[2,5], _[2,6],  _[2,7],  0,
↪ _[3,1],  _[3,2],  _[3,3], _[3,4], _[3,5], -ncgen(3), 0,      _[3,8],
↪ 0,      _[4,2],  _[4,3], _[4,4], _[4,5], -ncgen(4), -ncgen(4), _[4,8],
↪ ncgen(5), 0,      0,      _[5,4], _[5,5], 0,      0,      ncgen(5),
↪ 0,      ncgen(6), 0,      _[6,4], _[6,5], 0,      0,      0,
↪ ncgen(7), 0,      0,      0,      _[7,5], 0,      0,      0
// As we can see, intersections of the subbimodule GS with the free bimodules
// generated by all but first resp. all but first two bimodule generators
// are not empty and given by vectors having zero in the first resp.
// in the first two components.

```

See [Section 7.8 \[Functions \(letterplace\)\]](#), page 625 for the list of all available kernel functions.

There are various conversion routines in the library `freegb_lib` (see [Section 7.10.4 \[freegb\\_lib\]](#), page 666). Many algebras are predefined in the library `fpalgebras_lib` (see [Section 7.10.2 \[fpalgebras\\_lib\]](#), page 646). Important ring-theoretic properties can be established with the help of the library `fpaprops_lib` (see [Section 7.10.3 \[fpaprops\\_lib\]](#), page 660), while K-dimension and monomial bases and Hilbert data - with the help of the library `fpadim_lib` (see [Section 7.10.1 \[fpadim\\_lib\]](#), page 640). We work further on implementing more algorithms for non-commutative ideals and modules over free associative algebra.

### 7.7.2 Example of use of LETTERPLACE over $\mathbb{Z}$

Consider the following paradigmatic example:

```

LIB "freegb.lib";
ring r = integer,(x,y),Dp;
ring R = freeAlgebra(r,5); // length bound is 5

```

```

ideal I = 2*x, 3*y;
I = twostd(I);
print(matrix(I)); // pretty prints the generators
↳ 3*y, 2*x, y*x, x*y

```

As we can see, over  $Z \langle x, y \rangle$  the ideal  $\langle 2x, 3y \rangle$  has a finite Groebner basis and indeed

$$Z \langle x, y \rangle / \langle 2x, 3y \rangle =$$

$$Z \langle x, y \rangle / \langle 2x, 3y, yx, xy \rangle =$$

$$Z \langle x, y \rangle / \langle 2x, 3y, yx - xy, xy \rangle$$

and the later is naturally isomorphic to

$$Z[x, y] / \langle 2x, 3y, xy \rangle \text{ as a } Z \text{-algebra.}$$

Now, we analyze the same ideal in the ring with one more variable  $z$  :

```

LIB "freegb.lib";
ring r = integer, (x, y, z), Dp;
ring R = freeAlgebra(r, 5); // length bound is 5
ideal I = 2*x, 3*y;
I = twostd(I);
print(matrix(I)); // pretty prints the generators
↳ 3*y, 2*x, y*x, x*y, y*z*x, x*z*y, y*z*z*x, x*z*z*y, y*z*z*z*x, x*z*z*z*y

```

Now we see, that this Groebner basis is potentially infinite and the following argument delivers a proof. Namely,  $yz^i x$  and

$xz^i y$  are present in the ideal for all  $i \geq 0$ . How can we do this? We wish to express  $y * z^i * x$  and  $x * z^i * y$  via the original generators by means of lift:

```

LIB "freegb.lib";
ring r = integer, (x, y, z), Dp;
ring R = freeAlgebra(r, 5, 2); // length bound is 5, rank of the free bimodule is 2
ideal I = 2*x, 3*y;
matrix T1 = lift(I, ideal(y*z*x, x*z*y));
print(T1);
↳ -y*z*ncgen(1), -ncgen(1)*z*y,
↳ ncgen(2)*z*x, x*z*ncgen(2)
-y*z*I[1] + I[2]*z*x; // gives y*z*x
↳ y*z*x
matrix T2 = lift(I, ideal(y*z^2*x, x*z^2*y));
print(T2);
↳ -y*z*z*ncgen(1), -ncgen(1)*z*z*y,
↳ ncgen(2)*z*z*x, x*z*z*ncgen(2)
-y*z^2*I[1] + I[2]*z^2*x; // gives y*z^2*x
↳ y*z*z*x

```

The columns of matrices, returned by lift, encode the presentation of new elements in terms of generators. From this we conjecture, that in particular

$$-yz^i * (2x) + (3y) * z^i x = yz^i x \text{ holds for all } i \geq 0$$

and indeed, confirm it via a routine computation by hands.

### Comparing computations over $\mathbb{Q}$ with computations over $\mathbb{Z}$ .

In the next example, we first compute over the field of rationals  $\mathbb{Q}$  and a bit later compare the result with computations over the ring of integers  $\mathbb{Z}$ .

```

LIB "freegb.lib"; // initialization of free algebras
ring r = 0, (z, y, x), Dp; // degree left lex ord on z>y>x
ring R = freeAlgebra(r, 7); // length bound is 7

```

```

ideal I = y*x - 3*x*y - 3*z, z*x - 2*x*z + y, z*y-y*z-x;
option(redSB); option(redTail); // for minimal reduced GB
option(intStrategy); // avoid divisions by coefficients
ideal J = twostd(I); // compute a two-sided GB of I
J; // prints generators of J
⇒ J[1]=4*x*y+3*z
⇒ J[2]=3*x*z-y
⇒ J[3]=4*y*x-3*z
⇒ J[4]=2*y*y-3*x*x
⇒ J[5]=2*y*z+x
⇒ J[6]=3*z*x+y
⇒ J[7]=2*z*y-x
⇒ J[8]=3*z*z-2*x*x
⇒ J[9]=4*x*x*x+x
LIB "fpadim.lib"; // load the library for K-dimensions
lpMonomialBasis(7,0,J); // all monomials of length up to 7 in  $Q\langle x,y,z \rangle/J$ 
⇒ _[1]=1
⇒ _[2]=z
⇒ _[3]=y
⇒ _[4]=x
⇒ _[5]=x*x

```

As we see, we obtain a nice finite Groebner basis  $J$ . Moreover, from the form of its leading monomials, we conjecture that

$Q\langle x,y,z \rangle/J$  is finite dimensional  $Q$ -vector space. We check it with `lpMonomialBasis` and obtain an affirmative answer.

Now, for doing similar computations over  $Z$  one needs to change only the initialization of the ring, the rest stays the same

```

LIB "freegb.lib"; // initialization of free algebras
ring r = integer,(z,y,x),Dp; // Z and deg left lex ord on  $z>y>x$ 
ring R = freeAlgebra(r,7); // length bound is 7
ideal I = y*x - 3*x*y - 3*z, z*x - 2*x*z + y, z*y-y*z-x;
option(redSB); option(redTail); // for minimal reduced GB
option(intStrategy); // avoid divisions by coefficients
ideal J = twostd(I); // compute a two-sided GB of I
J; // prints generators of J
⇒ J[1]=12*x*y+9*z
⇒ J[2]=9*x*z-3*y
⇒ J[3]=y*x-3*x*y-3*z
⇒ J[4]=6*y*y-9*x*x
⇒ J[5]=6*y*z+3*x
⇒ J[6]=z*x-2*x*z+y
⇒ J[7]=z*y-y*z-x
⇒ J[8]=3*z*z+2*y*y-5*x*x
⇒ J[9]=6*x*x*x-3*y*z
⇒ J[10]=4*x*x*y+3*x*z
⇒ J[11]=3*x*x*z+3*x*y+3*z
⇒ J[12]=2*x*y*y+75*x*x*x+39*y*z+39*x
⇒ J[13]=3*x*y*z-3*y*y+6*x*x
⇒ J[14]=2*y*y*y+x*x*y+3*x*z
⇒ J[15]=2*x*x*x*x+y*y-x*x
⇒ J[16]=2*x*x*x*y+3*y*y*z+3*x*y+3*z
⇒ J[17]=x*x*y*z+x*y*y-x*x*x

```

```

⇒ J[18]=x*y*y*z-y*y*y+x*x*y
⇒ J[19]=x*x*x*x*x+y*y*y*z+x*x*x
⇒ J[20]=x*x*x*x*z+x*x*x*y+2*y*y*z+x*x*z+3*x*y+3*z
⇒ J[21]=x*y*y*y*z-y*y*y*y+x*x*x*x-y*y+x*x
⇒ J[22]=y*y*y*z*z-x*x*x*x*y
⇒ J[23]=x*y*y*y*y*z-y*y*y*y*y+x*x*y*y*y
⇒ J[24]=x*y*y*y*y*y*z-y*y*y*y*y*y+x*x*x*x*y*y+y*y*y*y+x*x*x*x+2*y*y-2*x*x

```

The output has plenty of elements in each degree (which is the same as length because of the degree ordering), what hints at potentially infinite Groebner basis.

Indeed, one can show that for every  $i \geq 2$  the ideal  $J$  contains an element with the leading monomial  $xy^iz$ .

### 7.7.3 Functionality and release notes of LETTERPLACE

Over free associative algebras over fields or over a ring  $Z$ , one can perform many different computations with arbitrary two-sided ideals. It is possible to define a free bimodule of a fixed finite rank and also work with subbimodules of such. Groebner bases and related tools are thoroughly implemented, with respect to a variety of monomial module orderings.

The variables can be weighted by nonnegative weights, which are determined by the monomial ordering.

Restrictions/conventions of the LETTERPLACE subsystem:

Since free algebra is not Noetherian, one has to work with explicitly fixed degree (length) bound, up to which a partial Groebner basis will be computed. The initialization routine `freeAlgebra (letterplace)` constructs the ring with this bound. For increasing the length bound one needs to define another ring and to use `imap` for mapping the objects back and forth.

All the computations happen up to the length bound, which is explicitly fixed during the definition of the current ring.

The options `redSB`, `redTail` are effective for computations involving Groebner bases,

The options `prot`, `mem` are effective for the whole LETTERPLACE subsystem.

For monomial orderings, which are not compatible with the length, the following error message might appear: `degree bound of Letterplace ring is 11, but at least 12 is needed for this multiplication`. In such a situation, activating `option(redSB)`, `option(redTail)` and increasing the length (degree) bound might help. Though there are situations, where nothing leads to a finite computation simply while the nature of non-Noetherian rings is so.

Operations for polynomials in Letterplace rings are the usual ones: `+` (addition), `-` (subtraction), `*` (multiplication) and `^` (power).

The functions [Section 7.3.2 \[bracket\], page 336](#), [Section 5.1.88 \[maxideal\], page 219](#) and [Section 5.1.151 \[std\], page 271](#) (an alias for [Section 7.8.14 \[twostd \(letterplace\)\], page 635](#)) also work within letterplace rings:

```

LIB "freegb.lib";
ring r = 0,(x,y,z),dp; // the ordering will be degree right lex
ring R = freeAlgebra(r, 5); // degree (length) bound is 5
// maxideal in a letterplace ring:
print(matrix(maxideal(2))); // all monomials of length 2
⇒ x*x,y*x,z*x,x*y,y*y,z*y,x*z,y*z,z*z
// bracket in a letterplace ring:
bracket(x,y);

```

```

 $\mapsto -y*x+x*y$ 
poly f = x*x + x*y - z;
bracket(f,x);
 $\mapsto x*y*x-x*x*y-z*x+x*z$ 
bracket(f,x,2); // left-normed iterated bracket [f,[f,x]]
 $\mapsto -x*y*x*x*x+x*x*y*x*x+x*x*x*y*x+x*y*x*y*x-x*x*x*x*y-2*x*y*x*x*y+x*x*y*x*y+\backslash$ 
 $z*x*x*x-x*z*x*x-z*x*y*x-x*x*z*x-x*y*z*x+2*z*x*x*y-x*z*x*y+x*x*x*z+2*x*y*x\backslash$ 
 $*z-x*x*y*z+z*z*x-2*z*x*z+x*z*z$ 

```

Further functionality is provided in the libraries for the LETTERPLACE subsystem: see [Section 7.10 \[LETTERPLACE libraries\]](#), page 640 for details.

In the [Section 7.10.4 \[freegb.lib\]](#), page 666 one finds e.g. Letterplace initialization together with legacy, conversion and convenience tools.

The [Section 7.10.1 \[fpadim.lib\]](#), page 640 contains procedures for computations with vector space basis of a factor algebra including finiteness check and dimension computation.

The [Section 7.10.3 \[fpaprops.lib\]](#), page 660 contains procedures for determining important ring-theoretic properties including Gelfand-Kirillov dimension.

The [Section 7.10.2 \[fpalgebras.lib\]](#), page 646 contains procedures for the generation of various algebras, including group algebras of finitely presented groups in the Letterplace ring.

The [Section 7.5.12 \[ncfactor.lib\]](#), page 487 contains the procedure `ncfactor` for factorizing polynomials in the Letterplace ring.

See [Section 7.3.2 \[bracket\]](#), page 336; [Section 5.1.88 \[maxideal\]](#), page 219; [Section 7.8.9 \[reduce \(letterplace\)\]](#), page 632; [Section 7.8.10 \[rightstd \(letterplace\)\]](#), page 633; [Section 7.8.11 \[std \(letterplace\)\]](#), page 633; [Section 7.8.14 \[twostd \(letterplace\)\]](#), page 635.

## 7.7.4 References and history of LETTERPLACE

LETTERPLACE has undergone several stages of development.

The first one, the pure Letterplace implementation for homogeneous ideals, was created by V. Levandovskyy and H. Schoenemann in 2007-2009.

Later in 2010-2014, experiments with advanced (among other, with shift-invariant) data structures were performed by V. Levandovskyy, B. Schnitzler and G. Studzinski, and new libraries for  $K$ -dimension,  $K$ -bases, and Ufnarovskij graph were written.

The next stage started in 2017, when K. Abou Zeid joined the team of H. Schoenemann and V. Levandovskyy. Those recent activities led to the change of interface to the one, usual in the free algebra. The Letterplace data structure is still at heart of the implementation, though not explicitly visible by default. It has been generalized to support  $Z$  as coefficient ring (together with T. Metzlauff (RWTH Aachen and INRIA Sophia Antipolis)); to support bimodules and compute syzygies and lifts, to name a few. We are grateful to L. Schmitz (RWTH Aachen) for his contributions to the development.

References:

[LL09]: Roberto La Scala and Viktor Levandovskyy, "Letterplace ideals and non-commutative Groebner bases", *Journal of Symbolic Computation*, Volume 44, Issue 10, October 2009, Pages 1374-1393, see <http://dx.doi.org/10.1016/j.jsc.2009.03.002>.

[LL13]: Roberto La Scala and Viktor Levandovskyy, "Skew polynomial rings, Groebner bases and the letterplace embedding of the free associative algebra", *Journal of Symbolic Computation*, Volume 48, Issue 1, January 2013, Pages 1374-1393, see <http://dx.doi.org/10.1016/j.jsc.2012.05.003> and also <http://arxiv.org/abs/1009.4152>.

[LSS13]: Viktor Levandovskyy, Grisha Studzinski and Benjamin Schnitzler, "Enhanced Computations of Groebner Bases in Free Algebras as a New Application of the Letterplace Paradigm", Proc. ISSAC 2013, ACM Press, 259-266, see <https://doi.org/10.1145/2465506.2465948>.

[L14]: Roberto La Scala, "Extended letterplace correspondence for nongraded noncommutative ideals and related algorithms", International Journal of Algebra and Computation, Volume 24, Number 08, Pages 1157-1182, 2014, see also <https://doi.org/10.1142/S0218196714500519>.

[Mora16]: Teo Mora, "Solving Polynomial Equation Systems IV: Volume 4, Buchberger Theory and Beyond.", Cambridge University Press, 2016.

[LMZ20]: Viktor Levandovskyy, Tobias Metzlaß and Karim Abou Zeid, "Computation of free non-commutative Groebner Bases over  $\mathbb{Z}$  with SINGULAR:LETTERPLACE", Proc. ISSAC 2020, Pages 312-319, ACM Press (2020), <https://dl.acm.org/doi/10.1145/3373207.3404052>. Video of the talk is at <https://av.tib.eu/media/50124>.

[LSZ20]: Viktor Levandovskyy, Hans Schoenemann and Karim Abou Zeid, "LETTERPLACE - a Subsystem of SINGULAR for computations with free algebras via Letterplace Embedding", Proc. ISSAC 2020, 305-311, ACM Press, <https://dl.acm.org/doi/10.1145/3373207.3404056>. Video of the talk is at <https://av.tib.eu/media/50123>.

[SL20]: Leonard Schmitz and Viktor Levandovskyy : Formally Verifying Proofs for Algebraic Identities of Matrices . In: Intelligent Computer Mathematics (Proceedings of the CICM 2020), Pages 222-236, Springer LNAI, LNCS (2020).

## 7.8 Functions (letterplace)

This chapter gives a complete reference of all functions and commands of the LETTERPLACE kernel, i.e. all built-in commands (for the numerous LETTERPLACE libraries see [Section 7.10 \[LETTERPLACE libraries\]](#), [page 640](#)).

The general syntax of a function is

```
[target =] function_name (<arguments>);
```

Note, that both **Control structures** and **System variables** of LETTERPLACE are the same as of SINGULAR (see [Section 5.2 \[Control structures\]](#), [page 290](#), [Section 5.3 \[System variables\]](#), [page 302](#)).

### 7.8.1 dim (letterplace)

**Syntax:** `dim( ideal_expression)`

**Type:** `int`

**Purpose:** Compute the Gelfand-Kirillov dimension of the algebra basering/(input ideal). Uses Ufnarovskij graph for computations.

**Note:** the input ideal must be given as a two-sided Groebner basis.

**Example:**

```
LIB "freegb.lib";
ring r = 0,(x,y,z),dp;
ring R = freeAlgebra(r, 5);
ideal I = z;
dim(twostd(I)); // GK dimension is infinite
↦ -1
I = x,y,z;
dim(twostd(I));
↦ 0
```



```

I = x*y, x*z, z*y, z*z;
dim(twostd(I));
↪ 2
I = y*x - x*y, z*x - x*z, z*y - y*z;
I = twostd(I); I;
↪ I[1]=z*y-y*z
↪ I[2]=z*x-x*z
↪ I[3]=y*x-x*y
dim(I); // 3, as expected for R/I = K[x,y,z]
↪ 3

```

See [Section 7.10.1 \[fpadim\\_lib\]](#), page 640.

## 7.8.2 fetch (letterplace)

**Syntax:**    `fetch ( ring_name, name )`  
               `fetch ( ring_name, name, intvec_expression )`

**Type:**     number, poly, vector, ideal, module, matrix or list (the same type as the second argument)

**Purpose:**    maps objects between rings. `fetch` is the identity map between rings and q rings, in the first case the *i*-th variable of the source ring is mapped to the *i*-th variable of the basering. If the basering has less variables than the source ring these variables are mapped to zero. The intvec in the 3rd argument describes the permutation of the variables: an *i* at position *j* maps the variable `var(j)` of the source to the variable `var(i)` of the destination.

A zero means that that variable/parameter is mapped to 0.

The coefficient fields must be compatible. (See [Section 4.12 \[map\]](#), page 106 for a description of possible mappings between different ground fields).

`fetch` offers a convenient way to change variable names or orderings, or to map objects from a ring to a factor ring of that ring or vice versa.

`option(Imap)`; reports the mapping.

**Note:**      Compared with `imap`, `fetch` uses the position of the ring variables, not their names.

**Example:**

```

LIB "freegb.lib";
ring r = (0,a),(x,y,z),dp;
ring R = freeAlgebra(r,4,2); // free bimodule of rank 2
poly p = z^2/a - a*y;
ideal I = x,y,z,a*z*y*x - x*y + 7;
module M = (x*y*a +3)*ncgen(1)*gen(1), ncgen(2)*gen(2)*z, ncgen(2)*gen(2)*z;
M; // note that a stands on the left
↪ M[1]=(a)*x*y*ncgen(1)*gen(1)+3*ncgen(1)*gen(1)
↪ M[2]=ncgen(2)*z*gen(2)
↪ M[3]=(a)*ncgen(2)*x*y*gen(2)-7*ncgen(2)*gen(2)
ring r2 = 0,(a,z,y,x),dp; // note: a is a variable in r2
ring R2 = freeAlgebra(r2,6,2);
fetch(R,p); // correctly processes incorrect input
↪ // ** Not defined: Cannot map a rational fraction and make a polynomial
    ut of it! Ignoring the denominator.
↪ y*y
fetch(R,I);

```



```

↳ _[1]=a
↳ _[2]=z
↳ _[3]=y
↳ _[4]=-a*z+7
  fetch(R,M);
↳ _[1]=3*ncgen(1)*gen(1)
↳ _[2]=ncgen(2)*y*gen(2)
↳ _[3]=-7*ncgen(2)*gen(2)
  setring R; // now we show the factor ring behavior
  ideal J = y*x-x*y,z; J = twostd(J); J;
↳ J[1]=z
↳ J[2]=y*x-x*y
  qring Q = J;
  fetch(R,p);
↳ 1/(a)*z*z+(-a)*y
  NF(_, twostd(0)); // the canonical representative in Q
↳ (-a)*y
  fetch(R,I);
↳ _[1]=x
↳ _[2]=y
↳ _[3]=z
↳ _[4]=(a)*z*y*x-x*y+7
  NF(_, twostd(0)); // the canonical representative in Q
↳ _[1]=x
↳ _[2]=y
↳ _[3]=0
↳ _[4]=-x*y+7
  fetch(R,M);
↳ _[1]=(a)*x*y*ncgen(1)*gen(1)+3*ncgen(1)*gen(1)
↳ _[2]=ncgen(2)*z*gen(2)
↳ _[3]=(a)*ncgen(2)*x*y*gen(2)-7*ncgen(2)*gen(2)
  NF(_, twostd(0)); // the canonical representative in Q
↳ _[1]=(a)*x*y*ncgen(1)*gen(1)+3*ncgen(1)*gen(1)
↳ _[2]=0
↳ _[3]=(a)*ncgen(2)*x*y*gen(2)-7*ncgen(2)*gen(2)

```

See [Section 7.8.4 \[imap \(letterplace\)\]](#), page 628; [Section 4.12 \[map\]](#), page 106; [Section 4.20.1 \[qring\]](#), page 127; [Section 4.20 \[ring\]](#), page 127.

### 7.8.3 freeAlgebra (letterplace)

**Syntax:**

```
freeAlgebra( ring_expression r, int_expression d )
```

**Type:** ring

**Purpose:** Creates a free (letterplace) ring with the variables of the ring `r` up to the degree (length) bound `d`, with the monomial ordering, determined by those on the ring `r`.

**Note:** A letterplace ring has an attribute called `isLetterplaceRing`, which is zero for non-letterplace rings and contains the number of variables of the free algebra it encodes, otherwise.

**Example:**

```

LIB "freegb.lib";
ring r = 0,(x,y,z),dp;
ring R = freeAlgebra(r, 7); // this ordering is degree right lex
R;
⇨ // coefficients: QQ
⇨ // number of vars : 21
⇨ //      block 1 : ordering dp
⇨ //      : names  x y z x y z x y z x y z x y z x y z x y z
⇨ //      block 2 : ordering C
⇨ // letterplace ring (block size 3, ncgen count 0)
attrib(R,"isLetterplaceRing");
⇨ 3
ring r2 = 0,(x,y,z),lp;
ring R2 = freeAlgebra(r2, 5); // note, that this ordering is NOT left or r
R2;
⇨ // coefficients: QQ
⇨ // number of vars : 15
⇨ //      block 1 : ordering a
⇨ //      : names  x y z x y z x y z x y z x y z
⇨ //      : weights 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0
⇨ //      block 2 : ordering a
⇨ //      : names  x y z x y z x y z x y z x y z
⇨ //      : weights 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0
⇨ //      block 3 : ordering a
⇨ //      : names  x y z x y z x y z x y z x y z
⇨ //      : weights 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1
⇨ //      block 4 : ordering lp
⇨ //      : names  x y z x y z x y z x y z x y z
⇨ //      block 5 : ordering C
⇨ // letterplace ring (block size 3, ncgen count 0)
attrib(R2,"isLetterplaceRing");
⇨ 3

```

See [Section 7.9.2 \[Monomial orderings on free algebras\]](#), page 637.

### 7.8.4 imap (letterplace)

**Syntax:** `imap ( ring_name, name )`

**Type:** number, poly, vector, ideal, module, matrix or list (the same type as the second argument)

**Purpose:** identity map on common subrings. `imap` is the map between rings and q rings with compatible ground fields which is the identity on variables and parameters of the same name and 0 otherwise. (See [Section 4.12 \[map\]](#), page 106 for a description of possible mappings between different ground fields). Useful for embeddings as well as for mappings from/to rings with/without parameters. Compared with `fetch`, `imap` uses the names of variables and parameters. Unlike `map` and `fetch`, `imap` can map parameters to variables, by forgetting the commutativity of parameters with each other and with variables.

Mapping rational functions which are not polynomials to polynomials is undefined.

**Example:**

```
ring r = (0,a),(x,y,z),dp;
```

```

LIB "freegb.lib";
ring R = freeAlgebra(r,4,2); // free bimodule of rank 2
poly p = z^2/a - a*y;
ideal I = x,y,z,a*z*y*x - x*y + 7;
module M = (x*y*a + 3)*ncgen(1)*gen(1), ncgen(2)*gen(2)*z, ncgen(2)*gen(2)*
M; // note that a stands on the left
↳ M[1]=(a)*x*y*ncgen(1)*gen(1)+3*ncgen(1)*gen(1)
↳ M[2]=ncgen(2)*z*gen(2)
↳ M[3]=(a)*ncgen(2)*x*y*gen(2)-7*ncgen(2)*gen(2)
ring r2 = 0,(a,z,y,x),dp; // note: a is a variable in r2
ring R2 = freeAlgebra(r2,6,2);
imap(R,p); // correctly processes incorrect input
↳ // ** Not defined: Cannot map a rational fraction and make a polynomial
  ut of it! Ignoring the denominator.
↳ z*z-a*y
  imap(R,I);
↳ _[1]=x
↳ _[2]=y
↳ _[3]=z
↳ _[4]=a*z*y*x-x*y+7
  imap(R,M);
↳ _[1]=a*x*y*ncgen(1)*gen(1)+3*ncgen(1)*gen(1)
↳ _[2]=ncgen(2)*z*gen(2)
↳ _[3]=a*ncgen(2)*x*y*gen(2)-7*ncgen(2)*gen(2)

```

See [Section 7.8.2 \[fetch \(letterplace\)\]](#), page 626; [Section 4.12 \[map\]](#), page 106; [Section 4.20.1 \[qring\]](#), page 127; [Section 4.20 \[ring\]](#), page 127.

### 7.8.5 lift (letterplace)

**Syntax:** lift ( ideal\_expression, subideal\_expression )  
 lift ( module\_expression, submodule\_expression )

**Type:** matrix

**Purpose:** computes the transformation matrix which expresses the generators of a subbimodule in terms of the generators of a bimodule.  
 More precisely, if *m* is the module (or ideal), *sm* the submodule (or ideal), and *T* the transformation matrix returned by lift, then the substitution of each *ncgen(i)* in *T* by the *m[i]* delivers a matrix, say *N*. The *i*-th generator of *sm* is equal to the sum of elements in the *i*-th column of *N*.

**Note:** Gives a warning if *sm* is not a submodule.

**Note:** The procedure [Section 7.10.4.12 \[testLift\]](#), page 672 can be used for testing the result.

**Example:**

```

LIB "freegb.lib";
ring r = 0,(x,y),(c,Dp);
ring R = freeAlgebra(r, 7, 2);
ideal I = std(x*y*x + 1);
print(matrix(I));
↳ x*y-y*x,y*x*x+1
ideal SI = x*I[1]*y + y*x*I[2], I[1]*y*x + I[2]*y;
matrix T = lift(I, SI);

```

```

print(T);
↪ y*ncgen(1)*x*x+x*ncgen(1)*y,y*x*ncgen(1)+y*ncgen(1)*x+ncgen(1)*y*x,
↪ y*ncgen(2)*x, y*ncgen(2)
print(matrix(SI)); // the original generators
↪ y*x*y*x*x+x*x*y*y-x*y*x*y+y*x,x*y*y*x+y*x*x*y-y*x*y*x+y
print(matrix(testLift(I,T))); // test for the result of lift
↪ y*x*y*x*x+x*x*y*y-x*y*x*y+y*x,x*y*y*x+y*x*x*y-y*x*y*x+y

```

See [Section 4.6 \[ideal\]](#), page 80; [Section 7.8.6 \[liftstd \(letterplace\)\]](#), page 630; [Section 7.8.13 \[syz \(letterplace\)\]](#), page 634; [Section 7.8.14 \[twostd \(letterplace\)\]](#), page 635.

## 7.8.6 liftstd (letterplace)

**Syntax:** liftstd ( ideal\_expression, matrix\_name )  
liftstd ( module\_expression, matrix\_name )  
liftstd ( ideal\_expression, matrix\_name, module\_name )  
liftstd ( module\_expression, matrix\_name, module\_name )

**Type:** ideal or module

**Purpose:** returns a Groebner basis of a two-sided ideal or a bimodule and the transformation matrix from the given ideal, resp. module, to the Groebner basis from the output. That is, if *m* is the module (or ideal), *sm* the submodule (or ideal), and *T* the transformation matrix returned by lift, then the substitution of each *ncgen(i)* in *T* by the *m[i]* delivers a matrix, say *N*. The *i*-th generator of *sm* is equal to the sum of elements in the *i*-th column of *N*.  
In an optional third argument the syzygy bimodule will be returned.

**Example:**

```

LIB "freegb.lib";
ring r = 0,(x,y),(c,Dp);
ring R = freeAlgebra(r, 8, 2);
ideal I = x*y*x + 1;
matrix T; module S;
ideal SI = liftstd(I,T,S);
print(matrix(SI));
↪ x*y-y*x,y*x*x+1
print(matrix(testLift(I,T))); // test for the result of lift
↪ x*y-y*x,y*x*x+1
S; // the bisyzygy module of I
↪ S[1]=[x*y*ncgen(1)*x*y+y*x*x*y*ncgen(1)-y*x*ncgen(1)*y*x-ncgen(1)*y*x*x
+y*ncgen(1)-ncgen(1)*y]
↪ S[2]=[x*x*y*ncgen(1)*x-x*ncgen(1)*y*x*x-x*ncgen(1)+ncgen(1)*x]
↪ S[3]=[x*y*ncgen(1)*x*x*y+y*x*x*x*y*ncgen(1)-y*x*x*ncgen(1)*y*x-ncgen(1)
*x*x*x*y+x*y*ncgen(1)+y*x*ncgen(1)-ncgen(1)*x*y-ncgen(1)*y*x]
↪ S[4]=[x*y*x*y*ncgen(1)*x-ncgen(1)*y*x*y*x*x+y*ncgen(1)*x-ncgen(1)*y*x]
testSyz(I,S);
↪ _[1]=0
↪ _[2]=0
↪ _[3]=0
↪ _[4]=0

```

See [Section 4.6 \[ideal\]](#), page 80; [Section 7.8.5 \[lift \(letterplace\)\]](#), page 629; [Section 7.8.13 \[syz \(letterplace\)\]](#), page 634; [Section 7.8.14 \[twostd \(letterplace\)\]](#), page 635.

### 7.8.7 modulo (letterplace)

**Syntax:**

**Syntax:**    modulo ( ideal\_expression, ideal\_expression )  
              modulo ( module\_expression, module\_expression )

**Type:**       module

**Purpose:**    computes the kernel of the bimodule homomorphism from the free bimodule (determined in basering) to its factor-bimodule modulo the second argument. The first argument determines the homomorphism via images of the canonical free bimodule generators.

If option(returnSB) is set, a Groebner basis is returned, otherwise a generating set.

**Example:**

```
LIB "freegb.lib";
ring r = 0,(x,y,z),dp;
ring R = freeAlgebra(r,7,2); // free bimodule of rank 2
ideal I = x*y*z - z*y*x;
I = twostd(I); I;
↪ I[1]=z*y*x-x*y*z
modulo(y,twostd(0)); // shows the canonical generator of the kernel
↪ _[1]=ncgen(1)*y*gen(1)-y*ncgen(1)*gen(1)
// which can be interpreted as (1 otimes y - y otimes 1)
module M = modulo(y, I);
print(M); // as we see (z E y - y E z) generates the kernel
↪ ncgen(1)*y-y*ncgen(1),z*ncgen(1)*x-x*ncgen(1)*z
// of bimodule homomorphism sending E to y
```

See [Section 4.6 \[ideal\]](#), page 80; [Section 7.8.5 \[lift \(letterplace\)\]](#), page 629; [Section 7.8.6 \[liftstd \(letterplace\)\]](#), page 630; [Section 4.14 \[module\]](#), page 112; [Section 7.8.8 \[ncgen\]](#), page 631; [Section 5.1.111 \[option\]](#), page 234; [Section 7.8.13 \[syz \(letterplace\)\]](#), page 634.

### 7.8.8 ncgen

**Syntax:**    ncgen ( int\_expression )

**Type:**       poly

**Purpose:**    returns the i-th free non-commutative generator of a free bimodule.

**Note:**       ncgen in bimodules is used together with the commutative gen, which encodes the component or the position in a vector.

**Example:**

```
LIB"freegb.lib";
ring r= 0,(x,y),dp;
ring R = freeAlgebra(r,4,3); // R supports free bimodule up to rank 3
typeof(ncgen(2));
↪ poly
vector v = [ncgen(1)*x, x*ncgen(2), y*ncgen(3)*x];
v;
↪ y*ncgen(3)*x*gen(3)+ncgen(1)*x*gen(1)+x*ncgen(2)*gen(2)
print(v*x);
↪ [ncgen(1)*x*x,x*ncgen(2)*x,y*ncgen(3)*x*x]
```

```

print(x*v);
↦ [x*ncgen(1)*x,x*x*ncgen(2),x*y*ncgen(3)*x]

```

See [Section 7.8.3 \[freeAlgebra \(letterplace\)\]](#), page 627; [Section 4.14 \[module\]](#), page 112; [Section 4.23 \[vector\]](#), page 134.

### 7.8.9 reduce (letterplace)

**Syntax:**    `reduce ( poly_expression, ideal_expression )`  
               `reduce ( poly_expression, ideal_expression, int_expression )`  
               `reduce ( vector_expression, ideal_expression )`  
               `reduce ( vector_expression, ideal_expression, int_expression )`  
               `reduce ( vector_expression, module_expression, int_expression )`  
               `reduce ( ideal_expression, ideal_expression )`  
               `reduce ( ideal_expression, ideal_expression, int_expression )`

**Type:**     the type of the first argument

**Purpose:**    reduces a polynomial, vector, or ideal (the first argument) to its **two-sided** normal form with respect to the second argument, meant to be an ideal, represented by its two-sided Groebner basis (otherwise, the result may have no meaning).  
 returns 0 if and only if the polynomial (resp. vector, ideal) is an element (resp. subideal) of the ideal.

The third (optional) argument of type `int` modifies the behavior:

0 default

1 consider only the leading term and do no tail reduction.

2 tail reduction: in the local/mixed ordering case: reduce also with bad ecart

4 reduce without division, return possibly a non-zero constant multiple of the remainder

**Note:**     The commands `reduce` and `NF` are synonymous.

**Note:**     A two-sided Groebner presentation of a polynomial with respect to a two-sided ideal can be computed by the procedure [Section 7.10.4.5 \[lpDivision\]](#), page 667 from [Section 7.10.4 \[freegb\\_lib\]](#), page 666.

**Example:**

```

LIB "freegb.lib";
ring r = 0,(x,y),dp;
ring R = freeAlgebra(r,5);
ideal I = x*x + y*y - 1; // 2D sphere
ideal J = twostd(I); // computes a two-sided Groebner basis
J; // it is finite and nice
↦ J[1]=x*x+y*y-1
↦ J[2]=y*y*x-x*y*y
poly g = x*y*y - y*y*x;
reduce(g,J); // 0, hence g belongs to J
↦ 0
poly h = x*y*y*x - y*x*x;
reduce(h,J); // the rest of two-sided division of h by J
↦ -y*y*y*y+y*y*y+y*y-y
qring Q = J; // switch to K<x,y>/J
reduce(x*y*y - y*y*x,twostd(0)); //image of g above
↦ 0

```

```

reduce(x*y*y*x - y*x*x,std(0)); //image of h above
↦ -y*y*y*y+y*y*y+y*y-y

```

See also [Section 7.8.5 \[lift \(letterplace\)\]](#), page 629.

### 7.8.10 rightstd (letterplace)

**Syntax:** `rightstd( ideal_expression); rightstd( module_expression);`

**Type:** ideal or module

**Purpose:** Compute a right Groebner basis of the set of generators of the input ideal/module.

**Note:** It is also effective in factor rings.

**Example:**

```

LIB "freegb.lib";
ring r = 0,(x,z),dp;
ring R = freeAlgebra(r,7);
ideal I = z, x*z, x*x*z;
rightstd(I); // a right GB of I in K<x,z>
↦ _[1]=z
↦ _[2]=x*z
↦ _[3]=x*x*z
qring Q = twostd(x*z); // now we change to the factor algebra modulo x*z
ideal I = imap(R,I);
rightstd(I); // a right GB in a factor algebra
↦ _[1]=z
reduce(I,twostd(0)); // an explanation for the latter
↦ _[1]=z
↦ _[2]=0
↦ _[3]=0

```

### 7.8.11 std (letterplace)

**Syntax:** `std( ideal_expression); std( module_expression);`

**Type:** ideal or module

**Purpose:** Alias to [Section 7.8.14 \[twostd \(letterplace\)\]](#), page 635.

### 7.8.12 subst (letterplace)

**Syntax:** `subst ( poly_expression, variable, poly_expression )`  
`subst ( poly_expression, variable, poly_expression , ... variable, poly_expression )`  
`subst ( vector_expression, variable, poly_expression )`  
`subst ( ideal_expression, variable, poly_expression )`  
`subst ( module_expression, variable, poly_expression )`

**Type:** poly, vector, ideal or module (corresponding to the first argument)

**Purpose:** substitutes one or more ring variable(s)/parameter variable(s) by (a) polynomial(s). Note that in the case of more than one substitution pair, the substitutions will be performed sequentially and not simultaneously. The below examples illustrate this behaviour.

Note, that the coefficients must be polynomial when substituting a parameter.

**Note:** When dealing with free non-commutative bimodules, their generators `ncgen(i)` can be used as variables in `subst` and therefore substituted in the corresponding vector component `gen(i)`.

**Example:**

```
LIB "freelib.lib";
ring r = 0,(x,y,z),dp;
ring R = freeAlgebra(r,5,2);
poly p = z^2 - y*x;
subst(p, x, -y);
↪ y*y+z*z
ideal I = z*y*x - x*y*z;
subst(I, y, p);
↪ _[1]=-z*y*x*x+z*z*z*x+x*y*x*z-x*z*z*z
subst(I, x, z); // produces zero
↪ _[1]=0
module M = I*ncgen(1)*gen(1), ncgen(1)*gen(1)*I, I*ncgen(2)*gen(2);
M;
↪ M[1]=z*y*x*ncgen(1)*gen(1)-x*y*z*ncgen(1)*gen(1)
↪ M[2]=ncgen(1)*z*y*x*gen(1)-ncgen(1)*x*y*z*gen(1)
↪ M[3]=z*y*x*ncgen(2)*gen(2)-x*y*z*ncgen(2)*gen(2)
subst(M, x, z); // produces zero
↪ _[1]=0
↪ _[2]=0
↪ _[3]=0
subst(M, ncgen(2), z); // evaluates ncgen(2) at z, see the 2nd component
↪ _[1]=z*y*x*ncgen(1)*gen(1)-x*y*z*ncgen(1)*gen(1)
↪ _[2]=ncgen(1)*z*y*x*gen(1)-ncgen(1)*x*y*z*gen(1)
↪ _[3]=z*y*x*z*gen(2)-x*y*z*z*gen(2)
```

See [Section 4.6 \[ideal\]](#), page 80; [Section 4.12 \[map\]](#), page 106; [Section 4.14 \[module\]](#), page 112.

### 7.8.13 syz (letterplace)

**Syntax:** `syz ( ideal_expression )`  
`syz ( module_expression )`

**Type:** module

**Purpose:** computes the first syzygy (i.e., the module of relations of the given generators) bimodule of the ideal, resp. module.  
 If `option(returnSB)` is set, a Groebner basis is returned, otherwise a generating set.

**Example:**

```
LIB "freelib.lib";
ring r = 0,(x,y),(c,Dp);
ring R = freeAlgebra(r, 7, 2);
ideal I = twostd(x*y*x + 1);
I;
↪ I[1]=x*y-y*x
↪ I[2]=y*x*x+1
module S = syz(I);
print(S);
↪ ncgen(1)*x*x,S[1,2],S[1,3],S[1,4],S[1,5],
↪ S[2,1], S[2,2],S[2,3],S[2,4],S[2,5]
```



```

testSyz(I,S);
↳ _[1]=0
↳ _[2]=0
↳ _[3]=0
↳ _[4]=0
↳ _[5]=0

```

See [Section 4.6 \[ideal\]](#), page 80; [Section 7.8.5 \[lift \(letterplace\)\]](#), page 629; [Section 7.8.6 \[liftstd \(letterplace\)\]](#), page 630; [Section 4.14 \[module\]](#), page 112; [Section 7.8.8 \[ncgen\]](#), page 631; [Section 5.1.111 \[option\]](#), page 234.

### 7.8.14 twostd (letterplace)

**Syntax:** `twostd( ideal_expression); twostd( module_expression);`

**Type:** `ideal`

**Purpose:** returns a two-sided Groebner basis of the two-sided ideal, generated by the input, which is treated as a set of two-sided generators.

**Example:**

```

LIB "freegb.lib";
ring r = 3,(x,d),dp; // notice: we work over Z/3Z
ring R = freeAlgebra(r,5);
ideal I = x^4, d^3, d*x - x*d - 1;
twostd(I); // a proper ideal, note x^3 as a generator
↳ _[1]=d*x-x*d-1
↳ _[2]=d*d*d
↳ _[3]=x*x*x
ideal J = x^2, d^3, d*x - x*d - 1;
twostd(J); // the whole ring
↳ _[1]=1
ideal T = twostd(ideal(d*x - x*d - 1));
T;
↳ T[1]=d*x-x*d-1
qring Q = T; // thus Q is the Weyl algebra over Z/3z
ideal I = x^4, d^3;
twostd(I);
↳ _[1]=d*d*d
↳ _[2]=x*x*x
ideal J = x^2, d^3;
twostd(J);
↳ _[1]=1

```

See [Section 4.6 \[ideal\]](#), page 80; [Section 7.8.5 \[lift \(letterplace\)\]](#), page 629; [Section 7.8.6 \[liftstd \(letterplace\)\]](#), page 630; [Section 4.14 \[module\]](#), page 112; [Section 7.8.8 \[ncgen\]](#), page 631; [Section 5.1.111 \[option\]](#), page 234; [Section 7.8.10 \[rightstd \(letterplace\)\]](#), page 633; [Section 7.8.13 \[syz \(letterplace\)\]](#), page 634.

### 7.8.15 vdim (letterplace)

**Syntax:** `vdim ( ideal_expression )`

**Type:** `int`

**Purpose:** computes the vector space dimension respective to the ground field of the ring modulo the ideal, generated by the leading terms of the given generators. If the generators form a standard basis, this is the same as the vector space dimension of the ring, resp. free module, modulo the ideal, resp. module.

If the ideal is not finite dimensional over the ground field, -1 is returned.

The non-commutative analog of the [Section 5.1.69 \[kbase\]](#), page 205 command is [Section 7.10.1.4 \[lpMonomialBasis\]](#), page 643 from [Section 7.10.1 \[fpadim.lib\]](#), page 640.

**Example:**

```
LIB "fpadim.lib";
ring r = 0,(x,y),dp;
ring R = freeAlgebra(r,5);
ideal I = x*x + x, y*y+y, x*y*x + x;
ideal G = twostd(I); G;
↳ G[1]=y*y+y
↳ G[2]=x*x+x
↳ G[3]=x*y*x+x
vdim(G); // 6
↳ 6
lpMonomialBasis(5,0,G); // lists the K-basis explicitly
↳ _[1]=1
↳ _[2]=x
↳ _[3]=y
↳ _[4]=y*x
↳ _[5]=x*y
↳ _[6]=y*x*y
```

See [Section 7.8.1 \[dim \(letterplace\)\]](#), page 625; [Section 7.10.1 \[fpadim.lib\]](#), page 640; [Section 7.8.14 \[twostd \(letterplace\)\]](#), page 635.

## 7.9 Mathematical background (letterplace)

### 7.9.1 Free associative algebras

Let  $V$  be a  $K$ -vector space, spanned by the symbols  $x_1, \dots, x_n$ . A free associative algebra in  $x_1, \dots, x_n$  over  $K$ , denoted by  $K$

$\langle x_1, \dots, x_n \rangle$

is also known as the tensor algebra  $T(V)$  of  $V$ ; it is also the monoid  $K$ -algebra of the free monoid  $\langle x_1, \dots, x_n \rangle$ . The elements of this free monoid constitute an infinite  $K$ -basis of  $K$

$\langle x_1, \dots, x_n \rangle$ , where the identity element (the empty word) of the free monoid is identified with the 1 in  $K$ . Yet in other words, the monomials of  $K$

$\langle x_1, \dots, x_n \rangle$  are the words of finite length in the finite alphabet  $\{x_1, \dots, x_n\}$ .

The algebra  $K$

$\langle x_1, \dots, x_n \rangle$  is an integral domain, which is not (left, right, weak or two-sided) Noetherian for  $n > 1$ ; hence, a Groebner basis of a finitely generated ideal might be infinite. Therefore, a general computation takes place **up to an explicit degree (length) bound**, provided by the user. The free associative algebra can be regarded as a graded algebra in a natural way.

**Definition.** An associative algebra  $A$  is called **finitely presented (f.p.)**, if it is isomorphic to  $K$

$\langle x_1, \dots, x_n \rangle / I$ , where  $I$  is a two-sided ideal.

$A$  is called **standard finitely presented (s.f.p.)**, if there exists a monomial ordering, such that  $I$  is given via its **finite** Groebner basis  $G$ .

### 7.9.2 Monomial orderings on free algebras

We provide many types of orderings for non-commutative Groebner bases up to a degree (length) bound. In general it is not clear, whether a given generating set has a finite Groebner bases with respect to some ordering.

Let  $X = \{x_1, \dots, x_n\}$  be a set of symbols. A total ordering  $<$  on the free monoid  $\langle X \rangle$  with 1 as the neutral element is called a **monomial ordering** if

- it is a well-ordering, i.e., every non empty subset has a least element with respect to  $<$ , and
- it is compatible with multiplication, that is  $u < v$  implies  $aub < avb$  for all  $u, v, a$  and  $b$  in  $\langle X \rangle$ .

Note that the latter implies  $1 \leq m$  for all  $m$  in  $\langle X \rangle$ .

The **left lexicographical ordering** on  $\langle X \rangle$  with  $x_1 > x_2 > \dots > x_n$  is defined as follows: For arbitrary  $a, b$  in  $\langle X \rangle$  we say that  $a < b$ , if

- $\exists u \in \langle X \rangle \setminus \{1\} : au = b$  or
- $\exists u, v, w \in \langle X \rangle \exists x_i, x_j \in X : a = ux_i v, b = ux_j w$  and  $i < j$  holds.

**Note:** left lex is **not** a monomial ordering, though it is a natural choice to break ties after, say, comparing elements by the total degree.

In a similar manner one can define the **right lexicographical ordering**.

On the monoid  $(N_0, +)$  define the **weight** homomorphism  $w : \langle X \rangle \rightarrow N_0$ , uniquely determined by  $w(x_i) = w_i$  in  $N_0$  for  $1 \leq i \leq n$ .

As a special case, define the **length**  $\text{len} : \langle X \rangle \rightarrow N_0$  by  $\text{len}(x_i) = 1$  for  $1 \leq i \leq n$ .

For any ordering  $<<$  on  $\langle X \rangle$  and any weight  $w : \langle X \rangle \rightarrow N_0$  define an ordering  $<$ , called the  **$w$ -weight extension of  $<<$**  as follows: For arbitrary  $a, b$  in  $\langle X \rangle$  we say that  $a < b$  if

- $w(a) < w(b)$  or
- $w(a) = w(b)$  and  $a << b$  holds.

An ordering  $<$  on  $\langle X \rangle$  **eliminates** a certain subset  $\emptyset \neq Y \subset X$  if for all  $f \in K\langle X \rangle \setminus \{0\}$  one has  $lm(f) \in K\langle X \setminus Y \rangle \Rightarrow f \in K\langle X \setminus Y \rangle \subseteq K\langle X \rangle$ .

In a ring declaration, LETTERPLACE supports the following monomial orderings.

We illustrate each of the available choices by an example on the free monoid  $\langle x_1, x_2, x_3 \rangle$ , where we order the monomials

$x_1 x_1 x_1, x_3 x_2 x_1, x_1 x_2 x_3, x_3 x_3 x_3, x_3 x_1, x_2 x_2, x_1 x_3, x_2 x_3, x_1, x_2$  and  $x_3$  correspondingly.

**‘dp’** The **degree right lexicographical ordering** is the length-weight extension of the right lexicographical ordering.

With respect to the ordering ‘dp’, the test monomials are ordered as follows:

$x_1 x_1 x_1 > x_3 x_2 x_1 > x_1 x_2 x_3 > x_3 x_3 x_3 > x_3 x_1 > x_2 x_2 > x_1 x_3 > x_2 x_3 > x_1 > x_2 > x_3$

**‘Dp’** The **degree left lexicographical ordering** is the length-weight extension of the left lexicographical ordering.

With respect to the ordering ‘Dp’, the test monomials are ordered as follows:

$x_1 x_1 x_1 > x_1 x_2 x_3 > x_3 x_2 x_1 > x_3 x_3 x_3 > x_1 x_3 > x_2 x_2 > x_2 x_3 > x_3 x_1 > x_1 > x_2 > x_3$

‘Wp(w) for intvec w’

The **weighted degree left lexicographical ordering** is the  $w$ -weight extension of the left lexicographical ordering with weight  $w : \langle X \rangle \rightarrow N_0$  uniquely determined by strict positive  $w(x_i) = w_i > 0$ .

With respect to the ordering ‘Wp(1, 2, 1)’, the test monomials are ordered as follows:

$$x_1x_2x_3 > x_2x_2 > x_3x_2x_1 > x_1x_1x_1 > x_2x_3 > x_3x_3x_3 > x_1x_3 > x_2 > x_3x_1 > x_1 > x_3$$

‘lp’

Let  $w^{(i)}$  be weights uniquely determined by  $w^{(i)}(x_j) = \delta_{i,j}$  for  $1 \leq i, j \leq n$  where  $\delta$  denotes the Kronecker delta. Let  $<_n$  be the  $w^{(n)}$ -weight extension of the left lexicographical ordering on  $\langle X \rangle$  and inductively  $<_i$  be the  $w^{(i)}$ -weight extension of  $<_{i+1}$  for all  $1 \leq i < n$ . The monomial ordering lp corresponds to  $<_1$  and eliminates  $x_1, \dots, x_j$  for all  $1 \leq j < n$ . We refer to it as to **left elimination ordering**.

The monomial ordering ‘lp’ corresponds to  $<_1$  and eliminates  $\{x_1, \dots, x_j\}$  for all  $1 \leq j < n$ . We refer to it as to **left elimination ordering**.

With respect to the ordering ‘lp’, the test monomials are ordered as follows:

$$x_1x_1x_1 > x_1x_2x_3 > x_3x_2x_1 > x_1x_3 > x_3x_1 > x_1 > x_2x_2 > x_2x_3 > x_2 > x_3x_3x_3 > x_3$$

‘rp’

Let  $w^{(i)}$  be weights uniquely determined by  $w^{(i)}(x_j) = \delta_{i,j}$  for  $1 \leq i, j \leq n$  where  $\delta$  denotes the Kronecker delta. Let  $<_1$  be the  $w^{(1)}$ -weight extension of the left lexicographical ordering on  $\langle X \rangle$  and inductively  $<_i$  be the  $w^{(i)}$ -weight extension of  $<_{i-1}$  for all  $1 < i \leq n$ . The monomial ordering rp corresponds to  $<_n$  and eliminates  $\{x_j, \dots, x_n\}$  for all  $1 < j \leq n$ . We refer to it as to **right elimination ordering**.

The monomial ordering ‘rp’ corresponds to  $<_n$  and eliminates  $\{x_j, \dots, x_n\}$  for all  $1 < j \leq n$ . We refer to it as to **right elimination ordering**.

With respect to the ordering ‘rp’, the test monomials are ordered as follows:

$$x_3x_3x_3 > x_1x_2x_3 > x_3x_2x_1 > x_2x_3 > x_1x_3 > x_3x_1 > x_3 > x_2x_2 > x_2 > x_1x_1x_1 > x_1$$

‘(a(v), ordering) for intvec v’

For weight  $v : \langle X \rangle \rightarrow N_0$  determined by  $v(x_i) = v_i \in N_0$  with  $1 \leq i \leq n$  and monomial ordering  $\prec$  on  $\langle X \rangle$ , the  $v$ -weight extension of  $\prec$  corresponds to  $(a(v), o)$ . As a choice for  $\prec$  there are currently two options implemented, which are dp and Dp. Notice that this ordering eliminates  $\{x_i \in X \mid v(x_i) \neq 0\}$ .

With respect to the ordering ‘(a(1, 0, 0), Dp)’, the test monomials are ordered as follows:

$$x_1x_1x_1 > x_1x_2x_3 > x_3x_2x_1 > x_1x_3 > x_3x_1 > x_1 > x_3x_3x_3 > x_2x_2 > x_2x_3 > x_2 > x_3$$

With ordering ‘(a(1, 1, 0), Dp)’ one obtains:

$$x_1x_1x_1 > x_1x_2x_3 > x_3x_2x_1 > x_2x_2 > x_1x_3 > x_2x_3 > x_3x_1 > x_1 > x_2 > x_3x_3x_3 > x_3$$

The examples are generated by the following code but with customized orderings denoted above.

```
LIB "freegb.lib";
ring r = 0, (x1,x2,x3), Dp; // variate ordering here
ring R = freeAlgebra(r, 4);
poly wr = x1*x1*x1+x3*x3*x3+x1*x2*x3+x3*x2*x1+x2*x2+x2*x3+x1*x3+x3*x1+x1+x2+x3;
wr; // polynomial will be automatically ordered according to the ordering on R
↦ x1*x1*x1+x1*x2*x3+x3*x2*x1+x3*x3*x3+x1*x3+x2*x2+x2*x3+x3*x1+x1+x2+x3
```

### 7.9.3 Groebner bases for two-sided ideals in free associative algebras

We say that a monomial  $v$  divides (two-sided or bilaterally) a monomial  $w$ , if there exist monomials  $p, s \in X$ , such that  $w = p \cdot v \cdot s$ , in other words  $v$  is a subword of  $w$ .

Let  $T := K\langle x_1, \dots, x_n \rangle$  be the free algebra and  $\prec$  be a fixed monomial ordering on  $T$ .

For a subset  $G \subset K\langle x_1, \dots, x_n \rangle$ , define the **leading ideal of  $G$**  to be the two-sided ideal  $LM(G) = \langle \{lm(g) \mid g \in G \setminus \{0\}\} \rangle_T \subseteq T$ .

**A subset  $G \subset I$  is a (two-sided) Groebner basis for the ideal  $I$  with respect to  $\prec$ , if  $LM(G) = LM(I)$ .**

**That is  $\forall f \in I \setminus \{0\}$  there exists  $g \in G$ , such that  $lm(g)$  divides  $lm(f)$ .**

The notion of **Groebner-Shirshov** basis applies to more general algebraic structures, but means the same as Groebner basis for associative algebras.

Suppose, that the weights of the ring variables are strictly positive. We can interpret these weights as defining a non-standard grading on the ring. If the set of input polynomials is weighted homogeneous with respect to the given weights of the ring variables, then computing up to a weighted degree (and thus, also length) bound  $d$

results in the **truncated Groebner basis**  $G(d)$ . In other words, by trimming elements of degree exceeding  $d$  from the complete Groebner basis  $G$ , one obtains precisely  $G(d)$ .

In general, given a set  $G(d)$ , which is the result of Groebner basis computation up to weighted degree bound  $d$ , then it is the complete finite Groebner basis, if and only if  $G(2d - 1) = G(d)$  holds.

**Note:** If the set of input polynomials is **not** weighted homogeneous with respect to the weights of the ring variables, and a Groebner is **not** finite,

then actually not much can be said precisely on the properties of the given ideal. By increasing the length bound bigger generating sets will be computed, but in contrast to the weighted homogeneous case some polynomials in of small length first enter the basis after computing up to a much higher length bound.

#### 7.9.4 Bimodules and syzygies and lifts

Let  $A = K$

$\langle x_1, \dots, x_n \rangle$  be the free algebra. A free bimodule of rank  $r$  over  $A$  is  $Ae_1A \oplus \dots \oplus Ae_rA$ , where  $e_i$  are the generators of the free bimodule.

NOTE: these  $e_i$  are freely non-commutative with respect to elements of  $A$  except constants from the ground field  $K$ .

The free bimodule of rank 1  $AeA$  surjects onto the algebra  $A$  itself. A two-sided ideal of the algebra  $A$  can be converted to a subbimodule of  $AeA$ .

The **syzygy bimodule** or even **module of bisyzygies** of the given finitely generated subbimodule  $N = \langle g_1, \dots, g_m \rangle \subset \bigoplus_{i=1}^r Ae_iA$  is the kernel of the natural homomorphism of  $A$ -bimodules  $\bigoplus_{j=1}^m Ae_jA \rightarrow \bigoplus_{i=1}^r Ae_iA$ ,  $e_j \mapsto g_j$ , that is  $\sum_{j=1}^m \sum_k \ell_{jk} e_j r_{jk} \mapsto \sum_{j=1}^m \sum_k \ell_{jk} g_j r_{jk}$ .

The syzygy bimodule is in general not finitely generated. Therefore as a bimodule, both the set of generators of the syzygy bimodule and its Groebner basis are computed up to a specified length bound.

Given a subbimodule  $N$  of a bimodule  $M$ , the **lift(ing)** process returns a matrix, which encodes the expression of generators  $N_1, \dots, N_s$

in terms of generators of  $M_1, \dots, M_m$  like this:  $N_i = \sum_{j=1}^m \sum_k \ell_{jk} M_j r_{jk} = \sum_{j=1}^m T_{ij} M_j$ ,

where  $T_{ij}$  are elements from the enveloping algebra  $R\langle X \rangle \otimes R\langle X \rangle$ , encoded as elements of the free bimodule of rank  $m$ , namely by using the non-commutative generators of the free bimodule which we call **ncgen**.

### 7.9.5 Letterplace correspondence

The name letterplace has been inspired by the work of Rota and, independently, Feynman.

Already Feynman and Rota encoded the monomials (words) of the free algebra  $x_{i_1}x_{i_2}\dots x_{i_m} \in K\langle x_1, \dots, x_n \rangle$  via the double-indexed letterplace (that is encoding the letter (= variable) and its place in the word) monomials  $x(i_1|1)x(i_2|2)\dots x(i_m|m) \in K[X \times N]$ , where  $X = \{x_1, \dots, x_n\}$  and  $N$  is the semigroup of natural numbers, starting with 1 as the first possible place. Note, that the letterplace algebra  $K[X \times N]$  is an infinitely generated commutative polynomial  $K$ -algebra. Since  $K\langle x_1, \dots, x_n \rangle$  is not Noetherian, it is common to perform the computations with its ideals and modules up to a given degree bound.

Subject to the given degree (length) bound  $d$ , the truncated letterplace algebra  $K[X \times (1, \dots, d)]$  is finitely generated commutative polynomial  $K$ -algebra.

In [LL09] a natural shifting on letterplace polynomials was introduced and used. Indeed, there is 1-to-1 correspondence between two-sided ideals of a free algebra and so-called letterplace ideals in the letterplace algebra, see [LL09], [LL13], [LSS13] and [L14] for details. Note, that first this correspondence was established for graded ideals, but holds more generally for arbitrary ideals and submodules of a free bimodule of a finite rank. All the computations internally take place in the Letterplace algebra.

A letterplace monomial of length  $m$  is a monomial of a letterplace algebra, such that its  $m$  places are exactly  $1, 2, \dots, m$ . In particular, such monomials are multilinear with respect to places (i.e. no place, smaller than the length is omitted or filled more than with one letter). A letterplace polynomial is an element of the  $K$ -vector space, spanned by letterplace monomials. A letterplace ideal is generated by letterplace polynomials subject to two kind of operations:

the  $K$ -algebra operations of the letterplace algebra **and simultaneous shifting of places by any natural number  $n$** .

**Note:** Letterplace correspondence naturally extends to the correspondence over

$R <$

$x_1, \dots, x_n$

$>$ , where  $R$  is a commutative unital ring. The case  $R = \mathbb{Z}$  is implemented, in addition to  $R$  being a field.

## 7.10 LETTERPLACE libraries

The content of libraries, created for LETTERPLACE is described in the following subsections.

Use the `LIB` command for loading of single libraries.

See also [Section 7.5.12 \[ncfactor.lib\]](#), [page 487](#) for the factorization of polynomials in noncommutative algebras.

### 7.10.1 fpadim.lib

**Library:** fpadim.lib

**Purpose:** Vector space dimension, basis and Hilbert series for finitely presented algebras (Letterplace)

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Support: Joint projects LE 2697/2-1 and KR 1907/3-1 of the Priority Programme SPP 1489: 'Algorithmische und Experimentelle Methoden in Algebra, Geometrie und Zahlentheorie' of the German DFG (2010-2013)

and Project II.6 of the transregional collaborative research centre SFB-TRR 195 'Symbolic Tools in Mathematics and their Application' of the German DFG (from 2017 on)

**Note:**

- basering is a Letterplace ring
- all intvecs correspond to Letterplace monomials
- if a degree bound  $d$  is specified,  $d \leq \text{attrib}(\text{basering}, \text{uptodeg})$  holds

In the procedures below, 'iv' stands for intvec representation and 'lp' for the letterplace representation of monomials

**Overview:** Given the free associative algebra  $A = K\langle x_1, \dots, x_n \rangle$  and a (finite or truncated) Groebner basis GB, one is interested in the following data:

- the  $K$ -dimension of  $A/\langle \text{GB} \rangle$  (check for finiteness or explicit value) - the Hilbert series of  $A/\langle \text{GB} \rangle$

- the explicit monomial  $K$ -basis of  $A/\langle \text{GB} \rangle$

In order to determine these, we need

- the Ufnarovskij graph induced by GB
- the mistletoes of  $A/\langle \text{GB} \rangle$  (which are special monomials in a basis)

The Ufnarovskij graph is used to determine whether  $A/\langle \text{GB} \rangle$  has finite  $K$ -dimension. One has to check if the graph contains cycles. For the whole theory we refer to [Ufn].

Given a

reduced set of monomials GB one can define the basis tree, whose vertex set  $V$  consists of all normal monomials w.r.t. GB. For every two monomials  $m_1, m_2$  in  $V$  there is a direct edge from  $m_1$  to  $m_2$ , if and only if there exists  $x_k$  in  $\{x_1, \dots, x_n\}$ , such that  $m_1 * x_k = m_2$ . The set  $M = \{m \text{ in } V \mid \text{there is no edge from } m \text{ to another monomial in } V\}$  is called the set of mistletoes. As one can easily see it consists of the endpoints of the graph. Since there is a unique path to every monomial in  $V$ , the whole graph can be described only from the knowledge of the mistletoes. Note that  $V$  corresponds to a basis of  $A/\langle \text{GB} \rangle$ , so knowing the mistletoes we know a  $K$ -basis. The name mistletoes was given to those points because of these miraculous value and the algorithm is named sickle, because a sickle is the tool to harvest mistletoes. For more details see [Stu]. This package uses the Letterplace format introduced by [LL09]. The algebra can either be represented as a Letterplace ring or via integer vectors: Every variable will only be represented by its number, so variable one is represented as 1, variable two as 2 and so on. The monomial  $x_1 * x_3 * x_2$  for example will be stored as (1,3,2). Multiplication is concatenation. Note that the approach in this library does not need an algorithm for computing the normal form. Note that fpa is an acronym for Finitely Presented Algebra.

## References:

[Ufn] V. Ufnarovskij: Combinatorial and asymptotic methods in algebra, 1990. [LL09] R. La Scala, V. Levandovskyy: Letterplace ideals and non-commutative Groebner bases, Journal of Symbolic Computation, 2009. [Stu] G. Studzinski: Dimension computations in non-commutative, associative algebras, Diploma thesis, RWTH Aachen, 2010.

**Procedures:** See also: [Section 7.10.3 \[fpaprops\\_lib\]](#), page 660; [Section 7.10.4 \[freegb\\_lib\]](#), page 666; [Section 7.10.5 \[ncHilb\\_lib\]](#), page 673.



### 7.10.1.1 teach\_lpKDimCheck

Procedure from library `fpadim.lib` (see [Section 7.10.1 \[fpadim.lib\]](#), page 640).

**Usage:** `teach_lpKDimCheck(G);`

**Return:** `int`, 1 if K-dimension of the factor algebra is infinite, 0 otherwise

**Purpose:** Checking a factor algebra for finiteness of the K-dimension

**Assume:** - basering is a Letterplace ring.

**Example:**

```
LIB "fpadim.lib";
ring r = 0,(x,y),dp;
def R = freeAlgebra(r, 5); // constructs a Letterplace ring
setring R; // sets basering to Letterplace ring
ideal G = x*x, y*y,x*y*x;
// Groebner basis
ideal I = x*x, y*x*y, x*y*x;
// Groebner basis
teach_lpKDimCheck(G); // invokes procedure, factor algebra is of finite K-dimension
↳ 0
teach_lpKDimCheck(I); // invokes procedure, factor algebra is of infinite Kdimension
↳ 1
```

### 7.10.1.2 lpKDim

Procedure from library `fpadim.lib` (see [Section 7.10.1 \[fpadim.lib\]](#), page 640).

**Usage:** `lpKDim(G);` G an ideal in a letterplace ring

**Return:** `int`

**Purpose:** Computes the K-dimension of  $A/\langle G \rangle$   
-1 means infinity

**Assume:** - basering is a Letterplace ring  
- G is a Groebner basis

**Note:** - Alias for `vdim(G)`

### 7.10.1.3 teach\_lpKDim

Procedure from library `fpadim.lib` (see [Section 7.10.1 \[fpadim.lib\]](#), page 640).

**Usage:** `teach_lpKDim(G[,degbound, n]);` G an ideal, degbound, n optional integers

**Return:** `int`, the K-dimension of the factor algebra

**Purpose:** Compute the K-dimension of a factor algebra, given via an ideal

**Assume:** - basering is a Letterplace ring  
- if you specify a different degree bound degbound,  
degbound  $\leq$  `attrib(basering,uptodeg)` holds.

**Note:** - If degbound is set, there will be a degree bound added. 0 means no degree bound. Default: `attrib(basering, uptodeg)`.  
- n is the number of variables, which can be set to a different number. Default: `attrib(basering, IV)`.  
- If the K-dimension is known to be infinite, a degree bound is needed



**Example:**

```

LIB "fpadim.lib";
ring r = 0,(x,y),dp;
def R = freeAlgebra(r, 5); // constructs a Letterplace ring
setring R; // sets basering to Letterplace ring
ideal G = x*x, y*y,x*y*x;
// ideal G contains a Groebner basis
teach_lpKDim(G); //procedure invoked with ring parameters
↪ 6
// the factor algebra is finite, so the degree bound given by the Letterplace
// ring is not necessary
teach_lpKDim(G,0); // procedure without any degree bound
↪ 6

```

**7.10.1.4 lpMonomialBasis**

Procedure from library `fpadim.lib` (see [Section 7.10.1 \[fpadim.lib\]](#), page 640).

**Usage:** `lpMonomialBasis(d, donly, J)`; `d`, `donly` integers, `J` an ideal

**Return:** ideal

**Purpose:** computes a list of free monomials in a Letterplace basering `R` of degree at most `d` and not contained in  $\langle \text{LM}(J) \rangle$  if `donly`  $\neq 0$ , only monomials of degree `d` are returned

**Assume:**

- basering is a Letterplace ring.
- $d \leq \text{attrib}(\text{basing}, \text{uptodeg})$  holds.
- `J` is a Groebner basis

**Note:** will be replaced with `reduce(maxideal(d), J)`; soon

**Example:**

```

LIB "fpadim.lib";
ring r = 0,(x,y),dp;
def R = freeAlgebra(r, 7); setring R;
ideal J = x*y*x - y*x*y;
option(redSB); option(redTail);
J = letplaceGBasis(J);
J;
↪ J[1]=x*y*x-y*x*y
↪ J[2]=y*x*y*y*x-x*y*y*x*y
↪ J[3]=y*x*y*y*y*x-x*x*y*y*x*y
↪ J[4]=y*x*y*y*y*y*x-x*x*x*y*y*x*y
//monomials of degree 2 only in K<x,y>:
lpMonomialBasis(2,1,ideal(0));
↪ _[1]=x*x
↪ _[2]=y*x
↪ _[3]=x*y
↪ _[4]=y*y
//monomials of degree <=2 in K<x,y>
lpMonomialBasis(2,0,ideal(0));
↪ _[1]=1
↪ _[2]=x
↪ _[3]=y

```

```

↳ _[4]=x*x
↳ _[5]=y*x
↳ _[6]=x*y
↳ _[7]=y*y
//monomials of degree 3 only in K<x,y>/J
lpMonomialBasis(3,1,J);
↳ _[1]=x*x*x
↳ _[2]=y*x*x
↳ _[3]=y*y*x
↳ _[4]=x*x*y
↳ _[5]=y*x*y
↳ _[6]=x*y*y
↳ _[7]=y*y*y
//monomials of degree <=3 in K<x,y>/J
lpMonomialBasis(3,0,J);
↳ _[1]=1
↳ _[2]=x
↳ _[3]=y
↳ _[4]=x*x
↳ _[5]=y*x
↳ _[6]=x*y
↳ _[7]=y*y
↳ _[8]=x*x*x
↳ _[9]=y*x*x
↳ _[10]=y*y*x
↳ _[11]=x*x*y
↳ _[12]=y*x*y
↳ _[13]=x*y*y
↳ _[14]=y*y*y

```

### 7.10.1.5 lpHilbert

Procedure from library `fpadim.lib` (see [Section 7.10.1 \[fpadim.lib\]](#), page 640).

**Usage:** `lpHilbert(G[,degbound,n]);` G an ideal, degbound, n optional integers

**Return:** intvec, containing the coefficients of the Hilbert series

**Purpose:** Compute the truncated Hilbert series of  $K\langle X \rangle / \langle G \rangle$  up to a degree bound

**Assume:** - basering is a Letterplace ring.  
 - if you specify a different degree bound degbound,  
 degbound  $\leq$  attrib(basering,uptodeg) holds.

**Theory:** Hilbert series of an algebra  $K\langle X \rangle / \langle G \rangle$  is  $\sum_{(i \geq 0)} h_i t^i$ , where  $h_i$  is the K-dimension of the space of monomials of degree i, not contained in  $\langle G \rangle$ . For finitely presented algebras Hilbert series NEED NOT be a rational function, though it happens often. Therefore in general there is no notion of a Hilbert polynomial.

**Note:** - If degbound is set, there will be a degree bound added. 0 means no degree bound. Default: attrib(basering,uptodeg).  
 - n is the number of variables, which can be set to a different number. Default: attrib(basering, IV).  
 - In the output intvec I,  $I[k]$  is the  $(k-1)$ -th coefficient of the Hilbert series, i.e.  $h_{(k-1)}$  as above.

**Example:**

```

LIB "fpadim.lib";
ring r = 0,(x,y),dp;
def R = freeAlgebra(r, 5); // constructs a Letterplace ring
setring R; // sets basering to Letterplace ring
ideal G = y*y,x*y*x; // G is a Groebner basis
lpHilbert(G); // procedure with default parameters
↪ 1,2,3,4,4,4
lpHilbert(G,3,2); // invokes procedure with degree bound 3 and (same) 2 variables
↪ 1,2,3,4

```

See also: [Section 7.10.5 \[ncHilb\\_lib\]](#), page 673.

**7.10.1.6 teach\_lpSickleDim**

Procedure from library `fpadim.lib` (see [Section 7.10.1 \[fpadim\\_lib\]](#), page 640).

**Usage:** `teach_lpSickleDim(G[,degbound,n]);` G an ideal, degbound, n optional integers

**Return:** list

**Purpose:** Compute the K-dimension and the mistletoes of  $K\langle X \rangle / \langle G \rangle$

**Assume:**

- basering is a Letterplace ring.
- if you specify a different degree bound degbound, degbound  $\leq$  attrib(basering,uptodeg) holds.

**Note:**

- If L is the list returned, then L[1] is an integer, the K-dimension, L[2] is an ideal, the mistletoes.
- If degbound is set, there will be a degree bound added. 0 means no degree bound. Default: attrib(basering,uptodeg).
- n is the number of variables, which can be set to a different number. Default: attrib(basering, IV).
- If the K-dimension is known to be infinite, a degree bound is needed

**Example:**

```

LIB "fpadim.lib";
ring r = 0,(x,y),dp;
def R = freeAlgebra(r, 5); // constructs a Letterplace ring
setring R; // sets basering to Letterplace ring
ideal G = x*x, y*y,x*y*x; // G is a monomial Groebner basis
teach_lpSickleDim(G); // invokes the procedure with ring parameters
↪ [1]:
↪ 6
↪ [2]:
↪ _[1]=x*y
↪ _[2]=y*x*y
// the factor algebra is finite, so the degree bound, given
// by the Letterplace ring is not necessary
teach_lpSickleDim(G,0); // procedure without any degree bound
↪ [1]:
↪ 6
↪ [2]:
↪ _[1]=x*y
↪ _[2]=y*x*y

```

## 7.10.2 fpalgebras.lib

**Library:** fpalgebras.lib

**Purpose:** Definitions of some finitely presented algebras and groups (Letterplace)

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 Support: Project II.6 in the transregional collaborative research centre SFB-TRR 195  
 'Symbolic Tools in Mathematics and their Application' of the German DFG

**Overview:** Generation of various algebras, including group algebras of finitely presented groups in the Letterplace ring. FPA stands for finitely presented algebra.

**Procedures:** See also: [Section 7.7 \[LETTERPLACE\]](#), page 617; [Section 7.10.1 \[fpadim.lib\]](#), page 640; [Section 7.10.3 \[fpaprops.lib\]](#), page 660; [Section 7.10.4 \[freegb.lib\]](#), page 666.

### 7.10.2.1 operatorAlgebra

Procedure from library fpalgebras.lib (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:** operatorAlgebra(a,d); a a string, d an integer

**Return:** ring

**Note:** - the ring contains the ideal I, which contains the required relations - a gives the name of the algebra  
 - d gives the degreebound for the Letterplace ring  
 a must be one of the following:  
 integrodifff3  
 toeplitz  
 weyl1  
 usl2  
 usl2h  
 shift1inverse  
 exterior2  
 quadrowmm  
 shift1  
 weyl1inverse  
 This is a collection of common algebras.

**Example:**

```
LIB "fpalgebras.lib";
def R = operatorAlgebra("integrodifff3",5); setring R;
I; //relations of the algebra
⇒ I[1]=-x*D+D*x-1
⇒ I[2]=II*II-x*II+II*x
⇒ I[3]=D*II-1
```

### 7.10.2.2 serreRelations

Procedure from library fpalgebras.lib (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:** serreRelations(A,z); A an intmat, z an int

**Return:** ideal

**Assume:** basering has a letterplace ring structure and  
A is a generalized Cartan matrix with integer entries

**Purpose:** compute the ideal of Serre's relations associated to A

**Example:**

```
LIB "fpalgebras.lib";
intmat A[3][3] =
2, -1, 0,
-1, 2, -3,
0, -1, 2; // G^1_2 Cartan matrix
ring r = 0,(f1,f2,f3),dp;
int uptodeg = 5;
def R = freeAlgebra(r, uptodeg);
setring R;
ideal I = serreRelations(A,1); I = simplify(I,1+2+8);
I;
↪ I[1]=f2*f2*f1-2*f2*f1*f2+f1*f2*f2
↪ I[2]=f3*f1-f1*f3
↪ I[3]=f2*f1*f1-2*f1*f2*f1+f1*f1*f2
↪ I[4]=f3*f3*f3*f3*f2-4*f3*f3*f3*f2*f3+6*f3*f3*f2*f3*f3-4*f3*f2*f3*f3*f3+f2\
*f3*f3*f3*f3
↪ I[5]=f3*f2*f2-2*f2*f3*f2+f2*f2*f3
```

### 7.10.2.3 fullSerreRelations

Procedure from library `fpalgebras.lib` (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:** fullSerreRelations(A,N,C,P,d); A an intmat, N,C,P ideals, d an int

**Return:** ring (and ideal)

**Purpose:** compute the inhomogeneous Serre's relations associated to A in given variable names

**Assume:** three ideals in the input are of the same sizes and contain merely variables which are interpreted as follows: N resp. P stand for negative resp. positive roots, C stand for Cartan elements. d is the degree bound for letterplace ring, which will be returned. The matrix A is a generalized Cartan matrix with integer entries The result is the ideal called 'fsRel' in the returned ring.

**Example:**

```
LIB "fpalgebras.lib";
intmat A[2][2] =
2, -1,
-1, 2; // A_2 = sl_3 Cartan matrix
ring r = 0,(f1,f2,h1,h2,e1,e2),dp;
ideal negroots = f1,f2; ideal cartans = h1,h2; ideal posroots = e1,e2;
int uptodeg = 5;
def RS = fullSerreRelations(A,negroots,cartans,posroots,uptodeg);
setring RS; fsRel;
↪ fsRel[1]=f2*f2*f1-2*f2*f1*f2+f1*f2*f2
↪ fsRel[2]=f2*f1*f1-2*f1*f2*f1+f1*f1*f2
↪ fsRel[3]=e2*e2*e1-2*e2*e1*e2+e1*e2*e2
↪ fsRel[4]=e2*e1*e1-2*e1*e2*e1+e1*e1*e2
```

```

⇒ fsRel[5]=e1*f2-f2*e1
⇒ fsRel[6]=e2*f1-f1*e2
⇒ fsRel[7]=e1*f1-f1*e1-h1
⇒ fsRel[8]=e2*f2-f2*e2-h2
⇒ fsRel[9]=h2*h1-h1*h2
⇒ fsRel[10]=e1*h1-h1*e1+2*e1
⇒ fsRel[11]=h1*f1-f1*h1+2*f1
⇒ fsRel[12]=e2*h1-h1*e2-e2
⇒ fsRel[13]=h1*f2-f2*h1-f2
⇒ fsRel[14]=e1*h2-h2*e1-e1
⇒ fsRel[15]=h2*f1-f1*h2-f1
⇒ fsRel[16]=e2*h2-h2*e2+2*e2
⇒ fsRel[17]=h2*f2-f2*h2+2*f2

```

#### 7.10.2.4 ademRelations

Procedure from library `fpalgebras.lib` (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:**      `ademRelations(i,j); i,j int`

**Return:**     ring (and exports ideal)

**Purpose:**     compute the ideal of Adem relations for  $i < 2j$  in characteristic 0 the ideal is exported under the name `AdemRel` in the output ring

**Example:**

```

LIB "fpalgebras.lib";
def A = ademRelations(2,5);
setring A;
AdemRel;
⇒ 6*s(7)*s(0)+s(6)*s(1)

```

#### 7.10.2.5 baumslagSolitar

Procedure from library `fpalgebras.lib` (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:**      `baumslagSolitar(m,n,d[,IsGroup]); n an integer, m an integer, d an integer, IsGroup an optional integer`

**Return:**     ring

**Note:**       - the ring contains the ideal I, which contains the required relations - in the group case:  
                   $A = a^{-1}$ ,  $B = b^{-1}$   
                  - negative input is only allowed in the group case!  
                  - d gives a degreebound and must be  $> m, n$   
                  - varying n and m produces a family of examples

**Example:**

```

LIB "fpalgebras.lib";
def R = baumslagSolitar(2,3,4); setring R;
I;
⇒ I[1]=-b*a*a*a+a*a*b
⇒ I[2]=a*A-1
⇒ I[3]=b*B-1
⇒ I[4]=a*A-A*a
⇒ I[5]=b*B-B*b

```

### 7.10.2.6 baumslagGroup

Procedure from library `fpalgebras.lib` (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:** `baumslagGroup(m,n,d)`; m an integer, n an integer, d an integer

**Return:** ring

**Note:** - the ring contains the ideal I, which contains the required relations - Baumslag group with the following presentation  
 $\langle a, b \mid a^m = b^n = 1 \rangle$   
 -d gives the degreebound for the Letterplace ring  
 - varying n and m produces a family of examples

**Example:**

```
LIB "fpalgebras.lib";
def R = baumslagGroup(2,3,4); setring R;
I;
⇨ I[1]=a*a-1
⇨ I[2]=b*b*b-1
```

### 7.10.2.7 crystallographicGroupP1

Procedure from library `fpalgebras.lib` (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:** `crystallographicGroupP1(d)`; d an integer

**Return:** ring

**Note:** - the ring contains the ideal I, which contains the required relations - p1 group with the following presentation  
 $\langle x, y \mid [x, y] = 1 \rangle$   
 -d gives the degreebound for the Letterplace ring

**Example:**

```
LIB "fpalgebras.lib";
def R = crystallographicGroupP1(5); setring R;
I;
⇨ I[1]=y*x+x*y+1
⇨ I[2]=X*x+1
⇨ I[3]=x*X+1
⇨ I[4]=y*Y+1
⇨ I[5]=Y*y+1
```

### 7.10.2.8 crystallographicGroupPM

Procedure from library `fpalgebras.lib` (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:** `crystallographicGroupPM(d)`; d an integer

**Return:** ring

**Note:** - the ring contains the ideal I, which contains the required relations - pm group with the following presentation  
 $\langle x, y, m \mid [x, y] = m^2 = 1, m^{(-1)}*x*m = x, m^{(-1)}*y*m = y^{(-1)} \rangle$  - d gives the degreebound for the Letterplace ring

**Example:**

```

LIB "fpalgebras.lib";
def R = crystallographicGroupPM(5); setring R;
I;
⇨ I[1]=y*x+x*y+1
⇨ I[2]=y*x+x*y+m*m
⇨ I[3]=m*m+1
⇨ I[4]=m*x*m+x
⇨ I[5]=m*y*m+Y
⇨ I[6]=x*X+1
⇨ I[7]=X*x+1
⇨ I[8]=Y*y+1
⇨ I[9]=y*Y+1

```

### 7.10.2.9 crystallographicGroupPG

Procedure from library `fpalgebras.lib` (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:** `crystallographicGroupPG(d)`;  $d$  an integer

**Return:** ring

**Note:** - the ring contains the ideal  $I$ , which contains the required relations - pg group with the following presentation  
 $\langle x, y, t \mid [x, y] = 1, t^2 = x, t^{(-1)}y^*t = y^{(-1)} \rangle$   
 -  $d$  gives the degreebound for the Letterplace ring

**Example:**

```

LIB "fpalgebras.lib";
def R = crystallographicGroupPG(5); setring R;
I;
⇨ I[1]=y*x+x*y+1
⇨ I[2]=t*t+x
⇨ I[3]=T*y*t+Y
⇨ I[4]=X*x+1
⇨ I[5]=x*X+1
⇨ I[6]=Y*y+1
⇨ I[7]=y*Y+1
⇨ I[8]=t*T+1
⇨ I[9]=T*t+1

```

### 7.10.2.10 crystallographicGroupP2MM

Procedure from library `fpalgebras.lib` (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:** `crystallographicGroupP2MM(d)`;  $d$  an integer

**Return:** ring

**Note:** - the ring contains the ideal  $I$ , which contains the required relations - p2mm group with the following presentation  
 $\langle x, y, p, q \mid [x, y] = [p, q] = p^2 = q^2 = 1, p^{(-1)}x^*p = x, q^{(-1)}x^*q = x^{(-1)}, p^{(-1)}y^*p = y^{(-1)}, q^{(-1)}y^*q = y \rangle$  -  $d$  gives the degreebound for the Letterplace ring

**Example:**



```

LIB "fpalgebras.lib";
def R = crystallographicGroupP2MM(5); setring R;
I;
⇨ I[1]=y*x+x*y+1
⇨ I[2]=q*p+p*q+1
⇨ I[3]=p*p+1
⇨ I[4]=q*q+1
⇨ I[5]=p*y*p+Y
⇨ I[6]=p*x*p+x
⇨ I[7]=q*y*q+y
⇨ I[8]=q*x*q+X
⇨ I[9]=X*x+1
⇨ I[10]=x*X+1
⇨ I[11]=Y*y+1
⇨ I[12]=y*Y+1
⇨ I[13]=y*x+x*y+p*p
⇨ I[14]=y*x+x*y+q*q
⇨ I[15]=p*p+q*q

```

### 7.10.2.11 crystallographicGroupP2

Procedure from library `fpalgebras.lib` (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:** `crystallographicGroupP2(d)`;  $d$  an integer

**Return:** ring

**Note:** - the ring contains the ideal  $I$ , which contains the required relations -  $p_2$  group with the following presentation  
 $\langle x, y, m, t \mid [x, y] = t^2 = 1, m^2 = y, t^{-1} * x * t = x, m^{-1} * x * m = x^{-1}, t^{-1} * y * t = y^{-1}, t^{-1} * m * t = m^{-1} \rangle$  -  $d$  gives the degreebound for the Letterplace ring

**Example:**

```

LIB "fpalgebras.lib";
def R = crystallographicGroupP2(5); setring R;
I;
⇨ I[1]=y*x+x*y+1
⇨ I[2]=y*x+x*y+t*t
⇨ I[3]=m*m+y
⇨ I[4]=t*t+1
⇨ I[5]=t*x*t+x
⇨ I[6]=M*x*m+X
⇨ I[7]=t*y*t+Y
⇨ I[8]=t*m*t+M
⇨ I[9]=X*x+1
⇨ I[10]=x*X+1
⇨ I[11]=Y*y+1
⇨ I[12]=y*Y+1
⇨ I[13]=m*M+1
⇨ I[14]=M*m+1

```

### 7.10.2.12 crystallographicGroupP2GG

Procedure from library `fpalgebras.lib` (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:** `crystallographicGroupP2GG(d)`; d an integer

**Return:** ring

**Note:** - the ring contains the ideal I, which contains the required relations - p2gg group with the following presentation  
 $\langle x, y, u, v \mid [x, y] = (u*v)^2 = 1, u^2 = x, v^2 = y, v^{-1}*x*v = x^{-1}, u^{-1}*y*u = y^{-1} \rangle$  - d gives the degreebound for the Letterplace ring

**Example:**

```
LIB "fpalgebras.lib";
def R = crystallographicGroupP2GG(5); setring R;
I;
⇨ I[1]=y*x+x*y+1
⇨ I[2]=u*v*u*v+y*x+x*y
⇨ I[3]=u*v*u*v+1
⇨ I[4]=u*u+x
⇨ I[5]=v*v+y
⇨ I[6]=V*x*v+X
⇨ I[7]=U*y*u+Y
⇨ I[8]=X*x+1
⇨ I[9]=x*X+1
⇨ I[10]=Y*y+1
⇨ I[11]=y*Y+1
⇨ I[12]=u*U+1
⇨ I[13]=U*u+1
⇨ I[14]=v*V+1
⇨ I[15]=V*v+1
```

### 7.10.2.13 crystallographicGroupCM

Procedure from library `fpalgebras.lib` (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:** `crystallographicGroupCM(d)`; d an integer

**Return:** ring

**Note:** - the ring contains the ideal I, which contains the required relations - cm group with the following presentation  
 $\langle x, y, t \mid [x, y] = t^2 = 1, t^{-1}*x*t = x*y, t^{-1}*y*t = y^{-1} \rangle$  - d gives the degreebound for the Letterplace ring

**Example:**

```
LIB "fpalgebras.lib";
def R = crystallographicGroupCM(5); setring R;
I;
⇨ I[1]=y*x+x*y+1
⇨ I[2]=y*x+x*y+t*t
⇨ I[3]=t*t+1
⇨ I[4]=t*x*t+x*y
⇨ I[5]=t*y*t+Y
⇨ I[6]=X*x+1
⇨ I[7]=x*X+1
⇨ I[8]=Y*y+1
⇨ I[9]=y*Y+1
```

### 7.10.2.14 crystallographicGroupC2MM

Procedure from library `fpalgebras.lib` (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:** `crystallographicGroupC2MM(d)`;  $d$  an integer

**Return:** ring

**Note:** - the ring contains the ideal  $I$ , which contains the required relations -  $c2mm$  group with the following presentation  
 $\langle x, y, m, r \mid [x, y] = m^2 = r^2 = 1, m^{-1}y^*m = y^{-1}, m^{-1}x^*m = x^*y, r^{-1}y^*r = y^{-1}, r^{-1}x^*r = x^{-1}, m^{-1}r^*m = r^{-1} \rangle$  -  $d$  gives the degreebound for the Letterplace ring

**Example:**

```
LIB "fpalgebras.lib";
def R = crystallographicGroupC2MM(5); setring R;
I;
⇨ I[1]=y*x+x*y+1
⇨ I[2]=y*x+x*y+m*m
⇨ I[3]=y*x+x*y+r*r
⇨ I[4]=m*m+1
⇨ I[5]=r*r+1
⇨ I[6]=m*m+r*r
⇨ I[7]=m*y*m+Y
⇨ I[8]=m*x*m+x*y
⇨ I[9]=r*y*r+Y
⇨ I[10]=r*x*r+X
⇨ I[11]=m*r*m+r
⇨ I[12]=X*x+1
⇨ I[13]=x*X+1
⇨ I[14]=Y*y+1
⇨ I[15]=y*Y+1
```

### 7.10.2.15 crystallographicGroupP4

Procedure from library `fpalgebras.lib` (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:** `crystallographicGroupP4(d)`;  $d$  an integer

**Return:** ring

**Note:** - the ring contains the ideal  $I$ , which contains the required relations -  $p4$  group with the following presentation  
 $\langle x, y, r \mid [x, y] = r^4 = 1, r^{-1}x^*r = x^{-1}, r^{-1}x^*r = y \rangle$  -  $d$  gives the degreebound for the Letterplace ring

**Example:**

```
LIB "fpalgebras.lib";
def R = crystallographicGroupP4(5); setring R;
I;
⇨ I[1]=y*x+x*y+1
⇨ I[2]=r*r*r*r+y*x+x*y
⇨ I[3]=r*r*r*r+1
⇨ I[4]=r*r*r*x*r+X
⇨ I[5]=r*r*r*x*r+y
```

```

↪ I[6]=X*x+1
↪ I[7]=x*X+1
↪ I[8]=Y*y+1
↪ I[9]=y*Y+1

```

### 7.10.2.16 crystallographicGroupP4MM

Procedure from library `fpalgebras.lib` (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:** `crystallographicGroupP4MM(d)`; d an integer

**Return:** ring

**Note:** - the ring contains the ideal I, which contains the required relations - p4mm group with the following presentation  
 $\langle x, y, r, m \mid [x, y] = r^4 = m^2 = 1, r^{(-1)}y^*r = x^{(-1)}, r^{(-1)}x^*r = y, m^{(-1)}x^*m = y, m^{(-1)}r^*m = r^{(-1)} \rangle$  - d gives the degreebound for the Letterplace ring

**Example:**

```

LIB "fpalgebras.lib";
def R = crystallographicGroupP4MM(5); setring R;
I;
↪ I[1]=y*x+x*y+1
↪ I[2]=r*r*r*r+y*x+x*y
↪ I[3]=r*r*r*r+1
↪ I[4]=r*r*r*x*r+X
↪ I[5]=r*r*r*x*r+y
↪ I[6]=X*x+1
↪ I[7]=x*X+1
↪ I[8]=Y*y+1
↪ I[9]=y*Y+1

```

### 7.10.2.17 crystallographicGroupP4GM

Procedure from library `fpalgebras.lib` (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:** `crystallographicGroupP4GM(d)`; d an integer

**Return:** ring

**Note:** - the ring contains the ideal I, which contains the required relations - p4gm group with the following presentation  
 $\langle x, y, r, t \mid [x, y] = r^4 = t^2 = 1, r^{(-1)}y^*r = x^{(-1)}, r^{(-1)}x^*r = y, t^{(-1)}x^*t = y, t^{(-1)}r^*t = x^{(-1)}r^{(-1)} \rangle$  - d gives the degreebound for the Letterplace ring

**Example:**

```

LIB "fpalgebras.lib";
def R = crystallographicGroupP4GM(5); setring R;
I;
↪ I[1]=y*x+x*y+1
↪ I[2]=r*r*r*r+y*x+x*y
↪ I[3]=r*r*r*r+1
↪ I[4]=y*x+x*y+t*t
↪ I[5]=t*t+1
↪ I[6]=r*r*r*r+t*t
↪ I[7]=r*r*r*y*r+X

```

```

⇒ I[8]=r*r*r*x*r+y
⇒ I[9]=X*r*r*r+t*r*t
⇒ I[10]=X*x+1
⇒ I[11]=x*X+1
⇒ I[12]=Y*y+1
⇒ I[13]=y*Y+1

```

### 7.10.2.18 crystallographicGroupP3

Procedure from library `fpalgebras.lib` (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:** `crystallographicGroupP3(d)`; d an integer

**Return:** ring

**Note:** - the ring contains the ideal I, which contains the required relations - p3 group with the following presentation  
 $\langle x, y, r \mid [x, y] = r^3 = 1, r^{(-1)} * x * r = x^{(-1)} * y, r^{(-1)} * y * r = x^{(-1)} \rangle$  - d gives the degreebound for the Letterplace ring

**Example:**

```

LIB "fpalgebras.lib";
def R = crystallographicGroupP3(5); setring R;
I;
⇒ I[1]=y*x+x*y+1
⇒ I[2]=r*r*r+y*x+x*y
⇒ I[3]=r*r*r+1
⇒ I[4]=r*r*x*r+X*y
⇒ I[5]=r*r*y*r+X
⇒ I[6]=X*x+1
⇒ I[7]=x*X+1
⇒ I[8]=Y*y+1
⇒ I[9]=y*Y+1

```

### 7.10.2.19 crystallographicGroupP31M

Procedure from library `fpalgebras.lib` (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:** `crystallographicGroupP31M(d)`; d an integer

**Return:** ring

**Note:** - the ring contains the ideal I, which contains the required relations - p31m group with the following presentation  
 $\langle x, y, r, t \mid [x, y] = r^2 = t^2 = (t*r)^3 = 1, r^{(-1)} * x * r = x, t^{(-1)} * y * t = y, t^{(-1)} * x * t = x^{(-1)} * y, r^{(-1)} * y * r = x * y^{(-1)} \rangle$  - d gives the degreebound for the Letterplace ring

**Example:**

```

LIB "fpalgebras.lib";
def R = crystallographicGroupP31M(6); setring R;
I;
⇒ I[1]=y*x+x*y+1
⇒ I[2]=y*x+x*y+r*r
⇒ I[3]=y*x+x*y+t*t
⇒ I[4]=r*r+1
⇒ I[5]=t*t+1

```

```

⇒ I[6]=t*r*t*r*t*r+1
⇒ I[7]=r*r+t*t
⇒ I[8]=t*r*t*r*t*r+y*x+x*y
⇒ I[9]=t*r*t*r*t*r+r*r
⇒ I[10]=t*r*t*r*t*r+t*t
⇒ I[11]=r*x*r+x
⇒ I[12]=t*y*t+y
⇒ I[13]=t*x*t+X*y
⇒ I[14]=r*y*r+x*Y
⇒ I[15]=X*x+1
⇒ I[16]=x*X+1
⇒ I[17]=Y*y+1
⇒ I[18]=y*Y+1

```

### 7.10.2.20 crystallographicGroupP3M1

Procedure from library `fpalgebras.lib` (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:** `crystallographicGroupP3M1(d)`;  $d$  an integer

**Return:** ring

**Note:** - the ring contains the ideal  $I$ , which contains the required relations - p3m1 group with the following presentation  
 $\langle x, y, r, m \mid [x, y] = r^3 = m^2 = 1, m^{(-1)}*r*m = r^2, r^{(-1)}*x*r = x^{(-1)}*y, r^{(-1)}*y*r = x^{(-1)}, m^{(-1)}*x*m = x^{(-1)}, m^{(-1)}*y*m = x^{(-1)}*y \rangle$  -  $d$  gives the degreebound for the Letterplace ring

**Example:**

```

LIB "fpalgebras.lib";
def R = crystallographicGroupP3M1(5); setring R;
I;
⇒ I[1]=y*x+x*y+1
⇒ I[2]=r*r*r+y*x+x*y
⇒ I[3]=y*x+x*y+m*m
⇒ I[4]=r*r*r+1
⇒ I[5]=m*m+1
⇒ I[6]=r*r*r+m*m
⇒ I[7]=m*r*m+r*r
⇒ I[8]=r*r*x*r+X*y
⇒ I[9]=r*r*y*r+X
⇒ I[10]=m*x*m+X
⇒ I[11]=m*y*m+X*y
⇒ I[12]=X*x+1
⇒ I[13]=x*X+1
⇒ I[14]=Y*y+1
⇒ I[15]=y*Y+1

```

### 7.10.2.21 crystallographicGroupP6

Procedure from library `fpalgebras.lib` (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:** `crystallographicGroupP6(d)`;  $d$  an integer

**Return:** ring

**Note:** - the ring contains the ideal  $I$ , which contains the required relations -  $p_6$  group with the following presentation  
 $\langle x, y, r \mid [x, y] = r^6 = 1, r^{-1}x^*r = y, r^{-1}y^*r = x^{-1}y^* \rangle$  -  $d$  gives the degreebound for the Letterplace ring

**Example:**

```
LIB "fpalgebras.lib";
def R = crystallographicGroupP6(7); setring R;
I;
⇨ I[1]=y*x+x*y+1
⇨ I[2]=r*r*r*r*r*r+y*x+x*y
⇨ I[3]=r*r*r*r*r*r+1
⇨ I[4]=r*r*r*r*r*r*x*r+y
⇨ I[5]=r*r*r*r*r*r*y*r+X*y
⇨ I[6]=X*x+1
⇨ I[7]=x*X+1
⇨ I[8]=Y*y+1
⇨ I[9]=y*Y+1
```

### 7.10.2.22 crystallographicGroupP6MM

Procedure from library `fpalgebras.lib` (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:** `crystallographicGroupP6MM(d)`;  $d$  an integer

**Return:** ring

**Note:** - the ring contains the ideal  $I$ , which contains the required relations -  $p_{6mm}$  group with the following presentation  
 $\langle x, y, r, m \mid [x, y] = r^6 = m^2 = 1, r^{-1}y^*r = x^{-1}y, r^{-1}x^*r = y, m^{-1}x^*m = x^{-1}, m^{-1}y^*m = x^{-1}y, m^{-1}r^*m = r^{-1}y^* \rangle$  -  $d$  gives the degreebound for the Letterplace ring

**Example:**

```
LIB "fpalgebras.lib";
def R = crystallographicGroupP6MM(7); setring R;
I;
⇨ I[1]=y*x+x*y+1
⇨ I[2]=r*r*r*r*r*r+y*x+x*y
⇨ I[3]=r*r*r*r*r*r+1
⇨ I[4]=y*x+x*y+m*m
⇨ I[5]=r*r*r*r*r*r+m*m
⇨ I[6]=m*m+1
⇨ I[7]=m*x*m+X
⇨ I[8]=m*y*m+X*y
⇨ I[9]=r*r*r*r*r*r*x*r+y
⇨ I[10]=r*r*r*r*r*r*y*r+X*y
⇨ I[11]=r*r*r*r*r*r*y+m*r*m
⇨ I[12]=X*x+1
⇨ I[13]=x*X+1
⇨ I[14]=Y*y+1
⇨ I[15]=y*Y+1
```

### 7.10.2.23 dyckGroup1

Procedure from library `fpalgebras.lib` (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:** `dyckGroup1(n,d,P)`;  $n$  an integer,  $d$  an integer,  $P$  an intvec

**Return:** ring

**Note:**

- the ring contains the ideal  $I$ , which contains the required relations - The Dyck group with the following presentation  
 $\langle x_1, x_2, \dots, x_n \mid (x_1)^{p_1} = (x_2)^{p_2} = \dots = (x_n)^{p_n} = x_1 * x_2 * \dots * x_n = 1 \rangle$  - negative exponents are allowed
- representation in the form  $x_i^{p_i} - x_{(i+1)}^{p_{(i+1)}}$
- $d$  gives the degreebound for the Letterplace ring
- varying  $n$  and  $P$  produces a family of examples

**Example:**

```
LIB "fpalgebras.lib";
intvec P = 1,2,3;
def R = dyckGroup1(3,5,P); setring R;
I;
⇒ I[1]=x(2)*x(2)+x(1)
⇒ I[2]=x(3)*x(3)*x(3)+x(2)*x(2)
⇒ I[3]=x(1)*x(2)*x(3)+x(3)*x(3)*x(3)
⇒ I[4]=x(1)*x(2)*x(3)+1
```

### 7.10.2.24 dyckGroup2

Procedure from library `fpalgebras.lib` (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:** `dyckGroup2(n,d,P)`;  $n$  an integer,  $d$  an integer,  $P$  an intvec

**Return:** ring

**Note:**

- the ring contains the ideal  $I$ , which contains the required relations - The Dyck group with the following presentation  
 $\langle x_1, x_2, \dots, x_n \mid (x_1)^{p_1} = (x_2)^{p_2} = \dots = (x_n)^{p_n} = x_1 * x_2 * \dots * x_n = 1 \rangle$  - negative exponents are allowed
- representation in the form  $x_i^{p_i} - 1$
- $d$  gives the degreebound for the Letterplace ring
- varying  $n$  and  $P$  produces a family of examples

**Example:**

```
LIB "fpalgebras.lib";
intvec P = 1,2,3;
def R = dyckGroup2(3,5,P); setring R;
I;
⇒ I[1]=x(1)+1
⇒ I[2]=x(2)*x(2)+1
⇒ I[3]=x(3)*x(3)*x(3)+1
⇒ I[4]=x(1)*x(2)*x(3)+1
```

### 7.10.2.25 dyckGroup3

Procedure from library `fpalgebras.lib` (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:** `dyckGroup2(n,d,P)`;  $n$  an integer,  $d$  an integer,  $P$  an intvec



**Return:** ring

**Note:** - the ring contains the ideal I, which contains the required relations - The Dyck group with the following presentation  $\langle x_1, x_2, \dots, x_n \mid (x_1)^{p_1} = (x_2)^{p_2} = \dots = (x_n)^{p_n} = x_1 * x_2 * \dots * x_n = 1 \rangle$  - only positive exponents are allowed  
 - no inverse generators needed  
 - d gives the degreebound for the Letterplace ring  
 - varying n and P produces a family of examples

**Example:**

```
LIB "fpalgebras.lib";
intvec P = 1,2,3;
def R = dyckGroup3(3,5,P); setring R;
I;
⇒ I[1]=x(1)+1
⇒ I[2]=x(2)*x(2)+1
⇒ I[3]=x(3)*x(3)*x(3)+1
⇒ I[4]=x(1)*x(2)*x(3)+1
```

### 7.10.2.26 fibonacciGroup

Procedure from library `fpalgebras.lib` (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:** fibonacciGroup(m,d); m an integer, d an integer

**Return:** ring

**Note:** - the ring contains the ideal I, which contains the required relations - The Fibonacci group  $F(2, m)$  with the following presentation  $\langle x_1, x_2, \dots, x_m \mid x_i * x_{(i+1)} = x_{(i+2)} \rangle$   
 - d gives the degreebound for the Letterplace ring  
 - varying m produces a family of examples

**Example:**

```
LIB "fpalgebras.lib";
def R = fibonacciGroup(3,5); setring R;
I;
⇒ I[1]=x(1)*x(2)+x(3)
⇒ I[2]=x(1)*Y(1)+1
⇒ I[3]=Y(1)*x(1)+1
⇒ I[4]=x(2)*Y(2)+1
⇒ I[5]=Y(2)*x(2)+1
⇒ I[6]=x(3)*Y(3)+1
⇒ I[7]=Y(3)*x(3)+1
```

### 7.10.2.27 tetrahedronGroup

Procedure from library `fpalgebras.lib` (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:** tetrahedronGroup(g,d); g an integer, d an integer

**Return:** ring

**Note:** - the ring contains the ideal I, which contains the required relations - g gives the number of the example (1 - 5)

- d gives the degreebound for the Letterplace ring
- varying g produces a family of examples

The examples are found in “Classification of the finite generalized tetrahedron groups” by Gerhard Rosenberger and Martin Scheer.

The 5 examples originate from Proposition 1.9 and describe finite generalized tetrahedron group in the Tsaranov-case, which are not equivalent to a presentation for an ordinary tetrahedron group.

**Example:**

```
LIB "fpalgebras.lib";
def R = tetrahedronGroup(3,5); setring R;
I;
⇨ I[1]=x*x*x+1
⇨ I[2]=y*y*y+1
⇨ I[3]=z*z*z+1
⇨ I[4]=x*y*x*y+1
⇨ I[5]=x*z*x*z+1
⇨ I[6]=y*z*y*z+1
```

### 7.10.2.28 triangularGroup

Procedure from library `fpalgebras.lib` (see [Section 7.10.2 \[fpalgebras.lib\]](#), page 646).

**Usage:** `triangularGroup(g,d)`; g an integer, d an integer

**Return:** ring

**Note:**

- the ring contains the ideal I, which contains the required relations - g gives the number of the example (1 - 14)
- d gives the degreebound for the Letterplace ring
- varying g produces a family of examples

The examples are found in

Classification of the finite generalized tetrahedron groups by Gerhard Rosenberger and Martin Scheer.

The 14 examples are denoted in theorem 2.12

**Example:**

```
LIB "fpalgebras.lib";
def R = triangularGroup(3,10); setring R;
I;
⇨ I[1]=a*a*a+1
⇨ I[2]=b*b*b+1
⇨ I[3]=a*b*a*b*b*a*b*a*b*b+1
```

### 7.10.3 fpaprops.lib

**Library:** `fpaprops.lib`

**Purpose:** Algorithmic ring-theoretic properties of finitely presented algebras (Letterplace)

**Authors:** Karim Abou Zeid, karim.abou.zeid at rwth-aachen.de

Support: Project II.6 in the transregional collaborative research centre SFB-TRR 195 ‘Symbolic Tools in Mathematics and their Application’ of the German DFG

**Overview:** In this library, algorithms for computing various ring-theoretic properties of finitely presented algebras are implemented.  
 Applicability: Letterplace rings.

**References:**

Huishi Li: Groebner bases in ring theory. World Scientific, 2010.

**Procedures:** See also: [Section 7.10.1 \[fpadim.lib\]](#), page 640; [Section 7.10.4 \[freegb.lib\]](#), page 666.

### 7.10.3.1 lpNoetherian

Procedure from library `fpaprops.lib` (see [Section 7.10.3 \[fpaprops.lib\]](#), page 660).

**Usage:** `lpNoetherian(G)`;  $G$  an ideal in a Letterplace ring

**Return:** `int`  
 0 not Noetherian  
 1 left Noetherian  
 2 right Noetherian  
 3 Noetherian  
 4 weak Noetherian

**Purpose:** Check whether the monomial algebra  $A/\langle LM(G) \rangle$  is (left/right) noetherian

**Assume:** - basering is a Letterplace ring  
 -  $G$  is a Groebner basis

**Theory:** `lpNoetherian` works with the monomial algebra  $A/\langle LM(G) \rangle$ . If it gives an affirmative answer for one of the properties, then it holds for both  $A/\langle LM(G) \rangle$  and  $A/\langle G \rangle$ . However, a negative answer applies only to  $A/\langle LM(G) \rangle$  and not necessarily to  $A/\langle G \rangle$ .

**Note:** Weak Noetherian means that two-sided ideals in  $A/\langle LM(G) \rangle$  satisfy the acc (ascending chain condition).

**Example:**

```
LIB "fpaprops.lib";
ring r = 0,(x,y),dp;
def R = freeAlgebra(r, 5);
setring R;
ideal G = x*x, y*x; // K<x,y>/<xx,yx> is right noetherian
lpNoetherian(G);
⇨ 2
```

### 7.10.3.2 lpIsSemiPrime

Procedure from library `fpaprops.lib` (see [Section 7.10.3 \[fpaprops.lib\]](#), page 660).

**Usage:** `lpIsSemiPrime(G)`;  $G$  an ideal in a Letterplace ring

**Return:** `boolean`

**Purpose:** Check whether  $A/\langle LM(G) \rangle$  is semi-prime ring,  
 alternatively whether  $\langle LM(G) \rangle$  is a semi-prime ideal in  $A$ .

**Assume:** - basering is a Letterplace ring  
 -  $G$  is a Groebner basis

**Theory:** A (two-sided) ideal  $I$  in the ring  $A$  is semi-prime, if for any  $a$  in  $A$  one has  $aAa \subseteq I$  implies  $a \in I$ .

**Note:** `lpIsSemiPrime` works with the monomial algebra  $A/\langle \text{LM}(G) \rangle$ . A positive answer holds for both  $A/\langle \text{LM}(G) \rangle$  and  $A/\langle G \rangle$ , while a negative answer applies only to  $A/\langle \text{LM}(G) \rangle$  and not necessarily to  $A/\langle G \rangle$ .

**Example:**

```
LIB "fpaprops.lib";
ring r = 0, (x1, x2), dp;
def R = freeAlgebra(r, 5);
setring R;
ideal G = x1*x2, x2*x1; // K<x1,x2>/<x1*x2,x2*x1> is semi prime
lpIsSemiPrime(G);
↪ 1
```

### 7.10.3.3 lpIsPrime

Procedure from library `fpaprops.lib` (see [Section 7.10.3 \[fpaprops.lib\]](#), page 660).

**Usage:** `lpIsPrime(G)`;  $G$  an ideal in a Letterplace ring

**Return:** boolean

**Purpose:** Check whether  $A/\langle \text{LM}(G) \rangle$  is prime ring, alternatively whether  $\langle \text{LM}(G) \rangle$  is a prime ideal in  $A$ .

**Assume:** - basering is a Letterplace ring  
-  $G$  is a Groebner basis

**Theory:** A (two-sided) ideal  $I$  in the ring  $A$  is prime, if for any  $a, b$  in  $A$  one has  $aAb \subseteq I$  implies  $a$  in  $I$  or  $b$  in  $I$ .

**Note:** `lpIsPrime` works with the monomial algebra  $A/\langle \text{LM}(G) \rangle$ . A positive answer holds for both  $A/\langle \text{LM}(G) \rangle$  and  $A/\langle G \rangle$ , while a negative answer applies only to  $A/\langle \text{LM}(G) \rangle$  and not necessarily to  $A/\langle G \rangle$ .

**Example:**

```
LIB "fpaprops.lib";
ring r = 0, (x, y), dp;
def R = freeAlgebra(r, 5);
setring R;
ideal G = x*x, y*y; // K<x,y>/<xx,yy> is prime
lpIsPrime(G);
↪ 1
```

### 7.10.3.4 lpGkDim

Procedure from library `fpaprops.lib` (see [Section 7.10.3 \[fpaprops.lib\]](#), page 660).

**Usage:** `lpGkDim(G)`;  $G$  an ideal in a letterplace ring

**Return:** int

**Purpose:** Determines the Gelfand Kirillov dimension of  $A/\langle G \rangle$   
-1 means positive infinite

**Assume:** - basering is a Letterplace ring  
-  $G$  is a Groebner basis

**Note:** Alias for `dim(G)`

**Example:**

```

LIB "fpaprops.lib";
ring r = 0,(x,y,z),dp;
ring R = freeAlgebra(r, 5);
ideal I = z; // infinite GK dimension (-1)
lpGkDim(I);
↳ WARNING: 'lpGkDim' is deprecated, you can use 'dim' instead.
↳ -1
I = x,y,z; I = std(I); // GK dimension 0
lpGkDim(I);
↳ WARNING: 'lpGkDim' is deprecated, you can use 'dim' instead.
↳ 0
I = x*y, x*z, z*y, z*z; I = std(I); // GK dimension 2
lpGkDim(I);
↳ WARNING: 'lpGkDim' is deprecated, you can use 'dim' instead.
↳ 2
ideal G = y*x - x*y, z*x - x*z, z*y - y*z; G = std(G);
G;
↳ G[1]=z*y-y*z
↳ G[2]=z*x-x*z
↳ G[3]=y*x-x*y
lpGkDim(G); // GK dimension 3
↳ WARNING: 'lpGkDim' is deprecated, you can use 'dim' instead.
↳ 3

```

**7.10.3.5 teach\_lpGkDim**

Procedure from library `fpaprops.lib` (see [Section 7.10.3 \[fpaprops.lib\]](#), page 660).

**Usage:** `teach_lpGkDim(G)`;  $G$  an ideal in a letterplace ring

**Return:** `int`

**Purpose:** Determines the Gelfand Kirillov dimension of  $A/\langle G \rangle$   
 -1 means positive infinite

**Assume:** - basering is a Letterplace ring  
 -  $G$  is a Groebner basis

**Example:**

```

LIB "fpaprops.lib";
ring r = 0,(x,y,z),dp;
def R = freeAlgebra(r, 5); // constructs a Letterplace ring
R;
↳ // coefficients: QQ
↳ // number of vars : 15
↳ //          block 1 : ordering dp
↳ //          : names  x y z x y z x y z x y z x y z
↳ //          block 2 : ordering C
↳ // letterplace ring (block size 3, ncgen count 0)
setring R; // sets basering to Letterplace ring
ideal I = z; // an example of infinite GK dimension
teach_lpGkDim(I);
↳ -1
I = x,y,z; // gkDim = 0

```

```

teach_lpGkDim(I);
↪ 0
I = x*y, x*z, z*y, z*z;//gkDim = 2
teach_lpGkDim(I);
↪ 2
ideal G = y*x - x*y, z*x - x*z, z*y - y*z; G = std(G);
G;
↪ G[1]=z*y-y*z
↪ G[2]=z*x-x*z
↪ G[3]=y*x-x*y
teach_lpGkDim(G); // 3, as expected for K[x,y,z]
↪ 3

```

### 7.10.3.6 lpGldimBound

Procedure from library `fpaprops.lib` (see [Section 7.10.3 \[fpaprops.lib\]](#), page 660).

**Usage:** `lpGldimBound(I)`; I an ideal

**Return:** int, an upper bound for the global dimension, -1 means infinity

**Purpose:** computing an upper bound for the global dimension

**Assume:** - basering is a Letterplace ring, G is a reduced Groebner Basis

**Note:** if  $I = \text{LM}(I)$ , then the global dimension is equal the Gelfand Kirillov dimension if it is finite  
Global dimension should be 0 for  $A/G = K$  and 1 for  $A/G = K\langle x_1 \dots x_n \rangle$

**Example:**

```

LIB "fpaprops.lib";
ring r = 0,(x,y),dp;
def R = freeAlgebra(r, 5); // constructs a Letterplace ring
setring R; // sets basering to Letterplace ring
ideal G = x*x, y*y,x*y*x; // it is a monomial Groebner basis
lpGldimBound(G);
↪ 0
ideal H = y*x - x*y; H = std(H); // H is a Groebner basis
lpGldimBound(H); // gl dim of K[x,y] is 2, as expected
↪ 2

```

### 7.10.3.7 lpSubstitute

Procedure from library `fpaprops.lib` (see [Section 7.10.3 \[fpaprops.lib\]](#), page 660).

**Usage:** `lpSubstitute(f,s1,s2[,G])`; f poly, s1 list (ideal) of variables to replace, s2 list (ideal) of polynomials to replace with, G optional ideal to reduce with.

**Return:** poly, the substituted polynomial

**Assume:** - basering is a Letterplace ring  
- s1 contains a subset of the set of variables  
- s2 and s1 are of the same size  
- G is a Groebner basis,  
- the current ring has a sufficient degbound (which also can be calculated with `lpCalcSubstDegBound()`)

**Note:** the procedure implements the image of a polynomial  $f$  under an endomorphism of a free algebra, defined by  $s1$  and  $s2$ : variables, not present in  $s1$ , are left unchanged;  
 variable  $s1[k]$  is mapped to a polynomial  $s2[k]$ .  
 - An optional ideal  $G$  extends the endomorphism as above to the morphism into the factor algebra  $K\langle X \rangle / G$ .

**Example:**

```
LIB "fpaprops.lib";
ring r = 0,(x,y,z),dp;
def R = freeAlgebra(r, 4);
setring R;
ideal G = x*y; // optional
poly f = 3*x*x+y*x;
ideal s1 = x, y;
ideal s2 = y*z*z, x; // i.e. x --> yzz and y --> x
// the substitution probably needs a higher degbound
int minDegBound = lpCalcSubstDegBound(f,s1,s2);
minDegBound; // thus the bound needs to be increased
↳ 9
setring r; // back to original r
def R1 = freeAlgebra(r, minDegBound);
setring R1;
lpSubstitute(imap(R,f), imap(R,s1), imap(R,s2));
↳ 3*y*z*z*y*z*z+x*y*z*z
// the last parameter is optional; above it was G=<xy>
// the output will be reduced with respect to G
lpSubstitute(imap(R,f), imap(R,s1), imap(R,s2), imap(R,G));
↳ 3*y*z*z*y*z*z
```

### 7.10.3.8 lpCalcSubstDegBound

Procedure from library `fpaprops.lib` (see [Section 7.10.3 \[fpaprops.lib\]](#), page 660).

**Usage:** `lpCalcSubstDegBound(I,s1,s2)`;  $I$  ideal of polynomials,  $s1$  ideal of variables to replace,  $s2$  ideal of polynomials to replace with

**Return:** int, the min degbound required to perform all of the substitutions

**Assume:** - basering is a Letterplace ring

**Note:** convenience method

**Example:**

```
LIB "fpaprops.lib";
ring r = 0,(x,y,z),dp;
def R = freeAlgebra(r, 4);
setring R;
ideal I = 3*x*x+y*x, x*y*x - z;
ideal s1 = x, y; // z --> z
ideal s2 = y*z*z, x; // i.e. x --> yzz and y --> x
// the substitution probably needs a higher degbound
lpCalcSubstDegBound(I,s1,s2);
↳ 10
lpCalcSubstDegBound(I[1],s1,s2);
↳ 9
```

### 7.10.4 freegb\_lib

**Library:** freegb.lib

**Purpose:** Two-sided Groebner bases in free algebras and tools via Letterplace approach

**Authors:** Viktor Levandovskyy, viktor.levandovskyy at math.rwth-aachen.de  
 Karim Abou Zeid, karim.abou.zeid at rwth-aachen.de  
 Grischa Studzinski, grischa.studzinski at math.rwth-aachen.de

**Overview:** For the theory, see chapter 'Letterplace' in the Singular Manual.

This library provides access to kernel functions and also contains legacy code (partially as static procedures) for compatibility reasons.

Support: Joint projects LE 2697/2-1 and KR 1907/3-1 of the Priority Programme SPP 1489: 'Algorithmische und Experimentelle Methoden in Algebra, Geometrie und Zahlentheorie' of the German DFG and Project II.6 of the transregional collaborative research centre SFB-TRR 195 'Symbolic Tools in Mathematics and their Application' of the German DFG

**Procedures:** See also: [Section 7.7 \[LETTERPLACE\]](#), page 617; [Section 7.10.1 \[fpadim\\_lib\]](#), page 640; [Section 7.10.2 \[fpalgebras\\_lib\]](#), page 646; [Section 7.10.3 \[fpaprops\\_lib\]](#), page 660.

#### 7.10.4.1 isFreeAlgebra

Procedure from library `freegb.lib` (see [Section 7.10.4 \[freegb\\_lib\]](#), page 666).

**Usage:** `isFreeAlgebra(r)`; `r` a ring

**Return:** boolean

**Purpose:** check whether `R` is a letterplace ring (free algebra)

**Example:**

```
LIB "freegb.lib";
ring r = 0,(x,y,z),dp;
isFreeAlgebra(r);
↳ 0
ring R = freeAlgebra(r, 7);
isFreeAlgebra(R);
↳ 1
```

#### 7.10.4.2 lpDegBound

Procedure from library `freegb.lib` (see [Section 7.10.4 \[freegb\\_lib\]](#), page 666).

**Usage:** `lpDegBound(R)`; `R` a letterplace ring

**Return:** int

**Purpose:** returns the degree bound of the letterplace ring

**Example:**

```
LIB "freegb.lib";
ring r = 0,(x,y,z),dp;
def R = freeAlgebra(r, 7);
lpDegBound(R);
↳ 7
```



### 7.10.4.3 lpVarBlockSize

Procedure from library `freegb.lib` (see [Section 7.10.4 \[freegb.lib\]](#), page 666).

**Usage:** `lpVarBlockSize(R)`;  $R$  a letterplace ring

**Return:** `int`

**Purpose:** returns the variable block size of the letterplace ring, that is the number of variables of the original ring.

**Example:**

```
LIB "freegb.lib";
ring r = 0,(x,y,z),dp;
ring R = freeAlgebra(r, 7);
lpVarBlockSize(R);
↦ 3
```

### 7.10.4.4 lpNcgenCount

Procedure from library `freegb.lib` (see [Section 7.10.4 \[freegb.lib\]](#), page 666).

**Usage:** `lpNcgenCount(R)`;  $R$  a letterplace ring

**Return:** `int`

**Purpose:** returns the number of ncgen variables in the letterplace ring.

**Example:**

```
LIB "freegb.lib";
ring r = 0,(x,y,z),dp;
ring R = freeAlgebra(r, 7, 3);
lpNcgenCount(R); // should be 3
↦ 3
```

### 7.10.4.5 lpDivision

Procedure from library `freegb.lib` (see [Section 7.10.4 \[freegb.lib\]](#), page 666).

**Usage:** `lpDivision(p,G)`; poly  $p$ , ideal  $G$

**Purpose:** compute a two-sided division with remainder of  $p$  wrt  $G$ ; two-sided noncommutative analogue of the procedure division

**Assume:**  $G = \{g_1, \dots, g_N\}$  is a Groebner basis, the original ring of the Letterplace ring has the name ' $r$ ' and no variable is called ' $\text{tag}_i$ ' for  $i$  in  $1 \dots N$

**Return:** list  $L$

**Note:**

- $L[1]$  is  $\text{NF}(p, I)$
- $L[2]$  is the list of expressions  $[i, l_{(ij)}, r_{(ij)}]$  with  $\sum_{(ij)} l_{(ij)} g_i r_{(ij)} = p - \text{NF}(p, I)$
- procedure `lpGBPres2Poly`, applied to  $L$ , reconstructs  $p$

**Example:**

```
LIB "freegb.lib";
ring r = 0,(x,y),dp;
ring R = freeAlgebra(r, 4);
ideal I = x*x + y*y - 1; // 2D sphere
ideal J = twostd(I); // compute a two-sided Groebner basis
```

```

J; // it is finite and nice
⇨ J[1]=x*x+y*y-1
⇨ J[2]=y*y*x-x*y*y
poly h = x*x*y-y*x*x+x*y;
list L = lpDivision(h,J); L; // what means that the NF of h wrt J is x*y
⇨ [1]:
⇨      x*y
⇨ [2]:
⇨      [1]:
⇨          1
⇨      [2]:
⇨      1
⇨      [3]:
⇨          y
⇨ [2]:
⇨      [1]:
⇨          1
⇨      [2]:
⇨          -y
⇨      [3]:
⇨          1
h - lpNF(h,J); // and this poly has the following two-sided Groebner presentation:
⇨ -y*x*x+x*x*y
⇨ -y*J[1] + J[1]*y;
⇨ -y*x*x+x*x*y
lpGBPres2Poly(L,J); // reconstructs the above automatically
⇨ -y*x*x+x*x*y+x*y

```

#### 7.10.4.6 lpGBPres2Poly

Procedure from library `freegb.lib` (see [Section 7.10.4 \[freegb.lib\]](#), page 666).

**Usage:** `lpGBPres2Poly(p,G);` poly  $p$ , ideal  $G$

**Assume:**  $L$  is a valid Groebner presentation like the result of `lpDivision`

**Return:** poly

**Note:** assembles  $p = \sum_{(i,j)} L_{(ij)} g_i r_{(ij)} + NF(p,I) = \sum_{(i)} L_{[2][i][2]} I_{[L[2][i][1]]} L_{[2][i][3]} + L[1]$

**Example:**

```

LIB "freegb.lib";
ring r = 0,(x,y),dp;
ring R = freeAlgebra(r, 4);
ideal I = x*x + y*y - 1; // 2D sphere
ideal J = twostd(I); // compute a two-sided Groebner basis
J; // it is finite and nice
⇨ J[1]=x*x+y*y-1
⇨ J[2]=y*y*x-x*y*y
poly h = x*x*y-y*x*x+x*y;
list L = lpDivision(h,J);
L[1]; // what means that the normal form (or the remainder) of h wrt J is x*y
⇨ x*y

```

```
lpGBPres2Poly(L,J); // we see, that it is equal to h from above
 $\mapsto -y*x*x+x*x*y+x*y$ 
```

#### 7.10.4.7 isOrderingShiftInvariant

Procedure from library `freegb.lib` (see [Section 7.10.4 \[freegb.lib\]](#), page 666).

**Usage:** `isOrderingShiftInvariant(b)`; `b` an integer interpreted as a boolean

**Return:** `int`

**Note:** Tests whether the ordering of the current ring is shift invariant, which is the case, when  $LM(p) > LM(p')$  for all  $p$  and  $p'$  where  $p'$  is  $p$  shifted by any number of places. If `withHoles != 0` even Letterplace polynomials with holes (eg.  $x(1)*y(4)$ ) are considered.

**Assume:** - basering is a Letterplace ring.

**Example:**

```
LIB "freegb.lib";
ring r = 0,(x,y,z),dp;
def R = freeAlgebra(r, 5);
setring R;
isOrderingShiftInvariant(0); // should be 1
 $\mapsto 1$ 
ring r2 = 0,(x,y,z),dp;
def R2 = freeAlgebra(r2, 5);
list RL = ringlist(R2);
RL[3][1][1] = "wp";
intvec weights = 1,1,1,1,1,1,1,2,3,1,1,1,1,1,1;
RL[3][1][2] = weights;
attrib(RL,"isLetterplaceRing",3);
attrib(RL,"maxExp",1);
def Rw = setLetterplaceAttributes(ring(RL),5,3);
setring Rw;
/* printlevel = voice + 1; */
isOrderingShiftInvariant(0);
 $\mapsto 0$ 
isOrderingShiftInvariant(1);
 $\mapsto 0$ 
```

#### 7.10.4.8 makeLetterplaceRing

Procedure from library `freegb.lib` (see [Section 7.10.4 \[freegb.lib\]](#), page 666).

**Usage:** `makeLetterplaceRing(d [,h])`; `d` an integer, `h` an optional integer (deprecated, use `freeAlgebra` instead)

**Return:** `ring`

**Purpose:** creates a ring with the ordering, used in letterplace computations

**Note:** `h = -1` (default) : the ordering of the current ring will be used `h = 0` : Dp ordering will be used  
`h = 2` : weights 1 used for all the variables, a tie breaker is a list of block of original ring  
`h = 1` : the pure homogeneous letterplace block ordering (applicable in the situation of homogeneous input ideals) will be used.

**Example:**

```

LIB "freegb.lib";
ring r = 0,(x,y,z),Dp;
def A = makeLetterplaceRing(2); // same as makeLetterplaceRing(2,0)
setring A; A;
⇒ // coefficients: QQ
⇒ // number of vars : 6
⇒ //      block 1 : ordering Dp
⇒ //      : names  x y z x y z
⇒ //      block 2 : ordering C
⇒ // letterplace ring (block size 3, ncgen count 0)
lpVarBlockSize(A);
⇒ 3
lpDegBound(A); // degree bound
⇒ 2
setring r; def B = makeLetterplaceRing(2,1); // to compare:
setring B; B;
⇒ // coefficients: QQ
⇒ // number of vars : 6
⇒ //      block 1 : ordering Dp
⇒ //      : names  x y z
⇒ //      block 2 : ordering Dp
⇒ //      : names  x y z
⇒ //      block 3 : ordering C
⇒ // letterplace ring (block size 3, ncgen count 0)
lpVarBlockSize(B);
⇒ 3
lpDegBound(B); // degree bound
⇒ 2
setring r; def C = makeLetterplaceRing(2,2); // to compare:
setring C; C;
⇒ // coefficients: QQ
⇒ // number of vars : 6
⇒ //      block 1 : ordering a
⇒ //      : names  x y z x y z
⇒ //      : weights 1 1 1 1 1 1
⇒ //      block 2 : ordering Dp
⇒ //      : names  x y z
⇒ //      block 3 : ordering Dp
⇒ //      : names  x y z
⇒ //      block 4 : ordering C
⇒ // letterplace ring (block size 3, ncgen count 0)
lpDegBound(C);
⇒ 2
lpDegBound(C); // degree bound
⇒ 2

```

**7.10.4.9 letplaceGBasis**

Procedure from library `freegb.lib` (see [Section 7.10.4 \[freegb.lib\]](#), page 666).

**Usage:**     `letplaceGBasis(I)`; I an ideal/module

**Return:**   ideal/module

**Assume:** basering is a Letterplace ring, input consists of Letterplace polynomials

**Purpose:** compute the two-sided Groebner basis of I via Letterplace algorithm (legacy routine)

**Note:** the degree bound for this computation is read off the letterplace structure of basering

**Example:**

```
LIB "freegb.lib";
ring r = 0,(x,y,z),Dp;
int degree_bound = 5;
def R = freeAlgebra(r, 5);
setring R;
ideal I = -x*y-7*y*y+3*x*x, x*y*x-y*x*y;
ideal J = letplaceGBasis(I);
J;
⇒ J[1]=3*x*x-x*y-7*y*y
⇒ J[2]=22*x*y*y-3*y*x*y-21*y*y*x+7*y*y*y
⇒ J[3]=3*x*y*x-22*x*y*y+21*y*y*x-7*y*y*y
⇒ J[4]=22803*y*y*y*x+19307*y*y*y*y
⇒ J[5]=1933*y*y*x*y+2751*y*y*y*x+161*y*y*y*y
⇒ J[6]=y*y*y*y*y
```

#### 7.10.4.10 lieBracket

Procedure from library `freegb.lib` (see [Section 7.10.4 \[freegb.lib\], page 666](#)).

**Usage:** `lieBracket(a,b[N])`; a,b letterplace polynomials, N an optional integer

**Return:** poly

**Assume:** basering has a letterplace ring structure

**Purpose:** compute the Lie bracket  $[a,b] = ab - ba$  between letterplace polynomials

**Note:** if  $N > 1$  is specified, then the left normed bracket  $[a, \dots [a,b]]$  is computed.

**Example:**

```
LIB "freegb.lib";
ring r = 0,(x,y),dp;
ring R = freeAlgebra(r, 4);
poly a = x*y; poly b = y;
lieBracket(a,b);
⇒ -y*x*y+x*y*y
lieBracket(x,y,2);
⇒ y*x*x-2*x*y*x+x*x*y
```

#### 7.10.4.11 setLetterplaceAttributes

Procedure from library `freegb.lib` (see [Section 7.10.4 \[freegb.lib\], page 666](#)).

**Usage:** `setLetterplaceAttributes(R, d, b)`; R a ring, b,d integers

**Return:** ring with special attributes set

**Purpose:** sets attributes for a letterplace ring:

'isLetterplaceRing' = 'IV' = b, 'uptodeg' = d, where 'uptodeg' stands for the degree bound,  
'IV' for the number of variables in the block 0.

**Note:** Activate the resulting ring by using `setring`

**Example:**

```
LIB "freegb.lib";
ring r = 0, (x(1), y(1), x(2), y(2), x(3), y(3), x(4), y(4)), dp;
def R = setLetterplaceAttributes(r, 4, 2); setring R;
lpVarBlockSize(R);
↪ 2
lieBracket(x(1), y(1), 2);
↪ y(1)*x(2)*x(3)-2*x(1)*y(2)*x(3)+x(1)*x(2)*y(3)
```

#### 7.10.4.12 testLift

Procedure from library `freegb.lib` (see [Section 7.10.4 \[freegb.lib\]](#), page 666).

**Usage:** `testLift(M, T)`; module M, matrix T

**Return:** module

**Purpose:** assembles the result of the lift procedure

**Assume:** T is the lift matrix of a submodule of M

**Note:** the inverse of the lift procedure

**Example:**

```
LIB "freegb.lib";
ring r = 0, (x, y), (c, Dp);
ring R = freeAlgebra(r, 7, 2);
ideal I = std(x*y*x + 1);
print(matrix(I)); // finite two-sided Groebner basis
↪ x*y-y*x, y*x*x+1
ideal SI = x*I[1]*y + y*x*I[2], I[1]*y*x + I[2]*y;
matrix T = lift(I, SI); // T is the lifting matrix of SI wrt I
print(T); //
↪ y*ncgen(1)*x*x+x*ncgen(1)*y, y*x*ncgen(1)+y*ncgen(1)*x+ncgen(1)*y*x,
↪ y*ncgen(2)*x, y*ncgen(2)
print(matrix(SI)); // the original generators of SI as a matrix
↪ y*x*y*x*x+x*x*y*y-x*y*x*y+y*x, x*y*y*x+y*x*x*y-y*x*y*x+y
print(matrix(testLift(I, T))); // and the result of testLift
↪ y*x*y*x*x+x*x*y*y-x*y*x*y+y*x, x*y*y*x+y*x*x*y-y*x*y*x+y
```

#### 7.10.4.13 testSyz

Procedure from library `freegb.lib` (see [Section 7.10.4 \[freegb.lib\]](#), page 666).

**Usage:** `testSyz(M, S)`; module M, S

**Return:** module

**Purpose:** tests the result of the syz procedure

**Assume:** S is the syzygy bimodule of M

**Example:**

```
LIB "freegb.lib";
ring r = 0, (x, y), (c, Dp);
ring R = freeAlgebra(r, 7, 2);
```

```

ideal I = twostd(x*y*x + 1);
print(matrix(I));
⇒ x*y-y*x,y*x*x+1
module S = syz(I);
print(S);
⇒ ncgen(1)*x*x,S[1,2],S[1,3],S[1,4],S[1,5],
⇒ S[2,1],S[2,2],S[2,3],S[2,4],S[2,5]
testSyz(I,S); // returns zero
⇒ _[1]=0
⇒ _[2]=0
⇒ _[3]=0
⇒ _[4]=0
⇒ _[5]=0

```

### 7.10.5 ncHilb\_lib

**Library:** ncHilb.lib

**Purpose:** Computation of graded and multi-graded Hilbert series of non-commutative algebras (Letterplace).

**Author:** Sharwan K. Tiwari shrawant@gmail.com  
 Roberto La Scala  
 Viktor Levandovskyy (adaptation to the new Letterplace release)

**References:**

La Scala R.: Monomial right ideals and the Hilbert series of non-commutative modules, Journal of Symbolic Computation (2016).  
 La Scala R., Tiwari Sharwan K.: Multigraded Hilbert Series of noncommutative modules, <https://arxiv.org/abs/1705.01083>.

**Procedures:**

#### 7.10.5.1 nchilb

Procedure from library `ncHilb.lib` (see [Section 7.10.5 \[ncHilb\\_lib\]](#), [page 673](#)).

**Usage:** nchilb(I, d[, L]), list I, int d, optional list L

**Purpose:** compute Hilbert series of a non-commutative algebra

**Assume:**

**Note:** d is an integer for the degree bound (maximal total degree of polynomials of the generating set of the input ideal),  
 $\#[] = 1$ , computation for non-finitely generated regular ideals,  $\#[] = 2$ , computation of multi-graded Hilbert series,  
 $\#[] = \text{tdeg}$ , for obtaining the truncated Hilbert series up to the total degree  $\text{tdeg}-1$  ( $\text{tdeg}$  should be  $> 2$ ), and  $\#[] = \text{string}(p)$ , to print the details about the orbit and system of equations. Let the orbit is  $O_I = \{T_{\{w_1\}}(I), \dots, T_{\{w_r\}}(I)\}$  ( $w_i \in W$ ), where we assume that if  $T_{\{w_i\}}(I) = T_{\{w'_i\}}(I)$  for some  $w'_i \in W$ , then  $\deg(w_i) \leq \deg(w'_i)$ .  
 Then, it prints words description of orbit:  $w_1, \dots, w_r$ . It also prints the maximal degree and the cardinality of  $\sum_j R(w_i, b_j)$  corresponding to each  $w_i$ , where  $\{b_j\}$  is a basis of I.  
 Moreover, it also prints the linear system (for the information about adjacency matrix) and its solving time.

**Note :** A Groebner basis of two-sided ideal of the input should be given in a special form. This form is a list of modules, where each generator of every module represents a monomial times a coefficient in the free associative algebra. The first entry, in each generator, represents a coefficient and every next entry is a variable.

Ex: module  $p1=[1,y,z],[-1,z,y]$ , represents the poly  $y*z-z*y$ ; module  $p2=[1,x,z,x],[-1,z,x,z]$ , represents the poly  $x*z*x-z*x*z$  for more details about the input, see examples.

**Example:**

```
LIB "ncHilb.lib";
ring r=0,(X,Y,Z),dp;
module p1=[1,Y,Z]; //represents the poly Y*Z
module p2=[1,Y,Z,X]; //represents the poly Y*Z*X
module p3=[1,Y,Z,Z,X,Z];
module p4=[1,Y,Z,Z,Z,X,Z];
module p5=[1,Y,Z,Z,Z,Z,X,Z];
module p6=[1,Y,Z,Z,Z,Z,Z,X,Z];
module p7=[1,Y,Z,Z,Z,Z,Z,Z,X,Z];
module p8=[1,Y,Z,Z,Z,Z,Z,Z,Z,X,Z];
list l1=list(p1,p2,p3,p4,p5,p6,p7,p8);
nchilb(l1,10);
↳
↳ maximal length of words = 2
↳
↳ length of the Orbit = 3
↳
↳
↳ Hilbert series:
↳ 1/(t2-3t+1)
ring r2=0,(x,y,z),dp;
module p1=[1,y,z],[-1,z,y]; //y*z-z*y
module p2=[1,x,z,x],[-1,z,x,z]; // x*z*x-z*x*z
module p3=[1,x,z,z,x,z],[-1,z,x,z,z,x]; // x*z^2*x*z-z*x*z^2*x
module p4=[1,x,z,z,z,x,z],[-1,z,x,z,z,x,x]; // x*z^3*x*z-z*x*z^2*x^2
list l2=list(p1,p2,p3,p4);
nchilb(l2,6,1); //third argument '1' is for non-finitely generated case
↳
↳ maximal length of words = 3
↳
↳ length of the Orbit = 7
↳
↳
↳ Hilbert series:
↳ (t3+t2+1)/(2t5-2t4-t3+2t2-3t+1)
ring r3=0,(a,b),dp;
module p1=[1,a,a,a];
module p2=[1,a,b,b];
module p3=[1,a,a,b];
list l3=list(p1,p2,p3);
nchilb(l3,5,2); //third argument '2' is to compute multi-graded HS
↳
↳ maximal length of words = 3
↳
↳ length of the Orbit = 5
```



```

⇒
⇒
⇒ Hilbert series:
⇒ (t1^2+t1+1)/(t1*t2^2-t1*t2-t2+1)
ring r4=0,(x,y,z),dp;
module p1=[1,x,z,y,z,x,z];
module p2=[1,x,z,x];
module p3=[1,x,z,y,z,z,x,z];
module p4=[1,y,z];
module p5=[1,x,z,z,x,z];
list l4=list(p1,p2,p3,p4,p5);
nchilb(l4,7,"p"); //third argument "p" is to print the details
⇒
⇒ maximal length of words = 3
⇒
⇒ length of the Orbit = 6
⇒ words description of the Orbit:
⇒ 1      x      y      x*z      y*z      x*z*z
⇒
⇒ maximal degree,  #(sum_j R(w,w_j))
⇒ NULL
⇒ 6,  4
⇒ 1,  1
⇒ 5,  4
⇒ 0,  1
⇒ 2,  1
⇒
⇒ linear system:
⇒ H(1) = (t)*H(2) + (t)*H(3) + (t)*H(1) + 1
⇒ H(2) = (t)*H(2) + (t)*H(3) + (t)*H(4) + 1
⇒ H(3) = (t)*H(2) + (t)*H(3) + (t)*H(5) + 1
⇒ H(4) = (t)*H(5) + (t)*H(3) + (t)*H(6) + 1
⇒ H(5) = (t)*H(5) + (t)*H(5) + (t)*H(5) + 0
⇒ H(6) = (t)*H(3) + (t)*H(3) + (t)*H(1) + 1
⇒ where H(1) represents the series corresp. to input ideal
⇒ and ith summand in the rhs of an eqn. is according
⇒ to the right colon map corresp. to the ith variable
⇒
⇒
⇒ Hilbert series:
⇒ (t3+t2+1)/(2t5-2t4-t3+2t2-3t+1)
// of the orbit and system

```

### 7.10.5.2 rcolon

Procedure from library `ncHilb.lib` (see [Section 7.10.5 \[ncHilb.lib\]](#), page 673).

**Usage:** `rcolon(list of relations, a monomial, an integer);`  
`L` is a list of modules (each module represents a monomial), `w` is a monomial  
`d` is an integer for the degree bound (maximal total degree of monomials of the generating set of the input monomial ideal),

**Note :** A two-sided monomial ideal and a monomial `w` for the input should be given in a special form. This form is a list of modules, where the generator of every module represents

a monomial times a coefficient in the free associative algebra. The first entry, in each generator, represents a coefficient, that is 1, and every next entry is a variable.

Ex: module  $p1=[1,y,z]$ , represents the monomial  $y*z$ ;  
 module  $p2=[1,x,z,x]$ , represents the monomial  $x*z*x$   
 for more details about the input, see examples.

**Example:**

```
LIB "ncHilb.lib";
ring r=0,(X,Y,Z),dp;
module w =[1,Y];
module p1 =[1,Y,Z];
module p2 =[1,Y,Z,X];
module p3 =[1,Y,Z,Z,X,Z];
module p4 =[1,Y,Z,Z,Z,X,Z];
module p5 =[1,Y,Z,Z,Z,Z,X,Z];
module p6 =[1,Y,Z,Z,Z,Z,Z,X,Z];
module p7 =[1,Y,Z,Z,Z,Z,Z,Z,X,Z];
module p8 =[1,Y,Z,Z,Z,Z,Z,Z,Z,X,Z];
list l1=list(p1,p2,p3,p4,p5,p6,p7,p8);
rcolon(l1,w,10);
↪ J[1]=Z
↪ + generators of the given ideal;
```

### 7.10.6 ncrat\_lib

Status: experimental

**Library:** ncrat.lib

**Purpose:** Framework for working with non-commutative rational functions

**Author:** Ricardo Schnur, email: ricardo.schnur@math.uni-sb.de

**Support:** This project has been funded by the SFB-TRR 195  
 'Symbolic Tools in Mathematics and their Application'.

**Overview:** This library provides a framework for working with  
 non-commutative rational functions (or rather, expressions) and their linearized representations

**References:**

T. Mai: On the analytic theory of non-commutative distributions in free probability.  
 Universitaet des Saarlandes, Dissertation, 2017

**Note:** an almost self-explaining introduction to the possibilities of the framework can be  
 achieved by running the example for the procedure `ncrepGetRegularMinimal`.

**Procedures:**

#### 7.10.6.1 ncInit

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat\\_lib\]](#), page 676).

**Usage:** `ncInit(vars);`  
 list vars containing strings

**Return:** datatypes `ncrat` and `ncrep` (and `token`, `tokenstream`, but they are not meant for users),  
sets ring as 'NCRING' with `nc` variables from list `l`

**Example:**

```
LIB "ncrat.lib";
ncInit(list("x", "y", "z"));
NCRING;
↪ // coefficients: QQ[I]/(I^2+1)
↪ // number of vars : 3
↪ //          block 1 : ordering dp
↪ //          : names  x y z
↪ //          block 2 : ordering C
```

### 7.10.6.2 ncVarsGet

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\], page 676](#)).

**Usage:** `ncVarsGet();`

**Returns:** `nc` variables that are in use

**Example:**

```
LIB "ncrat.lib";
ncInit(list("x", "y", "z"));
ncVarsGet();
↪ x,y,z
```

### 7.10.6.3 ncVarsAdd

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\], page 676](#)).

**Usage:** `ncVarsAdd(vars);`  
list `vars` contains variables

**Returns:** sets list elements as `nc` variables

**Example:**

```
LIB "ncrat.lib";
ncInit(list("x", "y", "z"));
ncVarsGet();
↪ x,y,z
ncVarsAdd(list("a", "b", "c"));
↪ // ** killing the basering for level 0
ncVarsGet();
↪ x,y,z,a,b,c
```

### 7.10.6.4 ncratDefine

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\], page 676](#)).

**Usage:** `ncrat f = ncratDefine(s, l);`  
string `s` contains kind, list `l` contains expressions

**Return:** `ncrat` with kind `s` and expressions `l`

**Note:** assignment operator '=' for ncrat is overloaded with this procedure, hence  
 ncrat f = s, l;  
 yields the same result as  
 ncrat f = ncratDefine(s, l);

**Example:**

```
LIB "ncrat.lib";
ncInit(list("x", "y", "z"));
number n = 5;
ncrat f = ncratDefine("Const", list(n));
typeof(f);
↳ ncrat
f.kind;
↳ Const
f.expr;
↳ [1]:
↳ 5
f;
↳ 5
↳
ncrat g = "Const", list(n);
g;
↳ 5
↳
```

### 7.10.6.5 ncratAdd

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\]](#), page 676).

**Usage:** ncrat h = ncratAdd(f, g);  
 f, g both of type ncrat

**Return:** h = f + g

**Note:** operator '+' for ncrat is overloaded with this procedure, hence  
 ncrat h = f + g;  
 yields the same result as  
 ncrat h = ncratAdd(f, g);

**Example:**

```
LIB "ncrat.lib";
ncInit(list("x", "y", "z"));
ncrat f = ncratFromString("2*x*y");
print(f);
↳ 2*x*y
ncrat g = ncratFromString("z");
print(g);
↳ z
ncrat h1, h2;
h1 = ncratAdd(f, g);
print(h1);
↳ 2*x*y+z
h2 = f + g;
```

```
print(h2);
↦ 2*x*y+z
```

### 7.10.6.6 ncratSubtract

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\], page 676](#)).

**Usage:**     `ncrat h = ncratSubtract(f, g);`  
               `f, g` both of type `ncrat`

**Return:**    `h = f - g`

**Note:**     operator `'-'` for `ncrat` is overloaded  
               with this procedure, hence  
               `ncrat h = f - g;`  
               yields the same result as  
               `ncrat h = ncratSubtract(f, g);`

**Example:**

```
LIB "ncrat.lib";
ncInit(list("x", "y", "z"));
ncrat f = ncratFromString("2*x*y");
print(f);
↦ 2*x*y
ncrat g = ncratFromString("z");
print(g);
↦ z
ncrat h1, h2;
h1 = ncratSubtract(f, g);
print(h1);
↦ 2*x*y-z
h2 = f - g;
print(h2);
↦ 2*x*y-z
```

### 7.10.6.7 ncratMultiply

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\], page 676](#)).

**Usage:**     `ncrat h = ncratMultiply(f, g);`  
               `f, g` both of type `ncrat`

**Return:**    `h = f * g`

**Note:**     operator `'*'` for `ncrat` is overloaded  
               with this procedure, hence  
               `ncrat h = f * g;`  
               yields the same result as  
               `ncrat h = ncratMultiply(f, g);`

**Example:**

```
LIB "ncrat.lib";
ncInit(list("x", "y", "z"));
ncrat f = ncratFromString("2*x*y");
print(f);
↦ 2*x*y
```

```

ncrat g = ncratFromString("z");
print(g);
 $\mapsto$  z
ncrat h1, h2;
h1 = ncratMultiply(f, g);
print(h1);
 $\mapsto$  2*x*y*z
h2 = f * g;
print(h2);
 $\mapsto$  2*x*y*z

```

### 7.10.6.8 ncratInvert

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\], page 676](#)).

**Usage:**     `ncrat h = ncratInvert(f);`  
               `f` of type `ncrat`

**Return:**    `h = inv(f)`

**Note:**     `ncrat h = f^-1;`  
               yields the same result as  
               `ncrat h = ncratInvert(f);`

**Example:**

```

LIB "ncrat.lib";
ncInit(list("x", "y", "z"));
ncrat f = ncratFromString("2*x*y");
print(f);
 $\mapsto$  2*x*y
ncrat h1, h2;
h1 = ncratInvert(f);
print(h1);
 $\mapsto$  inv(2*x*y)
h2 = f ^ -1;
print(h2);
 $\mapsto$  inv(2*x*y)

```

### 7.10.6.9 ncratSPrint

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\], page 676](#)).

**Usage:**     `string s = ncratSPrint(f);`  
               `f` of type `ncrat`

**Return:**    prints `f` to string

**Example:**

```

LIB "ncrat.lib";
ncInit(list("x", "y", "z"));
ncrat f = ncratFromString("2*x*y");
string s = ncratSPrint(f);
print(s);
 $\mapsto$  2*x*y

```

### 7.10.6.10 ncratPrint

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\], page 676](#)).

**Usage:**      `ncratPrint(f);`  
               `f` of type `ncrat`

**Return:**    prints `f`

**Note:**      `print(f);`  
               yields the same result as  
               `ncratPrint(f);`

**Example:**

```
LIB "ncrat.lib";
ncInit(list("x", "y", "z"));
ncrat f = ncratFromString("2*x*y");
ncratPrint(f);
↪ 2*x*y
print(f);
↪ 2*x*y
```

### 7.10.6.11 ncratFromString

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\], page 676](#)).

**Usage:**      `ncrat f = ncratFromString(s);`  
               `s` of type `string`

**Return:**    read string `s` into `ncrat f`

**Example:**

```
LIB "ncrat.lib";
ncInit(list("x", "y", "z"));
ncrat f = ncratFromString("2*x*y");
print(f);
↪ 2*x*y
```

### 7.10.6.12 ncratFromPoly

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\], page 676](#)).

**Usage:**      `ncrat f = ncratFromPoly(p);`  
               `p` of type `poly`

**Return:**    convert `poly` to `ncrat`

**Example:**

```
LIB "ncrat.lib";
ncInit(list("x", "y", "z"));
poly p = 2 * x * y;
ncrat f = ncratFromPoly(p);
print(f);
↪ 2*x*y
```

### 7.10.6.13 ncratPower

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\], page 676](#)).

**Usage:**      `ncrat h = ncratPower(f, n);`  
                  `f` of type `ncrat`, `n` integer

**Return:**     `h = f^n`

**Example:**

```
LIB "ncrat.lib";
ncInit(list("x", "y", "z"));
ncrat f = ncratFromString("2*x*y");
ncrat h = ncratPower(f, 3);
print(h);
↦ 2*x*y*2*x*y*2*x*y
```

### 7.10.6.14 ncratEvaluateAt

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\], page 676](#)).

**Usage:**      `matrix M = ncratEvaluateAt(f, vars, point);`

**Return:**     Evaluate the `ncrat f` by substituting in the  
                  matrices contained in `point` for the respective  
                  variables contained in `var`, that is, calculate  
                  `f(point)`.

**Example:**

```
LIB "ncrat.lib";
ncInit(list("x", "y"));
ncrat f = ncratFromString("x+y");
matrix A[2][2] = 1, 2, 3, 4;
matrix B[2][2] = 5, 6, 7, 8;
matrix M = ncratEvaluateAt(f, list(x, y), list(A, B));
print(M);
↦ 6, 8,
↦ 10,12
```

### 7.10.6.15 ncrepGet

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\], page 676](#)).

**Usage:**      `ncrep q = ncrepGet(f);`  
                  `f` of type `ncrat`

**Return:**     `q = (u, Q, v)` linear representation of `f`

**Example:**

```
LIB "ncrat.lib";
ncInit(list("x", "y", "z"));
ncrat f = ncratFromString("2*x*y");
ncrep q = ncrepGet(f);
print(q);
↦ lvec=
↦ 0,0,0,1
↦
```



```

↳ mat=
↳ 0, 0, 1/2*x,-1/2,
↳ 0, 1, -1/2, 0,
↳ y, -1,0, 0,
↳ -1,0, 0, 0
↳
↳ rvec=
↳ 0,
↳ 0,
↳ 0,
↳ 1

```

### 7.10.6.16 ncrepAdd

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\]](#), page 676).

**Usage:**      `ncrep s = ncrepAdd(q, r);`  
                  `q, r` both of type `ncrep`

**Return:**     representation `s` of  $h = f + g$   
                  if `q, r` are representations of `f, g`

**Note:**       operator '+' for `ncrep` is overloaded  
                  with this procedure, hence  
                  `ncrep s = q + r;`  
                  yields the same result as  
                  `ncrep s = ncrepAdd(q, r);`

**Example:**

```

LIB "ncrat.lib";
ncInit(list("x", "y", "z"));
ncrat f = ncratFromString("x");
ncrat g = ncratFromString("y");
ncrep q = ncrepGet(f);
ncrep r = ncrepGet(g);
ncrep s1, s2;
s1 = ncrepAdd(q, r);
print(s1);
↳ lvec=
↳ 0,1,0,1
↳
↳ mat=
↳ x, -1,0, 0,
↳ -1,0, 0, 0,
↳ 0, 0, y, -1,
↳ 0, 0, -1,0
↳
↳ rvec=
↳ 0,
↳ 1,
↳ 0,
↳ 1
s2 = q + r;
print(s2);
↳ lvec=

```

```

⇒ 0,1,0,1
⇒
⇒ mat=
⇒ x, -1,0, 0,
⇒ -1,0, 0, 0,
⇒ 0, 0, y, -1,
⇒ 0, 0, -1,0
⇒
⇒ rvec=
⇒ 0,
⇒ 1,
⇒ 0,
⇒ 1

```

### 7.10.6.17 ncrepSubtract

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\]](#), page 676).

**Usage:**      `ncrep s = ncrepSubtract(q, r);`  
                  `q, r` both of type `ncrep`

**Return:**     representation `s` of  $h = f - g$   
                  if `q, r` are representations of `f, g`

**Note:**       operator `'-'` for `ncrep` is overloaded  
                  with this procedure, hence  
                  `ncrep s = q - r;`  
                  yields the same result as  
                  `ncrep s = ncrepSubtract(q, r);`

**Example:**

```

LIB "ncrat.lib";
ncInit(list("x", "y", "z"));
ncrat f = ncratFromString("x");
ncrat g = ncratFromString("y");
ncrep q = ncrepGet(f);
ncrep r = ncrepGet(g);
ncrep s1, s2;
s1 = ncrepSubtract(q, r);
print(s1);
⇒ lvec=
⇒ 0,1,0,1
⇒
⇒ mat=
⇒ x, -1,0, 0,
⇒ -1,0, 0, 0,
⇒ 0, 0, -y,1,
⇒ 0, 0, 1, 0
⇒
⇒ rvec=
⇒ 0,
⇒ 1,
⇒ 0,
⇒ 1
s2 = q - r;

```

```

print(s2);
↳ lvec=
↳ 0,1,0,1
↳
↳ mat=
↳ x, -1,0, 0,
↳ -1,0, 0, 0,
↳ 0, 0, -y,1,
↳ 0, 0, 1, 0
↳
↳ rvec=
↳ 0,
↳ 1,
↳ 0,
↳ 1

```

### 7.10.6.18 ncrepMultiply

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\]](#), page 676).

- Usage:**     `ncrep s = ncrepMultiply(q, r);`  
               `q, r` both of type `ncrep`
- Return:**    representation `s` of  $h = f * g$   
               if `q, r` are representations of `f, g`
- Note:**     operator `'*'` for `ncrep` is overloaded  
               with this procedure, hence  
               `ncrep s = q * r;`  
               yields the same result as  
               `ncrep s = ncrepMultiply(q, r);`

**Example:**

```

LIB "ncrat.lib";
ncInit(list("x", "y", "z"));
ncrat f = ncratFromString("x");
ncrat g = ncratFromString("y");
ncrep q = ncrepGet(f);
ncrep r = ncrepGet(g);
ncrep s1, s2;
s1 = ncrepMultiply(q, r);
print(s1);
↳ lvec=
↳ 0,0,0,1
↳
↳ mat=
↳ 0, 0, x, -1,
↳ 0, 1, -1,0,
↳ y, -1,0, 0,
↳ -1,0, 0, 0
↳
↳ rvec=
↳ 0,
↳ 0,
↳ 0,

```

```

⇒ 1
s2 = q * r;
print(s2);
⇒ lvec=
⇒ 0,0,0,1
⇒
⇒ mat=
⇒ 0, 0, x, -1,
⇒ 0, 1, -1,0,
⇒ y, -1,0, 0,
⇒ -1,0, 0, 0
⇒
⇒ rvec=
⇒ 0,
⇒ 0,
⇒ 0,
⇒ 1

```

#### 7.10.6.19 ncrepInvert

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\]](#), page 676).

**Usage:**      `ncrep s = ncrepInvert(q);`  
                  `q` of type `ncrep`

**Return:**     representation of  $h = \text{inv}(f)$   
                  if `q` is a representation of `f`

**Example:**

```

LIB "ncrat.lib";
ncInit(list("x", "y", "z"));
ncrat f = ncratFromString("2*x*y");
ncrep q = ncrepGet(f);
ncrep s = ncrepInvert(q);
print(s);
⇒ lvec=
⇒ 1,0,0,0,0
⇒
⇒ mat=
⇒ 0,0, 0, 0,      1,
⇒ 0,0, 0, -1/2*x,1/2,
⇒ 0,0, -1,1/2,    0,
⇒ 0,-y,1, 0,      0,
⇒ 1,1, 0, 0,      0
⇒
⇒ rvec=
⇒ 1,
⇒ 0,
⇒ 0,
⇒ 0,
⇒ 0

```

#### 7.10.6.20 ncrepPrint

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\]](#), page 676).

**Usage:**     `ncrepPrint(q);`  
               `q` of type `ncrep`

**Return:**    prints `q`

**Note:**     `print(q);`  
               yields the same result as  
               `ncrepPrint(q);`

**Example:**

```
LIB "ncrat.lib";
ncInit(list("x", "y", "z"));
ncrat f = ncratFromString("2*x*y");
ncrep q = ncrepGet(f);
ncrepPrint(q);
⇒ lvec=
⇒ 0,0,0,1
⇒
⇒ mat=
⇒ 0, 0, 1/2*x,-1/2,
⇒ 0, 1, -1/2, 0,
⇒ y, -1,0, 0,
⇒ -1,0, 0, 0
⇒
⇒ rvec=
⇒ 0,
⇒ 0,
⇒ 0,
⇒ 1
print(q);
⇒ lvec=
⇒ 0,0,0,1
⇒
⇒ mat=
⇒ 0, 0, 1/2*x,-1/2,
⇒ 0, 1, -1/2, 0,
⇒ y, -1,0, 0,
⇒ -1,0, 0, 0
⇒
⇒ rvec=
⇒ 0,
⇒ 0,
⇒ 0,
⇒ 1
```

**7.10.6.21 ncrepDim**

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\]](#), page 676).

**Usage:**     `ncrepDim(q);`  
               `q` of type `ncrep`

**Return:**    dimension of `q`;  
               returns 0 if `q` represents the zero-function

**Example:**

```

LIB "ncrat.lib";
ncInit(list("x", "y", "z"));
ncrat f = ncratFromString("2*x*y");
ncrep q = ncrepGet(f);
print(q);
⇒ lvec=
⇒ 0,0,0,1
⇒
⇒ mat=
⇒ 0, 0, 1/2*x,-1/2,
⇒ 0, 1, -1/2, 0,
⇒ y, -1,0, 0,
⇒ -1,0, 0, 0
⇒
⇒ rvec=
⇒ 0,
⇒ 0,
⇒ 0,
⇒ 1
ncrepDim(q);
⇒ 4

```

### 7.10.6.22 ncrepSubstitute

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\]](#), page 676).

**Usage:**      `ncrep s = ncrepSubstitute(q, l);`  
                  `q` of type `ncrep`, `vars = (x1, ..., xg)`,  
                  `points = (A1, ... , Ag)` with `Ai` matrices of the  
                  same dimension and `xi` of type `poly` are `nc` variables

**Return:**      substitutes in `Ai` for `xi` in `q`

**Example:**

```

LIB "ncrat.lib";
ncInit(list("x", "y", "z"));
ncrat f = ncratFromString("x+y");
ncrep q = ncrepGet(f);
matrix A[2][2] = 1, 2, 3, 4;
matrix B[2][2] = 5, 6, 7, 8;
ncrep s = ncrepSubstitute(q, list(x, y), list(A, B));
print(q);
⇒ lvec=
⇒ 0,1,0,1
⇒
⇒ mat=
⇒ x, -1,0, 0,
⇒ -1,0, 0, 0,
⇒ 0, 0, y, -1,
⇒ 0, 0, -1,0
⇒
⇒ rvec=
⇒ 0,
⇒ 1,

```

```

↳ 0,
↳ 1
print(s);
↳ lvec=
↳ 0,0,1,0,0,0,1,0,
↳ 0,0,0,1,0,0,0,1
↳
↳ mat=
↳ 1, 2, -1,0, 0, 0, 0, 0,
↳ 3, 4, 0, -1,0, 0, 0, 0,
↳ -1,0, 0, 0, 0, 0, 0, 0,
↳ 0, -1,0, 0, 0, 0, 0, 0,
↳ 0, 0, 0, 0, 5, 6, -1,0,
↳ 0, 0, 0, 0, 7, 8, 0, -1,
↳ 0, 0, 0, 0, -1,0, 0, 0,
↳ 0, 0, 0, 0, 0, -1,0, 0
↳
↳ rvec=
↳ 0,0,
↳ 0,0,
↳ 1,0,
↳ 0,1,
↳ 0,0,
↳ 0,0,
↳ 1,0,
↳ 0,1

```

### 7.10.6.23 ncrepEvaluate

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\]](#), page 676).

**Usage:**      matrix M = ncrepEvaluate(q);

**Return:**     for  $q=(u, Q, v)$  calculate  $-u*Q^{(-1)}*v$

**Example:**

```

LIB "ncrat.lib";
ncInit(list("x", "y", "z"));
ncrat f = ncratFromString("x+y");
ncrep q = ncrepGet(f);
matrix A[2][2] = 1, 2, 3, 4;
matrix B[2][2] = 5, 6, 7, 8;
ncrep s = ncrepSubstitute(q, list(x, y), list(A, B));
matrix M = ncrepEvaluate(s);
print(M);
↳ 6, 8,
↳ 10,12

```

### 7.10.6.24 ncrepEvaluateAt

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\]](#), page 676).

**Usage:**      matrix M = ncrepEvaluateAt(q, vars, point);

**Return:** For  $q=(u, Q, v)$  calculate  $-u*Q(\text{point})^{(-1)*v}$ , that is to say, evaluate the ncrat represented by  $q$  at point (scalar or matrix point).

**Example:**

```
LIB "ncrat.lib";
ncInit(list("x", "y"));
ncrat f = ncratFromString("x+y");
ncrep q = ncrepGet(f);
matrix A[2][2] = 1, 2, 3, 4;
matrix B[2][2] = 5, 6, 7, 8;
matrix M = ncrepEvaluateAt(q, list(x, y), list(A, B));
print(M);
↪ 6, 8,
↪ 10,12
```

### 7.10.6.25 ncrepIsDefinedDim

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\]](#), page 676).

**Usage:** `list l = ncrepIsDefinedDim(q, N, vars, n, maxcoeff);`

**Return:** `list(k, list vars, list(A1, ..., Ak))`, where:  
 If  $k = N$  then there are matrices  $A_1, \dots, A_k$  of size  $N$  such that  $q$  is defined at  $A = (A_1, \dots, A_k)$ , i.e.,  $q.\text{mat}$  is invertible at  $A$ .  
 If  $k = 0$  then no such point was found.

**Note:** Test whether  $q.\text{mat}$  is invertible via evaluation at random matrix points with integer coefficients in  $[-\text{maxcoeff}, \text{maxcoeff}]$ . Stops after  $n$  tries.  
 Use square matrices of dimension  $N$ . The list `vars` contains the nc variables which occur in  $q$ .

**Example:**

```
LIB "ncrat.lib";
ncInit(list("x", "y"));
ncrat f = ncratFromString("inv(x*y-y*x)");
ncrep q = ncrepGet(f);
ncrepIsDefinedDim(q, 1, list(x, y), 10, 100);
↪ [1]:
↪ 0
↪ [2]:
↪ [1]:
↪ x
↪ [2]:
↪ y
↪ [3]:
↪ empty list
ncrepIsDefinedDim(q, 2, list(x, y), 10, 100);
↪ [1]:
↪ 2
↪ [2]:
↪ [1]:
↪ x
```



```

↳      [2]:
↳      y
↳ [3]:
↳      [1]:
↳      _[1,1]=-55
↳      _[1,2]=-24
↳      _[2,1]=39
↳      _[2,2]=-17
↳      [2]:
↳      _[1,1]=36
↳      _[1,2]=-58
↳      _[2,1]=-13
↳      _[2,2]=-55

```

### 7.10.6.26 ncrepIsDefined

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\]](#), page 676).

**Usage:** `list l = ncrepIsDefined(q, vars, n, maxcoeff);`

**Return:** `list(dim, list vars, list(A1, ..., Ak))`, where:  
 If `dim > 0` then there are matrices `A1, ..., Ak` of size `dim` such that `q` is defined at `A = (A1, ..., Ak)`, i.e.,  
`q.mat` is invertible at `A`.  
 If `dim = 0` then no such point was found.

**Note:** Test whether `q.mat` is invertible via evaluation  
 at random matrix points with integer coefficients  
 in `[-maxcoeff, maxcoeff]`. Stops after `n` tries.  
 Use `ixi`-matrix in `i`-th try. The list `vars` contains the  
`nc` variables which occur in `q`.

**Example:**

```

LIB "ncrat.lib";
ncInit(list("x", "y"));
ncrat f = ncratFromString("inv(x*y-y*x)");
ncrep q = ncrepGet(f);
ncrepIsDefined(q, list(x, y), 5, 10);
↳ [1]:
↳      2
↳ [2]:
↳      [1]:
↳      x
↳      [2]:
↳      y
↳ [3]:
↳      [1]:
↳      _[1,1]=0
↳      _[1,2]=-9
↳      _[2,1]=-2
↳      _[2,2]=7
↳      [2]:
↳      _[1,1]=8
↳      _[1,2]=-9

```

```

↳      _[2,1]=-4
↳      _[2,2]=-2
ncrat g = ncratFromString("inv(x-x)");
ncrep r = ncrepGet(g);
ncrepIsDefined(r, list(x), 5, 10);
↳ [1]:
↳      0
↳ [2]:
↳      [1]:
↳          x
↳ [3]:
↳      empty list

```

### 7.10.6.27 ncrepIsRegular

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\]](#), page 676).

**Usage:** `list l = ncrepIsRegular(q, vars, n, maxcoeff);`

**Return:** `list(k, list vars, list(a1, ..., ak))`, where:  
 If  $k = 1$  then there are scalars ( $1 \times 1$ -matrices)  $a_1, \dots, a_k$  such that  $q$  is defined at  $a = (a_1, \dots, a_k)$ , i.e.,  
 $q.mat$  is invertible at  $a$ .  
 If  $k = 0$  then no such point was found.

**Note:** Test whether  $q.mat$  is invertible via evaluation at random integers in  $[-maxcoeff, maxcoeff]$ .  
 Stops after  $n$  tries. The list `vars` contains the nc variables which occur in  $q$ .

**Example:**

```

LIB "ncrat.lib";
ncInit(list("x", "y"));
ncrat f = ncratFromString("inv(x*y-y*x)");
ncrep q = ncrepGet(f);
ncrepIsRegular(q, list(x, y), 10, 100);
↳ [1]:
↳      0
↳ [2]:
↳      [1]:
↳          x
↳      [2]:
↳          y
↳ [3]:
↳      empty list
ncrat g = ncratFromString("inv(1+x*y-y*x)");
ncrep r = ncrepGet(g);
ncrepIsRegular(r, list(x, y), 10, 100);
↳ [1]:
↳      1
↳ [2]:
↳      [1]:
↳          x
↳      [2]:

```

```

↳      y
↳ [3] :
↳      [1] :
↳      _[1,1]=-55
↳      [2] :
↳      _[1,1]=-24

```

### 7.10.6.28 ncrepRegularZeroMinimize

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\], page 676](#)).

**Usage:** `ncrep s = ncrepRegularZeroMinimize(q, l);`

**Return:** `ncrep s` representing the same rational function as `ncrep q`, where `s` is of minimal size

**Assumption:**

`q` is regular at zero, i.e.,  
if one substitutes in 0 for all nc variables in `q` then `q.mat` has to be invertible

**Note:** list `l = list(x1, ..., xn)` has to consist exactly of the nc variables occurring in `q`

**Example:**

```

LIB "ncrat.lib";
ncInit(list("x", "y"));
ncrat f = ncratFromString("inv(1+x*y-y*x)");
ncrep q = ncrepGet(f);
ncrepDim(q);
↳ 11
ncrep s = ncrepRegularZeroMinimize(q, list(x, y));
ncrepDim(s);
↳ 3
s;
↳ lvec=
↳ 0,1,0
↳
↳ mat=
↳ 1, y, 0,
↳ -x,1, -y,
↳ 0, -x,1
↳
↳ rvec=
↳ 0,
↳ -1,
↳ 0
↳

```

### 7.10.6.29 ncrepRegularMinimize

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\], page 676](#)).

**Usage:** `ncrep s = ncrepRegularMinimize(q, vars, point);`

**Return:** `ncrep s` representing the same rational function as `ncrep q`, where `s` is of minimal size

**Assumption:**

q is regular at scalar point a, i.e.,  
 if one substitutes in  $a_i$  for all nc variables  $x_i$  in q then q.mat has to be invertible

**Note:**

list vars = list( $x_1, \dots, x_n$ ) has to consist  
 exactly of the nc variables occurring in q and  
 list point = list( $a_1, \dots, a_n$ ) consists of scalars

**Example:**

```
LIB "ncrat.lib";
ncInit(list("x", "y"));
ncrat f = ncratFromString("inv(x*y)");
ncrep q = ncrepGet(f);
ncrepDim(q);
⇒ 5
ncrep s = ncrepRegularMinimize(q, list(x, y), list(1, 1));
ncrepDim(s);
⇒ 2
s;
⇒ lvec=
⇒ -1,0
⇒
⇒ mat=
⇒ -y,x+1,
⇒ 0, -x
⇒
⇒ rvec=
⇒ 1,
⇒ -1
⇒
```

**7.10.6.30 ncrepGetRegularZeroMinimal**

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\]](#), page 676).

**Usage:** ncrep q = ncrepGetRegularZeroMinimal(f, vars);

**Return:** q is a representation of f with  
 minimal dimension

**Assumption:**

f is regular at zero, i.e.,  
 $f(0)$  has to be defined

**Note:**

list vars = list( $x_1, \dots, x_n$ ) has to consist  
 exactly of the nc variables occurring in f

**Example:**

```
LIB "ncrat.lib";
ncInit(list("x", "y"));
ncrat f = ncratFromString("inv(1+x*y-y*x)");
list vars = list(x, y);
ncrep q = ncrepGetRegularZeroMinimal(f, vars);
q;
⇒ lvec=
⇒ 0,1,0
```

```

⇒
⇒ mat=
⇒ 1, y, 0,
⇒ -x, 1, -y,
⇒ 0, -x, 1
⇒
⇒ rvec=
⇒ 0,
⇒ -1,
⇒ 0
⇒

```

### 7.10.6.31 ncrepGetRegularMinimal

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\]](#), page 676).

**Usage:** `ncrep q = ncrepGetRegularMinimal(f, vars, point);`

**Return:** `q` is a representation of `f` with minimal dimension

**Assumption:**

`f` is regular at point, i.e.,  
`f(point)` has to be defined

**Note:** `list vars = list(x1, ..., xn)` has to consist exactly of the nc variables occurring in `f` and `list point = (p1, ..., pn)` of scalars such that `f(point)` is defined

**Example:**

```

LIB "ncrat.lib";
// We want to prove the Hua's identity, telling that for two
// invertible elements x,y from a division ring, one has
// inv(x+x*inv(y)*x)+inv(x+y) = inv(x)
// where inv(t) stands for the two-sided inverse of t
ncInit(list("x", "y"));
ncrat f = ncratFromString("inv(x+x*inv(y)*x)+inv(x+y)-inv(x)");
print(f);
⇒ inv(x+x*inv(y)*x)+inv(x+y)-inv(x)
ncrep r = ncrepGet(f);
ncrepDim(r);
⇒ 18
ncrep s = ncrepGetRegularMinimal(f, list(x, y), list(1, 1));
ncrepDim(s);
⇒ 0
print(s);
⇒ lvec=
⇒ 0
⇒
⇒ mat=
⇒ 1
⇒
⇒ rvec=
⇒ 0
// since s represents the zero element, Hua's identity holds.

```

### 7.10.6.32 ncrepPencilGet

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\]](#), page 676).

**Usage:**     `list pencil = ncrepPencilGet(r, vars);`

**Return:**     `pencil = list(vars, matrices)`,  
               where `vars = list(1, x1, ..., xg)` are the variables  
               occurring in `r` and `matrices = (Q0, ..., Qg)` is a list of matrices such that  

$$r.mat = Q_0 * x_0 + \dots + Q_g * x_g$$
  
               with  $x_0 = 1$

**Note:**     `list vars = list(x1, ..., xn)` has to consist  
               exactly of the nc variables occurring in `f`

**Example:**

```
LIB "ncrat.lib";
ncInit(list("x", "y"));
ncrat f = ncratFromString("x*y");
ncrep r = ncrepGet(f);
print(r.mat);
⇒ 0, 0, x, -1,
⇒ 0, 1, -1, 0,
⇒ y, -1, 0, 0,
⇒ -1, 0, 0, 0
list l = ncrepPencilGet(r, list(x, y));
print(l[1]);
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ x
⇒ [3]:
⇒ y
print(l[2][1]);
⇒ 0, 0, 0, -1,
⇒ 0, 1, -1, 0,
⇒ 0, -1, 0, 0,
⇒ -1, 0, 0, 0
print(l[2][2]);
⇒ 0, 0, 1, 0,
⇒ 0, 0, 0, 0,
⇒ 0, 0, 0, 0,
⇒ 0, 0, 0, 0
print(l[2][3]);
⇒ 0, 0, 0, 0,
⇒ 0, 0, 0, 0,
⇒ 1, 0, 0, 0,
⇒ 0, 0, 0, 0
```

### 7.10.6.33 ncrepPencilCombine

Procedure from library `ncrat.lib` (see [Section 7.10.6 \[ncrat.lib\]](#), page 676).

**Usage:**     `matrix Q = ncrepPencilCombine(pencil);`

**Return:** matrix  $Q = Q_0 \cdot x_0 + \dots + Q_g \cdot x_g$ ,  
 where  $\text{vars} = \text{list}(x_0, \dots, x_g)$  consists of polynomials and  $\text{matrices} = (Q_0, \dots, Q_g)$  is a  
 list of matrices

**Example:**

```
LIB "ncrat.lib";
ncInit(list("x", "y"));
ncrat f = ncratFromString("x*y");
ncrep r = ncrepGet(f);
print(r.mat);
⇒ 0, 0, x, -1,
⇒ 0, 1, -1, 0,
⇒ y, -1, 0, 0,
⇒ -1, 0, 0, 0
list l = ncrepPencilGet(r, list(x, y));
matrix Q = ncrepPencilCombine(l);
print(Q);
⇒ 0, 0, x, -1,
⇒ 0, 1, -1, 0,
⇒ y, -1, 0, 0,
⇒ -1, 0, 0, 0
```

## 7.11 Release Notes (letterplace)

# NEWS in SINGULAR:LETTERPLACE 4.4.0

## News for SINGULAR:LETTERPLACE version 4.4.0

New functions:

added support for free bimodules of a fixed rank ([Section 7.8.3 \[freeAlgebra \(letterplace\)\]](#), page 627, [Section 7.8.8 \[ncgen\]](#), page 631)

several types of monomial orderings become available, among them three types of elimination orderings

`twostd` ([Section 7.8.14 \[twostd \(letterplace\)\]](#), page 635), `reduce` ([Section 7.8.9 \[reduce \(letterplace\)\]](#), page 632) and other functions support subbimodules

`syz` ([Section 7.8.13 \[syz \(letterplace\)\]](#), page 634), `lift` ([Section 7.8.5 \[lift \(letterplace\)\]](#), page 629), `liftstd` ([Section 7.8.6 \[liftstd \(letterplace\)\]](#), page 630), `modulo` ([Section 7.8.7 \[modulo \(letterplace\)\]](#), page 631) implemented

`bracket` ([Section 7.3.2 \[bracket\]](#), page 336) and `maxideal` ([Section 5.1.88 \[maxideal\]](#), page 219) work in Letterplace

the options `redSB`, `redTail` are effective for computations related to Groebner bases

the options `prot`, `mem` are effective for the whole LETTERPLACE subsystem

New libraries:

`fpaprops.lib`: Algorithms for properties of quotient algebras ([Section 7.10.3 \[fpaprops.lib\]](#), page 660)

ncHilb.lib: Hilbert functions for non-commutative algebras ([Section 7.10.5 \[ncHilb\\_lib\]](#), [page 673](#))

Changed libraries:

fpadim.lib: Vector space dimension, basis and Hilbert series for finitely presented algebras ([Section 7.10.1 \[fpadim\\_lib\]](#), [page 640](#)), numerous enhancements, partially implemented in the kernel

freegb.lib: Main initialization and convenience tools ([Section 7.10.4 \[freegb\\_lib\]](#), [page 666](#))

Changes in the kernel/build system:

SINGULAR:LETTERPLACE is available as the dynamical module

adaptions/functions for Singular.jl(<https://github.com/oscar-system/Singular.jl>)

## News for SINGULAR:LETTERPLACE version 4-1-2

New libraries:

fpalgebras.lib: Generation of various algebras in the letterplace case ([Section 7.10.2 \[fpalgebras\\_lib\]](#), [page 646](#))

ncrat.lib: Manipulating non-commutative rational functions ([Section 7.10.6 \[ncrat\\_lib\]](#), [page 676](#))

Changed/updated libraries:

freegb.lib: lpDivision, lpPrint ([Section 7.10.4 \[freegb\\_lib\]](#), [page 666](#))

fpadim.lib ([Section 7.10.1 \[fpadim\\_lib\]](#), [page 640](#))

ncfactor.lib ([Section 7.5.12 \[ncfactor\\_lib\]](#), [page 487](#)) is available for Letterplace rings

Changes in the kernel/build system:

code for free algebras (letterplace rings) rewritten (using now the standard `+, -, *, ^, std, ...`) ([Section 7.7 \[LETTERPLACE\]](#), [page 617](#))

new command `rightstd` ([Section 7.8.10 \[rightstd \(letterplace\)\]](#), [page 633](#))

extended `twostd` to LETTERPLACE ([Section 7.8.14 \[twostd \(letterplace\)\]](#), [page 635](#), [Section 7.3.29 \[twostd \(plural\)\]](#), [page 364](#))



## Appendix A Examples

### A.1 Programming

#### A.1.1 Basic programming

We show in the example below the following:

- define the ring  $R$  of characteristic 32003, variables  $x, y, z$ , monomial ordering  $dp$  (implementing  $F_{32003}[x, y, z]$ )
- list the information about  $R$  by typing its name
- check the order of the variables
- define the integers  $a, b, c, t$
- define a polynomial  $f$  (depending on  $a, b, c, t$ ) and display it
- define the jacobian ideal  $i$  of  $f$
- compute a Groebner basis of  $i$
- compute the dimension of the algebraic set defined by  $i$  (requires the computation of a Groebner basis)
- create and display a string in order to comment the result (text between quotes " "; is a 'string')
- load a library (see [Section D.4.26 \[primdec\\_lib\]](#), page 1240)
- compute a primary decomposition for  $i$  and assign the result to a list  $L$  (which is a list of lists of ideals)
- display the number of primary components and the first primary and prime components (entries of the list  $L[1]$ )
- implement the localization of  $F_{32003}[x, y, z]$  at the homogeneous maximal ideal (generated by  $x, y, z$ ) by defining a ring with local monomial ordering ( $ds$  in place of  $dp$ )
- map  $i$  to this ring (see [Section 5.1.59 \[imap\]](#), page 198) - we may use the same name  $i$ , since ideals are ring dependent data
- compute the local dimension of the algebraic set defined by  $i$  at the origin (= dimension of the ideal generated by  $i$  in the localization)
- compute the local dimension of the algebraic set defined by  $i$  at the point  $(-2000, -6961, -7944)$  (by applying a linear coordinate transformation)

For a more basic introduction to programming in SINGULAR, we refer to [Section 2.3 \[Getting started\]](#), page 6.

```

ring R = 32003, (x, y, z), dp;
R;
⇒ // coefficients: ZZ/32003
⇒ // number of vars : 3
⇒ //      block   1 : ordering dp
⇒ //                : names   x y z
⇒ //      block   2 : ordering C
x > y;
⇒ 1
y > z;
⇒ 1

```

```

int a,b,c,t = 1,2,-1,4;
poly f = a*x3+b*xy3-c*xz3+t*xy2z2;
f;
↪ 4xy2z2+2xy3+xz3+x3
ideal i = jacob(f);    // Jacobian Ideal of f
ideal si = std(i);     // compute Groebner basis
int dimi = dim(si);
string s = "The dimension of V(i) is "+string(dimi)+".";
s;
↪ The dimension of V(i) is 1.
LIB "primdec.lib";     // load library primdec.lib
list L = primdecGTZ(i);
size(L);               // number of prime components
↪ 6
L[1][1];              // first primary component
↪ _[1]=2y2z2+y3-16001z3
↪ _[2]=x
L[1][2];              // corresponding prime component
↪ _[1]=2y2z2+y3-16001z3
↪ _[2]=x
ring Rloc = 32003,(x,y,z),ds; // ds = local monomial ordering
ideal i = imap(R,i);
dim(std(i));
↪ 1
map phi = R, x-2000, y-6961, z-7944;
dim(std(phi(i)));
↪ 0

```

### A.1.2 Writing procedures and libraries

The user may add their own commands to the commands already available in SINGULAR by writing SINGULAR procedures. There are basically two kinds of procedures:

- procedures written in the SINGULAR programming language (which are usually collected in SINGULAR libraries).
- procedures written in C/C++ (collected in dynamic modules).

At this point, we restrict ourselves to describing the first kind of (library) procedures, which are sufficient for most applications. The syntax and general structure of a library (procedure) is described in [Section 3.7 \[Procedures\], page 50](#), and [Section 3.8 \[Libraries\], page 55](#).

The probably most efficient way of writing a new library is to use one of the official SINGULAR libraries, say `ring.lib` as a sample. On a Unix-like operating system, type `LIB "ring.lib";` to get information on where the libraries are stored on your disk.

SINGULAR provides several commands and tools, which may be useful when writing a procedure, for instance, to have a look at intermediate results (see [Section 3.9 \[Debugging tools\], page 68](#)).

If such a library should be contributed to SINGULAR some formal requirements are needed:

the library header must explain the purpose of the library and (for non-trivial algorithm) a pointer to the algorithm (text book, article, etc.)

all global procedures must have a help string and an example which shows its usage.

it is strongly recommend also to provide test scripts which test the functionality: one should test the essential functionality of the library/command in a relatively short time (say, in no more than 30s), other tests should check the functionality of the library/command in detail so

that, if possible, all relevant cases/results are tested. Nevertheless, such a test should not run longer than, say, 10 minutes.

We give short examples of procedures to demonstrate the following:

- Write procedures which return an integer (ring independent), see also [Section A.4.1 \[Milnor and Tjurina number\]](#), page 735. (Here we restrict ourselves to the main body of the procedures).
  - The procedure `milnorNumber` must be called with one parameter, a polynomial. The name `g` is local to the procedure and is killed automatically when leaving the procedure. `milnorNumber` returns the Milnor number (and displays a comment).
  - The procedure `tjurinaNumber` has no specified number of parameters. Here, the parameters are referred to by `#[1]` for the 1st, `#[2]` for the 2nd parameter, etc. `tjurinaNumber` returns the Tjurina number (and displays a comment).
  - the procedure `milnor_tjurina` which returns a list consisting of two integers, the Milnor and the Tjurina number.
- Write a procedure which creates a new ring and returns data dependent on this new ring (two numbers) and an int. In this example, we also show how to write a help text for the procedure (which is optional, but recommended).

```
proc milnorNumber (poly g)
{
  "Milnor number:";
  return(vdim(std(jacob(g))));
}
```

```
proc tjurinaNumber
{
  "Tjurina number:";
  return(vdim(std(jacob(#[1])+#[1])));
}
```

```
proc milnor_tjurina (poly f)
{
  ideal j=jacob(f);
  list L=vdim(std(j)),vdim(std(j+f));
  return(L);
}
```

```
proc real_sols (number b, number c)
"USAGE: real_sols (b,c);  b,c number
ASSUME: active basering has characteristic 0
RETURN: list: first entry is an integer (the number of different real
        solutions). If this number is non-negative, the list has as second
        entry a ring in which the list SOL of real solutions of  $x^2+bx+c=0$ 
        is stored (as floating point number, precision 30 digits).
NOTE:   This procedure calls laguerre_solve from solve.lib.
"
{
  def oldring = basering; // assign name to the ring active when
                          // calling the procedure

  number disc = b^2-4*c;
  if (disc>0) { int n_of_sols = 2; }
  if (disc==0) { int n_of_sols = 1; }
  string s = nameof(var(1)); // name of first ring variable
```

```

if (disc>=0) {
  execute("ring rinC =(complex,30),("+s+"),lp;");
  if (not(defined(laguerre_solve))) { LIB "solve.lib"; }
  poly f = x2+imap(olddring,b)*x+imap(olddring,c);
  // f is a local ring-dependent variable
  list SOL = laguerre_solve(f,30);
  export SOL; // make SOL a global ring-dependent variable
  // such variables are still accessible when the
  // ring is among the return values of the proc

  setring olddring;
  return(list(n_of_sols,rinC));
}
else {
  return(list(0));
}
}

//
// We now apply the procedures which are defined by the
// lines of code above:
//
ring r = 0,(x,y),ds;
poly f = x7+y7+(x-y)^2*x2y2;

milnorNumber(f);
⇒ Milnor number:
⇒ 28
tjurinaNumber(f);
⇒ Tjurina number:
⇒ 24
milnor_tjurina(f); // a list containing Milnor and Tjurina number
⇒ [1]:
⇒ 28
⇒ [2]:
⇒ 24

def L=real_sols(2,1);
L[1]; // number of real solutions of x^2+2x+1
⇒ 1
def R1=L[2];
setring R1;
listvar(R1); // only global ring-dependent objects are still alive
⇒ // R1 [0] *ring
⇒ // SOL [0] list, size: 2
SOL; // the real solutions
⇒ [1]:
⇒ -1
⇒ [2]:
⇒ -1

setring r;
L=real_sols(1,1);
L[1]; // number of reals solutions of x^2+x+1

```

```

↦ 0

setring r;
L=real_sols(1,-5);
L[1];          // number of reals solutions of x^2+x-5
↦ 2
def R3=L[2];
setring R3; SOL;      // the real solutions
↦ [1]:
↦ -2.79128784747792000329402359686
↦ [2]:
↦ 1.79128784747792000329402359686

```

Writing a dynamic module is not as simple as writing a library procedure, since it does not only require some knowledge of C/C++, but also about the way the SINGULAR kernel works. See also [Section A.1.9 \[Dynamic modules\]](#), page 709.

### A.1.3 Rings associated to monomial orderings

In SINGULAR we may implement localizations of the polynomial ring by choosing an appropriate monomial ordering (when defining the ring by the `ring` command). We refer to [Section B.2 \[Monomial orderings\]](#), page 769 for a thorough discussion of the monomial orderings available in SINGULAR.

At this point, we restrict ourselves to describing the relation between a monomial ordering and the ring (as mathematical object) which is implemented by the ordering. This is most easily done by describing the set of units: if  $>$  is a monomial ordering then precisely those elements which have leading monomial 1 are considered as units (in all computations performed with respect to this ordering).

In mathematical terms: choosing a monomial ordering  $>$  implements the localization of the polynomial ring with respect to the multiplicatively closed set of polynomials with leading monomial 1.

That is, choosing  $>$  implements the ring

$$K[x]_{>} := \left\{ \frac{f}{u} \mid f, u \in K[x], LM(u) = 1 \right\}.$$

If  $>$  is global (that is, 1 is the smallest monomial), the implemented ring is just the polynomial ring. If  $>$  is local (that is, if 1 is the largest monomial), the implemented ring is the localization of the polynomial ring w.r.t. the homogeneous maximal ideal. For a mixed ordering, we obtain "something in between these two rings":

```

ring R = 0,(x,y,z),dp;      // polynomial ring (global ordering)
poly f = y4z3+2x2y2z2+4z4+5y2+1;
f;                          // display f in a degrevlex-ordered way
↦ y4z3+2x2y2z2+4z4+5y2+1
short=0;                   // avoid short notation
f;
↦ y^4*z^3+2*x^2*y^2*z^2+4*z^4+5*y^2+1
short=1;
leadmonom(f);              // leading monomial
↦ y4z3

ring r = 0,(x,y,z),ds;      // local ring (local ordering)

```

```

poly f = fetch(R,f);
f; // terms of f sorted by degree
 $\mapsto 1+5y^2+4z^4+2x^2y^2z^2+y^4z^3$ 
leadmonom(f); // leading monomial
 $\mapsto 1$ 

// Now we implement more "advanced" examples of rings:
//
// 1)  $(K[y]_{<y>})[x]$ 
//
int n,m=2,3;
ring A1 = 0,(x(1..n),y(1..m)),(dp(n),ds(m));
poly f = x(1)*x(2)^2+1+y(1)^10+x(1)*y(2)^5+y(3);
leadmonom(f);
 $\mapsto x(1)*x(2)^2$ 
leadmonom(1+y(1)); // unit
 $\mapsto 1$ 
leadmonom(1+x(1)); // no unit
 $\mapsto x(1)$ 

//
// 2) some ring in between  $(K[x]_{<x>})[y]$  and  $K[x,y]_{<x>}$ 
//
ring A2 = 0,(x(1..n),y(1..m)),(ds(n),dp(m));
leadmonom(1+x(1)); // unit
 $\mapsto 1$ 
leadmonom(1+x(1)*y(1)); // unit
 $\mapsto 1$ 
leadmonom(1+y(1)); // no unit
 $\mapsto y(1)$ 

//
// 3)  $K[x,y]_{<x>}$ 
//
ring A4 = (0,y(1..m)),(x(1..n)),ds;
leadmonom(1+y(1)); // in ground field
 $\mapsto 1$ 
leadmonom(1+x(1)*y(1)); // unit
 $\mapsto 1$ 
leadmonom(1+x(1)); // unit
 $\mapsto 1$ 

```

Note, that even if we implicitly compute over the localization of the polynomial ring, most computations are explicitly performed with polynomial data only. In particular,  $1/(1-x)$ ; does not return a power series expansion or a fraction but 0 (division with remainder in polynomial ring).

See [Section 5.1.26 \[division\]](#), page 174 for division with remainder in the localization and [Section D.6.17.2 \[invunit\]](#), page 1739 for a procedure returning a truncated power series expansion of the inverse of a unit.

### A.1.4 Long coefficients

The following innocent example produces in its standard basis extremely long coefficients in char 0 for the lexicographical ordering. But a very small deformation does not (the undeformed example is degenerated with respect to the Newton boundary). This example demonstrates that it might be wise, for complicated examples, to do the calculation first in positive char (e.g., 32003). It has been shown, that in complicated examples, more than 95 percent of the time needed for a standard basis computation is used in the computation of the coefficients (in char 0). The representation of long integers with real is demonstrated.

```

timer = 1;                                // activate the timer
ring R0 = 0,(x,y),lp;
poly f = x5+y11+xy9+x3y9;
ideal i = jacob(f);
ideal i1 = i,i[1]*i[2];                    // undeformed ideal
ideal i2 = i,i[1]*i[2]+1/1000000*x5y8;    // deformation of i1
i1; i2;
⇒ i1[1]=5x4+3x2y9+y9
⇒ i1[2]=9x3y8+9xy8+11y10
⇒ i1[3]=45x7y8+27x5y17+45x5y8+55x4y10+36x3y17+33x2y19+9xy17+11y19
⇒ i2[1]=5x4+3x2y9+y9
⇒ i2[2]=9x3y8+9xy8+11y10
⇒ i2[3]=45x7y8+27x5y17+45000001/1000000x5y8+55x4y10+36x3y17+33x2y19+9xy17+1\
1y19
ideal j = std(i1);
j;
⇒ j[1]=264627y39+26244y35-1323135y30-131220y26+1715175y21+164025y17+1830125\
y16
⇒ j[2]=12103947791971846719838321886393392913750065060875xy8-28639152114168\
3198701331939250003266767738632875y38-31954402206909026926764622877573565\
78554430672591y37+57436621420822663849721381265738895282846320y36+1657764\
214948799497573918210031067353932439400y35+213018481589308191195677223898\
98682697001205500y34+1822194158663066565585991976961565719648069806148y33\
-4701709279892816135156972313196394005220175y32-1351872269688192267600786\
97600850686824231975y31-3873063305929810816961516976025038053001141375y30\
+1325886675843874047990382005421144061861290080000y29+1597720195476063141\
9467945895542406089526966887310y28-26270181336309092660633348002625330426\
7126525y27-7586082690893335269027136248944859544727953125y26-867853074106\
49464602285843351672148965395945625y25-5545808143273594102173252331151835\
700278863924745y24+19075563013460437364679153779038394895638325y23+548562\
322715501761058348996776922561074021125y22+157465452677648386073957464715\
68100780933983125y21-1414279129721176222978654235817359505555191156250y20\
-20711190069445893615213399650035715378169943423125y19+272942733337472665\
573418092977905322984009750y18+789065115845334505801847294677413365720955\
3750y17+63554897038491686787729656061044724651089803125y16-22099251729923\
906699732244761028266074350255961625y14+147937139679655904353579489722585\
91339027857296625y10
⇒ j[3]=5x4+3x2y9+y9
// Compute average coefficient length (=51) by
//   - converting j[2] to a string in order to compute the number
//     of characters
//   - divide this by the number of monomials:
size(string(j[2])) div size(j[2]);
⇒ 51

```

```

vdim(j);
⇒ 63
// For a better representation normalize the long coefficients
// of the polynomial j[2] and map it to real:
poly p=(1/12103947791971846719838321886393392913750065060875)*j[2];
ring R1=real,(x,y),lp;
short=0; // force the long output format
poly p=imap(R0,p);
p;
⇒ x*y^8-(2.366e-02)*y^38-(2.640e-01)*y^37+(4.745e-06)*y^36+(1.370e-04)*y^35\
+(1.760e-03)*y^34+(1.505e-01)*y^33-(3.884e-07)*y^32-(1.117e-05)*y^31-(3.2\
00e-04)*y^30+(1.095e-01)*y^29+(1.320e+00)*y^28-(2.170e-05)*y^27-(6.267e-0\
4)*y^26-(7.170e-03)*y^25-(4.582e-01)*y^24+(1.576e-06)*y^23+(4.532e-05)*y^\
22+(1.301e-03)*y^21-(1.168e-01)*y^20-(1.711e+00)*y^19+(2.255e-05)*y^18+(6\
.519e-04)*y^17+(5.251e-03)*y^16-(1.826e+00)*y^14+(1.222e+00)*y^10
// Compute a standard basis for the deformed ideal:
setring R0; // return to the original ring R0
j = std(i2);
j;
⇒ j[1]=y16
⇒ j[2]=65610xy8+17393508y27+7223337y23+545292y19+6442040y18-119790y14+80190\
y10
⇒ j[3]=5x4+3x2y9+y9
vdim(j);
⇒ 40

```

### A.1.5 Parameters

Let us deform the ideal in [Section A.1.4 \[Long coefficients\], page 705](#) by introducing a parameter  $t$  and compute over the ground field  $Q(t)$ . We compute the dimension at the generic point, i.e.,  $\dim_{Q(t)} Q(t)[x, y]/j$ . (This gives the same result as for the deformed ideal above. Hence, the above small deformation was "generic".)

For almost all  $a \in Q$  this is the same as  $\dim_Q Q[x, y]/j_0$ , where  $j_0 = j|_{t=a}$ .

```

ring Rt = (0,t),(x,y),lp;
Rt;
⇒ // coefficients: QQ(t)
⇒ // number of vars : 2
⇒ //          block 1 : ordering lp
⇒ //          : names    x y
⇒ //          block 2 : ordering C
poly f = x5+y11+xy9+x3y9;
ideal i = jacob(f);
ideal j = i,i[1]*i[2]+t*x5y8; // deformed ideal, parameter t
vdim(std(j));
⇒ 40
ring R=0,(x,y),lp;
ideal i=imap(Rt,i);
int a=random(1,30000);
ideal j=i,i[1]*i[2]+a*x5y8; // deformed ideal, fixed integer a
vdim(std(j));
⇒ 40

```



### A.1.6 Formatting output

We show how to insert the result of a computation inside a text by using strings. First we compute the powers of 2 and comment the result with some text. Then we do the same and give the output a nice format by computing and adding appropriate space.

```
// The powers of 2:
int n;
for (n = 2; n <= 128; n = n * 2)
{"n = " + string (n);}
↪ n = 2
↪ n = 4
↪ n = 8
↪ n = 16
↪ n = 32
↪ n = 64
↪ n = 128
// The powers of 2 in a nice format
int j;
string space = "";
for (n = 2; n <= 128; n = n * 2)
{
    space = "";
    for (j = 1; j <= 5 - size (string (n)); j = j+1)
    { space = space + " "; }
    "n =" + space + string (n);
}
↪ n =    2
↪ n =    4
↪ n =    8
↪ n =   16
↪ n =   32
↪ n =   64
↪ n =  128
```

### A.1.7 Cyclic roots

We write a procedure returning a string that enables us to create automatically the ideal of cyclic roots over the basering with  $n$  variables. The procedure assumes that the variables consist of a single letter each (hence no indexed variables are allowed; the procedure `cyclic` in `polylib.lib` does not have this restriction). Then we compute a standard basis of this ideal and some numerical information. (This ideal is used as a classical benchmark for standard basis computations).

```
// We call the procedure 'cyclic':
proc cyclic (int n)
{
    string vs = varstr(basing)+varstr(basing);
    int c=find(vs,",");
    while ( c!=0 )
    {
        vs=vs[1,c-1]+vs[c+1,size(vs)];
        c=find(vs,",");
    }
    string t,s;
    int i,j;
```

```

    for ( j=1; j<=n-1; j=j+1 )
    {
        t="";
        for ( i=1; i <=n; i=i+1 )
        {
            t = t + vs[i,j] + "+";
        }
        t = t[1,size(t)-1] + "," + newline;
        s=s+t;
    }
    s=s+vs[1,n]+"-1";
    return (s);
}

ring r=0,(a,b,c,d,e),lp;          // basering, char 0, lex ordering
string sc=cyclic(nvars(basering));
sc;                                // the string of the ideal
↳ a+b+c+d+e,
↳ ab+bc+cd+de+ea,
↳ abc+bcd+cde+dea+eab,
↳ abcd+bcd+cde+dea+eab+eabc,
↳ abcde-1
execute("ideal i="+sc+";");      // this defines the ideal of cyclic roots
i;
↳ i[1]=a+b+c+d+e
↳ i[2]=ab+bc+cd+de+ea
↳ i[3]=abc+bcd+cde+dea+eab
↳ i[4]=abcd+bcd+cde+dea+eab+eabc
↳ i[5]=abcde-1
timer=1;
ideal j=std(i);
↳ //used time: 7.5 sec
size(j);                          // number of elements in the std basis
↳ 11
degree(j);
↳ // codimension = 5
↳ // dimension    = 0
↳ // degree       = 70

```

### A.1.8 Parallelization with ssi links

In this example, we demonstrate how ssi links can be used to parallelize computations.

To compute a standard basis for a zero-dimensional ideal in the lexicographical ordering, one of the two powerful routines `stdhilb` (see [Section 5.1.153 \[stdhilb\], page 273](#)) and `stdfglm` (see [Section 5.1.152 \[stdfglm\], page 272](#)) should be used. However, in general one cannot predict which one of the two commands is faster. This very much depends on the (input) example. Therefore, we use ssi links to let both commands work on the problem independently and in parallel, so that the one which finishes first delivers the result.

The example we use is the so-called "omndi example". See *Tim Wichmann; Der FGLM-Algorithmus: verallgemeinert und implementiert in Singular; Diplomarbeit Fachbereich Mathematik, Universitaet Kaiserslautern; 1997* for more details.

```

ring r=0,(a,b,c,u,v,w,x,y,z),lp;

```

```

ideal i=a+c+v+2x-1, ab+cu+2vw+2xy+2xz-2/3, ab2+cu2+2vw2+2xy2+2xz2-2/5,
ab3+cu3+2vw3+2xy3+2xz3-2/7, ab4+cu4+2vw4+2xy4+2xz4-2/9, vw2+2xyz-1/9,
vw4+2xy2z2-1/25, vw3+xyz2+xy2z-1/15, vw4+xyz3+xy3z-1/21;

link l_hilb,l_fglm = "ssi:fork","ssi:fork";          // 1.

open(l_fglm); open(l_hilb);

write(l_hilb, quote(stdhilb(i)));                    // 2.
write(l_fglm, quote(stdfglm(eval(i))));

list L=list(l_hilb,l_fglm);                          // 3.
int l_index=waitfirst(L);

if (l_index==1)
{
  "stdhilb won !!!!"; size(read(L[1]));
  close(L[1]); close(L[2]);
}
else
{
  "stdfglm won !!!!"; size(read(L[2]));
  close(L[1]); close(L[2]);
}
⇒ stdfglm won !!!!
⇒ 9

```

Some explanatory remarks are in order:

1. Instead of using links of the type `ssi:fork`, we alternatively could use `ssi:tcp` links such that the two "competing" SINGULAR processes run on different machines. This has the advantage of "true" parallel computing since no resource sharing is involved (as it usually is with forked processes).
2. Notice how quoting is used in order to prevent local evaluation (i.e., local computation of results). Since we "forked" the two competing processes, the identifier `i` is defined and has identical values in both child processes. Therefore, the innermost `eval` can be omitted (as is done for the `l_hilb` link), and only the identifier `i` needs to be communicated to the children. However, when `ssi:tcp` links are used, the inner evaluation must be applied so that actual values, and not the identifiers are communicated (as is done for the `l_fglm` link in our example).
3. We wait until one of the two children finished the computation. The main process sleeps (i.e., suspends its execution) in the intermediate time.
4. The child which has won delivers the result and is terminated with the usual `close` command. The other child which is still computing needs to be terminated by an explicit (i.e., system) kill command if running on a different computer. For `ssi:fork` a `close` is sufficient.

### A.1.9 Dynamic modules

The purpose of the following example is to illustrate the use of dynamic modules. Giving an example on how to write a dynamic module is beyond the scope of this manual. A technical reference is given at <https://www.singular.uni-kl.de/Manual/modules.pdf>.

In this example, we use a dynamic module, residing in the file `kstd.so`, which allows ignoring all but the first `j` entries of vectors when forming the pairs in the standard basis computation.

```

ring r=0,(x,y),dp;
module mo=[x^2-y^2,1,0,0],[xy+y^2,0,1,0],[y^2,0,0,1];
print(mo);
↳ x2-y2,xy+y2,y2,
↳ 1,    0,    0,
↳ 0,    1,    0,
↳ 0,    0,    1

// load dynamic module - at the same time creating package Kstd
// procedures will be available in the packages Top and Kstd
LIB("partialgb.so");
listvar(package);
↳ // Partialgb                                [0] package Partialgb (C,partialgb.so)
↳ // Singmathic                               [0] package Singmathic (C,singmathic.s\
    o)
↳ // Standard                                [0] package Standard (S,standard.lib)
↳ // Top                                      [0] package Top (T)

// set the number of components to be considered to 1
module mostd=partialStd(mo,1); // calling procedure in Top
                                // obviously computation ignored pairs with leading
                                // term in the second entry

print(mostd);
↳ 0,  0,  y2,xy,x2,
↳ -y, y,  0, 0, 1,
↳ x-y,-x, 0, 1, 0,
↳ 0,  x+y,1, -1,1

// now consider 2 components
module mostd2=Partialgb::partialStd(mo,2); // calling procedure in Partialgb
                                // this time the previously unconsidered pair was
                                // treated too

print(mostd2);
↳ 0,  0,  y2,xy,x2,
↳ 0,  y,  0, 0, 1,
↳ -y, -x+y,0, 1, 0,
↳ x+y,0,  1, -1,1

```

## A.2 Computing Groebner and Standard Bases

Several operations with ideals resp. modules uses Groebner bases to compute their result. Most allow an optional string argument to select the algorithm. The possible arguments for the algorithm are

default

std see [Section 5.1.151 \[std\]](#), page 271

slimgb see [Section 5.1.145 \[slimgb\]](#), page 265

sba see [Section 5.1.140 \[sba\]](#), page 259; not for module operations

singmatic Requires `singmatic.so`

groebner see [Section 5.1.53 \[groebner\]](#), page 191

modstd see [Section D.4.16.1 \[modStd\]](#), page 1147. Requires [Section D.4.16 \[modstd\\_lib\]](#), page 1147

**ffmod** see [Section D.4.9.7 \[ffmodStd\]](#), page 1087, Requires [Section D.4.9 \[ffmodstd\\_lib\]](#), page 1081

**nfmod** see [Section D.4.20.2 \[nfmodStd\]](#), page 1177. Requires [Section D.4.20 \[nfmodstd\\_lib\]](#), page 1175

**std:sat** Uses **satstd** instead of **std**, see [Section D.14.3.2 \[satstd\]](#), page 2239. Requires [Section D.14.3 \[customstd\\_lib\]](#), page 2238

Functions with such a choice of the algorithm:

[Section 5.1.28 \[eliminate\]](#), page 176

[Section 5.1.65 \[intersect\]](#), page 202

[Section 5.1.94 \[modulo\]](#), page 223

[Section 5.1.81 \[liftstd\]](#), page 212

[Section 5.1.156 \[syz\]](#), page 280

## A.2.1 groebner and std

The basic version of Buchberger's algorithm leaves a lot of freedom in carrying out the computational process. Considerable improvements are obtained by implementing criteria for reducing the number of S-polynomials to be actually considered (e.g., by applying the product criterion or the chain criterion). We refer to Cox, Little, and O'Shea [1997], Chapter 2 for more details and references on these criteria and on further strategies for improving the performance of Buchberger's algorithm (see also Greuel, Pfister [2002]).

SINGULAR's implementation of Buchberger's algorithm is available via the **std** command ('std' referring to **standard basis**). The computation of reduced Groebner and standard bases may be forced by setting **option(redSB)** (see [Section 5.1.111 \[option\]](#), page 234).

However, depending on the monomial ordering of the active basering, it may be advisable to use the **groebner** command instead. This command is provided by the SINGULAR library **standard.lib** which is automatically loaded when starting a SINGULAR session. Depending on some heuristics, **groebner** either refers to the **std** command (e.g., for rings with ordering **dp**), or to one of the algorithms described in the sections [Section A.2.2 \[Groebner basis conversion\]](#), page 713, [Section A.2.3 \[slim Groebner bases\]](#), page 715. For information on the heuristics behind **groebner**, see the library file **standard.lib** (see also [Section 2.3.3 \[Procedures and libraries\]](#), page 10).

We apply the commands **std** and **groebner** to compute a lexicographic Groebner basis for the ideal of cyclic roots over the basering with 6 variables (see [Section A.1.7 \[Cyclic roots\]](#), page 707). We set **option(prot)** to make SINGULAR display some information on the performed computations (see [Section 5.1.111 \[option\]](#), page 234 for an interpretation of the displayed symbols). For long running computations, it is always recommended to set this option.

```
LIB "polylib.lib";
ring r=32003,(a,b,c,d,e,f),lp;
ideal I=cyclic(6);
option(prot);
int t=timer;
system("--ticks-per-sec", 100);           // give time in 1/100 sec
ideal sI=std(I);
⇒ [1048575:2] 1(5)s2(4)s3(3)s4s(4)s5(6)s(9)s(11)s(14)s(17)-s6s(19)s(21)s(24)\
s(27)s(30)s(33)s(35)s(38)s(41)ss(42)-s----s7(41)s(43)s(46)s(48)s(51)s(54)\
s(56)s(59)s(62)s(63)s(65)s(66)s(68)s(70)s(73)s(75)s(78)---ss(81)-----\
--s(73)-----8-s(66)s(69)s(72)s(75)s(77)s(80)s(81)s(83)s(85)s(88)s(91)s\
(93)s(96)s(99)s(102)s(105)s(107)s(110)s(113)------(100)-----\
```

```

s(101)s(108)s(110)------(100)-----9-s(94)s(97)s(99)s(84)s(74)s(77)\
s(80)---ss(83)s(86)s(73)s(76)s(107)s(81)s(82)s(84)s(86)s(89)s(92)s(94)s(\
97)s(100)s(103)s(82)s(84)s(86)s(89)s(92)s(95)s(119)s(87)s(90)s(93)s(95)s\
(98)s(101)s(104)-----(100)---12-s(99)s(90)s(93)s(92)-----s(86)-----\
---13-s(74)s(77)s(79)s(82)s(85)s(88)-----14-s(64)s(67)ss(70)\
s(73)s(77)s(81)-----15-s(57)s(65)s(68)ss(71)-----\
-----s(57)----16-s(55)ss(56)-----17-s(34)s(32)-----\
-18-s(26)s(28)s-----19-s(25)s(28)s(31)-----20-s(27)s(30)s(35)-----21-s\
(23)s(26)-----22-s(22)-----23-s(15)24-s(17)-s(19)--25-s(18)s(19)s26-s(2\
1)-----27-s(11)28-s(13)--29-s(12)-30--s--31-s(11)---32-s33(7)s(10)---\
34-s-35----36-s37(6)s38s39s40---42-s43(5)s44s45--48-s49s50s51---54-s55(4)\
--67-86-
⇒ product criterion:664 chain criterion:2844
timer-t;                                     // used time (in 1/100 secs)
⇒ 15
size(sI);
⇒ 17
t=timer;
sI=groebner(I);
⇒ compute hilbert series with std in ring (ZZ/32003),(a,b,c,d,e,f,@),(dp(7)\
,C)
⇒ weights used for hilbert series: 1,1,1,1,1,1,1
⇒ [65535:2]1(5)s2(4)s3(3)s4ss5(4)s(5)s(7)-s6(8)s(9)s(11)s(13)s(16)s(18)s(21\
)--s7(22)s(23)s(24)s(27)s(29)s(31)s(32)s(35)-s(37)s(40)s(42)s(44)s(45)--s\
(46)s(48)-----8-s(44)s(47)s(50)s(52)s(55)s(57)s(59)s(61)-s(63)----s(62)--\
--s(61)s(64)-s(66)-----s(58)-----9-s(53)s(56)s(59)s(62)s(65)s(68)\
s(71)s(74)s(77)s(80)s(83)s(86)s(90)s(95)s(102)s(108)------(100)-----\
-----s(81)---10-s(83)s(88)s(90)s(94)s(99)s(104)s(109)s(114)-s(11\
6)s(121)s(126)s(128)s(132)------(100)-----\
----11-s(87)-----12-s(50)-----13-s(4\
4)s(47)s(51)s(55)-----14-s(45)s(48)s(51)s(55)s(58)s(61)s(64)s(67)\
s(70)-----15-s(52)s(55)s(58)s(61)s(64)s(67)s(70)s(73)s(76)\
s(79)s(82)-----16-----\
-----17-
⇒ product criterion:284 chain criterion:4184
⇒ std with hilb in (ZZ/32003),(a,b,c,d,e,f,@),(lp(6),dp(1),C)
⇒ [65535:2]1(98)s2(97)s3(96)s4s(97)-s5(98)s(101)s(103)s(106)s(109)---s6(107\
)s(109)s(111)s(114)s(117)s(120)s(123)s(125)s(128)s(131)ss(132)-s-----\
-s7(125)s(127)s(130)s(132)s(135)s(138)s(140)s(143)s(146)s(147)s(149)s(150\
)s(152)s(154)s(157)s(159)s(162)---ss(165)-----shhhhhhhhhhhhhhhhhhh\
hhh8(134)s(136)s(139)s(142)s(145)s(147)s(150)s(151)s(153)s(155)s(158)s(16\
1)s(163)s(166)s(169)s(172)s(175)s(177)s(180)s(183)-----\
-s(171)s(178)shhhhhhhhhhhhhhhhhhhhhhhhhhhhhhh9(147)s(150)s(153)s(155)s(\
181)s(184)s(187)s(190)s(203)s(208)s(213)s(217)s(218)s(220)s(222)s(225)---\
s-s(226)-----s(219)-----shhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhh\
hhhhhhhhhhhh10(163)s(166)s(168)s(171)s(177)s(180)s(183)s(186)shhhhhhhhhhh\
hhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhh11(125)s(128)s(13\
0)s(133)s(136)shhhhhhhhhhhhhhhhhhhhhhhhhhhhhhh12(110)s(113)s(120)s(123)s(12\
7)-----shhhhhhhhhhhhhhh13(102)s(106)s(109)s(111)s(114)s(117)---s\
hhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhh14(85)s(90)s(93)s(97)s(100)s(103)---(100)-\
s(103)shhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhh15(68)s(72)s(75)s(79)s(85)-\
---shhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhh16(42)s(45)s(49)shhhhhhhhh\
hhhhhhhhhhhhhh17(34)s(37)shhhhhhhhhhhhhhh18(27)s(30)s(32)-shhhhhhhhh19(26)s\

```

```

(29)s(32)shhhhhhhhhhh20(22)s(25)s(28)shhhhhhhhhhh21(20)s(26)shhhhhhh\
hhh22(18)shhhhhhhhh23(12)shhhhh24(11)s(14)-shhhh25(13)s(18)-s(21)shhhhh\
h26(18)shhhhhhhhhhh27(9)shhhhh28(8)shhhh29(7)shhhh30(8)-shhh31shhhhh32(7\
)shhhh33shhhh34(6)shhhhhhhhh36(2)s37(6)shhhh38shhhh39shhhhhhhhh42(2)s43(5)s\
hhh44shhhhhhhhh48s49shhhh50shhhhhhhhh54shhhh
⇒ product criterion:720 chain criterion:11620
⇒ hilbert series criterion:532
⇒ dehomogenization
⇒ simplification
⇒ imap to ring (ZZ/32003),(a,b,c,d,e,f),(lp(6),C)
timer-t; // used time (in 1/100 secs)
⇒ 10
size(sI);
⇒ 17
option(noprot);

```

### A.2.2 Groebner basis conversion

The performance of Buchberger's algorithm is sensitive to the chosen monomial order. A Groebner basis computation with respect to a less favorable order such as the lexicographic ordering may easily run out of time or memory even in cases where a Groebner basis computation with respect to a more efficient order such as the degree reverse lexicographic ordering is very well feasible. Groebner basis conversion algorithms and the Hilbert-driven Buchberger algorithm are based on this observation:

- Groebner basis conversion: Given an ideal  $I \subset K[x_1, \dots, x_n]$  and a slow monomial order, compute a Groebner basis with respect to an appropriately chosen fast order. Then convert the result to a Groebner basis with respect to the given slow order.
- Hilbert-driven Buchberger algorithm: Homogenize the given generators for  $I$  with respect to a new variable, say,  $x_0$ . Extend the given slow ordering on  $K[x_1, \dots, x_n]$  to a global product ordering on  $K[x_0, \dots, x_n]$ . Compute a Groebner basis for the ideal generated by the homogenized polynomials with respect to a fast ordering. Read the Hilbert function, and use this information when computing a Groebner basis with respect to the extended (slow) ordering. Finally, dehomogenize the elements of the resulting Groebner basis.

SINGULAR provides implementations for the FGLM conversion algorithm (which applies to zero-dimensional ideals only, see [Section 5.1.152 \[stdfglm\]](#), [page 272](#)) and variants of the Groebner walk conversion algorithm (which works for arbitrary ideals, See [Section 5.1.49 \[frwalk\]](#), [page 189](#), [Section D.4.10 \[grwalk\\_lib\]](#), [page 1088](#)). An implementation of the Hilbert-driven Buchberger algorithm is accessible via the `stdhilb` command (see also [Section 5.1.151 \[std\]](#), [page 271](#)).

For the ideal below, `stdfglm` is more than 100 times and `stdhilb` about 10 times faster than `std`.

```

ring r =32003,(a,b,c,d,e),lp;
ideal i=a+b+c+d, ab+bc+cd+ae+de, abc+bcd+abe+ade+cde,
      abc+abce+abde+acde+bcde, abcde-1;
int t=timer;
option(prot);
ideal j1=stdfglm(i);
⇒ std in (ZZ/32003),(a,b,c,d,e),(dp(5),C)
⇒ [1048575:3]1(4)s2(3)s3(2)s4s(3)s5s(4)s(5)s(6)6-ss(7)s(9)s(11)-7-ss(13)s(1\
5)s(17)--s--8-s(16)s(18)s(20)s(23)s(26)-s(23)-----9--s(16)s10(19)s(22)s\
(25)----s(24)--s11-----s12(17)s(19)s(21)-----s(17)s(19)s(21)s13(23)s\
--s-----s(20)-----14-s(12)-----15-s(6)--16-s(5)--17---
⇒ (S:21)-----

```

```
→ product criterion:109 chain criterion:322  
→ .....+....-...-..+-.....-...-.---...-++-----+--.....-...-++----.++------...--  
→ vdim= 45  
→ .....+-----  
timer-t;  
→ 0  
size(j1); // size (no. of polys) in computed GB  
→ 5  
t=timer;  
ideal j2=stdhilb(i);  
→ compute hilbert series with std in ring (ZZ/32003),(a,b,c,d,e,@),(dp(6),C\  
)  
→ weights used for hilbert series: 1,1,1,1,1,1  
→ [1048575:2]1(4)s2(3)s3(2)s4ss5(3)s(4)s(5)-s6s(6)s(7)s(9)s(11)-7-ss(13)s(1\  
5)s(17)--s--8-s(16)s(18)s(20)s(23)s(26)-s(29)-----9-s(25)s(28)--s(29)--\  
-s-----10-s(24)-----s(19)---11-s(17)s(19)s(21)-----s(18)-s(19)s12(21)\  
s(23)s(26)-s(27)-----s(23)-----13-s(15)-----14-s(6)--15-s(5)-\  
-16---  
→ product criterion:88 chain criterion:650  
→ std with hilb in (ZZ/32003),(a,b,c,d,e,@),(lp(5),dp(1),C)  
→ [1048575:2]1(41)s2(40)s3(39)s4s(40)-s5(41)s(44)s(46)s-s-sh6s(49)s(51)s(54\  
)s(55)s(56)s(58)s(59)--shhhhhh7(53)s(55)s(57)s(59)s(61)-s(62)s(68)s(70)s\  
(71)s(74)--shhhhhhhhhhhhhhhhh8(58)s(61)s(65)s(68)s(71)-s(72)s(75)-----s\  
hhhhhhhhhhhhhhhhhh9(51)s(53)s(56)s(58)s(61)s(64)-----s(61)s(64)shhhhhh\  
hhhhhhhh10(53)s(55)s(58)s(62)s(64)s(67)s(70)--s(71)-----s(68)s(71)s(73)-\  
-shhhhhhhhhhhhhhh11(58)s(60)s(63)s(66)s(69)s(72)s(74)---s-s(76)s(79)----\  
s(78)-----shhhhhhhhhhhhhhhhhhh12(51)s(54)s(57)s(58)s(60)s(63)s(65)s\  
(68)s(70)s(73)s(76)s(79)--s(80)----shhhhhhhhhhhhhhhhhhhhhhhhhhh13(48)s\  
(51)s(54)s(57)s(59)s(61)s(64)s(67)shhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhhh\`  
h14(31)s(33)s(36)s(39)s(42)s(45)shhhhhhhhhhhhhhhhhhhhhhhhhhh15(23)s(26)s(29\  
)s(32)s(35)shhhhhhhhhhhhhhhhhhh16(18)s(21)s(24)s(27)shhhhhhhhhhhhhhh17(15\  
)s(18)s(21)s(24)shhhhhhhhhhh18(15)s(18)s(21)s(24)shhhhhhhhhhh19(14)s(17\  
)s(20)shhhhhhhhhhh20(11)s(14)s(17)shhhhhhhhh21(11)s(14)s(17)shhhhhhhhh22\  
(11)s(14)s(16)shhhhhhhhh23(10)s(13)shhhhhhhhh24(7)s(10)shhhhhh25(7)s(10)s\  
hhhhhh26(7)s(10)shhhhhh27(7)s(10)shhhhhh28(7)s(10)shhhhhh29(7)s(10)shhhhh\  
h30(7)s(9)shhhhhh31(6)shhhhhh32(3)shhh33shhh34shhh35shhh36shhh37shhh38shh\  
h39shhh40shhh41shhh42shhh43shhh44shhh45shhh46shhh47shhh48shhh49shhh50shhh\  
51shhh52shhh53shhh54shhhhhh  
→ product criterion:491 chain criterion:11799  
→ hilbert series criterion:417  
→ dehomogenization  
→ simplification  
→ imap to ring (ZZ/32003),(a,b,c,d,e),(lp(5),C)  
timer-t;  
→ 0  
size(j2); // size (no. of polys) in computed GB  
→ 5  
// usual Groebner basis computation for lex ordering  
t=timer;  
ideal j0 =std(i);  
→ [1048575:2]1(4)s2(3)s3(2)s4s(3)s5(5)s(4)s6(6)s(7)s(9)s(8)sss7(10)s(11)s(1\  
0)s(11)s(13)s8(12)s(13)s(15).s.s(14).s.9.s(16)s(17)s(19).....10.s(20).s\  
(21)ss..11.s(23)s(25).ss(27)...s(28)s(26)...12.s(25)sss(23)sss.....s(22)
```



```

) ... 13.s(23)ssssssss(21)s(22)sssss(21)ss..14.ss(22)s.s.sssss(21)s(22)sss.\
s...15.ssss(21)s(22)ssssssssss(21)s(22)sss16.sssssss(21)s(22)ssssssssss\
17ss(21)s(22)ssssssssss(21)sss(22)ss(21)ss18(22)s(21)s(22)s.s.....\
..19.sssss(21)ss(22)ssssssssss(21)s(22)s20.sssssssss(21)s.....21.s(22\
)ssssssssssssss(21)s(22)ssss22ssssssssssss(21)s(22)ssssss23ssssssssss(\
21)s(22)ssssssss24ssssssssssss(21)s(22)ssssss25ssssssssss(21)s(22)ssssss\
sss26ssssssssss(21)s(20)ssssssss27.sssssssss.....s28.sssssss.....\
....29.sssssssssssssssssss30ssssssssssssssssss31.sssssssssssssssssss32.s\
ssssssssssssssssss33ssssssssssssssssss34ssssssssssssssssss35ssssssss\
ssssssssssss36ssssssssssssssssss37ssssssssssssssssss38ssssssssssssss\
ssss39ssssssssssssssssss40ssssssssssssssssss41ssss-----42-\
s(4)--43-s44s45s46s47s48s49s50s51s52s53s54s55s56s
⇒ product criterion:1395 chain criterion:904
option(noprot);
timer-t;
⇒ 0

```

### A.2.3 slim Groebner bases

The command `slimgb` calls an implementation of an algorithm to compute Groebner bases which is designed for keeping the polynomials slim (short with small coefficients) during a Groebner basis computation. It provides, in particular, a fast algorithm for computing Groebner bases over function fields or over the rational numbers, but also in several other cases. The algorithm which is still under development was developed in the diploma thesis of Michael Brickenstein. It has been published as [https://www.singular.uni-kl.de/reports/35/paper\\_35\\_full.ps.gz](https://www.singular.uni-kl.de/reports/35/paper_35_full.ps.gz).

In the example below (Groebner basis with respect to degree reverse lexicographic ordering over function field) `slimgb` is much faster than the `std` command.

```

ring r=(32003,u1, u2, u3, u4),(x1, x2, x3, x4, x5, x6, x7),dp;
timer=1;
ideal i=
-x4*u3+x5*u2,
x1*u3+2*x2*u1-2*x2*u2-2*x3*u3-u1*u4+u2*u4,
-2*x1*x5+4*x4*x6+4*x5*x7+x1*u3-2*x4*u1-2*x4*u4-2*x6*u2-2*x7*u3+u1*u2+u2*u4,
-x1*x5+x1*x7-x4*u1+x4*u2-x4*u4+x5*u3+x6*u1-x6*u2+x6*u4-x7*u3,
-x1*x4+x1*u1-x5*u1+x5*u4,
-2*x1*x3+x1*u3-2*x2*u4+u1*u4+u2*u4,
x1^2*u3+x1*u1*u2-x1*u2^2-x1*u3^2-u1*u3*u4+u3*u4^2;
i=slimgb(i);

```

For detailed information and limitations see [Section 5.1.145 \[slimgb\]](#), page 265.

## A.3 Commutative Algebra

### A.3.1 Saturation

For any two ideals  $i, j$  in the basering  $R$  let

$$\text{sat}(i, j) = \{x \in R \mid \exists n \text{ s.t. } x \cdot (j^n) \subseteq i\} = \bigcup_{n=1}^{\infty} i : j^n$$

denote the saturation of  $i$  with respect to  $j$ . This defines, geometrically, the closure of the complement of  $V(j)$  in  $V(i)$  (where  $V(i)$  denotes the variety defined by  $i$ ).

The saturation is computed by the procedure `sat` in `elim.lib` by computing iterated ideal quotients with the maximal ideal. `sat` returns a list of two elements: the saturated ideal and the number of iterations.

We apply saturation to show that a variety has no singular points outside the origin (see also [Section A.4.2 \[Critical points\], page 737](#)). We choose  $m$  to be the homogeneous maximal ideal (note that `maxideal(n)` denotes the  $n$ -th power of the maximal ideal). Then  $V(i)$  has no singular point outside the origin if and only if  $\text{sat}(j + (f), m)$  is the whole ring, that is, generated by 1.

```
LIB "elim.lib";          // loading library elim.lib
ring r2 = 32003,(x,y,z),dp;
poly f = x^11+y^5+z^(3*3)+x^(3+2)*y^(3-1)+x^(3-1)*y^(3-1)*z3+
        x^(3-2)*y^3*(y^2)^2;
ideal j=jacob(f);
sat(j+f,maxideal(1));
⇒ _[1]=1
   // list the variables defined so far:
   listvar();
⇒ // r2                                [0] *ring
⇒ //      j                            [0] ideal, 3 generator(s)
⇒ //      f                            [0] poly
```

### A.3.2 Finite fields

We define a variety in the  $n$ -space of codimension 2 defined by polynomials of degree  $d$  with generic coefficients over the prime field  $\mathbb{Z}/p$  and look for zeros on the torus. First over the prime field and then in the finite extension field with  $p^k$  elements. In general there will be many more solutions in the second case. (Since the SINGULAR language is interpreted, the evaluation of many `for`-loops is not very fast):

```
int p=3; int n=3; int d=5; int k=2;
ring rp = p,(x(1..n)),dp;
int s = size(maxideal(d));
s;
⇒ 21
   // create a dense homogeneous ideal m, all generators of degree d, with
   // generic (random) coefficients:
   ideal m = maxideal(d)*random(p,s,n-2);
m;
⇒ m[1]=x(1)^3*x(2)^2-x(1)*x(2)^4+x(1)^4*x(3)-x(1)^3*x(2)*x(3)+x(1)*x(2)^3*x\
  (3)+x(2)^4*x(3)+x(2)^3*x(3)^2+x(1)*x(2)*x(3)^3+x(1)*x(3)^4-x(3)^5
   // look for zeros on the torus by checking all points (with no component 0)
   // of the affine n-space over the field with p elements :
   ideal mt;
   int i(1..n);          // initialize integers i(1),...,i(n)
   int l;
   s=0;
   for (i(1)=1;i(1)<p;i(1)=i(1)+1)
   {
     for (i(2)=1;i(2)<p;i(2)=i(2)+1)
     {
       for (i(3)=1;i(3)<p;i(3)=i(3)+1)
       {
         mt=m;
         for (l=1;l<=n;l=l+1)
```

```

        {
            mt=subst(mt,x(1),i(1));
        }
        if (size(mt)==0)
        {
            "solution:",i(1..n);
            s=s+1;
        }
    }
}

⇒ solution: 1 1 2
⇒ solution: 1 2 1
⇒ solution: 1 2 2
⇒ solution: 2 1 1
⇒ solution: 2 1 2
⇒ solution: 2 2 1
    "/",s,"solutions over GF("+string(p)+")";
⇒ // 6 solutions over GF(3)
    // Now go to the field with p^3 elements:
    // As long as there is no map from Z/p to the field with p^3 elements
    // implemented, use the following trick: convert the ideal to be mapped
    // to the new ring to a string and then execute this string in the
    // new ring
    string ms="ideal m="+string(m)+" ";
    ms;
⇒ ideal m=x(1)^3*x(2)^2-x(1)*x(2)^4+x(1)^4*x(3)-x(1)^3*x(2)*x(3)+x(1)*x(2)^3*
    3*x(3)+x(2)^4*x(3)+x(2)^3*x(3)^2+x(1)*x(2)*x(3)^3+x(1)*x(3)^4-x(3)^5;
    // define a ring rpK with p^k elements, call the primitive element z. Hence
    // 'solution exponent: 0 1 5' means that (z^0,z^1,z^5) is a solution
    ring rpK=(p^k,z),(x(1..n)),dp;
    rpK;
⇒ // coefficients: ZZ/9[z]
⇒ // minpoly      : 1*z^2+2*z^1+2*z^0
⇒ // number of vars : 3
⇒ //          block 1 : ordering dp
⇒ //          : names  x(1) x(2) x(3)
⇒ //          block 2: ordering C
    execute(ms);
    s=0;
    ideal mt;
    for (i(1)=0;i(1)<p^k-1;i(1)=i(1)+1)
    {
        for (i(2)=0;i(2)<p^k-1;i(2)=i(2)+1)
        {
            for (i(3)=0;i(3)<p^k-1;i(3)=i(3)+1)
            {
                mt=m;
                for (l=1;l<=n;l=l+1)
                {
                    mt=subst(mt,x(l),z^i(l));
                }
                if (size(mt)==0)

```

```

→ solution exponent: 0 0 2
→ solution exponent: 0 0 4
→ solution exponent: 0 0 6
→ solution exponent: 0 4 0
→ solution exponent: 0 4 1
    "/"",s,"solutions over GF("+string(p^k)+)";
→ // 72 solutions over GF(9)

```

### A.3.3 Elimination

Elimination is the algebraic counterpart of the geometric concept of projection. If  $f = (f_1, \dots, f_n) : k^r \rightarrow k^n$  is a polynomial map, the Zariski-closure of the image is the zero-set of the ideal  $j = J \cap k[x_1, \dots, x_n]$ , where

$$J = (x_1 - f_1(t_1, \dots, t_r), \dots, x_n - f_n(t_1, \dots, t_r)) \subseteq k[t_1, \dots, t_r, x_1, \dots, x_n]$$

that is, of the ideal  $j$  obtained from  $J$  by eliminating the variables  $t_1, \dots, t_r$ . This can be done by computing a Groebner basis for  $J$  with respect to a (global) product ordering where the block of  $t$ -variables precedes the block of  $x$ -variables, and then selecting those polynomials which do not contain any  $t$ . Alternatively, we may use a global monomial ordering with extra weight vector (see [Section B.2.8 \[Extra weight vector\]](#), page 773), assigning to the  $t$ -variables a positive weight and to the  $x$ -variables weight 0.

Since elimination is expensive, it may be useful to use a Hilbert-driven approach to the elimination problem (see [Section A.2.2 \[Groebner basis conversion\]](#), page 713):

First compute the Hilbert function of the ideal w.r.t. a fast ordering (e.g., `dp`), then make use of it to speed up the computation by a Hilbert-driven elimination which uses the `intvec` provided as third argument.

In SINGULAR the most convenient way is to use the `eliminate` command. In contrast to the first method, with `eliminate` the result needs not be a standard basis in the given ordering. Hence, there may be cases where the first method is the preferred one.

**WARNING:** In the case of a local or a mixed ordering, elimination needs special care. `f` may be considered as a map of germs  $f : (k^r, 0) \rightarrow (k^n, 0)$ , but even if this map germ is finite, we are in general not able to compute the image germ because for this we would need an implementation of the Weierstrass preparation theorem. What we can compute, and what `eliminate` actually does, is the following: let  $V(J)$  be the zero-set of  $J$  in  $k^r \times (k^n, 0)$ , then the closure of the image of  $V(J)$  under the projection

$$\text{pr} : k^r \times (k^n, 0) \rightarrow (k^n, 0)$$

can be computed. (Note that this germ contains also those components of  $V(J)$  which meet the fiber of  $\text{pr}$  outside the origin.) This is achieved by an ordering with the block of  $t$ -variables having a global ordering (and preceding the  $x$ -variables) and the  $x$ -variables having a local ordering.

In any case, if the input is weighted homogeneous (=quasihomogeneous), the weights given to the variables should be chosen accordingly. SINGULAR offers a function `weight` which proposes,

given an ideal or module, integer weights for the variables, such that the ideal, resp. module, is as homogeneous as possible with respect to these weights. The function finds correct weights, if the input is weighted homogeneous (but is rather slow for many variables). In order to check, whether the input is quasihomogeneous, use the function `qhweight`, which returns an `intvec` of correct weights if the input is quasihomogeneous and an `intvec` of zeros otherwise.

Let us give three examples:

1. First we compute the equations of the simple space curve 'T[7]' consisting of two tangential cusps given in parametric form.
2. We compute weights for the equations such that the equations are quasihomogeneous w.r.t. these weights.
3. Then we compute the tangent developable of the rational normal curve in  $P^4$ .

```
// 1. Compute equations of curve given in parametric form:
// Two transversal cusps in (k^3,0):
ring r1 = 0,(t,x,y,z),ls;
ideal i1 = x-t2,y-t3,z;           // parametrization of the first branch
ideal i2 = y-t2,z-t3,x;           // parametrization of the second branch
ideal j1 = eliminate(i1,t);
j1;                                // equations of the first branch
⇨ j1[1]=z
⇨ j1[2]=y2-x3
ideal j2 = eliminate(i2,t);
j2;                                // equations of the second branch
⇨ j2[1]=z2-y3
⇨ j2[2]=x
// Now map to a ring with only x,y,z as variables and compute the
// intersection of j1 and j2 there:
ring r2 = 0,(x,y,z),ds;
ideal j1= imap(r1,j1);            // imap is a convenient ringmap for
ideal j2= imap(r1,j2);            // inclusions and projections of rings
ideal i = intersect(j1,j2);
i;                                // equations of both branches
⇨ i[1]=z2-y3+x3y
⇨ i[2]=xz
⇨ i[3]=xy2-x4
//
// 2. Compute the weights:
intvec v= qhweight(i);            // compute weights
v;
⇨ 4,6,9
//
// 3. Compute the tangent developable
// The tangent developable of a projective variety given parametrically
// by  $F=(f_1,\dots,f_n) : P^r \dashrightarrow P^n$  is the union of all tangent spaces
// of the image. The tangent space at a smooth point  $F(t_1,\dots,t_r)$ 
// is given as the image of the tangent space at  $(t_1,\dots,t_r)$  under
// the tangent map (affine coordinates)
//  $T(t_1,\dots,t_r): (y_1,\dots,y_r) \dashrightarrow \text{jacob}(f)*\text{transpose}((y_1,\dots,y_r))$ 
// where  $\text{jacob}(f)$  denotes the jacobian matrix of  $f$  with respect to the
//  $t$ 's evaluated at the point  $(t_1,\dots,t_r)$ .
// Hence we have to create the graph of this map and then to eliminate
// the  $t$ 's and  $y$ 's.
```

```

// The rational normal curve in  $P^4$  is given as the image of
//       $F(s,t) = (s^4, s^3t, s^2t^2, st^3, t^4)$ 
// each component being homogeneous of degree 4.
ring P = 0,(s,t,x,y,a,b,c,d,e),dp;
ideal M = maxideal(1);
ideal F = M[1..2];      // take the 1st two generators of M
F=F^4;
// simplify(...,2); deletes 0-columns
matrix jac = simplify(jacob(F),2);
ideal T = x,y;
ideal J = jac*transpose(T);
ideal H = M[5..9];
ideal i = matrix(H)-matrix(J); // this is tricky: difference between two
                                // ideals is not defined, but between two
                                // matrices. By type conversion
                                // the ideals are converted to matrices,
                                // subtracted and afterwards converted
                                // to an ideal. Note that '+' is defined
                                // and adds (concatenates) two ideals

i;
⇒ i[1]=-4s3x+a
⇒ i[2]=-3s2tx-s3y+b
⇒ i[3]=-2st2x-2s2ty+c
⇒ i[4]=-t3x-3st2y+d
⇒ i[5]=-4t3y+e
// Now we define a ring with product ordering and weights 4
// for the variables a,...,e.
// Then we map i from P to P1 and eliminate s,t,x,y from i.
ring P1 = 0,(s,t,x,y,a,b,c,d,e),(dp(4),wp(4,4,4,4,4));
ideal i = fetch(P,i);
ideal j= eliminate(i,stxy);    // equations of tangent developable
j;
⇒ j[1]=3c2-4bd+ae
⇒ j[2]=2bcd-3ad2-3b2e+4ace
⇒ j[3]=8b2d2-9acd2-9b2ce+14abde-4a2e2
// We can use the product ordering to eliminate s,t,x,y from i
// by a std-basis computation.
// We need proc 'nselect' from elim.lib.
LIB "elim.lib";
j = std(i);                    // compute a std basis j
j = nselect(j,1..4);           // select generators from j not
j;                              // containing variable 1,...,4
⇒ j[1]=3c2-4bd+ae
⇒ j[2]=2bcd-3ad2-3b2e+4ace
⇒ j[3]=8b2d2-9acd2-9b2ce+12ac2e-2abde

```

### A.3.4 Free resolution

In SINGULAR a free resolution of a module or ideal has its own type: **resolution**. It is a structure that stores all information related to free resolutions. This allows partial computations of resolutions via the command **res**. After applying **res**, only a pre-format of the resolution is computed which allows to determine invariants like Betti-numbers or homological dimension. To see the differentials of the complex, a resolution must be converted into the type list which yields a list of modules:

the  $k$ -th module in this list is the first syzygy-module (module of relations) of the  $(k-1)$ st module. There are the following commands to compute a resolution:

<b>res</b>	<a href="#">Section 5.1.134 [res]</a> , page 253 computes a free resolution of an ideal or module using a heuristically chosen method. This is the preferred method to compute free resolutions of ideals or modules.
<b>fres</b>	<a href="#">Section 5.1.48 [fres]</a> , page 188 improved version of <a href="#">Section 5.1.149 [sres]</a> , page 269, computes a free resolution of an ideal or module using Schreyer's method. The input has to be a standard basis.
<b>lres</b>	<a href="#">Section 5.1.83 [lres]</a> , page 215 computes a free resolution of an ideal or module with LaScala's method. The input needs to be homogeneous.
<b>mres</b>	<a href="#">Section 5.1.98 [mres]</a> , page 225 computes a minimal free resolution of an ideal or module with the syzygy method.
<b>sres</b>	<a href="#">Section 5.1.149 [sres]</a> , page 269 computes a free resolution of an ideal or module with Schreyer's method. The input has to be a standard basis.
<b>nres</b>	<a href="#">Section 5.1.106 [nres]</a> , page 232 computes a free resolution of an ideal or module with the standard basis method.
<b>minres</b>	<a href="#">Section 5.1.93 [minres]</a> , page 223 minimizes a free resolution of an ideal or module.
<b>syz</b>	<a href="#">Section 5.1.156 [syz]</a> , page 280 computes the first syzygy module.

`res(i,r)`, `lres(i,r)`, `sres(i,r)`, `mres(i,r)`, `nres(i,r)` compute the first  $r$  modules of the resolution of  $i$ , resp. the full resolution if  $r=0$  and the basering is not a qring. See the manual for a precise description of these commands.

Note: The command `betti` does not require a minimal resolution for the minimal Betti numbers.

Now let us take a look at an example which uses resolutions: The Hilbert-Burch theorem says that the ideal  $i$  of a reduced curve in  $K^3$  has a free resolution of length 2 and that  $i$  is given by the  $2 \times 2$  minors of the 2nd matrix in the resolution. We test this for two transversal cusps in  $K^3$ . Afterwards we compute the resolution of the ideal  $j$  of the tangent developable of the rational normal curve in  $P^4$  from above. Finally we demonstrate the use of the type `resolution` in connection with the `lres` command.

```
// Two transversal cusps in (k^3,0):
ring r2 =0,(x,y,z),ds;
ideal i =z2-1y3+x3y,xz,-1xy2+x4,x3z;
resolution rs=mres(i,0); // computes a minimal resolution
rs; // the standard representation of complexes
⇒ 1      3      2
⇒ r2 <-- r2 <-- r2
⇒
⇒ 0      1      2
⇒
list resi=rs; // conversion to a list
print(resi[1]); // the 1st module is i minimized
⇒ xz,
⇒ z2-y3+x3y,
⇒ xy2-x4
```

```

    print(resi[2]);          // the 1st syzygy module of i
    ↪ -z,-y2+x3,
    ↪ x, 0,
    ↪ y, z
    resi[3];                // the 2nd syzygy module of i
    ↪ _[1]=0
    ideal j=minor(resi[2],2);
    reduce(j,std(i));        // check whether j is contained in i
    ↪ _[1]=0
    ↪ _[2]=0
    ↪ _[3]=0
    size(reduce(i,std(j)));   // check whether i is contained in j
    ↪ 0
    // size(<ideal>) counts the non-zero generators
    // -----
    // The tangent developable of the rational normal curve in P4:
    ring P = 0,(a,b,c,d,e),dp;
    ideal j= 3c2-4bd+ae, -2bcd+3ad2+3b2e-4ace,
            8b2d2-9acd2-9b2ce+9ac2e+2abde-1a2e2;
    resolution rs=mres(j,0);
    rs;
    ↪ 1      2      1
    ↪ P <--  P <--  P
    ↪
    ↪ 0      1      2
    ↪
    list L=rs;
    print(L[2]);
    ↪ 2bcd-3ad2-3b2e+4ace,
    ↪ -3c2+4bd-ae
    // create an intmat with graded Betti numbers
    intmat B=betti(rs);
    // this gives a nice output of Betti numbers
    print(B,"betti");
    ↪          0      1      2
    ↪ -----
    ↪ 0:      1      -      -
    ↪ 1:      -      1      -
    ↪ 2:      -      1      -
    ↪ 3:      -      -      1
    ↪ -----
    ↪ total:   1      2      1
    ↪
    // the user has access to all Betti numbers
    // the 2-nd column of B:
    B[1..4,2];
    ↪ 0 1 1 0
    ring cyc5=32003,(a,b,c,d,e,h),dp;
    ideal i=
    a+b+c+d+e,
    ab+bc+cd+de+ea,
    abc+bcd+cde+dea+eab,
    abcd+bcde+cdea+deab+eabc,

```



```

h5-abcde;
resolution rs=lres(i,0); //computes the resolution according LaScala
rs;                       //the shape of the minimal resolution
⇒      1      5      10      10      5      1
⇒ cyc5 <-- cyc5 <-- cyc5 <-- cyc5 <-- cyc5 <-- cyc5
⇒
⇒ 0      1      2      3      4      5
⇒
print(betti(rs),"betti"); //shows the Betti-numbers of cyclic 5
⇒      0      1      2      3      4      5
⇒ -----
⇒ 0:      1      1      -      -      -      -
⇒ 1:      -      1      1      -      -      -
⇒ 2:      -      1      1      -      -      -
⇒ 3:      -      1      2      1      -      -
⇒ 4:      -      1      2      1      -      -
⇒ 5:      -      -      2      2      -      -
⇒ 6:      -      -      1      2      1      -
⇒ 7:      -      -      1      2      1      -
⇒ 8:      -      -      -      1      1      -
⇒ 9:      -      -      -      1      1      -
⇒ 10:     -      -      -      -      1      1
⇒ -----
⇒ total:      1      5      10      10      5      1
⇒
dim(rs); //the homological dimension
⇒ 4
size(list(rs)); //gets the full (non-reduced) resolution
⇒ 6
minres(rs); //minimizes the resolution
⇒      1      5      10      10      5      1
⇒ cyc5 <-- cyc5 <-- cyc5 <-- cyc5 <-- cyc5 <-- cyc5
⇒
⇒ 0      1      2      3      4      5
⇒
size(list(rs)); //gets the minimized resolution
⇒ 6

```

### A.3.5 Handling graded modules

How to deal with graded modules in SINGULAR is best explained by looking at an example:

```

ring R = 0, (w,x,y,z), dp;
module I = [-x,0,-z2,0,y2z], [0,-x,-yz,0,y3], [-w,0,0,yz,-z3],
           [0,-w,0,y2,-yz2], [0,-1,-w,0,xz], [0,-w,0,y2,-yz2],
           [x2,-y2,-wy2+xz2];
print(I);
⇒ -x, 0, -w, 0, 0, 0, x2,
⇒ 0, -x, 0, -w, -1,-w, -y2,
⇒ -z2,-yz,0, 0, -w,0, -wy2+xz2,
⇒ 0, 0, yz, y2, 0, y2, 0,
⇒ y2z,y3, -z3,-yz2,xz,-yz2,0

// (1) Check on degrees:

```

```
// =====
attrib(I,"isHomog"); // attribute not set => empty output
↳
homog(I);
↳ 1
attrib(I,"isHomog");
↳ 2,2,1,1,0

print(betti(I,0),"beti"); // read degrees from Betti diagram
↳          0      1
↳ -----
↳    0:      1      -
↳    1:      2      1
↳    2:      2      5
↳    3:      -      1
↳ -----
↳ total:      5      7
↳

// (2) Shift degrees:
// =====
def J=I;
intvec DV = 0,0,-1,-1,-2;
attrib(J,"isHomog",DV); // assign new weight vector
attrib(J,"isHomog");
↳ 0,0,-1,-1,-2
print(betti(J,0),"beti");
↳          0      1
↳ -----
↳   -2:      1      -
↳   -1:      2      1
↳    0:      2      5
↳    1:      -      1
↳ -----
↳ total:      5      7
↳

intmat bettiI=betti(I,0); // degree corresponding to first non-zero row
                           // of Betti diagram is accessible via
                           // attribute "rowShift"

attrib(bettiI);
↳ attr:rowShift, type int
intmat bettiJ=betti(J,0);
attrib(bettiJ);
↳ attr:rowShift, type int

// (3) Graded free resolutions:
// =====
resolution resJ = mres(J,0);
attrib(resJ);
↳ attr:isHomog, type intvec
print(betti(resJ),"beti");
↳          0      1      2
```

```

↳ -----
↳   -2:      1      -      -
↳   -1:      2      -      -
↳    0:      1      4      -
↳    1:      -      -      1
↳ -----
↳ total:      4      4      1
↳
attrib(betti(resJ));
↳ attr:rowShift, type int

```

A check on degrees ((1), by using the `homog` command) shows that this is a graded matrix. The `homog` command assigns an admissible weight vector (here: 2,2,1,1,0) to the module `I` which is accessible via the attribute `"isHomog"`. Thus, we may think of `I` as a graded submodule of the graded free  $R$ -module

$$F = R(-2)^2 \oplus R(-1)^2 \oplus R.$$

We may also read the degrees from the Betti diagram as shown above. The degree on the left of the first nonzero row of the Betti diagram is accessible via the attribute `"rowShift"` of the betti matrix (which is of type `intmat`):

(2) We may shift degrees by assigning another admissible degree vector. Note that SINGULAR does not check whether the assigned degree vector really is admissible. Moreover, note that all assigned attributes are lost under a type conversion (e.g. from `module` to `matrix`).

(3) These considerations may be applied when computing data from free resolutions (see [Section A.3.6 \[Computation of Ext\]](#), page 725).

### A.3.6 Computation of Ext

We start by showing how to calculate the  $n$ -th Ext group of an ideal. The ingredients to do this are by the definition of Ext the following: calculate a (minimal) resolution at least up to length  $n$ , apply the Hom functor, and calculate the  $n$ -th homology group, that is, form the quotient  $\ker/\text{im}$  in the resolution sequence.

The Hom functor is given simply by transposing (hence dualizing) the module or the corresponding matrix with the command `transpose`. The image of the  $(n-1)$ -st map is generated by the columns of the corresponding matrix. To calculate the kernel apply the command `syzy` at the  $(n-1)$ -st transposed entry of the resolution. Finally, the quotient is obtained by the command `modulo`, which gives for two modules  $A = \ker$ ,  $B = \text{Im}$  the module of relations of

$$A/(A \cap B)$$

in the usual way. As we have a chain complex, this is obviously the same as  $\ker/\text{Im}$ .

We collect these statements in the following short procedure:

```

proc ext(int n, ideal I)
{
  resolution rs = mres(I,n+1);
  module tAn    = transpose(rs[n+1]);
  module tAn_1  = transpose(rs[n]);
  module ext_n  = modulo(syz(tAn),tAn_1);
  return(ext_n);
}

```

Now consider the following example:

```

ring r5 = 32003,(a,b,c,d,e),dp;
ideal I = a2b2+ab2c+b2cd, a2c2+ac2d+c2de,a2d2+ad2e+bd2e,a2e2+abe2+bce2;
print(ext(2,I));
⇒ 1,0,0,0,0,0,0,
⇒ 0,1,0,0,0,0,0,
⇒ 0,0,1,0,0,0,0,
⇒ 0,0,0,1,0,0,0,
⇒ 0,0,0,0,1,0,0,
⇒ 0,0,0,0,0,1,0,
⇒ 0,0,0,0,0,0,1,
⇒ 0,0,0,0,0,0,1
ext(3,I); // too big to be displayed here

```

The library `homolog.lib` contains several procedures for computing Ext-modules and related modules, which are much more general and sophisticated than the above one. They are used in the following example:

If  $M$  is a module, then  $\text{Ext}^1(M, M)$ , resp.  $\text{Ext}^2(M, M)$ , are the modules of infinitesimal deformations, respectively of obstructions, of  $M$  (like T1 and T2 for a singularity). Similar to the treatment of singularities, the semiuniversal deformation of  $M$  can be computed (if  $\text{Ext}^1$  is finite dimensional) with the help of  $\text{Ext}^1$ ,  $\text{Ext}^2$  and the cup product. There is an extra procedure for  $\text{Ext}^k(R/J, R)$  if  $J$  is an ideal in  $R$ , since this is faster than the general `Ext`.

We compute

- the infinitesimal deformations ( $= \text{Ext}^1(K, K)$ ) and obstructions ( $= \text{Ext}^2(K, K)$ ) of the residue field  $K = R/m$  of an ordinary cusp,  $R = K[x, y]_m/(x^2 - y^3)$ ,  $m = (x, y)$ . To compute  $\text{Ext}^1(m, m)$  we have to apply `Ext(1,syz(m),syz(m))` with `syz(m)` the first syzygy module of  $m$ , which is isomorphic to  $\text{Ext}^2(K, K)$ .
- $\text{Ext}^k(R/i, R)$  for some ideal  $i$  and with an extra option.

```

LIB "homolog.lib";
ring R=0,(x,y),ds;
ideal i=x2-y3;
qring q = std(i); // defines the quotient ring k[x,y]_m/(x2-y3)
ideal m = maxideal(1);
module T1K = Ext(1,m,m); // computes Ext^1(R/m,R/m)
⇒ // dimension of Ext^1: 0
⇒ // vdim of Ext^1: 2
⇒
print(T1K);
⇒ 0,x,0,y,
⇒ x,0,y,0
printlevel=2; // gives more explanation
module T2K=Ext(2,m,m); // computes Ext^2(R/m,R/m)
⇒ // Computing Ext^2 (help Ext; gives an explanation):
⇒ // Let 0<--coker(M)<--F0<--F1<--F2<--... be a resolution of coker(M),
⇒ // and 0<--coker(N)<--G0<--G1 a presentation of coker(N),
⇒ // then Hom(F2,G0)-->Hom(F3,G0) is given by:
⇒ y2,x,
⇒ x, y
⇒ // and Hom(F1,G0) + Hom(F2,G1)-->Hom(F2,G0) is given by:
⇒ -y,x, x,0,y,0,
⇒ x, -y2,0,x,0,y
⇒
⇒ // dimension of Ext^2: 0
⇒ // vdim of Ext^2: 2

```

```

↳
    print(std(T2K));
↳ 0,x,0,y,
↳ x,0,y,0
    printlevel=0;
    module E = Ext(1,syz(m),syz(m));
↳ // dimension of Ext^1: 0
↳ // vdim of Ext^1: 2
↳
    print(std(E));
↳ 0,0,0,x,0,y,
↳ 0,0,x,0,y,0,
↳ 0,1,0,0,0,0,
↳ 1,0,0,0,0,0
    //The matrices which we have just computed are presentation matrices
    //of the modules T2K and E. Hence we may ignore those columns
    //containing 1 as an entry and see that T2K and E are isomorphic
    //as expected, but differently presented.
    //-----
    ring S=0,(x,y,z),dp;
    ideal i = x2y,y2z,z3x;
    module E = Ext_R(2,i);
↳ // dimension of Ext^2: 1
↳
    print(E);
↳ 0,y,0,z2,
↳ z,0,0,-x,
↳ 0,0,x,-y
    // if a 3-rd argument of type int is given,
    // a list of Ext^k(R/i,R), a SB of Ext^k(R/i,R) and a vector space basis
    // is returned:
    list LE = Ext_R(3,i,0);
↳ // dimension of Ext^3: 0
↳ // vdim of Ext^3: 2
↳
    LE;
↳ [1]:
↳ _[1]=y*gen(1)
↳ _[2]=x*gen(1)
↳ _[3]=z2*gen(1)
↳ [2]:
↳ _[1]=y*gen(1)
↳ _[2]=x*gen(1)
↳ _[3]=z2*gen(1)
↳ [3]:
↳ _[1,1]=z
↳ _[1,2]=1
    print(LE[2]);
↳ y,x,z2
    print(kbase(LE[2]));
↳ z,1

```

### A.3.7 Depth

We compute the depth of the module of Kaehler differentials  $D_k(R)$  of the variety defined by the  $(m+1)$ -minors of a generic symmetric  $(n \times n)$ -matrix. We do this by computing the resolution over the polynomial ring. Then, by the Auslander-Buchsbaum formula, the depth is equal to the number of variables minus the length of a minimal resolution. This example was suggested by U. Vetter in order to check whether his bound  $\text{depth}(D_k(R)) \geq m(m+1)/2 + m - 1$  could be improved.

```

LIB "matrix.lib"; LIB "sing.lib";
int n = 4;
int m = 3;
int N = n*(n+1) div 2;      // will become number of variables
ring R = 32003,x(1..N),dp;
matrix X = symmat(n);      // proc from matrix.lib
                             // creates the symmetric generic nxn matrix

print(X);
↪ x(1),x(2),x(3),x(4),
↪ x(2),x(5),x(6),x(7),
↪ x(3),x(6),x(8),x(9),
↪ x(4),x(7),x(9),x(10)
  ideal J = minor(X,m);
  J=std(J);
  // Kaehler differentials D_k(R)
  // of R=k[x1..xn]/J:
  module D = J*freemodule(N)+transpose(jacob(J));
  ncols(D);
↪ 110
  nrows(D);
↪ 10
  //
  // Note: D is a submodule with 110 generators of a free module
  // of rank 10 over a polynomial ring in 10 variables.
  // Compute a full resolution of D with sres.
  // This takes about 17 sec on a Mac PB 520c and 2 sec on a HP 735
  int time = timer;
  module sD = std(D);
  list Dres = sres(sD,0);    // the full resolution
  timer-time;               // time used for std + sres
↪ 0
  intmat B = betti(Dres);
  print(B,"beti");
↪
↪      0      1      2      3      4      5      6
↪ -----
↪  0:    10      -      -      -      -      -      -
↪  1:      -    10      -      -      -      -      -
↪  2:      -    84    144    60      -      -      -
↪  3:      -      -    35    80    60    16      1
↪ -----
↪ total:    10    94    179    140    60    16      1
↪
  N-ncols(B)+1;             // the desired depth
↪ 4

```

### A.3.8 Factorization

The factorization of polynomials is implemented in the C++ libraries `Factory` (written mainly by Ruediger Stobbe) and `libfac` (written by Michael Messollen) which are part of the `SINGULAR` system. For the factorization of univariate polynomials these libraries make use of the library `NTL` written by Victor Shoup.

```

ring r = 0,(x,y),dp;
poly f = 9x16-18x13y2-9x12y3+9x10y4-18x11y2+36x8y4
        +18x7y5-18x5y6+9x6y4-18x3y6-9x2y7+9y8;
// = 9 * (x5-1y2)^2 * (x6-2x3y2-1x2y3+y4)
factorize(f);
⇒ [1]:
⇒   _[1]=9
⇒   _[2]=x6-2x3y2-x2y3+y4
⇒   _[3]=-x5+y2
⇒ [2]:
⇒   1,1,2
    // returns factors and multiplicities,
    // first factor is a constant.
poly g = (y4+x8)*(x2+y2);
factorize(g);
⇒ [1]:
⇒   _[1]=1
⇒   _[2]=x2+y2
⇒   _[3]=x8+y4
⇒ [2]:
⇒   1,1,1
    // The same in characteristic 2:
ring s = 2,(x,y),dp;
poly g = (y4+x8)*(x2+y2);
factorize(g);
⇒ [1]:
⇒   _[1]=1
⇒   _[2]=x+y
⇒   _[3]=x2+y
⇒ [2]:
⇒   1,2,4
    // factorization over algebraic extension fields
ring rext = (0,i),(x,y),dp;
minpoly = i2+1;
poly g = (y4+x8)*(x2+y2);
factorize(g);
⇒ [1]:
⇒   _[1]=1
⇒   _[2]=x+(i)*y
⇒   _[3]=x+(-i)*y
⇒   _[4]=x4+(i)*y2
⇒   _[5]=x4+(-i)*y2
⇒ [2]:
⇒   1,1,1,1,1

```

### A.3.9 Primary decomposition

There are two algorithms implemented in SINGULAR which provide primary decomposition: `primdecGTZ`, based on Gianni/Trager/Zacharias (written by Gerhard Pfister) and `primdecSY`, based on Shimoyama/Yokoyama (written by Wolfram Decker and Hans Schoenemann).

The result of `primdecGTZ` and `primdecSY` is returned as a list of pairs of ideals, where the second ideal is the prime ideal and the first ideal the corresponding primary ideal.

```
LIB "primdec.lib";
ring r = 0,(a,b,c,d,e,f),dp;
ideal i= f3, ef2, e2f, bcf-adf, de+cf, be+af, e3;
primdecGTZ(i);
↳ [1]:
↳   [1]:
↳     _[1]=f
↳     _[2]=e
↳   [2]:
↳     _[1]=f
↳     _[2]=e
↳ [2]:
↳   [1]:
↳     _[1]=f3
↳     _[2]=ef2
↳     _[3]=e2f
↳     _[4]=e3
↳     _[5]=de+cf
↳     _[6]=be+af
↳     _[7]=-bc+ad
↳   [2]:
↳     _[1]=f
↳     _[2]=e
↳     _[3]=-bc+ad
// We consider now the ideal J of the base space of the
// miniversal deformation of the cone over the rational
// normal curve computed in section *8* and compute
// its primary decomposition.
ring R = 0,(A,B,C,D),dp;
ideal J = CD, BD+D2, AD;
primdecGTZ(J);
↳ [1]:
↳   [1]:
↳     _[1]=D
↳   [2]:
↳     _[1]=D
↳ [2]:
↳   [1]:
↳     _[1]=C
↳     _[2]=B+D
↳     _[3]=A
↳   [2]:
↳     _[1]=C
↳     _[2]=B+D
↳     _[3]=A
// We see that there are two components which are both
```



```

// prime, even linear subspaces, one 3-dimensional,
// the other 1-dimensional.
// (This is Pinkhams example and was the first known
// surface singularity with two components of
// different dimensions)
//
// Let us now produce an embedded component in the last
// example, compute the minimal associated primes and
// the radical. We use the Characteristic set methods
// from primdec.lib.
J = intersect(J,maxideal(3));
// The following shows that the maximal ideal defines an embedded
// (prime) component.
primdecSY(J);
⇒ [1]:
⇒ [1]:
⇒ _[1]=D
⇒ [2]:
⇒ _[1]=D
⇒ [2]:
⇒ [1]:
⇒ _[1]=C
⇒ _[2]=B+D
⇒ _[3]=A
⇒ [2]:
⇒ _[1]=C
⇒ _[2]=B+D
⇒ _[3]=A
⇒ [3]:
⇒ [1]:
⇒ _[1]=D2
⇒ _[2]=C2
⇒ _[3]=B2
⇒ _[4]=AB
⇒ _[5]=A2
⇒ _[6]=BCD
⇒ _[7]=ACD
⇒ [2]:
⇒ _[1]=D
⇒ _[2]=C
⇒ _[3]=B
⇒ _[4]=A
minAssChar(J);
⇒ [1]:
⇒ _[1]=C
⇒ _[2]=B+D
⇒ _[3]=A
⇒ [2]:
⇒ _[1]=D
radical(J);
⇒ _[1]=CD
⇒ _[2]=BD+D2
⇒ _[3]=AD

```

### A.3.10 Normalization

The normalization will be computed for a reduced ring  $R/I$ . The result is a list of rings; ideals are always called `norid` in the rings of this list. The normalization of  $R/I$  is the product of the factor rings of the rings in the list divided out by the ideals `norid`.

```
LIB "normal.lib";
// ----- first example: rational quadruple point -----
ring R=32003,(x,y,z),wp(3,5,15);
ideal I=z*(y3-x5)+x10;
list pr=normal(I);
⇒
⇒ // 'normal' created a list, say nor, of two elements.
⇒ // To see the list type
⇒     nor;
⇒
⇒ // * nor[1] is a list of 1 ring(s).
⇒ // To access the i-th ring nor[1][i], give it a name, say Ri, and type
⇒     def R1 = nor[1][1]; setring R1; norid; normap;
⇒ // For the other rings type first (if R is the name of your base ring)
⇒     setring R;
⇒ // and then continue as for R1.
⇒ // Ri/norid is the affine algebra of the normalization of R/P_i where
⇒ // P_i is the i-th component of a decomposition of the input ideal id
⇒ // and normap the normalization map from R to Ri/norid.
⇒
⇒ // * nor[2] is a list of 1 ideal(s). Let ci be the last generator
⇒ // of the ideal nor[2][i]. Then the integral closure of R/P_i is
⇒ // generated as R-submodule of the total ring of fractions by
⇒ // 1/ci * nor[2][i].
⇒     def S=pr[1][1];
⇒     setring S;
⇒     norid;
⇒ norid[1]=T(2)*x+y*z
⇒ norid[2]=T(1)*x^2-T(2)*y
⇒ norid[3]=-T(1)*y+x^7-x^2*z
⇒ norid[4]=T(1)*y^2*z+T(2)*x^8-T(2)*x^3*z
⇒ norid[5]=T(1)^2+T(2)*z+x^4*y*z
⇒ norid[6]=T(1)*T(2)+x^6*z-x*z^2
⇒ norid[7]=T(2)^2+T(1)*x*z
⇒ norid[8]=x^10-x^5*z+y^3*z
⇒ // ----- second example: union of straight lines -----
ring R1=0,(x,y,z),dp;
ideal I=(x-y)*(x-z)*(y-z);
list qr=normal(I);
⇒
⇒ // 'normal' created a list, say nor, of two elements.
⇒ // To see the list type
⇒     nor;
⇒
⇒ // * nor[1] is a list of 2 ring(s).
⇒ // To access the i-th ring nor[1][i], give it a name, say Ri, and type
⇒     def R1 = nor[1][1]; setring R1; norid; normap;
⇒ // For the other rings type first (if R is the name of your base ring)
```

```

⇒      setring R;
⇒ // and then continue as for R1.
⇒ // Ri/norid is the affine algebra of the normalization of R/P_i where
⇒ // P_i is the i-th component of a decomposition of the input ideal id
⇒ // and normap the normalization map from R to Ri/norid.
⇒
⇒ // * nor[2] is a list of 2 ideal(s). Let ci be the last generator
⇒ // of the ideal nor[2][i]. Then the integral closure of R/P_i is
⇒ // generated as R-submodule of the total ring of fractions by
⇒ // 1/ci * nor[2][i].
    def S1=qr[1][1]; def S2=qr[1][2];
    setring S1; norid;
⇒ norid[1]=-T(1)*y+T(1)*z+x-z
⇒ norid[2]=T(1)*x-T(1)*y
⇒ norid[3]=T(1)^2-T(1)
⇒ norid[4]=x^2-x*y-x*z+y*z
    setring S2; norid;
⇒ norid[1]=y-z

```

### A.3.11 Kernel of module homomorphisms

Let  $A, B$  be two matrices of size  $m \times r$  and  $m \times s$  over the ring  $R$  and consider the corresponding maps

$$R^r \xrightarrow{A} R^m \xleftarrow{B} R^s.$$

We want to compute the kernel of the map  $R^r \xrightarrow{A} R^m \longrightarrow R^m/\text{Im}(B)$ . This can be done using the `modulo` command:

$$\text{modulo}(A, B) = \ker(R^r \xrightarrow{A} R^m/\text{Im}(B)).$$

More precisely, the output of `modulo(A,B)` is a `module` such that the given generating vectors span the kernel on the right-hand side.

```

    ring r=0,(x,y,z),(c,dp);
    matrix A[2][2]=x,y,z,1;
    matrix B[2][2]=x2,y2,z2,xz;
    print(B);
⇒ x2,y2,
⇒ z2,xz
    def C=modulo(A,B);
    print(C); // matrix of generators for the kernel
⇒ yz2-x2, xyz-y2, x2z-xy, x3-y2z,
⇒ x2z-xz2,-x2z+y2z,xyz-yz2,0
    print(A*matrix(C)); // should be in Im(B)
⇒ x2yz-x3,y3z-xy2, x3z+xy2z-y2z2-x2y,x4-xy2z,
⇒ yz3-xz2,xyz2-x2z,x2z2-yz2, x3z-y2z2

```

### A.3.12 Algebraic dependence

Let  $g, f_1, \dots, f_r \in K[x_1, \dots, x_n]$ . We want to check whether

1.  $f_1, \dots, f_r$  are algebraically dependent.

Let  $I = \langle Y_1 - f_1, \dots, Y_r - f_r \rangle \subseteq K[x_1, \dots, x_n, Y_1, \dots, Y_r]$ . Then  $I \cap K[Y_1, \dots, Y_r]$  are the algebraic relations between  $f_1, \dots, f_r$ .

2.  $g \in K[f_1, \dots, f_r]$ .

$g \in K[f_1, \dots, f_r]$  if and only if the normal form of  $g$  with respect to  $I$  and a block ordering with respect to  $X = (x_1, \dots, x_n)$  and  $Y = (Y_1, \dots, Y_r)$  with  $X > Y$  is in  $K[Y]$ .

Both questions can be answered using the following procedure. If the second argument is zero, it checks for algebraic dependence and returns the ideal of relations between the generators of the given ideal. Otherwise it checks for subring membership and returns the normal form of the second argument with respect to the ideal  $I$ .

```

proc algebraicDep(ideal J, poly g)
{
  def R=basing;          // give a name to the basing
  int n=size(J);
  int k=nvars(R);
  int i;
  intvec v;

  // construction of the new ring:

  // construct a weight vector
  v[n+k]=0;              // gives a zero vector of length n+k
  for(i=1;i<=k;i++)
  {
    v[i]=1;
  }
  string orde="(a("+string(v)+"),dp)";
  string ri="ring Rhelp=("+charstr(R)+"),
              (" + varstr(R) + ",Y(1.." + string(n) + ")), "+orde;
              // ring definition as a string
  execute(ri);           // execution of the string

  // construction of the new ideal I=(J[1]-Y(1),...,J[n]-Y(n))
  ideal I=imap(R,J);
  for(i=1;i<=n;i++)
  {
    I[i]=I[i]-var(k+i);
  }
  poly g=imap(R,g);
  if(g==0)
  {
    // construction of the ideal of relations by elimination
    poly el=var(1);
    for(i=2;i<=k;i++)
    {
      el=el*var(i);
    }
    ideal KK=eliminate(I,el);
    keep(Rhelp);
    return(KK);
  }
  // reduction of g with respect to I
  ideal KK=reduce(g,std(I));
  keep(Rhelp);
  return(KK);
}

```

```

}

// applications of the procedure
ring r=0,(x,y,z),dp;
ideal i=xz,yz;
algebraicDep(i,0);
⇨ _[1]=0
// Note: after call of algebraicDep(), the basering is Rhelp.
setring r; kill Rhelp;
ideal j=xy+z2,z2+y2,x2y2-2xy3+y4;
algebraicDep(j,0);
⇨ _[1]=Y(1)^2-2*Y(1)*Y(2)+Y(2)^2-Y(3)
setring r; kill Rhelp;
poly g=y2z2-xz;
algebraicDep(i,g);
⇨ _[1]=Y(2)^2-Y(1)
// this shows that g is contained in i.
setring r; kill Rhelp;
algebraicDep(j,g);
⇨ _[1]=-z^4+z^2*Y(2)-x*z
// this shows that g is contained in j.

```

## A.4 Singularity Theory

### A.4.1 Milnor and Tjurina number

The Milnor number, resp. the Tjurina number, of a power series  $f$  in  $K[[x_1, \dots, x_n]]$  is

$$\text{milnor}(f) = \dim_K(K[[x_1, \dots, x_n]]/\text{jacob}(f)),$$

respectively

$$\text{tjurina}(f) = \dim_K(K[[x_1, \dots, x_n]]/((f) + \text{jacob}(f)))$$

where  $\text{jacob}(f)$  is the ideal generated by the partials of  $f$ .  $\text{tjurina}(f)$  is finite, if and only if  $f$  has an isolated singularity. The same holds for  $\text{milnor}(f)$  if  $K$  has characteristic 0. SINGULAR displays -1 if the dimension is infinite.

SINGULAR cannot compute with infinite power series. But it can work in  $\text{Loc}_{(x)}K[x_1, \dots, x_n]$ , the localization of  $K[x_1, \dots, x_n]$  at the maximal ideal  $(x_1, \dots, x_n)$ . To do this, one has to define a ring with a local monomial ordering such as *ds*, *Ds*, *ls*, *ws*, *Ws* (the second letter 's' referring to power 's'eries), or an appropriate matrix ordering. See [Section B.2 \[Monomial orderings\]](#), [page 769](#) for a menu of possible orderings.

For theoretical reasons, the vector space dimension computed over the localization ring coincides with the Milnor (resp. Tjurina) number as defined above (in the power series ring).

We show in the example below the following:

- set option `prot` to have a short protocol during standard basis computation
- define the ring `r1` of characteristic 32003 with variables `x,y,z`, monomial ordering `ds`, series ring (i.e.,  $K[x,y,z]$  localized at  $(x,y,z)$ )
- list the information about `r1` by typing its name
- define the integers `a,b,c,t`
- define a polynomial `f` (depending on `a,b,c,t`) and display it

- define the jacobian ideal  $i$  of  $f$
- compute a standard basis of  $i$
- compute the Milnor number ( $=250$ ) with `vdim` and create and display a string in order to comment the result (text between quotes " "; is a 'string')
- compute a standard basis of  $i+(f)$
- compute the Tjurina number ( $=195$ ) with `vdim`
- then compute the Milnor number ( $=248$ ) and the Tjurina number ( $=195$ ) for  $t=1$
- reset the option to `noprot`

See also [Section D.6.20 \[sing\\_lib\]](#), page 1756 for the library commands for the computation of the Milnor and Tjurina number.

```

option(prot);
ring r1 = 32003,(x,y,z),ds;
r1;
⇒ // coefficients: ZZ/32003
⇒ // number of vars : 3
⇒ //          block 1 : ordering ds
⇒ //          : names  x y z
⇒ //          block 2 : ordering C
int a,b,c,t=11,5,3,0;
poly f = x^a+y^b+z^(3*c)+x^(c+2)*y^(c-1)+x^(c-1)*y^(c-1)*z3+
        x^(c-2)*y^c*(y^2+t*x)^2;

f;
⇒ y5+x5y2+x2y2z3+xy7+z9+x11
ideal i=jacob(f);
i;
⇒ i[1]=5x4y2+2xy2z3+y7+11x10
⇒ i[2]=5y4+2x5y+2x2yz3+7xy6
⇒ i[3]=3x2y2z2+9z8
ideal j=std(i);
⇒ 7(2)s8s10s11s12s(3)s13(4)s(5)s14(6)s(7)15--.s(6)-16.-.s(5)17.s(7)s--s18(6\
).--19-..sH(24)20(3)...21....22....23.--24-
⇒ product criterion:10 chain criterion:69
"The Milnor number of f(11,5,3) for t=0 is", vdim(j);
⇒ The Milnor number of f(11,5,3) for t=0 is 250
j=i+f; // override j
j=std(j);
⇒ 7(3)s8(2)s10s11(3)ss12(4)s(5)s13(6)s(8)s14(9).s(10).15--sH(23)(8)...16...\
...17.....sH(21)18(9)sH(20)(8)s17(10)..18..-.....19..---sH(19)
⇒ product criterion:11 chain criterion:62
vdim(j); // compute the Tjurina number for t=0
⇒ 195
t=1;
f=x^a+y^b+z^(3*c)+x^(c+2)*y^(c-1)+x^(c-1)*y^(c-1)*z3
+x^(c-2)*y^c*(y^2+t*x)^2;
ideal i1=jacob(f);
ideal j1=std(i1);
⇒ 7(2)s8s10s11s12s13(3)ss(4)s14(5)s(6)s15(7).....s(8)16.s...s(9)..17.....\
.....s18(10).....s(11)..-..19.....sH(24)(10).....20.....21.....\
..22.....23.....24.----\
-----25.26
⇒ product criterion:11 chain criterion:83

```

```

    "The Milnor number of f(11,5,3) for t=1:",vdim(j1);
    ↪ The Milnor number of f(11,5,3) for t=1: 248
    vdim(std(j1+f)); // compute the Tjurina number for t=1
    ↪ 7(16)s8(15)s10s11ss(16)-12.s-s13s(17)s(18)s(19)-s(18).-14-s(17)-s(16)ss(1\
    7)s15(18)..-s...-.16....-.....s(16).sH(23)s(18)...17.....18.....\
    ...sH(20)17(17)18...19...-...-.-...-.....20.-----s17(9)..18...19...-2\
    0.-.....21.....sH(19)(4)----
    ↪ product criterion:15 chain criterion:174
    ↪ 195
    option(noprot);

```

## A.4.2 Critical points

The same computation which computes the Milnor, resp. the Tjurina, number, but with ordering **dp** instead of **ds** (i.e., in  $K[x_1, \dots, x_n]$  instead of  $\text{Loc}_{(x)}K[x_1, \dots, x_n]$ ) gives:

- the number of critical points of **f** in the affine space (counted with multiplicities)
- the number of singular points of **f** on the affine hypersurface **f**=0 (counted with multiplicities).

We start with the ring **r1** from section [Section A.4.1 \[Milnor and Tjurina number\], page 735](#) and its elements.

The following will be implemented below:

- reset the protocol option and activate the timer
- define the ring **r2** of characteristic 32003 with variables **x,y,z** and monomial ordering **dp** (= degrevlex) (i.e., the polynomial ring =  $K[x,y,z]$ ).
- Note that polynomials, ideals, matrices (of polys), vectors, modules belong to a ring, hence we have to define **f** and **jacob(f)** again in **r2**. Since these objects are local to a ring, we may use the same names. Instead of defining **f** again we map it from ring **r1** to **r2** by using the **imap** command (**imap** is a convenient way to map variables from some ring identically to variables with the same name in the basering, even if the ground field is different. Compare with **fetch** which works for almost identical rings, e.g., if the rings differ only by the ordering or by the names of the variables and which may be used to rename variables). Integers and strings, however, do not belong to any ring. Once defined they are globally known.
- The result of the computation here (together with the previous one in [Section A.4.1 \[Milnor and Tjurina number\], page 735](#)) shows that (for **t**=0)  $\dim_K(\text{Loc}_{(x,y,z)}K[x,y,z]/\text{jacob}(f)) = 250$  (previously computed) while  $\dim_K(K[x,y,z]/\text{jacob}(f)) = 536$ . Hence **f** has 286 critical points, counted with multiplicity, outside the origin. Moreover, since  $\dim_K(\text{Loc}_{(x,y,z)}K[x,y,z]/(\text{jacob}(f) + (f))) = 195 = \dim_K(K[x,y,z]/(\text{jacob}(f) + (f)))$ , the affine surface **f**=0 is smooth outside the origin.

```

    ring r1 = 32003,(x,y,z),ds;
    int a,b,c,t=11,5,3,0;
    poly f = x^a+y^b+z^(3*c)+x^(c+2)*y^(c-1)+x^(c-1)*y^(c-1)*z3+
            x^(c-2)*y^c*(y^2+t*x)^2;
    option(noprot);
    timer=1;
    ring r2 = 32003,(x,y,z),dp;
    poly f=imap(r1,f);
    ideal j=jacob(f);
    vdim(std(j));
    ↪ 536
    vdim(std(j+f));
    ↪ 195

```

```
timer=0; // reset timer
```

### A.4.3 Polar curves

The polar curve of a hypersurface given by a polynomial  $f \in k[x_1, \dots, x_n, t]$  with respect to  $t$  (we may consider  $f = 0$  as a family of hypersurfaces parametrized by  $t$ ) is defined as the Zariski closure of  $V(\partial f/\partial x_1, \dots, \partial f/\partial x_n) \setminus V(f)$  if this happens to be a curve. Some authors consider  $V(\partial f/\partial x_1, \dots, \partial f/\partial x_n)$  itself as polar curve.

We may consider projective hypersurfaces (in  $P^n$ ), affine hypersurfaces (in  $k^n$ ) or germs of hypersurfaces (in  $(k^n, 0)$ ), getting in this way projective, affine or local polar curves.

Now let us compute this for a family of curves. We need the library `elim.lib` for saturation and `sing.lib` for the singular locus.

```
LIB "elim.lib";
LIB "sing.lib";
// Affine polar curve:
ring R = 0,(x,z,t),dp;           // global ordering dp
poly f = z5+xz3+x2-tz6;
dim_slocus(f);                   // dimension of singular locus
⇨ 1
ideal j = diff(f,x),diff(f,z);
dim(std(j));                     // dim V(j)
⇨ 1
dim(std(j+ideal(f)));           // V(j,f) also 1-dimensional
⇨ 1
// j defines a curve, but to get the polar curve we must remove the
// branches contained in f=0 (they exist since dim V(j,f) = 1). This
// gives the polar curve set theoretically. But for the structure we
// may take either j:f or j:f^k for k sufficiently large. The first is
// just the ideal quotient, the second the iterated ideal quotient
// or saturation. In our case both coincide.
ideal q = quotient(j,ideal(f));  // ideal quotient
ideal qsat = sat(j,f);          // saturation, proc from elim.lib
ideal sq = std(q);
dim(sq);
⇨ 1
// 1-dimensional, hence q defines the affine polar curve
//
// to check that q and qsat are the same, we show both inclusions, i.e.,
// both reductions must give the 0-ideal
size(reduce(qsat,sq));
⇨ 0
size(reduce(q,std(qsat)));
⇨ 0
qsat;
⇨ qsat[1]=12zt+3z-10
⇨ qsat[2]=5z2+12xt+3x
⇨ qsat[3]=144xt2+72xt+9x+50z
// We see that the affine polar curve does not pass through the origin,
// hence we expect the local polar "curve" to be empty
// -----
// Local polar curve:
ring r = 0,(x,z,t),ds;          // local ordering ds
```



```

poly f = z5+xz3+x2-tz6;
ideal j = diff(f,x),diff(f,z);
dim(std(j)); // V(j) 1-dimensional
⇒ 1
dim(std(j+ideal(f))); // V(j,f) also 1-dimensional
⇒ 1
ideal q = quotient(j,ideal(f)); // ideal quotient
q;
⇒ q[1]=1
// The local polar "curve" is empty, i.e., V(j) is contained in V(f)
// -----
// Projective polar curve: (we need "sing.lib" and "elim.lib")
ring P = 0,(x,z,t,y),dp; // global ordering dp
poly f = z5y+xz3y2+x2y4-tz6;
// but consider t as parameter
dim_slocus(f); // projective 1-dimensional singular locus
⇒ 2
ideal j = diff(f,x),diff(f,z);
dim(std(j)); // V(j), projective 1-dimensional
⇒ 2
dim(std(j+ideal(f))); // V(j,f) also projective 1-dimensional
⇒ 2
ideal q = quotient(j,ideal(f));
ideal qsat = sat(j,f); // saturation, proc from elim.lib
dim(std(qsat));
⇒ 2
// projective 1-dimensional, hence q and/or qsat define the projective
// polar curve. In this case, q and qsat are not the same, we needed
// 2 quotients.
// Let us check both reductions:
size(reduce(qsat,std(q)));
⇒ 4
size(reduce(q,std(qsat)));
⇒ 0
// Hence q is contained in qsat but not conversely
q;
⇒ q[1]=12zty+3zy-10y2
⇒ q[2]=60z2t-36xty-9xy-50zy
⇒ q[3]=12xty2+5z2y+3xy2
⇒ q[4]=z3y+2xy3
qsat;
⇒ qsat[1]=12zt+3z-10y
⇒ qsat[2]=12xty+5z2+3xy
⇒ qsat[3]=144xt2+72xt+9x+50z
⇒ qsat[4]=z3+2xy2
//
// Now consider again the affine polar curve,
// homogenize it with respect to y (deg t=0) and compare:
// affine polar curve:
ideal qa = 12zt+3z-10,5z2+12xt+3x,-144xt2-72xt-9x-50z;
// homogenized:
ideal qh = 12zt+3z-10y,5z2+12xyt+3xy,-144xt2-72xt-9x-50z;
size(reduce(qh,std(qsat)));

```

```

↳ 0
  size(reduce(qsat,std(qh)));
↳ 0
  // both ideals coincide

```

#### A.4.4 T1 and T2

$T^1$ , resp.  $T^2$ , of an ideal  $j$  usually denote the modules of infinitesimal deformations, resp. of obstructions. In SINGULAR there are procedures `T_1` and `T_2` in `sing.lib` such that `T_1(j)` and `T_2(j)` compute a standard basis of a presentation of these modules. If  $T^1, T^2$  are finite dimensional  $K$ -vector spaces (e.g., for isolated singularities), a basis can be computed by applying `kbase(T_1(j))`, resp. `kbase(T_2(j))`; the dimensions by applying `vdim`. For a complete intersection  $j$  the procedure `Tjurina` also computes  $T^1$ , but faster ( $T^2 = 0$  in this case). For a non complete intersection, it is faster to use the procedure `T_12` instead of `T_1` and `T_2`. Type `help T_1`; (or `help T_2`; or `help T_12`;) to obtain more detailed information about these procedures.

We give three examples, the first being a hypersurface, the second a complete intersection, the third not a complete intersection:

- load `sing.lib`
- check whether the ideal  $j$  is a complete intersection. It is, if number of variables = dimension + minimal number of generators
- compute the Tjurina number
- compute a vector space basis (`kbase`) of  $T^1$
- compute the Hilbert function of  $T^1$
- create a polynomial encoding the Hilbert series
- compute the dimension of  $T^2$

```

LIB "sing.lib";
ring R=32003,(x,y,z),ds;
// -----
// hypersurface case (from series T[p,q,r]):
int p,q,r = 3,3,4;
poly f = x^p+y^q+z^r+xyz;
tjurina(f);
↳ 8
  // Tjurina number = 8
  kbase(Tjurina(f));
↳ // Tjurina number = 8
↳ _[1]=z3
↳ _[2]=z2
↳ _[3]=yz
↳ _[4]=xz
↳ _[5]=z
↳ _[6]=y
↳ _[7]=x
↳ _[8]=1
// -----
// complete intersection case (from series P[k,l]):
int k,l =3,2;
ideal j=xy,x^k+y^l+z2;
dim(std(j));          // Krull dimension
↳ 1

```

```

    size(minbase(j));      // minimal number of generators
⇒ 2
    tjurina(j);           // Tjurina number
⇒ 6
    module T=Tjurina(j);
⇒ // Tjurina number = 6
    kbase(T);             // a sparse output of the k-basis of T_1
⇒ _[1]=z*gen(1)
⇒ _[2]=gen(1)
⇒ _[3]=y*gen(2)
⇒ _[4]=x2*gen(2)
⇒ _[5]=x*gen(2)
⇒ _[6]=gen(2)
    print(kbase(T));      // columns of matrix are a k-basis of T_1
⇒ z,1,0,0, 0,0,
⇒ 0,0,y,x2,x,1
    // -----
    // general case (cone over rational normal curve of degree 4):
    ring r1=0,(x,y,z,u,v),ds;
    matrix m[2][4]=x,y,z,u,y,z,u,v;
    ideal i=minor(m,2);    // 2x2 minors of matrix m
    module M=T_1(i);      // a presentation matrix of T_1
⇒ // dim T_1 = 4
    vdim(M);              // Tjurina number
⇒ 4
    hilb(M);              // display of both Hilbert series
⇒ (-4t5+20t4-40t3+40t2-20t+4) / (1-t)^5
⇒ (4) / (1-t)^0
⇒ // dimension (local)   = 0
⇒ // multiplicity = 4
    bigintvec v1=hilb(M,1); // first Hilbert series as intvec
    bigintvec v2=hilb(M,2); // second Hilbert series as intvec
    v1;
⇒ 4,-20,40,-40,20,-4,0
    v2;
⇒ 4,0
    v1[3];                // 3rd coefficient of the 1st Hilbert series
⇒ 40
    module N=T_2(i);
⇒ // dim T_2 = 3

// In some cases it might be useful to have a polynomial in some ring
// encoding the Hilbert series. This polynomial can then be
// differentiated, evaluated etc. It can be done as follows:
ring H = 0,t,ls;
poly h1;
int ii;
for (ii=1; ii<=size(v1); ii=ii+1)
{
    h1=h1+v1[ii]*t^(ii-1);
}
h1;                      // 1st Hilbert series
⇒ 4-20t+40t2-40t3+20t4-4t5
diff(h1,t);              // differentiate h1

```

```

⇒ -20+80t-120t2+80t3-20t4
subst(h1,t,1);          // substitute t by 1
⇒ 0

// The procedures T_1, T_2, T_12 may be called with two arguments and then
// they return a list with more information (type help T_1; etc.)
// e.g., T_12(i,<any>); returns a list with 9 nonempty objects where
// _[1] = std basis of T_1-module, _[2] = std basis of T_2-module,
// _[3]= vdim of T_1, _[4]= vdim of T_2
setring r1;              // make r1 again the basering
list L = T_12(i,1);
⇒ // dim T_1 = 4
⇒ // dim T_2 = 3
kbase(L[1]);             // kbase of T_1
⇒ _[1]=1*gen(2)
⇒ _[2]=1*gen(3)
⇒ _[3]=1*gen(6)
⇒ _[4]=1*gen(7)
kbase(L[2]);             // kbase of T_2
⇒ _[1]=1*gen(6)
⇒ _[2]=1*gen(8)
⇒ _[3]=1*gen(9)
L[3];                   // vdim of T_1
⇒ 4
L[4];                   // vdim of T_2
⇒ 3

```

#### A.4.5 Deformations

- The libraries `sing.lib`, respectively `deform.lib`, contain procedures to compute total and base space of the miniversal (= semiuniversal) deformation of an isolated complete intersection singularity, respectively of an arbitrary isolated singularity.
- The procedure `deform` in `sing.lib` returns a matrix whose columns  $h_1, \dots, h_r$  represent all 1st order deformations. More precisely, if  $I \subset R$  is the ideal generated by  $f_1, \dots, f_s$ , then any infinitesimal deformation of  $R/I$  over  $K[\varepsilon]/(\varepsilon^2)$  is given by  $f + \varepsilon g$ , where  $f = (f_1, \dots, f_s)$ , and where  $g$  is a  $K$ -linear combination of the  $h_i$ .
- The procedure `versal` in `deform.lib` computes a formal miniversal deformation up to a certain order which can be prescribed by the user. For a complete intersection the 1st order part is already miniversal.
- The procedure `versal` extends the basering to a new ring with additional deformation parameters which contains the equations for the miniversal base space and the miniversal total space.
- There are default names for the objects created, but the user may also choose their own names.
- If the user sets `printlevel=2`; before running `versal`, some intermediate results are shown. This is useful since `versal` is already complicated and might run for some time on more complicated examples. (type `help versal`;) )

We compute for the same examples as in the section [Section A.4.4 \[T1 and T2\], page 740](#) the miniversal deformations:

```

LIB "deform.lib";
ring R=32003,(x,y,z),ds;

```

```

//-----
// hypersurface case (from series T[p,q,r]):
int p,q,r = 3,3,4;
poly f = x^p+y^q+z^r+xyz;
print(deform(f));
⇒ z3,z2,yz,xz,z,y,x,1
// the miniversal deformation of f=0 is the projection from the
// miniversal total space to the miniversal base space:
// { (A,B,C,D,E,F,G,H,x,y,z) | x3+y3+xyz+z4+A+Bx+Cxz+Dy+Eyz+Fz+Gz2+Hz3 =0 }
// --> { (A,B,C,D,E,F,G,H) }
//-----
// complete intersection case (from series P[k,l]):
int k,l =3,2;
ideal j=xy,x^k+y^l+z2;
print(deform(j));
⇒ 0,0, 0,0,z,1,
⇒ y,x2,x,1,0,0
def L=versal(j); // using default names
⇒ // smooth base space
⇒ // ready: T_1 and T_2
⇒
⇒
⇒ // 'versal' returned a list, say L, of four rings. In L[1] are stored:
⇒ // as matrix Fs: Equations of total space of the miniversal deformation\
,
⇒ // as matrix Js: Equations of miniversal base space,
⇒ // as matrix Rs: syzygies of Fs mod Js.
⇒ // To access these data, type
⇒ def Px=L[1]; setring Px; print(Fs); print(Js); print(Rs);
⇒
⇒ // L[2] = L[1]/Fo extending Qo=Po/Fo,
⇒ // L[3] = the embedding ring of the versal base space,
⇒ // L[4] = L[1]/Js extending L[3]/Js.
⇒
def Px=L[1]; setring Px;
show(Px); // show is a procedure from inout.lib
⇒ // ring: (ZZ/32003),(A,B,C,D,E,F,x,y,z),(ds(6),ds(3),C);
⇒ // minpoly = 0
⇒ // objects belonging to this ring:
⇒ // Rs [0] matrix 2 x 1
⇒ // Fs [0] matrix 1 x 2
⇒ // Js [0] matrix 1 x 0
listvar(matrix);
⇒ // Rs [0] matrix 2 x 1
⇒ // Fs [0] matrix 1 x 2
⇒ // Js [0] matrix 1 x 0
// ___ Equations of miniversal base space ___:
Js;
⇒
// ___ Equations of miniversal total space ___:
Fs;
⇒ Fs[1,1]=y2+z2+x3+Cy+Dx2+Ex+F
⇒ Fs[1,2]=xy+Az+B

```

```

// the miniversal deformation of V(j) is the projection from the
// miniversal total space to the miniversal base space:
// { (A,B,C,D,E,F,x,y,z) | xy+F+Ez=0, y2+z2+x3+D+Cx+Bx2+Ay=0 }
// --> { (A,B,C,D,E,F) }
//-----
// general case (cone over rational normal curve of degree 4):
kill L;
ring r1=0,(x,y,z,u,v),ds;
matrix m[2][4]=x,y,z,u,y,z,u,v;
ideal i=minor(m,2);          // 2x2 minors of matrix m
int time=timer;
// Call parameters of the miniversal base A(1),A(2),...:
def L=versal(i,0,"","A(");
⇒ // ready: T_1 and T_2
⇒ // start computation in degree 2.
⇒
⇒
⇒ // 'versal' returned a list, say L, of four rings. In L[1] are stored:
⇒ //   as matrix Fs: Equations of total space of the miniversal deformation\
,
⇒ //   as matrix Js: Equations of miniversal base space,
⇒ //   as matrix Rs: syzygies of Fs mod Js.
⇒ // To access these data, type
⇒   def Px=L[1]; setring Px; print(Fs); print(Js); print(Rs);
⇒
⇒ // L[2] = L[1]/Fo extending Qo=Po/Fo,
⇒ // L[3] = the embedding ring of the versal base space,
⇒ // L[4] = L[1]/Js extending L[3]/Js.
⇒
  "/// used time:",timer-time,"sec";    // time of last command
⇒ // used time: 0 sec
  def Def_rPx=L[1]; setring Def_rPx;
  Fs;
⇒ Fs[1,1]=u^2-z*v-A(2)*u+A(4)*v
⇒ Fs[1,2]=z*u-y*v-A(1)*u+A(4)*u
⇒ Fs[1,3]=y*u-x*v+A(3)*u+A(4)*z
⇒ Fs[1,4]=z^2-y*u-A(1)*z+A(2)*y
⇒ Fs[1,5]=y*z-x*u+A(2)*x+A(3)*z
⇒ Fs[1,6]=y^2-x*z+A(1)*x+A(3)*y
  Js;
⇒ Js[1,1]=A(2)*A(4)
⇒ Js[1,2]=-A(1)*A(4)+A(4)^2
⇒ Js[1,3]=A(3)*A(4)
// the miniversal deformation of V(i) is the projection from the
// miniversal total space to the miniversal base space:
// { (A(1..4),x,y,z,u,v) |
//   -u^2+x*v+A(2)*u+A(4)*v=0, -z*u+y*v-A(1)*u+A(3)*u=0,
//   -y*u+x*v+A(3)*u+A(4)*z=0, z^2-y*u+A(1)*z+A(2)*y=0,
//   y*z-x*u+A(2)*x-A(3)*z=0, -y^2+x*z+A(1)*x+A(3)*y=0 }
// --> { A(1..4) |
//   A(2)*A(4) = -A(3)*A(4) = -A(1)*A(4)+A(4)^2 = 0 }
//-----

```

### A.4.6 Invariants of plane curve singularities

The Puiseux pairs of an irreducible and reduced plane curve singularity are probably its most important invariants. They can be computed from its Hamburger-Noether expansion (which is the analogue of the Puiseux expansion in characteristic 0 for fields of arbitrary characteristic).

The library `hnoether.lib` (see [Section D.6.15 \[hnoether.lib\], page 1719](#)) uses the algorithm of Antonio Campillo in "Algebroid curves in positive characteristic" SLN 813, 1980. This algorithm has the advantage that it needs least possible field extensions and, moreover, works in any characteristic. This fact can be used to compute the invariants over a field of finite characteristic, say 32003, which will most probably be the same as in characteristic 0.

We compute the Hamburger-Noether expansion of a plane curve singularity given by a polynomial  $f$  in two variables. This expansion is given by a matrix, and it allows us to compute a primitive parametrization (up to a given order) for the curve singularity defined by  $f$  and numerical invariants such as the

- characteristic exponents,
- Puiseux pairs (of a complex model),
- degree of the conductor,
- delta invariant,
- generators of the semigroup.

Besides commands for computing a parametrization and the invariants mentioned above, the library `hnoether.lib` provides commands for the computation of the Newton polygon of  $f$ , the square-free part of  $f$  and a procedure to convert one set of invariants to another.

```
LIB "hnoether.lib";
// ===== The irreducible case =====
ring s = 0,(x,y),ds;
poly f = y4-2x3y2-4x5y+x6-x7;
list hn = develop(f);
show(hn[1]);      // Hamburger-Noether matrix
⇒ // matrix, 3x3
⇒ 0,x,  0,
⇒ 0,1,  x,
⇒ 0,1/4,-1/2
displayHNE(hn);   // Hamburger-Noether development
⇒ y = z(1)*x
⇒ x = z(1)^2+z(1)^2*z(2)
⇒ z(1) = 1/4*z(2)^2-1/2*z(2)^3 + ..... (terms of degree >=4)
setring s;
displayInvariants(hn);
⇒ characteristic exponents : 4,6,7
⇒ generators of semigroup : 4,6,13
⇒ Puiseux pairs           : (3,2)(7,2)
⇒ degree of the conductor : 16
⇒ delta invariant         : 8
⇒ sequence of multiplicities: 4,2,2,1,1
// invariants(hn); returns the invariants as list
// partial parametrization of f: param takes the first variable
// as infinite except the ring has more than 2 variables. Then
// the 3rd variable is chosen.
param(hn);
⇒ // ** Warning: result is exact up to order 5 in x and 7 in y !
```

```

⇒ _[1]=1/16x4-3/16x5+1/4x7
⇒ _[2]=1/64x6-5/64x7+3/32x8+1/16x9-1/8x10
  ring extring=0,(x,y,t),ds;
  poly f=x3+2xy2+y2;
  list hn=develop(f,-1);
  param(hn);          // partial parametrization of f
⇒ // ** Warning: result is exact up to order 2 in x and 3 in y !
⇒ _[1]=-t2
⇒ _[2]=-t3
  list hn1=develop(f,6);
  param(hn1);         // a better parametrization
⇒ // ** Warning: result is exact up to order 6 in x and 7 in y !
⇒ _[1]=-t2+2t4-4t6
⇒ _[2]=-t3+2t5-4t7
  // instead of recomputing you may extend the development:
  list hn2=extdevelop(hn,12);
  param(hn2);         // a still better parametrization
⇒ // ** Warning: result is exact up to order 12 in x and 13 in y !
⇒ _[1]=-t2+2t4-4t6+8t8-16t10+32t12
⇒ _[2]=-t3+2t5-4t7+8t9-16t11+32t13
  //
  // ===== The reducible case =====
  ring r = 0,(x,y),dp;
  poly f=x11-2y2x8-y3x7-y2x6+y4x5+2y4x3+y5x2-y6;
  // = (x5-1y2) * (x6-2x3y2-1x2y3+y4)
  list L=hnexpansion(f);
⇒ // No change of ring necessary, return value is HN expansion.
  show(L[1][1]);      // Hamburger-Noether matrix of 1st branch
⇒ // matrix, 3x3
⇒ 0,x,0,
⇒ 0,1,x,
⇒ 0,1,-1
  displayInvariants(L);
⇒ --- invariants of branch number 1 : ---
⇒ characteristic exponents   : 4,6,7
⇒ generators of semigroup    : 4,6,13
⇒ Puiseux pairs              : (3,2)(7,2)
⇒ degree of the conductor    : 16
⇒ delta invariant            : 8
⇒ sequence of multiplicities: 4,2,2,1,1
⇒
⇒ --- invariants of branch number 2 : ---
⇒ characteristic exponents   : 2,5
⇒ generators of semigroup    : 2,5
⇒ Puiseux pairs              : (5,2)
⇒ degree of the conductor    : 4
⇒ delta invariant            : 2
⇒ sequence of multiplicities: 2,2,1,1
⇒
⇒ ----- contact numbers : -----
⇒
⇒ branch |      2
⇒ -----+-----

```



```

⇒      1 |      2
⇒
⇒ ----- intersection multiplicities : -----
⇒
⇒ branch |      2
⇒ -----+-----
⇒      1 |     12
⇒
⇒ ----- delta invariant of the curve :  22
    param(L[2]);          // parametrization of 2nd branch
⇒ _[1]=x2
⇒ _[2]=x5

```

#### A.4.7 Branches of space curve singularities

In this example, the number of branches of a given quasihomogeneous isolated space curve singularity will be computed as an example of the pitfalls appearing in the use of primary decomposition. When dealing with singularities, two situations are possible in which the primary decomposition algorithm might not lead to a complete decomposition: first of all, one of the computed components could be globally irreducible, but analytically reducible (this is impossible for quasihomogeneous singularities) and, as a second possibility, a component might be irreducible over the rational numbers, but reducible over the complex numbers.

```

    ring r=0,(x,y,z),ds;
    ideal i=x^4-y*z^2,x*y-z^3,y^2-x^3*z; // the space curve singularity
    qhweight(i);
⇒ 1,2,1
    // The given space curve singularity is quasihomogeneous. Hence we can pass
    // to the polynomial ring.
    ring rr=0,(x,y,z),dp;
    ideal i=imap(r,i);
    resolution ires=mres(i,0);
    ires;
⇒      1      3      2
⇒ rr <-- rr <-- rr
⇒
⇒ 0      1      2
⇒
    // From the structure of the resolution, we see that the Cohen-Macaulay
    // type of the given singularity is 2
    //
    // Let us now look for the branches using the primdec library.
    LIB "primdec.lib";
    primdecSY(i);
⇒ [1]:
⇒      [1]:
⇒      _[1]=z3-xy
⇒      _[2]=x3+x2z+xz2+xy+yz
⇒      _[3]=x2z2+x2y+xyz+yz2+y2
⇒      [2]:
⇒      _[1]=z3-xy
⇒      _[2]=x3+x2z+xz2+xy+yz
⇒      _[3]=x2z2+x2y+xyz+yz2+y2
⇒ [2]:

```

```

⇒ [1]:
⇒   _[1]=x-z
⇒   _[2]=z2-y
⇒ [2]:
⇒   _[1]=x-z
⇒   _[2]=z2-y
  def li=_[1];
  ideal i2=li[2];      // call the first ideal i1
  // The curve seems to have 2 branches by what we computed using the
  // algorithm of Shimoyama-Yokoyama.
  // Now the same computation by the Gianni-Trager-Zacharias algorithm:
  primdecGTZ(i);
⇒ [1]:
⇒ [1]:
⇒   _[1]=-z2+y
⇒   _[2]=x-z
⇒ [2]:
⇒   _[1]=-z2+y
⇒   _[2]=x-z
⇒ [2]:
⇒ [1]:
⇒   _[1]=z8+yz6+y2z4+y3z2+y4
⇒   _[2]=xz5+z6+yz4+y2z2+y3
⇒   _[3]=-z3+xy
⇒   _[4]=x2z2+xz3+xyz+yz2+y2
⇒   _[5]=x3+x2z+xz2+xy+yz
⇒ [2]:
⇒   _[1]=z8+yz6+y2z4+y3z2+y4
⇒   _[2]=xz5+z6+yz4+y2z2+y3
⇒   _[3]=-z3+xy
⇒   _[4]=x2z2+xz3+xyz+yz2+y2
⇒   _[5]=x3+x2z+xz2+xy+yz
  // Having computed the primary decomposition in 2 different ways and
  // having obtained the same number of branches, we might expect that the
  // number of branches is really 2, but we can check this by formulae
  // for the invariants of space curve singularities:
  //
  // mu = tau - t + 1 (for quasihomogeneous curve singularities)
  // where mu denotes the Milnor number, tau the Tjurina number and
  // t the Cohen-Macaulay type
  //
  // mu = 2 delta - r + 1
  // where delta denotes the delta-Invariant and r the number of branches
  //
  // tau can be computed by using the corresponding procedure T1 from
  // sing.lib.
  setring r;
  LIB "sing.lib";
  T_1(i);
⇒ // dim T_1 = 13
⇒ _[1]=gen(6)+2z*gen(5)
⇒ _[2]=gen(4)+3x2*gen(2)
⇒ _[3]=gen(3)+gen(1)

```

```

⇒ _[4]=x*gen(5)-y*gen(2)-z*gen(1)
⇒ _[5]=x*gen(1)-z2*gen(2)
⇒ _[6]=y*gen(5)+3x2z*gen(2)
⇒ _[7]=y*gen(2)-z*gen(1)
⇒ _[8]=2y*gen(1)-z2*gen(5)
⇒ _[9]=z2*gen(5)
⇒ _[10]=z2*gen(1)
⇒ _[11]=x3*gen(2)
⇒ _[12]=x2z2*gen(2)
⇒ _[13]=xz3*gen(2)
⇒ _[14]=z4*gen(2)
  setring rr;
  // Hence tau is 13 and therefore mu is 12. But then it is impossible that
  // the singularity has two branches, since mu is even and delta is an
  // integer!
  // So obviously, we did not decompose completely. Because the second branch
  // is smooth, only the first ideal can be the one which can be decomposed
  // further.
  // Let us now consider the normalization of this first ideal i1.
  LIB "normal.lib";
  normal(i2);
⇒
⇒ // 'normal' created a list, say nor, of two elements.
⇒ // To see the list type
⇒     nor;
⇒
⇒ // * nor[1] is a list of 1 ring(s).
⇒ // To access the i-th ring nor[1][i], give it a name, say Ri, and type
⇒     def R1 = nor[1][1]; setring R1; norid; normap;
⇒ // For the other rings type first (if R is the name of your base ring)
⇒     setring R;
⇒ // and then continue as for R1.
⇒ // Ri/norid is the affine algebra of the normalization of R/P_i where
⇒ // P_i is the i-th component of a decomposition of the input ideal id
⇒ // and normap the normalization map from R to Ri/norid.
⇒
⇒ // * nor[2] is a list of 1 ideal(s). Let ci be the last generator
⇒ // of the ideal nor[2][i]. Then the integral closure of R/P_i is
⇒ // generated as R-submodule of the total ring of fractions by
⇒ // 1/ci * nor[2][i].
⇒ [1]:
⇒     [1]:
⇒         // coefficients: QQ
⇒ // number of vars : 6
⇒ //         block  1 : ordering dp
⇒ //                 : names      T(1) T(2) T(3)
⇒ //         block  2 : ordering dp
⇒ //                 : names      x y z
⇒ //         block  3 : ordering C
⇒ [2]:
⇒     [1]:
⇒         _[1]=y
⇒         _[2]=xz

```

```

⇒      _[3]=x2
⇒      _[4]=z2
      def rno=_[1][1];
      setring rno;
      norid;
⇒ norid[1]=-T(2)*z+x
⇒ norid[2]=T(1)*x-z
⇒ norid[3]=T(2)*x-T(3)*z
⇒ norid[4]=T(1)*z+T(2)*z+T(3)*x+T(3)*z+z
⇒ norid[5]=-T(2)*y+z^2
⇒ norid[6]=T(1)*z^2-y
⇒ norid[7]=T(2)*z^2-T(3)*y
⇒ norid[8]=T(1)*y+T(2)*y+T(3)*z^2+T(3)*y+y
⇒ norid[9]=T(1)^2+T(1)+T(2)+T(3)+1
⇒ norid[10]=T(1)*T(2)-1
⇒ norid[11]=T(2)^2-T(3)
⇒ norid[12]=T(1)*T(3)-T(2)
⇒ norid[13]=T(2)*T(3)+T(1)+T(2)+T(3)+1
⇒ norid[14]=T(3)^2-T(1)
⇒ norid[15]=z^3-x*y
⇒ norid[16]=x^3+x^2*z+x*z^2+x*y+y*z
⇒ norid[17]=x^2*z^2+x^2*y+x*y*z+y*z^2+y^2
      // The ideal is generated by a polynomial in one variable of degree 4 which
      // factors completely into 4 polynomials of type T(2)+a.
      // From this, we know that the ring of the normalization is the direct sum of
      // 4 polynomial rings in one variable.
      // Hence our original curve has these 4 branches plus a smooth one
      // which we already determined by primary decomposition.
      // Our final result is therefore: 5 branches.

```

#### A.4.8 Classification of hypersurface singularities

Classification of isolated hypersurface singularities with respect to right equivalence is provided by the command `classify` of the library `classify.lib`. The classification is done by using the algorithm of Arnold. Before entering this algorithm, a first guess based on the Hilbert polynomial of the Milnor algebra is made.

```

      LIB "classify.lib";
      ring r=0,(x,y,z),ds;
      poly p=singularity("E[6k+2]",2)[1];
      p=p+z^2;
      p;
⇒ z2+x3+xy6+y8
      // We received an E_14 singularity in normal form
      // from the database of normal forms. Since only the residual
      // part is saved in the database, we added z^2 to get an E_14
      // of embedding dimension 3.
      //
      // Now we apply a coordinate change in order to deal with a
      // singularity which is not in normal form:
      map phi=r,x+y,y+z,x;
      poly q=phi(p);
      // Yes, q really looks ugly, now:
      q;

```

```

⇒ x2+x3+3x2y+3xy2+y3+xy6+y7+6xy5z+6y6z+15xy4z2+15y5z2+20xy3z3+20y4z3+15xy2z\
  4+15y3z4+6xyz5+6y2z5+xz6+yz6+y8+8y7z+28y6z2+56y5z3+70y4z4+56y3z5+28y2z6+8\
  yz7+z8
  // Classification
  classify(q);
⇒ About the singularity :
⇒           Milnor number(f)    = 14
⇒           Corank(f)           = 2
⇒           Determinacy         <= 12
⇒ Guessing type via Milnorcode:  E[6k+2]=E[14]
⇒
⇒ Computing normal form ...
⇒ I have to apply the splitting lemma. This will take some time....:-)
⇒   Arnold step number 9
⇒ The singularity
⇒   x3-9/4x4+27/4x5-189/8x6+737/8x7+6x6y+15x5y2+20x4y3+15x3y4+6x2y5+xy6-24\
  089/64x8-x7y+11/2x6y2+26x5y3+95/2x4y4+47x3y5+53/2x2y6+8xy7+y8+104535/64x9\
  +27x8y+135/2x7y2+90x6y3+135/2x5y4+27x4y5+9/2x3y6-940383/128x10-405/4x9y-2\
  025/8x8y2-675/2x7y3-2025/8x6y4-405/4x5y5-135/8x4y6+4359015/128x11+1701/4x\
  10y+8505/8x9y2+2835/2x8y3+8505/8x7y4+1701/4x6y5+567/8x5y6-82812341/512x12\
  -15333/8x11y-76809/16x10y2-25735/4x9y3-78525/16x8y4-16893/8x7y5-8799/16x6\
  y6-198x5y7-495/4x4y8-55x3y9-33/2x2y10-3xy11-1/4y12
⇒ is R-equivalent to E[14].
⇒   Milnor number = 14
⇒   modality      = 1
⇒ 2z2+x3+xy6+y8
  // The library also provides routines to determine the corank of q
  // and its residual part without going through the whole
  // classification algorithm.
  corank(q);
⇒ 2
  morsesplit(q);
⇒ y3-9/4y4+27/4y5-189/8y6+737/8y7+6y6z+15y5z2+20y4z3+15y3z4+6y2z5+yz6-24089\
  /64y8-y7z+11/2y6z2+26y5z3+95/2y4z4+47y3z5+53/2y2z6+8yz7+z8+104535/64y9+27\
  y8z+135/2y7z2+90y6z3+135/2y5z4+27y4z5+9/2y3z6-940383/128y10-405/4y9z-2025\
  /8y8z2-675/2y7z3-2025/8y6z4-405/4y5z5-135/8y4z6+4359015/128y11+1701/4y10z\
  +8505/8y9z2+2835/2y8z3+8505/8y7z4+1701/4y6z5+567/8y5z6-82812341/512y12-15\
  333/8y11z-76809/16y10z2-25735/4y9z3-78525/16y8z4-16893/8y7z5-8799/16y6z6-\
  198y5z7-495/4y4z8-55y3z9-33/2y2z10-3yz11-1/4z12

```

#### A.4.9 Resolution of singularities

Resolution of singularities and applications thereof are provided by the libraries `resolve.lib` and `reszeta.lib`; graphical output may be generated automatically by using external programs `surf` and `dot` respectively to which a specialized interface is provided by the library `resgraph.lib`. In this example, the basic functionality of the resolution of singularities package is illustrated by the computation of the intersection matrix and genera of the exceptional curves on a surface obtained from resolving the A6 surface singularity. A separate tutorial, which introduces the complete functionality of the package and explains the rather complicated data structures appearing in intermediate results, can be found at [https://www.singular.uni-kl.de/tutor\\_resol.pdf](https://www.singular.uni-kl.de/tutor_resol.pdf).

```

LIB"resolve.lib";           // load the resolution algorithm
LIB"reszeta.lib";          // load its application algorithms

```

```

ring R=0,(x,y,z),dp;           // define the ring Q[x,y,z]
ideal I=x^7+y^2-z^2;           // an A6 surface singularity
list L=resolve(I);             // compute the resolution
list iD=intersectionDiv(L);     // compute intersection properties
iD;                             // show the output
⇒ [1]:
⇒   -2,0,1,0,0,0,
⇒   0,-2,0,1,0,0,
⇒   1,0,-2,0,1,0,
⇒   0,1,0,-2,0,1,
⇒   0,0,1,0,-2,1,
⇒   0,0,0,1,1,-2
⇒ [2]:
⇒   0,0,0,0,0,0
⇒ [3]:
⇒   [1]:
⇒       [1]:
⇒       2,1,1
⇒       [2]:
⇒       4,1,1
⇒   [2]:
⇒       [1]:
⇒       2,1,2
⇒       [2]:
⇒       4,1,2
⇒   [3]:
⇒       [1]:
⇒       4,2,1
⇒       [2]:
⇒       6,2,1
⇒   [4]:
⇒       [1]:
⇒       4,2,2
⇒       [2]:
⇒       6,2,2
⇒   [5]:
⇒       [1]:
⇒       6,3,1
⇒       [2]:
⇒       7,3,1
⇒   [6]:
⇒       [1]:
⇒       6,3,2
⇒       [2]:
⇒       7,3,2
⇒ [4]:
⇒   1,1,1,1,1,1
// The output is a list whose first entry contains the intersection matrix
// of the exceptional divisors. The second entry is the list of genera
// of these divisors. The third and fourth entry contain the information
// how to find the corresponding divisors in the respective charts.

```

## A.5 Invariant Theory

### A.5.1 G<sub>a</sub>-Invariants

We work in characteristic 0 and use the Lie algebra generated by one vectorfield of the form  $\sum x_i \partial / \partial x_{i+1}$ .

```
LIB "ainvar.lib";
int n=5;
int i;
ring s=32003,(x(1..n)),wp(1,2,3,4,5);
// definition of the vectorfield m=sum m[i,1]*d/dx(i)
matrix m[n][1];
for (i=1;i<=n-1;i=i+1)
{
    m[i+1,1]=x(i);
}
// computation of the ring of invariants
ideal in=invariantRing(m,x(2),x(1),0);
in; //invariant ring is generated by 5 invariants
⇒ in[1]=x(1)
⇒ in[2]=x(2)^2-2*x(1)*x(3)
⇒ in[3]=x(3)^2-2*x(2)*x(4)+2*x(1)*x(5)
⇒ in[4]=x(2)^3-3*x(1)*x(2)*x(3)+3*x(1)^2*x(4)
⇒ in[5]=x(3)^3-3*x(2)*x(3)*x(4)-15997*x(1)*x(4)^2+3*x(2)^2*x(5)-6*x(1)*x(3)\
    *x(5)
ring q=32003,(x,y,z,u,v,w),dp;
matrix m[6][1];
m[2,1]=x;
m[3,1]=y;
m[5,1]=u;
m[6,1]=v;
// the vectorfield is: xd/dy+yd/dz+ud/dv+vd/dw
ideal in=invariantRing(m,y,x,0);
in; //invariant ring is generated by 6 invariants
⇒ in[1]=x
⇒ in[2]=u
⇒ in[3]=v^2-2uw
⇒ in[4]=zu-yv+xw
⇒ in[5]=yu-xv
⇒ in[6]=y^2-2xz
```

### A.5.2 Invariants of a finite group

Two algorithms to compute the invariant ring are implemented in SINGULAR, `invariant_ring` and `invariant_ring_random`, both by Agnes E. Heydtmann ([agnes@math.uni-sb.de](mailto:agnes@math.uni-sb.de)).

Bases of homogeneous invariants are generated successively and those are chosen as primary invariants that lower the dimension of the ideal generated by the previously found invariants (see paper "Generating a Noetherian Normalization of the Invariant Ring of a Finite Group" by Decker, Heydtmann, Schreyer (J.Symb.Comput. 25, No.6, 727-731, 1998). In the non-modular case secondary invariants are calculated by finding a basis (in terms of monomials) of the basering modulo the primary invariants, mapping to invariants with the Reynolds operator and using those or their power products such that they are linearly independent modulo the primary invariants (see paper

"Some Algorithms in Invariant Theory of Finite Groups" by Kemper and Steel (In: Proceedings of the Euroconference in Essen 1997, Birkhäuser Prog. Math. 173, 267-285, 1999)). In the modular case they are generated according to "Calculating Invariant Rings of Finite Groups over Arbitrary Fields" by Kemper (J.Symb.Comput. 21, No.3, 351-366, 1996).

We calculate now an example from Sturmfels: "Algorithms in Invariant Theory 2.3.7":

```
LIB "finvar.lib";
ring R=0,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
// the group G is generated by A in Gl(3,Q);
print(A);
↪ 0, 1,0,
↪ -1,0,0,
↪ 0, 0,-1
print(A*A*A*A); // the fourth power of A is 1
↪ 1,0,0,
↪ 0,1,0,
↪ 0,0,1
// Use the first method to compute the invariants of G:
matrix B(1..3);
B(1..3)=invariant_ring(A);
// SINGULAR returns 2 matrices, the first containing
// primary invariants and the second secondary
// invariants, i.e., module generators over a Noetherian
// normalization
// the third result are the irreducible secondary invariants
// if the Molien series was available
print(B(1));
↪ z2,x2+y2,x2y2
print(B(2));
↪ 1,xyz,x2z-y2z,x3y-xy3
print(B(3));
↪ xyz,x2z-y2z,x3y-xy3
// Use the second method,
// with random numbers between -1 and 1:
B(1..3)=invariant_ring_random(A,1);
print(B(1..3));
↪ z2,x2+y2,x4+y4-z4
↪ 1,xyz,x2z-y2z,x3y-xy3
↪ xyz,x2z-y2z,x3y-xy3
```

## A.6 Geometric Invariant Theory

### A.6.1 GIT-Fans

Dolgachev/Hu and Thaddeus assigned to an algebraic variety with the action of an algebraic group the GIT-fan, a polyhedral fan enumerating the GIT-quotients in the sense of Mumford. The case of the action of an algebraic torus  $H$  on an affine variety  $X$  has been treated by Berchtold/Hausen. Based on their construction, an algorithm to compute the GIT-fan in this setting has been proposed by Keicher. Note that this setting is essential for many applications, since the torus case can be used to investigate the GIT-variation of the action of a connected reductive group  $G$ . In many important examples,  $X$  is symmetric under the action of a finite group which either is known directly from



its geometry or can be computed. A prominent instance is the Deligne-Mumford compactification  $M_{0,6}$  of the moduli space of 6-pointed stable curves of genus zero, which has a natural action of the symmetric group  $S_6$ . The library `gitfan.lib` implements an efficient algorithm for computing GIT-fans, which makes use of symmetries. We have applied this algorithm to determine the Mori chamber decomposition of the cone of movable divisor classes of  $M_{0,6}$ . Each cone is represented by a single integer. The algorithm relies on Groebner basis techniques, convex geometry and actions of finite symmetry groups. It demonstrates the strength of cross-boarder methods in computer algebra, and the efficiency of the algorithms implemented in all involved areas. The algorithm is also suitable for parallel computations.

As an example we address in the following the computation of the GIT-Fan of  $M_{0,5}$ .

We first compute the GIT-fan using the single line command provided by the library:

```
LIB "gitfan.lib";
setcores(4);
ring R = 0,T(1..10),wp(1,1,1,1,1,1,1,1,1,1);
ideal J =
    T(5)*T(10)-T(6)*T(9)+T(7)*T(8),
    T(1)*T(9)-T(2)*T(7)+T(4)*T(5),
    T(1)*T(8)-T(2)*T(6)+T(3)*T(5),
    T(1)*T(10)-T(3)*T(7)+T(4)*T(6),
    T(2)*T(10)-T(3)*T(9)+T(4)*T(8);
intmat Q[5][10] =
    1, 1, 1, 1, 0, 0, 0, 0, 0, 0,
    1, 0, 0, 0, 1, 1, 1, 0, 0, 0,
    0, 1, 1, 0, 0, 0, -1, 1, 0, 0,
    0, 1, 0, 1, 0, -1, 0, 0, 1, 0,
    0, 0, 1, 1, -1, 0, 0, 0, 0, 1;
fan GIT = GITfan(J,Q);
fVector(GIT);
```

The GIT-Fan can be computed using symmetries as follows:

```
LIB "gitfan.lib";
setcores(4);
ring R = 0,T(1..10),wp(1,1,1,1,1,1,1,1,1,1);
ideal J =
    T(5)*T(10)-T(6)*T(9)+T(7)*T(8),
    T(1)*T(9)-T(2)*T(7)+T(4)*T(5),
    T(1)*T(8)-T(2)*T(6)+T(3)*T(5),
    T(1)*T(10)-T(3)*T(7)+T(4)*T(6),
    T(2)*T(10)-T(3)*T(9)+T(4)*T(8);
intmat Q[5][10] =
    1, 1, 1, 1, 0, 0, 0, 0, 0, 0,
    1, 0, 0, 0, 1, 1, 1, 0, 0, 0,
    0, 1, 1, 0, 0, 0, -1, 1, 0, 0,
    0, 1, 0, 1, 0, -1, 0, 0, 1, 0,
    0, 0, 1, 1, -1, 0, 0, 0, 0, 1;
list simplexSymmetryGroup = G25Action();
fan GIT2 = GITfan(J,Q,simplexSymmetryGroup);
GIT2;
```

Although we provide a procedure to compute the orbit decomposition of the group action on the simplex of variables this is not fast in Singular. In the following we describe how to use GAP to obtain the orbit decomposition and then continue with this data in Singular. This is particularly useful for more complicated examples.

The file `orbits.gp` in the directory `doc` of the Singular source tree contains GAP code to do this computation. This result is provided in the file `doc/simplexOrbitRepresentativesG25.sing`.

The file `doc/simplexSymmetryGroupG25.sing` contains the symmetry group (which here is  $S_5$ ).

Moreover the file `doc/elementsInTermsOfGeneratorsG25.sing` contains a representation of the elements of the symmetry group in terms of generators.

```
LIB "gitfan.lib";
setcores(4);
ring R = 0,T(1..10),wp(1,1,1,1,1,1,1,1,1,1);
ideal J =
  T(5)*T(10)-T(6)*T(9)+T(7)*T(8),
  T(1)*T(9)-T(2)*T(7)+T(4)*T(5),
  T(1)*T(8)-T(2)*T(6)+T(3)*T(5),
  T(1)*T(10)-T(3)*T(7)+T(4)*T(6),
  T(2)*T(10)-T(3)*T(9)+T(4)*T(8);
intmat Q[5][10] =
  1, 1, 1, 1, 0, 0, 0, 0, 0, 0,
  1, 0, 0, 0, 1, 1, 1, 0, 0, 0,
  0, 1, 1, 0, 0, 0, -1, 1, 0, 0,
  0, 1, 0, 1, 0, -1, 0, 0, 1, 0,
  0, 0, 1, 1, -1, 0, 0, 0, 0, 1;
intmat Qt = transpose(Q);
<"doc/simplexOrbitRepresentativesG25.sing";
list afaceOrbitRepresentatives=afaces(J,simplexOrbitRepresentatives);
<"doc/simplexSymmetryGroupG25.sing";
list fulldimAfaceOrbitRepresentatives=fullDimImages(afaceOrbitRepresentatives,Q);
list afaceOrbits=computeAfaceOrbits(fulldimAfaceOrbitRepresentatives,simplexSymmetry);
apply(afaceOrbits,size);
list minAfaceOrbits = minimalAfaceOrbits(afaceOrbits);
apply(minAfaceOrbits,size);
list listOfOrbitConeOrbits = orbitConeOrbits(minAfaceOrbits,Q);
apply(listOfOrbitConeOrbits,size);
list listOfMinimalOrbitConeOrbits = minimalOrbitConeOrbits(listOfOrbitConeOrbits);
size(listOfMinimalOrbitConeOrbits);
cone mov = coneViaPoints(transpose(Q));
mov = canonicalizeCone(mov);
list OC = listOfOrbitConeOrbits;
<"doc/elementsInTermsOfGeneratorsG25.sing";
list Asigmagens = groupActionOnQImage(generatorsG,Q);
list actionOnOrbitconeIndicesForGenerators = groupActionOnHashes(Asigmagens,OC);
list actionOnOrbitconeIndices;
for (int i =1; i<=size(elementsInTermsOfGenerators);i++)
{
  actionOnOrbitconeIndices[i]=evaluateProduct(actionOnOrbitconeIndicesForGenerators,
}
list OClist = OC[1];
for (i =2;i<=size(OC);i++)
{
  OClist = OClist + OC[i];
}
list SigmaHashes = GITfanParallelSymmetric(OClist, Q, mov, actionOnOrbitconeIndices);
SigmaHashes;
fan Sigma = hashesToFan(SigmaHashes,OClist);
```

Note that the result is not the complete fan but only the fan generated by a minimal set of representatives of maximal cones for the group action (by the group generated by `Asigmagens`).

## A.7 Non-commutative Algebra

### A.7.1 Left and two-sided Groebner bases

For a set of polynomials (resp. vectors)  $S$  in a non-commutative  $G$ -algebra, `SINGULAR:PLURAL` provides two algorithms for computing Groebner bases.

The command `std` computes a left Groebner basis of a left module, generated by the set  $S$  (see [Section 7.3.26 \[std \(plural\)\], page 361](#)). The command `twostd (plural)` computes a two-sided Groebner basis (which is in particular also a left Groebner basis) of a two-sided ideal, generated by the set  $S$  (see [Section 7.3.29 \[twostd \(plural\)\], page 364](#)).

In the example below, we consider a particular set  $S$  in the algebra  $A := U(sl_2)$  with the degree reverse lexicographic ordering. We compute a left Groebner basis  $L$  of the left ideal generated by  $S$  and a two-sided Groebner basis  $T$  of the two-sided ideal generated by  $S$ .

Then, we read off the information on the vector space dimension of the factor modules  $A/L$  and  $A/T$  using the command `vdim` (see [Section 7.3.30 \[vdim \(plural\)\], page 365](#)).

Further on, we use the command `reduce` (see [Section 7.3.23 \[reduce \(plural\)\], page 357](#)) to compare the left ideals generated by  $L$  and  $T$ .

We set `option(redSB)` and `option(redTail)` to make `SINGULAR` compute completely reduced minimal bases of ideals (see [Section 5.1.111 \[option\], page 234](#) and [Section 7.4.2 \[Groebner bases in G-algebras\], page 367](#) for definitions and further details).

For long running computations, it is always recommended to set `option(prot)` to make `SINGULAR` display some information on the performed computations (see [Section 5.1.111 \[option\], page 234](#) for an interpretation of the displayed symbols).

```
// ----- 1.  setting up the algebra
ring R = 0,(e,f,h),dp;
matrix D[3][3];
D[1,2]=-h; D[1,3]=2*e; D[2,3]=-2*f;
def A=nc_algebra(1,D); setring A;
// ----- equivalently, you may use the following:
// LIB "ncalg.lib";
// def A = makeUsl2();
// setring A;
// ----- 2.  defining the set S
ideal S = e^3, f^3, h^3 - 4*h;
option(redSB);
option(redTail);
option(prot); // let us activate the protocol
ideal L = std(S);
⇒ 3(2)s
⇒ s
⇒ s
⇒ 5s
⇒ s
⇒ (4)s
⇒ 4(5)(4)s
⇒ (6)(5)(4)s
```

```

⇒ 3(7)4(5)(4)(3)s
⇒ 3(4)(3)4(2)s
⇒ (3)(2)s
⇒ 3(5)(4)4(2)5
⇒ (S:5)-----
⇒ product criterion:7 chain criterion:12
  L;
⇒ L[1]=h3-4h
⇒ L[2]=fh2-2fh
⇒ L[3]=eh2+2eh
⇒ L[4]=2efh-h2-2h
⇒ L[5]=f3
⇒ L[6]=e3
  vdim(L); // the vector space dimension of the module A/L
⇒ 15
  option(noprot); // turn off the protocol
  ideal T = twostd(S);
  T;
⇒ T[1]=h3-4h
⇒ T[2]=fh2-2fh
⇒ T[3]=eh2+2eh
⇒ T[4]=f2h-2f2
⇒ T[5]=2efh-h2-2h
⇒ T[6]=e2h+2e2
⇒ T[7]=f3
⇒ T[8]=ef2-fh
⇒ T[9]=e2f-eh-2e
⇒ T[10]=e3
  vdim(T); // the vector space dimension of the module A/T
⇒ 10
  print(matrix(reduce(L,T))); // reduce L with respect to T
⇒ 0,0,0,0,0,0
  // as we see, L is included in the left ideal generated by T
  print(matrix(reduce(T,L))); // reduce T with respect to L
⇒ 0,0,0,f2h-2f2,0,e2h+2e2,0,ef2-fh,e2f-eh-2e,0
  // the non-zero elements belong to T only
  ideal LT = twostd(L); // the two-sided Groebner basis of L
  // LT and T coincide as left ideals:
  size(reduce(LT,T));
⇒ 0
  size(reduce(T,LT));
⇒ 0

```

### A.7.2 Right Groebner bases and syzygies

Most of the SINGULAR:PLURAL commands correspond to the *left-sided* computations, that is left Groebner bases, left syzygies, left resolutions and so on. However, the *right-sided* computations can be done, using the *left-sided* functionality and *opposite* algebras.

In the example below, we consider the algebra  $A := U(sl_2)$  and a set of generators  $I = \{e^2, f\}$ .

We will compute a left Groebner basis LI and a left syzygy module LS of a left ideal, generated by the set  $I$ .

Then, we define the opposite algebra `Aop` of `A`, set it as a basering, and create opposite objects of already computed ones.

Further on, we compute a right Groebner basis `RI` and a right syzygy module `RS` of a right ideal, generated by the set  $I$  in  $A$ .

```
// ----- setting up the algebra:
LIB "ncalg.lib";
def A = makeUsl2();
setring A; A;
⇒ // coefficients: QQ
⇒ // number of vars : 3
⇒ //          block 1 : ordering dp
⇒ //          : names  e f h
⇒ //          block 2 : ordering C
⇒ // noncommutative relations:
⇒ //    fe=ef-h
⇒ //    he=eh+2e
⇒ //    hf=fh-2f
// ----- equivalently, you may use
// ring AA = 0,(e,f,h),dp;
// matrix D[3][3];
// D[1,2]=-h; D[1,3]=2*e; D[2,3]=-2*f;
// def A=nc_algebra(1,D); setring A;
option(redSB);
option(redTail);
matrix T;
// --- define a generating set
ideal I = e2,f;
ideal LI = std(I); // the left Groebner basis of I
LI; // we see that I was not a Groebner basis
⇒ LI[1]=f
⇒ LI[2]=h2+h
⇒ LI[3]=eh+e
⇒ LI[4]=e2
module LS = syz(I); // the left syzygy module of I
print(LS);
⇒ -ef-2h+6,-f3, -ef2-fh+4f, -e2f2-4efh+16ef-6h2+42h-72\
,
⇒ e3, e2f2-6efh-6ef+6h2+18h+12,e3f-3e2h-6e2,e4f
// check: LS is a left syzygy, if T=0:
T = transpose(LS)*transpose(I);
print(T);
⇒ 0,
⇒ 0,
⇒ 0,
⇒ 0
// --- let us define the opposite algebra of A
def Aop = opposite(A);
setring Aop; Aop; // see how Aop looks like
⇒ // coefficients: QQ
⇒ // number of vars : 3
⇒ //          block 1 : ordering a
⇒ //          : names  H F E
```

```

⇒ //                               : weights  1 1 1
⇒ //          block  2 : ordering ls
⇒ //                               : names    H F E
⇒ //          block  3 : ordering C
⇒ // noncommutative relations:
⇒ //      FH=HF-2F
⇒ //      EH=HE+2E
⇒ //      EF=FE-H
// --- we "oppose" (transfer) objects from A to Aop
ideal  Iop = oppose(A,I);
ideal  RIop = std(Iop); // the left Groebner basis of Iop in Aop
module RSop = syz(Iop); // the left syzygy module of Iop in Aop
module LSop = oppose(A,LS);
module RLS  = syz(transpose(LSop));
// RLS is the left syzygy of transposed LSop in Aop
// --- let us return to A and transfer (i.e. oppose)
// all the computed objects back
setring A;
ideal  RI = oppose(Aop,RIop); // the right Groebner basis of I
RI;
      // it differs from the left Groebner basis LI
⇒ RI[1]=f
⇒ RI[2]=h2-h
⇒ RI[3]=eh+e
⇒ RI[4]=e2
module RS = oppose(Aop,RSop); // the right syzygy module of I
print(RS);
⇒ -ef+3h+6,-f3,                -ef2+3fh,-e2f2+4efh+4ef,
⇒ e3,                e2f2+2efh-6ef+2h2-10h+12,e3f,        e4f
  // check: RS is a right syzygy, if T=0:
T = matrix(I)*RS;
T;
⇒ T[1,1]=0
⇒ T[1,2]=0
⇒ T[1,3]=0
⇒ T[1,4]=0
module RLS;
RLS = transpose(oppose(Aop,RLS));
// RLS is the right syzygy of a left syzygy of I
// it is I itself ?
print(RLS);
⇒ e2,f

```

## A.8 Applications

### A.8.1 Solving systems of polynomial equations

Here we turn our attention to the probably most popular aspect of the solving problem: given a system of complex polynomial equations with only finitely many solutions, compute floating point approximations for these solutions. This is widely considered as a task for numerical analysis. However, due to rounding errors, purely numerical methods are often unstable in an unpredictable way.

Therefore, in many cases, it is worth investing more computing power to derive additional knowledge on the geometric structure of the set of solutions (not to mention the question of how to decide whether the set of solutions is finite or not). The symbolic-numerical approach to the solving problem combines numerical methods with a symbolic preprocessing.

Depending on whether we want to preserve the multiplicities of the solutions or not, possible goals for a symbolic preprocessing are

- to find another system of generators (for instance, a reduced Groebner basis) for the ideal  $I$  generated by the polynomial equations. Alternatively, find a system of polynomials defining an ideal which has the same radical as  $I$  (see [Section A.2 \[Computing Groebner and Standard Bases\]](#), page 710, resp. [Section D.4.26.12 \[radical\]](#), page 1247).

In any case, the goal should be to find a system for which a numerical solution can be found more easily and in a more stable way. For systems with a large number of generators, the first step in a SINGULAR computation could be to reduce the number of generators by applying the `interred` command (see [Section 5.1.64 \[interred\]](#), page 201). Another goal might be

- to decompose the system into several smaller (or, at least, more accessible) systems of polynomial equations. Then, the set of solutions of the original system is obtained by taking the union of the sets of solutions of the new systems.

Such a decomposition can be obtained in several ways: for instance, by computing a triangular decomposition (see [Section D.8.5 \[triang\\_lib\]](#), page 1856) for the ideal  $I$ , or by applying the factorizing Buchberger algorithm (see [Section 5.1.34 \[facstd\]](#), page 179), or by computing a primary decomposition of  $I$  (see [Section D.4.26 \[primdec\\_lib\]](#), page 1240).

Moreover, the equational modelling of a problem frequently causes unwanted solutions, for instance, zero as a multiple solution. Not only for stability reasons, one is frequently interested to get rid of those. This can be done by computing the saturation of  $I$  with respect to an ideal having the excess components as set of solutions (see [Section D.4.7.7 \[sat\]](#), page 1066).

The SINGULAR libraries `solve.lib` and `triang.lib` provide several commands for solving systems of polynomial equations (based on a symbolic-numerical approach via Groebner bases, resp. resultants). In the example below, we show some of these commands at work.

```
LIB "solve.lib";
ring r=0,x(1..5),dp;
poly f0= x(1)^3+x(2)^2+x(3)^2+x(4)^2-x(5)^2;
poly f1= x(2)^3+x(1)^2+x(3)^2+x(4)^2-x(5)^2;
poly f2=x(3)^3+x(1)^2+x(2)^2+x(4)^2-x(5)^2;
poly f3=x(4)^2+x(1)^2+x(2)^2+x(3)^2-x(5)^2;
poly f4=x(5)^2+x(1)^2+x(2)^2+x(3)^2;
ideal i=f0,f1,f2,f3,f4;
ideal si=std(i);
//
// dimension of a solution set (here: 0) can be read from a Groebner bases
// (with respect to any global monomial ordering)
dim(si);
⇒ 0
//
// the number of complex solutions (counted with multiplicities) is:
vdim(si);
⇒ 108
//
// The given system has a multiple solution at the origin. We use facstd
// to compute equations for the non-zero solutions:
```

```

option(redSB);
ideal maxI=maxideal(1);
ideal j=sat(si,maxI);      // output is Groebner basis
vdim(j);                  // number of non-zero solutions (with mult's)
⇒ 76
//
// We compute a triangular decomposition for the ideal I. This requires first
// the computation of a lexicographic Groebner basis (we use the FGLM
// conversion algorithm):
ring R=0,x(1..5),lp;
ideal j=fglm(r,j);
list L=triangMH(j);
size(L);                  // number of triangular components
⇒ 7
L[1];                    // the first component
⇒ _[1]=x(5)^2+1
⇒ _[2]=x(4)^2+2
⇒ _[3]=x(3)-1
⇒ _[4]=x(2)^2
⇒ _[5]=x(1)^2
//
// We compute floating point approximations for the solutions (with 30 digits)
def S=triang_solve(L,30);
⇒
⇒ // 'triang_solve' created a ring, in which a list rlist of numbers (the
⇒ // complex solutions) is stored.
⇒ // To access the list of complex solutions, type (if the name R was assign\
    ned
⇒ // to the return value):
⇒      setring R; rlist;
setring S;
size(rlist);              // number of different non-zero solutions
⇒ 28
rlist[1];                 // the first solution
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ 1
⇒ [4]:
⇒ (-I*1.41421356237309504880168872421)
⇒ [5]:
⇒ -I
//
// Alternatively, we could have applied directly the solve command:
setring r;
def T=solve(i,30,1,"nodisplay"); // compute all solutions with mult's
⇒
⇒ // 'solve' created a ring, in which a list SOL of numbers (the complex so\
    lutions)
⇒ // is stored.
⇒ // To access the list of complex solutions, type (if the name R was assign\

```



```

    ned
    ↪ // to the return value):
    ↪      setring R; SOL;
    setring T;
    size(SOL);          // number of different solutions
    ↪ 4
    SOL[1][1]; SOL[1][2]; // first solution and its multiplicity
    ↪ [1]:
    ↪      [1]:
    ↪      1
    ↪      [2]:
    ↪      1
    ↪      [3]:
    ↪      1
    ↪      [4]:
    ↪      (i*2.44948974278317809819728407471)
    ↪      [5]:
    ↪      (i*1.73205080756887729352744634151)
    ↪ [2]:
    ↪      [1]:
    ↪      1
    ↪      [2]:
    ↪      1
    ↪      [3]:
    ↪      1
    ↪      [4]:
    ↪      (-i*2.44948974278317809819728407471)
    ↪      [5]:
    ↪      (i*1.73205080756887729352744634151)
    ↪ [3]:
    ↪      [1]:
    ↪      1
    ↪      [2]:
    ↪      1
    ↪      [3]:
    ↪      1
    ↪      [4]:
    ↪      (i*2.44948974278317809819728407471)
    ↪      [5]:
    ↪      (-i*1.73205080756887729352744634151)
    ↪ [4]:
    ↪      [1]:
    ↪      1
    ↪      [2]:
    ↪      1
    ↪      [3]:
    ↪      1
    ↪      [4]:
    ↪      (-i*2.44948974278317809819728407471)
    ↪      [5]:
    ↪      (-i*1.73205080756887729352744634151)
    ↪ 1
    SOL[size(SOL)];      // solutions of highest multiplicity

```

```

⇒ [1]:
⇒ [1]:
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ 0
⇒ [4]:
⇒ 0
⇒ [5]:
⇒ 0
⇒ [2]:
⇒ 32
//
// Or, we could remove the multiplicities first, by computing the
// radical:
setring r;
ideal k=std(radical(i));
vdim(k); // number of different complex solutions
⇒ 29
def T1=solve(k,30,"nodisplay"); // compute all solutions with mult's
⇒
⇒ // 'solve' created a ring, in which a list SOL of numbers (the complex so\
  lutions)
⇒ // is stored.
⇒ // To access the list of complex solutions, type (if the name R was assign\
  ed
⇒ // to the return value):
⇒ setring R; SOL;
setring T1;
size(SOL); // number of different solutions
⇒ 29
SOL[1];
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ 1
⇒ [4]:
⇒ (-i*2.44948974278317809819728407471)
⇒ [5]:
⇒ (-i*1.73205080756887729352744634151)

```

### A.8.2 AG codes

The library `brnoeth.lib` provides an implementation of the Brill-Noether algorithm for solving the Riemann-Roch problem and applications to Algebraic Geometry codes. The procedures can be applied to plane (singular) curves defined over a prime field of positive characteristic.

```

LIB "brnoeth.lib";
ring s=2,(x,y),lp; // characteristic 2
poly f=x3y+y3+x; // the Klein quartic

```

```

list KLEIN=Adj_div(f);          // compute the conductor
↳ Computing affine singular points ...
↳ Computing all points at infinity ...
↳ Computing affine singular places ...
↳ Computing singular places at infinity ...
↳ Computing non-singular places at infinity ...
↳ Adjunction divisor computed successfully
↳
↳ The genus of the curve is 3
KLEIN=NSplaces(1..3,KLEIN);    // computes places up to degree 3
↳ Computing non-singular affine places of degree 1 ...
↳ Computing non-singular affine places of degree 2 ...
↳ Computing non-singular affine places of degree 3 ...
KLEIN=extcurve(3,KLEIN);      // construct Klein quartic over F_8
↳
↳ Total number of rational places : NrRatPl = 24
↳
KLEIN[3];                      // display places (degree, number)
↳ [1]:
↳   1,1
↳ [2]:
↳   1,2
↳ [3]:
↳   1,3
↳ [4]:
↳   2,1
↳ [5]:
↳   3,1
↳ [6]:
↳   3,2
↳ [7]:
↳   3,3
↳ [8]:
↳   3,4
↳ [9]:
↳   3,5
↳ [10]:
↳   3,6
↳ [11]:
↳   3,7
// We define a divisor G of degree 14=6*1+4*2:
intvec G=6,0,0,4,0,0,0,0,0,0,0; // 6 * place #1 + 4 * place #4
// We compute an evaluation code which evaluates at all rational places
// outside the support of G (place #4 is not rational)
intvec D=2..24;
// in D, the number i refers to the i-th element of the list POINTS in
// the ring KLEIN[1][5].
def RR=KLEIN[1][5];
setring RR; POINTS[1];        // the place in the support of G (not in supp(D))
↳ [1]:
↳ 0
↳ [2]:
↳ 1

```

```

⇒ [3]:
⇒ 0
setring s;
def RR=KLEIN[1][4];
⇒ // ** redefining RR (def RR=KLEIN[1][4];) ./examples/AG_codes.sing:18
setring RR;
matrix C=AGcode_L(G,D,KLEIN); // generator matrix for the evaluation AG code
⇒ Forms of degree 5 :
⇒ 21
⇒
⇒ Vector basis successfully computed
⇒
nrows(C);
⇒ 12
ncols(C);
⇒ 23
//
// We can also compute a generator matrix for the residual AG code
matrix CO=AGcode_Omega(G,D,KLEIN);
⇒ Forms of degree 5 :
⇒ 21
⇒
⇒ Vector basis successfully computed
⇒
//
// Preparation for decoding:
// We need a divisor of degree at least 6 whose support is disjoint with the
// support of D:
intvec F=6; // F = 6*point #1
// in F, the i-th entry refers to the i-th element of the list POINTS in
// the ring KLEIN[1][5]
list K=prepSV(G,D,F,KLEIN);
⇒ Forms of degree 5 :
⇒ 21
⇒
⇒ Vector basis successfully computed
⇒
⇒ Forms of degree 4 :
⇒ 15
⇒
⇒ Vector basis successfully computed
⇒
⇒ Forms of degree 4 :
⇒ 15
⇒
⇒ Vector basis successfully computed
⇒
K[size(K)][1]; // error-correcting capacity
⇒ 3
//
// Encoding and Decoding:
matrix word[1][11]; // a word of length 11 is encoded
word = 1,1,1,1,1,1,1,1,1,1,1;

```

```

def y=word*C0;           // the code word (length: 23)
matrix disturb[1][23];
disturb[1,1]=1;
disturb[1,10]=a;
disturb[1,12]=1+a;
y=y+disturb;             // disturb the code word (3 errors)
def yy=decodeSV(y,K);    // error correction
yy-y;                    // display the error
⇨ _[1,1]=1
⇨ _[1,2]=0
⇨ _[1,3]=0
⇨ _[1,4]=0
⇨ _[1,5]=0
⇨ _[1,6]=0
⇨ _[1,7]=0
⇨ _[1,8]=0
⇨ _[1,9]=0
⇨ _[1,10]=(a)
⇨ _[1,11]=0
⇨ _[1,12]=(a+1)
⇨ _[1,13]=0
⇨ _[1,14]=0
⇨ _[1,15]=0
⇨ _[1,16]=0
⇨ _[1,17]=0
⇨ _[1,18]=0
⇨ _[1,19]=0
⇨ _[1,20]=0
⇨ _[1,21]=0
⇨ _[1,22]=0
⇨ _[1,23]=0

```

## Appendix B Polynomial data

### B.1 Representation of mathematical objects

SINGULAR distinguishes between objects which do not belong to a ring and those which belong to a specific ring (see [Section 3.3 \[Rings and orderings\]](#), page 30). We comment only on the latter ones.

Internally all ring-dependent objects are polynomials or structures built from polynomials (and some additional information). Note that SINGULAR stores (and hence prints) a polynomial automatically w.r.t. the monomial ordering.

The definition of ideals and matrices, respectively, is straight forward: The user gives a list of polynomials which generate the ideal, resp. which are the entries of the matrix. (The number of rows and columns need to be provided when creating the matrix.)

A vector in SINGULAR is always an element of a free module over the basering. It is given as a list of polynomials in one of the following formats  $[f_1, \dots, f_n]$  or  $f_1 * \text{gen}(1) + \dots + f_n * \text{gen}(n)$ , where  $\text{gen}(i)$  denotes the  $i$ -th canonical generator of a free module (with 1 at index  $i$  and 0 everywhere else). Both forms are equivalent. A vector is internally represented in the second form with the  $\text{gen}(i)$  being "special" ring variables, ordered accordingly to the monomial ordering. Therefore, the form  $[f_1, \dots, f_n]$  serves as output only if the monomial ordering gives priority to the component, i.e., is of the form  $(c, \dots)$  (see [Section B.2.5 \[Module orderings\]](#), page 770). However, in any case the procedure `show` from the library `inout.lib` displays the bracket format.

A vector  $v = [f_1, \dots, f_n]$  should always be considered as a column vector in a free module of rank equal to `nrows(v)` where `nrows(v)` is equal to the maximal index  $r$  such that  $f_r \neq 0$ . This is due to the fact, that internally  $v$  is a polynomial in a sparse representation, i.e.,  $f_i * \text{gen}(i)$  is not stored if  $f_i = 0$  (for reasons of efficiency), hence the last 0-entries of  $v$  are lost. Only more complex structures are able to keep the rank.

A module  $M$  in SINGULAR is given by a list of vectors  $v_1, \dots, v_k$  which generate the module as a submodule of the free module of rank equal to `nrows(M)` which is the maximum of `nrows(v_i)`.

If one wants to create a module with a larger rank than given by its generators, one has to use the command `attrib(M, "rank", r)` (see [Section 5.1.2 \[attrib\]](#), page 156, [Section 5.1.107 \[nrows\]](#), page 232) or to define a matrix first, then converting it into a module. Modules in SINGULAR are almost the same as matrices, they may be considered as sparse representations of matrices. A module of a matrix is generated by the columns of the matrix and a matrix of a module has as columns the generators of the module. These conversions preserve the rank and the number of generators, resp. the number of rows and columns.

By the above remarks it might appear that SINGULAR is only able to handle submodules of a free module. However, this is not true. SINGULAR can compute with any finitely generated module over the basering  $R$ . Such a module, say  $N$ , is not represented by its generators but by its (generators and) relations. This means that  $N = R^n/M$  where  $n$  is the number of generators of  $N$  and  $M \subseteq R^n$  is the module of relations. In other words, defining a module  $M$  as a submodule of a free module  $R^n$  can also be considered as the definition of  $N = R^n/M$ .

Note that most functions, when applied to a module  $M$ , really deal with  $M$ . However, there are some functions which deal with  $N = R^n/M$  instead of  $M$ .

For example, `std(M)` computes a standard basis of  $M$  (and thus gives another representation of  $N$  as  $N = R^n/\text{std}(M)$ ). However, `dim(M)`, resp. `vdim(M)`, return  $\dim(R^n/M)$ , resp.  $\dim_k(R^n/M)$  (if  $M$  is given by a standard basis).

The function `syz(M)` returns the first syzygy module of  $M$ , i.e., the module of relations of the given generators of  $M$  which is equal to the second syzygy module of  $N$ . Refer to the description of each

function in [Section 5.1 \[Functions\]](#), page 156 to get information which module the function deals with.

The numbering in `res` and other commands for computing resolutions refers to a resolution of  $N = R^n/M$  (see [Section 5.1.134 \[res\]](#), page 253; [Section C.3 \[Syzygies and resolutions\]](#), page 776). It is possible to compute in any field which is a valid ground field in SINGULAR. For doing so, one has to define a ring with the desired ground field and at least one variable. The elements of the field are of type number, but may also be considered as polynomials (of degree 0). Large computations should be faster if the elements of the field are defined as numbers.

The above remarks do also apply to quotient rings. Polynomial data are stored internally in the same manner, the only difference is that this polynomial representation is in general not unique. `reduce(f, std(0))` computes a normal form of a polynomial `f` in a quotient ring (cf. [Section 5.1.131 \[reduce\]](#), page 251).

## B.2 Monomial orderings

### B.2.1 Introduction to orderings

SINGULAR offers a great variety of monomial orderings which provide an enormous functionality, if used diligently. However, this flexibility might also be confusing for the novice user. Therefore, we recommend to those not familiar with monomial orderings to generally use the ordering `dp` for computations in the polynomial ring  $K[x_1, \dots, x_n]$ , resp. `ds` for computations in the localization  $\text{Loc}_{(x)}K[x_1, \dots, x_n]$ .

For inhomogeneous input ideals, standard (resp. groebner) bases computations are generally faster with the orderings  $\text{Wp}(w_1, \dots, w_n)$  (resp.  $\text{Ws}(w_1, \dots, w_n)$ ) if the input is quasihomogenous w.r.t. the weights  $w_1, \dots, w_n$  of  $x_1, \dots, x_n$ .

If the output needs to be "triangular" (resp. "block-triangular"), the lexicographical ordering `lp` (resp. lexicographical block-orderings) need to be used. However, these orderings usually result in much less efficient computations.

### B.2.2 General definitions for orderings

A monomial ordering (term ordering) on  $K[x_1, \dots, x_n]$  is a total ordering  $<$  on the set of monomials (power products)  $\{x^\alpha \mid \alpha \in \mathbb{N}^n\}$  which is compatible with the natural semigroup structure, i.e.,  $x^\alpha < x^\beta$  implies  $x^\gamma x^\alpha < x^\gamma x^\beta$  for any  $\gamma \in \mathbb{N}^n$ . We do not require  $<$  to be a wellordering. See the literature cited in [Section C.9 \[References\]](#), page 792.

It is known that any monomial ordering can be represented by a matrix  $M$  in  $GL(n, R)$ , but, of course, only integer coefficients are of relevance in practice.

Global orderings are wellorderings (i.e.,  $1 < x_i$  for each variable  $x_i$ ), local orderings satisfy  $1 > x_i$  for each variable. If some variables are ordered globally and others locally we call it a mixed ordering. Local or mixed orderings are not wellorderings.

Let  $K$  be the ground field,  $x = (x_1, \dots, x_n)$  the variables and  $<$  a monomial ordering, then  $\text{Loc } K[x]$  denotes the localization of  $K[x]$  with respect to the multiplicatively closed set

$$\{1 + g \mid g = 0 \text{ or } g \in K[x] \setminus \{0\} \text{ and } L(g) < 1\}.$$

Here,  $L(g)$  denotes the leading monomial of  $g$ , i.e., the biggest monomial of  $g$  with respect to  $<$ . The result of any computation which uses standard basis computations has to be interpreted in  $\text{Loc } K[x]$ .

Note that the definition of a ring includes the definition of its monomial ordering (see [Section 3.3 \[Rings and orderings\]](#), page 30). SINGULAR offers the monomial orderings described in the following sections.

### B.2.3 Global orderings

For all these orderings, we have  $\text{Loc } K[x] = K[x]$

- lp: lexicographical ordering:  
 $x^\alpha < x^\beta \Leftrightarrow \exists 1 \leq i \leq n : \alpha_1 = \beta_1, \dots, \alpha_{i-1} = \beta_{i-1}, \alpha_i < \beta_i.$
- rp: inverse lexicographical ordering:  
 $x^\alpha < x^\beta \Leftrightarrow \exists 1 \leq i \leq n : \alpha_n = \beta_n, \dots, \alpha_{i+1} = \beta_{i+1}, \alpha_i < \beta_i.$
- dp: degree reverse lexicographical ordering:  
 let  $\deg(x^\alpha) = \alpha_1 + \dots + \alpha_n$ , then  $x^\alpha < x^\beta \Leftrightarrow \deg(x^\alpha) < \deg(x^\beta)$  or  
 $\deg(x^\alpha) = \deg(x^\beta)$  and  $\exists 1 \leq i \leq n : \alpha_n = \beta_n, \dots, \alpha_{i+1} = \beta_{i+1}, \alpha_i > \beta_i.$
- Dp: degree lexicographical ordering:  
 let  $\deg(x^\alpha) = \alpha_1 + \dots + \alpha_n$ , then  $x^\alpha < x^\beta \Leftrightarrow \deg(x^\alpha) < \deg(x^\beta)$  or  
 $\deg(x^\alpha) = \deg(x^\beta)$  and  $\exists 1 \leq i \leq n : \alpha_1 = \beta_1, \dots, \alpha_{i-1} = \beta_{i-1}, \alpha_i < \beta_i.$
- wp: weighted reverse lexicographical ordering:  
 let  $w_1, \dots, w_n$  be positive integers. Then  $\mathbf{wp}(w_1, \dots, w_n)$  is defined as **dp** but with  
 $\deg(x^\alpha) = w_1\alpha_1 + \dots + w_n\alpha_n.$
- Wp: weighted lexicographical ordering:  
 let  $w_1, \dots, w_n$  be positive integers. Then  $\mathbf{Wp}(w_1, \dots, w_n)$  is defined as **Dp** but with  
 $\deg(x^\alpha) = w_1\alpha_1 + \dots + w_n\alpha_n.$

### B.2.4 Local orderings

For ls, ds, Ds and, if the weights are positive integers, also for ws and Ws, we have  $\text{Loc } K[x] = K[x]_{(x)}$ , the localization of  $K[x]$  at the maximal ideal  $(x) = (x_1, \dots, x_n)$ .

- ls: negative lexicographical ordering:  
 $x^\alpha < x^\beta \Leftrightarrow \exists 1 \leq i \leq n : \alpha_1 = \beta_1, \dots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i.$
- ds: negative degree reverse lexicographical ordering:  
 let  $\deg(x^\alpha) = \alpha_1 + \dots + \alpha_n$ , then  $x^\alpha < x^\beta \Leftrightarrow \deg(x^\alpha) > \deg(x^\beta)$  or  
 $\deg(x^\alpha) = \deg(x^\beta)$  and  $\exists 1 \leq i \leq n : \alpha_n = \beta_n, \dots, \alpha_{i+1} = \beta_{i+1}, \alpha_i > \beta_i.$
- Ds: negative degree lexicographical ordering:  
 let  $\deg(x^\alpha) = \alpha_1 + \dots + \alpha_n$ , then  $x^\alpha < x^\beta \Leftrightarrow \deg(x^\alpha) > \deg(x^\beta)$  or  
 $\deg(x^\alpha) = \deg(x^\beta)$  and  $\exists 1 \leq i \leq n : \alpha_1 = \beta_1, \dots, \alpha_{i-1} = \beta_{i-1}, \alpha_i < \beta_i.$
- ws: (general) weighted reverse lexicographical ordering:  
 $\mathbf{ws}(w_1, \dots, w_n)$ ,  $w_1$  a nonzero integer,  $w_2, \dots, w_n$  any integer (including 0), is defined  
 as **ds** but with  $\deg(x^\alpha) = w_1\alpha_1 + \dots + w_n\alpha_n.$
- Ws: (general) weighted lexicographical ordering:  
 $\mathbf{Ws}(w_1, \dots, w_n)$ ,  $w_1$  a nonzero integer,  $w_2, \dots, w_n$  any integer (including 0), is defined  
 as **Ds** but with  $\deg(x^\alpha) = w_1\alpha_1 + \dots + w_n\alpha_n.$

### B.2.5 Module orderings

SINGULAR offers also orderings on the set of “monomials”  $\{x^a e_i \mid a \in N^n, 1 \leq i \leq r\}$  in  $\text{Loc } K[x]^r = \text{Loc } K[x]e_1 + \dots + \text{Loc } K[x]e_r$ , where  $e_1, \dots, e_r$  denote the canonical generators of  $\text{Loc } K[x]^r$ , the  $r$ -fold direct sum of  $\text{Loc } K[x]$ . (The function **gen(i)** yields  $e_i$ ).

We have two possibilities: either to give priority to the component of a vector in  $\text{Loc } K[x]^r$  or (which is the default in SINGULAR) to give priority to the coefficients. The orderings **(<,c)** and



$(<, C)$  give priority to the coefficients; whereas  $(c, <)$  and  $(C, <)$  give priority to the components. Let  $<$  be any of the monomial orderings of  $\text{Loc } K[x]$  as above.

$(<, C)$ :  $<_m = (<, C)$  denotes the module ordering (giving priority to the coefficients):  
 $x^\alpha e_i <_m x^\beta e_j \Leftrightarrow x^\alpha < x^\beta \text{ or } (x^\alpha = x^\beta \text{ and } i < j).$

**Example:**

```
ring r = 0, (x,y,z), ds;
// the same as ring r = 0, (x,y,z), (ds, C);
[x+y2,z3+xy];
map x*gen(1)+xy*gen(2)+y2*gen(1)+z3*gen(2)
[x,x,x];
map x*gen(3)+x*gen(2)+x*gen(1)
```

$(C, <)$ :  $<_m = (C, <)$  denotes the module ordering (giving priority to the component):  
 $x^\alpha e_i <_m x^\beta e_j \Leftrightarrow i < j \text{ or } (i = j \text{ and } x^\alpha < x^\beta).$

**Example:**

```
ring r = 0, (x,y,z), (C,lp);
[x+y2,z3+xy];
map xy*gen(2)+z3*gen(2)+x*gen(1)+y2*gen(1)
[x,x,x];
map x*gen(3)+x*gen(2)+x*gen(1)
```

$(<, c)$ :  $<_m = (<, c)$  denotes the module ordering (giving priority to the coefficients):  
 $x^\alpha e_i <_m x^\beta e_j \Leftrightarrow x^\alpha < x^\beta \text{ or } (x^\alpha = x^\beta \text{ and } i > j).$

**Example:**

```
ring r = 0, (x,y,z), (lp,c);
[x+y2,z3+xy];
map xy*gen(2)+x*gen(1)+y2*gen(1)+z3*gen(2)
[x,x,x];
map x*gen(1)+x*gen(2)+x*gen(3)
```

$(c, <)$ :  $<_m = (c, <)$  denotes the module ordering (giving priority to the component):  
 $x^\alpha e_i <_m x^\beta e_j \Leftrightarrow i > j \text{ or } (i = j \text{ and } x^\alpha < x^\beta).$

**Example:**

```
ring r = 0, (x,y,z), (c,lp);
[x+y2,z3+xy];
map [x+y2,xy+z3]
[x,x,x];
map [x,x,x]
```

The output of a vector  $v$  in  $K[x]^r$  with components  $v_1, \dots, v_r$  has the format  $v_1 * \text{gen}(1) + \dots + v_r * \text{gen}(r)$  (up to permutation) unless the ordering starts with  $c$ . In this case a vector is written as  $[v_1, \dots, v_r]$ . In all cases SINGULAR can read input in both formats.

## B.2.6 Matrix orderings

Let  $M$  be an invertible  $(n \times n)$ -matrix with integer coefficients and  $M_1, \dots, M_n$  the rows of  $M$ .

The M-ordering  $<$  is defined as follows:

$$x^a < x^b \Leftrightarrow \exists 1 \leq i \leq n : M_1 a = M_1 b, \dots, M_{i-1} a = M_{i-1} b \text{ and } M_i a < M_i b.$$

Thus,  $x^a < x^b$  if and only if  $Ma$  is smaller than  $Mb$  with respect to the lexicographical ordering.

The following matrices represent (for 3 variables) the global and local orderings defined above (note that the matrix is not uniquely determined by the ordering):

$$\begin{aligned}
\text{lp: } & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{dp: } & \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} & \text{Dp: } & \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
\text{wp(1,2,3): } & \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} & \text{Wp(1,2,3): } & \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
\text{ls: } & \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} & \text{ds: } & \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} & \text{Ds: } & \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
\text{ws(1,2,3): } & \begin{pmatrix} -1 & -2 & -3 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} & \text{Ws(1,2,3): } & \begin{pmatrix} -1 & -2 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\end{aligned}$$

Product orderings (see next section) represented by a matrix:

$$\begin{aligned}
(\text{dp}(3), \text{wp}(1,2,3)) &: \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \\
(\text{Dp}(3), \text{ds}(3)) &: \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}
\end{aligned}$$

Orderings with extra weight vector (see below) represented by a matrix:

$$\begin{aligned}
(\text{dp}(3), \text{a}(1,2,3), \text{dp}(3)) &: \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \\
(\text{a}(1,2,3,4,5), \text{Dp}(3), \text{ds}(3)) &: \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}
\end{aligned}$$

**Example:**

```
ring r = 0, (x,y,z), M(1, 0, 0, 0, 1, 0, 0, 0, 1);
```

which may also be written as:

```
intmat m[3][3]=1, 0, 0, 0, 1, 0, 0, 0, 1;
m;
↦ 1,0,0,
↦ 0,1,0,
↦ 0,0,1
```

```

ring r = 0, (x,y,z), M(m);
r;
⇨ // coefficients: QQ
⇨ // number of vars : 3
⇨ //      block 1 : ordering M
⇨ //      : names   x y z
⇨ //      : weights 1 0 0
⇨ //      : weights 0 1 0
⇨ //      : weights 0 0 1
⇨ //      block 2 : ordering C

```

If the ring has  $n$  variables and the matrix does not contain  $n \times n$  entries, an error message is given.

**WARNING:** SINGULAR does not check whether the matrix has full rank. In such a case some computations might not terminate, others may not give a sensible result.

Having these matrix orderings SINGULAR can compute standard bases for any monomial ordering which is compatible with the natural semigroup structure. In practice the global and local orderings together with block orderings should be sufficient in most cases. These orderings are faster than the corresponding matrix orderings, since evaluating a matrix product is time consuming.

## B.2.7 Product orderings

Let  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_m)$  be two ordered sets of variables,  $<_1$  a monomial ordering on  $K[x]$  and  $<_2$  a monomial ordering on  $K[y]$ . The product ordering (or block ordering)  $< := (<_1, <_2)$  on  $K[x, y]$  is the following:

$$x^a y^b < x^A y^B \Leftrightarrow x^a <_1 x^A \text{ or } (x^a = x^A \text{ and } y^b <_2 y^B).$$

Inductively one defines the product ordering of more than two monomial orderings.

In SINGULAR, any of the above global orderings, local orderings or matrix orderings may be combined (in an arbitrary manner and length) to a product ordering. E.g., `(lp(3), M(1, 2, 3, 1, 1, 1, 1, 0, 0), ds(4), ws(1,2,3))` defines: `lp` on the first 3 variables, the matrix ordering `M(1, 2, 3, 1, 1, 1, 1, 0, 0)` on the next 3 variables, `ds` on the next 4 variables and `ws(1,2,3)` on the last 3 variables.

## B.2.8 Extra weight vector

`a(w1, ..., wn)`,  $w_1, \dots, w_n$  any integers (including 0), defines  $\deg(x^\alpha) = w_1 \alpha_1 + \dots + w_n \alpha_n$  and

$$\deg(x^\alpha) < \deg(x^\beta) \Rightarrow x^\alpha < x^\beta,$$

$$\deg(x^\alpha) > \deg(x^\beta) \Rightarrow x^\alpha > x^\beta.$$

An extra weight vector does not define a monomial ordering by itself: it can only be used in combination with other orderings to insert an extra line of weights into the ordering matrix.

**Example:**

```

ring r = 0, (x,y,z), (a(1,2,3), wp(4,5,2));
ring s = 0, (x,y,z), (a(1,2,3), dp);
ring q = 0, (a,b,c,d), (lp(1), a(1,2,3), ds);

```

### B.2.9 Pseudo ordering L

`L(max_exponent)` is not an ordering but sets the maximal allowed exponent for polynomial in this ring. The default is 32767. The current value for a ring is reflected in the attribute "maxExp". This attribute is also set (and acknowledged) for the list constructed by `ringlist` and the construction of a ring from such a list.

## Appendix C Mathematical background

This chapter introduces some of the mathematical notions and definitions used throughout the manual. It is mostly a collection of the most prominent definitions and properties. For details, please, refer to articles or text books (see [Section C.9 \[References\]](#), page 792).

### C.1 Standard bases

#### Definition

Let  $R = \text{Loc}_{<} K[\underline{x}]$  and let  $I$  be a submodule of  $R^r$ . Note that for  $r=1$  this means that  $I$  is an ideal in  $R$ . Denote by  $L(I)$  the submodule of  $R^r$  generated by the leading terms of elements of  $I$ , i.e. by  $\{L(f) \mid f \in I\}$ . Then  $f_1, \dots, f_s \in I$  is called a **standard basis** of  $I$  if  $L(f_1), \dots, L(f_s)$  generate  $L(I)$ .

A standard basis is **minimal** if  $\forall i : (f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_s) \neq I$ .

A minimal standard basis is **completely reduced** if  $\forall i : \text{reduce}(f_i, (f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_s)) = f_i$

#### Properties

normal form:

A function  $\text{NF} : R^r \times \{G \mid G \text{ a standard basis}\} \rightarrow R^r, (p, G) \mapsto \text{NF}(p|G)$ , is called a **normal form** if for any  $p \in R^r$  and any standard basis  $G$  the following holds: if  $\text{NF}(p|G) \neq 0$  then  $L(g)$  does not divide  $L(\text{NF}(p|G))$  for all  $g \in G$ . The function may also be applied to any generating set of an ideal: the result is then not uniquely defined.

$\text{NF}(p|G)$  is called a **normal form of  $p$  with respect to  $G$**

ideal membership:

For a standard basis  $G$  of  $I$  the following holds:  $f \in I$  if and only if  $\text{NF}(f, G) = 0$ .

Hilbert function:

Let  $I \subseteq K[\underline{x}]^r$  be a homogeneous module, then the Hilbert function  $H_I$  of  $I$  (see below) and the Hilbert function  $H_{L(I)}$  of the leading module  $L(I)$  coincide, i.e.,  $H_I = H_{L(I)}$ .

### C.2 Hilbert function

Let  $M = \bigoplus_{i \in \mathbb{Z}} M_i$  be a graded module over  $K[x_1, \dots, x_n]$  with respect to weights  $(w_1, \dots, w_n)$ . The **Hilbert function** of  $M$ ,  $H_M$ , is defined (on the integers) by

$$H_M(k) := \dim_K M_k.$$

The **Hilbert-Poincare series** of  $M$  is the power series

$$\text{HP}_M(t) := \sum_{i=-\infty}^{\infty} H_M(i) t^i = \sum_{i=-\infty}^{\infty} \dim_K M_i \cdot t^i.$$

It turns out that  $\text{HP}_M(t)$  can be written in two useful ways for weights  $(1, \dots, 1)$ :

$$\text{HP}_M(t) = \frac{Q(t)}{(1-t)^n} = \frac{P(t)}{(1-t)^{\dim(M)}}$$

where  $Q(t)$  and  $P(t)$  are polynomials in  $\mathbb{Z}[t]$ .  $Q(t)$  is called the **first Hilbert series**, and  $P(t)$  the **second Hilbert series**. If  $P(t) = \sum_{k=0}^N a_k t^k$ , and  $d = \dim(M)$ , then  $H_M(s) = \sum_{k=0}^N a_k \binom{d+s-k-1}{d-1}$  (the

**Hilbert polynomial**) for  $s \geq N$ .

Generalizing this to quasihomogeneous modules we get

$$\text{HP}_M(t) = \frac{Q(t)}{\prod_{i=1}^n (1 - t^{w_i})}$$

where  $Q(t)$  is a polynomial in  $\mathbf{Z}[t]$ .  $Q(t)$  is called the **first (weighted) Hilbert series** of  $M$ .

## C.3 Syzygies and resolutions

### Syzygies

Let  $R$  be a quotient of  $\text{Loc}_{<} K[\underline{x}]$  and let  $I = (g_1, \dots, g_s)$  be a submodule of  $R^r$ . Then the **module of syzygies** (or **1st syzygy module, module of relations**) of  $I$ ,  $\text{syzy}(I)$ , is defined to be the kernel of the map  $R^s \rightarrow R^r$ ,  $\sum_{i=1}^s w_i e_i \mapsto \sum_{i=1}^s w_i g_i$ .

The **k-th syzygy module** is defined inductively to be the module of syzygies of the  $(k-1)$ -st syzygy module.

Note, that the syzygy modules of  $I$  depend on a choice of generators  $g_1, \dots, g_s$ . But one can show that they depend on  $I$  uniquely up to direct summands.

**Example:**

```
ring R= 0,(u,v,x,y,z),dp;
ideal i=ux, vx, uy, vy;
print(syz(i));
↦ -y,0, -v,0,
↦ 0, -y,u, 0,
↦ x, 0, 0, -v,
↦ 0, x, 0, u
```

### Free resolutions

Let  $I = (g_1, \dots, g_s) \subseteq R^r$  and  $M = R^r/I$ . A **free resolution of  $M$**  is a long exact sequence

$$\dots \longrightarrow F_2 \xrightarrow{A_2} F_1 \xrightarrow{A_1} F_0 \longrightarrow M \longrightarrow 0,$$

where the columns of the matrix  $A_1$  generate  $I$ . Note that resolutions need not to be finite (i.e., of finite length). The Hilbert Syzygy Theorem states that for  $R = \text{Loc}_{<} K[\underline{x}]$  there exists a ("minimal") resolution of length not exceeding the number of variables.

**Example:**

```
ring R= 0,(u,v,x,y,z),dp;
ideal I = ux, vx, uy, vy;
resolution resI = mres(I,0); resI;
↦ 1      4      4      1
↦ R <--  R <--  R <--  R
↦
↦ 0      1      2      3
↦
// The matrix A_1 is given by
print(matrix(resI[1]));
```

```

↳ vy,uy,vx,ux
  // We see that the columns of A_1 generate I.
  // The matrix A_2 is given by
  print(matrix(resI[3]));
↳ u,
↳ -v,
↳ -x,
↳ y

```

## Betti numbers and regularity

Let  $R$  be a graded ring (e.g.,  $R = \text{Loc}_{<}K[x]$ ) and let  $I \subset R^r$  be a graded submodule. Let

$$R^r = \bigoplus_a R \cdot e_{a,0} \xleftarrow{A_1} \bigoplus_a R \cdot e_{a,1} \xleftarrow{\quad} \dots \xleftarrow{\quad} \bigoplus_a R \cdot e_{a,n} \xleftarrow{\quad} 0$$

be a minimal free resolution of  $R^r/I$  considered with homogeneous maps of degree 0. Then the **graded Betti number**  $b_{i,j}$  of  $R^r/I$  is the minimal number of generators  $e_{a,j}$  in degree  $i + j$  of the  $j$ -th syzygy module of  $R^r/I$  (i.e., the  $(j - 1)$ -st syzygy module of  $I$ ). Note that, by definition, the 0-th syzygy module of  $R^r/I$  is  $R^r$  and the 1st syzygy module of  $R^r/I$  is  $I$ .

The **regularity** of  $I$  is the smallest integer  $s$  such that

$$\deg(e_{a,j}) \leq s + j - 1 \quad \text{for all } j.$$

**Example:**

```

ring R= 0,(u,v,x,y,z),dp;
ideal I = ux, vx, uy, vy;
resolution resI = mres(I,0); resI;
↳ 1      4      4      1
↳ R <--  R <--  R <--  R
↳
↳ 0      1      2      3
↳
  // the betti number:
  print(betti(resI), "beti");
↳      0      1      2      3
↳ -----
↳ 0:    1      -      -      -
↳ 1:    -      4      4      1
↳ -----
↳ total: 1      4      4      1
↳
  // the regularity:
  regularity(resI);
↳ 2

```

## C.4 Characteristic sets

Let  $<$  be the lexicographical ordering on  $R = K[x_1, \dots, x_n]$  with  $x_1 < \dots < x_n$ . For  $f \in R$  let  $\text{lvar}(f)$  (the leading variable of  $f$ ) be the largest variable in  $f$ , i.e., if  $f = a_s(x_1, \dots, x_{k-1})x_k^s + \dots + a_0(x_1, \dots, x_{k-1})$  for some  $k \leq n$  then  $\text{lvar}(f) = x_k$ .

Moreover, let  $\text{ini}(f) := a_s(x_1, \dots, x_{k-1})$ . The pseudoremainder  $r = \text{prem}(g, f)$  of  $g$  with respect to  $f$  is defined by the equality  $\text{ini}(f)^a \cdot g = qf + r$  with  $\deg_{\text{lvar}(f)}(r) < \deg_{\text{lvar}(f)}(f)$  and  $a$  minimal.

A set  $T = \{f_1, \dots, f_r\} \subset R$  is called triangular if  $\text{lvar}(f_1) < \dots < \text{lvar}(f_r)$ . Moreover, let  $U \subset T$ , then  $(T, U)$  is called a triangular system, if  $T$  is a triangular set such that  $\text{ini}(T)$  does not vanish on  $V(T) \setminus V(U)(= V(T \setminus U))$ .

$T$  is called irreducible if for every  $i$  there are no  $d_i, f'_i, f''_i$  such that

$$\begin{aligned} \text{lvar}(d_i) < \text{lvar}(f_i) &= \text{lvar}(f'_i) = \text{lvar}(f''_i), \\ 0 &\notin \text{prem}(\{d_i, \text{ini}(f'_i), \text{ini}(f''_i)\}, \{f_1, \dots, f_{i-1}\}), \\ \text{prem}(d_i f_i - f'_i f''_i, \{f_1, \dots, f_{i-1}\}) &= 0. \end{aligned}$$

Furthermore,  $(T, U)$  is called irreducible if  $T$  is irreducible.

The main result on triangular sets is the following: Let  $G = \{g_1, \dots, g_s\} \subset R$ , then there are irreducible triangular sets  $T_1, \dots, T_l$  such that  $V(G) = \bigcup_{i=1}^l (V(T_i) \setminus I_i)$  where  $I_i = \{\text{ini}(f) \mid f \in T_i\}$ . Such a set  $\{T_1, \dots, T_l\}$  is called an **irreducible characteristic series** of the ideal  $(G)$ .

**Example:**

```
ring R= 0, (x,y,z,u), dp;
ideal i=-3zu+y2-2x+2,
      -3x2u-4yz-6xz+2y2+3xy,
      -3z2u-xu+y2z+y;
print(char_series(i));
⇒ _[1,1], 3x2z-y2+2yz, 3x2u-3xy-2y2+2yu,
⇒ x,      -y+2z,      -2y2+3yu-4
```

## C.5 Gauss-Manin connection

Let  $f: (C^{n+1}, 0) \rightarrow (C, 0)$  be a complex isolated hypersurface singularity given by a polynomial with algebraic coefficients which we also denote by  $f$ . Let  $O = C[x_0, \dots, x_n]_{(x_0, \dots, x_n)}$  be the local ring at the origin and  $J_f$  the Jacobian ideal of  $f$ .

A **Milnor representative** of  $f$  defines a differentiable fibre bundle over the punctured disc with fibres of homotopy type of  $\mu$   $n$ -spheres. The  $n$ -th cohomology bundle is a flat vector bundle of dimension  $n$  and carries a natural flat connection with covariant derivative  $\partial_t$ . The **monodromy operator** is the action of a positively oriented generator of the fundamental group of the punctured disc on the Milnor fibre. Sections in the cohomology bundle of **moderate growth** at 0 form a regular  $D = C\{t\}[\partial_t]$ -module  $G$ , the **Gauss-Manin connection**.

By integrating along flat multivalued families of cycles, one can consider fibrewise global holomorphic differential forms as elements of  $G$ . This factors through an inclusion of the **Brieskorn lattice**  $H'' := \Omega_{C^{n+1}, 0}^{n+1}/df \wedge d\Omega_{C^{n+1}, 0}^{n-1}$  in  $G$ .

The  $D$ -module structure defines the **V-filtration**  $V$  on  $G$  by  $V^\alpha := \sum_{\beta \geq \alpha} C\{t\} \ker(t\partial_t - \beta)^{n+1}$ . The Brieskorn lattice defines the **Hodge filtration**  $F$  on  $G$  by  $F_k = \partial_t^k H''$  which comes from the **mixed Hodge structure** on the Milnor fibre. Note that  $F_{-1} = H'$ .

The induced V-filtration on the Brieskorn lattice determines the **singularity spectrum**  $Sp$  by  $Sp(\alpha) := \dim_C Gr_V^\alpha Gr_0^F G$ . The spectrum consists of  $\mu$  rational numbers  $\alpha_1, \dots, \alpha_\mu$  such that  $e^{2\pi i \alpha_1}, \dots, e^{2\pi i \alpha_\mu}$  are the eigenvalues of the monodromy. These **spectral numbers** lie in the open interval  $(-1, n)$ , symmetric about the midpoint  $(n-1)/2$ .

The spectrum is constant under  $\mu$ -constant deformations and has the following semicontinuity property: The number of spectral numbers in an interval  $(a, a+1]$  of all singularities of a small deformation of  $f$  is greater than or equal to that of  $f$  in this interval. For semiquasihomogeneous singularities, this also holds for intervals of the form  $(a, a+1)$ .



Two given isolated singularities  $f$  and  $g$  determine two spectra and from these spectra we get an integer. This integer is the maximal positive integer  $k$  such that the semicontinuity holds for the spectrum of  $f$  and  $k$  times the spectrum of  $g$ . These numbers give bounds for the maximal number of isolated singularities of a specific type on a hypersurface  $X \subset P^n$  of degree  $d$ : such a hypersurface has a smooth hyperplane section, and the complement is a small deformation of a cone over this hyperplane section. The cone itself being a  $\mu$ -constant deformation of  $x_0^d + \dots + x_n^d = 0$ , the singularities are bounded by the spectrum of  $x_0^d + \dots + x_n^d$ .

Using the library `gmssing.lib` one can compute the **monodromy**, the V-filtration on  $H''/H'$ , and the spectrum.

Let us consider as an example  $f = x^5 + x^2y^2 + y^5$ . First, we compute a matrix  $M$  such that  $\exp(2\pi i M)$  is a monodromy matrix of  $f$  and the Jordan normal form of  $M$  :

```
LIB "mondromy.lib";
ring R=0,(x,y),ds;
poly f=x5+x2y2+y5;
matrix M=monodromyB(f);
print(M);
⇒ 11/10,0, 0, 0, 0, 0,-1/4,0, 0, 0, 0,
⇒ 0, 13/10,0, 0, 0, 0,0, 15/8,0, 0, 0,
⇒ 0, 0, 13/10,0, 0, 0,0, 0, 15/8,0, 0,
⇒ 0, 0, 0, 11/10,-1/4,0,0, 0, 0, 0, 0,
⇒ 0, 0, 0, 0, 9/10,0,0, 0, 0, 0, 0,
⇒ 0, 0, 0, 0, 0, 1,0, 0, 0, 0, 3/5,
⇒ 0, 0, 0, 0, 0, 0,9/10,0, 0, 0, 0,
⇒ 0, 0, 0, 0, 0, 0,0, 7/10,0, 0, 0,
⇒ 0, 0, 0, 0, 0, 0,0, 0, 7/10,0, 0,
⇒ 0, 0, 0, 0, 0, 0,0, 0, 0, 1, -2/5,
⇒ 0, 0, 0, 0, 0, 0,0, 0, 0, 5/8,0
```

Now, we compute the V-filtration on  $H''/H'$  and the spectrum:

```
LIB "gmssing.lib";
ring R=0,(x,y),ds;
poly f=x5+x2y2+y5;
list l=vfilt(f);
print(l[1]); // spectral numbers
⇒ -1/2,
⇒ -3/10,
⇒ -1/10,
⇒ 0,
⇒ 1/10,
⇒ 3/10,
⇒ 1/2
print(l[2]); // corresponding multiplicities
⇒ 1,
⇒ 2,
⇒ 2,
⇒ 1,
⇒ 2,
⇒ 2,
⇒ 1
print(l[3]); // vector space of i-th graded part
⇒ [1]:
⇒ _[1]=gen(11)
```

```

⇒ [2]:
⇒   _[1]=gen(10)
⇒   _[2]=gen(6)
⇒ [3]:
⇒   _[1]=gen(9)
⇒   _[2]=gen(4)
⇒ [4]:
⇒   _[1]=gen(5)
⇒ [5]:
⇒   _[1]=gen(3)
⇒   _[2]=gen(8)
⇒ [6]:
⇒   _[1]=gen(2)
⇒   _[2]=gen(7)
⇒ [7]:
⇒   _[1]=gen(1)
  print(l[4]); // monomial vector space basis of H''/s*H''
⇒ y5,
⇒ y4,
⇒ y3,
⇒ y2,
⇒ xy,
⇒ y,
⇒ x4,
⇒ x3,
⇒ x2,
⇒ x,
⇒ 1
  print(l[5]); // standard basis of Jacobian ideal
⇒ 2x2y+5y4,
⇒ 5x5-5y5,
⇒ 2xy2+5x4,
⇒ 10y6+25x3y4

```

Here  $l[1]$  contains the spectral numbers,  $l[2]$  the corresponding multiplicities,  $l[3]$  a  $C$ -basis of the  $V$ -filtration on  $H''/H'$  in terms of the monomial basis of  $O/J_f \cong H''/H'$  in  $l[4]$  (separated by degree).

If the principal part of  $f$  is  $C$ -nondegenerate, one can compute the spectrum using the library `spectrum.lib`. In this case, the  $V$ -filtration on  $H''$  coincides with the Newton-filtration on  $H''$  which allows to compute the spectrum more efficiently.

Let us calculate one specific example, the maximal number of triple points of type  $\tilde{E}_6$  on a surface  $X \subset P^3$  of degree seven. This calculation can be done over the rationals. We choose a local ordering on  $Q[x, y, z]$ . Here we take the negative degree lexicographical ordering, in SINGULAR denoted by `ds`:

```

ring r=0,(x,y,z),ds;
LIB "spectrum.lib";
poly f=x^7+y^7+z^7;
list s1=spectrumnd( f );
s1;
⇒ [1]:
⇒   _[1]=-4/7
⇒   _[2]=-3/7
⇒   _[3]=-2/7

```

```

⇒    _[4]=-1/7
⇒    _[5]=0
⇒    _[6]=1/7
⇒    _[7]=2/7
⇒    _[8]=3/7
⇒    _[9]=4/7
⇒    _[10]=5/7
⇒    _[11]=6/7
⇒    _[12]=1
⇒    _[13]=8/7
⇒    _[14]=9/7
⇒    _[15]=10/7
⇒    _[16]=11/7
⇒    [2]:
⇒    1,3,6,10,15,21,25,27,27,25,21,15,10,6,3,1

```

The command `spectrumnd(f)` computes the spectrum of  $f$  and returns a list with six entries: The Milnor number  $\mu(f)$ , the geometric genus  $p_g(f)$  and the number of different spectrum numbers. The other three entries are of type `intvec`. They contain the numerators, denominators and multiplicities of the spectrum numbers. So  $x^7 + y^7 + z^7 = 0$  has Milnor number 216 and geometrical genus 35. Its spectrum consists of the 16 different rationals

$\frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{1}{1}, \frac{8}{7}, \frac{9}{7}, \frac{10}{7}, \frac{11}{7}, \frac{12}{7}, \frac{13}{7}, \frac{2}{1}, \frac{15}{7}, \frac{16}{7}, \frac{17}{7}, \frac{18}{7}$   
 appearing with multiplicities

1,3,6,10,15,21,25,27,27,25,21,15,10,6,3,1.

The singularities of type  $\tilde{E}_6$  form a  $\mu$ -constant one parameter family given by  $x^3 + y^3 + z^3 + \lambda xyz = 0$ ,  $\lambda^3 \neq -27$ . Therefore they have all the same spectrum, which we compute for  $x^3 + y^3 + z^3$ .

```

poly g=x^3+y^3+z^3;
list s2=spectrumnd(g);
s2;
⇒    [1]:
⇒    8
⇒    [2]:
⇒    1
⇒    [3]:
⇒    4
⇒    [4]:
⇒    1,4,5,2
⇒    [5]:
⇒    1,3,3,1
⇒    [6]:
⇒    1,3,3,1

```

Evaluating semicontinuity is very easy:

```

semicont(s1,s2);
⇒    18

```

This tells us that there are at most 18 singularities of type  $\tilde{E}_6$  on a septic in  $P^3$ . But  $x^7 + y^7 + z^7$  is semiquasihomogeneous (sqh), so we can also apply the stronger form of semicontinuity:

```

semicontsqh(s1,s2);
⇒    17

```

So in fact a septic has at most 17 triple points of type  $\tilde{E}_6$ .

Note that `spectrumnd(f)` works only if  $f$  has a nondegenerate principal part. In fact `spectrumnd` will detect a degenerate principal part in many cases and print out an error message. However if it

is known in advance that  $f$  has nondegenerate principal part, then the spectrum may be computed much faster using `spectrumnd(f,1)`.

## C.6 Toric ideals and integer programming

### C.6.1 Toric ideals

Let  $A$  denote an  $m \times n$  matrix with integral coefficients. For  $u \in \mathbb{Z}^n$ , we define  $u^+, u^-$  to be the uniquely determined vectors with nonnegative coefficients and disjoint support (i.e.,  $u_i^+ = 0$  or  $u_i^- = 0$  for each component  $i$ ) such that  $u = u^+ - u^-$ . For  $u \geq 0$  component-wise, let  $x^u$  denote the monomial  $x_1^{u_1} \cdot \dots \cdot x_n^{u_n} \in K[x_1, \dots, x_n]$ .

The ideal

$$I_A := (x^{u^+} - x^{u^-} \mid u \in \ker(A) \cap \mathbb{Z}^n) \subset K[x_1, \dots, x_n]$$

is called a **toric ideal**.

The first problem in computing toric ideals is to find a finite generating set: Let  $v_1, \dots, v_r$  be a lattice basis of  $\ker(A) \cap \mathbb{Z}^n$  (i.e., a basis of the  $\mathbb{Z}$ -module). Then

$$I_A := I : (x_1 \cdot \dots \cdot x_n)^\infty$$

where

$$I = \langle x^{v_i^+} - x^{v_i^-} \mid i = 1, \dots, r \rangle$$

The required lattice basis can be computed using the LLL-algorithm ([Section 5.1.155 \[system\]](#), [page 275](#), see see [\[\[Coh93\]\]](#), [page 785](#)). For the computation of the saturation, there are various possibilities described in the section Algorithms.

### C.6.2 Algorithms

The following algorithms are implemented in [Section D.4.35 \[toric-lib\]](#), [page 1388](#).

#### C.6.2.1 The algorithm of Conti and Traverso

The algorithm of Conti and Traverso (see [\[\[CoTr91\]\]](#), [page 785](#)) computes  $I_A$  via the extended matrix  $B = (I_m \mid A)$ , where  $I_m$  is the  $m \times m$  unit matrix. A lattice basis of  $B$  is given by the set of vectors  $(a^j, -e_j) \in \mathbb{Z}^{m+n}$ , where  $a^j$  is the  $j$ -th row of  $A$  and  $e_j$  the  $j$ -th coordinate vector. We look at the ideal in  $K[y_1, \dots, y_m, x_1, \dots, x_n]$  corresponding to these vectors, namely

$$I_1 = \langle y^{a_j^+} - x_j y^{a_j^-} \mid j = 1, \dots, n \rangle.$$

We introduce a further variable  $t$  and adjoin the binomial  $t \cdot y_1 \cdot \dots \cdot y_m - 1$  to the generating set of  $I_1$ , obtaining an ideal  $I_2$  in the polynomial ring  $K[t, y_1, \dots, y_m, x_1, \dots, x_n]$ .  $I_2$  is saturated w.r.t. all variables because all variables are invertible modulo  $I_2$ . Now  $I_A$  can be computed from  $I_2$  by eliminating the variables  $t, y_1, \dots, y_m$ .

Because of the big number of auxiliary variables needed to compute a toric ideal, this algorithm is rather slow in practice. However, it has a special importance in the application to integer programming (see [Section C.6.4 \[Integer programming\]](#), [page 784](#)).

### C.6.2.2 The algorithm of Pottier

The algorithm of Pottier (see [\[\[Pot94\], page 785\]](#)) starts by computing a lattice basis  $v_1, \dots, v_r$  for the integer kernel of  $A$  using the LLL-algorithm ([Section 5.1.155 \[system\], page 275](#)). The ideal corresponding to the lattice basis vectors

$$I_1 = \langle x^{v_i^+} - x^{v_i^-} \mid i = 1, \dots, r \rangle$$

is saturated – as in the algorithm of Conti and Traverso – by inversion of all variables: One adds an auxiliary variable  $t$  and the generator  $t \cdot x_1 \cdot \dots \cdot x_n - 1$  to obtain an ideal  $I_2$  in  $K[t, x_1, \dots, x_n]$  from which one computes  $I_A$  by elimination of  $t$ .

### C.6.2.3 The algorithm of Hosten and Sturmfels

The algorithm of Hosten and Sturmfels (see [\[\[HoSt95\], page 785\]](#)) allows to compute  $I_A$  without any auxiliary variables, provided that  $A$  contains a vector  $w$  with positive coefficients in its row space. This is a real restriction, i.e., the algorithm will not necessarily work in the general case.

A lattice basis  $v_1, \dots, v_r$  is again computed via the LLL-algorithm. The saturation step is performed in the following way: First note that  $w$  induces a positive grading w.r.t. which the ideal

$$I = \langle x^{v_i^+} - x^{v_i^-} \mid i = 1, \dots, r \rangle$$

is homogeneous corresponding to our lattice basis. We use the following lemma:

Let  $I$  be a homogeneous ideal w.r.t. the weighted reverse lexicographical ordering with weight vector  $w$  and variable order  $x_1 > x_2 > \dots > x_n$ . Let  $G$  denote a Groebner basis of  $I$  w.r.t. this ordering. Then a Groebner basis of  $(I : x_n^\infty)$  is obtained by dividing each element of  $G$  by the highest possible power of  $x_n$ .

From this fact, we can successively compute

$$I_A = I : (x_1 \cdot \dots \cdot x_n)^\infty = (((I : x_1^\infty) : x_2^\infty) : \dots : x_n^\infty);$$

in the  $i$ -th step we take  $x_i$  as the smallest variable and apply the lemma with  $x_i$  instead of  $x_n$ .

This procedure involves  $n$  Groebner basis computations. Actually, this number can be reduced to at most  $n/2$  (see [\[\[HoSh98\], page 785\]](#)), and each computation – except for the first one – proves to be simple and fast in practice.

### C.6.2.4 The algorithm of Di Biase and Urbanke

Like the algorithm of Hosten and Sturmfels, the algorithm of Di Biase and Urbanke (see [\[\[DBUr95\], page 785\]](#)) performs up to  $n/2$  Groebner basis computations. It needs no auxiliary variables, but a supplementary precondition; namely, the existence of a vector without zero components in the kernel of  $A$ .

The main idea comes from the following observation:

Let  $B$  be an integer matrix,  $u_1, \dots, u_r$  a lattice basis of the integer kernel of  $B$ . Assume that all components of  $u_1$  are positive. Then

$$I_B = \langle x^{u_i^+} - x^{u_i^-} \mid i = 1, \dots, r \rangle,$$

i.e., the ideal on the right is already saturated w.r.t. all variables.

The algorithm starts by finding a lattice basis  $v_1, \dots, v_r$  of the kernel of  $A$  such that  $v_1$  has no zero component. Let  $\{i_1, \dots, i_l\}$  be the set of indices  $i$  with  $v_{1,i} < 0$ . Multiplying the components

$i_1, \dots, i_l$  of  $v_1, \dots, v_r$  and the columns  $i_1, \dots, i_l$  of  $A$  by  $-1$  yields a matrix  $B$  and a lattice basis  $u_1, \dots, u_r$  of the kernel of  $B$  that fulfill the assumption of the observation above. It is then possible to compute a generating set of  $I_A$  by applying the following “variable flip” successively to  $i = i_1, \dots, i_l$ : Let  $>$  be an elimination ordering for  $x_i$ . Let  $A_i$  be the matrix obtained by multiplying the  $i$ -th column of  $A$  by  $-1$ . Let

$$\{x_i^{r_j} x^{a_j} - x^{b_j} | j \in J\}$$

be a Groebner basis of  $I_{A_i}$  w.r.t.  $>$  (where  $x_i$  is neither involved in  $x^{a_j}$  nor in  $x^{b_j}$ ). Then

$$\{x^{a_j} - x_i^{r_j} x^{b_j} | j \in J\}$$

is a generating set for  $I_A$ .

### C.6.2.5 The algorithm of Bigatti, La Scala and Robbiano

The algorithm of Bigatti, La Scala and Robbiano (see [\[\[BLR98\], page 785\]](#)) combines the ideas of the algorithms of Pottier and of Hosten and Sturmfels. The computations are performed on a graded ideal with one auxiliary variable  $u$  and one supplementary generator  $x_1 \cdot \dots \cdot x_n - u$  (instead of the generator  $t \cdot x_1 \cdot \dots \cdot x_n - 1$  in the algorithm of Pottier). The algorithm uses a quite unusual technique to get rid of the variable  $u$  again.

There is another algorithm of the authors which tries to parallelize the computations (but which is not implemented in this library).

### C.6.3 The Buchberger algorithm for toric ideals

Toric ideals have a very special structure that allows us to improve the Buchberger algorithm in many aspects: They are prime ideals and generated by binomials. Pottier used this fact to describe all operations of the Buchberger algorithm on the ideal generators in terms of vector additions and subtractions. Some other strategies like multiple reduction (see [\[\[CoTr91\], page 785\]](#)) or the use of bit vectors to represent the support of a monomial (see [\[\[Big97\], page 785\]](#)) may be applied to more general ideals, but show to be especially useful in the toric case.

### C.6.4 Integer programming

Let  $A$  be an  $m \times n$  matrix with integral coefficients,  $b \in \mathbb{Z}^m$  and  $c \in \mathbb{Z}^n$ . The problem

$$\min\{c^T x | x \in \mathbb{Z}^n, Ax = b, x \geq 0 \text{ component-wise}\}$$

is called an instance of the **integer programming problem** or **IP problem**.

The IP problem is very hard; namely, it is NP-complete.

For the following discussion let  $c \geq 0$  (component-wise). We consider  $c$  as a weight vector; because of its nonnegativity,  $c$  can be refined into a monomial ordering  $>_c$ . It turns out that we can solve such an IP instance with the help of toric ideals:

First we assume that an initial solution  $v$  (i.e.,  $v \in \mathbb{Z}^n, v \geq 0, Av = b$ ) is already known. We obtain the optimal solution  $v_0$  (i.e., with  $c^T v_0$  minimal) by the following procedure:

- (1) Compute the toric ideal  $I(A)$  using one of the algorithms in the previous section.
- (2) Compute the reduced Groebner basis  $G(c)$  of  $I(A)$  w.r.t.  $>_c$ .
- (3) Reduce  $x^v$  modulo  $G(c)$  using the Hironaka division algorithm. If the result of this reduction is  $x^w$ , then  $w$  is an optimal solution of the given instance.

If no initial solution is known, we are nevertheless able to solve the problem with similar techniques. For this purpose we replace our instance by an extended instance with the matrix used in the Conti-Traverso algorithm. Indeed, the Conti-Traverso algorithm offers the possibility to verify solvability of a given instance and to find an initial solution in the case of existence (but none of the other algorithms does!). Details can be found in see [\[\[CoTr91\]\], page 785](#) and see [\[\[The99\]\], page 785](#).

An implementation of the above algorithm and some examples can be found in [Section D.4.13 \[intprog\\_lib\], page 1119](#).

In general, classical methods for solving IP instances like Branch-and-Bound methods seem to be faster than the methods using toric ideals. But the latter have one great advantage: If one wants to solve various instances that differ only by the vector  $b$ , one has to perform steps (1) and (2) above only once. As the running time of step (3) is very short, solving all the instances is not much harder than solving one single instance.

For a detailed discussion see [\[\[The99\]\], page 785](#).

### C.6.5 Relevant References

- [Big97] Bigatti, A.M.: Computation of Hilbert-Poincare series. *Journal of Pure and Applied Algebra* (1997) 199, 237-253
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## C.7 Non-commutative algebra

See [Section 7.4 \[Mathematical background \(plural\)\], page 366](#), [Section 7.9 \[Mathematical background \(letterplace\)\], page 636](#).

## C.8 Decoding codes with Groebner bases

This section introduces some of the mathematical notions, definitions, and results for solving decoding problems and finding the minimum distance of linear (and in particular cyclic) codes. The material presented here should assist the user in working with [Section D.10.2 \[decodegb\\_lib\], page 1938](#). More details can be obtained from [\[\[BP2008b\]\], page 791](#).

## C.8.1 Codes and the decoding problem

### Codes

- Let  $F_q$  be a field with  $q$  elements. A *linear code*  $C$  is a linear subspace of  $F_q^n$  endowed with the **Hamming metric**.
- **Hamming distance** between  $\mathbf{x}, \mathbf{y} \in F_q^n$  :  $d(x, y) = \#\{i | x_i \neq y_i\}$ . **Hamming weight** of  $\mathbf{x} \in F_q^n$  :  $wt(x) = \#\{i | x_i \neq 0\}$ .
- **Minimum distance** of the code  $C$  :  $d(C) := \min_{\mathbf{x}, \mathbf{y} \in C, \mathbf{x} \neq \mathbf{y}} (d(\mathbf{x}, \mathbf{y}))$ .
- The code  $C$  of dimension  $k$  and minimum distance  $d$  is denoted as  $[n, k, d]$ .
- A matrix  $G$  whose rows are the base vectors of  $C$  is the **generator matrix**.
- A matrix  $H$  with the property  $\mathbf{c} \in C \iff H\mathbf{c}^T = 0$  is the **check matrix**.

### Cyclic codes

The code  $C$  is **cyclic**, if for every codeword  $\mathbf{c} = (c_0, \dots, c_{n-1})$  in  $C$  its cyclic shift  $(c_{n-1}, c_0, \dots, c_{n-2})$  is again a codeword in  $C$ . When working with cyclic codes, vectors are usually presented as polynomials. So  $\mathbf{c}$  is represented by the polynomial  $c(x) = \sum_{i=0}^{n-1} c_i x^i$  with  $x^n = 1$ , more precisely  $c(x)$  is an element of the factor ring  $F_q[X]/\langle X^n - 1 \rangle$ . Cyclic codes over  $F_q$  of length  $n$  correspond one-to-one to ideals in this factor ring. We assume for cyclic codes that  $(q, n) = 1$ . Let  $F = F_{q^m}$  be the splitting field of  $X^n - 1$  over  $F_q$ . Then  $F$  has a **primitive  $n$ -th root of unity** which will be denoted by  $a$ . A cyclic code is uniquely given by a **defining set**  $S_C$  which is a subset of  $\mathbb{Z}_n$  such that

$$c(x) \in C \text{ if } c(a^i) = 0 \text{ for all } i \in S_C.$$

A cyclic code has several defining sets.

### Decoding problem

- **Complete decoding**: Given  $y \in F_q^n$  and a code  $C \subseteq F_q^n$ , so that  $y$  is at distance  $d(y, C)$  from the code, find  $c \in C$  :  $d(y, c) = d(y, C)$ .
- **Bounded up to half the minimum distance**: With the additional assumption  $d(\mathbf{y}, C) \leq (d(C) - 1)/2$ , a codeword with the above property is unique.

### Decoding via systems solving

One distinguishes between two concepts:

- **Generic decoding**: Solve some system  $S(C)$  and obtain some "closed" formulas  $CF$ . Evaluating these formulas at data specific to a received word  $\mathbf{r}$  should yield a solution to the decoding problem. For example for  $f \in CF$  :  $f(\text{syndrome}(\mathbf{r}), x) = \text{poly}(x)$ . The roots of  $\text{poly}(x) = 0$  yield error positions, see the section on the general error-locator polynomial.
- **Online decoding**: Solve some system  $S(C, \mathbf{r})$ . The solutions should solve the decoding problem.

### Computational effort

- Generic decoding. Here, preprocessing is very hard, whereas decoding is relatively simple (if the formulas are sparse).
- Online decoding. In this case, decoding is the hard part.



## C.8.2 Cooper philosophy

### Computing syndromes in cyclic code case

Let  $C$  be an  $[n, k]$  cyclic code over  $F_q$ ;  $F$  is a splitting field with  $a$  being a primitive  $n$ -th root of unity. Let  $S_C = \{i_1, \dots, i_{n-k}\}$  be the complete defining set of  $C$ . Let  $\mathbf{r} = \mathbf{c} + \mathbf{e}$  be a received word with  $\mathbf{c} \in C$  and  $\mathbf{e}$  an error vector. Denote the corresponding polynomials in  $F_q[x]/\langle x^n - 1 \rangle$  by  $r(x)$ ,  $c(x)$  and  $e(x)$ , resp. Compute syndromes

$$s_{i_m} = r(a^{i_m}) = e(a^{i_m}) = \sum_{l=1}^t e_{j_l} (a^{i_m})^{j_l}, \quad 1 \leq m \leq n - k,$$

where  $t$  is the number of errors,  $j_1, \dots, j_t$  are the **error positions** and  $e_{j_1}, \dots, e_{j_t}$  are the **error values**. Define  $z_l = a^{j_l}$  and  $y_l = e_{j_l}$ . Then  $z_1, \dots, z_t$  are the error locations and  $y_1, \dots, y_t$  are the error values and the syndromes above become **generalized power sum functions**  $s_{i_m} = \sum_{l=1}^t y_l z_l^{i_m}$ ,  $1 \leq m \leq n - k$ .

### CRHT-ideal

Replace the concrete values above by variables and add some natural restrictions. Introduce

- $f_u := \sum_{l=1}^e Y_l Z_l^{i_u} - X_u = 0, 1 \leq u \leq n - k$ ;
- $\epsilon_j := X_j^{q^m} - X_j = 0, 1 \leq j \leq n - k$ , since  $s_j \in F$ ;
- $\eta_i := Z_i^{n+1} - Z_i = 0, 1 \leq i \leq e$ , since  $a^{j_i}$  are either  $n$ -th roots of unity or zero;
- $\lambda_i := Y_i^{q-1} - 1 = 0, 1 \leq i \leq e$ , since  $y_l \in F_q \setminus \{0\}$ .

We obtain the following set of polynomials in the variables  $X = (X_1, \dots, X_{n-k})$ ,  $Z = (Z_1, \dots, Z_e)$  and  $Y = (Y_1, \dots, Y_e)$ :

$$F_C = \{f_j, \epsilon_j, \eta_i, \lambda_i : 1 \leq j \leq n - k, 1 \leq i \leq e\} \subset F_q[X, Z, Y].$$

The zero-dimensional ideal  $I_C$  generated by  $F_C$  is the **CRHT-syndrome ideal** associated to the code  $C$ , and the variety  $V(F_C)$  defined by  $F_C$  is the **CRHT-syndrome variety**, after Chen, Reed, Helleseht and Truong.

### General error-locator polynomial

Adding some more polynomials to  $F_C$ , thus obtaining some  $F'_C$ , it is possible to prove the following **Theorem**:

Every cyclic code  $C$  possesses a **general error-locator polynomial**  $L_C$  from  $F_q[X_1, \dots, X_{n-k}, Z]$  that satisfies the following two properties:

- $L_C = Z^e + a_{t-1}Z^{e-1} + \dots + a_0$  with  $a_j \in F_q[X_1, \dots, X_{n-k}]$ ,  $0 \leq j \leq e - 1$ , where  $e$  is the error-correcting capacity;
- given a syndrome  $\mathbf{s} = (s_{i_1}, \dots, s_{i_{n-k}}) \in F^{n-k}$  corresponding to an error of weight  $t \leq e$  and error locations  $\{k_1, \dots, k_t\}$ , if we evaluate the  $X_u = s_{i_u}$  for all  $1 \leq u \leq n - k$ , then the roots of  $L_C(\mathbf{s}, Z)$  are exactly  $a^{k_1}, \dots, a^{k_t}$  and 0 of multiplicity  $e - t$ , in other words  $L_C(\mathbf{s}, Z) = Z^{e-t} \prod_{i=1}^t (Z - a^{k_i})$ .

The general error-locator polynomial actually is an element of the reduced Gröbner basis of  $\langle F'_C \rangle$ . Having this polynomial, decoding of the cyclic code  $C$  reduces to univariate factorization.

For an example see **sysCRHT** in [Section D.10.2 \[decodegb\\_lib\]](#), [page 1938](#). More on Cooper's philosophy and the general error-locator polynomial can be found in [\[\[OS2005\]\]](#), [page 791](#).

## Finding the minimum distance

The method described above can be adapted to find the minimum distance of a code. More concretely, the following holds:

Let  $C$  be the binary  $[n, k, d]$  cyclic code with the defining set  $S_C = \{i_1, \dots, i_v\}$ . Let  $1 \leq w \leq n$  and let  $J_C(w)$  denote the system:

$$\begin{aligned} Z_1^{i_1} + \dots + Z_w^{i_1} &= 0, \\ &\vdots \\ Z_1^{i_v} + \dots + Z_w^{i_v} &= 0, \\ Z_1^n - 1 &= 0, \\ &\vdots \\ Z_w^n - 1 &= 0, \\ p(n, Z_i, Z_j) &= 0, 1 \leq i < j \leq w. \end{aligned}$$

Then the number of solutions of  $J_C(w)$  is equal to  $w!$  times the number of codewords of weight  $w$ . And for  $1 \leq w \leq d$ , either  $J_C(w)$  has no solutions, which is equivalent to  $w < d$ , or  $J_C(w)$  has some solutions, which is equivalent to  $w = d$ .

For an example see `sysCRHTMindist` in [Section D.10.2 \[decodegb\\_lib\], page 1938](#). More on finding the minimum distance with Groebner bases can be found in [\[\[S2007\]\], page 791](#). See [\[\[OS2005\]\], page 791](#), for the definition of the polynomial  $p$  above.

### C.8.3 Generalized Newton identities

The **error-locator polynomial** is defined by

$$\sigma(Z) = \prod_{l=1}^t (Z - z_l).$$

If this product is expanded,

$$\sigma(Z) = Z^t + \sigma_1 Z^{t-1} + \dots + \sigma_{t-1} Z + \sigma_t,$$

then the coefficients  $\sigma_i$  are the **elementary symmetric functions** in the error locations  $z_1, \dots, z_t$

$$\sigma_i = (-1)^i \sum_{1 \leq j_1 < j_2 < \dots < j_i \leq t} z_{j_1} z_{j_2} \dots z_{j_i}, \quad 1 \leq i \leq t.$$

### Generalized Newton identities

The syndromes  $s_i = r(a^i) = e(a^i)$  and the coefficients  $\sigma_i$  satisfy the following **generalized Newton identities**:

$$s_i + \sum_{j=1}^t \sigma_j s_{i-j} = 0, \quad \text{for all } i \in \mathbb{Z}_n.$$

## Decoding up to error-correcting capacity

We have  $s_{i+n} = s_i$ , for all  $i \in \mathbb{Z}_n$ , since  $s_{i+n} = r(a^{i+n}) = r(a^i)$ . Furthermore

$$s_i^q = (e(a^i))^q = e(a^{iq}) = s_{qi}, \text{ for all } i \in \mathbb{Z}_n,$$

and  $\sigma_i^{q^m} = \sigma_i$ , for all  $1 \leq i \leq t$ . Replace the syndromes by variables and obtain the following set of polynomials  $Newton_t$  in the variables  $S_1, \dots, S_n$  and  $\sigma_1, \dots, \sigma_t$ :

$$\sigma_i^{q^m} - \sigma_i, \quad \forall 1 \leq i \leq t,$$

$$S_{i+n} - S_i, \quad \forall i \in \mathbb{Z}_n,$$

$$S_i^q - S_{qi}, \quad \forall i \in \mathbb{Z}_n,$$

$$S_i + \sum_{j=1}^t \sigma_j S_{i-j}, \quad \forall i \in \mathbb{Z}_n,$$

$$S_i - s_i(r) \quad \forall i \in S_C.$$

For an example see `sysNewton` in [Section D.10.2 \[decodegb.lib\], page 1938](#). More on this method and the method based on Waring function can be found in [\[\[ABF2002\]\], page 791](#). See also [\[\[ABF2008\]\], page 791](#).

## C.8.4 Fitzgerald-Lax method

### Affine codes

Let  $I = \langle g_1, \dots, g_m \rangle \subseteq F_q[X_1, \dots, X_s]$  be an ideal. Define

$$I_q := I + \langle X_1^q - X_1, \dots, X_s^q - X_s \rangle.$$

So  $I_q$  is a zero-dimensional ideal. Define also  $V(I_q) = \{P_1, \dots, P_n\}$ . Every  $q$ -ary linear code  $C$  with parameters  $[n, k]$  can be seen as an **affine variety code**  $C(I, L)$ , that is, the image of a vector space  $L$  of the **evaluation map**

$$\phi : R \rightarrow F_q^n$$

$$\bar{f} \mapsto (f(P_1), \dots, f(P_n)),$$

where  $R := F_q[U_1, \dots, U_s]/I_q$ ,  $L$  is a vector subspace of  $R$  and  $\bar{f}$  the coset of  $f$  in  $F_q[U_1, \dots, U_s]$  modulo  $I_q$ .

## Decoding affine variety codes

Given a  $q$ -ary  $[n, k]$  code  $C$  with a generator matrix  $G = (g_{ij})$ :

1. choose  $s$ , such that  $q^s \geq n$ , and construct  $s$  distinct points  $P_1, \dots, P_s$  in  $F_q^s$ .
2. Construct a Gröbner basis  $\{g_1, \dots, g_m\}$  for an ideal  $I$  of polynomials from  $F_q[X_1, \dots, X_s]$  that vanish at the points  $P_1, \dots, P_s$ . Define  $\xi_i \in F_q[X_1, \dots, X_s]$  such that  $\xi_i(P_i) = 1, \xi_i(P_j) = 0, i \neq j$ .
3. Then  $f_i = \sum_{j=1}^n g_{ij} \xi_j$  span the space  $L$ , so that  $g_{ij} = f_i(P_j)$ .

In this way we obtain that the code  $C$  is the image of the evaluation above, thus  $C = C(I, L)$ . In the same way by considering a parity check matrix instead of a generator matrix we have that the dual code is also an affine variety code.

The method of decoding is a generalization of CRHT. One needs to add polynomials  $(g_l(X_{k1}, \dots, X_{ks}))_{l=1, \dots, m; k=1, \dots, t}$  for every error position. We also assume that field equations on  $X_{ij}$ 's are included among the polynomials above. Let  $C$  be a  $q$ -ary  $[n, k]$  linear code such that its dual is written as an affine variety code of the form  $C^\perp = C(I, L)$ . Let  $\mathbf{r} = \mathbf{c} + \mathbf{e}$  as usual and  $t \leq e$ . Then the syndromes are computed by  $s_i = \sum_{j=1}^n r_j f_i(P_j) = \sum_{j=1}^n e_j f_i(P_j)$  for  $i = 1, \dots, n - k$ .

Consider the ring  $F_q[X_{11}, \dots, X_{1s}, \dots, X_{t1}, \dots, X_{ts}, E_1, \dots, E_t]$ , where  $(X_{i1}, \dots, X_{is})$  correspond to the  $i$ -th error position and  $E_i$  to the  $i$ -th error value. Consider the ideal  $Id_C$  generated by

$$\begin{aligned} \sum_{j=1}^t E_j f_i(X_{j1}, \dots, X_{js}) - s_i, 1 \leq i \leq n - k, \\ g_l(X_{j1}, \dots, X_{js}), 1 \leq l \leq m, \\ E_k^{q-1} - 1. \end{aligned}$$

**Theorem:** Let  $G$  be the reduced Gröbner basis for  $Id_C$  with respect to an elimination order  $X_{11} < \dots < X_{1s} < E_1$ . Then we may solve for the error locations and values by applying elimination theory to the polynomials in  $G$ .

For an example see `sysFL` in [Section D.10.2 \[decodegb\\_lib\]](#), [page 1938](#). More on this method can be found in [\[\[FL1998\]\]](#), [page 791](#).

### C.8.5 Decoding method based on quadratic equations

#### Preliminary definitions

Let  $\mathbf{b}_1, \dots, \mathbf{b}_n$  be a basis of  $F_q^n$  and let  $B$  be the  $n \times n$  matrix with  $\mathbf{b}_1, \dots, \mathbf{b}_n$  as rows. The **unknown syndrome**  $\mathbf{u}(B, \mathbf{e})$  of a word  $\mathbf{e}$  w.r.t  $B$  is the column vector  $\mathbf{u}(B, \mathbf{e}) = B\mathbf{e}^T$  with entries  $u_i(B, \mathbf{e}) = \mathbf{b}_i \cdot \mathbf{e}$  for  $i = 1, \dots, n$ .

For two vectors  $\mathbf{x}, \mathbf{y} \in F_q^n$  define  $\mathbf{x} * \mathbf{y} = (x_1 y_1, \dots, x_n y_n)$ . Then  $\mathbf{b}_i * \mathbf{b}_j$  is a linear combination of  $\mathbf{b}_1, \dots, \mathbf{b}_n$ , so there are constants  $\mu_l^{ij} \in F_q$  such that  $\mathbf{b}_i * \mathbf{b}_j = \sum_{l=1}^n \mu_l^{ij} \mathbf{b}_l$ . The elements  $\mu_l^{ij} \in F_q$  are the **structure constants** of the basis  $\mathbf{b}_1, \dots, \mathbf{b}_n$ .

Let  $B_s$  be the  $s \times n$  matrix with  $\mathbf{b}_1, \dots, \mathbf{b}_s$  as rows ( $B = B_n$ ). Then  $\mathbf{b}_1, \dots, \mathbf{b}_n$  is an **ordered MDS basis** and  $B$  an **MDS matrix** if all the  $s \times s$  submatrices of  $B_s$  have rank  $s$  for all  $s = 1, \dots, n$ .

#### Expressing known syndromes

Let  $C$  be an  $F_q$ -linear code with parameters  $[n, k, d]$ . W.l.o.g  $n \leq q$ .  $H$  is a check matrix of  $C$ . Let  $\mathbf{h}_1, \dots, \mathbf{h}_{n-k}$  be the rows of  $H$ . One can express  $\mathbf{h}_i = \sum_{j=1}^n a_{ij} \mathbf{b}_j$  with some  $a_{ij} \in F_q$ . In other words  $H = AB$  where  $A$  is the  $(n - k) \times n$  matrix with entries  $a_{ij}$ .

Let  $\mathbf{r} = \mathbf{c} + \mathbf{e}$  be a received word with  $\mathbf{c} \in C$  and  $\mathbf{e}$  an error vector. The syndromes of  $\mathbf{r}$  and  $\mathbf{e}$  w.r.t  $H$  are equal and known:

$$s_i(\mathbf{r}) := \mathbf{h}_i \cdot \mathbf{r} = \mathbf{h}_i \cdot \mathbf{e} = s_i(\mathbf{e}).$$

They can be expressed in the unknown syndromes of  $\mathbf{e}$  w.r.t  $B$ :

$$s_i(\mathbf{r}) = s_i(\mathbf{e}) = \sum_{j=1}^n a_{ij} u_j(\mathbf{e})$$

since  $\mathbf{h}_i = \sum_{j=1}^n a_{ij} \mathbf{b}_j$  and  $\mathbf{b}_j \cdot \mathbf{e} = u_j(\mathbf{e})$ .

## Constructing the system

Let  $B$  be an MDS matrix with structure constants  $\mu_l^{ij}$ . Define  $U_{ij}$  in the variables  $U_1, \dots, U_n$  by

$$U_{ij} = \sum_{l=1}^n \mu_l^{ij} U_l.$$

The ideal  $J(\mathbf{r})$  in  $F_q[U_1, \dots, U_n]$  is generated by

$$\sum_{l=1}^n a_{jl} U_l - s_j(\mathbf{r}) \text{ for } j = 1, \dots, n-k.$$

The ideal  $I(t, U, V)$  in  $F_q[U_1, \dots, U_n, V_1, \dots, V_t]$  is generated by

$$\sum_{j=1}^t U_{ij} V_j - U_{i,t+1} \text{ for } i = 1, \dots, n$$

Let  $J(t, \mathbf{r})$  be the ideal in  $F_q[U_1, \dots, U_n, V_1, \dots, V_t]$  generated by  $J(\mathbf{r})$  and  $I(t, U, V)$ .

## Main theorem

Let  $B$  be an MDS matrix with structure constants  $\mu_l^{ij}$ . Let  $H$  be a check matrix of the code  $C$  such that  $H = AB$  as above. Let  $\mathbf{r} = \mathbf{c} + \mathbf{e}$  be a received word with  $\mathbf{c} \in C$  the codeword sent and  $\mathbf{e}$  the error vector. Suppose that  $wt(\mathbf{e}) \neq 0$  and  $wt(\mathbf{e}) \leq \lfloor (d(C) - 1)/2 \rfloor$ . Let  $t$  be the smallest positive integer such that  $J(t, \mathbf{r})$  has a solution  $(\mathbf{u}, \mathbf{v})$  over the algebraic closure of  $F_q$ . Then

- $wt(\mathbf{e}) = t$  and the solution is unique and of multiplicity one satisfying  $\mathbf{u} = \mathbf{u}(\mathbf{e})$ .
- the reduced Gröbner basis  $G$  for the ideal  $J(t, \mathbf{r})$  w.r.t any monomial ordering is

$$\begin{aligned} U_i - u_i(\mathbf{e}), i = 1, \dots, n, \\ V_j - v_j, j = 1, \dots, t, \end{aligned}$$

where  $(\mathbf{u}(\mathbf{e}), \mathbf{v})$  is the unique solution.

For an example see `sysQE` in [Section D.10.2 \[decodegb\\_lib\]](#), [page 1938](#). More on this method can be found in [\[BP2008a\]](#), [page 791](#).

## C.8.6 References for decoding with Groebner bases

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## C.9 References

The Centre for Computer Algebra Kaiserslautern publishes a series of preprints which are electronically available at <https://www.singular.uni-kl.de/reports>. Other sources to check are <http://symbolicnet.org/>, <http://www-sop.inria.fr/galaad/>,... and the following list of books.

For references on non-commutative algebras and algorithms, see [Section 7.4.4 \[References \(plural\)\]](#), [page 370](#).

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## Appendix D SINGULAR libraries

SINGULAR comes with a set of standard libraries. Their content is described in the following subsections.

Use the LIB command (see [Section 5.1.79 \[LIB\], page 211](#)) for loading of single libraries, and the command LIB "all.lib"; for loading all libraries.

### Interpreter libraries:

See also [Section 7.5 \[PLURAL libraries\], page 371](#) and [Section 7.10 \[LETTERPLACE libraries\], page 640](#).

### D.1 standard.lib

The library `standard.lib` provides extensions to the set of built-in commands and is automatically loaded during the start of SINGULAR, unless SINGULAR is started up with the `--no-stdlib` command line option (see [Section 3.1.6 \[Command line options\], page 19](#)).

**Library:** `standard.lib`

**Purpose:** Procedures which are always loaded at Start-up

**Procedures:**

#### D.1.1 qslimgb

Procedure from library `standard.lib` (see [Section D.1 \[standard.lib\], page 794](#)).

**Usage:** `qslimgb(i);` i ideal or module

**Return:** same type as input, a standard basis of i computed with `slimgb`

**Note:** Only as long as `slimgb` does not know qgrings `qslimgb` should be used in case the basering is (possibly) a quotient ring.  
The quotient ideal is added to the input and `slimgb` is applied.

**Example:**

```
ring R = (0,v),(x,y,z,u),dp;
qring Q = std(x2-y3);
ideal i = x+y2,xy+yz+zu+u*v,xyz*u*v-1;
ideal j = qslimgb(i); j;
⇒ j[1]=y-1
⇒ j[2]=x+1
⇒ j[3]=(v)*z+(v2)*u+(-v-1)
⇒ j[4]=(-v2)*u2+(v+1)*u+1
module m = [x+y2,1,0], [1,1,x2+y2+xyz];
print(qslimgb(m));
⇒ y2+x,x2+xy,1,      0,      0,      -x,      -xy-xz-x,
⇒ 1,      y,      1,      y3-x2,0,      y2-1,      y2z-xy-x-z,
⇒ 0,      0,      xyz+x2+y2,0,      y3-x2,x2y2+x3z+x2y,x3z2-x3y-2*x3-xy2
```

#### D.1.2 par2varRing

Procedure from library `standard.lib` (see [Section D.1 \[standard.lib\], page 794](#)).

**Usage:** `par2varRing([l]);` l list of ideals/modules [default:l=empty list]



**Return:** list, say  $L$ , with  $L[1]$  a ring where the parameters of the basering have been converted to an additional last block of variables, all of weight 1, and ordering  $dp$ .  
 If a list  $l$  with  $l[i]$  an ideal/module is given, then  
 $l[i] + \text{minpoly} * \text{freemodule}(\text{nrows}(l[i]))$  is mapped to an ideal/module in  $L[1]$  with name  $Id[i]$ .  
 If the basering has no parameters then  $L[1]$  is the basering.

**Example:**

```

ring R = (0,x),(y,z,u,v),lp;
minpoly = x2+1;
ideal i = x3,x2+y+z+u+v,xyzuv-1; i;
⇒ i[1]=(-x)
⇒ i[2]=y+z+u+v-1
⇒ i[3]=(x)*yzuv-1
def P = par2varRing(i)[1]; P;
⇒ // coefficients: QQ
⇒ // number of vars : 5
⇒ //      block 1 : ordering lp
⇒ //      : names  y z u v
⇒ //      block 2 : ordering dp
⇒ //      : names  x
⇒ //      block 3 : ordering C
setring(P);
Id[1];
⇒ _[1]=-x
⇒ _[2]=y+z+u+v-1
⇒ _[3]=yzuvx-1
⇒ _[4]=x2+1
setring R;
module m = x3*[1,1,1], (xyzuv-1)*[1,0,1];
def Q = par2varRing(m)[1]; Q;
⇒ // coefficients: QQ
⇒ // number of vars : 5
⇒ //      block 1 : ordering lp
⇒ //      : names  y z u v
⇒ //      block 2 : ordering dp
⇒ //      : names  x
⇒ //      block 3 : ordering C
setring(Q);
print(Id[1]);
⇒ -x,yzuvx-1,x2+1,0, 0,
⇒ -x,0, 0, x2+1,0,
⇒ -x,yzuvx-1,0, 0, x2+1

```

## D.2 General purpose

### D.2.1 all.lib

The library `all.lib` provides a convenient way to load all libraries of the SINGULAR distribution.

**Example:**

```

option(loadLib);
LIB "all.lib";

```

```

⇒ // ** loaded all.lib (4.1.1.0,Jan_2018)
⇒ // ** loaded ratgb.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded qmatrix.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded purityfiltration.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded perron.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded nctools.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded ncpream.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded dmodloc.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded ncModslimgb.lib (4.1.3.0,Apr_2020)
⇒ // ** loaded resources.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded parallel.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded tasks.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded nchomolog.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded ncfactor.lib (4.3.1.3,Jan_2023)
⇒ // ** loaded ncdecomp.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded ncalg.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded ncall.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded involut.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded gkdim.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded freegb.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded Singular/.libs/./MOD/freealgebra.so
⇒ // ** loaded fpaprops.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded fpalgebras.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded fpadim.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded dmodideal.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded dmodvar.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded dmodapp.lib (4.1.3.0,Mar_2020)
⇒ // ** loaded dmod.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded central.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded bfun.lib (4.3.1.3,Feb_2023)
⇒ // ** loaded bimodules.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded zeroset.lib (4.3.2.3,Jun_2023)
⇒ // ** loaded weierstr.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded triang.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded toric.lib (4.3.1.3,Feb_2023)
⇒ // ** loaded teachstd.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded surfex.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded surfacesignature.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded surf_jupyter.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded surf.lib (4.3.1.2,Sep_2022)
⇒ // ** loaded stratify.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded stanleyreisner.lib (4.2.0.0,Dec_2020)
⇒ // ** loaded Singular/.libs/./MOD/cohomo.so
⇒ // ** loaded spectrum.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded spcurve.lib (4.3.1.3,Jan_2023)
⇒ // ** loaded solve.lib (4.3.1.3,Feb_2023)
⇒ // ** loaded sagbiNormaliz.lib (4.3.2.0,June)
⇒ // ** loaded general.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded normaliz.lib (4.3.2.14,Apr_2023)
⇒ // ** loaded signcond.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded sing4ti2.lib (4.2.1,Nov_2021)
⇒ // ** loaded sing.lib (4.4.0.0,Nov_2023)
⇒ // ** loaded sheafcoh.lib (4.3.2.9,Oct_2023)

```

```

⇒ // ** loaded sagbi.lib (4.3.1.2,Nov_2022)
⇒ // ** loaded rootsur.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded rootsmr.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded rinvar.lib (4.3.1.3,Feb_2023)
⇒ // ** loaded ringgb.lib (4.2.0.0,Dec_2020)
⇒ // ** loaded ring.lib (4.3.1.3,Feb_2023)
⇒ // ** loaded reszeta.lib (4.3.1.3,Feb_2023)
⇒ // ** loaded resolve.lib (4.3.1.3,Feb_2023)
⇒ // ** loaded resjung.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded resgraph.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded resbinomial.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded reesclos.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded redcgs.lib (4.2.0.0,Dec_2020)
⇒ // ** loaded realrad.lib (4.4.0.0,Nov_2023)
⇒ // ** loaded realclassify.lib (4.2.1.0,Jul_2021)
⇒ // ** loaded rootisolation.lib (4.2.1.0,Jul_2021)
⇒ // ** loaded Singular/.libs/./MOD/interval.so
⇒ // ** loaded classify2.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded gfa.lib.so
⇒ // ** loaded polyclass.lib (4.2.0.0,Dec_2020)
⇒ // ** loaded random.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded qhmoduli.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded primitiv.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded primdecint.lib (4.3.2.2,Nov_2022)
⇒ // ** loaded primdec.lib (4.4.0.0,Dec_2023)
⇒ // ** loaded modfinduni.lib (4.4.0.0,Dec_2023)
⇒ // ** loaded presolve.lib (4.4.0.0,Nov_2023)
⇒ // ** loaded polylib.lib (4.4.0.0,Nov_2023)
⇒ // ** loaded pointid.lib (4.4.0.0,Nov_2023)
⇒ // ** loaded phindex.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded pfd.lib (4.1.3.2,Aug_2020)
⇒ // ** loaded paraplanecurves.lib (4.4.0.0,Nov_2023)
⇒ // ** loaded ntsolve.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded normal.lib (4.3.2.10,Nov_2023)
⇒ // ** loaded curveInv.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded noether.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded nfmodsyzy.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded nfmodstd.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded mregular.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded mprimdec.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded monomialideal.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded mondromy.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded modstd.lib (4.3.0.1,Mar_2022)
⇒ // ** loaded modular.lib (4.2.0.0,Dec_2020)
⇒ // ** loaded moddiq.lib (4.3.2.2,Jun_2023)
⇒ // ** loaded matrix.lib (4.3.2.3,Aug_2023)
⇒ // ** loaded makedbm.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded linalg.lib (4.3.1.3,Feb_2023)
⇒ // ** loaded latex.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded kskernel.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded jacobson.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded intprog.lib (4.2.1.3,Dec_2021)
⇒ // ** loaded inout.lib (4.1.2.0,Feb_2019)

```

```

⇒ // ** loaded integralbasis.lib (4.3.2.10,Nov_2023)
⇒ // ** loaded puiseuxexpansions.lib (4.3.1.0,Jul_2022)
⇒ // ** loaded hyperel.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded homolog.lib (4.3.1.3,Feb_2023)
⇒ // ** loaded hnoether.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded grwalk.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded groups.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded grobcov.lib (4.2.0,February_2021)
⇒ // ** loaded graphics.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded gmssing.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded gmspoly.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded finvar.lib (4.3.1.3,Feb_2023)
⇒ // ** loaded equising.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded elim.lib (4.3.2.5,Jul_2023)
⇒ // ** loaded deform.lib (4.3.1.3,Jan_2023)
⇒ // ** loaded decodegb.lib (4.3.1.3,Feb_2023)
⇒ // ** loaded curvepar.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded crypto.lib (4.2.1.0,Jul_2021)
⇒ // ** loaded control.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded compregb.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded classify.lib (4.4.0.0,Nov_2023)
⇒ // ** loaded cisimplicial.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded brnoeth.lib (4.3.1.3,Jan_2023)
⇒ // ** loaded assprimeszerodim.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded arcpoint.lib (4.3.1.3,Jan_2023)
⇒ // ** loaded algebra.lib (4.3.1.3,Feb_2023)
⇒ // ** loaded alexpoly.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded aksaka.lib (4.1.2.0,Feb_2019)
⇒ // ** loaded ainvar.lib (4.3.1.3,Feb_2023)
⇒ // ** loaded absfact.lib (4.1.2.0,Feb_2019)

```

## D.2.2 compregb.lib

**Library:** compregb.lib

**Purpose:** experimental implementation for comprehensive Groebner systems

**Author:** Akira Suzuki (<http://kurt.scitec.kobe-u.ac.jp/~sakira/CGBusingGB/>) (<sakira@kobe-u.ac.jp>)

**Overview:** see "A Simple Algorithm to compute Comprehensive Groebner Bases using Groebner Bases" by Akira Suzuki and Yosuke Sato for details.

**Procedures:**

### D.2.2.1 cgs

Procedure from library compregb.lib (see [Section D.2.2 \[compregb.lib\]](#), page 798).

**Usage:** cgs(Polys,Vars,Paras,RingVar,RingAll); Polys an ideal, Vars, the list of variables, Paras the list of parameters, RingVar the ring with Paras as parameters, RingAll the ring with Paras as variables (RingAll should be the current ring)

**Return:** a list L of lists L[i] of a polynomial and an ideal:  
 L[i][1] the polynomial giving the condition on the parameters L[i][2] the Groebner basis for this case

**Example:**

```

LIB "compregb.lib";
ring RingVar=(0,a,b),(x,y,t),lp;
ring RingAll=0,(x,y,t,a,b),(lp(3),dp);
ideal polys=x^3-a,y^4-b,x+y-t;
list vars=x,y,t;
list paras=a,b;
list G = cgs(polys,vars,paras, RingVar, RingAll);
G;
↳ [1]:
↳   [1]:
↳     1
↳   [2]:
↳     _[1]=b
↳     _[2]=a
↳     _[3]=t6
↳     _[4]=5yt4-3t5
↳     _[5]=6y2t2-8yt3+3t4
↳     _[6]=y3-3y2t+3yt2-t3
↳     _[7]=x+y-t
↳ [2]:
↳   [1]:
↳     a
↳   [2]:
↳     _[1]=b
↳     _[2]=a4
↳     _[3]=t6a3
↳     _[4]=5t8a2-28t5a3
↳     _[5]=14t10a-60t7a2+105t4a3
↳     _[6]=t12-4t9a+6t6a2-4t3a3
↳     _[7]=81ya3-14t10+60t7a-105t4a2+59ta3
↳     _[8]=81yt2a2+4t9-21t6a+3t3a2+14a3
↳     _[9]=21yt3a+6ya2-t7-7t4a+8ta2
↳     _[10]=12yt5+15yt2a-7t6+5t3a+2a2
↳     _[11]=3y2a+5yt4+4yta-3t5+3t2a
↳     _[12]=6y2t2-8yt3-ya+3t4-3ta
↳     _[13]=y3-3y2t+3yt2-t3+a
↳     _[14]=x+y-t
↳ [3]:
↳   [1]:
↳     1
↳   [2]:
↳     _[1]=b
↳     _[2]=a
↳     _[3]=t6
↳     _[4]=5yt4-3t5
↳     _[5]=6y2t2-8yt3+3t4
↳     _[6]=y3-3y2t+3yt2-t3
↳     _[7]=x+y-t
↳ [4]:
↳   [1]:
↳     b
↳   [2]:

```

```

⇒      _[1]=a
⇒      _[2]=b3
⇒      _[3]=t6b2
⇒      _[4]=5t9b-18t5b2
⇒      _[5]=t12-3t8b+3t4b2
⇒      _[6]=32yb2-5t9+18t5b-45tb2
⇒      _[7]=32yt3b+3t8-30t4b-5b2
⇒      _[8]=5yt4+3yb-3t5-5tb
⇒      _[9]=10y2b-24ytb-t6+15t2b
⇒      _[10]=6y2t2-8yt3+3t4-b
⇒      _[11]=y3-3y2t+3yt2-t3
⇒      _[12]=x+y-t
⇒ [5]:
⇒      [1]:
⇒      ab
⇒      [2]:
⇒      _[1]=729a4-4096b3
⇒      _[2]=41472t11b2-6561t10a3+5832t9a2b-171072t8ab2+27648t7b3-4374t6a3b\
+252720t5a2b2-2215296t4ab3+2093568t3b4-497097t2a3b2-802296ta2b3+215488ab4
⇒      _[3]=46656t11ab-41472t10b2+6561t9a3-192456t8a2b+31104t7ab2-27648t6b\
3+284310t5a3b-2492208t4a2b2+2355264t3ab3-3142144t2b4-902583ta3b2+242424a2\
b3
⇒      _[4]=52488t11a2-46656t10ab+41472t9b2-216513t8a3+34992t7a2b-31104t6a\
b2+1797120t5b3-2803734t4a3b+2649672t3a2b2-3534912t2ab3-5705216tb4+272727a\
3b2
⇒      _[5]=729t12-2916t9a-2187t8b+4374t6a2-34992t5ab+2187t4b2-2916t3a3-21\
870t2a2b-8748tab2+3367b3
⇒      _[6]=3594240ytb3+568620ya3b-99144t11a-1728t10b+426465t8a2+327888t7a\
b+17280t6b2-752328t5a3+4509270t4a2b-366984t3ab2+2206528t2b3+1791180ta3b+6\
59529a2b2
⇒      _[7]=1137240yta2b2+1010880yab3-28431t10a2+24786t9ab+31104t8b2+12465\
9t7a3-13122t6a2b-263412t5ab2-1398528t4b3+1467477t3a3b-1414503t2a2b2-22543\
8tab3+2088320b4
⇒      _[8]=1705860yta3b+1516320ya2b2-269568t11b-729t9a2+1158624t8ab+87091\
2t7b2+8748t6a3-2037798t5a2b+12301632t4ab2-1240320t3b3+1109376t2a3b+487676\
7ta2b2+1731808ab3
⇒      _[9]=12130560yt2ab2-1705860ya3b-425736t11a+642816t10b+1782405t8a2-1\
403568t7ab-2612736t6b2-2956824t5a3+24555150t4a2b-35184456t3ab2+19255040t2\
b3+6714252ta3b-4160403a2b2
⇒      _[10]=3411720yt2a2b-1516320ytab2-4043520yb3+112266t10a+61560t9b-481\
140t7a2-788292t6ab-221616t5b2+841995t4a3-5807700t3a2b-762534t2ab2-2104264\
tb3-1043523a3b
⇒      _[11]=171072yt3b2+413343yt2a3+393660yta2b+44712yab2+20412t9a+16038t\
8b-107163t6a2-163296t5ab-160380t4b2+15309t3a3-817209t2a2b-329508tab2+3746\
78b3
⇒      _[12]=552yt3ab-448yt2b2-405yta3-228ya2b+70t11-300t8a-252t7b+525t5a2\
-3384t4ab+630t3b2-295t2a3-1089ta2b-228ab2
⇒      _[13]=2052yt3a2-648yt2ab-320ytb2+297ya3+50t10-312t7a-180t6b-309t4a2\
-1440t3ab+450t2b2+571ta3+297a2b
⇒      _[14]=66yt4b+81yt2a2+96ytab+14yb2+4t9-21t6a-54t5b+3t3a2-135t2ab-30t\
b2+14a3
⇒      _[15]=63yt4a-32yt3b+18yta2+5yab-3t8-21t5a+30t4b+24t2a2+33tab+5b2
⇒      _[16]=10yt6+16yt3a+6yt2b+ya2-6t7+3t4a-10t3b+3ta2+ab

```

```

⇒      _[17]=2y2b-12yt5-15yt2a-12ytb+7t6-5t3a+15t2b-2a2
⇒      _[18]=3y2a+5yt4+4yta+3yb-3t5+3t2a-5tb
⇒      _[19]=6y2t2-8yt3-ya+3t4-3ta-b
⇒      _[20]=y3-3y2t+3yt2-t3+a
⇒      _[21]=x+y-t
⇒ [6] :
⇒      [1] :
⇒      1
⇒      [2] :
⇒      _[1]=b
⇒      _[2]=a
⇒      _[3]=t6
⇒      _[4]=5yt4-3t5
⇒      _[5]=6y2t2-8yt3+3t4
⇒      _[6]=y3-3y2t+3yt2-t3
⇒      _[7]=x+y-t
⇒ [7] :
⇒      [1] :
⇒      a
⇒      [2] :
⇒      _[1]=b
⇒      _[2]=a4
⇒      _[3]=t6a3
⇒      _[4]=5t8a2-28t5a3
⇒      _[5]=14t10a-60t7a2+105t4a3
⇒      _[6]=t12-4t9a+6t6a2-4t3a3
⇒      _[7]=81ya3-14t10+60t7a-105t4a2+59ta3
⇒      _[8]=81yt2a2+4t9-21t6a+3t3a2+14a3
⇒      _[9]=21yt3a+6ya2-t7-7t4a+8ta2
⇒      _[10]=12yt5+15yt2a-7t6+5t3a+2a2
⇒      _[11]=3y2a+5yt4+4yta-3t5+3t2a
⇒      _[12]=6y2t2-8yt3-ya+3t4-3ta
⇒      _[13]=y3-3y2t+3yt2-t3+a
⇒      _[14]=x+y-t
⇒ [8] :
⇒      [1] :
⇒      1
⇒      [2] :
⇒      _[1]=b
⇒      _[2]=a
⇒      _[3]=t6
⇒      _[4]=5yt4-3t5
⇒      _[5]=6y2t2-8yt3+3t4
⇒      _[6]=y3-3y2t+3yt2-t3
⇒      _[7]=x+y-t
⇒ [9] :
⇒      [1] :
⇒      b
⇒      [2] :
⇒      _[1]=a
⇒      _[2]=b3
⇒      _[3]=t6b2
⇒      _[4]=5t9b-18t5b2

```

```

⇒      _[5]=t12-3t8b+3t4b2
⇒      _[6]=32yb2-5t9+18t5b-45tb2
⇒      _[7]=32yt3b+3t8-30t4b-5b2
⇒      _[8]=5yt4+3yb-3t5-5tb
⇒      _[9]=10y2b-24ytb-t6+15t2b
⇒      _[10]=6y2t2-8yt3+3t4-b
⇒      _[11]=y3-3y2t+3yt2-t3
⇒      _[12]=x+y-t
⇒ [10]:
⇒      [1]:
⇒      ab
⇒      [2]:
⇒      _[1]=729a4+64b3
⇒      _[2]=432t10b2-2187t9a3-1458t8a2b-2592t7ab2-2592t6b3+16038t5a3b+1263\
6t4a2b2-13536t3ab3-3472t2b4+31077ta3b2+4758a2b3
⇒      _[3]=5832t10ab+2592t9b2-19683t8a3-34992t7a2b-34992t6ab2-19008t5b3+1\
70586t4a3b-182736t3a2b2-46872t2ab3-36832tb4+64233a3b2
⇒      _[4]=6561t10a2+2916t9ab+1944t8b2-39366t7a3-39366t6a2b-21384t5ab2-16\
848t4b3-205578t3a3b-52731t2a2b2-41436tab3-6344b4
⇒      _[5]=648t11b-729t9a2-2916t8ab-2160t7b2+4374t6a3+8262t5a2b-28728t4ab\
2+3816t3b3+20250t2a3b-13581ta2b2-3172ab3
⇒      _[6]=2916t11a-648t10b-10935t8a2-5832t7ab+2160t6b2+13122t5a3-148230t\
4a2b+37476t3ab2-2792t2b3-107730ta3b-21411a2b2
⇒      _[7]=729t12-2916t9a-2187t8b+4374t6a2-34992t5ab+2187t4b2-2916t3a3-21\
870t2a2b-8748tab2-793b3
⇒      _[8]=112320yt2b3+568620yta3b+126360ya2b2-4374t9a2+12474t8ab+6336t7b\
2+43011t6a3-54108t5a2b-80388t4ab2-75392t3b3-52407t2a3b-489222ta2b2-75062a\
b3
⇒      _[9]=505440yt2ab2-224640ytb3+568620ya3b-3888t10b+69255t8a2+51840t7a\
b+6336t6b2-387828t5a3-475470t4a2b-217440t3ab2+51952t2b3-2481192ta3b-38060\
1a2b2
⇒      _[10]=3411720yt2a2b-1516320ytab2-336960yb3+112266t10a+61560t9b-4811\
40t7a2-788292t6ab-221616t5b2+841995t4a3-5807700t3a2b-762534t2ab2+595576tb\
3-1043523a3b
⇒      _[11]=171072yt3b2+413343yt2a3+393660yta2b+44712yab2+20412t9a+16038t\
8b-107163t6a2-163296t5ab-160380t4b2+15309t3a3-817209t2a2b-329508tab2-3300\
2b3
⇒      _[12]=552yt3ab-448yt2b2-405yta3-228ya2b+70t11-300t8a-252t7b+525t5a2\
-3384t4ab+630t3b2-295t2a3-1089ta2b-228ab2
⇒      _[13]=2052yt3a2-648yt2ab-320ytb2+297ya3+50t10-312t7a-180t6b-309t4a2\
-1440t3ab+450t2b2+571ta3+297a2b
⇒      _[14]=66yt4b+81yt2a2+96ytab+14yb2+4t9-21t6a-54t5b+3t3a2-135t2ab-30t\
b2+14a3
⇒      _[15]=63yt4a-32yt3b+18yta2+5yab-3t8-21t5a+30t4b+24t2a2+33tab+5b2
⇒      _[16]=10yt6+16yt3a+6yt2b+ya2-6t7+3t4a-10t3b+3ta2+ab
⇒      _[17]=2y2b-12yt5-15yt2a-12ytb+7t6-5t3a+15t2b-2a2
⇒      _[18]=3y2a+5yt4+4yta+3yb-3t5+3t2a-5tb
⇒      _[19]=6y2t2-8yt3-ya+3t4-3ta-b
⇒      _[20]=y3-3y2t+3yt2-t3+a
⇒      _[21]=x+y-t
⇒ [11]:
⇒      [1]:
⇒      1

```



```

⇒ [2]:
⇒   _[1]=b
⇒   _[2]=a
⇒   _[3]=t6
⇒   _[4]=5yt4-3t5
⇒   _[5]=6y2t2-8yt3+3t4
⇒   _[6]=y3-3y2t+3yt2-t3
⇒   _[7]=x+y-t
⇒ [12]:
⇒   [1]:
⇒   a
⇒   [2]:
⇒   _[1]=b
⇒   _[2]=a4
⇒   _[3]=t6a3
⇒   _[4]=5t8a2-28t5a3
⇒   _[5]=14t10a-60t7a2+105t4a3
⇒   _[6]=t12-4t9a+6t6a2-4t3a3
⇒   _[7]=81ya3-14t10+60t7a-105t4a2+59ta3
⇒   _[8]=81yt2a2+4t9-21t6a+3t3a2+14a3
⇒   _[9]=21yt3a+6ya2-t7-7t4a+8ta2
⇒   _[10]=12yt5+15yt2a-7t6+5t3a+2a2
⇒   _[11]=3y2a+5yt4+4yta-3t5+3t2a
⇒   _[12]=6y2t2-8yt3-ya+3t4-3ta
⇒   _[13]=y3-3y2t+3yt2-t3+a
⇒   _[14]=x+y-t
⇒ [13]:
⇒   [1]:
⇒   1
⇒   [2]:
⇒   _[1]=b
⇒   _[2]=a
⇒   _[3]=t6
⇒   _[4]=5yt4-3t5
⇒   _[5]=6y2t2-8yt3+3t4
⇒   _[6]=y3-3y2t+3yt2-t3
⇒   _[7]=x+y-t
⇒ [14]:
⇒   [1]:
⇒   b
⇒   [2]:
⇒   _[1]=a
⇒   _[2]=b3
⇒   _[3]=t6b2
⇒   _[4]=5t9b-18t5b2
⇒   _[5]=t12-3t8b+3t4b2
⇒   _[6]=32yb2-5t9+18t5b-45tb2
⇒   _[7]=32yt3b+3t8-30t4b-5b2
⇒   _[8]=5yt4+3yb-3t5-5tb
⇒   _[9]=10y2b-24ytb-t6+15t2b
⇒   _[10]=6y2t2-8yt3+3t4-b
⇒   _[11]=y3-3y2t+3yt2-t3
⇒   _[12]=x+y-t

```

```

⇒ [15]:
⇒   [1]:
⇒   ab
⇒   [2]:
⇒   _[1]=16767a4+5632b3
⇒   _[2]=16767t12-67068t9a-50301t8b+100602t6a2-804816t5ab+50301t4b2-670\
68t3a3-503010t2a2b-201204tab2-22399b3
⇒   _[3]=32348160yb4+27766152t11a2-2146176t10ab+476928t9b2-114535377t8a\
3-78067152t7a2b+2861568t6ab2-63272448t5b3-1314163926t4a3b+210548808t3a2b2\
+27688320t2ab3+228423424tb4-183555801a3b2
⇒   _[4]=2274480ya2b3-655776t11b2-150903t10a3+33534t9a2b+2705076t8ab2+1\
843776t7b3+201204t6a3b-4448844t5a2b2+31037688t4ab3-4972704t3b4+1946835t2a\
3b2+16061022ta2b3+4335188ab4
⇒   _[5]=10235160ya3b2-2950992t11ab+228096t10b2+150903t9a3+12172842t8a2\
b+8296992t7ab2-304128t6b3-20019798t5a3b+139669596t4a2b2-22377168t3ab3-294\
2720t2b4+72274599ta3b2+19508346a2b3
⇒   _[6]=1797120ytb3+13078260ya3b-3889944t11a+317952t10b+15844815t8a2+1\
0760688t7ab-794880t6b2-24546888t5a3+185536170t4a2b-29127384t3ab2-6614272t\
2b3+100664100ta3b+26988039a2b2
⇒   _[7]=26156520yta2b2+2021760yab3+251505t10a2+972486t9ab+983664t8b2-2\
565351t7a3-5734314t6a2b-9009468t5ab2-6972768t4b3+5382207t3a3b-39810447t2a\
2b2-26365962tab3-6221072b4
⇒   _[8]=1705860yta3b+379080ya2b2-71280t11b+67068t9a2+358182t8ab+256608\
t7b2-352107t6a3-1071144t5a2b+2918916t4ab2-658416t3b3-2384721t2a3b+26244ta\
2b2+65494ab3
⇒   _[9]=6065280yt2ab2+42646500ya3b-12206376t11a+1168992t10b+50049495t8\
a2+33199632t7ab-3250368t6b2-78871968t5a3+581360490t4a2b-102330216t3ab2-17\
630624t2b3+303270156ta3b+80747199a2b2
⇒   _[10]=78469560yt2a2b-34875360ytab2-4043520yb3+2582118t10a+1415880t9\
b-11066220t7a2-18130716t6ab-5097168t5b2+19365885t4a3-133577100t3a2b-17538\
282t2ab2+16398088tb3-24001029a3b
⇒   _[11]=3934656yt3b2+9506889yt2a3+9054180yta2b+1028376yab2+469476t9a+\
368874t8b-2464749t6a2-3755808t5ab-3688740t4b2+352107t3a3-18795807t2a2b-75\
78684tab2-1166726b3
⇒   _[12]=552yt3ab-448yt2b2-405yta3-228ya2b+70t11-300t8a-252t7b+525t5a2\
-3384t4ab+630t3b2-295t2a3-1089ta2b-228ab2
⇒   _[13]=2052yt3a2-648yt2ab-320ybt2+297ya3+50t10-312t7a-180t6b-309t4a2\
-1440t3ab+450t2b2+571ta3+297a2b
⇒   _[14]=66yt4b+81yt2a2+96ytab+14yb2+4t9-21t6a-54t5b+3t3a2-135t2ab-30t\
b2+14a3
⇒   _[15]=63yt4a-32yt3b+18yta2+5yab-3t8-21t5a+30t4b+24t2a2+33tab+5b2
⇒   _[16]=10yt6+16yt3a+6yt2b+ya2-6t7+3t4a-10t3b+3ta2+ab
⇒   _[17]=2y2b-12yt5-15yt2a-12ybt+7t6-5t3a+15t2b-2a2
⇒   _[18]=3y2a+5yt4+4yta+3yb-3t5+3t2a-5tb
⇒   _[19]=6y2t2-8yt3-ya+3t4-3ta-b
⇒   _[20]=y3-3y2t+3yt2-t3+a
⇒   _[21]=x+y-t
⇒ [16]:
⇒   [1]:
⇒   1
⇒   [2]:
⇒   _[1]=b
⇒   _[2]=a

```

```

⇒      _[3]=t6
⇒      _[4]=5yt4-3t5
⇒      _[5]=6y2t2-8yt3+3t4
⇒      _[6]=y3-3y2t+3yt2-t3
⇒      _[7]=x+y-t
⇒ [17]:
⇒      [1]:
⇒      a
⇒      [2]:
⇒      _[1]=b
⇒      _[2]=t12-4t9a+6t6a2-4t3a3+a4
⇒      _[3]=81ya3-14t10+60t7a-105t4a2+59ta3
⇒      _[4]=81yt2a2+4t9-21t6a+3t3a2+14a3
⇒      _[5]=21yt3a+6ya2-t7-7t4a+8ta2
⇒      _[6]=12yt5+15yt2a-7t6+5t3a+2a2
⇒      _[7]=3y2a+5yt4+4yta-3t5+3t2a
⇒      _[8]=6y2t2-8yt3-ya+3t4-3ta
⇒      _[9]=y3-3y2t+3yt2-t3+a
⇒      _[10]=x+y-t
⇒ [18]:
⇒      [1]:
⇒      1
⇒      [2]:
⇒      _[1]=b
⇒      _[2]=a
⇒      _[3]=t6
⇒      _[4]=5yt4-3t5
⇒      _[5]=6y2t2-8yt3+3t4
⇒      _[6]=y3-3y2t+3yt2-t3
⇒      _[7]=x+y-t
⇒ [19]:
⇒      [1]:
⇒      b
⇒      [2]:
⇒      _[1]=a
⇒      _[2]=t12-3t8b+3t4b2-b3
⇒      _[3]=32yb2-5t9+18t5b-45tb2
⇒      _[4]=32yt3b+3t8-30t4b-5b2
⇒      _[5]=5yt4+3yb-3t5-5tb
⇒      _[6]=10y2b-24ytb-t6+15t2b
⇒      _[7]=6y2t2-8yt3+3t4-b
⇒      _[8]=y3-3y2t+3yt2-t3
⇒      _[9]=x+y-t
⇒ [20]:
⇒      [1]:
⇒      -8910671247a13b+46290636864a9b4+20949663744a5b7+1476395008ab10
⇒      [2]:
⇒      _[1]=t12-4t9a-3t8b+6t6a2-48t5ab+3t4b2-4t3a3-30t2a2b-12tab2+a4-b3
⇒      _[2]=531441ya8-2939328ya4b3-262144yb6+673920t11a2b2-91854t10a5-8294\
4t10ab3+87480t9a4b+40960t9b4-2779920t8a3b2+393660t7a6-1762560t7a2b3-78732\
t6a5b+43008t6ab4+4132944t5a4b2-147456t5b5-688905t4a7-32127840t4a3b3+48551\
40t3a6b+6741120t3a2b4-7735014t2a5b2-1926144t2ab5+387099ta8-15277896ta4b3+\
368640tb6+1336257a7b-4006288a3b4

```

```

⇒      _[3]=6561yta5+576ytab3+5832ya4b+512yb4-1134t11a2+72t10ab-80t9b2+486\
0t8a3+3348t7a2b+240t6ab2-8505t5a4+288t5b3+52380t4a3b-8934t3a2b2+4779t2a5+\
2952t2ab3+20745ta4b-720tb4+6344a3b2
⇒      _[4]=373248yta4b2+32768ytb5+59049ya7+5184ya3b3+10368t11ab2-10206t10\
a4-5120t10b3+9720t9a3b-32400t8a2b2+43740t7a5-5376t7ab3-8748t6a4b+18432t6b\
4-24624t5a3b2-76545t4a6-589920t4a2b3+539460t3a5b+240768t3ab4-587574t2a4b2\
-46080t2b5+43011ta7-517384ta3b3+148473a6b-84240a2b4
⇒      _[5]=9360yt2ab2+13851yta4-2944ytb3+10530ya3b-2394t11a+460t10b+10260\
t8a2+5748t7ab-1656t6b2-17955t5a3+112890t4a2b-34794t3ab2+10089t2a4+4140t2b\
3+42497ta3b+10530a2b2
⇒      _[6]=42120yt2a2b-18720ytab2-8019ya4-4864yb3+1386t10a+760t9b-5940t7a\
2-9732t6ab-2736t5b2+10395t4a3-71700t3a2b-9414t2ab2-5841ta4+6840tb3-12883a\
3b
⇒      _[7]=266240yt2b4+1347840yta3b2+150903ya6+312768ya2b3-41600t11b2-260\
82t10a3+24840t9a2b+209040t8ab2+111780t7a4+149760t7b3-22356t6a3b-573648t5a\
2b2-195615t4a5+1703520t4ab3+1378620t3a4b-374400t3b4-1214602t2a3b2+109917t\
a6-113400ta2b3+379431a5b+84240ab4
⇒      _[8]=16767yt2a4+5632yt2b3+21060yta3b+4680ya2b2-880t11b+828t9a2+4422\
t8ab+3168t7b2-4347t6a3-13224t5a2b+36036t4ab2+621t3a4-7920t3b3-29441t2a3b+\
324ta2b2+2898a5+1782ab3
⇒      _[9]=704yt3b2+1701yt2a3+1620yta2b+184yab2+84t9a+66t8b-441t6a2-672t5\
ab-660t4b2+63t3a3-3363t2a2b-1356tab2+294a4-110b3
⇒      _[10]=552yt3ab-448yt2b2-405yta3-228ya2b+70t11-300t8a-252t7b+525t5a2\
-3384t4ab+630t3b2-295t2a3-1089ta2b-228ab2
⇒      _[11]=2052yt3a2-648yt2ab-320ytb2+297ya3+50t10-312t7a-180t6b-309t4a2\
-1440t3ab+450t2b2+571ta3+297a2b
⇒      _[12]=66yt4b+81yt2a2+96ytab+14yb2+4t9-21t6a-54t5b+3t3a2-135t2ab-30t\
b2+14a3
⇒      _[13]=63yt4a-32yt3b+18yta2+5yab-3t8-21t5a+30t4b+24t2a2+33tab+5b2
⇒      _[14]=10yt6+16yt3a+6yt2b+ya2-6t7+3t4a-10t3b+3ta2+ab
⇒      _[15]=2y2b-12yt5-15yt2a-12ytb+7t6-5t3a+15t2b-2a2
⇒      _[16]=3y2a+5yt4+4yta+3yb-3t5+3t2a-5tb
⇒      _[17]=6y2t2-8yt3-ya+3t4-3ta-b
⇒      _[18]=y3-3y2t+3yt2-t3+a
⇒      _[19]=x+y-t

```

### D.2.2.2 base2str

Procedure from library `compregb.lib` (see [Section D.2.2 \[compregb.lib\]](#), page 798).

### D.2.3 general.lib

**Library:** `general.lib`

**Purpose:** Elementary Computations of General Type

**Procedures:**

#### D.2.3.1 A\_Z

Procedure from library `general.lib` (see [Section D.2.3 \[general.lib\]](#), page 806).

**Usage:** `A_Z("a",n);` a any letter, n integer ( $-26 \leq n \leq 26$ ,  $! = 0$ )

**Return:** string of n small (if a is small) or capital (if a is capital) letters, comma separated, beginning with a, in alphabetical order (or reverse alphabetical order if  $n < 0$ )

**Example:**

```

LIB "general.lib";
A_Z("c",5);
⇒ c,d,e,f,g
A_Z("Z",-5);
⇒ Z,Y,X,W,V
ring R = create_ring("(0,"+A_Z("A",6)+")", "("+A_Z("a",10)+")", "dp");
R;
⇒ // coefficients: QQ(A, B, C, D, E, F)
⇒ // number of vars : 10
⇒ //          block 1 : ordering dp
⇒ //          : names  a b c d e f g h i j
⇒ //          block 2 : ordering C

```

**D.2.3.2 A\_Z\_L**

Procedure from library `general.lib` (see [Section D.2.3 \[general.lib\]](#), page 806).

**Usage:** `A_Z_L("a",n)`; a any letter, n integer ( $-26 \leq n \leq 26, n \neq 0$ )

**Return:** list of n small (if a is small) or capital (if a is capital) letters, beginning with a, in alphabetical order (or reverse alphabetical order if  $n < 0$ )

**Example:**

```

LIB "general.lib";
A_Z_L("c",5);
⇒ [1]:
⇒ c
⇒ [2]:
⇒ d
⇒ [3]:
⇒ e
⇒ [4]:
⇒ f
⇒ [5]:
⇒ g
A_Z_L("Z",-5);
⇒ [1]:
⇒ Z
⇒ [2]:
⇒ Y
⇒ [3]:
⇒ X
⇒ [4]:
⇒ W
⇒ [5]:
⇒ V
ring r;
list L=list(0,A_Z_L("A",6),list(list("dp",1:6),list("C",0)),ideal(0));
ring R=ring(L);
R;
⇒ // coefficients: QQ
⇒ // number of vars : 6

```

```

⇒ //      block   1 : ordering dp
⇒ //      : names   A B C D E F
⇒ //      block   2 : ordering C

```

### D.2.3.3 ASCII

Procedure from library `general.lib` (see [Section D.2.3 \[general.lib\], page 806](#)).

**Usage:**     `ASCII([n,m]);`  $n, m = \text{integers } (32 \leq n \leq m \leq 126)$

**Return:**    string of printable ASCII characters (no native language support)  
               `ASCII()`: string of all ASCII characters with its numbers,  
               `ASCII(n)`:  $n$ -th ASCII character  
               `ASCII(n,m)`:  $n$ -th up to  $m$ -th ASCII character (inclusive)

**Example:**

```

LIB "general.lib";
ASCII();
⇒      !      "      #      $      %      &      '      (      )      *      +      ,      -      .
⇒ 32      33      34      35      36      37      38      39      40      41      42      43      44      45      46
⇒ /      0      1      2      3      4      5      6      7      8      9      :      ;      <      =
⇒ 47      48      49      50      51      52      53      54      55      56      57      58      59      60      61
⇒ >      ?      @      A      B      C      D      E      F      G      H      I      J      K      L
⇒ 62      63      64      65      66      67      68      69      70      71      72      73      74      75      76
⇒ M      N      O      P      Q      R      S      T      U      V      W      X      Y      Z      [
⇒ 77      78      79      80      81      82      83      84      85      86      87      88      89      90      91
⇒ \      ]      ^      _      `      a      b      c      d      e      f      g      h      i      j
⇒ 92      93      94      95      96      97      98      99      100      101      102      103      104      105      106
⇒ k      l      m      n      o      p      q      r      s      t      u      v      w      x      y
⇒ 107      108      109      110      111      112      113      114      115      116      117      118      119      120      121
⇒ z      {      |      }      ~
⇒ 122      123      124      125      126
⇒
ASCII(42);
⇒ *
ASCII(32,126);
⇒ !"#$%&'()*+,-./0123456789:;<=>?@ABCDEFGHIJKLMNopQRSTUVWXYZ[\]^_`abcdefghijklmnopqrstuvwxyz{|}~

```

### D.2.3.4 absValue

Procedure from library `general.lib` (see [Section D.2.3 \[general.lib\], page 806](#)).

**Usage:**     `absValue(c);`  $c$  int, number or poly

**Return:**    `absValue(c);` the absolute value of  $c$

**Note:**     `absValue(c)=c` if  $c \geq 0$ ; `absValue=-c` if  $c \leq 0$ .  
               So the function can be applied to any type, for which comparison  
               operators are defined.

**Example:**

```

LIB "general.lib";
ring r1 = 0,x,dp;
absValue(-2002);
⇒ 2002

```

```
poly f=-4;
absValue(f);
↦ 4
```

### D.2.3.5 binomial

Procedure from library `general.lib` (see [Section D.2.3 \[general.lib\]](#), page 806).

**Usage:** `binomial(n,k)`;  $n, k$  integers

**Return:** `binomial(n,k)`; binomial coefficient  $n$  choose  $k$

**Example:**

```
LIB "general.lib";
binomial(200,100);""; //type bigint
↦ 90548514656103281165404177077484163874504589675413336841320
↦
int n,k = 200,100;
bigint b1 = binomial(n,k);
ring r = 0,x,dp;
poly b2 = coeffs((x+1)^n,x)[k+1,1]; //coefficient of x^k in (x+1)^n
b1-b2; //b1 and b2 should coincide
↦ 0
```

### D.2.3.6 deleteSublist

Procedure from library `general.lib` (see [Section D.2.3 \[general.lib\]](#), page 806).

**Usage:** `deleteSublist(v,l)`; intvec  $v$ ; list  $l$   
 where the entries of the integer vector  $v$  correspond to the positions of the elements to be deleted

**Return:** list without the deleted elements

**Example:**

```
LIB "general.lib";
list l=1,2,3,4,5;
intvec v=1,3,4;
l=deleteSublist(v,l);
l;
↦ [1] :
↦ 2
↦ [2] :
↦ 5
```

### D.2.3.7 factorial

Procedure from library `general.lib` (see [Section D.2.3 \[general.lib\]](#), page 806).

**Usage:** `factorial(n)`;  $n$  integer

**Return:** `factorial(n)`;  $n!$  of type bigint.

**Example:**

```
LIB "general.lib";
factorial(37);
↦ 13763753091226345046315979581580902400000000
```

See also: [Section 5.1.118 \[prime\]](#), page 241.

### D.2.3.8 fibonacci

Procedure from library `general.lib` (see [Section D.2.3 \[general.lib\]](#), page 806).

**Usage:** `fibonacci(n)`;  $n, p$  integers

**Return:** `fibonacci(n)`:  $n$ th Fibonacci number,  $f(0)=f(1)=1$ ,  $f(i+1)=f(i-1)+f(i)$   
 - computed in characteristic 0, of type `bigint`

**Example:**

```
LIB "general.lib";
fibonacci(42);
⇒ 267914296
```

See also: [Section 5.1.118 \[prime\]](#), page 241.

### D.2.3.9 kmemory

Procedure from library `general.lib` (see [Section D.2.3 \[general.lib\]](#), page 806).

**Usage:** `kmemory([n,[v]])`;  $n, v$  integers

**Return:** memory in kilobyte of type `bigint`  
 $n=0$ : memory used by active variables (same as no parameters)  
 $n=1$ : total memory allocated by Singular

**Display:** detailed information about allocated and used memory if  $v!=0$

**Note:** `kmemory` uses internal function 'memory' to compute kilobyte, and is the same as 'memory' for  $n!=0,1,2$

**Example:**

```
LIB "general.lib";
kmemory();
⇒ 172
kmemory(1,1);
⇒ // total memory allocated, at the moment, by SINGULAR (kilobyte):
⇒ 2156
```

### D.2.3.10 killall

Procedure from library `general.lib` (see [Section D.2.3 \[general.lib\]](#), page 806).

**Usage:** `killall()`; (no parameter)  
`killall("type_name")`;  
`killall("not", "type_name")`;

**Return:** `killall()`; kills all user-defined variables except loaded procedures, no return value.  
 - `killall("type_name")`; kills all user-defined variables, of type "type\_name"  
 - `killall("not", "type_name")`; kills all user-defined variables, except those of type "type\_name" and except loaded procedures  
 - `killall("not", "name_1", "name_2", ...)`; kills all user-defined variables, except those of name "name\_i" and except loaded procedures

**Note:** `killall` should never be used inside a procedure

**Example:**



```

LIB "general.lib";
ring rtest; ideal i=x,y,z; string str="hi"; int j = 3;
export rtest,i,str,j;          //this makes the local variables global
listvar();
⇒ // j                        [0] int 3
⇒ // str                      [0] string hi
⇒ // rtest                    [0] *ring
⇒ // i                        [0] ideal, 3 generator(s)
killall("ring");              // kills all rings
⇒ // ** killing the basering for level 0
listvar();
⇒ // j                        [0] int 3
⇒ // str                      [0] string hi
killall("not", "int");        // kills all variables except int's (and procs)
⇒ // ** cannot kill 'Singmathic'
listvar();
⇒ // j                        [0] int 3
⇒ // str                      [0] string hi
killall();                    // kills all vars except loaded procs
listvar();
⇒ // j                        [0] int 3

```

### D.2.3.11 number\_e

Procedure from library `general.lib` (see [Section D.2.3 \[general.lib\], page 806](#)).

**Usage:**     `number_e(n)`;  $n$  integer

**Return:**    Euler number  $e=\exp(1)$  up to  $n$  decimal digits (no rounding)  
               - of type string if no basering of char 0 is defined  
               - of type number if a basering of char 0 is defined

**Display:**   decimal format of  $e$  if `printlevel > 0` (default: `printlevel=0` )

**Note:**       procedure uses algorithm of A.H.J. Sale

**Example:**

```

LIB "general.lib";
number_e(30);"";
⇒ 2.71828182845904523536028747135
⇒
ring R = 0,t,lp;
number e = number_e(30);
e;
⇒ 13591409142295226176801437356763/50000000000000000000000000000000

```

### D.2.3.12 number\_pi

Procedure from library `general.lib` (see [Section D.2.3 \[general.lib\], page 806](#)).

**Usage:**     `number_pi(n)`;  $n$  positive integer

**Return:**     $\pi$  (area of unit circle) up to  $n$  decimal digits (no rounding)  
               - of type string if no basering of char 0 is defined,  
               - of type number, if a basering of char 0 is defined

**Display:**   decimal format of  $\pi$  if `printlevel > 0` (default: `printlevel=0` )

**Note:** procedure uses algorithm of S. Rabinowitz

**Example:**

```
LIB "general.lib";
number_pi(11);"";
↦ 3.1415926535
↦
ring r = (real,10),t,dp;
number_pi = number_pi(11); pi;
↦ 3.141592654
```

### D.2.3.13 primes

Procedure from library `general.lib` (see [Section D.2.3 \[general.lib\], page 806](#)).

**Usage:** `primes(n,m)`;  $n,m$  integers

**Return:** `intvec`, consisting of all primes  $p$ ,  $\text{prime}(n) \leq p \leq m$ , in increasing order if  $n \leq m$ , resp.  $\text{prime}(m) \leq p \leq n$ , in decreasing order if  $m < n$ .

**Note:** `prime(n)`; returns the biggest prime number  $\leq \min(n, 32003)$  if  $n \geq 2$ , else 2

**Example:**

```
LIB "general.lib";
primes(50,100);"";
↦ 47,53,59,61,67,71,73,79,83,89,97
↦
intvec v = primes(37,1); v;
↦ 37,31,29,23,19,17,13,11,7,5,3,2
```

### D.2.3.14 product

Procedure from library `general.lib` (see [Section D.2.3 \[general.lib\], page 806](#)).

**Usage:** `product(id[v])`;  $\text{id}$  ideal/vector/module/matrix/intvec/intmat/list,  $v$  `intvec` (default:  $v=1..$ number of entries of  $\text{id}$ )

**Assume:** list members can be multiplied.

**Return:** The product of all entries of  $\text{id}$  [with index given by  $v$ ] of type depending on the entries of  $\text{id}$ .

**Note:** If  $\text{id}$  is not a list,  $\text{id}$  is treated as a list of polys resp. integers. A module  $m$  is identified with the corresponding matrix  $M$  (columns of  $M$  generate  $m$ ).  
If  $v$  is outside the range of  $\text{id}$ , we have the empty product and the result will be 1 (of type `int`).

**Example:**

```
LIB "general.lib";
ring r= 0,(x,y,z),dp;
ideal m = maxideal(1);
product(m);
↦ xyz
product(m[2..3]);
↦ yz
matrix M[2][3] = 1,x,2,y,3,z;
product(M);
```

```

↳ 6xyz
intvec v=2,4,6;
product(M,v);
↳ xyz
intvec iv = 1,2,3,4,5,6,7,8,9;
v=1..5,7,9;
product(iv,v);
↳ 7560
intmat A[2][3] = 1,1,1,2,2,2;
product(A,3..5);
↳ 4

```

### D.2.3.15 sort

Procedure from library `general.lib` (see [Section D.2.3 \[general.lib\]](#), page 806).

**Usage:**     `sort(id[,v,o,n]);` id = ideal/module/intvec/list  
               sort may be called with 1, 2 or 3 arguments in the following way:  
               `sort(id[,v,n]);` v=intvec of positive integers, n=integer,  
               `sort(id[,o,n]);` o=string (any allowed ordstr of a ring), n=integer

**Return:**     a list l of two elements:  
               l[1]: object of same type as input but sorted in the following way:  
                   - if id=ideal/module: generators of id are sorted w.r.t. intvec v  
                     (id[v[1]] becomes 1-st, id[v[2]] 2-nd element, etc.). If no v is  
                     present, id is sorted w.r.t. ordering o (if o is given) or w.r.t.  
                     actual monomial ordering (if no o is given):  
                     NOTE: generators with SMALLER(!) leading term come FIRST  
                     (e.g. `sort(id);` sorts backwards to actual monomial ordering)  
                   - if id=list or intvec: sorted w.r.t. < (indep. of other arguments)  
                   - if n!=0 the ordering is inverse, i.e. w.r.t. v(size(v)..1)  
                     default: n=0  
               l[2]: intvec, describing the permutation of the input (hence l[2]=v  
                     if v is given (with positive integers))

**Note:**       If v is given id may be any simply indexed object (e.g. any list or string); if v[i]<0  
               and i<=size(id) v[i] is set internally to i; entries of v must be pairwise distinct to get a  
               permutation id. Zero generators of ideal/module are deleted  
               If 'o' is given, the input is sorted by considering leading terms w.r.t. the new ring  
               ordering given by 'o'

**Example:**

```

LIB "general.lib";
ring r0 = 0,(x,y,z,t),lp;
ideal i = x3,z3,xyz;
sort(i);           //sorts using lex ordering, smaller polys come first
↳ [1]:
↳   _[1]=z3
↳   _[2]=xyz
↳   _[3]=x3
↳ [2]:
↳   2,3,1
sort(i,3..1);
↳ [1]:

```

```

⇒  _[1]=xyz
⇒  _[2]=z3
⇒  _[3]=x3
⇒  [2]:
⇒    3,2,1
sort(i,"ls")[1];      //sort w.r.t. negative lex ordering
⇒  _[1]=x3
⇒  _[2]=xyz
⇒  _[3]=z3
intvec v =1,10..5,2..4;v;
⇒  1,10,9,8,7,6,5,2,3,4
sort(v)[1];          // sort v lexicographically
⇒  1,2,3,4,5,6,7,8,9,10
sort(v,"Dp",1)[1];   // sort v w.r.t (total sum, reverse lex)
⇒  1,2,3,4,5,6,7,8,9,10
// Note that in general: lead(sort(M)) != sort(lead(M)), e.g:
module M = [0, 1, 1, 0], [1, 0, 0, 1]; M;
⇒  M[1]=gen(3)+gen(2)
⇒  M[2]=gen(4)+gen(1)
sort(lead(M), "c, dp")[1];
⇒  _[1]=gen(4)
⇒  _[2]=gen(3)
lead(sort(M, "c, dp")[1]);
⇒  _[1]=gen(3)
⇒  _[2]=gen(4)
// In order to sort M wrt a NEW ordering by considering OLD leading
// terms use one of the following equivalent commands:
module( M[ sort(lead(M), "c,dp")[2] ] );
⇒  _[1]=gen(4)+gen(1)
⇒  _[2]=gen(3)+gen(2)
sort( M, sort(lead(M), "c,dp")[2] )[1];
⇒  _[1]=gen(4)+gen(1)
⇒  _[2]=gen(3)+gen(2)
// BUG: Please, don't use this sort for integer vectors or lists
// with them if there can be negative integers!
// TODO: for some HiWi
sort(3..-3)[1];
⇒  -3,-2,-1,0,1,2,3
sort(list(-v, v))[1];
⇒  [1]:
⇒    -1,-10,-9,-8,-7,-6,-5,-2,-3,-4
⇒  [2]:
⇒    1,10,9,8,7,6,5,2,3,4

```

### D.2.3.16 sum

Procedure from library `general.lib` (see [Section D.2.3 \[general.lib\]](#), page 806).

**Usage:** `sum(id[v]);` id ideal/vector/module/matrix/intvec/intmat/list, v intvec (default: v=1..number of entries of id)

**Assume:** list members can be added.

**Return:** The sum of all entries of id [with index given by v] of type depending on the entries of id.

**Note:** If `id` is not a list, `id` is treated as a list of polys resp. integers. A module `m` is identified with the corresponding matrix `M` (columns of `M` generate `m`).  
If `v` is outside the range of `id`, we have the empty sum and the result will be 0 (of type `int`).

**Example:**

```
LIB "general.lib";
ring r1 = 0,(x,y,z),dp;
vector pv = [xy,xz,yz,x2,y2,z2];
sum(pv);
↪ x2+xy+y2+xz+yz+z2
sum(pv,2..5);
↪ x2+y2+xz+yz
matrix M[2][3] = 1,x,2,y,3,z;
intvec w=2,4,6;
sum(M,w);
↪ x+y+z
intvec iv = 1,2,3,4,5,6,7,8,9;
sum(iv,2..4);
↪ 9
iv = intvec(1..100);
sum(iv);
↪ 5050
ring r2 = 0,(x(1..10)),dp;
sum(x(3..7),intvec(1,3,5));
↪ x(3)+x(5)+x(7)
```

### D.2.3.17 watchdog

Procedure from library `general.lib` (see [Section D.2.3 \[general.lib\], page 806](#)).

**Return:** Result of `cmd`, if the result can be computed in `i` seconds. Otherwise the computation is interrupted after `i` seconds, the string "Killed" is returned and the global variable 'watchdog\_interrupt' is defined.

**Note:** \* the current basering should not be `watchdog_rneu`, since `watchdog_rneu` will be killed  
\* if there are variable names of the structure `x(i)` all polynomials have to be put into `eval(...)` in order to be interpreted correctly  
\* a second Singular process is started by this procedure

**Example:**

```
LIB "general.lib";
proc sleep(int s) {return(system("sh","sleep "+string(s)));}
watchdog(1,"sleep(5)");
↪ Killed
watchdog(10,"sleep(5)");
↪ 0
```

### D.2.3.18 primecoeffs

Procedure from library `general.lib` (see [Section D.2.3 \[general.lib\], page 806](#)).

**Usage:** `primecoeffs(J[p]);` `J` any type which can be converted to a matrix e.g. ideal, matrix, vector, module, `int`, `intvec`  
`p` = integer

**Compute:** primefactors  $\leq p$  of coeffs of J (default  $p = 32003$ )

**Return:** a list, say l, of two intvectors:  
 l[1] : the different primefactors of all coefficients of J  
 l[2] : the different remaining factors

**Example:**

```
LIB "general.lib";
primecoeffs(intvec(7*8*121,7*8));"";
↳ [1]:
↳ 2,7,11
↳ [2]:
↳ 1
↳
ring r = 0,(b,c,t),dp;
ideal I = -13b6c3t+4b5c4t,-10b4c2t-5b4ct2;
primecoeffs(I);
↳ [1]:
↳ 2,5,13
↳ [2]:
↳ _[1]=-1
↳ _[2]=1
```

### D.2.3.19 timeStd

Procedure from library `general.lib` (see [Section D.2.3 \[general.lib\]](#), page 806).

**Usage:** timeStd(i,d), i ideal, d integer

**Return:** std(i) if the standard basis computation finished after d-1 seconds and i otherwise

**Example:**

```
LIB "general.lib";
ring r=32003,(a,b,c,d,e),dp;
int n=7;
ideal i=
a^n-b^n,
b^n-c^n,
c^n-d^n,
d^n-e^n,
a^(n-1)*b+b^(n-1)*c+c^(n-1)*d+d^(n-1)*e+e^(n-1)*a;
def i1=timeStd(i,1);
def i2=timeStd(i,20);
listvar();
↳ // n [0] int 7
↳ // r [0] *ring
↳ // i2 [0] ideal (SB), 746 generator(s)
↳ // i1 [0] ideal, 5 generator(s)
↳ // i [0] ideal, 5 generator(s)
```

### D.2.3.20 timeFactorize

Procedure from library `general.lib` (see [Section D.2.3 \[general.lib\]](#), page 806).

**Usage:** timeFactorize(p,d); poly p , integer d

**Return:** factorize(p) if the factorization finished after d-1 seconds otherwise f is considered to be irreducible

**Example:**

```
LIB "general.lib";
ring r=0,(x,y),dp;
poly p=((x2+y3)^2+xy6)*((x3+y2)^2+x10y);
p=p^2;
timeFactorize(p,2);
⇒ [1]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ x22y14+2x21y14+4x23y11+x20y14+2x25y8+4x22y11+6x24y8+4x26y5+2x18y13+\
x28y2+4x17y13+4x15y15+8x19y10+2x16y13+8x14y15+2x12y17+4x21y7+8x18y10+16x1\
6y12+4x13y15+4x11y17+12x20y7+8x18y9+16x15y12+8x13y14+2x10y17+8x22y4+24x17\
y9+4x15y11+x14y12+8x12y14+2x24y+16x19y6+12x14y11+2x13y12+4x11y14+4x21y3+8\
x16y8+4x15y9+x12y12+8x10y14+6x8y16+2x18y5+2x17y6+4x14y9+16x12y11+4x9y14+1\
2x7y16+4x5y18+6x16y6+8x14y8+16x11y11+24x9y13+6x6y16+8x4y18+x2y20+4x18y3+2\
4x13y8+12x11y10+24x8y13+16x6y15+4x3y18+2xy20+x20+16x15y5+36x10y10+8x8y12+\
16x5y15+4x3y17+y20+4x17y2+24x12y7+24x7y12+2x5y14+4x2y17+6x14y4+16x9y9+6x4\
y14+4x11y6+4x6y11+x8y8
⇒ [2]:
⇒ 1,1
//timeFactorize(p,20);
```

### D.2.3.21 factorH

Procedure from library `general.lib` (see [Section D.2.3 \[general.lib\]](#), page 806).

**Usage:** factorH(p) p poly

**Return:** factorize(p)

**Note:** changes variables to make the last variable the principal one in the multivariate factorization and factorizes then the polynomial

**Example:**

```
LIB "general.lib";
system("random",992851144);
ring r=32003,(x,y,z,w,t),lp;
poly p=y2w9+yz7t-yz5w4-z2w4t4-w8t3;
factorize(p); //fast
⇒ [1]:
⇒ _[1]=-1
⇒ _[2]=-y2w9-yz7t+yz5w4+z2w4t4+w8t3
⇒ [2]:
⇒ 1,1
system("random",992851262);
//factorize(p); //slow
system("random",992851262);
factorH(p);
⇒ [1]:
⇒ _[1]=1
⇒ _[2]=y2w9+yz7t-yz5w4-z2w4t4-w8t3
```

$\mapsto [2] :$   
 $\mapsto 1,1$

### D.2.4 grobcov\_lib

**Library:** grobcov.lib

**Purpose:**

"Groebner Cover for parametric ideals.", Comprehensive Groebner Systems, Groebner Cover, Canonical Forms, Parametric Polynomial Systems, Automatic Deduction of Geometric Theorems, Dynamic Geometry, Loci, Envelope, Constructible sets. See: A. Montes A, M. Wibmer, "Groebner Bases for Polynomial Systems with parameters", Journal of Symbolic Computation 45 (2010) 1391-1425. (<https://www.mat.upc.edu/en/people/antonio.montes/>).

**Important:**

Recently published book:

A. Montes. "The Groebner Cover":

Springer, Algorithms and Computation in Mathematics 27 (2019) ISSN 1431-1550

ISBN 978-3-030-03903-5

ISBN 978-3-030-03904-2 (e-Book)

Springer Nature Switzerland AG 2018

<https://www.springer.com/gp/book/9783030039035>

The book can also be used as a user manual of all the routines included in this library.

It defines and proves all the theoretic results used in the library, and shows examples of all the routines. There are many previous papers related to the subject, and the book actualices all the contents.

**Authors:** Antonio Montes (Universitat Politecnica de Catalunya), Hans Schoenemann (Technische Universitaet Kaiserslautern).

**Overview:** In 2010, the library was designed to contain Montes-Wibmer's algorithm for computing the Canonical Groebner Cover of a parametric ideal. The central routine is grobcov. Given a parametric ideal, grobcov outputs its Canonical Groebner Cover, consisting of a set of triplets of (lpp, basis, segment). The basis (after normalization) is the reduced Groebner basis for each point of the segment. The segments are disjoint, locally closed and correspond to constant lpp (leading power product) of the basis, and are represented in canonical representation. The segments cover the whole parameter space. The output is canonical, it only depends on the given parametric ideal and the monomial order, because the segments have different lpph of the homogenized system. This is much more than a simple Comprehensive Groebner System. The algorithm grobcov allows options to solve partially the problem when the



whole automatic algorithm does not finish in reasonable time. Its existence was proved for the first time by Michael Wibmer "Groebner bases for families of affine or projective schemes", JSC, 42,803-834 (2007).

grobcov uses a first algorithm cgsdr that outputs a disjoint reduced Comprehensive Groebner System with constant lpp. For this purpose, in this library, the implemented algorithm is Kapur-Sun-Wang algorithm, because it is actually the most efficient algorithm known for this purpose. D. Kapur, Y. Sun, and D.K. Wang "A New Algorithm for Computing Comprehensive Groebner Systems". Proceedings of ISSAC'2010, ACM Press, (2010), 29-36.

The library has evolved to include new applications of the Groebner Cover, and new theoretical developments have been done.

A routine locus has been included to compute loci of points, and determining the taxonomy of the components. Additional routines to transform the output to string (locusdg, locusto) are also included and used in the Dynamic Geometry software GeoGebra. They were described in:

M.A. Abanades, F. Botana, A. Montes, T. Recio:  
"An Algebraic Taxonomy for Locus Computation in Dynamic Geometry".  
Computer-Aided Design 56 (2014) 22-33.

Routines for determining the generalized envelope of a family of hypersurfaces (envelop, AssocTanToEnv, FamElementsToEnvCompPoints) are also included.

It also includes procedures for Automatic Deduction of Geometric Theorems (ADGT).

The actual version also includes a routine (ConsLevels) for computing the canonical form of a constructible set, given as a union of locally closed sets. It determines the canonical levels of a constructible set. It is described in:

J.M. Brunat, A. Montes, "Computing the canonical representation of constructible sets".  
Math. Comput. Sci. (2016) 19: 165-178.

A complementary routine Levels transforms the output of ConsLevels into the proper locally closed sets forming the levels of the constructible.

Another complementary routine Grob1Levels has been included to select the locally closed sets of the segments of the grobcov that correspond to basis different from 1, add them together and return the canonical form of this constructible set.

More recently (2019) given two locally closed sets in canonical form the new routine DifConsLCSets determines a set of locally closed sets equivalent to the difference them. The description of the routine is submitted to the Journal of Symbolic Computation. This routine can be also used internally by ADGT

with the option "neg",1 . With this option DifConsLCSets is used for the negative hypothesis and thesis in ADGT.

The last version N11 (2021) has improved the routines for locus and allows to determine a parametric locus.

This version was finished on 1/2/2021,

**Notations:** Before calling any routine of the library grobcov, the user must define the ideal  $Q[a][x]$ , and all the input polynomials and ideals defined on it. Internally the routines define and use also other ideals:  $Q[a]$ ,  $Q[x,a]$  and so on.

**Procedures:** See also: [Section D.2.2 \[compregb.lib\]](#), page 798.

### D.2.4.1 grobcov

Procedure from library `grobcov.lib` (see [Section D.2.4 \[grobcov.lib\]](#), page 818).

**Usage:** `grobcov(ideal F[,options]);`  
 $F$ : ideal in  $Q[a][x]$  ( $a$ =parameters,  $x$ =variables) to be discussed. This is the fundamental routine of the library. It computes the Groebner Cover of a parametric ideal  $F$  in  $Q[a][x]$ . See A. Montes , M. Wibmer, "Groebner Bases for Polynomial Systems with parameters". JSC 45 (2010) 1391-1425.)  
 or the not yet published book  
 A. Montes. " The Groebner Cover" (Discussing Parametric Polynomial Systems).  
 The Groebner Cover of a parametric ideal  $F$  consist of a set of pairs  $(S_i, B_i)$ , where the  $S_i$  are disjoint locally closed segments of the parameter space,  
 and the  $B_i$  are the reduced Groebner bases of the ideal on every point of  $S_i$ . The ideal  $F$  must be defined on a parametric ring  $Q[a][x]$  ( $a$ =parameters,  $x$ =variables).

**Return:** The list `[[lpp_1,basis_1,segment_1], ..., [lpp_s,basis_s,segment_s]]`  
 optionally `[[ lpp_1,basis_1,segment_1,lpph_1], ..., [lpp_s,basis_s,segment_s,lpph_s]]`  
 The `lpp` are constant over a segment and correspond to the set of `lpp` of the reduced Groebner basis for each point of the segment.  
 With option ("showhom",1) the `lpph` will be shown: The `lpph` corresponds to the `lpp` of the homogenized ideal and is different for each segment. It is given as a string, and shown only for information. With the default option "can",1, the segments have different `lpph`.  
 Basis: to each element of `lpp` corresponds an I-regular function given in full representation (by option ("ext",1)) or in generic representation (default option ("ext",0)). The I-regular function is the corresponding

element of the reduced Groebner basis for each point of the segment with the given lpp. For each point in the segment, the polynomial or the set of polynomials representing it, if they do not specialize to 0, then after normalization, specializes to the corresponding element of the reduced Groebner basis. In the full representation at least one of the polynomials representing the I-regular function specializes to non-zero.

With the default option ("rep",0) the representation of the segment is the P-representation.

With option ("rep",1) the representation of the segment is the C-representation.

With option ("rep",2) both representations of the segment are given.

The P-representation of a segment is of the form  $[[p_1, [p_{11}, \dots, p_{1k1}]], \dots, [p_r, [p_{r1}, \dots, p_{rkr}]]]$  representing the segment

$\text{Union}_i ( V(p_i) \setminus ( \text{Union}_j V(p_{ij}) ) )$ , where the p's are prime ideals.

The C-representation of a segment is of the form (E,N) representing  $V(E) \setminus V(N)$ , and the ideals E and N are radical and N contains E.

**Options:** An option is a pair of arguments: string, integer. To modify the default options, pairs of arguments -option name, value- of valid options must be added to the call.

"null",ideal E: The default is ("null",ideal(0)).

"nonnull",ideal N: The default is ("nonnull",ideal(1)).

When options "null" and/or "nonnull" are given, then the parameter space is restricted to  $V(E) \setminus V(N)$ . "can",0-1: The default is ("can",1).

With the default option the homogenized ideal is computed before obtaining the Groebner Cover, so that the result is the canonical Groebner Cover. Setting ("can",0) only homogenizes the basis so the result is not exactly canonical, but the computation is shorter.

"ext",0-1: The default is ("ext",0).

With the default ("ext",0), only the generic representation of the bases is computed (single polynomials, but not specializing to non-zero for every point of the segment).

With option ("ext",1) the full representation of the bases is computed (possible sheaves) and sometimes a simpler result is obtained, but the computation is more time consuming.

"rep",0-1-2: The default is ("rep",0) and then the segments are given in canonical P-representation.  
Option ("rep",1) represents the segments in canonical C-representation, and option ("rep",2) gives both representations.  
"comment",0-3: The default is ("comment",0). Setting "comment" higher will provide information about the development of the computation.  
"showhom",0-1: The default is ("showhom",0). Setting "showhom",1 will output the set of lpp of the homogenized ideal of each segment as last element. One can give none or whatever of these options.

**Note:** The basering R, must be of the form  $\mathbb{Q}[a][x]$ , (a=parameters, x=variables), and should be defined previously. The ideal must be defined on R.

**Example:**

```
LIB "grobcov.lib";
// EXAMPLE 1:
// Casas conjecture for degree 4:
// Casas-Alvero conjecture states that on a field of characteristic 0,
// if a polynomial of degree n in x has a common root with each of its
// n-1 derivatives (not assumed to be the same), then it is of the form
//  $P(x) = k(x + a)^n$ , i.e. the common roots must all be the same.
if(defined(R)){kill R;}
ring R=(0,a0,a1,a2,a3,a4),(x1,x2,x3),dp;
short=0;
ideal F=x1^4+(4*a3)*x1^3+(6*a2)*x1^2+(4*a1)*x1+(a0),
x1^3+(3*a3)*x1^2+(3*a2)*x1+(a1),
x2^4+(4*a3)*x2^3+(6*a2)*x2^2+(4*a1)*x2+(a0),
x2^3+(3*a3)*x2^2+(3*a2)*x2+(a1),
x3^4+(4*a3)*x3^3+(6*a2)*x3^2+(4*a1)*x3+(a0),
x3^3+(3*a3)*x3^2+(3*a2)*x3+(a1);
grobcov(F);
--> [1]:
--> [1]:
--> _[1]=1
--> [2]:
--> _[1]=1
--> [3]:
--> [1]:
--> [1]:
--> _[1]=0
--> [2]:
--> [1]:
--> _[1]=(a2-a3^2)
--> _[2]=(a1-a3^3)
--> _[3]=(a0-a3^4)
```

```

⇒ [2]:
⇒   [1]:
⇒     _[1]=x3
⇒     _[2]=x2^2
⇒     _[3]=x1^3
⇒   [2]:
⇒     _[1]=x3+(a3)
⇒     _[2]=x2^2+(2*a3)*x2+(a3^2)
⇒     _[3]=x1^3+(3*a3)*x1^2+(3*a3^2)*x1+(a3^3)
⇒   [3]:
⇒     [1]:
⇒       [1]:
⇒         _[1]=(a2-a3^2)
⇒         _[2]=(a1-a3^3)
⇒         _[3]=(a0-a3^4)
⇒     [2]:
⇒       [1]:
⇒         _[1]=1
// EXAMPLE 2
// M. Rychlik robot;
// Complexity and Applications of Parametric Algorithms of
// Computational Algebraic Geometry.;
// In: Dynamics of Algorithms, R. de la Llave, L. Petzold and J. Lorenz eds.;
// IMA Volumes in Mathematics and its Applications,
// Springer-Verlag 118: 1-29 (2000).;
// (18. Mathematical robotics: Problem 4, two-arm robot).
if (defined(R)){kill R;}
ring R=(0,a,b,l2,l3),(c3,s3,c1,s1), dp;
short=0;
ideal S12=a-l3*c3-l2*c1,b-l3*s3-l2*s1,c1^2+s1^2-1,c3^2+s3^2-1;
S12;
⇒ S12[1]=(-l3)*c3+(-l2)*c1+(a)
⇒ S12[2]=(-l3)*s3+(-l2)*s1+(b)
⇒ S12[3]=c1^2+s1^2-1
⇒ S12[4]=c3^2+s3^2-1
grobcov(S12);
⇒ [1]:
⇒   [1]:
⇒     _[1]=c1
⇒     _[2]=s3
⇒     _[3]=c3
⇒     _[4]=s1^2
⇒   [2]:
⇒     _[1]=(2*a*l2)*c1+(2*b*l2)*s1+(-a^2-b^2-l2^2+l3^2)
⇒     _[2]=(l3)*s3+(l2)*s1+(-b)
⇒     _[3]=(2*a*l3)*c3+(-2*b*l2)*s1+(-a^2+b^2+l2^2-l3^2)
⇒     _[4]=(4*a^2*l2^2+4*b^2*l2^2)*s1^2+(-4*a^2*b*l2-4*b^3*l2-4*b*l2^3+4*\
b*l2*l3^2)*s1+(a^4+2*a^2*b^2-2*a^2*l2^2-2*a^2*l3^2+b^4+2*b^2*l2^2-2*b^2*\
3^2+l2^4-2*l2^2*l3^2+l3^4)
⇒   [3]:
⇒     [1]:
⇒       [1]:
⇒         _[1]=0

```

```

    ↪      [2]:
    ↪      [1]:
    ↪      _[1]=(13)
    ↪      [2]:
    ↪      _[1]=(12)
    ↪      [3]:
    ↪      _[1]=(a^2+b^2)
    ↪      [4]:
    ↪      _[1]=(a)
    ↪ [2]:
    ↪ [1]:
    ↪ _[1]=s1
    ↪ _[2]=s3
    ↪ _[3]=c3
    ↪ _[4]=c1^2
    ↪ [2]:
    ↪ _[1]=(2*b*12)*s1+(-b^2-12^2+13^2)
    ↪ _[2]=(2*b*13)*s3+(-b^2+12^2-13^2)
    ↪ _[3]=(13)*c3+(12)*c1
    ↪ _[4]=(4*b^2*12^2)*c1^2+(b^4-2*b^2*12^2-2*b^2*13^2+12^4-2*12^2*13^2+\
13^4)
    ↪ [3]:
    ↪ [1]:
    ↪ [1]:
    ↪ _[1]=(a)
    ↪ [2]:
    ↪ [1]:
    ↪ _[1]=(13)
    ↪ _[2]=(a)
    ↪ [2]:
    ↪ _[1]=(12)
    ↪ _[2]=(a)
    ↪ [3]:
    ↪ _[1]=(b)
    ↪ _[2]=(a)
    ↪ [3]:
    ↪ [1]:
    ↪ _[1]=1
    ↪ [2]:
    ↪ _[1]=1
    ↪ [3]:
    ↪ [1]:
    ↪ [1]:
    ↪ _[1]=(b)
    ↪ _[2]=(a)
    ↪ [2]:
    ↪ [1]:
    ↪ _[1]=(12+13)
    ↪ _[2]=(b)
    ↪ _[3]=(a)
    ↪ [2]:
    ↪ _[1]=(13)
    ↪ _[2]=(b)

```

```

↳          _[3]=(a)
↳      [3]:
↳          _[1]=(12-13)
↳          _[2]=(b)
↳          _[3]=(a)
↳      [2]:
↳      [1]:
↳          _[1]=(12)
↳      [2]:
↳      [1]:
↳          _[1]=(12)
↳          _[2]=(a^2+b^2-13^2)
↳      [2]:
↳          _[1]=(13)
↳          _[2]=(12)
↳ [4]:
↳ [1]:
↳     _[1]=s3
↳     _[2]=c3
↳     _[3]=c1^2
↳ [2]:
↳     _[1]=(12^2*13+2*12^2-13^3)*s3+(2*12*13)*s1+(b*13^2)
↳     _[2]=(12^2*13+2*12^2-13^3)*c3+(2*12*13)*c1+(a*13^2)
↳     _[3]=c1^2+s1^2-1
↳ [3]:
↳ [1]:
↳     [1]:
↳     _[1]=(12-13)
↳     _[2]=(b)
↳     _[3]=(a)
↳     [2]:
↳     [1]:
↳     _[1]=(13)
↳     _[2]=(12)
↳     _[3]=(b)
↳     _[4]=(a)
↳     [2]:
↳     [1]:
↳     _[1]=(12+13)
↳     _[2]=(b)
↳     _[3]=(a)
↳     [2]:
↳     [1]:
↳     _[1]=(13)
↳     _[2]=(12)
↳     _[3]=(b)
↳     _[4]=(a)
↳     [3]:
↳     [1]:
↳     _[1]=(12)
↳     _[2]=(a^2+b^2-13^2)
↳     [2]:
↳     [1]:

```

```

↳          _[1]=(13)
↳          _[2]=(12)
↳          _[3]=(a^2+b^2)
↳ [5]:
↳   [1]:
↳     _[1]=c1^2
↳     _[2]=c3^2
↳   [2]:
↳     _[1]=c1^2+s1^2-1
↳     _[2]=c3^2+s3^2-1
↳   [3]:
↳     [1]:
↳       [1]:
↳         _[1]=(13)
↳         _[2]=(12)
↳         _[3]=(b)
↳         _[4]=(a)
↳     [2]:
↳       [1]:
↳         _[1]=1
↳ [6]:
↳   [1]:
↳     _[1]=1
↳   [2]:
↳     _[1]=1
↳   [3]:
↳     [1]:
↳       [1]:
↳         _[1]=(13)
↳       [2]:
↳         [1]:
↳           _[1]=(13)
↳           _[2]=(a^2+b^2-12^2)
↳         [2]:
↳           _[1]=(13)
↳           _[2]=(12)
↳ [7]:
↳   [1]:
↳     _[1]=1
↳   [2]:
↳     _[1]=1
↳   [3]:
↳     [1]:
↳       [1]:
↳         _[1]=(13)
↳         _[2]=(12)
↳       [2]:
↳         [1]:
↳           _[1]=(13)
↳           _[2]=(12)
↳           _[3]=(a^2+b^2)
↳ [8]:
↳   [1]:

```



```

⇒      _[1]=s1
⇒      _[2]=c1
⇒      _[3]=c3^2
⇒      [2]:
⇒      _[1]=(12)*s1+(-b)
⇒      _[2]=(12)*c1+(-a)
⇒      _[3]=c3^2+s3^2-1
⇒      [3]:
⇒      [1]:
⇒      [1]:
⇒      _[1]=(13)
⇒      _[2]=(a^2+b^2-12^2)
⇒      [2]:
⇒      [1]:
⇒      _[1]=(13)
⇒      _[2]=(12)
⇒      _[3]=(a^2+b^2)
⇒ [9]:
⇒ [1]:
⇒ _[1]=1
⇒ [2]:
⇒ _[1]=1
⇒ [3]:
⇒ [1]:
⇒ [1]:
⇒ _[1]=(13)
⇒ _[2]=(12)
⇒ _[3]=(a^2+b^2)
⇒ [2]:
⇒ [1]:
⇒ _[1]=(13)
⇒ _[2]=(12)
⇒ _[3]=(b)
⇒ _[4]=(a)
⇒ [10]:
⇒ [1]:
⇒ _[1]=s1
⇒ _[2]=c1
⇒ _[3]=s3
⇒ _[4]=c3
⇒ [2]:
⇒ _[1]=(4*b*12^3-4*b*12*13^2)*s1+(-4*b^2*12^2-12^4+2*12^2*13^2-13^4)
⇒ _[2]=(4*b^2*12^3-4*b^2*12*13^2)*c1+(-4*a*b^2*12^2+a*12^4-2*a*12^2*1\
3^2+a*13^4)
⇒ _[3]=(4*b*12^2*13-4*b*13^3)*s3+(4*b^2*13^2+12^4-2*12^2*13^2+13^4)
⇒ _[4]=(4*b^2*12^2*13-4*b^2*13^3)*c3+(4*a*b^2*13^2-a*12^4+2*a*12^2*13\
^2-a*13^4)
⇒ [3]:
⇒ [1]:
⇒ [1]:
⇒ _[1]=(a^2+b^2)
⇒ [2]:
⇒ [1]:

```

```

↳          _[1]=(12+13)
↳          _[2]=(a^2+b^2)
↳      [2]:
↳          _[1]=(13)
↳          _[2]=(a^2+b^2)
↳      [3]:
↳          _[1]=(12-13)
↳          _[2]=(a^2+b^2)
↳      [4]:
↳          _[1]=(12)
↳          _[2]=(a^2+b^2)
↳      [5]:
↳          _[1]=(b)
↳          _[2]=(a)
↳ [11]:
↳     [1]:
↳         _[1]=1
↳     [2]:
↳         _[1]=1
↳     [3]:
↳         [1]:
↳             [1]:
↳                 _[1]=(12-13)
↳                 _[2]=(a^2+b^2)
↳             [2]:
↳                 [1]:
↳                     _[1]=(13)
↳                     _[2]=(12)
↳                     _[3]=(a^2+b^2)
↳                 [2]:
↳                     _[1]=(12-13)
↳                     _[2]=(b)
↳                     _[3]=(a)
↳         [2]:
↳             [1]:
↳                 _[1]=(12+13)
↳                 _[2]=(a^2+b^2)
↳             [2]:
↳                 [1]:
↳                     _[1]=(13)
↳                     _[2]=(12)
↳                     _[3]=(a^2+b^2)
↳                 [2]:
↳                     _[1]=(12+13)
↳                     _[2]=(b)
↳                     _[3]=(a)

```

#### D.2.4.2 cgsdr

Procedure from library `grobcov.lib` (see [Section D.2.4 \[grobcov.lib\]](#), page 818).

**Usage:** `cgsdr(ideal F);`  
 F: ideal in  $\mathbb{Q}[a][x]$  (a=parameters, x=variables) to be  
 discussed. Computes a disjoint, reduced Comprehensive

Groebner System (CGS). cgsdr is the starting point of the fundamental routine grobcov.

The basering R, must be of the form  $\mathbb{Q}[a][x]$ ,  
(a=parameters, x=variables),  
and should be defined previously.

**Return:** Returns a list T describing a reduced and disjoint Comprehensive Groebner System (CGS). The output is a list of (full,hole,basis), where the ideals full and hole represent the segment  $V(\text{full}) \setminus V(\text{hole})$ . With option ("out",0) the segments are grouped by leading power products (lpp) of the reduced Groebner basis and given in P-representation. The returned list is of the form:  

$$[ [\text{lpp}, [\text{num}, \text{basis}, \text{segment}], \dots, [\text{num}, \text{basis}, \text{segment}], \text{lpph}], \dots, [\text{lpp}, [\text{num}, \text{basis}, \text{segment}], \dots, [\text{num}, \text{basis}, \text{segment}], \text{lpph}] ]$$
The bases are the reduced Groebner bases (after normalization) for each point of the corresponding segment. The third element lpph of each lpp segment is the lpp of the homogenized ideal used ideal in the CGS as a string, that is shown only when option ("can",1) is used.  
With option ("can",0) the homogenized basis is used.  
With option ("can",1) the homogenized ideal is used.  
With option ("can",2) the given basis is used.  
With option ("out",1) (default) only KSW is applied and segments are given as difference of varieties and are not grouped The returned list is of the form:  

$$[[E, N, B], \dots, [E, N, B]]$$
E is the top variety  
N is the hole variety.  
Segment =  $V(E) \setminus V(N)$   
B is the reduced Groebner basis

**Options:** An option is a pair of arguments: string, integer.  
To modify the default options, pairs of arguments -option name, value- of valid options must be added to the call. Inside grobcov the default option is "can",1. It can be used also with option "can",0 but then the output is not the canonical Groebner Cover. grobcov cannot be used with option "can",2.  
When cgsdr is called directly, the options are "can",0-1-2: The default value is "can",2.  
In this case no homogenization is done. With option ("can",0) the given basis is homogenized, and with option ("can",1) the whole given ideal is homogenized before computing the cgs and dehomogenized after.  
With option ("can",0) the homogenized basis is used. With option ("can",1) the ho-

mogenized ideal is used. With option ("can",2) the given basis is used.  
 "null",ideal E: The default is ("null",ideal(0)).  
 "nonnull",ideal N: The default ("nonnull",ideal(1)). When options "null" and/or "nonnull" are given,  
 then the parameter space is restricted to  $V(E) \setminus V(N)$ . "comment",0-1: The default is ("comment",0).  
 Setting ("comment",1) will provide information about the development of the computation.  
 "out",0-1: (default is 1) the output segments are given as as difference of varieties.  
 With option "out",0 the output segments are given in P-representation and the segments grouped by lpp.  
 With options ("can",0) and ("can",1) the option ("out",1) is set to ("out",0) because it is not compatible.  
 One can give none or whatever of these options.  
 With the default options ("can",2,"out",1), only the Kapur-Sun-Wang algorithm is computed.  
 The algorithm used is:  
 D. Kapur, Y. Sun, and D.K. Wang "A New Algorithm for Computing Comprehensive Groebner Systems".  
 Proceedings of ISSAC'2010, ACM Press, (2010), 29-36. It is very efficient but is only the starting point  
 for the computation of grobcov.  
 When grobcov is computed, the call to cgsdr inside uses specific options that are more expensive ("can",0-1,"out",0).

**Example:**

```
LIB "grobcov.lib";
// EXAMPLE:
// Casas conjecture for degree 4:
// Casas-Alvero conjecture states that on a field of characteristic 0,
// if a polynomial of degree n in x has a common root whith each of its
// n-1 derivatives (not assumed to be the same), then it is of the form
// P(x) = k(x + a)^n, i.e. the common roots must all be the same.
if(defined(R)){kill R;}
ring R=(0,a0,a1,a2,a3,a4),(x1,x2,x3),dp;
short=0;
ideal F=x1^4+(4*a3)*x1^3+(6*a2)*x1^2+(4*a1)*x1+(a0),
x1^3+(3*a3)*x1^2+(3*a2)*x1+(a1),
x2^4+(4*a3)*x2^3+(6*a2)*x2^2+(4*a1)*x2+(a0),
x2^2+(2*a3)*x2+(a2),
x3^4+(4*a3)*x3^3+(6*a2)*x3^2+(4*a1)*x3+(a0),
x3+(a3);
cgsdr(F);
=> [1]:
=>      [1]:
=>      _[1]=0
=>      [2]:
=>      _[1]=(-a2+a3^2)
```

```

⇒      _[2]=(-a1*a2+a1*a3^2+3*a2^2*a3-5*a2*a3^3+2*a3^5)
⇒      _[3]=(23*a1^2-138*a1*a2*a3+92*a1*a3^3+25*a2^3+132*a2^2*a3^2-201*a2*\
a3^4+67*a3^6)
⇒      _[4]=(a0-4*a1*a3+6*a2*a3^2-3*a3^4)
⇒      [3]:
⇒      _[1]=1
⇒      [2]:
⇒      [1]:
⇒      _[1]=(a2-a3^2)
⇒      _[2]=(a1-a3^3)
⇒      _[3]=(a0-a3^4)
⇒      [2]:
⇒      _[1]=1
⇒      [3]:
⇒      _[1]=x3+(a3)
⇒      _[2]=x2^2+(2*a3)*x2+(a3^2)
⇒      _[3]=x1^3+(3*a3)*x1^2+(3*a3^2)*x1+(a3^3)

```

### D.2.4.3 pdivi

Procedure from library `grobcov.lib` (see [Section D.2.4 \[grobcov.lib\]](#), page 818).

- Usage:** `pdivi(poly f, ideal F);`  
 poly f: the polynomial in  $\mathbb{Q}[a][x]$  to be divided  
 ideal F: the divisor ideal in  $\mathbb{Q}[a][x]$ .  
 (a=parameters, x=variables).
- Return:** A list (poly r, ideal q, poly m). r is the remainder of the pseudodivision, q is the set of quotients, and m is the coefficient factor by which f is to be multiplied.
- Note:** pseudodivision of a poly f by an ideal F in  $\mathbb{Q}[a][x]$ .  
 Returns a list (r,q,m) such that  
 $m*f=r+\text{sum}(q.F)$ ,  
 and no lpp of a divisor divides a pp of r.

**Example:**

```

LIB "grobcov.lib";
"RXAMPLE:";
⇒ RXAMPLE:
// Division of a polynom by an ideal
if(defined(R)){kill R;}
ring R=(0,a,b,c),(x,y),dp;
short=0;
// Divisor
poly f=(ab-ac)*xy+(ab)*x+(5c);
// Dividends
ideal F=ax+b,
cy+a;
// (Remainder, quotients, factor)
def r=pdivi(f,F);
r;
⇒ [1]:

```

```

↳      (a*b^2-a*b*c-b^2*c+5*c^2)
↳ [2]:
↳      _[1]=(b*c-c^2)*y+(b*c)
↳      _[2]=(-b^2+b*c)
↳ [3]:
↳      (c)
// Verifying the division
r[3]*f-(r[2][1]*F[1]+r[2][2]*F[2]+r[1]);
↳ 0

```

#### D.2.4.4 pnormalf

Procedure from library `grobcov.lib` (see [Section D.2.4 \[grobcov.lib\]](#), page 818).

**Usage:** `pnormalf(poly f, ideal E, ideal N);`  
`f`: the polynomial in  $\mathbb{Q}[a][x]$  ( $a$ =parameters,  $x$ =variables) to be reduced modulo  $V(E) \setminus V(N)$  of a segment in  $\mathbb{Q}[a]$ .  
`E`: the null conditions ideal in  $\mathbb{Q}[a]$   
`N`: the non-null conditions in  $\mathbb{Q}[a]$

**Return:** a reduced polynomial  $g$  of  $f$ , whose coefficients are reduced modulo  $E$  and having no factor in  $N$ .

**Note:** Should be called from ring  $\mathbb{Q}[a][x]$ . Ideals  $E$  and  $N$  must be given by polynomials in  $\mathbb{Q}[a]$ .

**Example:**

```

LIB "grobcov.lib";
if(defined(R)){kill R;}
ring R=(0,a,b,c),(x,y),dp;
short=0;
// parametric polynom
poly f=(b^2-1)*x^3*y+(c^2-1)*x*y^2+(c^2*b-b)*x+(a-bc)*y;
// ideals defining V(p)\V(q)
ideal p=c-1;
ideal q=a-b;
// pnormaform of f on V(p) \ V(q)
pnormalf(f,p,q);
↳ (b^2-1)*x^3*y+(a-b)*y

```

#### D.2.4.5 Crep

Procedure from library `grobcov.lib` (see [Section D.2.4 \[grobcov.lib\]](#), page 818).

**Usage:** `Crep(ideal N, ideal M);`  
`ideal N` (null ideal) (not necessarily radical nor maximal) in  $\mathbb{Q}[a]$ . ( $a$ =parameters,  $x$ =variables).  
`ideal M` (hole ideal) (not necessarily containing  $N$ ) in  $\mathbb{Q}[a]$ . To be called in a ring  $\mathbb{Q}[a][x]$  or a ring  $\mathbb{Q}[a]$ . But the ideals can contain only the parameters in  $\mathbb{Q}[a]$ .

**Return:** The canonical C-representation  $[P, Q]$  of the locally closed set, formed by a pair of radical

ideals with  $P$  included in  $Q$ , representing the set  
 $V(P) \setminus V(Q) = V(N) \setminus V(M)$

**Example:**

```
LIB "grobcov.lib";
short=0;
if(defined(R)){kill R;}
ring R=0,(a,b,c),lp;
ideal p=a*b;
ideal q=a,b-2;
// C-representation of  $V(p) \setminus V(q)$ 
Crep(p,q);
⇒ [1]:
⇒   _[1]=ab
⇒ [2]:
⇒   _[1]=b-2
⇒   _[2]=a
```

**D.2.4.6 Prep**

Procedure from library `grobcov.lib` (see [Section D.2.4 \[grobcov.lib\]](#), page 818).

**Usage:** Prep(ideal N,ideal M);  
 ideal N (null ideal) (not necessarily radical nor maximal) in  $Q[a]$ . ( $a$ =parameters,  $x$ =variables).  
 ideal M (hole ideal) (not necessarily containing N) in  $Q[a]$ . To be called in a ring  $Q[a][x]$  or a ring  $Q[a]$ . But the ideals can contain only the parameters in  $Q[a]$ .

**Return:** The canonical P-representation of the locally closed set  $V(N) \setminus V(M)$   
 Output: [Comp\_1, .. , Comp\_s ] where  
 Comp\_i=[p\_i,[p\_i1,...,p\_is\_i]]

**Example:**

```
LIB "grobcov.lib";
short=0;
if(defined(R)){kill R;}
ring R=0,(a,b,c),lp;
ideal p=a*b;;
ideal q=a,b-1;
// P-representation of  $V(p) \setminus V(q)$ 
Prep(p,q);
⇒ [1]:
⇒   [1]:
⇒   _[1]=b
⇒   [2]:
⇒   [1]:
⇒   _[1]=1
⇒ [2]:
⇒   [1]:
⇒   _[1]=a
⇒   [2]:
```

```

⇒      [1]:
⇒      _[1]=b-1
⇒      _[2]=a

```

### D.2.4.7 PtoCrep

Procedure from library `grobcov.lib` (see [Section D.2.4 \[grobcov.lib\]](#), page 818).

**Usage:** `PtoCrep(list L)`  
 list `L = [ Comp_1, .. , Comp_s ]` where  
 list `Comp_i = [p_i, [p_i1, ..., p_is_i] ]`, is the  
 P-representation of a locally closed set  
 $V(N) \setminus V(M)$ . To be called in a ring  $Q[a][x]$   
 or a ring  $Q[a]$ . But the ideals can contain  
 only the parameters in  $Q[a]$ .

**Return:** The canonical C-representation  $[P, Q]$  of the  
 locally closed set. A pair of radical ideals with  
 $P$  included in  $Q$ , representing the  
 set  $V(P) \setminus V(Q)$

**Example:**

```

LIB "grobcov.lib";
//EXAMPLE:
if(defined(R)){kill R;}
ring R=0,(a,b,c),lp;
short=0;
ideal p=a*(a^2+b^2+c^2-25);
ideal q=a*(a-3),b-4;
// C-representaion of V(p) \ V(q)
def Cr=Crep(p,q);
Cr;
⇒ [1]:
⇒ _[1]=a^3+a*b^2+a*c^2-25*a
⇒ [2]:
⇒ _[1]=b-4
⇒ _[2]=a*c
⇒ _[3]=a^2-3*a
// P-representation of V(p) \ V(q)
def L=Prep(p,q);
L;
⇒ [1]:
⇒ [1]:
⇒ _[1]=a^2+b^2+c^2-25
⇒ [2]:
⇒ [1]:
⇒ _[1]=c-3
⇒ _[2]=b-4
⇒ _[3]=a
⇒ [2]:
⇒ _[1]=c+3
⇒ _[2]=b-4
⇒ _[3]=a
⇒ [3]:

```



```

⇒      _[1]=c
⇒      _[2]=b-4
⇒      _[3]=a-3
⇒ [2]:
⇒      [1]:
⇒      _[1]=a
⇒      [2]:
⇒      [1]:
⇒      _[1]=b-4
⇒      _[2]=a
PtoCrep(L);
⇒ [1]:
⇒      _[1]=a^3+a*b^2+a*c^2-25*a
⇒ [2]:
⇒      _[1]=b-4
⇒      _[2]=a*c
⇒      _[3]=a^2-3*a

```

### D.2.4.8 extendpoly

Procedure from library `grobpcov.lib` (see [Section D.2.4 \[grobpcov.lib\]](#), page 818).

**Usage:**     `extendGC(poly f,ideal p,ideal q);`  
 $f$  is a polynomial in  $Q[a][x]$  in generic representation of an I-regular function  $F$  defined on the locally closed segment  $S=V(p) \setminus V(q)$ .  
 $p, q$  are ideals in  $Q[a]$ , representing the Crep of segment  $S$ .

**Return:**    the extended representation of  $F$  in  $S$ .  
It can consist of a single polynomial or a set of polynomials when needed.

**Note:**      The basering  $R$ , must be of the form  $Q[a][x]$ ,  
( $a$ =parameters, $x$ =variables), and should be defined previously. The ideals must be defined on  $R$ .

**Example:**

```

LIB "grobpcov.lib";
// EXAMPLE 1
if (defined(R)) {kill R;}
ring R=(0,a1,a2),(x),lp;
short=0;
poly f=(a1^2-4*a1+a2^2-a2)*x+(a1^4-16*a1+a2^3-4*a2);
ideal p=a1*a2;
ideal q=a2^2-a2,a1*a2,a1^2-4*a1;
extendpoly(f,p,q);
⇒ (a1+4*a2-4)*x+(a1^3+4*a2^2-16)
// EXAMPLE 2
if (defined(R)) {kill R;}
ring R=(0,a0,b0,c0,a1,b1,c1,a2,b2,c2),(x),dp;
short=0;
poly f=(b1*a2*c2-c1*a2*b2)*x+(-a1*c2^2+b1*b2*c2+c1*a2*c2-c1*b2^2);
ideal p=

```

```

(-a0*b1*c2+a0*c1*b2+b0*a1*c2-b0*c1*a2-c0*a1*b2+c0*b1*a2),
(a1^2*c2^2-a1*b1*b2*c2-2*a1*c1*a2*c2+a1*c1*b2^2+b1^2*a2*c2-b1*c1*a2*b2+c1^2*a2^2),
(a0*a1*c2^2-a0*b1*b2*c2-a0*c1*a2*c2+a0*c1*b2^2+b0*b1*a2*c2-b0*c1*a2*b2
- c0*a1*a2*c2+c0*c1*a2^2),
(a0^2*c2^2-a0*b0*b2*c2-2*a0*c0*a2*c2+a0*c0*b2^2+b0^2*a2*c2-b0*c0*a2*b2+c0^2*a2^2),
(a0*a1*c1*c2-a0*b1^2*c2+a0*b1*c1*b2-a0*c1^2*a2+b0*a1*b1*c2-b0*a1*c1*b2
-c0*a1^2*c2+c0*a1*c1*a2),
(a0^2*c1*c2-a0*b0*b1*c2-a0*c0*a1*c2+a0*c0*b1*b2-a0*c0*c1*a2+b0^2*a1*c2
-b0*c0*a1*b2+c0^2*a1*a2),
(a0^2*c1^2-a0*b0*b1*c1-2*a0*c0*a1*c1+a0*c0*b1^2+b0^2*a1*c1-b0*c0*a1*b1+c0^2*a1^2),
(2*a0*a1*b1*c1*c2-a0*a1*c1^2*b2-a0*b1^3*c2+a0*b1^2*c1*b2-a0*b1*c1^2*a2
-b0*a1^2*c1*c2+b0*a1*b1^2*c2-b0*a1*b1*c1*b2+b0*a1*c1^2*a2-c0*a1^2*b1*c2+c0*a1^2*c1*b2)
ideal q=
(-a1*c2+c1*a2),
(-a1*b2+b1*a2),
(-a0*c2+c0*a2),
(-a0*b2+b0*a2),
(-a0*c1+c0*a1),
(-a0*b1+b0*a1),
(-a1*b1*c2+a1*c1*b2),
(-a0*b1*c2+a0*c1*b2),
(-a0*b0*c2+a0*c0*b2),
(-a0*b0*c1+a0*c0*b1);
extendpoly(f,p,q);
→ _[1]=(-a1*b2+b1*a2)*x+(-a1*c2+c1*a2)
→ _[2]=(-a0*b2+b0*a2)*x+(-a0*c2+c0*a2)
→ _[3]=(-a0*b1+b0*a1)*x+(-a0*c1+c0*a1)

```

#### D.2.4.9 extendGC

Procedure from library `grobcov.lib` (see [Section D.2.4 \[grobcov.lib\]](#), page 818).

**Usage:**    `extendGC(list GC);`  
           `list GC` must be the `grobcov` of a parametric ideal computed with option "rep",2. It determines the full representation.  
           The default option of `grobcov` provides the bases in generic representation (the I-regular functions forming the bases are then given by a single polynomial.  
           They can specialize to zero for some points of the segments, but in general, it is sufficient for many purposes. Nevertheless the I-regular functions allow a full representation given by a set of polynomials  
           specializing to the value of the function (after normalization) or to zero, but at least one of the polynomials specializes to non-zero. The full representation can be obtained by computing the `grobcov` with option "ext",1. (The default option there is "ext",0).  
           With option "ext",1 the computation can be much more time consuming, but the result can be simpler.  
           Alternatively, one can compute the full representation of the bases after computing

grobpcov with the default option for "ext" and the option "rep",2, that outputs both the Prep and the Crep of the segments, and then call "extendGC" to its output.

**Return:** When calling `extendGC(grobpcov(S,"rep",2))` the result is of the form  
 $[[[lpp\_1, basis\_1, segment\_1, lpph\_1], \dots, [lpp\_s, basis\_s, segment\_s, lpph\_s]]]$ ,  
 where each function of the basis can be given by an ideal of representants.

**Note:** The basering R, must be of the form  $\mathbb{Q}[a][x]$ , (a=parameters, x=variables), and should be defined previously. The ideal must be defined on R.  
 discussion of parametric ideal

**Example:**

```
LIB "grobpcov.lib";
// EXAMPLE
if(defined(R)){kill R;}
ring R=(0,a0,b0,c0,a1,b1,c1),(x), dp;
short=0;
ideal S=a0*x^2+b0*x+c0,
a1*x^2+b1*x+c1;
def GCS=grobpcov(S,"rep",2);
// grobpcov(S) with both P and C representations
GCS;
⇒ [1]:
⇒ [1]:
⇒ _[1]=1
⇒ [2]:
⇒ _[1]=1
⇒ [3]:
⇒ [1]:
⇒ [1]:
⇒ _[1]=0
⇒ [2]:
⇒ [1]:
⇒ _[1]=(a0^2*c1^2-a0*b0*b1*c1-2*a0*c0*a1*c1+a0*c0*b1^2+b0^2*\
a1*c1-b0*c0*a1*b1+c0^2*a1^2)
⇒ [4]:
⇒ [1]:
⇒ _[1]=0
⇒ [2]:
⇒ _[1]=(a0^2*c1^2-a0*b0*b1*c1-2*a0*c0*a1*c1+a0*c0*b1^2+b0^2*a1*c1-\
b0*c0*a1*b1+c0^2*a1^2)
⇒ [2]:
⇒ [1]:
⇒ _[1]=x
⇒ [2]:
⇒ _[1]=(b0*a1*c1-c0*a1*b1)*x+(-a0*c1^2+b0*b1*c1+c0*a1*c1-c0*b1^2)
⇒ [3]:
```

```

⇒      [1]:
⇒      [1]:
⇒      _[1]=(a0^2*c1^2-a0*b0*b1*c1-2*a0*c0*a1*c1+a0*c0*b1^2+b0^2*a1*\
c1-b0*c0*a1*b1+c0^2*a1^2)
⇒      [2]:
⇒      [1]:
⇒      _[1]=(b0*c1-c0*b1)
⇒      _[2]=(a0*c1-c0*a1)
⇒      _[3]=(a0*b1-b0*a1)
⇒      [2]:
⇒      _[1]=(a1)
⇒      _[2]=(a0)
⇒      [4]:
⇒      [1]:
⇒      _[1]=(a0^2*c1^2-a0*b0*b1*c1-2*a0*c0*a1*c1+a0*c0*b1^2+b0^2*a1*c1-\
b0*c0*a1*b1+c0^2*a1^2)
⇒      [2]:
⇒      _[1]=(-a0*c1+c0*a1)
⇒      _[2]=(-a0*b1+b0*a1)
⇒      _[3]=(-a0*b0*c1+a0*c0*b1)
⇒      [3]:
⇒      [1]:
⇒      _[1]=x^2
⇒      [2]:
⇒      _[1]=(a1)*x^2+(b1)*x+(c1)
⇒      [3]:
⇒      [1]:
⇒      [1]:
⇒      _[1]=(b0*c1-c0*b1)
⇒      _[2]=(a0*c1-c0*a1)
⇒      _[3]=(a0*b1-b0*a1)
⇒      [2]:
⇒      [1]:
⇒      _[1]=(a1)
⇒      _[2]=(b0*c1-c0*b1)
⇒      _[3]=(a0)
⇒      [4]:
⇒      [1]:
⇒      _[1]=(-b0*c1+c0*b1)
⇒      _[2]=(-a0*c1+c0*a1)
⇒      _[3]=(-a0*b1+b0*a1)
⇒      [2]:
⇒      _[1]=(a1)
⇒      _[2]=(a0)
⇒      _[3]=(-b0*c1+c0*b1)
⇒      [4]:
⇒      [1]:
⇒      _[1]=1
⇒      [2]:
⇒      _[1]=1
⇒      [3]:
⇒      [1]:
⇒      [1]:

```

```

↳          _[1]=(b1)
↳          _[2]=(a1)
↳          _[3]=(b0)
↳          _[4]=(a0)
↳      [2]:
↳          [1]:
↳              _[1]=(c1)
↳              _[2]=(b1)
↳              _[3]=(a1)
↳              _[4]=(c0)
↳              _[5]=(b0)
↳              _[6]=(a0)
↳      [4]:
↳          [1]:
↳              _[1]=(b1)
↳              _[2]=(a1)
↳              _[3]=(b0)
↳              _[4]=(a0)
↳          [2]:
↳              _[1]=(c1)
↳              _[2]=(b1)
↳              _[3]=(a1)
↳              _[4]=(c0)
↳              _[5]=(b0)
↳              _[6]=(a0)
↳      [5]:
↳          [1]:
↳              _[1]=0
↳          [2]:
↳              _[1]=0
↳          [3]:
↳              [1]:
↳                  [1]:
↳                      _[1]=(c1)
↳                      _[2]=(b1)
↳                      _[3]=(a1)
↳                      _[4]=(c0)
↳                      _[5]=(b0)
↳                      _[6]=(a0)
↳                  [2]:
↳                      [1]:
↳                          _[1]=1
↳          [4]:
↳              [1]:
↳                  _[1]=(c1)
↳                  _[2]=(b1)
↳                  _[3]=(a1)
↳                  _[4]=(c0)
↳                  _[5]=(b0)
↳                  _[6]=(a0)
↳              [2]:
↳                  _[1]=1
↳      [6]:

```

```

⇒ [1]:
⇒   _[1]=x
⇒ [2]:
⇒   _[1]=(b1)*x+(c1)
⇒ [3]:
⇒   [1]:
⇒     [1]:
⇒       _[1]=(a1)
⇒       _[2]=(b0*c1-c0*b1)
⇒       _[3]=(a0)
⇒     [2]:
⇒       [1]:
⇒         _[1]=(b1)
⇒         _[2]=(a1)
⇒         _[3]=(b0)
⇒         _[4]=(a0)
⇒   [4]:
⇒     [1]:
⇒       _[1]=(a1)
⇒       _[2]=(a0)
⇒       _[3]=(-b0*c1+c0*b1)
⇒     [2]:
⇒       _[1]=(b1)
⇒       _[2]=(a1)
⇒       _[3]=(b0)
⇒       _[4]=(a0)
⇒ [7]:
⇒ [1]:
⇒   _[1]=1
⇒ [2]:
⇒   _[1]=1
⇒ [3]:
⇒ [1]:
⇒   [1]:
⇒     _[1]=(a1)
⇒     _[2]=(a0)
⇒   [2]:
⇒     [1]:
⇒       _[1]=(a1)
⇒       _[2]=(b0*c1-c0*b1)
⇒       _[3]=(a0)
⇒ [4]:
⇒ [1]:
⇒   _[1]=(a1)
⇒   _[2]=(a0)
⇒ [2]:
⇒   _[1]=(a1)
⇒   _[2]=(a0)
⇒   _[3]=(-b0*c1+c0*b1)
def FGC=extendGC(GCS,"rep",1);
// Full representation
FGC;
⇒ [1]:

```

```

⇒ [1]:
⇒   _[1]=1
⇒ [2]:
⇒   _[1]=1
⇒ [3]:
⇒   [1]:
⇒     [1]:
⇒     _[1]=0
⇒     [2]:
⇒     [1]:
⇒     _[1]=(a0^2*c1^2-a0*b0*b1*c1-2*a0*c0*a1*c1+a0*c0*b1^2+b0^2*
a1*c1-b0*c0*a1*b1+c0^2*a1^2)
⇒ [4]:
⇒   [1]:
⇒   _[1]=0
⇒   [2]:
⇒   _[1]=(a0^2*c1^2-a0*b0*b1*c1-2*a0*c0*a1*c1+a0*c0*b1^2+b0^2*a1*c1-
b0*c0*a1*b1+c0^2*a1^2)
⇒ [2]:
⇒   [1]:
⇒   _[1]=x
⇒   [2]:
⇒   _[1]=(a0*b1-b0*a1)*x+(a0*c1-c0*a1)
⇒ [3]:
⇒   [1]:
⇒   [1]:
⇒   _[1]=(a0^2*c1^2-a0*b0*b1*c1-2*a0*c0*a1*c1+a0*c0*b1^2+b0^2*a1*
c1-b0*c0*a1*b1+c0^2*a1^2)
⇒   [2]:
⇒   [1]:
⇒   _[1]=(b0*c1-c0*b1)
⇒   _[2]=(a0*c1-c0*a1)
⇒   _[3]=(a0*b1-b0*a1)
⇒   [2]:
⇒   _[1]=(a1)
⇒   _[2]=(a0)
⇒ [4]:
⇒   [1]:
⇒   _[1]=(a0^2*c1^2-a0*b0*b1*c1-2*a0*c0*a1*c1+a0*c0*b1^2+b0^2*a1*c1-
b0*c0*a1*b1+c0^2*a1^2)
⇒   [2]:
⇒   _[1]=(-a0*c1+c0*a1)
⇒   _[2]=(-a0*b1+b0*a1)
⇒   _[3]=(-a0*b0*c1+a0*c0*b1)
⇒ [3]:
⇒   [1]:
⇒   _[1]=x^2
⇒   [2]:
⇒   [1]:
⇒   _[1]=(a1)*x^2+(b1)*x+(c1)
⇒   _[2]=(a0*a1)*x^2+(b0*a1)*x+(c0*a1)
⇒   [3]:
⇒   [1]:

```

```

⇒      [1]:
⇒      _[1]=(b0*c1-c0*b1)
⇒      _[2]=(a0*c1-c0*a1)
⇒      _[3]=(a0*b1-b0*a1)
⇒      [2]:
⇒      [1]:
⇒      _[1]=(a1)
⇒      _[2]=(b0*c1-c0*b1)
⇒      _[3]=(a0)
⇒      [4]:
⇒      [1]:
⇒      _[1]=(-b0*c1+c0*b1)
⇒      _[2]=(-a0*c1+c0*a1)
⇒      _[3]=(-a0*b1+b0*a1)
⇒      [2]:
⇒      _[1]=(a1)
⇒      _[2]=(a0)
⇒      _[3]=(-b0*c1+c0*b1)
⇒      [4]:
⇒      [1]:
⇒      _[1]=1
⇒      [2]:
⇒      _[1]=1
⇒      [3]:
⇒      [1]:
⇒      [1]:
⇒      _[1]=(b1)
⇒      _[2]=(a1)
⇒      _[3]=(b0)
⇒      _[4]=(a0)
⇒      [2]:
⇒      [1]:
⇒      _[1]=(c1)
⇒      _[2]=(b1)
⇒      _[3]=(a1)
⇒      _[4]=(c0)
⇒      _[5]=(b0)
⇒      _[6]=(a0)
⇒      [4]:
⇒      [1]:
⇒      _[1]=(b1)
⇒      _[2]=(a1)
⇒      _[3]=(b0)
⇒      _[4]=(a0)
⇒      [2]:
⇒      _[1]=(c1)
⇒      _[2]=(b1)
⇒      _[3]=(a1)
⇒      _[4]=(c0)
⇒      _[5]=(b0)
⇒      _[6]=(a0)
⇒      [5]:
⇒      [1]:

```



```

↳      _[1]=0
↳      [2]:
↳      _[1]=0
↳      [3]:
↳      [1]:
↳      [1]:
↳      _[1]=(c1)
↳      _[2]=(b1)
↳      _[3]=(a1)
↳      _[4]=(c0)
↳      _[5]=(b0)
↳      _[6]=(a0)
↳      [2]:
↳      [1]:
↳      _[1]=1
↳      [4]:
↳      [1]:
↳      _[1]=(c1)
↳      _[2]=(b1)
↳      _[3]=(a1)
↳      _[4]=(c0)
↳      _[5]=(b0)
↳      _[6]=(a0)
↳      [2]:
↳      _[1]=1
↳      [6]:
↳      [1]:
↳      _[1]=x
↳      [2]:
↳      [1]:
↳      _[1]=(b1)*x+(c1)
↳      _[2]=(b0)*x+(c0)
↳      [3]:
↳      [1]:
↳      [1]:
↳      _[1]=(a1)
↳      _[2]=(b0*c1-c0*b1)
↳      _[3]=(a0)
↳      [2]:
↳      [1]:
↳      _[1]=(b1)
↳      _[2]=(a1)
↳      _[3]=(b0)
↳      _[4]=(a0)
↳      [4]:
↳      [1]:
↳      _[1]=(a1)
↳      _[2]=(a0)
↳      _[3]=(-b0*c1+c0*b1)
↳      [2]:
↳      _[1]=(b1)
↳      _[2]=(a1)
↳      _[3]=(b0)

```

```

⇒      _[4]=(a0)
⇒ [7]:
⇒ [1]:
⇒      _[1]=1
⇒ [2]:
⇒      _[1]=1
⇒ [3]:
⇒ [1]:
⇒ [1]:
⇒      _[1]=(a1)
⇒      _[2]=(a0)
⇒ [2]:
⇒ [1]:
⇒      _[1]=(a1)
⇒      _[2]=(b0*c1-c0*b1)
⇒      _[3]=(a0)
⇒ [4]:
⇒ [1]:
⇒      _[1]=(a1)
⇒      _[2]=(a0)
⇒ [2]:
⇒      _[1]=(a1)
⇒      _[2]=(a0)
⇒      _[3]=(-b0*c1+c0*b1)

```

#### D.2.4.10 locus

Procedure from library `grobcov.lib` (see [Section D.2.4 \[grobcov.lib\]](#), page 818).

**Usage:** `locus(ideal F [,options])`

Special routine for determining the locus of points of a geometrical construction.

**Input:** The input ideal must be the ideal of the set equations defining the locus, defined in the ring

`ring Q(0,a1,...,ap,x1,...,xn)(u1,...,um,v1,...,vn),lp;`

Calling sequence:

`locus(F [,options]);`

`a`=fixed parameters, `x`=tracer variables, `u`=auxiliary variables, `v`=mover variables. The parameters `a` are optative. If they are used, then the option "`numpar\=,np`" must be declared, being `np` the number of fixed parameters. The tracer variables are `x1,...,xn`, where `n` is the dimension of the space. By default, the mover variables are the last `n` variables. Its number can be forced by the user to the last `k` variables by adding the option "`moverdim,k`".

Nevertheless, this option is recommended only to experiment, and can provide incorrect taxonomies.

The remaining variables are auxiliary.

**Options:** An option is a pair of arguments: string, integer.

To modify the default options, pairs of arguments

-option name, value- of valid options must be added to the call. The algorithm allows the following options as pair of arguments:

"`numpar`", `np` in order to consider the first `np` parameters of the ring to be fixed parameters of the locus, being the tracer variables the remaining parameters.

To be used for a parametric locus. (New in release N12).

"moverdim", k to force the mover-variables to be the last k variables. This determines the antiimage and its dimension. By default k is equal to the last n variables, We can experiment with a different value, but this can produce an error in the character "Normal" or "Special" of a locus component.

"grobcov", G, where G is the list of a previous computed grobcov(F). It is to be used when we modify externally the grobcov, for example to obtain the real grobcov.

"comments", c: by default it is 0, but it can be set to 1.

**Return:** The output is a list of the components:  
 $((p1, (p11, \dots, p1s\_1), tax\_1), \dots, (pk, (pk1, \dots, pks\_k), tax\_k))$  Elements 1 and 2 of a component represent the P-canonical form of the component.  
 The third element tax is:  
 for normal point components,  
 $tax = (d, taxonomy, anti-image)$  being  
 $d$  = dimension of the anti-image on the mover variables,  $taxonomy = "Normal"$  or  $"Special"$  and  
 $anti-image$  = ideal of the anti-image over the mover variables.  
 for non-normal point components,  
 $tax = (d, taxonomy)$  being  
 $d$  = dimension of the component and  
 $taxonomy = "Accumulation"$  or  $"Degenerate"$ .  
 The components are given in canonical P-representation. The normal locus has two kind of components:  
 Normal and Special.  
 Normal component:  
 - each point in the component has 0-dimensional anti-image.  
 - the anti-image in the mover coordinates is equal to the dimension of the component  
 Special component:  
 - each point in the component has 0-dimensional anti-image.  
 - the anti-image in the mover coordinates has dimension smaller than the dimension of the component  
 The non-normal locus has two kind of components:  
 Accumulation and Degenerate.  
 Accumulation component:  
 - each point in the component has anti-image of dimension greater than 0.  
 - the component has dimension less than  $n-1$ .  
 Degenerate components:  
 - each point in the component has anti-image of dimension greater than 0.  
 - the component has dimension  $n-1$ .  
 When a normal point component has degree greater than 9, then the taxonomy is not determined, and  $(n, 'normal', 0)$  is returned as third element of the component. ( $n$  is the dimension of the tracer space).

Given a parametric ideal  $F$  representing the system  $F$  determining the locus of points  $(x)$  which verify certain properties, the call to `locus(F)` determines the different classes of locus components, following the taxonomy defined in the book:

A. Montes. "The Groebner Cover"

A previous paper gives particular definitions for loci in 2d.

M. Abanades, F. Botana, A. Montes, T. Recio, "An Algebraic Taxonomy for Locus Computation in Dynamic Geometry", Computer-Aided Design 56 (2014) 22-33.

**Note:** The input must be the locus system.

**Example:**

```
LIB "grobpcov.lib";
// EXAMPLE 1
// Conchoid, Pascal's Limacon.
// 1. Given a circle:  $x_1^2+y_1^2-4$ 
// 2. and a mover point  $M(x_1,y_1)$  on it
// 3. Consider the fix point  $P(0,2)$  on the circle
// 4. Consider the line  $l$  passing through  $M$  and  $P$ 
// 5. The tracer  $T(x,y)$  are the points on  $l$  at fixed distance 1 to  $M$ .
if(defined(R)){kill R;}
ring R=(0,x,y),(x1,y1),dp;
short=0;
// Conchoid
ideal S96=x1^2+y1^2-4,(x-2)*x1-x*y1+2*x,(x-x1)^2+(y-y1)^2-1;
locus(S96);
=> [1]:
=> [1]:
=> _[1]=(x^4+2*x^3+x^2*y^2-3*x^2-2*x*y^2-8*x*y-6*x+2*y^2+8*y+6)
=> [2]:
=> [1]:
=> _[1]=(y^2+2*y+2)
=> _[2]=(x-y-2)
=> [3]:
=> [1]:
=> 1
=> [2]:
=> Normal
=> [3]:
=> _[1]=x1^2+y1^2-4
=> [2]:
=> [1]:
=> _[1]=(x^2+y^2-4*y+3)
=> [2]:
=> [1]:
=> _[1]=1
=> [3]:
=> [1]:
=> 0
=> [2]:
```

```

⇒      Special
⇒      [3]:
⇒      _[1]=65*y1^5-86*y1^4-180*y1^3+120*y1^2+256*y1-256
⇒      _[2]=-2015*y1^4+5916*y1^3+5700*y1^2+9216*x1-14400*y1-9088
// EXAMPLE 2
// Consider two parallel planes z1=-1 and z1=1, and two orthogonal parabolas on them
// Determine the locus generated by the lines that rely the two parabolas
// through the points having parallel tangent vectors.
if(defined(R)){kill R;}
ring R=(0,x,y,z),(x2,y2,z2,z1,y1,x1,lam), lp;
short=0;
ideal L=z1+1,
x1^2-y1,
z2-1,
y2^2-x2,
4*x1*y2-1,
x-x1-lam*(x2-x1),
y-y1-lam*(y2-y1),
z-z1-lam*(z2-z1);
locus(L); // uses "moverdim",3
⇒ [1]:
⇒ [1]:
⇒ _[1]=(2048*x^3*z+2048*x^3-4096*x^2*y^2+1152*x*y*z^2-1152*x*y-2048*y\
^3*z+2048*y^3+27*z^4-54*z^2+27)
⇒ [2]:
⇒ [1]:
⇒ _[1]=(z+1)
⇒ _[2]=(y)
⇒ [2]:
⇒ _[1]=(z-1)
⇒ _[2]=(x)
⇒ [3]:
⇒ [1]:
⇒ 2
⇒ [2]:
⇒ Normal
⇒ [3]:
⇒ _[1]=y1-x1^2
// Observe the choose of the mover variables: the last 3 variables y1,x1,lam
// If we choose x1,y1,z1 instead, the taxonomy becomes "Special" because
// z1=-1 is fix and do not really correspond to the mover variables.
// EXAMPLE 3 of parametric locus:
// Determining the equation of a general ellipse;
// Uncentered ellipse;
// Parameters (a,b,a0,b0,p):
// a=large semiaxis, b=small semiaxis,
// (a0,b0) = center of the ellipse,
// (a1,b1) and (2*a0-a1,2*b0-b1) the focus,
// p the slope of the line of the a-axis of the ellipse.
// Determine the equation of the ellipse.
// We must use the option "numpar",5 in order to consider
// the first 5 parameters as free parameters for the locus
// Auxiliary variabes:

```

```

// d1=distance from focus (a1,b1) to the mover point M(x1,y1),
// d2=distance from focus (a2,b2) to the mover point M(x1,y1),
// f=focus distance= distance from (a0,b0) to (a1,b1).
// Mover point (x1,y1) = tracer point (x,y).
if(defined(R1)){kill R1;}
ring R1=(0,a,b,a0,b0,p,x,y),(d1,d2,f,a1,b1,x1,y1),lp;
ideal F3=b1-b0-p*(a1-a0),
//b2-b0+p*(a1-a0),
//a1+a2-2*a0,
//b1+b2-2*b0,
f^2-(a1-a0)^2-(b1-b0)^2,
f^2-a^2-b^2,
(x1-a1)^2+(y1-b1)^2-d1^2,
(x1-2*a0+a1)^2+(y1-2*b0+b1)^2-d2^2,
d1+d2-2*a,
x-x1,
y-y1;
def G3=grobcov(F3);
def Loc3=locus(F3,"grobcov",G3,"numpar",5); Loc3;
⇒ [1]:
⇒ [1]:
⇒ _[1]=(a^2*b^2*p^2+a^2*b^2+a^2*a0^2*p^2-2*a^2*a0*b0*p-2*a^2*a0*p^2*x\
+2*a^2*a0*p*y+a^2*b0^2+2*a^2*b0*p*x-2*a^2*b0*y+a^2*p^2*x^2-2*a^2*p*x*y+a^2\
2*y^2-b^2*a0^2-2*b^2*a0*b0*p+2*b^2*a0*p*y+2*b^2*a0*x-b^2*b0^2*p^2+2*b^2*b\
0*p^2*y+2*b^2*b0*p*x-b^2*p^2*y^2-2*b^2*p*x*y-b^2*x^2)
⇒ [2]:
⇒ [1]:
⇒ _[1]=(p^2+1)
⇒ _[2]=(a0+b0*p-p*y-x)
⇒ [3]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ Normal
⇒ [3]:
⇒ _[1]=(a^2*p^2-b^2)*x1^2+(-2*a^2*p-2*b^2*p)*x1*y1+(-2*a^2*a0*p^2+\
2*a^2*b0*p+2*b^2*a0+2*b^2*b0*p)*x1+(a^2-b^2*p^2)*y1^2+(2*a^2*a0*p-2*a^2*b\
0+2*b^2*a0*p+2*b^2*b0*p^2)*y1+(a^2*b^2*p^2+a^2*b^2+a^2*a0^2*p^2-2*a^2*a0*\
b0*p+a^2*b0^2-b^2*a0^2-2*b^2*a0*b0*p-b^2*b0^2*p^2)
⇒ [2]:
⇒ [1]:
⇒ _[1]=(p^2+1)
⇒ _[2]=(a0+b0*p-p*y-x)
⇒ _[3]=(b)
⇒ _[4]=(a)
⇒ [2]:
⇒ [1]:
⇒ _[1]=1
⇒ [3]:
⇒ [1]:
⇒ -1
⇒ [2]:
⇒ Accumulation

```

```

// General ellipse:
def C=Loc3[1][1][1];
C;
⇒ (a^2*b^2*p^2+a^2*b^2+a^2*a0^2*p^2-2*a^2*a0*b0*p-2*a^2*a0*p^2*x+2*a^2*a0*p\
*y+a^2*b0^2+2*a^2*b0*p*x-2*a^2*b0*y+a^2*p^2*x^2-2*a^2*p*x*y+a^2*y^2-b^2*a\
0^2-2*b^2*a0*b0*p+2*b^2*a0*p*y+2*b^2*a0*x-b^2*b0^2*p^2+2*b^2*b0*p^2*y+2*b\
^2*b0*p*x-b^2*p^2*y^2-2*b^2*p*x*y-b^2*x^2)
// Centered ellipse of semiaxes (a,b):
def C0=subst(C,a0,0,b0,0,p,0);
C0;
⇒ (a^2*b^2+a^2*y^2-b^2*x^2)

```

### D.2.4.11 locusdg

Procedure from library `grobcov.lib` (see [Section D.2.4 \[grobcov.lib\]](#), page 818).

**Usage:**      locusdg(list L)  
                 Calling sequence:  
                 locusdg(locus(S)).

**Return:**     The output is the list of the "Relevant" components of the locus in Dynamic Geometry  
                  $[C_1, \dots, C_m]$ , where  
                  $C_i = [p_i, [p_{i1}, \dots, p_{i s_i}], \text{"Relevant"}, \text{level}_i]$   
                 The "Relevant" components are "Normal" and  
                 "Accumulation" components of the locus. (See help  
                 for locus).

**Example:**

```

LIB "grobcov.lib";
if(defined(R)){kill R;};
ring R=(0,a,b),(x,y),dp;
short=0;
// Concoind
ideal S96=x^2+y^2-4,(b-2)*x-a*y+2*a,(a-x)^2+(b-y)^2-1;
def L96=locus(S96);
L96;
⇒ [1]:
⇒ [1]:
⇒ _[1]=(a^4+2*a^2*b^2-9*a^2+b^4-9*b^2+4*b+12)
⇒ [2]:
⇒ [1]:
⇒ _[1]=1
⇒ [3]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ Normal
⇒ [3]:
⇒ _[1]=x^2+y^2-4
⇒ [2]:
⇒ [1]:
⇒ _[1]=(a^2+b^2-4*b+3)
⇒ [2]:
⇒ [1]:

```

```

⇒      _[1]=1
⇒      [3]:
⇒      [1]:
⇒      0
⇒      [2]:
⇒      Special
⇒      [3]:
⇒      _[1]=y^2-3*y+2
⇒      _[2]=x*y-x
⇒      _[3]=x^2+3*y-6
locusdg(L96);
⇒ [1]:
⇒ [1]:
⇒ _[1]=(a^4+2*a^2*b^2-9*a^2+b^4-9*b^2+4*b+12)
⇒ [2]:
⇒ [1]:
⇒ _[1]=1
⇒ [3]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ Relevant
⇒ [3]:
⇒ _[1]=x^2+y^2-4

```

#### D.2.4.12 envelop

Procedure from library `grobcov.lib` (see [Section D.2.4 \[grobcov.lib\]](#), page 818).

**Usage:**    `envelop(poly F,ideal C,[options]);`  
poly F must represent the family of hyper-surfaces for which on want to compute its envelop. ideal C must be the ideal of restrictions on the variables defining the family, and should contain less polynomials than the number of variables.  $(x_1, \dots, x_n)$  are the variables of the hyper-surfaces of F, that are considered as parameters of the parametric ring.  $(u_1, \dots, u_m)$  are the parameteres of the hyper-surfaces, that are considered as variables of the parametric ring.  
In the actual version, parametric envelope are accepted. To include fixed parameters  $a_1, \dots, a_p$ , to the problem, one must declare them as the first parameters of the ring. if the the number of free parameters is p, the option "numpar",p is required.  
Calling sequence:  
`ring R=(0,a1,...,ap,x1,...,xn),(u1,...,um),lp;`  
`poly F=F(a1,...,ap,x1,...,xn,u1,...,um);`  
`ideal C=g1(a1,...,ap,u1,...,um),...,gs(a1,...,ap,u1,...,um); envelop(F,C,[options]);` where  $s < m$ .  
 $x_1, \dots, x_n$  are the tracer variables.  
 $u_1, \dots, u_m$  are the auxiliary variables.  
 $a_1, \dots, a_p$  are the fixed parameters if they exist  
If the problem is a parametric envelope, and a's exist, then the option "numpar",p m must be given.



By default the last  $n$  variables are the mover variables. See the EXAMPLE of parametric envelop by calling  
 example envelop,

**Return:** The output is a list of the components  $[C_1, \dots, C_n]$  of the locus. Each component is given by  
 $C_i = [p_i, [p_{i1}, \dots, p_{is_i}], \text{tax}]$  where  
 $p_i, [p_{i1}, \dots, p_{is_i}]$  is the canonical P-representation of the component.  
 Concerning tax: (see help for locus)  
 For normal-point components is  
 $\text{tax} = [d, \text{taxonomy}, \text{anti-image}]$ , being  
 $d$  = dimension of the anti-image  
 $\text{taxonomy} = \text{"Normal"}$  or  $\text{"Special"}$   
 $\text{anti-image}$  = values of the mover corresponding to the component  
 For non-normal-point components is  
 $\text{tax} = [d, \text{taxonomy}]$   
 $d$  = dimension of the component  
 $\text{taxonomy} = \text{"Accumulation"}$  or  $\text{"Degenerate"}$ .

**Options:** An option is a pair of arguments: string, integer.  
 To modify the default options,  
 pairs of arguments -option name, value- of valid options must be added to the call.  
 The algorithm allows the following options as pair of arguments: "comments", c: by default it is 0, but it can be set to 1. "anti-image", a: by default  $a=1$  and the anti-image is shown also for "Normal" components.  
 For  $a=0$ , it is not shown.  
 "moverdim", k: by default it is equal to  $n$ , the number of x-tracer variables.  
 "numpar", p when fixed parameters are included

**Note:** grobcov and locus are called internally.  
 The basering R, must be of the form  $\mathbb{Q}[a, x][u]$   
 ( $x$ =variables,  $u$ =auxiliary variables), (a fixed parameters). This routine uses the generalized definition of envelop introduced in the book  
 A. Montes. "The Groebner Cover" (Discussing Parametric Polynomial Systems) not yet published.

**Example:**

```
LIB "grobcov.lib";
"EXAMPLE:";
⇨ EXAMPLE:
// EXAMPLE 1
// Steiner Deltoid
// 1. Consider the circle  $x_1^2 + y_1^2 - 1 = 0$ , and a mover point  $M(x_1, y_1)$  on it.
// 2. Consider the triangle  $A(0, 1)$ ,  $B(-1, 0)$ ,  $C(1, 0)$ .
// 3. Consider lines passing through  $M$  perpendicular to two sides of ABC triangle.
// 4. Determine the envelope of the lines above.
if(defined(R)){kill R;}
ring R=(0,x,y),(x1,y1,x2,y2),lp;
short=0;
ideal C=(x1)^2+(y1)^2-1,
x2+y2-1,
x2-y2-x1+y1;
```

```

matrix M[3][3]=x,y,1,x2,y2,1,x1,0,1;
poly F=det(M);
// The lines of family F are
F;
⇒ -x1*y2+(y)*x1+(-y)*x2+(x)*y2
// The conditions C are
C;
⇒ C[1]=x1^2+y1^2-1
⇒ C[2]=x2+y2-1
⇒ C[3]=-x1+y1+x2-y2
envelop(F,C);
⇒ [1]:
⇒ [1]:
⇒ _[1]=(x^4+2*x^2*y^2+10*x^2*y-x^2+y^4-6*y^3+12*y^2-8*y)
⇒ [2]:
⇒ [1]:
⇒ _[1]=1
⇒ [3]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ Normal
⇒ [3]:
⇒ _[1]=x2+y2-1
⇒ [2]:
⇒ [1]:
⇒ _[1]=(x+y-1)
⇒ [2]:
⇒ [1]:
⇒ _[1]=1
⇒ [3]:
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ Special
⇒ [3]:
⇒ _[1]=4*y2^4-4*y2^3-y2^2
⇒ _[2]=x2+y2-1
// EXAMPLE 2
// Parametric envelope
// Let c be the circle centered at the origin 0(0,0) and having radius 1.
// M(x1,y1) be a mover point gliding on c.
// Let A(a0,b0) be a parametric fixed point:
// Consider the set of lines parallel to the line A0 passing thoug M.
// Determine the envelope of these lines
// We let the fixed point A coordinates as free parameters of the envelope.
// We have to declare the existence of two parameters when
// defining the ring in which we call envelop,
// and set a0,b0 as the first variables of the parametric ring
// The ring is thus
if(defined(R1)){kill R1;}
ring R1=(0,a0,b0,x,y),(x1,y1),lp;
short=0;

```

```

// The lines are F1
poly F1=b0*(x-x1)-a0*(y-y1);
// and the mover is on the circle c
ideal C1=x1^2+y1^2-1;
// The call is thus
def E1=envelop(F1,C1,"numpar",2);
E1;
⇒ [1]:
⇒ [1]:
⇒ _[1]=(a0^2*y^2-a0^2-2*a0*b0*x*y+b0^2*x^2-b0^2)
⇒ [2]:
⇒ [1]:
⇒ _[1]=(x^2+y^2)
⇒ _[2]=(a0*y-b0*x)
⇒ _[3]=(a0*x+b0*y)
⇒ _[4]=(a0^2+b0^2)
⇒ [2]:
⇒ _[1]=(b0)
⇒ _[2]=(a0)
⇒ [3]:
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ Special
⇒ [3]:
⇒ _[1]=(a0^2+b0^2)*y1^2+(-a0^2)
⇒ _[2]=(a0)*x1+(b0)*y1
⇒ [2]:
⇒ [1]:
⇒ _[1]=(b0)
⇒ _[2]=(a0)
⇒ [2]:
⇒ [1]:
⇒ _[1]=1
⇒ [3]:
⇒ [1]:
⇒ -1
⇒ [2]:
⇒ Accumulation
// The interesting first component EC1 is
def EC1=E1[1][1][1];
EC1;
⇒ (a0^2*y^2-a0^2-2*a0*b0*x*y+b0^2*x^2-b0^2)
// that is equivalent to (a0*y-b0*x)^2-a0^2-b0^2.
// As expected it consists of the two lines
// a0*y-b0*x - sqrt(a0^2+b0^2),
// a0*y-b0*x + sqrt(a0^2+b0^2),
// parallel to the line OM passing at the
// points of the circle in the line perpendicular to OA.
// EXAMPLE 3
// Parametric envelope
// Let c be the circle centered at the origin O(a1,b1) and having radiusr,
// where a1,b1,r are fixed parameters

```

```

// M(x1,y1) be a mover point gliding on c.
// Let A(a0,b0) be a parametric fixed point:
// Consider the set of lines parallel to the line A0 passing though M.
// Determine the envelope of these lines
// We let the fixed point A, point M and r as free parameters of the envelope.
// We have to declare the existence of 5 parameters when
// defining the ring in which we call envelop,
// and set a0,b0,a1,b1,r as the first variables of the parametric ring
// The ring is thus
if(defined(R1)){kill R1;}
ring R1=(0,a0,b0,a1,b1,r,x,y),(x1,y1),lp;
short=0;
// The lines are F1
poly F1=b0*(x-x1)-a0*(y-y1);
// and the mover is on the circle c
ideal C1=(x1-a1)^2+(y1-b1)^2-r^2;
// The call is thus
def E1=envelop(F1,C1,"numpar",5);
E1;
⇒ [1]:
⇒ [1]:
⇒ _[1]=(a0^2*b1^2-2*a0^2*b1*y-a0^2*r^2+a0^2*y^2-2*a0*b0*a1*b1+2*a0*b0\
*a1*y+2*a0*b0*b1*x-2*a0*b0*x*y+b0^2*a1^2-2*b0^2*a1*x-b0^2*r^2+b0^2*x^2)
⇒ [2]:
⇒ [1]:
⇒ _[1]=(a1^2-2*a1*x+b1^2-2*b1*y+x^2+y^2)
⇒ _[2]=(a0*b1-a0*y-b0*a1+b0*x)
⇒ _[3]=(a0*a1-a0*x+b0*b1-b0*y)
⇒ _[4]=(a0^2+b0^2)
⇒ [2]:
⇒ _[1]=(b0)
⇒ _[2]=(a0)
⇒ [3]:
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ Special
⇒ [3]:
⇒ _[1]=(a0^2+b0^2)*y1^2+(-2*a0^2*b1-2*b0^2*b1)*y1+(a0^2*b1^2-a0^2*\
r^2+b0^2*b1^2)
⇒ _[2]=(a0)*x1+(b0)*y1+(-a0*a1-b0*b1)
⇒ [2]:
⇒ [1]:
⇒ _[1]=(b0)
⇒ _[2]=(a0)
⇒ [2]:
⇒ [1]:
⇒ _[1]=1
⇒ [3]:
⇒ [1]:
⇒ -1
⇒ [2]:
⇒ Accumulation

```

```

⇒ [3]:
⇒ [1]:
⇒ [1]=(r)
⇒ [2]=(a1^2-2*a1*x+b1^2-2*b1*y+x^2+y^2)
⇒ [3]=(a0*b1-a0*y-b0*a1+b0*x)
⇒ [4]=(a0*a1-a0*x+b0*b1-b0*y)
⇒ [5]=(a0^2+b0^2)
⇒ [2]:
⇒ [1]:
⇒ [1]=1
⇒ [3]:
⇒ [1]:
⇒ -1
⇒ [2]:
⇒ Accumulation
// The interesting first component EC1 is
def EC1=E1[1][1][1];
EC1;
⇒ (a0^2*b1^2-2*a0^2*b1*y-a0^2*r^2+a0^2*y^2-2*a0*b0*a1*b1+2*a0*b0*a1*y+2*a0*\
b0*b1*x-2*a0*b0*x*y+b0^2*a1^2-2*b0^2*a1*x-b0^2*r^2+b0^2*x^2)
// which corresponds to the product of two lines
// parallel to the line AM and intercepting the circle
// on the intersection of the line perpendicuar
// to line AM passing through A

```

### D.2.4.13 locusto

Procedure from library `grobcov.lib` (see [Section D.2.4 \[grobcov.lib\]](#), page 818).

**Usage:** `locusto(list L);`  
The argument must be the output of `locus` or `locusdg` or `envelop`. It transforms the output into a string in standard form readable in other languages, not only Singular (Geogebra).

**Return:** The locus in string standard form

**Note:** It can only be called after computing either  
- `locus(F) -> locusto( locus(F) )`  
- `locusdg(locus(F)) -> locusto( locusdg(locus(F)) )`  
- `envelop(F,C) -> locusto( envelop(F,C) )`

**Example:**

```

LIB "grobcov.lib";
if(defined(R)){kill R;}
ring R=(0,x,y),(x1,y1),dp;
short=0;
ideal S=x1^2+y1^2-4,(y-2)*x1-x*y1+2*x,(x-x1)^2+(y-y1)^2-1;
def L=locus(S);
locusto(L);
⇒ [[[(x^4+2*x^2*y^2-9*x^2+y^4-9*y^2+4*y+12)],[[1]]],[[1],[Normal],[x1^2+y1^2-4]]],[[[x^2+y^2-4*y+3]],[[1]]],[[0],[Special],[y1^2-3*y1+2,x1*y1-x1,x1^2+3*y1-6]]]]
locusto(locusdg(L));
⇒ [[[(x^4+2*x^2*y^2-9*x^2+y^4-9*y^2+4*y+12)],[[1]]],[[1],[Relevant],[x1^2+y1^2-4]]]]

```

### D.2.4.14 stdlocus

Procedure from library `grobcov.lib` (see [Section D.2.4 \[grobcov.lib\]](#), page 818).

- Usage:** `stdlocus(ideal F)`  
 The input ideal must be the set equations defining the locus. Calling sequence:  
`locus(F)`;  
 The input ring must be a parametrical ideal in  $\mathbb{Q}[x][u]$ , ( $x$ =tracer variables,  
 $u$ =remaining variables).  
 (Inverts the concept of parameters and variables of the ring). Special routine for  
 determining the locus of points of a geometrical construction. Given a parametric  
 ideal  $F$  representing the system determining the locus of points ( $x$ ) which verify certain  
 properties, the call to `stdlocus(F)` determines the different irreducible components of  
 the locus. This is a simple routine, using only standard Groebner basis computation,  
 elimination and prime decomposition instead of using `grobcov`. It does not determine  
 the taxonomy, nor the holes of the components
- Return:** The output is a list of the tops of the components  $[C_1, \dots, C_n]$  of the locus. Each  
 component is given its top ideal  $p_i$ .
- Note:** The input must be the locus system.

**Example:**

```
LIB "grobcov.lib";
if(defined(R)){kill R;}
ring R=(0,x,y),(x1,y1),dp;
short=0;
// Concoind
ideal S96=x1^2+y1^2-4,(x-2)*x1-x*y1+2*x,(x-x1)^2+(y-y1)^2-1;
stdlocus(S96);
⇒ [1]:
⇒ _[1]=(x^4+2*x^3+x^2*y^2-3*x^2-2*x*y^2-8*x*y-6*x+2*y^2+8*y+6)
⇒ [2]:
⇒ _[1]=(x^2+y^2-4*y+3)
```

### D.2.4.15 AssocTanToEnv

Procedure from library `grobcov.lib` (see [Section D.2.4 \[grobcov.lib\]](#), page 818).

- Usage:** `AssocTanToEnv(poly F,ideal C,ideal E)`;  
 $\text{poly } F$  must be the family of hyper-surfaces whose  
 envelope is analyzed. It must be defined in the ring  
 $R=\mathbb{Q}[x_1, \dots, x_n][u_1, \dots, u_m]$ ,  
 ideal  $C$  must be the ideal of restrictions  
 in the variables  $u_1, \dots, u_m$  for defining the family.  
 $C$  must contain less polynomials than  $m$ .  
 ideal  $E$  must be a component of  
 $\text{envelop}(F,C)$ , previously computed.  
 $(x_1, \dots, x_n)$  are the variables of the hypersurfaces  
 of  $F$ , that are considered as parameters of the  
 parametric ring.  $(u_1, \dots, u_m)$  are the parameteres  
 of the hyper-surfaces, that are considered as variables of the parametric ring. Having  
 computed an envelop  
 component  $E$  of a family of hyper-surfaces  $F$ ,

with constraints  $C$ , it returns the parameter values of the associated tangent hyper-surface of the family passing at one point of the envelop component  $E$ . Calling sequence:  $(s < m)$

```

ring R=(0,x_1,...,x_n),(u_1,...,u_m),lp;
poly F=F(x_1,...,x_n,u_1,...,u_m);
ideal C=g_1(u_1,...,u_m),...,g_s(u_1,...,u_m);
poly E(x_1,...,x_n);
AssocTanToEnv(F,C,E,[options]);

```

**Return:** list [lpp,basis,segment]. The basis determines the associated tangent hyper-surface at a point of the envelop component  $E$ . The segment is given in Prep. See book A. Montes. "The Groebner Cover":

**Options:** "moreinfo",  $n$   $n=0$  is the default option, and only the segment of the top of the component is shown.  $n=1$  makes the result to shown all the segments.

**Note:** grobcov is called internally.

**Example:**

```

LIB "grobcov.lib";
if(defined(R)){kill R;}
ring R=(0,x,y),(r,s,y1,x1),lp;
poly F=(x-x1)^2+(y-y1)^2-r;
ideal g=(x1-2*(s+r))^2+(y1-s)^2-s;
def E=envelop(F,g);
E;
⇒ [1]:
⇒ [1]:
⇒ _[1]=(512*x^3-1024*x^2*y^2-2560*x^2*y-640*x^2+4096*x*y^3+4864*x*y^2\
-704*x*y+984*x-4096*y^4+1536*y^3-16*y^2+144*y+289)
⇒ [2]:
⇒ [1]:
⇒ _[1]=1
⇒ [3]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ Normal
⇒ [3]:
⇒ _[1]=1024*y1^4-1024*y1^3*x1+896*y1^3+256*y1^2*x1^2-1344*y1^2*x1+\
132*y1^2+128*y1*x1^2-560*y1*x1-28*y1+16*x1^2-72*x1+1
def A=AssocTanToEnv(F,g,E[1][1][1]);
A;
⇒ [1]:
⇒ _[1]=x1
⇒ _[2]=y1
⇒ _[3]=s
⇒ _[4]=r
⇒ [2]:
⇒ _[1]=4*x1+(-4*x-1)
⇒ _[2]=(1024*y^3-768*y^2+192*y-556)*y1+(192*x^2+128*x*y^2-832*x*y-136*x-\
1024*y^4+1024*y^3+272*y^2+48*y-29)

```

```

⇒ _[3]=(5120*y^3-3840*y^2+960*y-2780)*s+(1024*x^2*y+704*x^2-2048*x*y^3-1\
920*x*y^2-4288*x*y+984*x-1024*y^4+4352*y^3-816*y^2-880*y+357)
⇒ _[4]=(2048*y^3-1536*y^2+384*y-1112)*r+(-512*x^2*y-256*x^2+1792*x*y^2+1\
536*x*y-4*x-2176*y^3+928*y^2+368*y+85)
⇒ [3]:
⇒ [1]:
⇒ [1]:
⇒ _[1]=(512*x^3-1024*x^2*y^2-2560*x^2*y-640*x^2+4096*x*y^3+4864*x*\
y^2-704*x*y+984*x-4096*y^4+1536*y^3-16*y^2+144*y+289)
⇒ [2]:
⇒ [1]:
⇒ _[1]=(256*y^3-192*y^2+48*y-139)
⇒ _[2]=(24*x-64*y^2-16*y-13)
def M1=coef(A[2][1],x1);
def M2=coef(A[2][2],y1);
def M3=coef(A[2][3],s);
def M4=coef(A[2][4],r);
"x1=";-M1[2,2]/M1[2,1];
⇒ x1=
⇒ (4*x+1)/4
"y1=";-M2[2,2]/M2[2,1];
⇒ y1=
⇒ (-192*x^2-128*x*y^2+832*x*y+136*x+1024*y^4-1024*y^3-272*y^2-48*y+29)/(102\
4*y^3-768*y^2+192*y-556)
"s=";-M3[2,2]/M3[2,1];
⇒ s=
⇒ (-1024*x^2*y-704*x^2+2048*x*y^3+1920*x*y^2+4288*x*y-984*x+1024*y^4-4352*y\
^3+816*y^2+880*y-357)/(5120*y^3-3840*y^2+960*y-2780)
"r=";-M4[2,2]/M4[2,1];
⇒ r=
⇒ (512*x^2*y+256*x^2-1792*x*y^2-1536*x*y+4*x+2176*y^3-928*y^2-368*y-85)/(20\
48*y^3-1536*y^2+384*y-1112)

```

#### D.2.4.16 FamElemsAtEnvCompPoints

Procedure from library `grobpcov.lib` (see [Section D.2.4 \[grobpcov.lib\]](#), page 818).

**Usage:** FamElemsAtEnvCompPoints(poly F,ideal C,poly E);  
poly F must be the family of hyper-surfaces whose  
envelope is analyzed. It must be defined in the ring  
 $R=Q[x_1, \dots, x_n][u_1, \dots, u_m]$ ,  
ideal C must be the ideal of restrictions on the  
variables  $u_1, \dots, u_m$ .  
Must contain less polynomials than m.  
ideal E must be a component of  
envelop(F,C), previously computed.  
After computing the envelop of a family of  
hyper-surfaces F, with constraints C,  
Consider a component with top E. The call to  
FamElemsAtEnvCompPoints(F,C,E)  
returns the parameter values of the  
set of all hyper-surfaces of the family passing at  
one point of the envelop component E.



Calling sequence:

```
ring R=(0,x_1,...,x_n),(u_1,...,u_m),lp;
poly F=F(x_1,...,x_n,u_1,...,u_m);
ideal C=g_1(u_1,...,u_m),...,g_s(u_1,...,u_m);
poly E(x_1,...,x_n);
FamElemsAtEnvCompPoints(F,C,E[,options]);
```

**Return:** list [lpp,basis,segment]. The basis determines the parameter values of the of hyper-surfaces that pass at a fixed point of the envelop component E. The lpp determines the dimension of the set. The segment is the component and is given in Prep. Fixing the values of  $(x_1, \dots, x_n)$  inside E, the basis allows to determine the values of the parameters  $(u_1, \dots, u_m)$ , of the hyper-surfaces passing at a point of E. See the book A. Montes. "The Groebner Cover" (Discussing Parametric Polynomial Systems).

**Options:** "moreinfo", n n=0 is the default option, and only the segment of the top of the component is shown. n=1 makes the result to shown all the segments.

**Note:** grobcov is called internally.  
The basering R, must be of the form  $\mathbb{Q}[a][x]$  (a=parameters, x=variables).

**Example:**

```
LIB "grobcov.lib";
if(defined(R)){kill R;}
ring R=(0,x,y),(t),dp;
short=0;
poly F=(x-5*t)^2+y^2-9*t^2;
ideal C;
def Env=envelop(F,C);
Env;
⇒ [1]:
⇒ [1]:
⇒ _[1]=(3*x+4*y)
⇒ [2]:
⇒ [1]:
⇒ _[1]=1
⇒ [3]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ Normal
⇒ [3]:
⇒ _[1]=0
⇒ [2]:
⇒ [1]:
⇒ _[1]=(3*x-4*y)
⇒ [2]:
⇒ [1]:
⇒ _[1]=1
```

```

⇒      [3]:
⇒      [1]:
⇒      1
⇒      [2]:
⇒      Normal
⇒      [3]:
⇒      _[1]=0
// E is a component of the envelope:
def E=Env[1][1][1];
E;
⇒ (3*x+4*y)
def A=AssocTanToEnv(F,C,E);
A;
⇒ [1]:
⇒ _[1]=t
⇒ [2]:
⇒ _[1]=12*t+(5*y)
⇒ [3]:
⇒ [1]:
⇒ [1]:
⇒ _[1]=(3*x+4*y)
⇒ [2]:
⇒ [1]:
⇒ _[1]=1
// The basis of the parameter values of the associated
// tangent component is
A[2][1];
⇒ 12*t+(5*y)
// Thus  $t=-(5/12)*y$ , and the associated tangent family
// element at (x,y) is
subst(F,t,-(5/12)*y);
⇒ (18*x^2+75*x*y+68*y^2)/18
def FE=FamElemsAtEnvCompPoints(F,C,E);
FE;
⇒ [1]:
⇒ _[1]=t^2
⇒ [2]:
⇒ _[1]=144*t^2+(120*y)*t+(25*y^2)
⇒ [3]:
⇒ [1]:
⇒ [1]:
⇒ _[1]=(3*x+4*y)
⇒ [2]:
⇒ [1]:
⇒ _[1]=1
factorize(FE[2][1]);
⇒ [1]:
⇒ _[1]=1
⇒ _[2]=12*t+(5*y)
⇒ [2]:
⇒ 1,2
// Thus the unique family element passing through the envelope point (x,y)
// corresponds to the value of t of the Associated Tangent

```

```

// EXAMPLE:
if(defined(R)){kill R;}
ring R=(0,x,y),(r,s,y1,x1),lp;
poly F=(x-x1)^2+(y-y1)^2-r;
ideal g=(x1-2*(s+r))^2+(y1-s)^2-s;
def E=envelop(F,g);
E;
⇒ [1]:
⇒ [1]:
⇒ _[1]=(512*x^3-1024*x^2*y^2-2560*x^2*y-640*x^2+4096*x*y^3+4864*x*y^2\
-704*x*y+984*x-4096*y^4+1536*y^3-16*y^2+144*y+289)
⇒ [2]:
⇒ [1]:
⇒ _[1]=1
⇒ [3]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ Normal
⇒ [3]:
⇒ _[1]=1024*y1^4-1024*y1^3*x1+896*y1^3+256*y1^2*x1^2-1344*y1^2*x1+\
132*y1^2+128*y1*x1^2-560*y1*x1-28*y1+16*x1^2-72*x1+1
def A=AssocTanToEnv(F,g,E[1][1][1]);
A;
⇒ [1]:
⇒ _[1]=x1
⇒ _[2]=y1
⇒ _[3]=s
⇒ _[4]=r
⇒ [2]:
⇒ _[1]=4*x1+(-4*x-1)
⇒ _[2]=(1024*y^3-768*y^2+192*y-556)*y1+(192*x^2+128*x*y^2-832*x*y-136*x-\
1024*y^4+1024*y^3+272*y^2+48*y-29)
⇒ _[3]=(5120*y^3-3840*y^2+960*y-2780)*s+(1024*x^2*y+704*x^2-2048*x*y^3-1\
920*x*y^2-4288*x*y+984*x-1024*y^4+4352*y^3-816*y^2-880*y+357)
⇒ _[4]=(2048*y^3-1536*y^2+384*y-1112)*r+(-512*x^2*y-256*x^2+1792*x*y^2+1\
536*x*y-4*x-2176*y^3+928*y^2+368*y+85)
⇒ [3]:
⇒ [1]:
⇒ [1]:
⇒ _[1]=(512*x^3-1024*x^2*y^2-2560*x^2*y-640*x^2+4096*x*y^3+4864*x*\
y^2-704*x*y+984*x-4096*y^4+1536*y^3-16*y^2+144*y+289)
⇒ [2]:
⇒ [1]:
⇒ _[1]=(256*y^3-192*y^2+48*y-139)
⇒ _[2]=(24*x-64*y^2-16*y-13)
def M1=coef(A[2][1],x1);
def M2=coef(A[2][2],y1);
def M3=coef(A[2][3],s);
def M4=coef(A[2][4],r);
// The parameter values corresponding to the family
// element tangent at point (x,y) of the envelope are:
"x1=";-M1[2,2]/M1[2,1];

```

```

⇒ x1=
⇒ (4*x+1)/4
"y1=";-M2[2,2]/M2[2,1];
⇒ y1=
⇒ (-192*x^2-128*x*y^2+832*x*y+136*x+1024*y^4-1024*y^3-272*y^2-48*y+29)/(102\
4*y^3-768*y^2+192*y-556)
"s=";-M3[2,2]/M3[2,1];
⇒ s=
⇒ (-1024*x^2*y-704*x^2+2048*x*y^3+1920*x*y^2+4288*x*y-984*x+1024*y^4-4352*y\
^3+816*y^2+880*y-357)/(5120*y^3-3840*y^2+960*y-2780)
"r=";-M4[2,2]/M4[2,1];
⇒ r=
⇒ (512*x^2*y+256*x^2-1792*x*y^2-1536*x*y+4*x+2176*y^3-928*y^2-368*y-85)/(20\
48*y^3-1536*y^2+384*y-1112)
// Now detect if there are other family elements passing at this point:
def FE=FamElemsAtEnvCompPoints(F,g,E[1][1][1]);
FE;
⇒ [1]:
⇒ _[1]=s^2
⇒ _[2]=r
⇒ [2]:
⇒ _[1]=2560*s^2+4096*s*y1^2+(-8192*y-1024)*s*y1+4096*s*x1^2+(-8192*x-204\
8)*s*x1+(4096*x^2+4096*y^2-512)*s+2048*y1^4+(-8192*y)*y1^3+4096*y1^2*x1^2\
+(-8192*x-2048)*y1^2*x1+(4096*x^2+12288*y^2+512)*y1^2+(-8192*y)*y1*x1^2+(\
16384*x*y+4096*y)*y1*x1+(-8192*x^2*y-8192*y^3)*y1+2048*x1^4+(-8192*x-2048\
)*x1^3+(12288*x^2+4096*x+4096*y^2+512)*x1^2+(-16384*x^2*y^2-40960*x^2*y-1\
2288*x^2+65536*x*y^3+69632*x*y^2-11264*x*y+15744*x-65536*y^4+24576*y^3-23\
04*y^2+2304*y+4624)*x1+(8192*x^2*y^4+24576*x^2*y^3+46080*x^2*y^2+28416*x\
^2*y-736*x^2-32768*x*y^5-104448*x*y^4-118272*x*y^3-18048*x*y^2-16736*x*y-6\
076*x+32768*y^6+69632*y^5-8064*y^4-8512*y^3-5112*y^2-6500*y-1445)
⇒ _[2]=r-y1^2+(2*y)*y1-x1^2+(2*x)*x1+(-x^2-y^2)
⇒ [3]:
⇒ [1]:
⇒ [1]:
⇒ _[1]=(512*x^3-1024*x^2*y^2-2560*x^2*y-640*x^2+4096*x*y^3+4864*x*\
y^2-704*x*y+984*x-4096*y^4+1536*y^3-16*y^2+144*y+289)
⇒ [2]:
⇒ [1]:
⇒ _[1]=1
// FE[1] is the set of lpp. It has dimension 4-2=2.
// Thus there are points of the envelope at which
// they pass infinitely many circles of the family.
// To separate the points of the envelope further analysis must be done.

```

### D.2.4.17 discrim

Procedure from library `grobccov.lib` (see [Section D.2.4 \[grobccov.lib\]](#), page 818).

**Usage:**      `discrim(f,x);`  
               poly f: the polynomial in  $\mathbb{Q}[a][x]$  or  $\mathbb{Q}[x]$  of degree 2 in x  
               poly x: can be a variable or a parameter of the ring.

**Return:**     the factorized discriminant of f wrt x for discussing its sign

**Example:**

```

LIB "grobcov.lib";
if(defined(R)){kill R;}
ring R=(0,a,b,c),(x,y),dp;
short=0;
poly f=a*x^2*y+b*x*y+c*y;
discrim(f,x);
⇒ [1]:
⇒   _[1]=-1
⇒   _[2]=(4*a*c-b^2)
⇒   _[3]=y
⇒ [2]:
⇒   1,1,2

```

### D.2.4.18 WLemma

Procedure from library `grobcov.lib` (see [Section D.2.4 \[grobcov.lib\]](#), page 818).

**Usage:** `WLemma(F,A[,options]);`  
The first argument ideal  $F$  in  $\mathbb{Q}[x_1, \dots, x_n][u_1, \dots, u_m]$ ; The second argument ideal  $A$  in  $\mathbb{Q}[x_1, \dots, x_n]$ .  
Calling sequence:  
`ring R=(0,x_1,...,x_n),(u_1,...,u_m),lp;`  
`ideal F=f_1(x_1,...,x_n,u_1,...,u_m),...`  
`f_s(x_1,...,x_n,u_1,...,u_m);`  
`ideal A=g_1(u_1,...,u_m),...,g_s(u_1,...,u_m);`  
`list # : Options`  
Calling sequence:  
`WLemma(F,A[,options]);`  
Given the ideal  $F$  and ideal  $A$   
it returns the list  $(lpp, B, S)$  where  $B$  is the  
reduced Groebner basis of the specialized  $F$  over  
the segment  $S$ , subset of  $V(A)$  with top  $A$ ,  
determined by Wibmer's Lemma.  
 $S$  is determined in  $P$ -representation  
(or optionally in  $C$ -representation). The basis is  
given by  $I$ -regular functions.

**Options:** either `("rep", 0)` or `("rep", 1)` the representation of the resulting segment, by default is  
`0 = P-representation`, (default) but can be set to  
`1 = C-representation`.

**Return:** list of `[lpp, B, S] =`  
`[leading power product, basis, segment]`,  
being  $B$  the reduced Groebner Basis given by  
 $I$ -regular functions in full representation, of  
the specialized ideal  $F$  on the segment  $S$ ,  
subset of  $V(A)$  with top  $A$ .  
given in  $P$ - or  $C$ -representation.  
It is the result of Wibmer's Lemma. See  
A. Montes , M. Wibmer, "Groebner Bases for  
Polynomial Systems with parameters".  
JSC 45 (2010) 1391-1425.)  
or the book

A. Montes. "The Groebner Cover" (Discussing Parametric Polynomial Systems).

**Note:** The basering  $R$ , must be of the form  $\mathbb{Q}[a][x]$  ( $a$ =parameters,  $x$ =variables).

**Example:**

```
LIB "grobcov.lib";
if(defined(RE)){kill RE;}
ring RE=(0,a,b,c,d,e,f),(x,y),lp;
ideal F=a*x^2+b*x*y+c*y^2,d*x^2+e*x*y+f*y^2;
ideal A=a*e-b*d;
WLemma(F,A);
↳ [1]:
↳   _[1]=y2
↳   _[2]=x2
↳ [2]:
↳   [1]:
↳   _[1]=y2
↳   [2]:
↳   _[1]=(d)*x2+(e)*xy+(f)*y2
↳   _[2]=(a)*x2+(b)*xy+(c)*y2
↳ [3]:
↳   [1]:
↳   [1]:
↳   _[1]=(ae-bd)
↳   [2]:
↳   [1]:
↳   _[1]=(bf-ce)
↳   _[2]=(af-cd)
↳   _[3]=(ae-bd)
↳   [2]:
↳   _[1]=(d)
↳   _[2]=(a)
WLemma(F,A,"rep",1);
↳ [1]:
↳   _[1]=y2
↳   _[2]=x2
↳ [2]:
↳   [1]:
↳   _[1]=y2
↳   [2]:
↳   _[1]=(d)*x2+(e)*xy+(f)*y2
↳   _[2]=(a)*x2+(b)*xy+(c)*y2
↳ [3]:
↳   [1]:
↳   _[1]=(ae-bd)
↳   [2]:
↳   _[1]=(bdf-cde)
↳   _[2]=(af-cd)
↳   _[3]=(ae-bd)
```

### D.2.4.19 WLCGS

Procedure from library `grobcov.lib` (see [Section D.2.4 \[grobcov.lib\], page 818](#)).

### D.2.4.20 intersectpar

Procedure from library `grobcov.lib` (see [Section D.2.4 \[grobcov.lib\], page 818](#)).

**Usage:**      `intersectpar(list of ideals S)`  
                  `list S=ideal I1,...,ideal Ik;`

**Return:**     The intersection of the ideals  $I_1 \dots I_k$  in  $Q[x,a]$

**Note:**        The routine is called in  $Q[a][x]$ ,  
                  The ideals  $I_1, \dots, I_k$  can be ideals depending only on  $[a]$  or on  $[x,a]$

**Example:**

```
LIB "grobcov.lib";
"EXAMPLE:";
⇒ EXAMPLE:
if(defined(R)){kill R;}
ring R=(0,x,y,z),(x1,y1,z1),lp;
ideal I1=x+y*z*x1;
ideal I2=x-y*z*y1;
ideal I3=x+y+z*z1;
list S=I1,I2,I3;
S;
⇒ [1]:
⇒ _[1]=(y*z)*x1+(x)
⇒ [2]:
⇒ _[1]=(-y*z)*y1+(x)
⇒ [3]:
⇒ _[1]=(z)*z1+(x+y)
intersectpar(S);
⇒ _[1]=(y^2*z^3)*x1*y1*z1+(x*y^2*z^2+y^3*z^2)*x1*y1+(-x*y*z^2)*x1*z1+(-x^2*\
y*z-x*y^2*z)*x1+(x*y*z^2)*y1*z1+(x^2*y*z+x*y^2*z)*y1+(-x^2*z)*z1+(-x^3-x^\
2*y)
```

### D.2.4.21 ADGT

Procedure from library `grobcov.lib` (see [Section D.2.4 \[grobcov.lib\], page 818](#)).

**Usage:**      `ADGT(ideal H, ideal T, ideal H1,ideal T1[,options]);`  
                  `H: ideal in  $Q[a][x]$  hypothesis`  
                  `T: ideal in  $Q[a][x]$  thesis`  
                  `H1: ideal in  $Q[a][x]$  negative hypothesis, only dependent on  $[a]$  T1: ideal in  $Q[a][x]$`   
                  `negative thesis`  
                  `of the Proposition  $(H \text{ and not } H1) \Rightarrow (T \text{ and not } T1)$`

**Return:**     The list `[[1,S1],[2,S2],...]`,  
                  `S1, S2, .. represent the set of parameter values`  
                  `that must be verified as supplementary`  
                  `conditions for the Proposition to become a Theorem.`  
                  `They are given by default in Prep.`  
                  `If the proposition is generally true,`

(the proposition is already a theorem), then the generic segment of the internal grobcov called is also returned to provide information about the values of the variables determined for every value of the parameters. If the proposition is false for every values of the parameters, then the empty list is returned.

- Options:** An option is a pair of arguments: string, integer. To modify the default options, pairs of arguments -option name, value- of valid options must be added to the call.
- Option "rep",0-1: The default is ("rep",0) and then the segments are given in canonical P-representation.
- Option ("rep",1) represents the segments in canonical C-representation,
- Option "gseg",0-1: The default is "gseg",1 and then when the proposition is generally true, ADGT outputs a second element which is the "generic segment" to provide supplementary information. With option "gseg",0 this is avoided.
- Option "neg", 0,1: The default is "neg",0 With option "neg",0 Rabinovitch trick is used for negative hypothesis and thesis With option "neg",1 Difference of constructible sets is used instead.
- Note:** The basering R, must be of the form  $\mathbb{Q}[a][x]$ , (a=parameters, x=variables), and should be defined previously. The ideals must be defined on R.

**Example:**

```
LIB "grobcov.lib";
// Determine the supplementary conditions
// for the non-degenerate triangle A(-1,0), B(1,0), C(x,y)
// to have an orthic non-degenerate isosceles triangle
if(defined(R)){kill R;}
ring R=(0,x,y),(x2,y2,x1,y1),lp;
// Hypothesis H: the triangle A1(x1,y1), B1(x2,y2), C1(x,0), is the
// orthic triangle of ABC
ideal H=-y*x1+(x-1)*y1+y,
(x-1)*(x1+1)+y*y1,
-y*x2+(x+1)*y2-y,
(x+1)*(x2-1)+y*y2;
// Thesis T: the orthic triangle is isosceles
ideal T=(x1-x)^2+y1^2-(x2-x)^2-y2^2;
// Negative Hypothesis H1: ABC is non-degenerate
ideal H1=y;
// Negative Thesis T1: the orthic triangle is non-degenerate
ideal T1=x*(y1-y2)-y*(x1-x2)+x1*y2-x2*y1;
// Complementary conditions for the
// Proposition (H and not H1) => (T and not T1)
// to be true
```



```

ADGT(H,T,H1,T1);
⇨ [1]:
⇨   [1]:
⇨     1
⇨   [2]:
⇨     [1]:
⇨       [1]:
⇨          $_{[1]}=(x^2-y^2-1)$ 
⇨       [2]:
⇨         [1]:
⇨            $_{[1]}=(y)$ 
⇨            $_{[2]}=(x-1)$ 
⇨         [2]:
⇨            $_{[1]}=(y)$ 
⇨            $_{[2]}=(x+1)$ 
⇨         [3]:
⇨            $_{[1]}=(y^2+1)$ 
⇨            $_{[2]}=(x)$ 
⇨     [2]:
⇨       [1]:
⇨          $_{[1]}=(x)$ 
⇨       [2]:
⇨         [1]:
⇨            $_{[1]}=(y)$ 
⇨            $_{[2]}=(x)$ 
⇨         [2]:
⇨            $_{[1]}=(y-1)$ 
⇨            $_{[2]}=(x)$ 
⇨         [3]:
⇨            $_{[1]}=(y+1)$ 
⇨            $_{[2]}=(x)$ 
⇨         [4]:
⇨            $_{[1]}=(y^2+1)$ 
⇨            $_{[2]}=(x)$ 
// Now using difference of constructible sets for negative hypothesis and thesis
ADGT(H,T,H1,T1,"neg",1);
⇨ [1]:
⇨   [1]:
⇨     1
⇨   [2]:
⇨     [1]:
⇨       [1]:
⇨          $_{[1]}=(x^2-y^2-1)$ 
⇨       [2]:
⇨         [1]:
⇨            $_{[1]}=(y)$ 
⇨            $_{[2]}=(x-1)$ 
⇨         [2]:
⇨            $_{[1]}=(y)$ 
⇨            $_{[2]}=(x+1)$ 
⇨         [3]:
⇨            $_{[1]}=(y^2+1)$ 
⇨            $_{[2]}=(x)$ 

```

```

↳      [2]:
↳      [1]:
↳      _[1]=(x)
↳      [2]:
↳      [1]:
↳      _[1]=(y)
↳      _[2]=(x)
↳      [2]:
↳      _[1]=(y-1)
↳      _[2]=(x)
↳      [3]:
↳      _[1]=(y+1)
↳      _[2]=(x)
↳      [4]:
↳      _[1]=(y^2+1)
↳      _[2]=(x)
// The results are identical using both methods for the negative propositions
// - Rabinovitch or
// - DifConsLCSets
// EXAMPLE 2
// Automatic Theorem Proving.
// The nine points circle theorem.
// Vertices of the triangle: A(-2,0), B(2,0), C(2a,2b)
// Height foot: A1(x1,y1),
// Height foot: B1(x2,y2),
// Height foot: C1(2a,0)
// Middle point BC: A2(a+1,b)
// Middle point CA: B2(a-1,b)
// Middle point AB: C2(0,0)
// Ortocenter: O(2x0,2y0)
// Middle point of A and O: A3(x0-1,y0)
// Middle point of B and O: B3(x0+1,y0)
// Middle point of C and O: C3(x0+a,y0+b)
// Nine points circle: c:=(X-x3)^2+(Y-y3)^2-r2
if (defined(R1)){kill R1;}
ring R1=(0,a,b),(x1,y1,x2,y2,x0,y0,x3,y3,r2),dp;
short=0;
ideal H=-x1*b+(a-1)*y1+2*b,
(a-1)*x1+b*y1+2*a-2,
-x2*b+(a+1)*y2-2*b,
(a+1)*x2+b*y2-2*a-2,
-x0*y1+(x1+2)*y0-y1,
-x0*y2+(x2-2)*y0+y2;
ideal T=(x1-2*x3)^2+(y1-2*y3)^2-r2,
(a+1-2*x3)^2+(b-2*y3)^2-r2,
(x0-1-2*x3)^2+(y0-2*y3)^2-r2,
(x2-2*x3)^2+(y2-2*y3)^2-r2,
(a-1-2*x3)^2+(b-2*y3)^2-r2,
(x0+1-2*x3)^2+(y0-2*y3)^2-r2,
(2*a-2*x3)^2+4*y3^2-r2,
4*x3^2+4*y3^2-r2,
(x0+a-2*x3)^2+(y0+b-2*y3)^2-r2;
ADGT(H,T,b,1);

```

```

⇒ [1]:
⇒   [1]:
⇒   1
⇒   [2]:
⇒   [1]:
⇒   [1]:
⇒   _[1]=0
⇒   [2]:
⇒   [1]:
⇒   _[1]=(a^2+2*a+b^2+1)
⇒   [2]:
⇒   _[1]=(a^2-2*a+b^2+1)
⇒   [3]:
⇒   _[1]=(b)
⇒ [2]:
⇒   [1]:
⇒   Generic segment
⇒   [2]:
⇒   [1]:
⇒   _[1]=r2
⇒   _[2]=y3
⇒   _[3]=x3
⇒   _[4]=y0
⇒   _[5]=x0
⇒   _[6]=y2
⇒   _[7]=x2
⇒   _[8]=y1
⇒   _[9]=x1
⇒   [2]:
⇒   _[1]=(4*b^2)*r2+(-a^4-2*a^2*b^2+2*a^2-b^4-2*b^2-1)
⇒   _[2]=(4*b)*y3+(a^2-b^2-1)
⇒   _[3]=2*x3+(-a)
⇒   _[4]=(b)*y0+(a^2-1)
⇒   _[5]=x0+(-a)
⇒   _[6]=(a^2+2*a+b^2+1)*y2+(-4*a*b-4*b)
⇒   _[7]=(a^2+2*a+b^2+1)*x2+(-2*a^2-4*a+2*b^2-2)
⇒   _[8]=(a^2-2*a+b^2+1)*y1+(4*a*b-4*b)
⇒   _[9]=(a^2-2*a+b^2+1)*x1+(2*a^2-4*a-2*b^2+2)
⇒   [3]:
⇒   [1]:
⇒   [1]:
⇒   _[1]=0
⇒   [2]:
⇒   [1]:
⇒   _[1]=(a^2+2*a+b^2+1)
⇒   [2]:
⇒   _[1]=(a^2-2*a+b^2+1)
⇒   [3]:
⇒   _[1]=(b)
// Thus the nine point circle theorem is true for all real points excepting b=0.

```

### D.2.4.22 ConsLevels

Procedure from library `grobccov.lib` (see [Section D.2.4 \[grobccov.lib\], page 818](#)).

- Usage:**     `ConsLevels(list L);`  
 $L = [[P_1, Q_1], \dots, [P_s, Q_s]]$  is a list of lists of pairs of ideals representing the constructible set  
 $S = V(P_1) \setminus V(Q_1) \cup \dots \cup V(P_s) \setminus V(Q_s)$ .  
 To be called in a ring  $Q[a][x]$  or a ring  $Q[a]$ . But the ideals can contain only the parameters in  $Q[a]$ .
- Return:**    The list of ideals  $[a_1, a_2, \dots, a_t]$  representing the closures of the canonical levels of  $S$  and its complement  $C$  wrt to the closure of  $S$ . The canonical levels of  $S$  are represented by their Crep. So we have:  
 Levels of  $S$ :  $[a_1, a_2], [a_3, a_4], \dots$   
 Levels of  $C$ :  $[a_2, a_3], [a_4, a_5], \dots$   
 $S = V(a_1) \setminus V(a_2) \cup V(a_3) \setminus V(a_4) \cup \dots$   
 $C = V(a_2 \setminus V(a_3) \cup V(a_4) \setminus V(a_5) \cup \dots$   
 The expression of  $S$  can be obtained from the output of `ConsLevels` by the call to `Levels`.
- Note:**       The algorithm was described in  
 J.M. Brunat, A. Montes. "Computing the canonical representation of constructible sets."  
 Math. Comput. Sci. (2016) 19: 165-178.

**Example:**

```
LIB "grobccov.lib";
// EXAMPLE:
if(defined(R)){kill R;}
ring R=0,(x,y,z),lp;
short=0;
ideal P1=(x^2+y^2+z^2-1);
ideal Q1=z,x^2+y^2-1;
ideal P2=y,x^2+z^2-1;
ideal Q2=z*(z+1),y,x*(x+1);
ideal P3=x;
ideal Q3=5*z-4,5*y-3,x;
list Cr1=Crep(P1,Q1);
list Cr2=Crep(P2,Q2);
list Cr3=Crep(P3,Q3);
list L=list(Cr1,Cr2,Cr3);
L;
⇒ [1]:
⇒ [1]:
⇒ [1]=x^2+y^2+z^2-1
⇒ [2]:
⇒ [1]=z
⇒ [2]=x^2+y^2-1
⇒ [2]:
⇒ [1]:
```

```

↳      _[1]=y
↳      _[2]=x^2+z^2-1
↳      [2]:
↳      _[1]=z^2+z
↳      _[2]=y
↳      _[3]=x+z+1
↳ [3]:
↳      [1]:
↳      _[1]=x
↳      [2]:
↳      _[1]=5*z-4
↳      _[2]=5*y-3
↳      _[3]=x
def LL=ConsLevels(L);
LL;
↳ [1]:
↳      _[1]=x^3+x*y^2+x*z^2-x
↳ [2]:
↳      _[1]=z
↳      _[2]=x^2+y^2-1
↳ [3]:
↳      _[1]=z
↳      _[2]=x+y^2-1
↳      _[3]=x*y
↳      _[4]=x^2-x
↳ [4]:
↳      _[1]=1
def LLL=Levels(LL);
LLL;
↳ [1]:
↳      [1]:
↳      1
↳      [2]:
↳      [1]:
↳      _[1]=x^3+x*y^2+x*z^2-x
↳      [2]:
↳      _[1]=z
↳      _[2]=x^2+y^2-1
↳ [2]:
↳      [1]:
↳      3
↳      [2]:
↳      [1]:
↳      _[1]=z
↳      _[2]=x+y^2-1
↳      _[3]=x*y
↳      _[4]=x^2-x
↳      [2]:
↳      _[1]=1

```

### D.2.4.23 Levels

Procedure from library `grobcov.lib` (see [Section D.2.4 \[grobcov.lib\]](#), page 818).

**Usage:** Levels(list L);  
 The input list L must be the output of the call to the routine ConsLevels of a constructible set:  
 $L = [a_1, a_2, \dots, a_k]$ , where the  $a$ 's are the closures of the levels, determined by ConsLevels.  
 Levels selects the levels of the constructible set. To be called in a ring  $\mathbb{Q}[a][x]$  or a ring  $\mathbb{Q}[a]$ . But the ideals can contain only the parameters in  $\mathbb{Q}[a]$ .

**Return:** The levels of the constructible set:  
 $Lc = [ [1, [a_1, a_2]], [3, [a_3, a_4]], \dots, [2l-1, [a_{\{2l-1\}}, a_{\{2l\}}]] ]$   
 the list of levels of S

**Example:**

```
LIB "grobcov.lib";
// EXAMPLE:
if(defined(R)){kill R;}
ring R=0,(x,y,z),lp;
short=0;
ideal P1=(x^2+y^2+z^2-1);
ideal Q1=z,x^2+y^2-1;
ideal P2=y,x^2+z^2-1;
ideal Q2=z*(z+1),y,x*(x+1);
ideal P3=x;
ideal Q3=5*z-4,5*y-3,x;
list Cr1=Crep(P1,Q1);
list Cr2=Crep(P2,Q2);
list Cr3=Crep(P3,Q3);
list L=list(Cr1,Cr2,Cr3);
L;
⇒ [1]:
⇒ [1]:
⇒ _[1]=x^2+y^2+z^2-1
⇒ [2]:
⇒ _[1]=z
⇒ _[2]=x^2+y^2-1
⇒ [2]:
⇒ [1]:
⇒ _[1]=y
⇒ _[2]=x^2+z^2-1
⇒ [2]:
⇒ _[1]=z^2+z
⇒ _[2]=y
⇒ _[3]=x+z+1
⇒ [3]:
⇒ [1]:
⇒ _[1]=x
⇒ [2]:
⇒ _[1]=5*z-4
⇒ _[2]=5*y-3
⇒ _[3]=x
```

```

def LL=ConsLevels(L);
LL;
↳ [1]:
↳   _[1]=x^3+x*y^2+x*z^2-x
↳ [2]:
↳   _[1]=z
↳   _[2]=x^2+y^2-1
↳ [3]:
↳   _[1]=z
↳   _[2]=x+y^2-1
↳   _[3]=x*y
↳   _[4]=x^2-x
↳ [4]:
↳   _[1]=1
def LLL=Levels(LL);
LLL;
↳ [1]:
↳   [1]:
↳   1
↳   [2]:
↳   [1]:
↳   _[1]=x^3+x*y^2+x*z^2-x
↳   [2]:
↳   _[1]=z
↳   _[2]=x^2+y^2-1
↳ [2]:
↳   [1]:
↳   3
↳   [2]:
↳   [1]:
↳   _[1]=z
↳   _[2]=x+y^2-1
↳   _[3]=x*y
↳   _[4]=x^2-x
↳   [2]:
↳   _[1]=1

```

#### D.2.4.24 Grob1Levels

Procedure from library `grobcov.lib` (see [Section D.2.4 \[grobcov.lib\]](#), page 818).

**Usage:**     `Grob1Levels(list G);`  
              `G` is the output of `grobcov(F,"rep",1)`  
              for obtaining the segments in `C`-rep.  
              Then `Grob1Levels`, selects the set of segments `S` of `G` having solutions (i.e. with basis  
              different from 1), and determines the canonical levels of this constructible set.  
              To be called in a ring `Q[a][x]`.

**Return:**    The list of ideals  
              `[a1,a2,...,at]`  
              representing the closures of the canonical levels of `S` and its complement `C` wrt to the  
              closure of `S`.  
              The levels of `S` and `C` are  
              Levels of `S`: `[a1,a2],[a3,a4],...`

**Note:** The algorithm was described in  
J.M. Brunat, A. Montes. "Computing the canonical  
representation of constructible sets."  
Math. Comput. Sci. (2016) 19: 165-178.

```
LIB "grobco.lib";
"EXAMPLE:";
⇒ EXAMPLE:
if (defined(R)) {kill R;}
ring R=(0,x,y),(x1,y1,x2,y2),lp;
ideal F=-y*x1+(x-1)*y1+y,
(x-1)*(x1+1)+y*y1,
-y*x2+(x+1)*y2-y,
(x+1)*(x2-1)+y*y2,
(x1-x)^2+y1^2-(x1-x)^2-y2^2;
def G=grobco(F,"rep",1);
G;
⇒ [1]:
⇒ [1]:
⇒ _[1]=1
⇒ [2]:
⇒ _[1]=1
⇒ [3]:
⇒ [1]:
⇒ _[1]=0
⇒ [2]:
⇒ _[1]=(x^5*y-2*x^3*y-x*y^5+x*y)
⇒ [2]:
⇒ [1]:
⇒ _[1]=y2
⇒ _[2]=x2
⇒ _[3]=y1
⇒ _[4]=x1
⇒ [2]:
⇒ _[1]=(x^5+2*x^4*y^6+4*x^4*y^4+2*x^4*y^2+4*x^3*y^6+8*x^3*y^4+4*x^3*y^2-2*x^3-4*x^2*y^6-8*x^2*y^4-4*x^2*y^2-4*x*y^6-9*x*y^4-4*x*y^2+x-2*y^10-4*y^8+4*y^4+2*y^2)*y2+(-6*x^4*y^5-6*x^4*y^3-2*x^3*y^7-4*x^3*y^5-2*x^3*y^3-2*x^2*y^7+8*x^2*y^5+10*x^2*y^3+2*x*y^9+6*x*y^7+6*x*y^5+2*x*y^3+4*y^9+4*y^7-4*y^5-4*y^3)
⇒ _[2]=(x^5+2*x^4*y^7+4*x^4*y^5+2*x^4*y^3+4*x^3*y^7+8*x^3*y^5+4*x^3*y^3-2*x^3-4*x^2*y^7-8*x^2*y^5-4*x^2*y^3-4*x*y^7-8*x*y^5-x*y^4-4*x*y^3+x-2*y^11-4*y^9+4*y^5+2*y^3)*x2+(-x^5-6*x^4*y^5-6*x^4*y^3+2*x^3+2*x^2*y^9+6*x^2*y^5+8*x^2*y^3+x*y^4-x-2*y^11+4*y^7-2*y^3)
⇒ _[3]=(x^5+2*x^4*y^6+4*x^4*y^4+2*x^4*y^2+4*x^3*y^6+8*x^3*y^4+4*x^3*y^2-2*x^3-4*x^2*y^6-8*x^2*y^4-4*x^2*y^2-4*x*y^6-9*x*y^4-4*x*y^2+x-2*y^10-4*y^8+4*y^4+2*y^2)*y2+(-6*x^4*y^5-6*x^4*y^3-2*x^3*y^7-4*x^3*y^5-2*x^3*y^3-2*x^2*y^7+8*x^2*y^5+10*x^2*y^3+2*x*y^9+6*x*y^7+6*x*y^5+2*x*y^3+4*y^9+4*y^7-4*y^5-4*y^3)
```



```

*y^8+4*y^4+2*y^2)*y1+(-2*x^4*y^5-2*x^4*y^3-2*x^3*y^7-4*x^3*y^5-2*x^3*y^3+\
2*x^2*y^7+8*x^2*y^5+6*x^2*y^3+2*x*y^9+6*x*y^7+6*x*y^5+2*x*y^3+4*y^9+4*y^7\
-4*y^5-4*y^3)
↳   _[4]=(x^5+2*x^4*y^7+4*x^4*y^5+2*x^4*y^3+4*x^3*y^7+8*x^3*y^5+4*x^3*y\
^3-2*x^3-4*x^2*y^7-8*x^2*y^5-4*x^2*y^3-4*x*y^7-8*x*y^5-x*y^4-4*x*y^3+x-2*\
y^11-4*y^9+4*y^5+2*y^3)*x1+(x^5-4*x^4*y^7-6*x^4*y^5-2*x^4*y^3-2*x^3+2*x^2\
*y^9+8*x^2*y^7+6*x^2*y^5-x*y^4+x+2*y^11-4*y^7+2*y^3)
↳   [3]:
↳   [1]:
↳   _[1]=(x^5*y-2*x^3*y-x*y^5+x*y)
↳   [2]:
↳   _[1]=(x*y)
↳   _[2]=(x^2-y^2-1)
↳   _[3]=(y^3+y)
↳   [3]:
↳   [1]:
↳   _[1]=y2^2
↳   _[2]=y1
↳   _[3]=x1
↳   [2]:
↳   _[1]=y2^2
↳   _[2]=y1
↳   _[3]=x1+1
↳   [3]:
↳   [1]:
↳   _[1]=(y)
↳   _[2]=(x+1)
↳   [2]:
↳   _[1]=1
↳   [4]:
↳   [1]:
↳   _[1]=y2
↳   _[2]=x2
↳   _[3]=y1^2
↳   [2]:
↳   _[1]=y2
↳   _[2]=x2-1
↳   _[3]=y1^2
↳   [3]:
↳   [1]:
↳   _[1]=(y)
↳   _[2]=(x-1)
↳   [2]:
↳   _[1]=1
↳   [5]:
↳   [1]:
↳   _[1]=1
↳   [2]:
↳   _[1]=1
↳   [3]:
↳   [1]:
↳   _[1]=(x)
↳   _[2]=(y^2+1)

```

```

⇒      [2]:
⇒      _[1]=1
def L=Grob1Levels(G);
L;
⇒ [1]:
⇒ _[1]=(x^5*y-2*x^3*y-x*y^5+x*y)
⇒ [2]:
⇒ _[1]=(x)
⇒ _[2]=(y^2+1)
⇒ [3]:
⇒ _[1]=1
def LL=Levels(L);
LL;
⇒ [1]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ [1]:
⇒ _[1]=(x^5*y-2*x^3*y-x*y^5+x*y)
⇒ [2]:
⇒ _[1]=(x)
⇒ _[2]=(y^2+1)

```

#### D.2.4.25 DifConsLCSets

Procedure from library `grobcov.lib` (see [Section D.2.4 \[grobcov.lib\]](#), page 818).

**Return:** A list of locally closed sets equivalent to the difference  $S = A \setminus B$ .  $Lc = [ [1][p1,q1] [2][p2,q2] ]$ .  
For obtaining the canonical representation into levels of the constructible  $A \setminus B$  one have to apply `ConsLevels` and then optatively `Levels`.

**Example:**

```

LIB "grobcov.lib";
// EXAMPLE:
if(defined(R)){kill R;}
ring R=(0,x,y,z,t),(x1,y1),lp;
ideal a1=x;
ideal a2=x,y;
ideal a3=x,y,z;
ideal a4=x,y,z,t;
ideal b1=y;
ideal b2=y,z;
ideal b3=y,z,t;
ideal b4=1;
list L1=a1,a2,a3,a4;
list L2=b1,b2,b3,b4;
L1;
⇒ [1]:
⇒ _[1]=(x)
⇒ [2]:
⇒ _[1]=(x)
⇒ _[2]=(y)
⇒ [3]:

```

```

↳   _[1]=(x)
↳   _[2]=(y)
↳   _[3]=(z)
↳ [4]:
↳   _[1]=(x)
↳   _[2]=(y)
↳   _[3]=(z)
↳   _[4]=(t)
L2;
↳ [1]:
↳   _[1]=(y)
↳ [2]:
↳   _[1]=(y)
↳   _[2]=(z)
↳ [3]:
↳   _[1]=(y)
↳   _[2]=(z)
↳   _[3]=(t)
↳ [4]:
↳   _[1]=1
def LL=DifConsLCSets(L1,L2);
LL;
↳ [1]:
↳   [1]:
↳   _[1]=(x)
↳   [2]:
↳   _[1]=(y)
↳   _[2]=(x)
↳ [2]:
↳   [1]:
↳   _[1]=(z)
↳   _[2]=(y)
↳   _[3]=(x)
↳   [2]:
↳   _[1]=(t)
↳   _[2]=(z)
↳   _[3]=(y)
↳   _[4]=(x)
def LLL=ConsLevels(LL);
LLL;
↳ [1]:
↳   _[1]=(x)
↳ [2]:
↳   _[1]=(y)
↳   _[2]=(x)
↳ [3]:
↳   _[1]=(z)
↳   _[2]=(y)
↳   _[3]=(x)
↳ [4]:
↳   _[1]=(t)
↳   _[2]=(z)
↳   _[3]=(y)

```

```

↳      _[4]=(x)
↳ [5]:
↳      _[1]=1
def LLLL=Levels(LLL);
LLLL;
↳ [1]:
↳      [1]:
↳      1
↳      [2]:
↳      [1]:
↳      _[1]=(x)
↳      [2]:
↳      _[1]=(y)
↳      _[2]=(x)
↳ [2]:
↳      [1]:
↳      3
↳      [2]:
↳      [1]:
↳      _[1]=(z)
↳      _[2]=(y)
↳      _[3]=(x)
↳      [2]:
↳      _[1]=(t)
↳      _[2]=(z)
↳      _[3]=(y)
↳      _[4]=(x)

```

### D.2.5 inout.lib

**Library:** inout.lib

**Purpose:** Printing and Manipulating In- and Output

**Procedures:**

#### D.2.5.1 allprint

Procedure from library `inout.lib` (see [Section D.2.5 \[inout.lib\]](#), page 878).

**Usage:** allprint(L); L list

**Display:** prints L[1], L[2], ... if an integer with name ALLprint is defined.  
makes "pause", if ALLprint > 0

**Return:** no return value

**Example:**

```

LIB "inout.lib";
ring S;
matrix M=matrix(freemodule(2),3,3);
int ALLprint; export ALLprint;
allprint("M =",M);
↳ M =
↳ 1,0,0,
↳ 0,1,0,

```

```

    ↦ 0,0,0
    kill ALLprint;

```

### D.2.5.2 lprint

Procedure from library `inout.lib` (see [Section D.2.5 \[inout.lib\], page 878](#)).

**Usage:** `lprint(id[,n]);` id poly/ideal/vector/module/matrix, n integer

**Return:** string of id in a format fitting into lines of size n, such that no monomial gets destroyed, i.e. the new line starts with + or -; (default: n = pagewidth).

**Note:** id is printed columnwise, each column separated by a blank line; hence `lprint(transpose(id));` displays a matrix id in a format which can be used as input.

**Example:**

```

LIB "inout.lib";
ring r= 0,(x,y,z),ds;
poly f=((x+y)*(x-y)*(x+z)*(y+z)^2);
lprint(f,40);
↦ x3y2-xy4+2x3yz+x2y2z-2xy3z-y4z+x3z2
↦ +2x2yz2-xy2z2-2y3z2+x2z3-y2z3
module m = [f*(x-y)], [0,f*(x-y)];
string s=lprint(m); s;"";
↦ x4y2-x3y3-x2y4+xy5+2x4yz-x3y2z-3x2y3z+xy4z+y5z+x4z2+x3yz2-3x2y2z2-xy3z2
↦ +2y4z2+x3z3-x2yz3-xy2z3+y3z3,
↦ 0,
↦
↦ 0,
↦ x4y2-x3y3-x2y4+xy5+2x4yz-x3y2z-3x2y3z+xy4z+y5z+x4z2+x3yz2-3x2y2z2-xy3z2
↦ +2y4z2+x3z3-x2yz3-xy2z3+y3z3
↦
execute("matrix M[2][2]="+s+""); //use the string s as input
module m1 = transpose(M); //should be the same as m
print(matrix(m)-matrix(m1));
↦ 0,0,
↦ 0,0

```

### D.2.5.3 pmat

Procedure from library `inout.lib` (see [Section D.2.5 \[inout.lib\], page 878](#)).

**Usage:** `pmat(M[,n]);` M matrix, n integer

**Return:** A string representing M in array format if it fits into pagewidth; if n is given, only the first n characters of each column are shown (n>1 required), where a truncation of a column is indicated by two dots (‘\..\’)

**Example:**

```

LIB "inout.lib";
ring r=0,(x,y,z),ls;
ideal i= x,z+3y,x+y,z;
matrix m[3][3]=i^2;
pmat(m);
↦ x2,      xz+3xy,      xy+x2,
↦ xz,      z2+6yz+9y2,  yz+3y2+xz+3xy,

```

```

    ↪ z2+3yz, y2+2xy+x2, yz+xz
    pmat(m,5);
    ↪ x2,      xz+.., xy+x2,
    ↪ xz,      z2+.., yz+..,
    ↪ z2+.., y2+.., yz+xz

```

#### D.2.5.4 rMacaulay

Procedure from library `inout.lib` (see [Section D.2.5 \[inout.lib\]](#), page 878).

**Usage:** `rMacaulay(s[n]);` s string, n integer

**Return:** A string denoting a file which should be readable by Singular and it should be produced by Macaulay Classic.  
If a second argument is present the first n lines of the file are deleted (which is useful if the file was produced e.g. by the `putstd` command of Macaulay).

**Note:** This does not always work with 'cut and paste' since the character `\` is treated differently

**Example:**

```

LIB "inout.lib";
// Assume there exists a file 'Macid' with the following ideal in
// Macaulay format:
// x[0]^3-101/74x[0]^2x[1]+7371x[0]x[1]^2-13/83x[1]^3-x[0]^2x[2] \
//      -4/71x[0]x[1]x[2]
// Read this file into Singular and assign it to the string s1 by:
// string s1 = read("Macid");
// This is equivalent to";
string s1 =
"x[0]^3-101/74x[0]^2x[1]+7371x[0]x[1]^2-13/83x[1]^3-x[0]^2x[2]-4/71x[0]x[1]x[2]";
rMacaulay(s1);
↪ x(0)^3-101/74*x(0)^2*x(1)+7371*x(0)*x(1)^2-13/83*x(1)^3-x(0)^2*x(2)-4/71*
  x(0)*x(1)*x(2)
// You may wish to assign s1 to a Singular ideal:
string sid = "ideal id =",rMacaulay(s1),";";
ring r = 0,x(0..3),dp;
execute(sid);
id; ""
↪ id[1]=x(0)^3-101/74*x(0)^2*x(1)+7371*x(0)*x(1)^2-13/83*x(1)^3-x(0)^2*x(2)\
  -4/71*x(0)*x(1)*x(2)
↪
// Now treat a matrix in Macaulay format. Using the execute
// command, this could be assigned to a Singular matrix as above.
string s2 = "
0 0 0 0 0
a3 0 0 0 0
0 b3 0 0 0
0 0 c3 0 0
0 0 0 d3 0
0 0 0 0 e3 ";
rMacaulay(s2);
↪ 0, 0, 0, 0, 0,
↪ a3,0, 0, 0, 0,

```

```

⇒ 0, b3,0, 0, 0,
⇒ 0, 0, c3,0, 0,
⇒ 0, 0, 0, d3,0,
⇒ 0, 0, 0, 0, e3

```

### D.2.5.5 show

Procedure from library `inout.lib` (see [Section D.2.5 \[inout.lib\]](#), page 878).

**Usage:**      `show(id)`; `id` any object of basering or of type ring/qring  
               `show(R,s)`; `R=ring`, `s=string` (`s` = name of an object belonging to `R`)

**Display:**    display `id/s` in a compact format together with some information

**Return:**     no return value

**Note:**       objects of type `string`, `int`, `intvec`, `intmat` belong to any ring. `id` may be a ring or a qring. In this case the minimal polynomial is displayed, and, for a qring, also the defining ideal.  
               `id` may be of type list but the list must not contain a ring.  
               `show(R,s)` does not work inside a procedure!

**Example:**

```

LIB "inout.lib";
ring r;
show(r);
⇒ // ring: (ZZ/32003),(x,y,z),(dp(3),C);
⇒ // minpoly = 0
⇒ // objects belonging to this ring:
ideal i=x^3+y^5-6*z^3,xy,x3-y2;
show(i,3);           // introduce 3 space tabs before information
⇒ // ideal, 3 generator(s)
⇒ y5+x3-6z3,
⇒ xy,
⇒ x3-y2
vector v=x*gen(1)+y*gen(3);
module m=v,2*v+gen(4);
list L = i,v,m;
show(L);
⇒ // list, 3 element(s):
⇒ [1]:
⇒ // ideal, 3 generator(s)
⇒ y5+x3-6z3,
⇒ xy,
⇒ x3-y2
⇒ [2]:
⇒ // vector
⇒ [x,0,y]
⇒ [3]:
⇒ // module, 2 generator(s)
⇒ [x,0,y]
⇒ [2x,0,2y,1]
ring S=(0,T),(a,b,c,d),ws(1,2,3,4);
minpoly = T^2+1;
ideal i=a2+b,c2+T^2*d2; i=std(i);

```

```

qring Q=i;
show(Q);
⇨ // ring: (0,T),(a,b,c,d),(ws(1,2,3,4),C);
⇨ // minpoly = (T2+1)
⇨ // quotient ring from ideal:
⇨ _[1]=a2+b
⇨ _[2]=c2-d2
⇨ // objects belonging to this ring:
map F=r,a2,b^2,3*c3;
show(F);
⇨ // i-th variable of preimage ring is mapped to @map[i]
⇨ // @map map from r
⇨ @map[1]=a2
⇨ @map[2]=b2
⇨ @map[3]=3*c3
// Apply 'show' to i (which does not belong to the basering) by typing
// ring r; ideal i=xy,x3-y2; ring Q; show(r,"i");

```

### D.2.5.6 showrecursive

Procedure from library `inout.lib` (see [Section D.2.5 \[inout.lib\], page 878](#)).

**Usage:** `showrecursive(id,p[,ord]);` `id` any object of basering, `p`= product of variables and `ord`=string (any allowed `ordstr`)

**Display:** display 'id' in a recursive format as a polynomial in the variables occurring in `p` with coefficients in the remaining variables. This is done by mapping to a ring with parameters [and ordering 'ord', if a 3rd argument is present (default: `ord="dp"`)] and applying procedure 'show'

**Return:** no return value

**Example:**

```

LIB "inout.lib";
ring r=2,(a,b,c,d,x,y),ds;
poly f=y+ax2+bx3+cx2y2+dxy3;
showrecursive(f,x);
⇨ // poly, 4 monomial(s)
⇨ (b)*x3+(a+cy2)*x2+(dy3)*x+(y)
showrecursive(f,xy,"lp");
⇨ // poly, 5 monomial(s)
⇨ (b)*x3+(c)*x2y2+(a)*x2+(d)*xy3+y

```

### D.2.5.7 split

Procedure from library `inout.lib` (see [Section D.2.5 \[inout.lib\], page 878](#)).

**Usage:** `split(s[,n]);` `s` string, `n` integer

**Return:** same string, split into lines of length `n` separated by `\` (default: `n`=pagewidth)

**Note:** may be used in connection with `lprint`

**Example:**

```

LIB "inout.lib";
ring r= 0,(x,y,z),ds;

```



```

poly f = (x+y+z)^4;
split(string(f),50);
↳ x4+4x3y+6x2y2+4xy3+y4+4x3z+12x2yz+12xy2z+4y3z+6x\
↳ 2z2+12xyz2+6y2z2+4xz3+4yz3+z4
split(lprint(f));
↳ x4+4x3y+6x2y2+4xy3+y4+4x3z+12x2yz+12xy2z+4y3z+6x2z2+12xyz2+6y2z2+4xz3+4\
  yz3\
↳ +z4

```

### D.2.5.8 tab

Procedure from library `inout.lib` (see [Section D.2.5 \[inout.lib\], page 878](#)).

**Usage:** `tab(n);`  $n$  integer

**Return:** string of  $n$  space tabs

**Example:**

```

LIB "inout.lib";
for(int n=0; n<=5; n=n+1)
{ tab(5-n)+"*"+tab(n)+" "+tab(n)+"*"; }
↳      **
↳      * + *
↳      * + *
↳      * + *
↳      * + *
↳      * + *
↳      * + *

```

### D.2.5.9 pause

Procedure from library `inout.lib` (see [Section D.2.5 \[inout.lib\], page 878](#)).

**Usage:** `pause([ prompt ])` prompt string

**Return:** none

**Purpose:** interrupt the execution of commands, displays prompt or pause and waits for user input

**Note:** `pause` is useful in procedures in connection with `printlevel` to interrupt the computation and to display intermediate results.

**Example:**

```

LIB "inout.lib";
// can only be shown interactively, try the following commands:
// pause("press <return> to continue");
// pause();
// In the following pocedure TTT, xxx is printed and the execution of
// TTT is stopped until the return-key is pressed, if printlevel>0.
// xxx may be any result of a previous computation or a comment, etc:
//
// proc TTT
// { int pp = printlevel-voice+2; //pp=0 if printlevel=0 and if TTT is
//   ....                       //not called from another procedure
//   if( pp>0 )
//   {
//       print( xxx );
//       pause("press <return> to continue");

```

```
// }
// ....
// }
```

See also: [Section 5.3.6 \[printlevel\]](#), page 304; [Section 5.1.130 \[read\]](#), page 250.

## D.2.6 modular\_lib

**Library:** modular.lib

**Purpose:** An abstraction layer for modular techniques

**Author:** Andreas Steenpass, e-mail: [steenpass@mathematik.uni-kl.de](mailto:steenpass@mathematik.uni-kl.de)

**Overview:** This library is an abstraction layer for modular techniques which are well-known to speed up many computations and to be easy parallelizable.

The basic idea is to execute some computation modulo several primes and then to lift the result back to characteristic zero via the farey rational map and chinese remaining. It is thus possible to overcome the often problematic coefficient swell and to run the modular computations in parallel.

In Singular, modular techniques have been quite successfully employed for several applications. A first implementation was done for Groebner bases in Singular's [Section D.4.16 \[modstd\\_lib\]](#), page 1147, a pioneering work by Stefan Steidel. Since the algorithm is basically the same for all applications, this library aims at preventing library authors from writing the same code over and over again by providing an appropriate abstraction layer. It also offers one-line commands for ordinary Singular users who want to take advantage of modular techniques for their own calculations. Thus modular techniques can be regarded as a parallel skeleton of their own.

The terminology (such as 'pTest' and 'finalTest') follows Singular's [Section D.4.16 \[modstd\\_lib\]](#), page 1147 and [1].

### References:

- [1] Nazeran Idrees, Gerhard Pfister, Stefan Steidel: Parallelization of Modular Algorithms. *Journal of Symbolic Computation* 46, 672-684 (2011). <http://arxiv.org/abs/1005.5663>

**Procedures:** See also: [Section D.4.3 \[assprimeszerodim\\_lib\]](#), page 1014; [Section 4.10 \[link\]](#), page 96; [Section D.4.16 \[modstd\\_lib\]](#), page 1147; [Section D.2.7 \[parallel\\_lib\]](#), page 885; [Section D.2.13 \[tasks\\_lib\]](#), page 959.

### D.2.6.1 modular

Procedure from library `modular.lib` (see [Section D.2.6 \[modular\\_lib\]](#), page 884).

**Usage:** `modular(command, arguments[, primeTest, deleteUnluckyPrimes, pTest, finalTest, pmax], command string, arguments list, primeTest proc, deleteUnluckyPrimes proc, pTest proc, finalTest proc, pmax int)`

**Return:** the result of `command` applied to `arguments`, computed using modular methods.

**Note:** For the general algorithm and the role of the optional arguments `primeTest`, `deleteUnluckyPrimes`, `pTest`, and `finalTest`, see [Section D.4.16.1 \[modStd\]](#), page 1147 and the reference given in [Section D.2.6 \[modular\\_lib\]](#), page 884. The default for these arguments is that all tests succeed and that all primes are assumed to be lucky.

The type of the result when `command` is applied to `arguments` must be either `bigint`, `ideal`, `module`, or `matrix`.

The optional argument `pmax` is an upper bound for the prime numbers to be used for the modular computations. The default is 2147483647 (largest prime which can be represented as an `int` in Singular), or 536870909 (largest prime below  $2^{29}$  for baserings with parameters).

**Example:**

```
LIB "modular.lib";
ring R = 0, (x,y), dp;
ideal I = x9y2+x10, x2y7-y8;
modular("std", list(I));
⇒ _[1]=x2y7-y8
⇒ _[2]=x9y2+x10
⇒ _[3]=x12y+xy11
⇒ _[4]=x13-xy12
⇒ _[5]=y14+xy12
⇒ _[6]=xy13+y12
```

See also: [Section D.4.16.1 \[modStd\]](#), page 1147.

## D.2.7 parallel\_lib

**Library:** parallel.lib

**Purpose:** An abstraction layer for parallel skeletons

**Author:** Andreas Steenpass, e-mail: steenpass@mathematik.uni-kl.de

**Overview:** This library provides implementations of several parallel 'skeletons' (i.e. ways in which parallel tasks rely upon and interact with each other). It is based on the library `tasks.lib` and aims at both ordinary Singular users as well as authors of Singular libraries.

**Procedures:** See also: [Section D.4.16 \[modstd\\_lib\]](#), page 1147; [Section D.4.23 \[normal\\_lib\]](#), page 1184; [Section D.2.11 \[resources\\_lib\]](#), page 942; [Section D.2.13 \[tasks\\_lib\]](#), page 959.

### D.2.7.1 parallelWaitN

Procedure from library `parallel.lib` (see [Section D.2.7 \[parallel\\_lib\]](#), page 885).

**Usage:** parallelWaitN(commands, arguments, N[, timeout]); commands list, arguments list, N int, timeout int

**Return:** a list, containing the results of `commands[i]` applied to `arguments[i]`,  $i = 1, \dots, \text{size}(\text{arguments})$ .

The procedure waits for N jobs to finish.

An optional timeout in ms can be provided. Default is 0 which disables the timeout.

**Note:** The entries of the list `commands` must be strings. The entries of the list `arguments` must be lists.

The type of any entry of the returned list whose corresponding task did not finish (due to timeout or error) is "none".

The returned list may contain more than N results if several jobs finished "at the same time". It may contain less than N results in the case of timeout or errors occurring.

**Example:**

```
LIB "parallel.lib";
ring R = 0, (x,y,z), lp;
ideal I = 3x3y+x3+xy3+y2z2, 2x3z-xy-xz3-y4-z2, 2x2yz-2xy2+xz2-y4;
```

```

ideal J = x10+x9y2, x2y7-y8;
list commands = list("std", "std");
list arguments = list(list(I), list(J));
parallelWaitN(commands, arguments, 1);
↳ [2]:
↳   _[1]=y15-y12
↳   _[2]=xy12+y14
↳   _[3]=x2y7-y8
↳   _[4]=x10+x9y2

```

See also: [Section D.2.7.3 \[parallelWaitAll\]](#), page 886; [Section D.2.7.2 \[parallelWaitFirst\]](#), page 886; [Section D.2.13 \[tasks\\_lib\]](#), page 959.

### D.2.7.2 parallelWaitFirst

Procedure from library `parallel.lib` (see [Section D.2.7 \[parallel\\_lib\]](#), page 885).

**Usage:** `parallelWaitFirst(commands, args[, timeout]);` commands list, arguments list, timeout  
int

**Return:** a list, containing at least one (if no timeout occurs) of the results of `commands[i]` applied to `arguments[i]`,  $i = 1, \dots, \text{size}(\text{arguments})$ .  
The command `parallelWaitFirst(commands, arguments[, timeout])` is synonymous to `parallelWaitN(commands, arguments, 1[, timeout])`. See [Section D.2.7.1 \[parallelWaitN\]](#), page 885 for details on optional arguments and other remarks.

**Example:**

```

LIB "parallel.lib";
ring R = 0, (x,y,z), lp;
ideal I = 3x3y+x3+xy3+y2z2, 2x3z-xy-xz3-y4-z2, 2x2yz-2xy2+xz2-y4;
ideal J = x10+x9y2, x2y7-y8;
list commands = list("std", "std");
list arguments = list(list(I), list(J));
parallelWaitFirst(commands, arguments);
↳ [2]:
↳   _[1]=y15-y12
↳   _[2]=xy12+y14
↳   _[3]=x2y7-y8
↳   _[4]=x10+x9y2

```

See also: [Section D.2.7.3 \[parallelWaitAll\]](#), page 886; [Section D.2.7.1 \[parallelWaitN\]](#), page 885; [Section D.2.13 \[tasks\\_lib\]](#), page 959.

### D.2.7.3 parallelWaitAll

Procedure from library `parallel.lib` (see [Section D.2.7 \[parallel\\_lib\]](#), page 885).

**Usage:** `parallelWaitAll(commands, arguments[, timeout]);` commands list or string, arguments list, timeout int

**Return:** a list, containing the results of `commands[i]` applied to `arguments[i]`,  $i = 1, \dots, \text{size}(\text{arguments})$ .  
The command `parallelWaitAll(commands, arguments[, timeout])` is synonymous to `@code{parallelWaitN(commands, arguments, size(arguments)[, timeout])}`. See [Section D.2.7.1 \[parallelWaitN\]](#), page 885 for details on optional arguments and other remarks.

**Note:** As a shortcut, `commands` can be a string. This is synonymous to providing a list of `size(arguments)` copies of this string.

**Example:**

```
LIB "parallel.lib";
ring R = 0, (x,y,z), dp;
ideal I1 = z8+z6+4z5+4z3+4z2+4, -z2+y;
ideal I2 = x9y2+x10, x2y7-y8;
ideal I3 = x3-2xy, x2y-2y2+x;
string command = "std";
list arguments = list(list(I1), list(I2), list(I3));
parallelWaitAll(command, arguments);
⇒ [1]:
⇒   _[1]=z2-y
⇒   _[2]=y4+y3+4y2z+4yz+4y+4
⇒ [2]:
⇒   _[1]=x2y7-y8
⇒   _[2]=x9y2+x10
⇒   _[3]=x12y+xy11
⇒   _[4]=x13-xy12
⇒   _[5]=y14+xy12
⇒   _[6]=xy13+y12
⇒ [3]:
⇒   _[1]=2y2-x
⇒   _[2]=xy
⇒   _[3]=x2
```

See also: [Section D.2.7.2 \[parallelWaitFirst\], page 886](#); [Section D.2.7.1 \[parallelWaitN\], page 885](#); [Section D.2.13 \[tasks\\_lib\], page 959](#).

#### D.2.7.4 parallelTestAND

Procedure from library `parallel.lib` (see [Section D.2.7 \[parallel\\_lib\], page 885](#)).

**Usage:** `parallelTestAND(commands, arguments[, timeout]);` `commands` list or string, `arguments` list, `timeout` int

**Return:** 1, if `commands[i]` applied to `arguments[i]` is not equal to zero for all  $i = 1, \dots, \text{size}(\text{arguments})$ ;  
0, otherwise.  
An optional timeout in ms can be provided. Default is 0 which disables the timeout. In case of timeout, -1 is returned.

**Note:** The entries of the list `commands` must be strings. The entries of the list `arguments` must be lists.  
`commands[i]` applied to `arguments[i]` must evaluate to an integer for  $i = 1, \dots, \text{size}(\text{arguments})$ .  
As a shortcut, `commands` can be a string. This is synonymous to providing a list of `size(arguments)` copies of this string.

**Example:**

```
LIB "parallel.lib";
ring R = 0, (x,y,z), dp;
ideal I = x, y, z;
intvec v = 0:3;
```

```

list l = list(I, v);
module m1 = x*gen(1);
module m2;
string command = "size";
list arguments1 = list(list(I), list(v), list(l), list(m1));
list arguments2 = list(list(I), list(v), list(l), list(m2));
// test if all the arguments have non-zero size
parallelTestAND(command, arguments1);
⇒ 1
parallelTestAND(command, arguments2);
⇒ 0

```

See also: [Section D.2.7.5 \[parallelTestOR\], page 888](#); [Section D.2.13 \[tasks\\_lib\], page 959](#).

### D.2.7.5 parallelTestOR

Procedure from library `parallel.lib` (see [Section D.2.7 \[parallel.lib\], page 885](#)).

- Usage:** `parallelTestOR(commands, arguments[, timeout]);` `commands` list or string, `arguments` list, `timeout` int
- Return:** 1, if `commands[i]` applied to `arguments[i]` is not equal to zero for any  $i = 1, \dots, \text{size}(\text{arguments})$ ;  
0, otherwise.  
An optional timeout in ms can be provided. Default is 0 which disables the timeout. In case of timeout, -1 is returned.
- Note:** The entries of the list `commands` must be strings. The entries of the list `arguments` must be lists.  
`commands[i]` applied to `arguments[i]` must evaluate to an integer for  $i = 1, \dots, \text{size}(\text{arguments})$ .  
As a shortcut, `commands` can be a string. This is synonymous to providing a list of `size(arguments)` copies of this string.

**Example:**

```

LIB "parallel.lib";
ring R = 0, (x,y,z), dp;
ideal I;
string s;
list l;
module m1 = x*gen(1);
module m2;
string command = "size";
list arguments1 = list(list(I), list(s), list(l), list(m1));
list arguments2 = list(list(I), list(s), list(l), list(m2));
// test if any of the arguments has non-zero size
parallelTestOR(command, arguments1);
⇒ 1
parallelTestOR(command, arguments2);
⇒ 0

```

See also: [Section D.2.7.4 \[parallelTestAND\], page 887](#); [Section D.2.13 \[tasks\\_lib\], page 959](#).

### D.2.8 polylib.lib

**Library:** `polylib.lib`

**Purpose:** Procedures for Manipulating Polys, Ideals, Modules

**Authors:** O. Bachmann, G.-M. Greuel, A. Fruehbis

**Procedures:**

### D.2.8.1 cyclic

Procedure from library `polylib.lib` (see [Section D.2.8 \[polylib\\_lib\]](#), page 888).

**Usage:** `cyclic(n)`;  $n$  integer

**Return:** ideal of cyclic  $n$ -roots from 1-st  $n$  variables of basering

**Example:**

```
LIB "polylib.lib";
ring r=0,(u,v,w,x,y,z),lp;
cyclic(nvars(basering));
⇒ _[1]=u+v+w+x+y+z
⇒ _[2]=uv+uz+vw+wx+xy+yz
⇒ _[3]=uvw+uvz+uyz+vwx+wxy+xyz
⇒ _[4]=uvwx+uvwz+uvyz+uxyz+vwxy+wxyz
⇒ _[5]=uvwxy+uvwxz+uvwyz+uvxyz+uwxyz+vwxyz
⇒ _[6]=uvwxyz-1
homog(cyclic(5),z);
⇒ _[1]=u+v+w+x+y
⇒ _[2]=uv+uy+vw+wx+xy
⇒ _[3]=uvw+uvy+uxy+vwx+wxy
⇒ _[4]=uvwx+uvwxy+uvwy+uwxy+vwxy
⇒ _[5]=uvwxy-z5
```

### D.2.8.2 elemSymmId

Procedure from library `polylib.lib` (see [Section D.2.8 \[polylib\\_lib\]](#), page 888).

**Return:** ideal of elementary symmetric polynomials for 1-st  $n$  variables of basering

**Example:**

```
LIB "polylib.lib";
ring R = 0, (v,w,x,y,z), lp;
elemSymmId(3);
⇒ _[1]=v+w+x
⇒ _[2]=vw+vx+wx
⇒ _[3]=vwx
elemSymmId(nvars(basering));
⇒ _[1]=v+w+x+y+z
⇒ _[2]=vw+vx+vy+vz+wx+wy+wz+xy+xz+yz
⇒ _[3]=vwx+vwy+vwz+vxy+vzx+vyz+wxy+wxz+wyz+xyz
⇒ _[4]=vwxy+vwxz+vwyz+vxyz+wxyz
⇒ _[5]=vwxyz
```

### D.2.8.3 katsura

Procedure from library `polylib.lib` (see [Section D.2.8 \[polylib\\_lib\]](#), page 888).

**Usage:** `katsura([n])`;  $n$  integer

**Return:** katsura(n) : n-th katsura ideal of  
 (1) newly created and set ring (32003, x(0..n), dp), if nvars(basering) < n  
 (2) basering, if nvars(basering) >= n  
 katsura() : katsura ideal of basering

**Example:**

```
LIB "polylib.lib";
ring r; basering;
⇨ // coefficients: ZZ/32003
⇨ // number of vars : 3
⇨ //          block 1 : ordering dp
⇨ //          : names  x y z
⇨ //          block 2 : ordering C
katsura();
⇨ _[1]=x+2y+2z-1
⇨ _[2]=x2+2y2+2z2-x
⇨ _[3]=2xy+2yz-y
katsura(4); basering;
⇨ _[1]=x(0)+2*x(1)+2*x(2)+2*x(3)-1
⇨ _[2]=x(0)^2+2*x(1)^2+2*x(2)^2+2*x(3)^2-x(0)
⇨ _[3]=2*x(0)*x(1)+2*x(1)*x(2)+2*x(2)*x(3)-x(1)
⇨ _[4]=x(1)^2+2*x(0)*x(2)+2*x(1)*x(3)-x(2)
⇨ // coefficients: ZZ/32003
⇨ // number of vars : 5
⇨ //          block 1 : ordering dp
⇨ //          : names  x(0) x(1) x(2) x(3) x(4)
⇨ //          block 2 : ordering C
```

#### D.2.8.4 freerank

Procedure from library polylib.lib (see [Section D.2.8 \[polylib\\_lib\]](#), page 888).

**Usage:** freerank(M[,any]); M=poly/ideal/vector/module/matrix

**Compute:** rank of module presented by M in case it is free.  
 By definition this is  $\text{vdim}(\text{coker}(M)/\mathfrak{m}*\text{coker}(M))$  if  $\text{coker}(M)$  is free, where  $\mathfrak{m}$  is the maximal ideal of the variables of the basering and M is considered to be a matrix.  
 (the 0-module is free of rank 0)

**Return:** rank of  $\text{coker}(M)$  if  $\text{coker}(M)$  is free and -1 else;  
 in case of a second argument return a list:  
 L[1] = rank of  $\text{coker}(M)$  or -1  
 L[2] = minbase(M)

**Note:** freerank(syz(M)); computes the rank of M if M is free (and -1 else)

**Example:**

```
LIB "polylib.lib";
ring r;
ideal i=x;
module M=[x,0,1],[-x,0,-1];
freerank(M); // should be 2, coker(M) is not free
⇨ 2
freerank(syz (M),"");
⇨ [1]:
```



```

↳ 1
↳ [2]:
↳ _[1]=gen(2)+gen(1)
// [1] should be 1, coker(syz(M))=M is free of rank 1
// [2] should be gen(2)+gen(1) (minimal relation of M)
freerank(i);
↳ -1
freerank(syz(i)); // should be 1, coker(syz(i))=i is free of rank 1
↳ 1

```

### D.2.8.5 is\_zero

Procedure from library `polylib.lib` (see [Section D.2.8 \[polylib.lib\], page 888](#)).

**Usage:** `is_zero(M[,any]);` `M=poly/ideal/vector/module/matrix`

**Return:** integer, 1 if `coker(M)=0` resp. 0 if `coker(M)≠0`, where `M` is considered as matrix.  
 If a second argument is given, return a list:  
`L[1] = 1` if `coker(M)=0` resp. 0 if `coker(M)≠0`  
`L[2] = dim(M)`

**Example:**

```

LIB "polylib.lib";
ring r;
module m = [x],[y],[1,z];
is_zero(m,1);
↳ [1]:
↳ 0
↳ [2]:
↳ 2
qring q = std(ideal(x2+y3+z2));
ideal j = x2+y3+z2-37;
is_zero(j);
↳ 1

```

### D.2.8.6 lcm

Procedure from library `polylib.lib` (see [Section D.2.8 \[polylib.lib\], page 888](#)).

**Usage:** `lcm(p[,q]);` `p int/intvec` `q` a list of integers or  
`p poly/ideal` `q` a list of polynomials

**Return:** the least common multiple of `p` and `q`:  
 - of type `int` if `p` is an `int/intvec`  
 - of type `poly` if `p` is a `poly/ideal`

**Example:**

```

LIB "polylib.lib";
ring r = 0,(x,y,z),lp;
poly p = (x+y)*(y+z);
poly q = (z4+2)*(y+z);
lcm(p,q);
↳ xyz4+2xy+xz5+2xz+y2z4+2y2+yz5+2yz
ideal i=p,q,y+z;
lcm(i,p);

```

```

↳ xyz4+2xy+xz5+2xz+y2z4+2y2+yz5+2yz
lcm(2,3,6);
↳ 6
lcm(2..6);
↳ 60

```

### D.2.8.7 maxcoef

Procedure from library `polylib.lib` (see [Section D.2.8 \[polylib.lib\], page 888](#)).

**Usage:** `maxcoef(f);` `f` poly/ideal/vector/module/matrix

**Return:** maximal length of coefficient of `f` of type `int` (by measuring the length of the string of each coefficient)

**Example:**

```

LIB "polylib.lib";
ring r= 0,(x,y,z),ds;
poly g = 345x2-1234567890y+7/4z;
maxcoef(g);
↳ 10
ideal i = g,10/1234567890;
maxcoef(i);
↳ 11
// since i[2]=1/123456789

```

### D.2.8.8 maxdeg

Procedure from library `polylib.lib` (see [Section D.2.8 \[polylib.lib\], page 888](#)).

**Usage:** `maxdeg(id);` `id` poly/ideal/vector/module/matrix

**Return:** `int/intmat`, each component equals maximal degree of monomials in the corresponding component of `id`, independent of ring ordering (`maxdeg` of each var is 1).  
Of type `int`, if `id` is of type `poly`; of type `intmat` otherwise

**Example:**

```

LIB "polylib.lib";
ring r = 0,(x,y,z),wp(1,2,3);
poly f = x+y2+z3;
deg(f); //deg; returns weighted degree (in case of 1 block)!
↳ 9
maxdeg(f);
↳ 3
matrix m[2][2]=f+x10,1,0,f^2;
maxdeg(m);
↳ 10,0,
↳ -1,6

```

See also: [Section D.2.8.9 \[maxdeg1\], page 892](#).

### D.2.8.9 maxdeg1

Procedure from library `polylib.lib` (see [Section D.2.8 \[polylib.lib\], page 888](#)).

**Usage:** `maxdeg1(id[,v]);` `id`=poly/ideal/vector/module/matrix, `v`=intvec

**Return:** integer, maximal [weighted] degree of monomials of `id` independent of ring ordering, `maxdeg1` of  $i$ -th variable is  $v[i]$  (default:  $v=1..1$ ).

**Note:** This proc returns one integer while `maxdeg` returns, in general, a matrix of integers. For one polynomial and if no intvec  $v$  is given `maxdeg` is faster

**Example:**

```
LIB "polylib.lib";
ring r = 0,(x,y,z),wp(1,2,3);
poly f = x+y^2+z^3;
deg(f);           //deg returns weighted degree (in case of 1 block)!
↳ 9
maxdeg1(f);
↳ 3
intvec v = ringweights(r);
maxdeg1(f,v);      //weighted maximal degree
↳ 9
matrix m[2][2]=f+x^10,1,0,f^2;
maxdeg1(m,v);      //absolute weighted maximal degree
↳ 18
```

### D.2.8.10 mindeg

Procedure from library `polylib.lib` (see [Section D.2.8 \[polylib.lib\]](#), page 888).

**Usage:** `mindeg(id)`; `id` poly/ideal/vector/module/matrix

**Return:** minimal degree/s of monomials of `id`, independent of ring ordering (mindeg of each variable is 1) of type int if `id` of type poly, else of type intmat.

**Example:**

```
LIB "polylib.lib";
ring r = 0,(x,y,z),ls;
poly f = x^5+y^2+z^3;
ord(f);           // ord returns weighted order of leading term!
↳ 3
mindeg(f);        // computes minimal degree
↳ 2
matrix m[2][2]=x^10,1,0,f^2;
mindeg(m);        // computes matrix of minimum degrees
↳ 10,0,
↳ -1,4
```

See also: [Section D.2.8.11 \[mindeg1\]](#), page 893.

### D.2.8.11 mindeg1

Procedure from library `polylib.lib` (see [Section D.2.8 \[polylib.lib\]](#), page 888).

**Usage:** `mindeg1(id[v])`; `id`=poly/ideal/vector/module/matrix,  $v$ =intvec

**Return:** integer, minimal [weighted] degree of monomials of `id` independent of ring ordering, `mindeg1` of  $i$ -th variable is  $v[i]$  (default  $v=1..1$ ).

**Note:** This proc returns one integer while `mindeg` returns, in general, a matrix of integers. For one polynomial and if no intvec  $v$  is given `mindeg` is faster.

**Example:**

```

LIB "polylib.lib";
ring r = 0,(x,y,z),ls;
poly f = x5+y2+z3;
ord(f);                      // ord returns weighted order of leading term!
↳ 3
intvec v = 1,-3,2;
mindeg1(f,v);                // computes minimal weighted degree
↳ -6
matrix m[2][2]=x10,1,0,f^2;
mindeg1(m,1..3);            // computes absolute minimum of weighted degrees
↳ -1

```

### D.2.8.12 normalize

Procedure from library `polylib.lib` (see [Section D.2.8 \[polylib.lib\], page 888](#)).

**Usage:** `normalize(id); id=poly/vector/ideal/module`

**Return:** object of same type  
each element is normalized with leading coefficient equal to 1

**Example:**

```

LIB "polylib.lib";
ring r = 0,(x,y,z),ls;
poly f = 2x5+3y2+4z3;
normalize(f);
↳ z3+3/4y2+1/2x5
module m=[9xy,0,3z3],[4z,6y,2x];
normalize(m);
↳ _[1]=z3*gen(3)+3xy*gen(1)
↳ _[2]=z*gen(1)+3/2y*gen(2)+1/2x*gen(3)
ring s = 0,(x,y,z),(c,ls);
module m=[9xy,0,3z3],[4z,6y,2x];
normalize(m);
↳ _[1]=[xy,0,1/3z3]
↳ _[2]=[z,3/2y,1/2x]

```

### D.2.8.13 rad\_con

Procedure from library `polylib.lib` (see [Section D.2.8 \[polylib.lib\], page 888](#)).

**Usage:** `rad_con(g,I); g polynomial, I ideal`

**Return:** 1 (TRUE) (type int) if  $g$  is contained in the radical of  $I$   
0 (FALSE) (type int) otherwise

**Example:**

```

LIB "polylib.lib";
ring R=0,(x,y,z),dp;
ideal I=x2+y2,z2;
poly f=x4+y4;
rad_con(f,I);
↳ 0
ideal J=x2+y2,z2,x4+y4;
poly g=z;
rad_con(g,I);
↳ 1

```

### D.2.8.14 content

Procedure from library `polylib.lib` (see [Section D.2.8 \[polylib\\_lib\]](#), page 888).

**Usage:** `content(f)`;  $f$  polynomial/vector

**Return:** number, the content (greatest common factor of coefficients) of the polynomial/vector  $f$

**Example:**

```
LIB "polylib.lib";
ring r=0,(x,y,z),(c,lp);
content(3x2+18xy-27xyz);
↪ 3
vector v=[3x2+18xy-27xyz,15x2+12y4,3];
content(v);
↪ 3
```

See also: [Section 5.1.9 \[cleardenom\]](#), page 163.

### D.2.8.15 mod2id

Procedure from library `polylib.lib` (see [Section D.2.8 \[polylib\\_lib\]](#), page 888).

**Usage:** `mod2id(M,vpos)`;  $M$  matrix,  $vpos$  intvec

**Assume:**  $vpos$  is an integer vector such that  $gen(i)$  corresponds to  $var(vpos[i])$ .  
The basering contains variables  $var(vpos[i])$  which do not occur in  $M$ .

**Return:** ideal  $I$  in which each  $gen(i)$  from the module is replaced by  $var(vpos[i])$  and all monomials  $var(vpos[i])*var(vpos[j])$  have been added to the generating set of  $I$ .

**Note:** This procedure should be used in the following situation: one wants to pass to a ring with new variables, say  $e(1), \dots, e(s)$ , which correspond to the components  $gen(1), \dots, gen(s)$  of the module  $M$  such that  $e(i)*e(j)=0$  for all  $i,j$ .  
The new ring should already exist and be the current ring

**Example:**

```
LIB "polylib.lib";
ring r=0,(e(1),e(2),x,y,z),(dp(2),ds(3));
module mo=x*gen(1)+y*gen(2);
intvec iv=2,1;
mod2id(mo,iv);
↪ _[1]=e(2)^2
↪ _[2]=e(1)*e(2)
↪ _[3]=e(1)^2
↪ _[4]=e(1)*y+e(2)*x
```

### D.2.8.16 id2mod

Procedure from library `polylib.lib` (see [Section D.2.8 \[polylib\\_lib\]](#), page 888).

**Usage:** `id2mod(I,vpos)`;  $I$  ideal,  $vpos$  intvec

**Return:** module corresponding to the ideal by replacing  $var(vpos[i])$  by  $gen(i)$  and omitting all generators  $var(vpos[i])*var(vpos[j])$

**Note:** \* This procedure only makes sense if the ideal contains all  $\text{var}(\text{vpos}[i]) \cdot \text{var}(\text{vpos}[j])$  as monomial generators and all other generators of  $I$  are linear combinations of the  $\text{var}(\text{vpos}[i])$  over the ring in the other variables.  
 \* This is the inverse procedure to `mod2id` and should be applied only to ideals created by `mod2id` using the same `intvec vpos` (possibly after a standard basis computation)

**Example:**

```
LIB "polylib.lib";
ring r=0,(e(1),e(2),x,y,z),(dp(2),ds(3));
ideal i=e(2)^2,e(1)*e(2),e(1)^2,e(1)*y+e(2)*x;
intvec iv=2,1;
id2mod(i,iv);
↦ _[1]=x*gen(1)+y*gen(2)
```

### D.2.8.17 substitute

Procedure from library `polylib.lib` (see [Section D.2.8 \[polylib.lib\]](#), page 888).

**Usage:** - case 1: `typeof(#[1]) == poly`:  
`substitute(I,v,f[,v1,f1,v2,f2,...])`;  $I$  object of basering which can be mapped,  $v, v_1, v_2, \dots$  ring variables,  $f, f_1, f_2, \dots$  poly  
 - case 2: `typeof(#[1]) == ideal`: `substitute(I,v,f)`;  $I$  object of basering which can be mapped,  $v$  ideal of ring variables,  $f$  ideal

**Return:** object of same type as  $I$ ,  
 - case 1: ring variable  $v, v_1, v_2, \dots$  substituted by polynomials  $f, f_1, f_2, \dots$ , in this order  
 - case 2: ring variables in  $v$  substituted by polynomials in  $f$ :  $v[i]$  is substituted by  $f[i]$ ,  $i=1, \dots, i=\min(\text{size}(v), \text{ncols}(f))$

**Note:** this procedure extends the built-in command `subst` via maps

**Example:**

```
LIB "polylib.lib";
ring r = 0, (b, c, t), dp;
ideal I = -bc+4b2c2t, bc2t-5b2c;
substitute(I, c, b+c, t, 0, b, b-1);
↦ _[1] = -b2-bc+2b+c-1
↦ _[2] = -5b3-5b2c+15b2+10bc-15b-5c+5
ideal v = c, t, b;
ideal f = b+c, 0, b-1;
substitute(I, v, f);
↦ _[1] = -b2-bc+b+c
↦ _[2] = -5b3-5b2c+10b2+10bc-5b-5c
```

### D.2.8.18 subrInterred

Procedure from library `polylib.lib` (see [Section D.2.8 \[polylib.lib\]](#), page 888).

**Usage:** `subrInterred(mon, sm, iv)`;  
 $sm$ : ideal in a ring  $r$  with  $n + s$  variables,  
 e.g.  $x_1, \dots, x_n$  and  $t_1, \dots, t_s$   
 $mon$ : ideal with monomial generators (not divisible by any of the  $t_i$ ) such that  $sm$  is contained in the module  $k[t_1, \dots, t_s] \cdot mon[1] + \dots + k[t_1, \dots, t_s] \cdot mon[\text{size}(mon)]$   
 $iv$ : `intvec` listing the variables which are supposed to be used as  $x_i$

**Return:** list l:  
 $l[1]$ =the monomials from mon in the order used  
 $l[2]$ =their coefficients after interreduction  
 $l[3]=l[1]*l[2]$

**Purpose:** Do interred only w.r.t. a subset of variables.  
 The procedure returns an interreduced system of generators of sm considered as a  $k[t_1, \dots, t_s]$ -submodule of the free module  $k[t_1, \dots, t_s]*\text{mon}[1] + \dots + k[t_1, \dots, t_s]*\text{mon}[\text{size}(\text{mon})]$ .

**Example:**

```
LIB "polylib.lib";
ring r=0,(x,y,z),dp;
ideal i=x^2+x*y^2,x*y+x^2*y,z;
ideal j=x^2+x*y^2,x*y,z;
ideal mon=x^2,z,x*y;
intvec iv=1,3;
subrInterred(mon,i,iv);
⇒ [1]:
⇒ _[1,1]=z
⇒ _[1,2]=xy
⇒ _[1,3]=x2
⇒ [2]:
⇒ _[1]=gen(1)
⇒ _[2]=y2*gen(2)-gen(2)
⇒ _[3]=y*gen(2)+gen(3)
⇒ [3]:
⇒ _[1,1]=z
⇒ _[1,2]=xy3-xy
⇒ _[1,3]=xy2+x2
subrInterred(mon,j,iv);
⇒ [1]:
⇒ _[1,1]=z
⇒ _[1,2]=xy
⇒ _[1,3]=x2
⇒ [2]:
⇒ _[1]=gen(1)
⇒ _[2]=gen(2)
⇒ _[3]=gen(3)
⇒ [3]:
⇒ _[1,1]=z
⇒ _[1,2]=xy
⇒ _[1,3]=x2
```

### D.2.8.19 newtonDiag

Procedure from library `polylib.lib` (see [Section D.2.8 \[polylib.lib\]](#), page 888).

**Usage:** `newtonDiag(f)`; f a poly

**Return:** intmat

**Purpose:** compute the Newton diagram of f

**Note:** each row is the exponent of a monomial of f

**Example:**

```
LIB "polylib.lib";
ring r = 0,(x,y,z),lp;
poly f = x2y+3xz-5y+3;
newtonDiag(f);
⇒ 2,1,0,
⇒ 1,0,1,
⇒ 0,1,0,
⇒ 0,0,0
```

**D.2.8.20 hilbPoly**

Procedure from library `polylib.lib` (see [Section D.2.8 \[polylib.lib\]](#), page 888).

**Usage:** `hilbPoly(I)`; I a homogeneous ideal

**Return:** the Hilbert polynomial of `basing/I` as an intvec  $v=v_0, \dots, v_r$  such that the Hilbert polynomial is  $(v_0+v_1*t+\dots+v_r*t^r)/r!$

**Example:**

```
LIB "polylib.lib";
ring r = 0,(b,c,t,h),dp;
ideal I=
bct-t2h+2th2+h3,
bt3-ct3-t4+b2th+c2th-2bt2h+2ct2h+2t3h-bch2-2bth2+2cth2+2th3,
b2c2+bt2h-ct2h-t3h+b2h2+2bch2+c2h2-2bth2+2cth2+t2h2-2bh3+2ch3+2th3+3h4,
c2t3+ct4-c3th-2c2t2h-2ct3h-t4h+bc2h2-2c2th2-bt2h2+4t3h2+2bth3-2cth3-t2h3
+bh4-6th4-2h5;
hilbPoly(I);
⇒ -11,10
```

**D.2.9 redcgs\_lib**

**Library:** `redcgs.lib`

**Purpose:** Reduced Comprehensive Groebner Systems.

**Overview:** Comprehensive Groebner Systems. Canonical Forms.

The library contains Monte's algorithms to compute disjoint, reduced Comprehensive Groebner Systems (CGS). A CGS is a set of pairs of (segment,basis). The segments  $S_i$  are subsets of the parameter space, and the bases  $B_i$  are sets of polynomials specializing to Groebner bases of the specialized ideal for every point in  $S_i$ .

The purpose of the routines in this library is to obtain CGS with better properties, namely disjoint segments forming a partition of the parameter space and reduced bases. Reduced bases are sets of polynomials that specialize to the reduced Groebner basis of the specialized ideal preserving the leading power products (lpp). The lpp characterize the type of solution in each segment.

A further objective is to summarize as much as possible the segments with the same lpp into a single segment, and if possible to obtain a final result that is canonical, i.e. independent of the algorithm and only attached to the given ideal.

There are three fundamental routines in the library: `mrcgs`, `rcgs` and `crcgs`. `mrcgs` (Minimal Reduced CGS) is an algorithm that packs so much as it is able to do (using algorithms `adhoc`) the segments with the same lpp, obtaining the minimal number of



segments. The hypothesis is that the result is also canonical, but for the moment there is no proof of the uniqueness of this minimal packing. Moreover, the segments that are obtained are not locally closed, i.e. there are not difference of two varieties.

On the other side, Michael Wibmer has proved that for homogeneous ideals, all the segments with reduced bases having the same lpp admit a unique basis specializing well. For this purpose it is necessary to extend the description of the elements of the bases to functions, forming sheaves of polynomials instead of simple polynomials, so that the polynomials in a sheaf either preserve the lpp of the corresponding polynomial of the specialized Groebner basis (and then it specializes well) or it specializes to 0. Moreover, in a sheaf, for every point in the corresponding segment, at least one of the polynomials specializes well. And moreover Wibmer's Theorem ensures that the packed segments are locally closed, that is can be described as the difference of two varieties.

Using Wibmer's Theorem we proved that an affine ideal can be homogenized, than discussed by mrcgs and finally de-homogenized. The bases so obtained can be reduced and specialize well in the segment. If the theoretic objective is reached, and all the segments of the homogenized ideal have been packed, locally closed segments will be obtained.

If we only homogenize the given basis of the ideal, then we cannot ensure the canonicity of the partition obtained, because there are many different bases of the given ideal that can be homogenized, and the homogenized ideals are not identical. This corresponds to the algorithm rcgs and is recommended as the most practical routine. It provides locally closed segments and is usually faster than mrcgs and crcgs. But the given partition is not always canonical.

Finally it is possible to homogenize the whole affine ideal, and then the packing algorithm will provide canonical segments by dehomogenizing. This corresponds to crcgs routine. It provides the best description of the segments and bases. In contrast crcgs algorithm is usually much more time consuming and it will not always finish in a reasonable time. Moreover it will contain more segments than mrcgs and possibly also more than rcgs.

But the actual algorithms in the library to pack segments have some lacks. They are not theoretically always able to pack the segments that we know that can be packed. Nevertheless, thanks to Wibmer's Theorem, the algorithms rcgs and crcgs are able to detect if the objective has not been reached, and if so, to give a Warning. The warning does not invalidate the output, but it only recognizes that the theoretical objective is not completely reached by the actual computing methods and that some segments that can be packed have not been packed with a single basis.

The routine buildtree is the first algorithm used in all the previous methods providing a first disjoint CGS, and can be used if none of the three fundamental algorithms of the library finishes in a reasonable time.

There are also routines to visualize better the output of the previous algorithms: finalcases can be applied to the list provided by buildtree to obtain the CGS. The list provided by buildtree contains the whole discussion, and finalcases extracts the CGS. The output of buildtree can also be transformed into a file using buildtreetoMaple routine that can be read in Maple. Using Monte's dpgb library in Maple the output can be plotted (with the routine tplot). To plot the output of mrcgs, rcgs or crcgs in Maple, the library also provides the routine cantreetoMaple. The file written using it and read in Maple can then be plotted with the command plotcantree and printed with printcantree from the Monte's dpgb library in Maple. The output of mrcgs,

rcgs and crcgs is given in form of tree using prime ideals in a canonical form that is described in the papers. Nevertheless this canonical form is somewhat uncomfortable to be interpreted. When the segments are all locally closed (and this is always the case for rcgs and crcgs) the routine cantodiffcgs transforms the output into a simpler form having only one list element for each segment and providing the two varieties whose difference represent the segment also in a canonical form.

**Authors:** Antonio Montes , Hans Schoenemann.

**Overview:** see "Minimal Reduced Comprehensive Groebner Systems" by Antonio Montes. (<http://www-ma2.upc.edu/~montes/>).

**Notations:** All given and determined polynomials and ideals are in the basering  $K[a][x]$ ; (a=parameters, x=variables)  
After defining the ring and calling setglobalrings(); the rings  
 $\mathcal{O}R$  ( $K[a][x]$ ),  
 $\mathcal{O}P$  ( $K[a]$ ),  
 $\mathcal{O}RP$  ( $K[x,a]$ ) are defined globally  
They are used internally and can also be used by the user.  
The fundamental routines are: buildtree, mrcgs, rcgs and crcgs

**Procedures:** See also: [Section D.2.2 \[compregb\\_lib\]](#), page 798.

### D.2.9.1 setglobalrings

Procedure from library redcgs.lib (see [Section D.2.9 \[redcgs\\_lib\]](#), page 898).

**Usage:** setglobalrings();  
No arguments

**Return:** After its call the rings  $\mathcal{O}R=K[a][x]$ ,  $\mathcal{O}P=K[a]$ ,  $\mathcal{O}RP=K[x,a]$  are defined as global variables.

**Note:** It is called by the fundamental routines of the library. The user does not need to call it, except when none of the fundamental routines have been called and some other routines of the library are used.  
The basering R, must be of the form  $K[a][x]$ , a=parameters, x=variables, and should be defined previously.

**Example:**

```
LIB "redcgs.lib";
ring R=(0,a,b),(x,y,z),dp;
setglobalrings();
OR;
⇒ // coefficients: QQ(a, b)
⇒ // number of vars : 3
⇒ //      block 1 : ordering dp
⇒ //      : names x y z
⇒ //      block 2 : ordering C
OP;
⇒ // coefficients: QQ
⇒ // number of vars : 2
⇒ //      block 1 : ordering lp
⇒ //      : names a b
⇒ //      block 2 : ordering C
```

```

@RP;
↪ // coefficients: QQ
↪ // number of vars : 5
↪ //      block 1 : ordering dp
↪ //      : names x y z
↪ //      block 2 : ordering lp
↪ //      : names a b
↪ //      block 3 : ordering C

```

### D.2.9.2 memberpos

Procedure from library `redcgs.lib` (see [Section D.2.9 \[redcgs.lib\]](#), page 898).

**Usage:**      `memberpos(f,J);`  
               `(f,J)` expected (polynomial,ideal)  
               or `(int,list(int))`  
               or `(int,intvec)`  
               or `(intvec,list(intvec))`  
               or `(list(int),list(list(int)))`  
               or `(ideal,list(ideal))`  
               or `(list(intvec), list(list(intvec)))`.  
               The ring can be `@R` or `@P` or `@RP` or any other.

**Return:**    The list `(t,pos)` `t` int; `pos` int;  
               `t` is 1 if `f` belongs to `J` and 0 if not.  
               `pos` gives the position in `J` (or 0 if `f` does not belong).

**Example:**

```

LIB "redcgs.lib";
list L=(7,4,5,1,1,4,9);
memberpos(1,L);
↪ [1]:
↪ 1
↪ [2]:
↪ 4

```

### D.2.9.3 subset

Procedure from library `redcgs.lib` (see [Section D.2.9 \[redcgs.lib\]](#), page 898).

**Usage:**      `subset(J,K);`  
               `(J,K)` expected (ideal,ideal)  
               or `(list, list)`

**Return:**    1 if all the elements of `J` are in `K`, 0 if not.

**Example:**

```

LIB "redcgs.lib";
list J=list(7,3,2);
list K=list(1,2,3,5,7,8);
subset(J,K);
↪ 1

```

### D.2.9.4 pdivi2

Procedure from library `redcgs.lib` (see [Section D.2.9 \[redcgs.lib\]](#), page 898).

**Usage:**     `pdivi2(f,F);`  
               poly f: the polynomial to be divided  
               ideal F: the divisor ideal

**Return:**    A list (poly r, ideal q, poly m). r is the remainder of the pseudodivision, q is the ideal of quotients, and m is the factor by which f is to be multiplied.

**Note:**      Pseudodivision of a polynomial f by an ideal F in  $\mathcal{O}R$ . Returns a list (r,q,m) such that  $m*f=r+\text{sum}(q.G)$ .

**Example:**

```
LIB "redcgs.lib";
ring R=(0,a,b,c),(x,y),dp;
setglobalrings();
poly f=(ab-ac)*xy+(ab)*x+(5c);
ideal F=ax+b,cy+a;
def r=pdivi2(f,F);
r;
⇒ [1]:
⇒ (ab2-abc-b2c+5c2)
⇒ [2]:
⇒ _[1]=(bc-c2)*y+(bc)
⇒ _[2]=(-b2+bc)
⇒ [3]:
⇒ (c)
r[3]*f-(r[2][1]*F[1]+r[2][2]*F[2])-r[1];
⇒ 0
```

### D.2.9.5 facvar

Procedure from library `redcgs.lib` (see [Section D.2.9 \[redcgs.lib\]](#), page 898).

**Usage:**     `facvar(J);`  
               J: an ideal in the parameters

**Return:**    all the free-square factors of the elements  
               of ideal J (non repeated). Integer factors are ignored, even 0 is ignored. It can be called  
               from ideal  $\mathcal{O}R$ , but the given ideal J must only contain poynomials in the parameters.

**Note:**      Operates in the ring  $\mathcal{O}P$ , and the ideal J must contain only polynomials in the param-  
               eters, but can be called from ring  $\mathcal{O}R$ .

**Example:**

```
LIB "redcgs.lib";
ring R=(0,a,b,c),(x,y,z),dp;
setglobalrings();
ideal J=a2-b2,a2-2ab+b2,abc-bc;
facvar(J);
⇒ _[1]=(a-b)
⇒ _[2]=(a+b)
⇒ _[3]=(a-1)
⇒ _[4]=(b)
⇒ _[5]=(c)
```

### D.2.9.6 redspec

Procedure from library `redcgs.lib` (see [Section D.2.9 \[redcgs.lib\]](#), page 898).

- Usage:**      `redspec(N,W);`  
               N: null conditions ideal  
               W: set of non-null polynomials (ideal)
- Return:**     a list (N1,W1,L1) containing a red-specification of the segment (N,W). N1 is the radical reduced ideal characterizing the segment.  $V(N1)$  is the Zarisky closure of the segment (N,W).  
               The segment  $S=V(N1) \setminus V(h)$ , where  $h=\text{prod}(w \text{ in } W1)$   
               N1 is uniquely determined and no prime component of N1 contains none of the polynomials in W1. The polynomials in W1 are prime and reduced w.r.t. N1, and are considered non-null on the segment. L1 contains the list of prime components of N1.
- Note:**        can be called from ring @R but it works in ring @P.

**Example:**

```
LIB "redcgs.lib";
ring r=(0,a,b,c),(x,y),dp;
setglobalrings();
ideal N=(ab-c)*(a-b),(a-bc)*(a-b);
ideal W=a^2-b^2,bc;
redspec(N,W);
⇒ [1]:
⇒   _[1]=(b2-1)
⇒   _[2]=(a-bc)
⇒ [2]:
⇒   _[1]=(b)
⇒   _[2]=(c-1)
⇒   _[3]=(c+1)
⇒   _[4]=(c)
⇒ [3]:
⇒   [1]:
⇒     _[1]=(b+1)
⇒     _[2]=(a+c)
⇒   [2]:
⇒     _[1]=(b-1)
⇒     _[2]=(a-c)
```

### D.2.9.7 pnormalform

Procedure from library `redcgs.lib` (see [Section D.2.9 \[redcgs.lib\]](#), page 898).

- Usage:**      `pnormalform(f,N,W);`  
               f: the polynomial to be reduced modulo N,W (in parameters and variables)  
               N: the null conditions ideal  
               W: the non-null conditions (set of irreducible polynomials, ideal)
- Return:**     a reduced polynomial g of f, whose coefficients are reduced modulo N and having no factor in W.
- Note:**        Should be called from ring @R. Ideals N and W must be polynomials in the parameters forming a red-specification (see definition) the papers).

**Example:**

```

LIB "redcgs.lib";
ring R=(0,a,b,c),(x,y),dp;
setglobalrings();
poly f=(b^2-1)*x^3*y+(c^2-1)*x*y^2+(c^2*b-b)*x+(a-bc)*y;
ideal N=(ab-c)*(a-b),(a-bc)*(a-b);
ideal W=a^2-b^2,bc;
def r=redspec(N,W);
pnormalform(f,r[1],r[2]);
 $\mapsto xy^2+(b)*x$ 

```

**D.2.9.8 buildtree**

Procedure from library `redcgs.lib` (see [Section D.2.9 \[redcgs.lib\], page 898](#)).

**Usage:** `buildtree(F);`

F: ideal in  $K[a][x]$  (parameters and variables) to be discussed

**Return:** Returns a list T describing a dichotomic discussion tree, whose content is the first discussion of the ideal F of  $K[a][x]$ . The first element of the list is the root, and contains

- [1] label: `intvec(-1)`
- [2] number of children : `int`
- [3] the ideal F
- [4], [5], [6] the red-spec of the null and non-null conditions given (as option). ideal (0), ideal (0), `list(ideal(0))` if no optional conditions are given.
- [7] the set of lpp of ideal F
- [8] condition that was taken to reach the vertex

(poly 1, for the root).

The remaining elements of the list represent vertices of the tree: with the same structure:

- [1] label: `intvec (1,0,0,1,...)` gives its position in the tree: first branch condition is taken non-null, second null,...
- [2] number of children (0 if it is a terminal vertex)
- [3] the specialized ideal with the previous assumed conditions to reach the vertex
- [4],[5],[6] the red-spec of the previous assumed conditions to reach the vertex
- [7] the set of lpp of the specialized ideal at this stage
- [8] condition that was taken to reach the vertex from the father's vertex (that was taken non-null if the last integer in the label is 1, and null if it is 0)

The terminal vertices form a disjoint partition of the parameter space whose bases specialize to the reduced Groebner basis of the specialized ideal on each point of the segment and preserve the lpp. So they form a disjoint reduced CGS.

**Note:** The basering R, must be of the form  $K[a][x]$ , a=parameters, x=variables, and should be defined previously. The ideal must be defined on R.

The disjoint and reduced CGS built by `buildtree` can be obtained from the output of `buildtree` by calling `finalcases(T)`; this selects the terminal vertices.

The content of `buildtree` can be written in a file that is readable by Maple in order to plot its content using `buildtreetoMaple`; The file written by `buildtreetoMaple` when read in a Maple worksheet can be plotted using the `dbg` routine `tplot`;

**Example:**

```

LIB "redcgs.lib";
ring R=(0,a1,a2,a3,a4),(x1,x2,x3,x4),dp;
ideal F=x4-a4+a2,

```

```

x1+x2+x3+x4-a1-a3-a4,
x1*x3*x4-a1*a3*a4,
x1*x3+x1*x4+x2*x3+x3*x4-a1*a4-a1*a3-a3*a4;
def T=buildtree(F);
finalcases(T);
↳ [1]:
↳   [1]:
↳   0,0
↳   [2]:
↳   1
↳   [3]:
↳   _[1]=x4
↳   _[2]=-x1-x2-x3+(a1+a3+a4)
↳   _[3]=x3^2+(-a1-a3-a4)*x3+(a1*a3+a1*a4+a3*a4)
↳   [4]:
↳   _[1]=(a2-a4)
↳   _[2]=(a1*a3*a4)
↳   [5]:
↳   _[1]=0
↳   [6]:
↳   [1]:
↳   _[1]=(a4)
↳   _[2]=(a2)
↳   [2]:
↳   _[1]=(a3)
↳   _[2]=(a2-a4)
↳   [3]:
↳   _[1]=(a2-a4)
↳   _[2]=(a1)
↳   [7]:
↳   _[1]=x4
↳   _[2]=x1
↳   _[3]=x3^2
↳   [8]:
↳   1
↳ [2]:
↳   [1]:
↳   0,1
↳   [2]:
↳   1
↳   [3]:
↳   _[1]=1
↳   [4]:
↳   _[1]=(a2-a4)
↳   [5]:
↳   _[1]=(a1)
↳   _[2]=(a3)
↳   _[3]=(a4)
↳   [6]:
↳   [1]:
↳   _[1]=(a2-a4)
↳   [7]:
↳   _[1]=1

```

```

⇒      [8]:
⇒      1
⇒      [3]:
⇒      [1]:
⇒      1
⇒      [2]:
⇒      1
⇒      [3]:
⇒      _[1]=x4+(a2-a4)
⇒      _[2]=-x1-x2-x3+(a1+a2+a3)
⇒      _[3]=x3^2+(-a2+a4)*x2+(-a1-a2-a3)*x3+(a1*a2+a1*a3+a2^2+a2*a3-a2*a4)
⇒      _[4]=(a2-a4)*x2*x3+(a2^2-2*a2*a4+a4^2)*x2+(-a1*a2^2-a1*a2*a3+a1*a2*\
a4-a2^3-a2^2*a3+2*a2^2*a4+a2*a3*a4-a2*a4^2)
⇒      _[5]=(a2^2-2*a2*a4+a4^2)*x2^2+(-2*a1*a2^2-a1*a2*a3+3*a1*a2*a4+a1*a3\
*a4-a1*a4^2-3*a2^3-2*a2^2*a3+7*a2^2*a4+3*a2*a3*a4-5*a2*a4^2-a3*a4^2+a4^3)\
*x2+(-a1*a2^2-a1*a2*a3+a1*a2*a4-a2^3-a2^2*a3+2*a2^2*a4+a2*a3*a4-a2*a4^2)*\
x3+(a1^2*a2^2+a1^2*a2*a3-a1^2*a2*a4+3*a1*a2^3+4*a1*a2^2*a3-5*a1*a2^2*a4+a\
1*a2*a3^2-3*a1*a2*a3*a4+2*a1*a2*a4^2+2*a2^4+3*a2^3*a3-5*a2^3*a4+a2^2*a3^2\
-5*a2^2*a3*a4+4*a2^2*a4^2-a2*a3^2*a4+2*a2*a3*a4^2-a2*a4^3)
⇒      [4]:
⇒      _[1]=0
⇒      [5]:
⇒      _[1]=(a2-a4)
⇒      [6]:
⇒      [1]:
⇒      _[1]=0
⇒      [7]:
⇒      _[1]=x4
⇒      _[2]=x1
⇒      _[3]=x3^2
⇒      _[4]=x2*x3
⇒      _[5]=x2^2
⇒      [8]:
⇒      1
buildtreetoMaple(T,"Tb","Tb.txt");

```

### D.2.9.9 buildtreetoMaple

Procedure from library `redcgs.lib` (see [Section D.2.9 \[redcgs.lib\]](#), page 898).

**Usage:** `buildtreetoMaple(T, TM, writefile);`  
T: is the list provided by `buildtree`,  
TM: is the name (string) of the table variable in Maple that will represent the output of `buildtree`,  
writefile: is the name (string) of the file where to write the content.

**Return:** writes the list provided by `buildtree` to a file containing the table representing it in Maple.

**Example:**

```

LIB "redcgs.lib";
ring R=(0,a1,a2,a3,a4),(x1,x2,x3,x4),dp;
ideal F=x4-a4+a2,
x1+x2+x3+x4-a1-a3-a4,

```



```

x1*x3*x4-a1*a3*a4,
x1*x3+x1*x4+x2*x3+x3*x4-a1*a4-a1*a3-a3*a4;
def T=buildtree(F);
finalcases(T);
↳ [1]:
↳   [1]:
↳   0,0
↳   [2]:
↳   1
↳   [3]:
↳   _[1]=x4
↳   _[2]=-x1-x2-x3+(a1+a3+a4)
↳   _[3]=x3^2+(-a1-a3-a4)*x3+(a1*a3+a1*a4+a3*a4)
↳   [4]:
↳   _[1]=(a2-a4)
↳   _[2]=(a1*a3*a4)
↳   [5]:
↳   _[1]=0
↳   [6]:
↳   [1]:
↳   _[1]=(a4)
↳   _[2]=(a2)
↳   [2]:
↳   _[1]=(a3)
↳   _[2]=(a2-a4)
↳   [3]:
↳   _[1]=(a2-a4)
↳   _[2]=(a1)
↳   [7]:
↳   _[1]=x4
↳   _[2]=x1
↳   _[3]=x3^2
↳   [8]:
↳   1
↳ [2]:
↳   [1]:
↳   0,1
↳   [2]:
↳   1
↳   [3]:
↳   _[1]=1
↳   [4]:
↳   _[1]=(a2-a4)
↳   [5]:
↳   _[1]=(a1)
↳   _[2]=(a3)
↳   _[3]=(a4)
↳   [6]:
↳   [1]:
↳   _[1]=(a2-a4)
↳   [7]:
↳   _[1]=1
↳   [8]:

```

```

⇒      1
⇒ [3]:
⇒      [1]:
⇒      1
⇒      [2]:
⇒      1
⇒      [3]:
⇒      _[1]=x4+(a2-a4)
⇒      _[2]=-x1-x2-x3+(a1+a2+a3)
⇒      _[3]=x3^2+(-a2+a4)*x2+(-a1-a2-a3)*x3+(a1*a2+a1*a3+a2^2+a2*a3-a2*a4)
⇒      _[4]=(a2-a4)*x2*x3+(a2^2-2*a2*a4+a4^2)*x2+(-a1*a2^2-a1*a2*a3+a1*a2*\
a4-a2^3-a2^2*a3+2*a2^2*a4+a2*a3*a4-a2*a4^2)
⇒      _[5]=(a2^2-2*a2*a4+a4^2)*x2^2+(-2*a1*a2^2-a1*a2*a3+3*a1*a2*a4+a1*a3\
*a4-a1*a4^2-3*a2^3-2*a2^2*a3+7*a2^2*a4+3*a2*a3*a4-5*a2*a4^2-a3*a4^2+a4^3)\
*x2+(-a1*a2^2-a1*a2*a3+a1*a2*a4-a2^3-a2^2*a3+2*a2^2*a4+a2*a3*a4-a2*a4^2)*\
x3+(a1^2*a2^2+a1^2*a2*a3-a1^2*a2*a4+3*a1*a2^3+4*a1*a2^2*a3-5*a1*a2^2*a4+a\
1*a2*a3^2-3*a1*a2*a3*a4+2*a1*a2*a4^2+2*a2^4+3*a2^3*a3-5*a2^3*a4+a2^2*a3^2\
-5*a2^2*a3*a4+4*a2^2*a4^2-a2*a3^2*a4+2*a2*a3*a4^2-a2*a4^3)
⇒      [4]:
⇒      _[1]=0
⇒      [5]:
⇒      _[1]=(a2-a4)
⇒      [6]:
⇒      [1]:
⇒      _[1]=0
⇒      [7]:
⇒      _[1]=x4
⇒      _[2]=x1
⇒      _[3]=x3^2
⇒      _[4]=x2*x3
⇒      _[5]=x2^2
⇒      [8]:
⇒      1
buildtreetoMaple(T,"Tb","Tb.txt");

```

### D.2.9.10 finalcases

Procedure from library `redcgs.lib` (see [Section D.2.9 \[redcgs.lib\]](#), page 898).

**Usage:** `finalcases(T);`

T is the list provided by `buildtree`

**Return:** A list with the CGS determined by `buildtree`.

Each element of the list represents one segment of the `buildtree` CGS.

The list elements have the following structure:

[1]: label (an `intvec(1,0,..)`) that indicates the position in the `buildtree` but that is irrelevant for the CGS

[2]: 1 (integer) it is also irrelevant and indicates that this was a terminal vertex in `buildtree`.

[3]: the reduced basis of the segment.

[4], [5], [6]: the red-spec of the null and non-null conditions of the segment.

[4] is the null-conditions radical ideal N,

[5] is the non-null polynomials set (ideal) W,

[6] is the set of prime components (ideals) of  $N$ .

[7]: is the set of lpp

[8]: poly 1 (irrelevant) is the condition to branch (but no more branch is necessary in the discussion, so 1 is the result).

**Note:** It can be called having as argument the list output by buildtree

**Example:**

```
LIB "redcgs.lib";
ring R=(0,a1,a2,a3,a4),(x1,x2,x3,x4),dp;
ideal F=x4-a4+a2, x1+x2+x3+x4-a1-a3-a4, x1*x3*x4-a1*a3*a4, x1*x3+x1*x4+x2*x3+x3*x4-a1*a3*a4;
def T=buildtree(F);
finalcases(T);
⇒ [1]:
⇒ [1]:
⇒ 0,0
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ _[1]=x4
⇒ _[2]=-x1-x2-x3+(a1+a3+a4)
⇒ _[3]=x3^2+(-a1-a3-a4)*x3+(a1*a3+a1*a4+a3*a4)
⇒ [4]:
⇒ _[1]=(a2-a4)
⇒ _[2]=(a1*a3*a4)
⇒ [5]:
⇒ _[1]=0
⇒ [6]:
⇒ [1]:
⇒ _[1]=(a4)
⇒ _[2]=(a2)
⇒ [2]:
⇒ _[1]=(a3)
⇒ _[2]=(a2-a4)
⇒ [3]:
⇒ _[1]=(a2-a4)
⇒ _[2]=(a1)
⇒ [7]:
⇒ _[1]=x4
⇒ _[2]=x1
⇒ _[3]=x3^2
⇒ [8]:
⇒ 1
⇒ [2]:
⇒ [1]:
⇒ 0,1
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ _[1]=1
⇒ [4]:
⇒ _[1]=(a2-a4)
⇒ [5]:
⇒ _[1]=(a1)
```

```

⇒      _[2]=(a3)
⇒      _[3]=(a4)
⇒      [6]:
⇒      [1]:
⇒      _[1]=(a2-a4)
⇒      [7]:
⇒      _[1]=1
⇒      [8]:
⇒      1
⇒      [3]:
⇒      [1]:
⇒      1
⇒      [2]:
⇒      1
⇒      [3]:
⇒      _[1]=x4+(a2-a4)
⇒      _[2]=-x1-x2-x3+(a1+a2+a3)
⇒      _[3]=x3^2+(-a2+a4)*x2+(-a1-a2-a3)*x3+(a1*a2+a1*a3+a2^2+a2*a3-a2*a4)
⇒      _[4]=(a2-a4)*x2*x3+(a2^2-2*a2*a4+a4^2)*x2+(-a1*a2^2-a1*a2*a3+a1*a2*\
a4-a2^3-a2^2*a3+2*a2^2*a4+a2*a3*a4-a2*a4^2)
⇒      _[5]=(a2^2-2*a2*a4+a4^2)*x2^2+(-2*a1*a2^2-a1*a2*a3+3*a1*a2*a4+a1*a3\
*a4-a1*a4^2-3*a2^3-2*a2^2*a3+7*a2^2*a4+3*a2*a3*a4-5*a2*a4^2-a3*a4^2+a4^3)\
*x2+(-a1*a2^2-a1*a2*a3+a1*a2*a4-a2^3-a2^2*a3+2*a2^2*a4+a2*a3*a4-a2*a4^2)*\
x3+(a1^2*a2^2+a1^2*a2*a3-a1^2*a2*a4+3*a1*a2^3+4*a1*a2^2*a3-5*a1*a2^2*a4+a\
1*a2*a3^2-3*a1*a2*a3*a4+2*a1*a2*a4^2+2*a2^4+3*a2^3*a3-5*a2^3*a4+a2^2*a3^2\
-5*a2^2*a3*a4+4*a2^2*a4^2-a2*a3^2*a4+2*a2*a3*a4^2-a2*a4^3)
⇒      [4]:
⇒      _[1]=0
⇒      [5]:
⇒      _[1]=(a2-a4)
⇒      [6]:
⇒      [1]:
⇒      _[1]=0
⇒      [7]:
⇒      _[1]=x4
⇒      _[2]=x1
⇒      _[3]=x3^2
⇒      _[4]=x2*x3
⇒      _[5]=x2^2
⇒      [8]:
⇒      1

```

### D.2.9.11 mrcgs

Procedure from library `redcgs.lib` (see [Section D.2.9 \[redcgs.lib\]](#), page 898).

**Usage:** `mrcgs(F);`  
`F` is the ideal from which to obtain the Minimal Reduced CGS. Alternatively, as option:  
`mrcgs(F,L);`  
where `L` is a list of the null conditions ideal `N`, and `W` the set of non-null polynomials (ideal). If this option is set, the ideals `N` and `W` must depend only on the parameters and the parameter space is reduced to  $V(N) \setminus V(h)$ , where  $h = \text{prod}(w)$ , for  $w$  in `W`.

A reduced specification of  $(N, W)$  will be computed and used to restrict the parameter-space. The output will omit the known restrictions given as option.

**Return:** The list representing the Minimal Reduced CGS.  
The description given here is identical for rcgs and crcgs. The elements of the list T computed by mrcgs are lists representing a rooted tree.  
Each element has as the two first entries with the following content:  
[1]: The label (intvec) representing the position in the rooted tree: 0 for the root (and this is a special element) i for the root of the segment i  
(i,...) for the children of the segment i  
[2]: the number of children (int) of the vertex.  
There thus three kind of vertices:  
1) the root (first element labelled 0),  
2) the vertices labelled with a single integer i,  
3) the rest of vertices labelled with more indices.  
Description of the root. Vertex type 1)  
There is a special vertex (the first one) whose content is the following:  
[3] lpp of the given ideal  
[4] the given ideal  
[5] the red-spec of the (optional) given null and non-null conditions (see redspec for the description)  
[6] MRCGS (to remember which algorithm has been used). If the algorithm used is rcgs or crcgs then this will be stated at this vertex (RCGS or CRCGS).  
Description of vertices type 2). These are the vertices that initiate a segment, and are labelled with a single integer. [3] lpp (ideal) of the reduced basis. If they are repeated lpp's this will correspond to a sheaf.  
[4] the reduced basis (ideal) of the segment.  
Description of vertices type 3). These vertices have as first label i and descend form vertex i in the position of the label (i,...). They contain moreover a unique prime ideal in the parameters and form ascending chains of ideals.  
How is to be read the mrcgs tree? The vertices with an even number of integers in the label are to be considered as additive and those with an odd number of integers in the label are to be considered as subtraction. As an example consider the following vertices:  $v1=((i),2,lpp,B)$ ,  
 $v2=((i,1),2,P_{-}\{(i,1)\})$ ,  
 $v3=((i,1,1),2,P_{-}\{(i,1,1)\})$ ,  
 $v4=((i,1,1,1),1,P_{-}\{(i,1,1,1)\})$ ,  
 $v5=((i,1,1,1,1),0,P_{-}\{(i,1,1,1,1)\})$ ,  
 $v6=((i,1,1,2),1,P_{-}\{(i,1,1,2)\})$ ,  
 $v7=((i,1,1,2,1),0,P_{-}\{(i,1,1,2,1)\})$ ,  
 $v8=((i,1,2),0,P_{-}\{(i,1,2)\})$ ,  
 $v9=((i,2),1,P_{-}\{(i,2)\})$ ,  
 $v10=((i,2,1),0,P_{-}\{(i,2,1)\})$ ,  
They represent the segment:  
 $(V(i,1) \setminus (((V(i,1,1) \setminus ((V(i,1,1,1) \setminus V(i,1,1,1,1)) \cup (V(i,1,1,2) \setminus V(i,1,1,2,1)))) \cup V(i,1,2)) \cup (V(i,2) \setminus V(i,2,1)))$   
and can also be represented by  
 $(V(i,1) \setminus (V(i,1,1) \cup V(i,1,2))) \cup$   
 $(V(i,1,1,1) \setminus V(i,1,1,1)) \cup$   
 $(V(i,1,1,2) \setminus V(i,1,1,2,1)) \cup$

$(V(i,2) \setminus V(i,2,1))$   
 where  $V(i,j,..) = V(P_{-\{i,j,..\}})$

**Note:** There are three fundamental routines in the library: `mrcgs`, `rcgs` and `crcgs`. `mrcgs` (Minimal Reduced CGS) is an algorithm that packs so much as it is able to do (using algorithms `adhoc`) the segments with the same `lpp`, obtaining the minimal number of segments. The hypothesis is that this is also canonical, but for the moment there is no proof of the uniqueness of that minimal packing. Moreover, the segments that are obtained are not locally closed, i.e. there are not always the difference of two varieties, but can be a union of differences.

The output can be visualized using `cantreetoMaple`, that will write a file with the content of `mrcgs` that can be read in Maple and plotted using the Maple `plotcantree` routine of the Monte's `dpgb` library. You can also try the routine `cantodiffcgs` when the segments are all difference of two varieties to have a simpler view of the output. But it will give an error if the output is not locally closed.

**Example:**

```
LIB "redcgs.lib";
ring R=(0,b,c,d,e,f),(x,y),dp;
ideal F=x^2+b*y^2+2*c*x*y+2*d*x+2*e*y+f, 2*x+2*c*y+2*d, 2*b*y+2*c*x+2*e;
def T=mrcgs(F);
T;
↳ [1]:
↳ [1]:
↳ 0
↳ [2]:
↳ 3
↳ [3]:
↳ _[1]=x2
↳ _[2]=x
↳ _[3]=x
↳ [4]:
↳ _[1]=x2+(2c)*xy+(b)*y2+(2d)*x+(2e)*y+(f)
↳ _[2]=2*x+(2c)*y+(2d)
↳ _[3]=(2c)*x+(2b)*y+(2e)
↳ [5]:
↳ [1]:
↳ _[1]=0
↳ [2]:
↳ _[1]=0
↳ [3]:
↳ [1]:
↳ _[1]=0
↳ [6]:
↳ MRCGS
↳ [2]:
↳ [1]:
↳ 1
↳ [2]:
↳ 1
↳ [3]:
↳ _[1]=1
↳ [4]:
```

```

⇒      _[1]=1
⇒ [3]:
⇒      [1]:
⇒      1,1
⇒      [2]:
⇒      1
⇒      [3]:
⇒      _[1]=0
⇒ [4]:
⇒      [1]:
⇒      1,1,1
⇒      [2]:
⇒      1
⇒      [3]:
⇒      _[1]=(bd2-bf+c2f-2cde+e2)
⇒ [5]:
⇒      [1]:
⇒      1,1,1,1
⇒      [2]:
⇒      1
⇒      [3]:
⇒      _[1]=(cd-e)
⇒      _[2]=(b-c2)
⇒ [6]:
⇒      [1]:
⇒      1,1,1,1,1
⇒      [2]:
⇒      0
⇒      [3]:
⇒      _[1]=(d2-f)
⇒      _[2]=(cf-de)
⇒      _[3]=(cd-e)
⇒      _[4]=(b-c2)
⇒ [7]:
⇒      [1]:
⇒      2
⇒      [2]:
⇒      1
⇒      [3]:
⇒      _[1]=y
⇒      _[2]=x
⇒      [4]:
⇒      _[1]=(b-c2)*y+(-cd+e)
⇒      _[2]=(-b+c2)*x+(-bd+ce)
⇒ [8]:
⇒      [1]:
⇒      2,1
⇒      [2]:
⇒      1
⇒      [3]:
⇒      _[1]=(bd2-bf+c2f-2cde+e2)
⇒ [9]:
⇒      [1]:

```

```

⇒      2,1,1
⇒      [2]:
⇒      0
⇒      [3]:
⇒      _[1]=(cd-e)
⇒      _[2]=(b-c2)
⇒ [10]:
⇒      [1]:
⇒      3
⇒      [2]:
⇒      1
⇒      [3]:
⇒      _[1]=x
⇒      [4]:
⇒      _[1]=x+(c)*y+(d)
⇒ [11]:
⇒      [1]:
⇒      3,1
⇒      [2]:
⇒      1
⇒      [3]:
⇒      _[1]=(d2-f)
⇒      _[2]=(cf-de)
⇒      _[3]=(cd-e)
⇒      _[4]=(b-c2)
⇒ [12]:
⇒      [1]:
⇒      3,1,1
⇒      [2]:
⇒      0
⇒      [3]:
⇒      _[1]=1
cantreetoMaple(T,"Tm","Tm.txt");
//cantodiffcgs(T); // has non locally closed segments
ring R=(0,a1,a2,a3,a4),(x1,x2,x3,x4),dp;
⇒ // ** redefining R (ring R=(0,a1,a2,a3,a4),(x1,x2,x3,x4),dp;) ./examples/\
    mrcgs.sing:8
ideal F2=x4-a4+a2, x1+x2+x3+x4-a1-a3-a4, x1*x3*x4-a1*a3*a4, x1*x3+x1*x4+x2*x3+x3*x4-a4;
def T2=mrcgs(F2);
⇒ // ** redefining @R ( exportto(Top,@R); // global ring K[a][x])
⇒ // ** redefining @P ( exportto(Top,@P); // global ring K[a])
⇒ // ** redefining @RP ( exportto(Top,@RP); // global ring K[x,a] with\
    product order)
T2;
⇒ [1]:
⇒      [1]:
⇒      0
⇒      [2]:
⇒      3
⇒      [3]:
⇒      _[1]=x4
⇒      _[2]=x1
⇒      _[3]=x1*x3*x4

```



```

⇒      _[4]=x1*x3
⇒      [4]:
⇒      _[1]=x4+(a2-a4)
⇒      _[2]=x1+x2+x3+x4+(-a1-a3-a4)
⇒      _[3]=x1*x3*x4+(-a1*a3*a4)
⇒      _[4]=x1*x3+x2*x3+x1*x4+x3*x4+(-a1*a3-a1*a4-a3*a4)
⇒      [5]:
⇒      [1]:
⇒      _[1]=0
⇒      [2]:
⇒      _[1]=0
⇒      [3]:
⇒      [1]:
⇒      _[1]=0
⇒      [6]:
⇒      MRCGS
⇒      [2]:
⇒      [1]:
⇒      1
⇒      [2]:
⇒      1
⇒      [3]:
⇒      _[1]=x4
⇒      _[2]=x1
⇒      _[3]=x3^2
⇒      _[4]=x2*x3
⇒      _[5]=x2^2
⇒      [4]:
⇒      _[1]=x4+(a2-a4)
⇒      _[2]=-x1-x2-x3+(a1+a2+a3)
⇒      _[3]=x3^2+(-a2+a4)*x2+(-a1-a2-a3)*x3+(a1*a2+a1*a3+a2^2+a2*a3-a2*a4)
⇒      _[4]=(a2-a4)*x2*x3+(a2^2-2*a2*a4+a4^2)*x2+(-a1*a2^2-a1*a2*a3+a1*a2*
a4-a2^3-a2^2*a3+2*a2^2*a4+a2*a3*a4-a2*a4^2)
⇒      _[5]=(a2^2-2*a2*a4+a4^2)*x2^2+(-2*a1*a2^2-a1*a2*a3+3*a1*a2*a4+a1*a3\
*a4-a1*a4^2-3*a2^3-2*a2^2*a3+7*a2^2*a4+3*a2*a3*a4-5*a2*a4^2-a3*a4^2+a4^3)\
*x2+(-a1*a2^2-a1*a2*a3+a1*a2*a4-a2^3-a2^2*a3+2*a2^2*a4+a2*a3*a4-a2*a4^2)*\
x3+(a1^2*a2^2+a1^2*a2*a3-a1^2*a2*a4+3*a1*a2^3+4*a1*a2^2*a3-5*a1*a2^2*a4+a\
1*a2*a3^2-3*a1*a2*a3*a4+2*a1*a2*a4^2+2*a2^4+3*a2^3*a3-5*a2^3*a4+a2^2*a3^2\
-5*a2^2*a3*a4+4*a2^2*a4^2-a2*a3^2*a4+2*a2*a3*a4^2-a2*a4^3)
⇒      [3]:
⇒      [1]:
⇒      1,1
⇒      [2]:
⇒      1
⇒      [3]:
⇒      _[1]=0
⇒      [4]:
⇒      [1]:
⇒      1,1,1
⇒      [2]:
⇒      0
⇒      [3]:
⇒      _[1]=(a2-a4)

```

```

⇒ [5]:
⇒ [1]:
⇒ 2
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ _[1]=1
⇒ [4]:
⇒ _[1]=1
⇒ [6]:
⇒ [1]:
⇒ 2,1
⇒ [2]:
⇒ 3
⇒ [3]:
⇒ _[1]=(a2-a4)
⇒ [7]:
⇒ [1]:
⇒ 2,1,1
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ _[1]=(a4)
⇒ _[2]=(a2)
⇒ [8]:
⇒ [1]:
⇒ 2,1,2
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ _[1]=(a3)
⇒ _[2]=(a2-a4)
⇒ [9]:
⇒ [1]:
⇒ 2,1,3
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ _[1]=(a2-a4)
⇒ _[2]=(a1)
⇒ [10]:
⇒ [1]:
⇒ 3
⇒ [2]:
⇒ 3
⇒ [3]:
⇒ _[1]=x4
⇒ _[2]=x1
⇒ _[3]=x3^2
⇒ [4]:
⇒ _[1]=x4
⇒ _[2]=-x1-x2-x3+(a1+a3+a4)
⇒ _[3]=x3^2+(-a1-a3-a4)*x3+(a1*a3+a1*a4+a3*a4)

```

```

⇒ [11]:
⇒   [1]:
⇒     3,1
⇒   [2]:
⇒     1
⇒   [3]:
⇒     _[1]=(a4)
⇒     _[2]=(a2)
⇒ [12]:
⇒   [1]:
⇒     3,1,1
⇒   [2]:
⇒     0
⇒   [3]:
⇒     _[1]=1
⇒ [13]:
⇒   [1]:
⇒     3,2
⇒   [2]:
⇒     1
⇒   [3]:
⇒     _[1]=(a3)
⇒     _[2]=(a2-a4)
⇒ [14]:
⇒   [1]:
⇒     3,2,1
⇒   [2]:
⇒     0
⇒   [3]:
⇒     _[1]=1
⇒ [15]:
⇒   [1]:
⇒     3,3
⇒   [2]:
⇒     1
⇒   [3]:
⇒     _[1]=(a2-a4)
⇒     _[2]=(a1)
⇒ [16]:
⇒   [1]:
⇒     3,3,1
⇒   [2]:
⇒     0
⇒   [3]:
⇒     _[1]=1
cantreetoMaple(T2,"T2m","T2m.txt");
cantodiffcgs(T2);
⇒ // ** redefining NP (    def NP=imap(RR,N);) redcgs.lib::cantodiffcgs:396\
0
⇒ // ** redefining MP (    def MP=imap(RR,M);) redcgs.lib::cantodiffcgs:396\
1
⇒ // ** redefining NP (    def NP=imap(RR,N);) redcgs.lib::cantodiffcgs:396\
0

```

```

⇒ // ** redefining MP (    def MP=imap(RR,M);) redcgs.lib::cantodiffcgs:396\
1
⇒ // ** redefining NP (    def NP=imap(RR,N);) redcgs.lib::cantodiffcgs:396\
0
⇒ // ** redefining MP (    def MP=imap(RR,M);) redcgs.lib::cantodiffcgs:396\
1
⇒ [1]:
⇒   [1]:
⇒   0
⇒   [2]:
⇒   3
⇒   [3]:
⇒   _[1]=x4
⇒   _[2]=x1
⇒   _[3]=x1*x3*x4
⇒   _[4]=x1*x3
⇒   [4]:
⇒   _[1]=x4+(a2-a4)
⇒   _[2]=x1+x2+x3+x4+(-a1-a3-a4)
⇒   _[3]=x1*x3*x4+(-a1*a3*a4)
⇒   _[4]=x1*x3+x2*x3+x1*x4+x3*x4+(-a1*a3-a1*a4-a3*a4)
⇒   [5]:
⇒   [1]:
⇒   _[1]=0
⇒   [2]:
⇒   _[1]=0
⇒   [3]:
⇒   [1]:
⇒   _[1]=0
⇒   [6]:
⇒   MRCGS
⇒ [2]:
⇒   [1]:
⇒   _[1]=x4
⇒   _[2]=x1
⇒   _[3]=x3^2
⇒   _[4]=x2*x3
⇒   _[5]=x2^2
⇒   [2]:
⇒   _[1]=x4+(a2-a4)
⇒   _[2]=-x1-x2-x3+(a1+a2+a3)
⇒   _[3]=x3^2+(-a2+a4)*x2+(-a1-a2-a3)*x3+(a1*a2+a1*a3+a2^2+a2*a3-a2*a4)
⇒   _[4]=(a2-a4)*x2*x3+(a2^2-2*a2*a4+a4^2)*x2+(-a1*a2^2-a1*a2*a3+a1*a2*
a4-a2^3-a2^2*a3+2*a2^2*a4+a2*a3*a4-a2*a4^2)
⇒   _[5]=(a2^2-2*a2*a4+a4^2)*x2^2+(-2*a1*a2^2-a1*a2*a3+3*a1*a2*a4+a1*a3\
*a4-a1*a4^2-3*a2^3-2*a2^2*a3+7*a2^2*a4+3*a2*a3*a4-5*a2*a4^2-a3*a4^2+a4^3)\
*x2+(-a1*a2^2-a1*a2*a3+a1*a2*a4-a2^3-a2^2*a3+2*a2^2*a4+a2*a3*a4-a2*a4^2)*\
x3+(a1^2*a2^2+a1^2*a2*a3-a1^2*a2*a4+3*a1*a2^3+4*a1*a2^2*a3-5*a1*a2^2*a4+a\
1*a2*a3^2-3*a1*a2*a3*a4+2*a1*a2*a4^2+2*a2^4+3*a2^3*a3-5*a2^3*a4+a2^2*a3^2\
-5*a2^2*a3*a4+4*a2^2*a4^2-a2*a3^2*a4+2*a2*a3*a4^2-a2*a4^3)
⇒   [3]:
⇒   _[1]=0
⇒   [4]:

```

```

⇒      _[1]=(a2-a4)
⇒ [3] :
⇒      [1] :
⇒      _[1]=1
⇒      [2] :
⇒      _[1]=1
⇒      [3] :
⇒      _[1]=(a2-a4)
⇒      [4] :
⇒      _[1]=(a2-a4)
⇒      _[2]=(a1*a3*a4)
⇒ [4] :
⇒      [1] :
⇒      _[1]=x4
⇒      _[2]=x1
⇒      _[3]=x3^2
⇒      [2] :
⇒      _[1]=x4
⇒      _[2]=-x1-x2-x3+(a1+a3+a4)
⇒      _[3]=x3^2+(-a1-a3-a4)*x3+(a1*a3+a1*a4+a3*a4)
⇒      [3] :
⇒      _[1]=(a2-a4)
⇒      _[2]=(a1*a3*a4)
⇒      [4] :
⇒      _[1]=1

```

### D.2.9.12 rcgs

Procedure from library `redcgs.lib` (see [Section D.2.9 \[redcgs.lib\]](#), page 898).

**Usage:** `rcgs(F);`  
`F` is the ideal from which to obtain the Reduced CGS.  
`rcgs(F,L);`  
where `L` is a list of the null conditions ideal `N`, and `W` the set of non-null polynomials (ideal). If this option is set, the ideals `N` and `W` must depend only on the parameters and the parameter space is reduced to  $V(N) \setminus V(h)$ , where  $h = \text{prod}(w)$ , for  $w$  in `W`. A reduced specification of  $(N,W)$  will be computed and used to restrict the parameter-space. The output will omit the known restrictions given as option.

**Return:** The list representing the Reduced CGS.  
The description given here is analogous as for `mrcgs` and `crcgs`. The elements of the list `T` computed by `rcgs` are lists representing a rooted tree.  
Each element has as the two first entries with the following content:  
[1]: The label (intvec) representing the position in the rooted tree: 0 for the root (and this is a special element)  $i$  for the root of the segment  $i$   
 $(i, \dots)$  for the children of the segment  $i$   
[2]: the number of children (int) of the vertex.  
There thus three kind of vertices:  
1) the root (first element labelled 0),  
2) the vertices labelled with a single integer  $i$ ,  
3) the rest of vertices labelled with more indices.  
Description of the root. Vertex type 1)  
There is a special vertex (the first one) whose content is the following:

[3] lpp of the given ideal  
 [4] the given ideal  
 [5] the red-spec of the (optional) given null and non-null conditions (see redspec for the description)  
 [6] RCGS (to remember which algorithm has been used). If the algorithm used is mrcgs or crcgs then this will be stated at this vertex (mrcgs or CRCGS).  
 Description of vertices type 2). These are the vertices that initiate a segment, and are labelled with a single integer. [3] lpp (ideal) of the reduced basis. If they are repeated lpp's this will correspond to a sheaf.  
 [4] the reduced basis (ideal) of the segment.  
 Description of vertices type 3). These vertices have as first label i and descend from vertex i in the position of the label (i,...). They contain moreover a unique prime ideal in the parameters and form ascending chains of ideals.  
 How is to be read the rcgs tree? The vertices with an even number of integers in the label are to be considered as additive and those with an odd number of integers in the label are to be considered as subtraction. As an example consider the following vertices:  $v1=((i),2,lpp,B)$ ,  
 $v2=((i,1),2,P_{-\{i,1\}})$ ,  
 $v3=((i,1,1),0,P_{-\{i,1,1\}})$ ,  $v4=((i,1,2),0,P_{-\{i,1,1\}})$ ,  $v5=((i,2),2,P_{-\{i,2\}})$ ,  
 $v6=((i,2,1),0,P_{-\{i,2,1\}})$ ,  $v7=((i,2,2),0,P_{-\{i,2,2\}})$   
 They represent the segment:  
 $(V(i,1) \setminus (V(i,1,1) \cup V(i,1,2))) \cup$   
 $(V(i,2) \setminus (V(i,2,1) \cup V(i,2,2)))$   
 where  $V(i,j,...) = V(P_{-\{i,j,...\}})$

**Note:** There are three fundamental routines in the library: mrcgs, rcgs and crcgs. rcgs (Reduced CGS) is an algorithm that first homogenizes the basis of the given ideal then applies mrcgs and finally de-homogenizes and reduces the resulting bases. (See the note of mrcgs). As a result of Wibmer's Theorem, the resulting segments are locally closed (i.e. difference of varieties). Nevertheless, the output is not completely canonical as the homogeneous ideal considered is not the homogenized ideal of the given ideal but only the ideal obtained by homogenizing the given basis.

The output can be visualized using cantreetoMaple, that will write a file with the content of mrcgs that can be read in Maple and plotted using the Maple plotcantree routine of the Monte's dpbg library. You can also use the routine cantodiffcgs as the segments are all difference of two varieties to have a simpler view of the output.

**Example:**

```
LIB "redcgs.lib";
ring R=(0,b,c,d,e,f),(x,y),dp;
ideal F=x^2+b*y^2+2*c*x*y+2*d*x+2*e*y+f, 2*x+2*c*y+2*d, 2*b*y+2*c*x+2*e;
def T=rcgs(F);
⇨ // ** redefining @R ( exportto(Top,@R); // global ring K[a][x])
⇨ // ** redefining @P ( exportto(Top,@P); // global ring K[a])
⇨ // ** redefining @RP ( exportto(Top,@RP); // global ring K[x,a] with\
    product order)
⇨ // ** redefining @R ( exportto(Top,@R); // global ring K[a][x])
⇨ // ** redefining @P ( exportto(Top,@P); // global ring K[a])
⇨ // ** redefining @RP ( exportto(Top,@RP); // global ring K[x,a] with\
    product order)
T;
⇨ [1]:
```

```

⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 5
⇒ [3]:
⇒ _[1]=x2
⇒ _[2]=x
⇒ _[3]=x
⇒ [4]:
⇒ _[1]=x2+(2c)*xy+(b)*y2+(2d)*x+(2e)*y+(f)
⇒ _[2]=2*x+(2c)*y+(2d)
⇒ _[3]=(2c)*x+(2b)*y+(2e)
⇒ [5]:
⇒ [1]:
⇒ _[1]=0
⇒ [2]:
⇒ _[1]=0
⇒ [3]:
⇒ [1]:
⇒ _[1]=0
⇒ [6]:
⇒ RCGS
⇒ [2]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ _[1]=1
⇒ [4]:
⇒ _[1]=1
⇒ [3]:
⇒ [1]:
⇒ 1,1
⇒ [2]:
⇒ 2
⇒ [3]:
⇒ _[1]=0
⇒ [4]:
⇒ [1]:
⇒ 1,1,1
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ _[1]=(bd2-bf+c2f-2cde+e2)
⇒ [5]:
⇒ [1]:
⇒ 1,1,2
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ _[1]=(b-c2)
⇒ [6]:

```

```

⇒ [1]:
⇒ 2
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ _[1]=y
⇒ _[2]=x
⇒ [4]:
⇒ _[1]=(b-c2)*y+(-cd+e)
⇒ _[2]=(-b+c2)*x+(-bd+ce)
⇒ [7]:
⇒ [1]:
⇒ 2,1
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ _[1]=(bd2-bf+c2f-2cde+e2)
⇒ [8]:
⇒ [1]:
⇒ 2,1,1
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ _[1]=(cd-e)
⇒ _[2]=(b-c2)
⇒ [9]:
⇒ [1]:
⇒ 3
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ _[1]=1
⇒ [4]:
⇒ _[1]=1
⇒ [10]:
⇒ [1]:
⇒ 3,1
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ _[1]=(b-c2)
⇒ [11]:
⇒ [1]:
⇒ 3,1,1
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ _[1]=(cd-e)
⇒ _[2]=(b-c2)
⇒ [12]:
⇒ [1]:
⇒ 4
⇒ [2]:

```



```

⇒      1
⇒      [3]:
⇒      _[1]=1
⇒      [4]:
⇒      _[1]=1
⇒ [13]:
⇒      [1]:
⇒      4,1
⇒      [2]:
⇒      1
⇒      [3]:
⇒      _[1]=(cd-e)
⇒      _[2]=(b-c2)
⇒ [14]:
⇒      [1]:
⇒      4,1,1
⇒      [2]:
⇒      0
⇒      [3]:
⇒      _[1]=(d2-f)
⇒      _[2]=(cf-de)
⇒      _[3]=(cd-e)
⇒      _[4]=(b-c2)
⇒ [15]:
⇒      [1]:
⇒      5
⇒      [2]:
⇒      1
⇒      [3]:
⇒      _[1]=x
⇒      [4]:
⇒      _[1]=x+(c)*y+(d)
⇒ [16]:
⇒      [1]:
⇒      5,1
⇒      [2]:
⇒      1
⇒      [3]:
⇒      _[1]=(d2-f)
⇒      _[2]=(cf-de)
⇒      _[3]=(cd-e)
⇒      _[4]=(b-c2)
⇒ [17]:
⇒      [1]:
⇒      5,1,1
⇒      [2]:
⇒      0
⇒      [3]:
⇒      _[1]=1
cantreetoMaple(T,"Tr","Tr.txt");
cantodiffcgs(T);
⇒ // ** redefining NP (      def NP=imap(RR,N);) redcgs.lib::cantodiffcgs:396\
0

```

```

⇒ // ** redefining MP (    def MP=imap(RR,M);) redcgs.lib::cantodiffcgs:396\
    1
⇒ // ** redefining NP (    def NP=imap(RR,N);) redcgs.lib::cantodiffcgs:396\
    0
⇒ // ** redefining MP (    def MP=imap(RR,M);) redcgs.lib::cantodiffcgs:396\
    1
⇒ // ** redefining NP (    def NP=imap(RR,N);) redcgs.lib::cantodiffcgs:396\
    0
⇒ // ** redefining MP (    def MP=imap(RR,M);) redcgs.lib::cantodiffcgs:396\
    1
⇒ // ** redefining NP (    def NP=imap(RR,N);) redcgs.lib::cantodiffcgs:396\
    0
⇒ // ** redefining MP (    def MP=imap(RR,M);) redcgs.lib::cantodiffcgs:396\
    1
⇒ // ** redefining NP (    def NP=imap(RR,N);) redcgs.lib::cantodiffcgs:396\
    0
⇒ // ** redefining MP (    def MP=imap(RR,M);) redcgs.lib::cantodiffcgs:396\
    1
⇒ [1]:
⇒   [1]:
⇒     0
⇒   [2]:
⇒     5
⇒   [3]:
⇒     _[1]=x^2
⇒     _[2]=x
⇒     _[3]=x
⇒   [4]:
⇒     _[1]=x^2+(2*c)*x*y+(b)*y^2+(2*d)*x+(2*e)*y+(f)
⇒     _[2]=2*x+(2*c)*y+(2*d)
⇒     _[3]=(2*c)*x+(2*b)*y+(2*e)
⇒   [5]:
⇒     [1]:
⇒       _[1]=0
⇒     [2]:
⇒       _[1]=0
⇒     [3]:
⇒       [1]:
⇒         _[1]=0
⇒   [6]:
⇒     RCGS
⇒ [2]:
⇒   [1]:
⇒     _[1]=1
⇒   [2]:
⇒     _[1]=1
⇒   [3]:
⇒     _[1]=0
⇒   [4]:
⇒     _[1]=(b^2*d^2-b^2*f-b*c^2*d^2+2*b*c^2*f-2*b*c*d*e+b*e^2-c^4*f+2*c^3\
      *d*e-c^2*e^2)
⇒ [3]:
⇒   [1]:

```

```

⇒      _[1]=y
⇒      _[2]=x
⇒      [2]:
⇒      _[1]=(b-c^2)*y+(-c*d+e)
⇒      _[2]=(-b+c^2)*x+(-b*d+c*e)
⇒      [3]:
⇒      _[1]=(b*d^2-b*f+c^2*f-2*c*d*e+e^2)
⇒      [4]:
⇒      _[1]=(c*d-e)
⇒      _[2]=(b-c^2)
⇒      [4]:
⇒      [1]:
⇒      _[1]=1
⇒      [2]:
⇒      _[1]=1
⇒      [3]:
⇒      _[1]=(b-c^2)
⇒      [4]:
⇒      _[1]=(c*d-e)
⇒      _[2]=(b-c^2)
⇒      [5]:
⇒      [1]:
⇒      _[1]=1
⇒      [2]:
⇒      _[1]=1
⇒      [3]:
⇒      _[1]=(c*d-e)
⇒      _[2]=(b-c^2)
⇒      [4]:
⇒      _[1]=(d^2-f)
⇒      _[2]=(c*f-d*e)
⇒      _[3]=(c*d-e)
⇒      _[4]=(b-c^2)
⇒      [6]:
⇒      [1]:
⇒      _[1]=x
⇒      [2]:
⇒      _[1]=x+(c)*y+(d)
⇒      [3]:
⇒      _[1]=(d^2-f)
⇒      _[2]=(c*f-d*e)
⇒      _[3]=(c*d-e)
⇒      _[4]=(b-c^2)
⇒      [4]:
⇒      _[1]=1

```

### D.2.9.13 crcgs

Procedure from library `redcgs.lib` (see [Section D.2.9 \[redcgs.lib\]](#), page 898).

**Usage:** `crcgs(F);`  
`F` is the ideal from which to obtain the Canonical Reduced CGS. `crcgs(F,L);`  
where `L` is a list of the null conditions ideal `N`, and `W` the set of non-null polynomials (ideal). If this option is set, the ideals `N` and `W` must depend only on the parameters

and the parameter space is reduced to  $V(N) \setminus V(h)$ , where  $h = \text{prod}(w)$ , for  $w$  in  $W$ . A reduced specification of  $(N, W)$  will be computed and used to restrict the parameter-space. The output will omit the known restrictions given as option.

**Return:** The list representing the Canonical Reduced CGS.  
 The description given here is identical for `mrcgs` and `rcgs`. The elements of the list `T` computed by `crcgs` are lists representing a rooted tree.  
 Each element has as the two first entries with the following content:  
 [1]: The label (intvec) representing the position in the rooted tree: 0 for the root (and this is a special element)  $i$  for the root of the segment  $i$   
 $(i, \dots)$  for the children of the segment  $i$   
 [2]: the number of children (int) of the vertex.  
 There thus three kind of vertices:  
 1) the root (first element labelled 0),  
 2) the vertices labelled with a single integer  $i$ ,  
 3) the rest of vertices labelled with more indices.  
 Description of the root. Vertex type 1)  
 There is a special vertex (the first one) whose content is the following:  
 [3] lpp of the given ideal  
 [4] the given ideal  
 [5] the red-spec of the (optional) given null and non-null conditions (see `redspec` for the description)  
 [6] `mrcgs` (to remember which algorithm has been used). If the algorithm used is `rcgs` of `crcgs` then this will be stated at this vertex (`RCGS` or `CRCGS`).  
 Description of vertices type 2). These are the vertices that initiate a segment, and are labelled with a single integer. [3] lpp (ideal) of the reduced basis. If they are repeated lpp's this will correspond to a sheaf.  
 [4] the reduced basis (ideal) of the segment.  
 Description of vertices type 3). These vertices have as first label  $i$  and descend from vertex  $i$  in the position of the label  $(i, \dots)$ . They contain moreover a unique prime ideal in the parameters and form ascending chains of ideals.  
 How is to be read the `mrcgs` tree? The vertices with an even number of integers in the label are to be considered as additive and those with an odd number of integers in the label are to be considered as subtraction. As an example consider the following vertices:  $v1 = ((i), 2, \text{lpp}, B)$ ,  
 $v2 = ((i, 1), 2, P_{\{i, 1\}})$ ,  
 $v3 = ((i, 1, 1), 0, P_{\{i, 1, 1\}})$ ,  $v4 = ((i, 1, 2), 0, P_{\{i, 1, 1\}})$ ,  $v5 = ((i, 2), 2, P_{\{i, 2\}})$ ,  
 $v6 = ((i, 2, 1), 0, P_{\{i, 2, 1\}})$ ,  $v7 = ((i, 2, 2), 0, P_{\{i, 2, 2\}})$   
 They represent the segment:  
 $(V(i, 1) \setminus (V(i, 1, 1) \cup V(i, 1, 2))) \cup$   
 $(V(i, 2) \setminus (V(i, 2, 1) \cup V(i, 2, 2)))$   
 where  $V(i, j, \dots) = V(P_{\{i, j, \dots\}})$

**Note:** There are three fundamental routines in the library: `mrcgs`, `rcgs` and `crcgs`. `crcgs` (Canonical Reduced CGS) is an algorithm that first homogenizes the the given ideal then applies `mrcgs` and finally de-homogenizes and reduces the resulting bases. (See the note of `mrcgs`). As a result of Wibmer's Theorem, the resulting segments are locally closed (i.e. difference of varieties) and the partition is canonical as the homogenized ideal is uniquely associated to the given ideal not depending of the given basis.

Nevertheless the computations to do are usually more time consuming and so it is preferable to compute first the `rcgs` and only if it success you can try `crcgs`.

The output can be visualized using `cantreetoMaple`, that will write a file with the content of `crcgs` that can be read in Maple and plotted using the Maple `plotcantree` routine of the Monte's `dpgb` library. You can also use the routine `cantodiffcgs` as the segments are all difference of two varieties to have a simpler view of the output.

**Example:**

```
LIB "redcgs.lib";
ring R=(0,b,c,d,e,f),(x,y),dp;
ideal F=x^2+b*y^2+2*c*x*y+2*d*x+2*e*y+f, 2*x+2*c*y+2*d, 2*b*y+2*c*x+2*e;
def T=crcgs(F);
⇒ // ** redefining @R ( exportto(Top,@R); // global ring K[a][x])
⇒ // ** redefining @R ( exportto(Top,@R); // global ring K[a][x])
⇒ // ** redefining @P ( exportto(Top,@P); // global ring K[a])
⇒ // ** redefining @RP ( exportto(Top,@RP); // global ring K[x,a] with\
    product order)
⇒ // ** NP2 is no standard basis
⇒ // ** NP2 is no standard basis
⇒ // ** redefining @R ( exportto(Top,@R); // global ring K[a][x])
⇒ // ** redefining @P ( exportto(Top,@P); // global ring K[a])
⇒ // ** redefining @RP ( exportto(Top,@RP); // global ring K[x,a] with\
    product order)
T;
⇒ [1]:
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 4
⇒ [3]:
⇒ _[1]=1
⇒ _[2]=y
⇒ _[3]=y
⇒ _[4]=x
⇒ [4]:
⇒ _[1]=(bd2-bf+c2f-2cde+e2)
⇒ _[2]=(cd-e)*y+(d2-f)
⇒ _[3]=(b-c2)*y+(-cd+e)
⇒ _[4]=x+(c)*y+(d)
⇒ [5]:
⇒ [1]:
⇒ _[1]=0
⇒ [2]:
⇒ _[1]=0
⇒ [3]:
⇒ [1]:
⇒ _[1]=0
⇒ [6]:
⇒ CRCGS
⇒ [2]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 1
⇒ [3]:
```

```

↳      _[1]=1
↳      [4]:
↳      _[1]=1
↳      [3]:
↳      [1]:
↳      1,1
↳      [2]:
↳      1
↳      [3]:
↳      _[1]=0
↳      [4]:
↳      [1]:
↳      1,1,1
↳      [2]:
↳      0
↳      [3]:
↳      _[1]=(bd2-bf+c2f-2cde+e2)
↳      [5]:
↳      [1]:
↳      2
↳      [2]:
↳      1
↳      [3]:
↳      _[1]=y
↳      _[2]=x
↳      [4]:
↳      _[1]=(b-c2)*y+(-cd+e)
↳      _[2]=(b-c2)*x+(bd-ce)
↳      [6]:
↳      [1]:
↳      2,1
↳      [2]:
↳      1
↳      [3]:
↳      _[1]=(bd2-bf+c2f-2cde+e2)
↳      [7]:
↳      [1]:
↳      2,1,1
↳      [2]:
↳      0
↳      [3]:
↳      _[1]=(cd-e)
↳      _[2]=(b-c2)
↳      [8]:
↳      [1]:
↳      3
↳      [2]:
↳      1
↳      [3]:
↳      _[1]=1
↳      [4]:
↳      _[1]=1
↳      [9]:

```

```

⇒ [1]:
⇒ 3,1
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ _[1]=(cd-e)
⇒ _[2]=(b-c2)
⇒ [10]:
⇒ [1]:
⇒ 3,1,1
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ _[1]=(d2-f)
⇒ _[2]=(cf-de)
⇒ _[3]=(cd-e)
⇒ _[4]=(b-c2)
⇒ [11]:
⇒ [1]:
⇒ 4
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ _[1]=x
⇒ [4]:
⇒ _[1]=x+(c)*y+(d)
⇒ [12]:
⇒ [1]:
⇒ 4,1
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ _[1]=(d2-f)
⇒ _[2]=(cf-de)
⇒ _[3]=(cd-e)
⇒ _[4]=(b-c2)
⇒ [13]:
⇒ [1]:
⇒ 4,1,1
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ _[1]=1
cantreetoMaple(T,"Tc","Tc.txt");
cantodiffcgs(T);
⇒ // ** redefining NP ( def NP=imap(RR,N);) redcgs.lib::cantodiffcgs:396\
0
⇒ // ** redefining MP ( def MP=imap(RR,M);) redcgs.lib::cantodiffcgs:396\
1
⇒ // ** redefining NP ( def NP=imap(RR,N);) redcgs.lib::cantodiffcgs:396\
0
⇒ // ** redefining MP ( def MP=imap(RR,M);) redcgs.lib::cantodiffcgs:396\
1

```

```

⇒ // ** redefining NP (    def NP=imap(RR,N);) redcgs.lib::cantodiffcgs:396\
0
⇒ // ** redefining MP (    def MP=imap(RR,M);) redcgs.lib::cantodiffcgs:396\
1
⇒ // ** redefining NP (    def NP=imap(RR,N);) redcgs.lib::cantodiffcgs:396\
0
⇒ // ** redefining MP (    def MP=imap(RR,M);) redcgs.lib::cantodiffcgs:396\
1
⇒ [1]:
⇒   [1]:
⇒     0
⇒   [2]:
⇒     4
⇒   [3]:
⇒     _[1]=1
⇒     _[2]=y
⇒     _[3]=y
⇒     _[4]=x
⇒   [4]:
⇒     _[1]=(b*d^2-b*f+c^2*f-2*c*d*e+e^2)
⇒     _[2]=(c*d-e)*y+(d^2-f)
⇒     _[3]=(b-c^2)*y+(-c*d+e)
⇒     _[4]=x+(c)*y+(d)
⇒   [5]:
⇒     [1]:
⇒       _[1]=0
⇒     [2]:
⇒       _[1]=0
⇒     [3]:
⇒       [1]:
⇒         _[1]=0
⇒   [6]:
⇒     CRCGS
⇒ [2]:
⇒   [1]:
⇒     _[1]=1
⇒   [2]:
⇒     _[1]=1
⇒   [3]:
⇒     _[1]=0
⇒   [4]:
⇒     _[1]=(b*d^2-b*f+c^2*f-2*c*d*e+e^2)
⇒ [3]:
⇒   [1]:
⇒     _[1]=y
⇒     _[2]=x
⇒   [2]:
⇒     _[1]=(b-c^2)*y+(-c*d+e)
⇒     _[2]=(b-c^2)*x+(b*d-c*e)
⇒   [3]:
⇒     _[1]=(b*d^2-b*f+c^2*f-2*c*d*e+e^2)
⇒   [4]:
⇒     _[1]=(c*d-e)

```



```

⇒      _[2]=(b-c^2)
⇒ [4] :
⇒      [1]:
⇒      _[1]=1
⇒      [2]:
⇒      _[1]=1
⇒      [3]:
⇒      _[1]=(c*d-e)
⇒      _[2]=(b-c^2)
⇒      [4]:
⇒      _[1]=(d^2-f)
⇒      _[2]=(c*f-d*e)
⇒      _[3]=(c*d-e)
⇒      _[4]=(b-c^2)
⇒ [5] :
⇒      [1]:
⇒      _[1]=x
⇒      [2]:
⇒      _[1]=x+(c)*y+(d)
⇒      [3]:
⇒      _[1]=(d^2-f)
⇒      _[2]=(c*f-d*e)
⇒      _[3]=(c*d-e)
⇒      _[4]=(b-c^2)
⇒      [4]:
⇒      _[1]=1

```

#### D.2.9.14 cantodiffcgs

Procedure from library `redcgs.lib` (see [Section D.2.9 \[redcgs.lib\]](#), page 898).

**Usage:**      `cantodiffcgs(T);`  
               T: is the list provided by `mrcgs` or `rcgs` or `crcgs`,

**Return:**     The list transforming the content of these routines to a simpler output where each segment corresponds to a single element of the list that is described as difference of two varieties.

The first element of the list is identical to the first element of the list provided by the corresponding `cgs` algorithm, and contains general information on the call (see `mrcgs`).

The remaining elements are lists of 4 elements, representing segments. These elements are

[1]: the lpp of the segment

[2]: the basis of the segment

[3]: the ideal of the first variety (radical)

[4]: the ideal of the second variety (radical)

The segment is  $V([3]) \setminus V([4])$ .

**Note:**        It can be called from the output of `mrcgs` or `rcgs` of `crcgs`

**Example:**

```

LIB "redcgs.lib";
ring R=(0,b,c,d,e,f),(x,y),dp;
ideal F=x^2+b*y^2+2*c*x*y+2*d*x+2*e*y+f, 2*x+2*c*y+2*d, 2*b*y+2*c*x+2*e;
def T=crcgs(F);

```

```

⇒ // ** redefining @R ( exportto(Top,@R); // global ring K[a][x])
⇒ // ** redefining @R ( exportto(Top,@R); // global ring K[a][x])
⇒ // ** redefining @P ( exportto(Top,@P); // global ring K[a])
⇒ // ** redefining @RP ( exportto(Top,@RP); // global ring K[x,a] with\
    product order)
⇒ // ** NP2 is no standard basis
⇒ // ** NP2 is no standard basis
⇒ // ** redefining @R ( exportto(Top,@R); // global ring K[a][x])
⇒ // ** redefining @P ( exportto(Top,@P); // global ring K[a])
⇒ // ** redefining @RP ( exportto(Top,@RP); // global ring K[x,a] with\
    product order)
T;
⇒ [1]:
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 4
⇒ [3]:
⇒ _[1]=1
⇒ _[2]=y
⇒ _[3]=y
⇒ _[4]=x
⇒ [4]:
⇒ _[1]=(bd2-bf+c2f-2cde+e2)
⇒ _[2]=(cd-e)*y+(d2-f)
⇒ _[3]=(b-c2)*y+(-cd+e)
⇒ _[4]=x+(c)*y+(d)
⇒ [5]:
⇒ [1]:
⇒ _[1]=0
⇒ [2]:
⇒ _[1]=0
⇒ [3]:
⇒ [1]:
⇒ _[1]=0
⇒ [6]:
⇒ CRCGS
⇒ [2]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ _[1]=1
⇒ [4]:
⇒ _[1]=1
⇒ [3]:
⇒ [1]:
⇒ 1,1
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ _[1]=0

```

```

⇒ [4]:
⇒ [1]:
⇒ 1,1,1
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ _[1]=(bd2-bf+c2f-2cde+e2)
⇒ [5]:
⇒ [1]:
⇒ 2
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ _[1]=y
⇒ _[2]=x
⇒ [4]:
⇒ _[1]=(b-c2)*y+(-cd+e)
⇒ _[2]=(b-c2)*x+(bd-ce)
⇒ [6]:
⇒ [1]:
⇒ 2,1
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ _[1]=(bd2-bf+c2f-2cde+e2)
⇒ [7]:
⇒ [1]:
⇒ 2,1,1
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ _[1]=(cd-e)
⇒ _[2]=(b-c2)
⇒ [8]:
⇒ [1]:
⇒ 3
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ _[1]=1
⇒ [4]:
⇒ _[1]=1
⇒ [9]:
⇒ [1]:
⇒ 3,1
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ _[1]=(cd-e)
⇒ _[2]=(b-c2)
⇒ [10]:
⇒ [1]:
⇒ 3,1,1

```

```

⇒ [2]:
⇒ 0
⇒ [3]:
⇒ _[1]=(d2-f)
⇒ _[2]=(cf-de)
⇒ _[3]=(cd-e)
⇒ _[4]=(b-c2)
⇒ [11]:
⇒ [1]:
⇒ 4
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ _[1]=x
⇒ [4]:
⇒ _[1]=x+(c)*y+(d)
⇒ [12]:
⇒ [1]:
⇒ 4,1
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ _[1]=(d2-f)
⇒ _[2]=(cf-de)
⇒ _[3]=(cd-e)
⇒ _[4]=(b-c2)
⇒ [13]:
⇒ [1]:
⇒ 4,1,1
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ _[1]=1
cantreetoMaple(T,"Tc","Tc.txt");
cantodiffcgs(T);
⇒ // ** redefining NP ( def NP=imap(RR,N);) redcgs.lib::cantodiffcgs:396\
0
⇒ // ** redefining MP ( def MP=imap(RR,M);) redcgs.lib::cantodiffcgs:396\
1
⇒ // ** redefining NP ( def NP=imap(RR,N);) redcgs.lib::cantodiffcgs:396\
0
⇒ // ** redefining MP ( def MP=imap(RR,M);) redcgs.lib::cantodiffcgs:396\
1
⇒ // ** redefining NP ( def NP=imap(RR,N);) redcgs.lib::cantodiffcgs:396\
0
⇒ // ** redefining MP ( def MP=imap(RR,M);) redcgs.lib::cantodiffcgs:396\
1
⇒ // ** redefining NP ( def NP=imap(RR,N);) redcgs.lib::cantodiffcgs:396\
0
⇒ // ** redefining MP ( def MP=imap(RR,M);) redcgs.lib::cantodiffcgs:396\
1
⇒ [1]:
⇒ [1]:

```

```

↳      0
↳      [2]:
↳      4
↳      [3]:
↳      _[1]=1
↳      _[2]=y
↳      _[3]=y
↳      _[4]=x
↳      [4]:
↳      _[1]=(b*d^2-b*f+c^2*f-2*c*d*e+e^2)
↳      _[2]=(c*d-e)*y+(d^2-f)
↳      _[3]=(b-c^2)*y+(-c*d+e)
↳      _[4]=x+(c)*y+(d)
↳      [5]:
↳      [1]:
↳      _[1]=0
↳      [2]:
↳      _[1]=0
↳      [3]:
↳      [1]:
↳      _[1]=0
↳      [6]:
↳      CRCGS
↳      [2]:
↳      [1]:
↳      _[1]=1
↳      [2]:
↳      _[1]=1
↳      [3]:
↳      _[1]=0
↳      [4]:
↳      _[1]=(b*d^2-b*f+c^2*f-2*c*d*e+e^2)
↳      [3]:
↳      [1]:
↳      _[1]=y
↳      _[2]=x
↳      [2]:
↳      _[1]=(b-c^2)*y+(-c*d+e)
↳      _[2]=(b-c^2)*x+(b*d-c*e)
↳      [3]:
↳      _[1]=(b*d^2-b*f+c^2*f-2*c*d*e+e^2)
↳      [4]:
↳      _[1]=(c*d-e)
↳      _[2]=(b-c^2)
↳      [4]:
↳      [1]:
↳      _[1]=1
↳      [2]:
↳      _[1]=1
↳      [3]:
↳      _[1]=(c*d-e)
↳      _[2]=(b-c^2)
↳      [4]:

```

```

⇒      _[1]=(d^2-f)
⇒      _[2]=(c*f-d*e)
⇒      _[3]=(c*d-e)
⇒      _[4]=(b-c^2)
⇒ [5] :
⇒      [1] :
⇒      _[1]=x
⇒      [2] :
⇒      _[1]=x+(c)*y+(d)
⇒      [3] :
⇒      _[1]=(d^2-f)
⇒      _[2]=(c*f-d*e)
⇒      _[3]=(c*d-e)
⇒      _[4]=(b-c^2)
⇒      [4] :
⇒      _[1]=1

```

## D.2.10 random.lib

**Library:** random.lib

**Purpose:** Creating Random and Sparse Matrices, Ideals, Polys

**Procedures:**

### D.2.10.1 genericid

Procedure from library `random.lib` (see [Section D.2.10 \[random.lib\]](#), page 936).

**Usage:** `genericid(id[p,b]);` id ideal/module, p,b integers

**Return:** system of generators of id which are generic, sparse, triangular linear combinations of given generators with coefficients in  $[1,b]$  and sparsity p percent, bigger p being sparser (default: p=75, b=30000)

**Note:** For performance reasons try small bound b in characteristic 0

**Example:**

```

LIB "random.lib";
ring r=0,(t,x,y,z),ds;
ideal i= x3+y4,z4+yx,t+x+y+z;
genericid(i,0,10);
⇒ _[1]=3t+3x+3y+3z+2xy+x3+y4+2z4
⇒ _[2]=4t+4x+4y+4z+xy+z4
⇒ _[3]=t+x+y+z
module m=[x,0,0,0],[0,y2,0,0],[0,0,z3,0],[0,0,0,t4];
print(genericid(m));
⇒ x,      0,      0, 0,
⇒ 17904y2,y2,      0, 0,
⇒ 0,      24170z3,z3,0,
⇒ 0,      0,      0, t4

```

### D.2.10.2 randomid

Procedure from library `random.lib` (see [Section D.2.10 \[random.lib\]](#), page 936).

- Usage:** `randomid(id[,k,b]);` id ideal/module, b,k integers
- Return:** ideal/module having k generators which are random linear combinations of generators of id with coefficients in the interval  $[-b,b]$  (default:  $b=30000$ ,  $k=\text{size}(\text{id})$ )
- Note:** For performance reasons try small bound b in characteristic 0

**Example:**

```
LIB "random.lib";
ring r=0,(x,y,z),dp;
randomid(maxideal(2),2,9);
↪ _[1]=-5x2-9xy+6y2-8xz-8yz+4z2
↪ _[2]=-9xy+2y2+xz+yz-z2
module m=[x,0,1],[0,y2,0],[y,0,z3];
show(randomid(m));
↪ // module, 3 generator(s)
↪ [1369x-11685y,-4481y2,-11685z3+1369]
↪ [-642x-13756y,25342y2,-13756z3-642]
↪ [2536x-6355y,8285y2,-6355z3+2536]
```

**D.2.10.3 randommat**

Procedure from library `random.lib` (see [Section D.2.10 \[random.lib\]](#), page 936).

- Usage:** `randommat(n,m[id,b]);` n,m,b integers, id ideal
- Return:**  $n \times m$  matrix, entries are random linear combinations of elements of id and coefficients in  $[-b,b]$   
[default:  $(\text{id},b) = (\text{maxideal}(1),30000)$ ]
- Note:** For performance reasons try small bound b in char 0

**Example:**

```
LIB "random.lib";
ring r=0,(x,y,z),dp;
matrix A=randommat(3,3,maxideal(2),9);
print(A);
↪ 9x2-2xy-8y2-9xz+yz+4z2, 9x2-4xy+y2-5xz+6yz-z2, 8x2+xy-9y2+2yz-8z2,
↪ -x2+5xy-8y2-7xz+4yz-3z2, x2+xy-4y2-xz+5z2, 5x2-8xy+8y2+6xz+yz+7z2,
↪ 4x2-5xy-6y2-4yz-5z2, -4x2-6xy-4y2-8xz+3yz+5z2, 2x2+3xy+y2+4xz-3yz+2z2
A=randommat(2,3);
print(A);
↪ 15276x+9897y+7526z, 6495x-24178y+11295z,-5745x-14754y+15979z,
↪ 20788x-28366y-20283z,24911x-10978y+3341z,12412x+11216y+15344z
```

**D.2.10.4 sparseid**

Procedure from library `random.lib` (see [Section D.2.10 \[random.lib\]](#), page 936).

- Usage:** `sparseid(k,u[o,p,b]);` k,u,o,p,b integers
- Return:** ideal having k generators, each of degree d,  $u \leq d \leq o$ , p percent of terms in degree d are 0, the remaining have random coefficients in the interval  $[1,b]$ , (default:  $o=u$ ,  $p=75$ ,  $b=30000$ )

**Example:**

```

LIB "random.lib";
ring r = 0,(a,b,c,d),ds;
sparseid(2,3);"";
↳ _[1]=12773a3+24263a2c+20030abc+17904b2c+26359c3
↳ _[2]=24004a3+6204b2c+24170bc2+19505c2d+21962bd2
↳
sparseid(3,0,4,90,9);
↳ _[1]=1+4a2+8b2c+3c3+4a3b+4a2b2+5abc2+3ac3
↳ _[2]=a+a2+7ab2+6a2c+3c3+5a3b+9ab3+2c4+3c3d+8ad3
↳ _[3]=5a+ab+2ac2+2b3c+8abcd

```

### D.2.10.5 sparsematrix

Procedure from library `random.lib` (see [Section D.2.10 \[random.lib\]](#), page 936).

**Usage:** `sparsematrix(n,m,o[,u,pe,pp,b]);`  $n,m,o,u,pe,pp,b$  integers

**Return:**  $n \times m$  matrix, about  $pe$  percent of the entries are 0, the remaining are random polynomials of degree  $d$ ,  $u \leq d \leq o$ , with  $pp$  percent of the terms being 0, the remaining have random coefficients in the interval  $[1,b]$  [default:  $(pe,u,pp,b) = (0,50,75,100)$ ]

**Example:**

```

LIB "random.lib";
ring r = 0,(a,b,c,d),dp;
// sparse matrix of sparse polys of degree <=2:
print(sparsematrix(3,4,2));"";
↳ 17a2+30ab+94bc+19b+45d,88a2+44bc+13d2+31a,0,
↳ 0, 0, 6c2+16b+64c+76, 0,
↳ 14ab+20bc+79cd+30b, 32a2+97bc+5b, 23bc+73c2+ad+48cd+73b+59d+25,0
↳
// dense matrix of sparse linear forms:
print(sparsematrix(3,3,1,1,0,55,9));
↳ 9a+5b+9c,2a+9d,2d,
↳ 7c+d, a+6b, 2b+2d,
↳ 9b+7c+8d,9b+9d,5a

```

### D.2.10.6 sparsemat

Procedure from library `random.lib` (see [Section D.2.10 \[random.lib\]](#), page 936).

**Usage:** `sparsemat(n,m[,p,b]);`  $n,m,p,b$  integers

**Return:**  $n \times m$  integer matrix,  $p$  percent of the entries are 0, the remaining are random coefficients  $\geq 1$  and  $\leq b$ ; [defaults:  $(p,b) = (75,1)$ ]

**Example:**

```

LIB "random.lib";
sparsemat(5,5);"";
↳ 0,0,0,0,0,
↳ 0,1,0,0,1,
↳ 0,0,0,1,0,
↳ 0,1,0,0,0,
↳ 0,1,0,1,1
↳
sparsemat(5,5,95);"";

```



```

⇒ 1,0,0,0,0,
⇒ 0,0,0,0,0,
⇒ 0,0,0,0,0,
⇒ 0,0,0,0,0,
⇒ 0,0,0,1,0
⇒
sparsemat(5,5,5);"";
⇒ 1,1,1,1,1,
⇒ 1,1,1,1,1,
⇒ 1,1,1,1,1,
⇒ 1,0,1,1,1,
⇒ 1,1,1,1,0
⇒
sparsemat(5,5,50,100);
⇒ 0,17,24,80,0,
⇒ 0,13,30,45,0,
⇒ 19,0,0,0,0,
⇒ 93,0,23,0,69,
⇒ 0,88,44,31,0

```

### D.2.10.7 sparsepoly

Procedure from library `random.lib` (see [Section D.2.10 \[random.lib\], page 936](#)).

**Usage:** `sparsepoly(u[o,p,b]);` `u,o,p,b` integers

**Return:** poly having only terms in degree `d`,  $u \leq d \leq o$ , `p` percentage of the terms in degree `d` are 0, the remaining have random coefficients in  $[1,b)$ , (defaults: `o=u`, `p=75`, `b=30000`)

**Example:**

```

LIB "random.lib";
ring r=0,(x,y,z),dp;
sparsepoly(5);"";
⇒ 24263xy4+24170x4z+21962x3yz+26642xy3z+5664xy2z2+17904xz4
⇒
sparsepoly(3,5,90,9);
⇒ 8x3z2+2y3z2+3xyz3+2xy3+yz3+xy2

```

### D.2.10.8 sparsetriag

Procedure from library `random.lib` (see [Section D.2.10 \[random.lib\], page 936](#)).

**Usage:** `sparsetriag(n,m[,p,b]);` `n,m,p,b` integers

**Return:** `nxm` lower triangular integer matrix, diagonal entries equal to 1, about `p` percent of lower diagonal entries are 0, the remaining are random integers  $\geq 1$  and  $\leq b$ ; [defaults: `(p,b) = (75,1)`]

**Example:**

```

LIB "random.lib";
sparsetriag(5,7);"";
⇒ 1,0,0,0,0,0,0,
⇒ 0,1,0,0,0,0,0,
⇒ 0,1,1,0,0,0,0,
⇒ 0,0,0,1,0,0,0,

```

```

⇒ 1,1,0,0,1,0,0
⇒
sparsetriag(7,5,90);"";
⇒ 1,0,0,0,0,0,
⇒ 0,1,0,0,0,0,
⇒ 0,1,1,0,0,0,
⇒ 0,0,0,1,0,0,
⇒ 0,0,0,0,1,0,
⇒ 0,0,0,1,0,0,
⇒ 0,0,0,1,0,0,
⇒ 0,1,0,0,0,0,
⇒
sparsetriag(5,5,0);"";
⇒ 1,0,0,0,0,0,
⇒ 1,1,0,0,0,0,
⇒ 1,1,1,0,0,0,
⇒ 1,1,1,1,0,0,
⇒ 1,1,1,1,1,0,
⇒ 1,1,1,1,1,1
⇒
sparsetriag(5,5,50,100);
⇒ 1,0,0,0,0,0,
⇒ 73,1,0,0,0,0,
⇒ 0,79,1,0,0,0,
⇒ 14,0,0,1,0,0,
⇒ 0,48,23,0,1,0

```

### D.2.10.9 sparseHomogIdeal

Procedure from library `random.lib` (see [Section D.2.10 \[random.lib\]](#), page 936).

**Usage:** `sparseid(k,u[o,p,b]);`  $k, u, o, p, b$  integers

**Return:** ideal having  $k$  homogeneous generators, each of random degree in the interval  $[u, o]$ ,  $p$  percent of terms in degree  $d$  are 0, the remaining have random coefficients in the interval  $[1, b]$ , (default:  $o=u$ ,  $p=75$ ,  $b=30000$ )

**Example:**

```

LIB "random.lib";
ring r = 0,(a,b,c,d),dp;
sparseHomogIdeal(2,3);"";
⇒ _[1]=24004a3+12773a2b+6204a2c+20030b2c+19505bcd
⇒ _[2]=817b3+9650c3+28857c2d+7247bd2+22567cd2
⇒
sparseHomogIdeal(3,0,4,90,9);
⇒ _[1]=5d
⇒ _[2]=abc2+4ab2d+c3d+c2d2
⇒ _[3]=3a

```

### D.2.10.10 triagmatrix

Procedure from library `random.lib` (see [Section D.2.10 \[random.lib\]](#), page 936).

**Usage:** `triagmatrix(n,m,o[u,pe,pp,b]);`  $n, m, o, u, pe, pp, b$  integers

**Return:**  $n \times m$  lower triangular matrix, diagonal entries equal to 1, about  $p$  percent of lower diagonal entries are 0, the remaining are random polynomials of degree  $d$ ,  $u \leq d \leq o$ ,

with  $pp$  percent of the terms being 0, the remaining have random coefficients in the interval  $[1,b]$  [default:  $(pe,u,pp,b) = (0,50,75,100)$ ]

**Example:**

```
LIB "random.lib";
ring r = 0,(a,b,c,d),dp;
// sparse triangular matrix of sparse polys of degree <=2:
print(triagmatrix(3,4,2));"";
↪ 1,                                0,0,0,
↪ 52ac+54cd+14c,                    1,0,0,
↪ 17a2+19b2+45ac+94bc+50b+87c+54d+21,0,1,0
↪
// dense triangular matrix of sparse linear forms:
print(triagmatrix(3,3,1,1,0,55,9));
↪ 1,      0,      0,
↪ 7a+8d,   1,      0,
↪ 9b+7c+4d,7b+9d,1
```

**D.2.10.11 randomLast**

Procedure from library `random.lib` (see [Section D.2.10 \[random.lib\]](#), page 936).

**Usage:** `randomLast(b); b int`

**Return:** `ideal = maxideal(1)`, but the last variable is exchanged by a random linear combination of all variables, with coefficients in the interval  $[-b,b]$ , except for the last variable which always has coefficient 1

**Example:**

```
LIB "random.lib";
ring r = 0,(x,y,z),lp;
ideal i = randomLast(10);
i;
↪ i[1]=x
↪ i[2]=y
↪ i[3]=-x+z
```

**D.2.10.12 randomBinomial**

Procedure from library `random.lib` (see [Section D.2.10 \[random.lib\]](#), page 936).

**Usage:** `randomBinomial(k,u[o,b]); k,u,o,b integers`

**Return:** binomial ideal,  $k$  homogeneous generators of degree  $d$ ,  $u \leq d \leq o$ , with randomly chosen monomials and coefficients in the interval  $[-b,b]$  (default:  $u=o$ ,  $b=10$ ).

**Example:**

```
LIB "random.lib";
ring r = 0,(x,y,z),lp;
ideal i = randomBinomial(4,5,6);
i;
↪ i[1]=-x4z-xz4
↪ i[2]=8x2y3+8xy3z
↪ i[3]=-4x2y2z2-4xy5
↪ i[4]=5x3yz2+5xz5
```

## D.2.11 resources.lib

**Library:** resources.lib

**Purpose:** Tools to manage the computational resources

**Author:** Andreas Steenpass, e-mail: steenpass@mathematik.uni-kl.de

**Overview:** The purpose of this library is to manage the computational resources of a Singular session. The library tasks.lib and any library build upon tasks.lib respect these settings, i.e. they will not use more computational resources than provided via resources.lib.

The provided procedures and their implementation are currently quite simple. The library can be extended later on to support, e.g., distributed computations on several servers.

**Procedures:** See also: [Section D.2.7 \[parallel.lib\]](#), page 885; [Section D.2.13 \[tasks.lib\]](#), page 959.

### D.2.11.1 addcores

Procedure from library `resources.lib` (see [Section D.2.11 \[resources.lib\]](#), page 942).

**Usage:** addcores(n), n int

**Return:** the adjusted number of available processor cores, after n has been added to it. If n is negative, this number is reduced.

**Note:** The number of available processor cores must be at least 1. Reducing this number may take some time.

This procedure should only be called in the main process of a Singular session and not within any task defined via tasks.lib.

**Example:**

```
LIB "resources.lib";
setcores(4);
↦ 4
addcores(-2);
↦ 2
```

See also: [Section D.2.11.3 \[getcores\]](#), page 943; [Section D.2.7 \[parallel.lib\]](#), page 885; [Section D.2.11.2 \[setcores\]](#), page 942; [Section D.2.13 \[tasks.lib\]](#), page 959.

### D.2.11.2 setcores

Procedure from library `resources.lib` (see [Section D.2.11 \[resources.lib\]](#), page 942).

**Usage:** setcores(n), n int

**Return:** n. The number of available processor cores is set to n and n is returned.

**Note:** The number of available processor cores must be at least 1. Reducing this number may take some time.

This procedure should only be called in the main process of a Singular session and not within any task defined via tasks.lib.

**Example:**

```
LIB "resources.lib";
setcores(2);
↦ 2
```

```
setcores(4);
↦ 4
```

See also: [Section D.2.11.1 \[addcores\]](#), page 942; [Section D.2.11.3 \[getcores\]](#), page 943; [Section D.2.7 \[parallel\\_lib\]](#), page 885; [Section D.2.13 \[tasks\\_lib\]](#), page 959.

### D.2.11.3 getcores

Procedure from library `resources.lib` (see [Section D.2.11 \[resources\\_lib\]](#), page 942).

**Usage:** `getcores(n)`, `n` int

**Return:** the number of available processor cores.

**Note:** This procedure should only be called in the main process of a Singular session and not within any task defined via `tasks.lib`.

**Example:**

```
LIB "resources.lib";
setcores(4);
↦ 4
getcores();
↦ 4
```

See also: [Section D.2.11.1 \[addcores\]](#), page 942; [Section D.2.7 \[parallel\\_lib\]](#), page 885; [Section D.2.11.2 \[setcores\]](#), page 942; [Section D.2.13 \[tasks\\_lib\]](#), page 959.

### D.2.11.4 semaphore

Procedure from library `resources.lib` (see [Section D.2.11 \[resources\\_lib\]](#), page 942).

**Usage:** `semaphore(n)`, `n` int

**Return:** the index of a new semaphore initialized with `n`.

**Example:**

```
LIB "resources.lib";
int sem = semaphore(1);
system("semaphore", "acquire", sem);
↦ 1
system("semaphore", "try_acquire", sem);
↦ 0
system("semaphore", "release", sem);
↦ 1
system("semaphore", "try_acquire", sem);
↦ 1
```

## D.2.12 ring\_lib

**Library:** `ring.lib`

**Purpose:** Manipulating Rings and Maps

**Authors:** Singular team

**Procedures:**

### D.2.12.1 changechar

Procedure from library `ring.lib` (see [Section D.2.12 \[ring\\_lib\], page 943](#)).

**Usage:** `changechar(c[r]); c=list, r=ring`

**Return:** ring `R`, obtained from the ring `r` [default: `r=basing`], by changing `ring_list(r)[1]` to `c`.

**Example:**

```
LIB "ring.lib";
ring rr=2,A,dp;
ring r=0,(x,y,u,v),(dp(2),ds);
def R=changechar(ringlist(rr)); R;"";
⇒ // coefficients: ZZ/2(A)
⇒ // number of vars : 4
⇒ //      block 1 : ordering dp
⇒ //      : names x y
⇒ //      block 2 : ordering ds
⇒ //      : names u v
⇒ //      block 3 : ordering C
⇒
def R1=changechar(32003,R); setring R1; R1;
⇒ // coefficients: ZZ/32003
⇒ // number of vars : 4
⇒ //      block 1 : ordering dp
⇒ //      : names x y
⇒ //      block 2 : ordering ds
⇒ //      : names u v
⇒ //      block 3 : ordering C
kill R,R1;
```

### D.2.12.2 changeord

Procedure from library `ring.lib` (see [Section D.2.12 \[ring\\_lib\], page 943](#)).

**Usage:** `changeord(newword[r]); newwordstr=list, r=ring/qring`

**Return:** ring `R`, obtained from the ring `r` [default: `r=basing`], by changing `order(r)` to `newword`.  
If, say, `newword=list(list("wp",intvec(2,3)),list(list("dp",1:(n-2))))`; and if the ring `r` exists and has `n` variables, the ring `R` will be equipped with the monomial ordering `wp(2,3),dp`.

**Example:**

```
LIB "ring.lib";
ring r=0,(x,y,u,v),(dp(2),ds);
def R=changeord(list(list("wp",intvec(2,3)),list("dp",1:2))); R; "";
⇒ // coefficients: QQ
⇒ // number of vars : 4
⇒ //      block 1 : ordering wp
⇒ //      : names x y
⇒ //      : weights 2 3
⇒ //      block 2 : ordering dp
⇒ //      : names u v
⇒ //      block 3 : ordering C
⇒
ideal i = x^2,y^2-u^3,v;
```

```

qring Q = std(i);
def Q'=changeord(list(list("lp",nvars(Q))),Q); setring Q'; Q';
↳ // coefficients: QQ
↳ // number of vars : 4
↳ //          block 1 : ordering lp
↳ //          : names  x y u v
↳ //          block 2 : ordering C
↳ // quotient ring from ideal
↳ _[1]=v
↳ _[2]=x2
↳ _[3]=y2-u3
kill R,Q,Q';

```

### D.2.12.3 changevar

Procedure from library `ring.lib` (see [Section D.2.12 \[ring.lib\]](#), page 943).

**Usage:** `changevar(vars[,r]);` vars=string, r=ring/qring

**Return:** ring R, obtained from the ring r [default: r=basing], by changing varstr(r) according to the value of vars.

If, say, vars = "t()" and the ring r exists and has n variables, the new basering will have name R and variables t(1),...,t(n).

If vars = "a,b,c,d", the new ring will have the variables a,b,c,d.

**Note:** This procedure is useful in connection with the procedure `ringtensor`, when a conflict between variable names must be avoided. This proc uses 'execute' or calls a procedure using 'execute'.

**Example:**

```

LIB "ring.lib";
ring r=0,(x,y,u,v),(dp(2),ds);
ideal i = x^2,y^2-u^3,v;
qring Q = std(i);
setring(r);
def R=changevar("A()"); R; "";
↳ // coefficients: QQ
↳ // number of vars : 4
↳ //          block 1 : ordering dp
↳ //          : names  A(1) A(2)
↳ //          block 2 : ordering ds
↳ //          : names  A(3) A(4)
↳ //          block 3 : ordering C
↳
def Q'=changevar("a,b,c,d",Q); setring Q'; Q';
↳ // coefficients: QQ
↳ // number of vars : 4
↳ //          block 1 : ordering dp
↳ //          : names  a b
↳ //          block 2 : ordering ds
↳ //          : names  c d
↳ //          block 3 : ordering C
↳ // quotient ring from ideal
↳ _[1]=d
↳ _[2]=a2

```

```

    ↪ _[3]=b2-c3
    kill R,Q,Q';

```

### D.2.12.4 defring

Procedure from library `ring.lib` (see [Section D.2.12 \[ring\\_lib\], page 943](#)).

**Usage:** `defring(ch,n,va,or);` `ch,va,or=strings, n=integer`

**Return:** ring `R` with characteristic '`ch`', ordering '`or`' and `n` variables with names derived from `va`.  
 If `va` is a single letter, say `va="a"`, and if `n<=26` then `a` and the following `n-1` letters from the alphabet (cyclic order) are taken as variables. If `n>26` or if `va` is a single letter followed by a bracket, say `va="T("`, the variables are `T(1),...,T(n)`.

**Note:** This proc is useful for defining a ring in a procedure. This proc uses '`execute`' or calls a procedure using '`execute`'.

**Example:**

```

LIB "ring.lib";
def r=defring("0",5,"u","ls"); r; setring r;"";
↪ // coefficients: QQ
↪ // number of vars : 5
↪ //      block 1 : ordering ls
↪ //      : names  u v w x y
↪ //      block 2 : ordering C
↪
def R=defring("(2,A)",10,"x(", "(dp(3),ws(1,2,3),ds)"); R; setring R;
↪ // coefficients: ZZ/2(A)
↪ // number of vars : 10
↪ //      block 1 : ordering dp
↪ //      : names  x(1) x(2) x(3)
↪ //      block 2 : ordering ws
↪ //      : names  x(4) x(5) x(6)
↪ //      : weights 1 2 3
↪ //      block 3 : ordering ds
↪ //      : names  x(7) x(8) x(9) x(10)
↪ //      block 4 : ordering C

```

### D.2.12.5 defrings

Procedure from library `ring.lib` (see [Section D.2.12 \[ring\\_lib\], page 943](#)).

**Usage:** `defrings(n,[p]);` `n,p integers`

**Return:** ring `R` with characteristic `p` [default: `p=32003`], ordering `ds` and `n` variables `x,y,z,a,b,...` if `n<=26` (resp. `x(1..n)` if `n>26`)

**Note:** This proc uses '`execute`' or calls a procedure using '`execute`'.

**Example:**

```

LIB "ring.lib";
def S5=defrings(5,0); S5; "";
↪ // coefficients: QQ
↪ // number of vars : 5
↪ //      block 1 : ordering ds

```



```

⇒ //          : names    x y z a b
⇒ //          block    2 : ordering C
⇒
def S30=defrings(30); S30;
⇒ // coefficients: ZZ/32003
⇒ // number of vars : 30
⇒ //          block    1 : ordering ds
⇒ //          : names    x(1) x(2) x(3) x(4) x(5) x(6) x(7) x(8) x(\
    9) x(10) x(11) x(12) x(13) x(14) x(15) x(16) x(17) x(18) x(19) x(20) x(21\
    ) x(22) x(23) x(24) x(25) x(26) x(27) x(28) x(29) x(30)
⇒ //          block    2 : ordering C
kill S5,S30;

```

### D.2.12.6 defringp

Procedure from library `ring.lib` (see [Section D.2.12 \[ring\\_lib\], page 943](#)).

**Usage:** `defringp(n,[p]);`  $n, p = \text{integers}$

**Return:** ring  $R$  with characteristic  $p$  [default:  $p=32003$ ], ordering  $dp$  and  $n$  variables  $x, y, z, a, b, \dots$  if  $n \leq 26$  (resp.  $x(1..n)$  if  $n > 26$ )

**Note:** This proc uses 'execute' or calls a procedure using 'execute'.

**Example:**

```

LIB "ring.lib";
def P5=defringp(5,0); P5; "";
⇒ // coefficients: QQ
⇒ // number of vars : 5
⇒ //          block    1 : ordering dp
⇒ //          : names    x y z a b
⇒ //          block    2 : ordering C
⇒
def P30=defringp(30); P30;
⇒ // coefficients: ZZ/32003
⇒ // number of vars : 30
⇒ //          block    1 : ordering dp
⇒ //          : names    x(1) x(2) x(3) x(4) x(5) x(6) x(7) x(8) x(\
    9) x(10) x(11) x(12) x(13) x(14) x(15) x(16) x(17) x(18) x(19) x(20) x(21\
    ) x(22) x(23) x(24) x(25) x(26) x(27) x(28) x(29) x(30)
⇒ //          block    2 : ordering C
kill P5,P30;

```

### D.2.12.7 extendring

Procedure from library `ring.lib` (see [Section D.2.12 \[ring\\_lib\], page 943](#)).

**Usage:** `extendring(n,va,o[,iv,i,r]);`  $va, o = \text{strings}, n, i = \text{integers}, r = \text{ring}, iv = \text{intvec of positive integers or } iv=0$

**Return:** ring  $R$ , which extends the ring  $r$  by adding  $n$  new variables in front of (resp. after, if  $i \neq 0$ ) the old variables.  
 [default:  $(i,r)=(0,\text{basing})$ ].

- The characteristic is the characteristic of  $r$ .
- The new vars are derived from  $va$ . If  $va$  is a single letter, say  $va="T"$ , and if  $n \leq 26$

then T and the following n-1 letters from T..Z..T (resp. T(1..n) if  $n > 26$ ) are taken as additional variables. If va is a single letter followed by a bracket, say va="x(", the new variables are x(1),...,x(n).

– The ordering is the product ordering of the ordering of r and of an ordering derived from 'o' [and iv].

– If o contains a 'c' or a 'C' in front resp. at the end, this is taken for the whole ordering in front, resp. at the end. If o does not contain a 'c' or a 'C' the same rule applies to ordstr(r).

– If no intvec iv is given, or if iv=0, o may be any allowed ordstr, like "ds" or "dp(2),wp(1,2,3),Ds(2)" or "ds(a),dp(b),ls" if a and b are globally (!) defined integers and if  $a+b+1 \leq n$ . If, however, a and b are local to a proc calling extendring, the intvec iv must be used to let extendring know the values of a and b

– If a non-zero intvec iv is given, iv[1],iv[2],... are taken for the 1st, 2nd,... block of o, if o contains no substring "w" or "W" i.e. no weighted ordering (in the above case o="ds,dp,ls" and iv=a,b).

If o contains a weighted ordering (only one (!) weighted block is allowed) iv[1] is taken as size for the weight-vector, the next iv[1] values of iv are taken as weights and the remaining values of iv as block size for the remaining non-weighted blocks. e.g. o="dp,ws,Dp,ds", iv=3,2,3,4,2,5 creates the ordering dp(2),ws(2,3,4),Dp(5),ds

**Note:** This proc is useful for adding deformation parameters.

This proc uses 'execute' or calls a procedure using 'execute'. If you use it in your own proc, it may be advisable to let the local names of your proc start with a @

**Example:**

```
LIB "ring.lib";
ring r=0,(x,y,z),ds;
show(r);"";
⇒ // ring: (QQ),(x,y,z),(ds(3),C);
⇒ // minpoly = 0
⇒ // objects belonging to this ring:
⇒
// blocksize is derived from no of vars:
int t=5;
def R1=extendring(t,"a","dp");          //t global: "dp" -> "dp(5)"
show(R1); setring R1; "";
⇒ // ring: (QQ),(a,b,c,d,e,x,y,z),(dp(5),ds(3),C);
⇒ // minpoly = 0
⇒ // objects belonging to this ring:
⇒
def R2=extendring(4,"T(","c,dp",1,r);    //"dp" -> "c,...,dp(4)"
show(R2); setring R2; "";
⇒ // ring: (QQ),(x,y,z,T(1),T(2),T(3),T(4)),(c,ds(3),dp(4));
⇒ // minpoly = 0
⇒ // objects belonging to this ring:
⇒
// no intvec given, blocksize given: given blocksize is used:
def R3=extendring(4,"T(","dp(2)",0,r);   // "dp(2)" -> "dp(2)"
show(R3); setring R3; "";
⇒ // ring: (QQ),(T(1),T(2),T(3),T(4),x,y,z),(dp(2),ds(5),C);
⇒ // minpoly = 0
⇒ // objects belonging to this ring:
⇒
```

```

// intvec given: weights and blocksize is derived from given intvec
// (no specification of a blocksize in the given ordstr is allowed!)
// if intvec does not cover all given blocks, the last block is used
// for the remaining variables, if intvec has too many components,
// the last ones are ignored
intvec v=3,2,3,4,1,3;
def R4=extendingring(10,"A","ds,ws,Dp,dp",v,0,r);
// v covers 3 blocks: v[1] (=3) : no of components of ws
// next v[1] values (=v[2..4]) give weights
// remaining components of v are used for the remaining blocks
show(R4);
⇒ // ring: (QQ),(A,B,C,D,E,F,G,H,I,J,x,y,z),(ds(1),ws(2,3,4),Dp(3),dp(3),ds\
(3),C);
⇒ // minpoly = 0
⇒ // objects belonging to this ring:
kill r,R1,R2,R3,R4;

```

### D.2.12.8 fetchall

Procedure from library `ring.lib` (see [Section D.2.12 \[ring.lib\]](#), page 943).

**Usage:** `fetchall(R[,s]);` `R=ring/qring`, `s=string`

**Create:** fetch all objects of ring `R` (of type `poly/ideal/vector/module/number/matrix`) into the basering.  
If no 2nd argument is present, the names are the same as in `R`. If, say, `f` is a polynomial in `R` and the 2nd argument is the string `"R"`, then `f` is mapped to `f_R` etc.

**Return:** no return value

**Note:** As `fetch`, this procedure maps the 1st, 2nd, ... variable of `R` to the 1st, 2nd, ... variable of the basering.  
The 2nd argument is useful in order to avoid conflicts of names, the empty string is allowed

**Caution:** `fetchall` does not work for locally defined names.  
It does not work if `R` contains a map.

**Example:**

```

LIB "ring.lib";
// The example is not shown since fetchall does not work in a procedure;
// (and hence not in the example procedure). Try the following commands:
// ring R=0,(x,y,z),dp;
// ideal j=x,y2,z2;
// matrix M[2][3]=1,2,3,x,y,z;
// j; print(M);
// ring S=0,(a,b,c),ds;
// fetchall(R); //map from R to S: x->a, y->b, z->c;
// names(S);
// j; print(M);
// fetchall(S,"1"); //identity map of S: copy objects, change names
// names(S);
// kill R,S;

```

See also: [Section D.2.12.9 \[imapall\]](#), page 950.

### D.2.12.9 imapall

Procedure from library `ring.lib` (see [Section D.2.12 \[ring\\_lib\]](#), page 943).

**Usage:** `imapall(R[,s]); R=ring/qring, s=string`

**Create:** map all objects of ring `R` (of type poly/ideal/vector/module/number/matrix) into the basering by applying `imap` to all objects of `R`. If no 2nd argument is present, the names are the same as in `R`. If, say, `f` is a polynomial in `R` and the 3rd argument is the string "`R`", then `f` is mapped to `f_R` etc.

**Return:** no return value

**Note:** As `imap`, this procedure maps the variables of `R` to the variables with the same name in the basering, the other variables are mapped to 0. The 2nd argument is useful in order to avoid conflicts of names, the empty string is allowed

**Caution:** `imapall` does not work for locally defined names.  
It does not work if `R` contains a map

**Example:**

```
LIB "ring.lib";
// The example is not shown since imapall does not work in a procedure
// (and hence not in the example procedure). Try the following commands:
// ring R=0,(x,y,z,u),dp;
// ideal j=x,y,z,u2+ux+z;
// matrix M[2][3]=1,2,3,x,y,uz;
// j; print(M);
// ring S=0,(a,b,c,x,z,y),ds;
// imapall(R); //map from R to S: x->x, y->y, z->z, u->0
// names(S);
// j; print(M);
// imapall(S,"1"); //identity map of S: copy objects, change names
// names(S);
// kill R,S;
```

See also: [Section D.2.12.8 \[fetchall\]](#), page 949.

### D.2.12.10 mapall

Procedure from library `ring.lib` (see [Section D.2.12 \[ring\\_lib\]](#), page 943).

**Usage:** `mapall(R,i[,s]); R=ring/qring, i=ideal of basering, s=string`

**Create:** map all objects of ring `R` (of type poly/ideal/vector/module/number/ matrix, map) into the basering by mapping the `j`-th variable of `R` to the `j`-th generator of the ideal `i`. If no 3rd argument is present, the names are the same as in `R`. If, say, `f` is a polynomial in `R` and the 3rd argument is the string "`R`", then `f` is mapped to `f_R` etc.

**Return:** no return value.

**Note:** This procedure has the same effect as defining a map, say `psi`, by `map psi=R,i;` and then applying `psi` to all objects of `R`. In particular, maps from `R` to some ring `S` are composed with `psi`, creating thus a map from the basering to `S`.  
`mapall` may be combined with `copyring` to change vars for all objects. The 3rd argument is useful in order to avoid conflicts of names, the empty string is allowed.

**Caution:** `mapall` does not work for locally defined names.

**Example:**

```

LIB "ring.lib";
// The example is not shown since mapall does not work in a procedure
// (and hence not in the example procedure). Try the following commands:
//   ring R=0,(x,y,z),dp;
//   ideal j=x,y,z;
//   matrix M[2][3]=1,2,3,x,y,z;
//   map phi=R,x2,y2,z2;
//   ring S=0,(a,b,c),ds;
//   ideal i=c,a,b;
//   mapall(R,i);           //map from R to S: x->c, y->a, z->b
//   names(S);
//   j; print(M); phi;     //phi maps R to S: x->c2, y->a2, z->b2
//   ideal i1=a2,a+b,1;
//   mapall(R,i1,"");      //map from R to S: x->a2, y->a+b, z->1
//   names(S);
//   j_; print(M_); phi_;
//   changevar("T","x()",R); //change vars in R and call result T
//   mapall(R,maxideal(1)); //identity map from R to T
//   names(T);
//   j; print(M); phi;
//   kill R,S,T;

```

**D.2.12.11 ord\_test**

Procedure from library `ring.lib` (see [Section D.2.12 \[ring\\_lib\], page 943](#)).

**Usage:** `ord_test(r);` `r` ring/qring

**Return:** `int` 1 (resp. -1, resp. 0) if ordering of `r` is global (resp. local, resp. mixed)

**Example:**

```

LIB "ring.lib";
ring R = 0,(x,y),dp;
ring S = 0,(u,v),ls;
ord_test(R);
⇒ 1
ord_test(S);
⇒ -1
ord_test(R+S);
⇒ 0

```

See also: [Section 5.1.2 \[attrib\], page 156](#).

**D.2.12.12 ringtensor**

Procedure from library `ring.lib` (see [Section D.2.12 \[ring\\_lib\], page 943](#)).

**Usage:** `ringtensor(r1,r2,...);` `r1,r2,...=rings`

**Return:** ring `R` whose variables are the variables from all rings `r1,r2,...` and whose monomial ordering is the block (product) ordering of the respective monomial orderings of `r1,r2,...`. Hence, `R` is the tensor product of the rings `r1,r2,...` with ordering matrix equal to the direct sum of the ordering matrices of `r1,r2,...`

**Note:** The characteristic of the new ring will be  $p$  if one ring has characteristic  $p$ . The names of variables in the rings  $r_1, r_2, \dots$  must differ.  
 The procedure works also for quotient rings  $r_i$ , if the characteristic of  $r_i$  is compatible with the characteristic of the result (i.e. if  $\text{imap}$  from  $r_i$  to the result is implemented)

**Example:**

```
LIB "ring.lib";
ring r=32003,(x,y,u,v),dp;
ring s=0,(a,b,c),wp(1,2,3);
ring t=0,x(1..5),(c,ls);
def R=ringtensor(r,s,t);
type R;
⇒ // R ring
⇒ // coefficients: ZZ/32003
⇒ // number of vars : 12
⇒ //      block 1 : ordering dp
⇒ //      : names  x y u v
⇒ //      block 2 : ordering wp
⇒ //      : names  a b c
⇒ //      : weights 1 2 3
⇒ //      block 3 : ordering ls
⇒ //      : names  x(1) x(2) x(3) x(4) x(5)
⇒ //      block 4 : ordering C
setring s;
ideal i = a2+b3+c5;
def S=changevar("x,y,z");      //change vars of s
setring S;
qring qS =std(fetch(s,i));      //create qring of S mod i (mapped to S)
def T=changevar("d,e,f,g,h",t); //change vars of t
setring T;
qring qT=std(d2+e2-f3);        //create qring of T mod d2+e2-f3
def Q=ringtensor(s,qS,t,qT);
setring Q; type Q;
⇒ // Q ring
⇒ // coefficients: QQ
⇒ // number of vars : 16
⇒ //      block 1: ordering wp
⇒ //      : names  a b c
⇒ //      : weights 1 2 3
⇒ //      block 2 : ordering wp
⇒ //      : names  x y z
⇒ //      : weights 1 2 3
⇒ //      block 3 : ordering ls
⇒ //      : names  x(1) x(2) x(3) x(4) x(5)
⇒ //      block 4 : ordering ls
⇒ //      : names  d e f g h
⇒ //      block 5 : ordering C
⇒ // quotient ring from ideal
⇒ _[1]=z5+y3+x2
⇒ _[2]=f3-e2-d2
kill R,S,T,Q;
```

See also: [Section 4.20.4 \[ring operations\]](#), page 128.

### D.2.12.13 ringweights

Procedure from library `ring.lib` (see [Section D.2.12 \[ring\\_lib\], page 943](#)).

**Usage:** `ringweights(P)`;  $P$ =name of an existing ring (true name, not a string)

**Return:** `intvec` consisting of the weights of the variables of  $P$ , as they appear when typing  $P$ ;

**Note:** This is useful when enlarging  $P$  but keeping the weights of the old variables.

**Example:**

```
LIB "ring.lib";
ring r0 = 0,(x,y,z),dp;
ringweights(r0);
↪ 1,1,1
ring r1 = 0,x(1..5),(ds(3),wp(2,3));
ringweights(r1);"";
↪ 1,1,1,2,3
↪
// an example for enlarging the ring, keeping the first weights:
intvec v = ringweights(r1),6,2,3,4,5;
ring R = 0,x(1..10),(a(v),dp);
ordstr(R);
↪ a(1,1,1,2,3,6,2,3,4,5),dp(10),C
```

### D.2.12.14 preimageLoc

Procedure from library `ring.lib` (see [Section D.2.12 \[ring\\_lib\], page 943](#)).

**Usage:** `preimageLoc ( ring_name, map_name, ideal_name );`  
all input parameters of type string

**Return:** ideal

**Purpose:** compute the preimage of an ideal under a given map for non-global orderings.  
The 2nd argument has to be the name of a map from the basering to the given ring (or the name of an ideal defining such a map), and the ideal has to be an ideal in the given ring.

**Example:**

```
LIB "ring.lib";
ring S =0,(x,y,z),dp;
ring R0=0,(x,y,z),ds;
qring R=std(x+x2); if(voice>1) {export R;}
map psi=S,x,y,z; if(voice>1) {export psi;}
ideal null; if(voice>1) {export null;}
setring S;
ideal nu=preimageLoc("R","psi","null");
nu;
↪ nu[1]=x
```

See also: [Section 5.1.117 \[preimage\], page 240](#).

### D.2.12.15 rootofUnity

Procedure from library `ring.lib` (see [Section D.2.12 \[ring\\_lib\], page 943](#)).

**Usage:** `rootofUnity(n)`;  $n$  an integer

**Return:** number

**Purpose:** compute the minimal polynomial for the n-th primitive root of unity

**Note:** works only in field extensions by one element

**Example:**

```
LIB "ring.lib";
ring r = (0,q),(x,y,z),dp;
rootofUnity(6);
↪ (q2-q+1)
rootofUnity(7);
↪ (q6+q5+q4+q3+q2+q+1)
minpoly = rootofUnity(8);
r;
↪ // coefficients: QQ[q]/(q4+1)
↪ // number of vars : 3
↪ //      block 1 : ordering dp
↪ //      : names x y z
↪ //      block 2 : ordering C
```

### D.2.12.16 optionIsSet

Procedure from library `ring.lib` (see [Section D.2.12 \[ring\\_lib\], page 943](#)).

**Usage:** optionIsSet( optionName )

**Parameters:**

optionName: a name as string of an option of interest

**Return:** true, if the by optionName given option is active, false otherwise.

**Example:**

```
LIB "ring.lib";
// check if the option "warn" is set.
optionIsSet("warn");
↪ 0
option("warn");
// now the option is set
optionIsSet("warn");
↪ 1
option("nowarn");
// now the option is unset
optionIsSet("warn");
↪ 0
```

### D.2.12.17 hasNumericCoeffs

Procedure from library `ring.lib` (see [Section D.2.12 \[ring\\_lib\], page 943](#)).

**Usage:** hasNumericCoeffs ( rng );

**Return:** 1 if rng has inexact coefficients, 0 otherwise.

**Example:**



```

LIB "ring.lib";
ring r1 = integer,x,dp;
hasNumericCoeffs(r1);
↪ 0
ring r2 = complex,x,dp;
hasNumericCoeffs(r2);
↪ 1

```

### D.2.12.18 hasCommutativeVars

Procedure from library `ring.lib` (see [Section D.2.12 \[ring\\_lib\], page 943](#)).

**Usage:**     `hasCommutativeVars ( rng );`

**Return:**    1 if `rng` is a commutative polynomial ring, 0 otherwise.

**Example:**

```

LIB "ring.lib";
ring r=0,(x,y,z),dp;
hasCommutativeVars(r);
↪ 1

```

### D.2.12.19 hasGlobalOrdering

Procedure from library `ring.lib` (see [Section D.2.12 \[ring\\_lib\], page 943](#)).

**Usage:**     `hasGlobalOrdering ( rng );`

**Return:**    1 if `rng` has a global monomial ordering, 0 otherwise.

**Example:**

```

LIB "ring.lib";
ring rng = integer,x,dp;
hasGlobalOrdering(rng); //yes
↪ 1
ring rng2 = 0, x, ds;
hasGlobalOrdering(rng2); // no
↪ 0

```

### D.2.12.20 hasMixedOrdering

Procedure from library `ring.lib` (see [Section D.2.12 \[ring\\_lib\], page 943](#)).

**Usage:**     `hasMixedOrdering();`

**Return:**    1 if ordering of basering is mixed, 0 else

**Example:**

```

LIB "ring.lib";
ring R1 = 0,(x,y,z),dp;
hasMixedOrdering();
↪ 0
ring R2 = 31,(x(1..4),y(1..3)),(ds(4),lp(3));
hasMixedOrdering();
↪ 1
ring R3 = 181,x(1..9),(dp(5),lp(4));
hasMixedOrdering();
↪ 0

```

### D.2.12.21 hasAlgExtensionCoefficient

Procedure from library `ring.lib` (see [Section D.2.12 \[ring\\_lib\], page 943](#)).

**Usage:**     `hasAlgExtensionCoefficient ( rng );`

**Return:**    1 if the coefficients are an algebraic extension, 0 otherwise.

**Example:**

```
LIB "ring.lib";
ring rng = integer,x,dp;
hasAlgExtensionCoefficient(rng); //no
↳ 0
ring rng2 = (0,a), x, dp; minpoly=a2-1;
hasAlgExtensionCoefficient(rng2); // yes
↳ 1
ring rng3=(49,a),x,dp;
hasAlgExtensionCoefficient(rng3); // no
↳ 0
```

### D.2.12.22 hasTransExtensionCoefficient

Procedure from library `ring.lib` (see [Section D.2.12 \[ring\\_lib\], page 943](#)).

**Usage:**     `hasTransExtensionCoefficient ( rng );`

**Return:**    1 if the coefficients are rational functions, 0 otherwise.

**Example:**

```
LIB "ring.lib";
ring rng = integer,x,dp;
hasTransExtensionCoefficient(rng); //no
↳ 0
ring rng2 = (0,a), x, dp;
hasTransExtensionCoefficient(rng2); // yes
↳ 1
ring rng3=(49,a),x,dp;
hasTransExtensionCoefficient(rng3); // no
↳ 0
```

### D.2.12.23 isQuotientRing

Procedure from library `ring.lib` (see [Section D.2.12 \[ring\\_lib\], page 943](#)).

**Return:**    1 if `rng` is a quotient ring, 0 otherwise.

**Purpose:**    check if type of a `rng` "qring"

**Example:**

```
LIB "ring.lib";
ring rng = 0,x,dp;
isQuotientRing(rng); //no
↳ 0
// if a certain method does not support quotient rings,
// then a parameter test could be performed:
ASSUME( 0, 0==isQuotientRing(basing));
qring q= ideal(x); // constructs rng/ideal(x)
isQuotientRing(q); // yes
↳ 1
```

**D.2.12.24 isSubModule**

Procedure from library `ring.lib` (see [Section D.2.12 \[ring\\_lib\], page 943](#)).

**Return:** 1 if `module(I)` is in `module(J)`, 0 otherwise

**Example:**

```
LIB "ring.lib";
ring r=0,x,dp;
ideal I1=x2;
ideal I2=x3;
isSubModule(I1, I2);
↪ 0
isSubModule(I2, I1);
↪ 1
```

**D.2.12.25 changeordTo**

Procedure from library `ring.lib` (see [Section D.2.12 \[ring\\_lib\], page 943](#)).

**Usage:** `changeordTo(ring, string s);`

**Return:** a ring with the ordering changed to the (simple) ordering `s`

**Example:**

```
LIB "ring.lib";
ring r=0,(x,y),lp;
def rr=changeordTo(r,"dp");
rr;
↪ // coefficients: QQ
↪ // number of vars : 2
↪ //          block 1 : ordering C
↪ //          block 2 : ordering dp
↪ //          : names    x y
```

**D.2.12.26 addvarsTo**

Procedure from library `ring.lib` (see [Section D.2.12 \[ring\\_lib\], page 943](#)).

**Usage:** `addvarsTo(ring,list_of_strings, int);`  
 int may be: 0:ordering: dp  
 1:ordering dp,dp  
 2:oring.ordering,dp

**Return:** a ring with the additional variables

**Example:**

```
LIB "ring.lib";
ring r=0,(x,y),lp;
def rr=addvarsTo(r,list("a","b"),0);
rr; kill rr;
↪ // coefficients: QQ
↪ // number of vars : 4
↪ //          block 1 : ordering C
↪ //          block 2 : ordering dp
↪ //          : names    x y a b
```

```

def rr=addvarsTo(r,list("a","b"),1);
rr; kill rr;
⇨ // coefficients: QQ
⇨ // number of vars : 4
⇨ //          block 1 : ordering C
⇨ //          block 2 : ordering dp
⇨ //          : names  x y
⇨ //          block 3 : ordering dp
⇨ //          : names  a b
def rr=addvarsTo(r,list("a","b"),2);
rr;
⇨ // coefficients: QQ
⇨ // number of vars : 4
⇨ //          block 1 : ordering lp
⇨ //          : names  x y
⇨ //          block 2 : ordering C
⇨ //          block 3 : ordering dp
⇨ //          : names  a b

```

### D.2.12.27 addNvarsTo

Procedure from library `ring.lib` (see [Section D.2.12 \[ring.lib\]](#), page 943).

**Usage:**     `addNvarsTo(ring,int N, string name, int b);`  
               `b` may be: 0:ordering: dp  
                   1:ordering dp,dp  
                   2:oring.ordering,dp

**Return:**    a ring with `N` additional variables

**Example:**

```

LIB "ring.lib";
ring r=0,(x,y),lp;
def rr=addNvarsTo(r,2,"@",0);
rr; kill rr;
⇨ // coefficients: QQ
⇨ // number of vars : 4
⇨ //          block 1 : ordering C
⇨ //          block 2 : ordering dp
⇨ //          : names  x y @(1) @(2)
def rr=addNvarsTo(r,2,"@",1);
rr; kill rr;
⇨ // coefficients: QQ
⇨ // number of vars : 4
⇨ //          block 1 : ordering C
⇨ //          block 2 : ordering dp
⇨ //          : names  x y
⇨ //          block 3 : ordering dp
⇨ //          : names  @(1) @(2)
def rr=addNvarsTo(r,2,"@",2);
rr;
⇨ // coefficients: QQ
⇨ // number of vars : 4
⇨ //          block 1 : ordering lp

```

```

⇒ //                : names      x y
⇒ //                block 2 : ordering C
⇒ //                block 3 : ordering dp
⇒ //                : names      @(1) @(2)

```

### D.2.13 tasks.lib

**Library:** tasks.lib

**Purpose:** A parallel framework based on tasks

**Author:** Andreas Steenpass, e-mail: steenpass@mathematik.uni-kl.de

**Overview:** This library provides a parallel framework based on tasks. It introduces a new Singular type `task`; an object of this type is a command (given by a string) applied to a list of arguments. Tasks can be computed in parallel via the procedures in this library and they can even be started recursively, i.e. from within other tasks.

tasks.lib respects the limits for computational resources defined in [Section D.2.11 \[resources.lib\]](#), page 942, i.e., all tasks within the same Singular session will not use more computational resources than provided via resources.lib, even if tasks are started recursively.

The Singular library [Section D.2.7 \[parallel.lib\]](#), page 885 provides implementations of several parallel 'skeletons' based on tasks.lib.

**Procedures:** See also: [Section D.2.7 \[parallel.lib\]](#), page 885; [Section D.2.11 \[resources.lib\]](#), page 942.

#### D.2.13.1 createTask

Procedure from library `tasks.lib` (see [Section D.2.13 \[tasks.lib\]](#), page 959).

**Usage:** `createTask(command, arguments)`, command string, arguments list

**Return:** a task with the given command and arguments whose state is 'created'.

**Note:** `'t = command, arguments;'` is a shortcut for  
`'t = createTask(command, arguments);'`

**Example:**

```

LIB "tasks.lib";
ring R = 0, (x,y), dp;
ideal I = x9y2+x10, x2y7-y8;
task t = createTask("std", list(I));
// This is the same as:
// task t = "std", list(I);
t;
⇒ A task with the following properties:
⇒ command:          std
⇒ no. of arguments: 1
⇒ state:            created
⇒
killTask(t);

```

See also: [Section D.2.13.4 \[compareTasks\]](#), page 961; [Section D.2.13.3 \[copyTask\]](#), page 960; [Section D.2.13.12 \[getArguments\]](#), page 967; [Section D.2.13.11 \[getCommand\]](#), page 966; [Section D.2.13.14 \[getState\]](#), page 968; [Section D.2.13.2 \[killTask\]](#), page 960; [Section D.2.13.5 \[printTask\]](#), page 962; [Section D.2.13.6 \[startTasks\]](#), page 962.

### D.2.13.2 killTask

Procedure from library `tasks.lib` (see [Section D.2.13 \[tasks.lib\]](#), page 959).

**Usage:** `killTask(t)`, `t` task

**Return:** nothing. If the state of `t` is 'started', then `t` is stopped first. The internal data structures of `t` are erased and its state is set to 'uninitialized'.

**Note:** '`killTask(t)`;' is not the same as '`kill t`;' . The latter command does not erase the internal data structures of `t`. Hence `killTask()` should be called for any no longer needed task in order to free memory.

**Example:**

```
LIB "tasks.lib";
ring R = 0, (x,y), dp;
ideal I = x9y2+x10, x2y7-y8;
task t = "std", list(I);
startTasks(t);
t;
⇒ A task with the following properties:
⇒ command:          std
⇒ no. of arguments: 1
⇒ state:            started
⇒
killTask(t);
t;
⇒ An uninitialized task
⇒
getState(t);
⇒ uninitialized
```

See also: [Section D.2.13.1 \[createTask\]](#), page 959; [Section D.2.13.14 \[getState\]](#), page 968; [Section D.2.13.5 \[printTask\]](#), page 962; [Section D.2.13.7 \[stopTask\]](#), page 963.

### D.2.13.3 copyTask

Procedure from library `tasks.lib` (see [Section D.2.13 \[tasks.lib\]](#), page 959).

**Usage:** `copyTask(t)`, `t` task

**Return:** a copy of `t`.

**Note:** '`task t1 = copyTask(t2)`;' is not the same as '`task t1 = t2`;' . After the latter command, `t1` points to the same object as `t2`; any changes to `t2` will also effect `t1`. In contrast to this, `copyTask()` creates a new independent task.  
A task whose state is 'started' cannot be copied.

**Example:**

```
LIB "tasks.lib";
ring R = 0, (x,y), dp;
ideal I = x9y2+x10, x2y7-y8;
task t1 = "std", list(I);
startTasks(t1);
waitAllTasks(t1);
task t2 = copyTask(t1);
killTask(t1);
```

```

t2;    // t2 survived
⇒ A task with the following properties:
⇒ command:          std
⇒ no. of arguments: 1
⇒ state:            completed
⇒
getResult(t2);
⇒ _[1]=x2y7-y8
⇒ _[2]=x9y2+x10
⇒ _[3]=x12y+xy11
⇒ _[4]=x13-xy12
⇒ _[5]=y14+xy12
⇒ _[6]=xy13+y12
killTask(t2);

```

See also: [Section D.2.13.4 \[compareTasks\]](#), page 961; [Section D.2.13.1 \[createTask\]](#), page 959; [Section D.2.13.12 \[getArguments\]](#), page 967; [Section D.2.13.11 \[getCommand\]](#), page 966; [Section D.2.13.13 \[getResult\]](#), page 967; [Section D.2.13.14 \[getState\]](#), page 968; [Section D.2.13.2 \[killTask\]](#), page 960; [Section D.2.13.5 \[printTask\]](#), page 962.

### D.2.13.4 compareTasks

Procedure from library `tasks.lib` (see [Section D.2.13 \[tasks.lib\]](#), page 959).

**Usage:**      `compareTasks(t1, t2)`, `t1`, `t2` tasks

**Return:**    1, if `t1` and `t2` coincide;  
               0, otherwise.

**Note:**      The arguments and the results of `t1` and `t2` are not compared.  
               '`t1 == t2`' is a shortcut for '`compareTasks(t1, t2)`'.

**Example:**

```

LIB "tasks.lib";
ring R = 0, (x,y), dp;
ideal I = x9y2+x10, x2y7-y8;
task t1 = "std", list(I);
task t2 = "std", list(I);
compareTasks(t1, t2);
⇒ 1
startTasks(t1);
waitAllTasks(t1);
t1 == t2;    // the same as compareTasks(t1, t2);
⇒ 0
killTask(t1);
killTask(t2);
// The arguments and the result are not compared!
ideal J = x;
task t3 = "std", list(I);
task t4 = "std", list(J);
t3 == t4;
⇒ 1
killTask(t3);
killTask(t4);

```

See also: [Section D.2.13.3 \[copyTask\]](#), page 960; [Section D.2.13.12 \[getArguments\]](#), page 967; [Section D.2.13.11 \[getCommand\]](#), page 966; [Section D.2.13.13 \[getResult\]](#), page 967; [Section D.2.13.14 \[getState\]](#), page 968; [Section D.2.13.5 \[printTask\]](#), page 962.

### D.2.13.5 printTask

Procedure from library `tasks.lib` (see [Section D.2.13 \[tasks.lib\]](#), page 959).

**Usage:** `printTask(t)`, `t` task

**Return:** nothing. Prints information about `t`.

**Note:** `'print(t);'` and `'t;'` are shortcuts for `'printTask(t)'`.

**Example:**

```
LIB "tasks.lib";
ring R = 0, (x,y), dp;
ideal I = x9y2+x10, x2y7-y8;
task t;
printTask(t);
  ↳ An uninitialized task
t = "std", list(I);
t; // the same as printTask(t);
  ↳ A task with the following properties:
  ↳ command:          std
  ↳ no. of arguments: 1
  ↳ state:             created
  ↳
startTasks(t);
waitAllTasks(t);
t;
  ↳ A task with the following properties:
  ↳ command:          std
  ↳ no. of arguments: 1
  ↳ state:             completed
  ↳
killTask(t);
```

See also: [Section D.2.13.1 \[createTask\]](#), page 959; [Section D.2.13.12 \[getArguments\]](#), page 967; [Section D.2.13.11 \[getCommand\]](#), page 966; [Section D.2.13.13 \[getResult\]](#), page 967; [Section D.2.13.14 \[getState\]](#), page 968; [Section D.2.13.2 \[killTask\]](#), page 960.

### D.2.13.6 startTasks

Procedure from library `tasks.lib` (see [Section D.2.13 \[tasks.lib\]](#), page 959).

**Usage:** `startTasks(t1, t2, ...)`, `t1, t2, ...` tasks

**Return:** nothing. Starts the tasks `t1, t2, ...` and sets their states to `'started'`.

**Note:** A task whose state is neither `'created'` nor `'stopped'` cannot be started. If `startTasks()` is applied to a task whose state is `'stopped'`, then the computation of this task will be restarted from the beginning. Tasks can be started from within other tasks. A started task should not be accessed from within any task other than the one within which it was started. For each task, the start of its computation is subject to the internal scheduling.



**Example:**

```

LIB "tasks.lib";
ring R = 0, (x,y), dp;
ideal I = x9y2+x10, x2y7-y8;
task t1 = "std", list(I);
task t2 = "slimgb", list(I);
startTasks(t1, t2);
waitAllTasks(t1, t2);
getResult(t1);
↳ _[1]=x2y7-y8
↳ _[2]=x9y2+x10
↳ _[3]=x12y+xy11
↳ _[4]=x13-xy12
↳ _[5]=y14+xy12
↳ _[6]=xy13+y12
getResult(t2);
↳ _[1]=x2y7-y8
↳ _[2]=x9y2+x10
↳ _[3]=x12y+xy11
↳ _[4]=x13-xy12
↳ _[5]=xy13+y12
↳ _[6]=y14+xy12
killTask(t1);
killTask(t2);

```

See also: [Section D.2.13.1 \[createTask\]](#), page 959; [Section D.2.13.14 \[getState\]](#), page 968; [Section D.2.13.10 \[pollTask\]](#), page 965; [Section D.2.13.5 \[printTask\]](#), page 962; [Section D.2.13.7 \[stopTask\]](#), page 963; [Section D.2.13.8 \[waitTasks\]](#), page 964.

**D.2.13.7 stopTask**

Procedure from library `tasks.lib` (see [Section D.2.13 \[tasks.lib\]](#), page 959).

**Usage:**      `stopTask(t)`, `t` task

**Return:**    nothing. Stops the `t` and sets its state to 'stopped'.

**Note:**      A task whose state is not 'started' cannot be stopped.  
Intermediate results are discarded when a task is stopped.  
`killTask()` should be called for any no longer needed task.

**Example:**

```

LIB "tasks.lib";
ring R = 0, (x,y), dp;
ideal I = x9y2+x10, x2y7-y8;
task t = "std", list(I);
startTasks(t);
stopTask(t);
t;
↳ A task with the following properties:
↳ command:                std
↳ no. of arguments: 1
↳ state:                  stopped
↳
killTask(t);

```

See also: [Section D.2.13.14 \[getState\]](#), page 968; [Section D.2.13.2 \[killTask\]](#), page 960; [Section D.2.13.10 \[pollTask\]](#), page 965; [Section D.2.13.5 \[printTask\]](#), page 962; [Section D.2.13.6 \[startTasks\]](#), page 962; [Section D.2.13.8 \[waitTasks\]](#), page 964.

### D.2.13.8 waitTasks

Procedure from library `tasks.lib` (see [Section D.2.13 \[tasks.lib\]](#), page 959).

**Usage:** `waitTasks(T, N[, timeout])`, T list of tasks, N int, timeout int

**Return:** an ordered list of the indices of those tasks which have been successfully completed. The state of these tasks is set to 'completed'.  
The procedure waits for N tasks to complete.  
An optional timeout in ms can be provided. Default is 0 which disables the timeout.

**Note:** A task whose state is neither 'started' nor 'completed' cannot be waited for.  
The result of any completed task can be accessed via [Section D.2.13.13 \[getResult\]](#), page 967.

The returned list may contain more than N entries if the computation of some tasks has already finished and/or if several tasks finish "at the same time". It may contain less than N entries in the case of timeout or errors occurring.

Polling is guaranteed, i.e. the index of any task t for which 'pollTask(t);' would return 1 will appear in the returned list.

#### Example:

```
LIB "tasks.lib";
ring R = 0, (x,y), dp;
ideal I = x9y2+x10, x2y7-y8;
task t1 = "std", list(I);
task t2 = "slimgb", list(I);
startTasks(t1, t2);
waitTasks(list(t1, t2), 2);    // wait for both tasks
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 2
getResult(t1);
⇒ _[1]=x2y7-y8
⇒ _[2]=x9y2+x10
⇒ _[3]=x12y+xy11
⇒ _[4]=x13-xy12
⇒ _[5]=y14+xy12
⇒ _[6]=xy13+y12
getResult(t2);
⇒ _[1]=x2y7-y8
⇒ _[2]=x9y2+x10
⇒ _[3]=x12y+xy11
⇒ _[4]=x13-xy12
⇒ _[5]=xy13+y12
⇒ _[6]=y14+xy12
killTask(t1);
killTask(t2);
```

See also: [Section D.2.13.13 \[getResult\]](#), page 967; [Section D.2.13.14 \[getState\]](#), page 968; [Section D.2.13.10 \[pollTask\]](#), page 965; [Section D.2.13.5 \[printTask\]](#), page 962; [Section D.2.13.6 \[startTasks\]](#), page 962.

### D.2.13.9 waitAllTasks

Procedure from library `tasks.lib` (see [Section D.2.13 \[tasks.lib\]](#), page 959).

**Usage:**      `waitAllTasks(t1, t2, ...), t1, t2, ... tasks`

**Return:**      nothing. Waits for all the tasks `t1, t2, ...` to complete. The state of the tasks is set to 'completed'.

**Note:**      A task whose state is neither 'started' nor 'completed' cannot be waited for. The result of any completed task can be accessed via [Section D.2.13.13 \[getResult\]](#), page 967.   
`'waitAllTasks(t1, t2, ...);'` is a shortcut for `'waitTasks(list(t1, t2, ...), size(list(t1, t2, ...)))'`. Since returning a list of the indices of the completed tasks does not make sense in this case, nothing is returned.

**Example:**

```
LIB "tasks.lib";
ring R = 0, (x,y), dp;
ideal I = x9y2+x10, x2y7-y8;
task t1 = "std", list(I);
task t2 = "slimgb", list(I);
startTasks(t1, t2);
waitAllTasks(t1, t2); // the same as 'waitTasks(list(t1, t2), 2);',
// but without return value
getResult(t1);
⇒ _[1]=x2y7-y8
⇒ _[2]=x9y2+x10
⇒ _[3]=x12y+xy11
⇒ _[4]=x13-xy12
⇒ _[5]=y14+xy12
⇒ _[6]=xy13+y12
getResult(t2);
⇒ _[1]=x2y7-y8
⇒ _[2]=x9y2+x10
⇒ _[3]=x12y+xy11
⇒ _[4]=x13-xy12
⇒ _[5]=xy13+y12
⇒ _[6]=y14+xy12
killTask(t1);
killTask(t2);
```

See also: [Section D.2.13.13 \[getResult\]](#), page 967; [Section D.2.13.14 \[getState\]](#), page 968; [Section D.2.13.10 \[pollTask\]](#), page 965; [Section D.2.13.5 \[printTask\]](#), page 962; [Section D.2.13.6 \[startTasks\]](#), page 962; [Section D.2.13.8 \[waitTasks\]](#), page 964.

### D.2.13.10 pollTask

Procedure from library `tasks.lib` (see [Section D.2.13 \[tasks.lib\]](#), page 959).

**Usage:**      `pollTask(t), t task`

**Return:** 1, if the computation of the task `t` has successfully finished; 0, otherwise.  
The state of any task whose computation has successfully finished is set to 'completed'.

**Note:** A task whose state is neither 'started' nor 'completed' cannot be polled.  
The result of any completed task can be accessed via [Section D.2.13.13 \[getResult\]](#), page 967.  
`pollTask()` should return immediately. However, receiving the result of the task may take some time.

**Example:**

```
LIB "tasks.lib";
ring R = 0, (x,y), dp;
ideal I = x9y2+x10, x2y7-y8;
task t = "std", list(I);
startTasks(t);
waitAllTasks(t);
pollTask(t);    // task already completed
⇒ 1
t;
⇒ A task with the following properties:
⇒ command:          std
⇒ no. of arguments: 1
⇒ state:             completed
⇒
getResult(t);
⇒ _[1]=x2y7-y8
⇒ _[2]=x9y2+x10
⇒ _[3]=x12y+xy11
⇒ _[4]=x13-xy12
⇒ _[5]=y14+xy12
⇒ _[6]=xy13+y12
killTask(t);
```

See also: [Section D.2.13.13 \[getResult\]](#), page 967; [Section D.2.13.14 \[getState\]](#), page 968; [Section D.2.13.5 \[printTask\]](#), page 962; [Section D.2.13.6 \[startTasks\]](#), page 962; [Section D.2.13.8 \[waitTasks\]](#), page 964.

### D.2.13.11 getCommand

Procedure from library `tasks.lib` (see [Section D.2.13 \[tasks.lib\]](#), page 959).

**Usage:** `getCommand(t)`, `t` task

**Return:** a string, the command of `t`.

**Note:** This command cannot be applied to tasks whose state is 'uninitialized'.

**Example:**

```
LIB "tasks.lib";
ring R = 0, (x,y), dp;
ideal I = x9y2+x10, x2y7-y8;
task t = "std", list(I);
getCommand(t);
⇒ std
killTask(t);
```

See also: [Section D.2.13.1 \[createTask\]](#), page 959; [Section D.2.13.12 \[getArguments\]](#), page 967; [Section D.2.13.13 \[getResult\]](#), page 967; [Section D.2.13.14 \[getState\]](#), page 968; [Section D.2.13.5 \[printTask\]](#), page 962.

### D.2.13.12 getArguments

Procedure from library `tasks.lib` (see [Section D.2.13 \[tasks.lib\]](#), page 959).

**Usage:**      `getArguments(t)`, `t` task

**Return:**     a list, the arguments of `t`.

**Note:**        This command cannot be applied to tasks whose state is 'uninitialized'.

**Example:**

```
LIB "tasks.lib";
ring R = 0, (x,y), dp;
ideal I = x9y2+x10, x2y7-y8;
task t = "std", list(I);
getArguments(t);
⇒ [1]:
⇒   _[1]=x9y2+x10
⇒   _[2]=x2y7-y8
killTask(t);
```

See also: [Section D.2.13.1 \[createTask\]](#), page 959; [Section D.2.13.11 \[getCommand\]](#), page 966; [Section D.2.13.13 \[getResult\]](#), page 967; [Section D.2.13.14 \[getState\]](#), page 968; [Section D.2.13.5 \[printTask\]](#), page 962.

### D.2.13.13 getResult

Procedure from library `tasks.lib` (see [Section D.2.13 \[tasks.lib\]](#), page 959).

**Usage:**      `getResult(t)`, `t` task

**Return:**     the result of `t`.

**Note:**        This command cannot be applied to tasks whose state is not 'completed'.

**Example:**

```
LIB "tasks.lib";
ring R = 0, (x,y), dp;
ideal I = x9y2+x10, x2y7-y8;
task t = "std", list(I);
startTasks(t);
waitAllTasks(t);
getResult(t);
⇒ _[1]=x2y7-y8
⇒ _[2]=x9y2+x10
⇒ _[3]=x12y+xy11
⇒ _[4]=x13-xy12
⇒ _[5]=y14+xy12
⇒ _[6]=xy13+y12
killTask(t);
```

See also: [Section D.2.13.12 \[getArguments\]](#), page 967; [Section D.2.13.11 \[getCommand\]](#), page 966; [Section D.2.13.14 \[getState\]](#), page 968; [Section D.2.13.10 \[pollTask\]](#), page 965; [Section D.2.13.5 \[printTask\]](#), page 962; [Section D.2.13.8 \[waitTasks\]](#), page 964.

### D.2.13.14 getState

Procedure from library `tasks.lib` (see [Section D.2.13 \[tasks.lib\]](#), page 959).

**Usage:**      `getState(t)`, `t` task

**Return:**     a string, the state of `t`.

**Example:**

```
LIB "tasks.lib";
ring R = 0, (x,y), dp;
ideal I = x9y2+x10, x2y7-y8;
task t = "std", list(I);
getState(t);
↳ created
startTasks(t);
getState(t);
↳ started
waitAllTasks(t);
getState(t);
↳ completed
killTask(t);
getState(t);
↳ uninitialized
```

See also: [Section D.2.13.1 \[createTask\]](#), page 959; [Section D.2.13.12 \[getArguments\]](#), page 967; [Section D.2.13.11 \[getCommand\]](#), page 966; [Section D.2.13.13 \[getResult\]](#), page 967; [Section D.2.13.2 \[killTask\]](#), page 960; [Section D.2.13.10 \[pollTask\]](#), page 965; [Section D.2.13.5 \[printTask\]](#), page 962; [Section D.2.13.6 \[startTasks\]](#), page 962; [Section D.2.13.7 \[stopTask\]](#), page 963; [Section D.2.13.8 \[waitTasks\]](#), page 964.

## D.3 Linear algebra

### D.3.1 matrix\_lib

**Library:**    `matrix.lib`

**Purpose:**     Elementary Matrix Operations

**Procedures:**

#### D.3.1.1 compress

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix\\_lib\]](#), page 968).

**Usage:**      `compress(A)`; `A` matrix/ideal/module/intmat/intvec

**Return:**     same type, zero columns/generators from `A` deleted  
(if `A=intvec`, zero elements are deleted)

**Example:**

```
LIB "matrix.lib";
ring r=0,(x,y,z),ds;
matrix A[3][4]=1,0,3,0,x,0,z,0,x2,0,z2,0;
print(A);
↳ 1, 0,3, 0,
```

```

↳ x, 0,z, 0,
↳ x2,0,z2,0
print(compress(A));
↳ 1, 3,
↳ x, z,
↳ x2,z2
module m=module(A); show(m);
↳ // module, 4 generator(s)
↳ [1,x,x2]
↳ [0]
↳ [3,z,z2]
↳ [0]
show(compress(m));
↳ // module, 2 generator(s)
↳ [1,x,x2]
↳ [3,z,z2]
intmat B[3][4]=1,0,3,0,4,0,5,0,6,0,7,0;
compress(B);
↳ 1,3,
↳ 4,5,
↳ 6,7
intvec C=0,0,1,2,0,3;
compress(C);
↳ 1,2,3

```

### D.3.1.2 concat

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix.lib\]](#), page 968).

**Usage:**      `concat(A1,A2,..)`;  $A_1, A_2, \dots$  matrices

**Return:**    matrix, concatenation of  $A_1, A_2, \dots$       Number of rows of result matrix is  $\max(\text{nrows}(A_1), \text{nrows}(A_2), \dots)$

**Example:**

```

LIB "matrix.lib";
ring r=0,(x,y,z),ds;
matrix A[3][3]=1,2,3,x,y,z,x2,y2,z2;
matrix B[2][2]=1,0,2,0; matrix C[1][4]=4,5,x,y;
print(A);
↳ 1, 2, 3,
↳ x, y, z,
↳ x2,y2,z2
print(B);
↳ 1,0,
↳ 2,0
print(C);
↳ 4,5,x,y
print(concat(A,B,C));
↳ 1, 2, 3, 1,0,4,5,x,y,
↳ x, y, z, 2,0,0,0,0,0,
↳ x2,y2,z2,0,0,0,0,0,0

```

### D.3.1.3 diag

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix.lib\], page 968](#)).

**Usage:**      `diag(p,n)`;  $p$  poly,  $n$  integer  
               `diag(A)`;  $A$  matrix

**Return:**    `diag(p,n)`: diagonal matrix,  $p$  times unit matrix of size  $n$ .  
               `diag(A)` :  $n*m \times n*m$  diagonal matrix with entries all the entries of the  $n \times m$  matrix  $A$ , taken from the 1st row, 2nd row etc of  $A$

**Example:**

```
LIB "matrix.lib";
ring r = 0,(x,y,z),ds;
print(diag(xy,4));
↪ xy,0, 0, 0,
↪ 0, xy,0, 0,
↪ 0, 0, xy,0,
↪ 0, 0, 0, xy
matrix A[3][2] = 1,2,3,4,5,6;
print(A);
↪ 1,2,
↪ 3,4,
↪ 5,6
print(diag(A));
↪ 1,0,0,0,0,0,
↪ 0,2,0,0,0,0,
↪ 0,0,3,0,0,0,
↪ 0,0,0,4,0,0,
↪ 0,0,0,0,5,0,
↪ 0,0,0,0,0,6
```

### D.3.1.4 dsum

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix.lib\], page 968](#)).

**Usage:**      `dsum(A1,A2,...)`;  $A_1, A_2, \dots$  matrices

**Return:**    matrix, direct sum of  $A_1, A_2, \dots$

**Example:**

```
LIB "matrix.lib";
ring r = 0,(x,y,z),ds;
matrix A[3][3] = 1,2,3,4,5,6,7,8,9;
matrix B[2][2] = 1,x,y,z;
print(A);
↪ 1,2,3,
↪ 4,5,6,
↪ 7,8,9
print(B);
↪ 1,x,
↪ y,z
print(dsum(A,B));
↪ 1,2,3,0,0,
↪ 4,5,6,0,0,
↪ 7,8,9,0,0,
```



```

    ↦ 0,0,0,1,x,
    ↦ 0,0,0,y,z

```

### D.3.1.5 flatten

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix.lib\], page 968](#)).

**Usage:**      `flatten(A)`; A matrix/smatrix

**Return:**    ideal, generated by all entries from A resp. all columns of A appended

**Example:**

```

LIB "matrix.lib";
ring r = 0,(x,y,z),ds;
matrix A[2][3] = 1,2,x,y,z,7;
print(A);
    ↦ 1,2,x,
    ↦ y,z,7
flatten(A);
    ↦ _[1]=1
    ↦ _[2]=2
    ↦ _[3]=x
    ↦ _[4]=y
    ↦ _[5]=z
    ↦ _[6]=7
flatten(smatrix(A));
    ↦ 1,
    ↦ y,
    ↦ 2,
    ↦ z,
    ↦ x,
    ↦ 7

```

### D.3.1.6 genericmat

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix.lib\], page 968](#)).

**Usage:**      `genericmat(n,m[,id])`; n,m=integers, id=ideal

**Return:**    nxm matrix, with entries from id.

**Note:**      if id has less than nxm elements, the matrix is filled with 0's, (default: id=maxideal(1)).  
               `genericmat(n,m)`; creates the generic nxm matrix

**Example:**

```

LIB "matrix.lib";
ring R = 0,x(1..16),lp;
print(genericmat(3,3));           // the generic 3x3 matrix
    ↦ x(1),x(2),x(3),
    ↦ x(4),x(5),x(6),
    ↦ x(7),x(8),x(9)
ring R1 = 0,(a,b,c,d),dp;
matrix A = genericmat(3,4,maxideal(1)^3);
print(A);
    ↦ a3, a2b,a2c,a2d,

```

```

⇒ ab2,abc,abd,ac2,
⇒ acd,ad2,b3, b2c
int n,m = 3,2;
ideal i = ideal(randommat(1,n*m,maxideal(1),9));
print(genericmat(n,m,i)); // matrix of generic linear forms
⇒ 4a-8b-2c-3d,-a+b-4c+5d,
⇒ -8a-9b+c+7d,a-9b+9c+4d,
⇒ 6a-5b+9c, 2a+8c+d

```

### D.3.1.7 is\_complex

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix.lib\], page 968](#)).

**Usage:** `is_complex(c)`; `c` = list of size-compatible modules or matrices

**Return:** 1 if  $c[i]*c[i+1]=0$  for all  $i$ , 0 if not, hence checking whether the list of matrices forms a complex.

**Note:** Ideals are treated internally as 1-line matrices.  
If `printlevel > 0`, the position where `c` is not a complex is shown.

**Example:**

```

LIB "matrix.lib";
ring r = 32003,(x,y,z),ds;
ideal i = x4+y5+z6,xyz,yx2+xz2+zy7;
list L = nres(i,0);
is_complex(L);
⇒ 1
L[4] = matrix(i);
is_complex(L);
⇒ 0

```

### D.3.1.8 outer

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix.lib\], page 968](#)).

**Usage:** `outer(A,B)`; `A,B` matrices

**Return:** matrix, outer (tensor) product of `A` and `B`

**Example:**

```

LIB "matrix.lib";
ring r=32003,(x,y,z),ds;
matrix A[3][3]=1,2,3,4,5,6,7,8,9;
matrix B[2][2]=x,y,0,z;
print(A);
⇒ 1,2,3,
⇒ 4,5,6,
⇒ 7,8,9
print(B);
⇒ x,y,
⇒ 0,z
print(outer(A,B));
⇒ x, y, 2x,2y,3x,3y,
⇒ 0, z, 0, 2z,0, 3z,
⇒ 4x,4y,5x,5y,6x,6y,

```

```

⇒ 0, 4z,0, 5z,0, 6z,
⇒ 7x,7y,8x,8y,9x,9y,
⇒ 0, 7z,0, 8z,0, 9z

```

### D.3.1.9 power

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix.lib\], page 968](#)).

**Usage:** `power(A,n)`; A a square-matrix of type `intmat` or `matrix`, `n=integer`  $\geq 0$

**Return:** `intmat` resp. `matrix`, the `n`-th power of A

**Note:** for A=`intmat` and big `n` the result may be wrong because of int overflow

**Example:**

```

LIB "matrix.lib";
intmat A[3][3]=1,2,3,4,5,6,7,8,9;
print(power(A,3));"";
⇒ 468 576 684
⇒ 1062 1305 1548
⇒ 1656 2034 2412
⇒
ring r=0,(x,y,z),dp;
matrix B[3][3]=0,x,y,z,0,0,y,z,0;
print(power(B,3));"";
⇒ yz2, xy2+x2z,y3+xyz,
⇒ y2z+xz2,yz2, 0,
⇒ y3+xyz, y2z+xz2,yz2
⇒

```

### D.3.1.10 skewmat

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix.lib\], page 968](#)).

**Usage:** `skewmat(n[id])`; `n` integer, `id` ideal

**Return:** skew-symmetric `nxn` matrix, with entries from `id`  
 (default: `id=maxideal(1)`)  
`skewmat(n)`; creates the generic skew-symmetric matrix

**Note:** if `id` has less than  $n*(n-1)/2$  elements, the matrix is filled with 0's,

**Example:**

```

LIB "matrix.lib";
ring R=0,x(1..5),lp;
print(skewmat(4)); // the generic skew-symmetric matrix
⇒ 0, x(1), x(2),x(3),
⇒ -x(1),0, x(4),x(5),
⇒ -x(2),-x(4),0, 0,
⇒ -x(3),-x(5),0, 0
ring R1 = 0,(a,b,c),dp;
matrix A=skewmat(4,maxideal(1)^2);
print(A);
⇒ 0, a2, ab, ac,
⇒ -a2,0, b2, bc,

```

```

↳ -ab,-b2,0, c2,
↳ -ac,-bc,-c2,0
int n=3;
ideal i = ideal(randommat(1,n*(n-1) div 2,maxideal(1),9));
print(skewmat(n,i)); // skew matrix of generic linear forms
↳ 0, 4a+b-8c, -a+6b+c,
↳ -4a-b+8c,0, -8a+2b-9c,
↳ a-6b-c, 8a-2b+9c,0
kill R1;

```

### D.3.1.11 submat

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix.lib\], page 968](#)).

**Usage:** `submat(A,r,c);` A=matrix, r,c=intvec

**Return:** matrix, submatrix of A with rows specified by intvec r and columns specified by intvec c.

**Example:**

```

LIB "matrix.lib";
ring R=32003,(x,y,z),lp;
matrix A[4][4]=x,y,z,0,1,2,3,4,5,6,7,8,9,x2,y2,z2;
print(A);
↳ x,y, z, 0,
↳ 1,2, 3, 4,
↳ 5,6, 7, 8,
↳ 9,x2,y2,z2
intvec v=1,3,4;
matrix B=submat(A,v,1..3);
print(B);
↳ x,y, z,
↳ 5,6, 7,
↳ 9,x2,y2

```

### D.3.1.12 symmat

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix.lib\], page 968](#)).

**Usage:** `symmat(n[id]);` n integer, id ideal

**Return:** symmetric nxn matrix, with entries from id (default: id=maxideal(1))

**Note:** if id has less than  $n*(n+1)/2$  elements, the matrix is filled with 0's, `symmat(n)`; creates the generic symmetric matrix

**Example:**

```

LIB "matrix.lib";
ring R=0,x(1..10),lp;
print(symmat(4)); // the generic symmetric matrix
↳ x(1),x(2),x(3),x(4),
↳ x(2),x(5),x(6),x(7),
↳ x(3),x(6),x(8),x(9),
↳ x(4),x(7),x(9),x(10)
ring R1 = 0,(a,b,c),dp;
matrix A=symmat(4,maxideal(1)^3);

```

```

print(A);
↳ a3, a2b,a2c,ab2,
↳ a2b,abc,ac2,b3,
↳ a2c,ac2,b2c,bc2,
↳ ab2,b3, bc2,c3
int n=3;
ideal i = ideal(randommat(1,n*(n+1) div 2,maxideal(1),9));
print(symmat(n,i)); // symmetric matrix of generic linear forms
↳ 4a-8b-2c,-a+b-4c, -8a-9b+c,
↳ -a+b-4c, a-9b+9c, 6a-5b+9c,
↳ -8a-9b+c,6a-5b+9c,2a+8c
kill R1;

```

### D.3.1.13 unitmat

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix.lib\], page 968](#)).

**Usage:**      `unitmat(n)`;  $n$  integer  $\geq 0$

**Return:**      $n \times n$  unit matrix

**Note:**        needs a basering, diagonal entries are numbers (=1) in the basering

**Example:**

```

LIB "matrix.lib";
ring r=32003,(x,y,z),lp;
print(xyz*unitmat(4));
↳ xyz,0, 0, 0,
↳ 0, xyz,0, 0,
↳ 0, 0, xyz,0,
↳ 0, 0, 0, xyz
print(unitmat(5));
↳ 1,0,0,0,0,
↳ 0,1,0,0,0,
↳ 0,0,1,0,0,
↳ 0,0,0,1,0,
↳ 0,0,0,0,1

```

### D.3.1.14 gauss\_col

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix.lib\], page 968](#)).

**Usage:**      `gauss_col(A[,e])`;  $A$  a matrix,  $e$  any type

**Return:**     - a matrix  $B$ , if called with one argument;  $B$  is the complete column- reduced upper-triangular normal form of  $A$  if  $A$  is constant, (resp. as far as this is possible if  $A$  is a polynomial matrix; no division by polynomials).  
 - a list  $L$  of two matrices, if called with two arguments;  $L$  satisfies  $L[1] = A * L[2]$  with  $L[1]$  the column-reduced form of  $A$  and  $L[2]$  the transformation matrix.

**Note:**        \* The procedure just applies interred to  $A$  with ordering  $(C,dp)$ . The transformation matrix is obtained by applying 'lift'. This should be faster than the procedure `colred`.  
 \* It should only be used with exact coefficient field (there is no pivoting and rounding error treatment).  
 \* Parameters are allowed. Hence, if the entries of  $A$  are parameters,  $B$  is the column-reduced form of  $A$  over the rational function field.

**Example:**

```

LIB "matrix.lib";
ring r=(0,a,b),(A,B,C),dp;
matrix m[8][6]=
0,    2*C, 0,    0, 0, 0,
0,    -4*C,a*A, 0, 0, 0,
b*B,  -A,  0,    0, 0, 0,
-A,   B,   0,    0, 0, 0,
-4*C, 0,   B,    2, 0, 0,
2*A,  B,   0,    0, 0, 0,
0,    3*B, 0,    0, 2b, 0,
0,    AB,  0,    2*A,A, 2a;"";
↳
list L=gauss_col(m,1);
print(L[1]);
↳ 0,0,2*C, 0,    0,0,
↳ A,0,-4*C,0,    0,0,
↳ 0,0,-A,   (b)/2*B,0,0,
↳ 0,0,B,   -1/2*A, 0,0,
↳ 0,1,0,   0,    0,0,
↳ 0,0,B,   A,    0,0,
↳ 0,0,0,   0,    1,0,
↳ 0,0,0,   0,    0,1
print(L[2]);
↳ 0,    0,    0,    1/2,    0,    0,
↳ 0,    0,    1,    0,    0,    0,
↳ 1/(a), 0,    0,    0,    0,    0,
↳ -1/(2a)*B, 1/2, 0,    C,    0,    0,
↳ 0,    0,    -3/(2b)*B, 0,    1/(2b), 0,
↳ 1/(2a2)*AB,-1/(2a)*A,(-2b+3)/(4ab)*AB,-1/(a)*AC,-1/(4ab)*A,1/(2a)
ring S=0,x,(c,dp);
matrix A[5][4] =
3, 1, 1, 1,
13, 8, 6,-7,
14,10, 6,-7,
7, 4, 3,-3,
2, 1, 0, 3;
print(gauss_col(A));
↳ 8/9,-5/9,-1/3,7/9,
↳ 1, 0, 0, 0,
↳ 0, 1, 0, 0,
↳ 0, 0, 1, 0,
↳ 0, 0, 0, 1

```

See also: [Section D.3.1.23 \[colred\]](#), page 981.

**D.3.1.15 gauss\_row**

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix.lib\]](#), page 968).

**Usage:** `gauss_row(A [e]);` A matrix, e any type

**Return:** - a matrix B, if called with one argument; B is the complete row- reduced lower-triangular normal form of A if A is constant, (resp. as far as this is possible if A is a polynomial matrix; no division by polynomials).

- a list  $L$  of two matrices, if called with two arguments;  $L$  satisfies  $\text{transpose}(L[2]) \cdot A = \text{transpose}(L[1])$  with  $L[1]$  the row-reduced form of  $A$  and  $L[2]$  the transformation matrix.

**Note:** \* This procedure just applies `gauss_col` to the transposed matrix. The transformation matrix is obtained by applying `lift`. This should be faster than the procedure `rowred`.  
 \* It should only be used with exact coefficient field (there is no pivoting and rounding error treatment).  
 \* Parameters are allowed. Hence, if the entries of  $A$  are parameters,  $B$  is the row-reduced form of  $A$  over the rational function field.

**Example:**

```
LIB "matrix.lib";
ring r=(0,a,b),(A,B,C),dp;
matrix m[6][8]=
0, 0,  b*B, -A,-4C,2A,0, 0,
2C,-4C,-A,B, 0,  B, 3B,AB,
0,a*A,  0, 0, B,  0, 0, 0,
0, 0,  0, 0, 2,  0, 0, 2A,
0, 0,  0, 0, 0,  0, 2b, A,
0, 0,  0, 0, 0,  0, 0, 2a;"";
↳
print(gauss_row(m));"";
↳ 0,  A,  0,  0,  0,0,0,0,
↳ 0,  0,  0,  0,  1,0,0,0,
↳ 2*C,-4*C,-A,  B,  0,B,0,0,
↳ 0,  0,  (b)/2*B,-1/2*A,0,A,0,0,
↳ 0,  0,  0,  0,  0,0,1,0,
↳ 0,  0,  0,  0,  0,0,0,1
↳
ring S=0,x,dp;
matrix A[4][5] = 3, 1,1,-1,2,
13, 8,6,-7,1,
14,10,6,-7,1,
7, 4,3,-3,3;
list L=gauss_row(A,1);
print(L[1]);
↳ 1/2,-7/3,-19/6,5/6,
↳ 1,  0,  0,  0,
↳ 0,  1,  0,  0,
↳ 0,  0,  1,  0,
↳ 0,  0,  0,  1
print(L[2]);
↳ 0,  -6,  -5,  1,
↳ -1/2,2/3, -1/6,-1/6,
↳ 1/2, -5/3,-5/6,1/6,
↳ 0,  13/3,11/3,-1/3
```

See also: [Section D.3.1.22 \[rowred\]](#), page 980.

### D.3.1.16 addcol

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix\\_lib\]](#), page 968).

**Usage:** `addcol(A,c1,p,c2)`; A matrix, p poly, c1, c2 positive integers

**Return:** matrix, A being modified by adding p times column c1 to column c2

**Example:**

```
LIB "matrix.lib";
ring r=32003,(x,y,z),lp;
matrix A[3][3]=1,2,3,4,5,6,7,8,9;
print(A);
  ↪ 1,2,3,
  ↪ 4,5,6,
  ↪ 7,8,9
print(addcol(A,1,xy,2));
  ↪ 1,xy+2, 3,
  ↪ 4,4xy+5,6,
  ↪ 7,7xy+8,9
```

### D.3.1.17 addrow

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix.lib\], page 968](#)).

**Usage:** `addrow(A,r1,p,r2)`; A matrix, p poly, r1, r2 positive integers

**Return:** matrix, A being modified by adding p times row r1 to row r2

**Example:**

```
LIB "matrix.lib";
ring r=32003,(x,y,z),lp;
matrix A[3][3]=1,2,3,4,5,6,7,8,9;
print(A);
  ↪ 1,2,3,
  ↪ 4,5,6,
  ↪ 7,8,9
print(addrow(A,1,xy,3));
  ↪ 1, 2, 3,
  ↪ 4, 5, 6,
  ↪ xy+7,2xy+8,3xy+9
```

### D.3.1.18 multcol

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix.lib\], page 968](#)).

**Usage:** `multcol(A,c,p)`; A matrix, p poly, c positive integer

**Return:** matrix, A being modified by multiplying column c by p

**Example:**

```
LIB "matrix.lib";
ring r=32003,(x,y,z),lp;
matrix A[3][3]=1,2,3,4,5,6,7,8,9;
print(A);
  ↪ 1,2,3,
  ↪ 4,5,6,
  ↪ 7,8,9
print(multcol(A,2,xy));
  ↪ 1,2xy,3,
  ↪ 4,5xy,6,
  ↪ 7,8xy,9
```



### D.3.1.19 multrow

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix.lib\], page 968](#)).

**Usage:** `multrow(A,r,p)`; A matrix, p poly, r positive integer

**Return:** matrix, A being modified by multiplying row r by p

**Example:**

```
LIB "matrix.lib";
ring r=32003,(x,y,z),lp;
matrix A[3][3]=1,2,3,4,5,6,7,8,9;
print(A);
  ↪ 1,2,3,
  ↪ 4,5,6,
  ↪ 7,8,9
print(multrow(A,2,xy));
  ↪ 1, 2, 3,
  ↪ 4xy,5xy,6xy,
  ↪ 7, 8, 9
```

### D.3.1.20 permcol

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix.lib\], page 968](#)).

**Usage:** `permcol(A,c1,c2)`; A matrix, c1,c2 positive integers

**Return:** matrix, A being modified by permuting columns c1 and c2

**Example:**

```
LIB "matrix.lib";
ring r=32003,(x,y,z),lp;
matrix A[3][3]=1,x,3,4,y,6,7,z,9;
print(A);
  ↪ 1,x,3,
  ↪ 4,y,6,
  ↪ 7,z,9
print(permcol(A,2,3));
  ↪ 1,3,x,
  ↪ 4,6,y,
  ↪ 7,9,z
```

### D.3.1.21 permrow

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix.lib\], page 968](#)).

**Usage:** `permrow(A,r1,r2)`; A matrix, r1,r2 positive integers

**Return:** matrix, A being modified by permuting rows r1 and r2

**Example:**

```
LIB "matrix.lib";
ring r=32003,(x,y,z),lp;
matrix A[3][3]=1,2,3,x,y,z,7,8,9;
print(A);
  ↪ 1,2,3,
  ↪ x,y,z,
```

```

    ↪ 7,8,9
    print(permrow(A,2,1));
    ↪ x,y,z,
    ↪ 1,2,3,
    ↪ 7,8,9

```

### D.3.1.22 rowred

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix.lib\]](#), page 968).

**Usage:** `rowred(A[,e]);` A matrix, e any type

**Return:** - a matrix B, being the row reduced form of A, if rowred is called with one argument.  
 (as far as this is possible over the polynomial ring; no division by polynomials)  
 - a list L of two matrices, such that  $L[1] = L[2] * A$  with L[1] the row-reduced form of A and L[2] the transformation matrix (if rowred is called with two arguments).

**Assume:** The entries of A are in the base field. It is not verified whether this assumption holds.

**Note:** \* This procedure is designed for teaching purposes mainly.  
 \* The straight forward Gaussian algorithm is implemented in the library (no standard basis computation).  
 The transformation matrix is obtained by concatenating a unit matrix to A. `proc gauss_row` should be faster.  
 \* It should only be used with exact coefficient field (there is no pivoting) over the polynomial ring (ordering lp or dp).  
 \* Parameters are allowed. Hence, if the entries of A are parameters the computation takes place over the field of rational functions.

**Example:**

```

LIB "matrix.lib";
ring r=(0,a,b),(A,B,C),dp;
matrix m[6][8]=
0, 0,  b*B, -A,-4C,2A,0, 0,
2C,-4C,-A,B, 0,  B, 3B,AB,
0,a*A,  0, 0, B,  0, 0, 0,
0, 0,  0, 0, 2,  0, 0, 2A,
0, 0,  0, 0, 0,  0, 2b, A,
0, 0,  0, 0, 0,  0, 0, 2a;"";
↪
print(rowred(m));"";
↪ 0,  0,  0,  0, 1,0,  0,0,
↪ 0,  0,  0,  0, 0,0,  1,0,
↪ 0,  0,  0,  0, 0,0,  0,1,
↪ 0,  0,  (b)*B,-A,0,2*A,0,0,
↪ 2*C,-4*C, -A,  B, 0,B,  0,0,
↪ 0,  (a)*A,0,  0, 0,0,  0,0
↪
list L=rowred(m,1);
print(L[1]);
↪ 0,  0,  0,  0, 1,0,  0,0,
↪ 0,  0,  0,  0, 0,0,  1,0,
↪ 0,  0,  0,  0, 0,0,  0,1,
↪ 0,  0,  (b)*B,-A,0,2*A,0,0,
↪ 2*C,-4*C, -A,  B, 0,B,  0,0,

```

```

⇒ 0, (a)*A,0, 0, 0,0, 0,0
print(L[2]);
⇒ 0,0,0,1/2, 0, -1/(2a)*A,
⇒ 0,0,0,0, 1/(2b), -1/(4ab)*A,
⇒ 0,0,0,0, 0, 1/(2a),
⇒ 1,0,0,2*C, 0, -2/(a)*AC,
⇒ 0,1,0,0, -3/(2b)*B, (-2b+3)/(4ab)*AB,
⇒ 0,0,1,-1/2*B,0, 1/(2a)*AB

```

See also: [Section D.3.1.15 \[gauss\\_row\]](#), page 976.

### D.3.1.23 colred

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix.lib\]](#), page 968).

**Usage:** `colred(A[e]);` A matrix, e any type

**Return:** - a matrix B, being the column reduced form of A, if `colred` is called with one argument.  
 (as far as this is possible over the polynomial ring; no division by polynomials)  
 - a list L of two matrices, such that  $L[1] = A * L[2]$  with  $L[1]$  the column-reduced form of A and  $L[2]$  the transformation matrix (if `colred` is called with two arguments).

**Assume:** The entries of A are in the base field. It is not verified whether this assumption holds.

**Note:** \* This procedure is designed for teaching purposes mainly.  
 \* It applies `rowred` to the transposed matrix. `proc gauss_col` should be faster.  
 \* It should only be used with exact coefficient field (there is no pivoting) over the polynomial ring (ordering `lp` or `dp`).  
 \* Parameters are allowed. Hence, if the entries of A are parameters the computation takes place over the field of rational functions.

**Example:**

```

LIB "matrix.lib";
ring r=(0,a,b),(A,B,C),dp;
matrix m[8][6]=
0, 2*C, 0, 0, 0, 0,
0, -4*C,a*A, 0, 0, 0,
b*B, -A, 0, 0, 0, 0,
-A, B, 0, 0, 0, 0,
-4*C, 0, B, 2, 0, 0,
2*A, B, 0, 0, 0, 0,
0, 3*B, 0, 0, 2b, 0,
0, AB, 0, 2*A,A, 2a;"";
⇒
print(colred(m));"";
⇒ 0,0,0,0, 2*C, 0,
⇒ 0,0,0,0, -4*C,(a)*A,
⇒ 0,0,0,(b)*B,-A, 0,
⇒ 0,0,0,-A, B, 0,
⇒ 1,0,0,0, 0, 0,
⇒ 0,0,0,2*A, B, 0,
⇒ 0,1,0,0, 0, 0,
⇒ 0,0,1,0, 0, 0
⇒
list L=colred(m,1);
print(L[1]);

```

```

⇒ 0,0,0,0, 2*C, 0,
⇒ 0,0,0,0, -4*C, (a)*A,
⇒ 0,0,0,(b)*B,-A, 0,
⇒ 0,0,0,-A, B, 0,
⇒ 1,0,0,0, 0, 0,
⇒ 0,0,0,2*A, B, 0,
⇒ 0,1,0,0, 0, 0,
⇒ 0,0,1,0, 0, 0
print(L[2]);
⇒ 0, 0, 0, 1, 0, 0,
⇒ 0, 0, 0, 0, 1, 0,
⇒ 0, 0, 0, 0, 0, 1,
⇒ 1/2, 0, 0, 2*C, 0, -1/2*B,
⇒ 0, 1/(2b), 0, 0, -3/(2b)*B, 0,
⇒ -1/(2a)*A, -1/(4ab)*A, 1/(2a), -2/(a)*AC, (-2b+3)/(4ab)*AB, 1/(2a)*AB

```

See also: [Section D.3.1.14 \[gauss\\_col\]](#), page 975.

### D.3.1.24 linear\_relations

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix\\_lib\]](#), page 968).

**Usage:** linear\_relations(M);  
M: a module

**Assume:** All non-zero entries of M are homogeneous polynomials of the same positive degree.  
The base field must be an exact field (not real or complex).  
It is not checked whether these assumptions hold.

**Return:** a maximal module R such that  $M \cdot R$  is formed by zero vectors.

**Example:**

```

LIB "matrix.lib";
ring r = (3,w), (a,b,c,d),dp;
minpoly = w2-w-1;
module M = [a2,b2],[wab,w2c2+2b2],[(w-2)*a2+wab,wb2+w2c2];
module REL = linear_relations(M);
pmat(REL);
⇒ (-w-1),
⇒ -1,
⇒ 1
pmat(matrix(M)*matrix(REL));
⇒ 0,
⇒ 0

```

### D.3.1.25 rm\_unitrow

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix\\_lib\]](#), page 968).

**Usage:** rm\_unitrow(A); A matrix (being col-reduced)

**Return:** matrix, obtained from A by deleting unit rows (having just one 1 and else 0 as entries) and associated columns

**Example:**

```

LIB "matrix.lib";
ring r=0,(A,B,C),dp;
matrix m[8][6]=
0,0, 0, 0, 2C, 0,
0,0, 0, 0, -4C,A,
A,-C2,0, B, -A, 0,
0,0, 1/2B,-A,B, 0,
1,0, 0, 0, 0, 0,
0,0, 0, 2A,B, 0,
0,1, 0, 0, 0, 0,
0,0, 1, 0, 0, 0;
print(rm_unitrow(m));
↪ 0, 2C, 0,
↪ 0, -4C,A,
↪ B, -A, 0,
↪ -A,B, 0,
↪ 2A,B, 0

```

### D.3.1.26 rm\_unitcol

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix.lib\], page 968](#)).

**Usage:** `rm_unitcol(A)`; A matrix (being row-reduced)

**Return:** matrix, obtained from A by deleting unit columns (having just one 1 and else 0 as entries) and associated rows

**Example:**

```

LIB "matrix.lib";
ring r=0,(A,B,C),dp;
matrix m[6][8]=
0, 0, A, 0, 1,0, 0,0,
0, 0, -C2, 0, 0,0, 1,0,
0, 0, 0,1/2B, 0,0, 0,1,
0, 0, B, -A, 0,2A, 0,0,
2C,-4C, -A, B, 0,B, 0,0,
0, A, 0, 0, 0,0, 0,0;
print(rm_unitcol(m));
↪ 0, 0, B, -A,2A,
↪ 2C,-4C,-A,B, B,
↪ 0, A, 0, 0, 0

```

### D.3.1.27 headStand

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix.lib\], page 968](#)).

**Usage:** `headStand(M)`; M matrix

**Return:** matrix B such that  $B[i][j]=M[n-i+1,m-j+1]$ ,  $n=\text{nrows}(M)$ ,  $m=\text{ncols}(M)$

**Example:**

```

LIB "matrix.lib";
ring r=0,(A,B,C),dp;
matrix M[2][3]=
0,A, B,
A2, B2, C;

```

```

print(M);
↳ 0, A, B,
↳ A2,B2,C
print(headStand(M));
↳ C,B2,A2,
↳ B,A, 0

```

### D.3.1.28 symmetricBasis

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix.lib\]](#), page 968).

**Return:** ring, poynomial ring containing the ideal "symBasis", being a basis of the k-th symmetric power of an n-dim vector space.

**Note:** The output polynomial ring has characteristics 0 and n variables named "S(i)", where the base variable name S is either given by the optional string argument (which must not contain brackets) or equal to "e" by default.

**Example:**

```

LIB "matrix.lib";
// basis of the 3-rd symmetricPower of a 4-dim vector space:
def R = symmetricBasis(4, 3, "@e"); setring R;
R; // container ring:
↳ // coefficients: QQ
↳ // number of vars : 4
↳ //          block 1 : ordering dp
↳ //          : names    @e(1) @e(2) @e(3) @e(4)
↳ //          block 2 : ordering C
symBasis; // symmetric basis:
↳ symBasis[1]=@e(4)^3
↳ symBasis[2]=@e(3)*@e(4)^2
↳ symBasis[3]=@e(3)^2*@e(4)
↳ symBasis[4]=@e(3)^3
↳ symBasis[5]=@e(2)*@e(4)^2
↳ symBasis[6]=@e(2)*@e(3)*@e(4)
↳ symBasis[7]=@e(2)*@e(3)^2
↳ symBasis[8]=@e(2)^2*@e(4)
↳ symBasis[9]=@e(2)^2*@e(3)
↳ symBasis[10]=@e(2)^3
↳ symBasis[11]=@e(1)*@e(4)^2
↳ symBasis[12]=@e(1)*@e(3)*@e(4)
↳ symBasis[13]=@e(1)*@e(3)^2
↳ symBasis[14]=@e(1)*@e(2)*@e(4)
↳ symBasis[15]=@e(1)*@e(2)*@e(3)
↳ symBasis[16]=@e(1)*@e(2)^2
↳ symBasis[17]=@e(1)^2*@e(4)
↳ symBasis[18]=@e(1)^2*@e(3)
↳ symBasis[19]=@e(1)^2*@e(2)
↳ symBasis[20]=@e(1)^3

```

See also: [Section D.3.1.29 \[exteriorBasis\]](#), page 984.

### D.3.1.29 exteriorBasis

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix.lib\]](#), page 968).

**Return:** qring, an exterior algebra containing the ideal "extBasis", being a basis of the k-th exterior power of an n-dim vector space.

**Note:** The output polynomial ring has characteristics 0 and n variables named "S(i)", where the base variable name S is either given by the optional string argument (which must not contain brackets) or equal to "e" by default.

**Example:**

```
LIB "matrix.lib";
// basis of the 3-rd symmetricPower of a 4-dim vector space:
def r = exteriorBasis(4, 3, "@e"); setring r;
r; // container ring:
⇒ // coefficients: QQ
⇒ // number of vars : 4
⇒ //          block 1 : ordering dp
⇒ //          : names  @e(1) @e(2) @e(3) @e(4)
⇒ //          block 2 : ordering C
⇒ // noncommutative relations:
⇒ //      @e(2)@e(1)=-@e(1)*@e(2)
⇒ //      @e(3)@e(1)=-@e(1)*@e(3)
⇒ //      @e(4)@e(1)=-@e(1)*@e(4)
⇒ //      @e(3)@e(2)=-@e(2)*@e(3)
⇒ //      @e(4)@e(2)=-@e(2)*@e(4)
⇒ //      @e(4)@e(3)=-@e(3)*@e(4)
⇒ // quotient ring from ideal
⇒ _[1]=@e(4)^2
⇒ _[2]=@e(3)^2
⇒ _[3]=@e(2)^2
⇒ _[4]=@e(1)^2
extBasis; // exterior basis:
⇒ extBasis[1]=@e(2)*@e(3)*@e(4)
⇒ extBasis[2]=@e(1)*@e(3)*@e(4)
⇒ extBasis[3]=@e(1)*@e(2)*@e(4)
⇒ extBasis[4]=@e(1)*@e(2)*@e(3)
```

See also: [Section D.3.1.28 \[symmetricBasis\]](#), page 984.

### D.3.1.30 symmetricPower

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix.lib\]](#), page 968).

**Usage:** symmetricPower(A, k); A module, k int

**Return:** module: the k-th symmetric power of A

**Note:** the chosen bases and most of intermediate data will be shown if printlevel is big enough

**Example:**

```
LIB "matrix.lib";
ring r = (0),(a, b, c, d), dp; r;
⇒ // coefficients: QQ
⇒ // number of vars : 4
⇒ //          block 1 : ordering dp
⇒ //          : names  a b c d
⇒ //          block 2 : ordering C
module B = a*gen(1) + c* gen(2), b * gen(1) + d * gen(2); print(B);
```

```

↳ a,b,
↳ c,d
// symmetric power over a commutative K-algebra:
print(symmetricPower(B, 2));
↳ d2, cd,    c2,
↳ 2bd,bc+ad,2ac,
↳ b2, ab,    a2
print(symmetricPower(B, 3));
↳ d3, cd2,    c2d,    c3,
↳ 3bd2,2bcd+ad2,bc2+2acd,3ac2,
↳ 3b2d,b2c+2abd,2abc+a2d,3a2c,
↳ b3, ab2,    a2b,    a3
// symmetric power over an exterior algebra:
def g = superCommutative(); setring g; g;
↳ // coefficients: QQ
↳ // number of vars : 4
↳ //          block 1 : ordering dp
↳ //          : names  a b c d
↳ //          block 2 : ordering C
↳ // noncommutative relations:
↳ //      ba=-ab
↳ //      ca=-ac
↳ //      da=-ad
↳ //      cb=-bc
↳ //      db=-bd
↳ //      dc=-cd
↳ // quotient ring from ideal
↳ _[1]=d2
↳ _[2]=c2
↳ _[3]=b2
↳ _[4]=a2
module B = a*gen(1) + c* gen(2), b * gen(1) + d * gen(2); print(B);
↳ a,b,
↳ c,d
print(symmetricPower(B, 2)); // much smaller!
↳ 0,cd,    0,
↳ 0,-bc+ad,0,
↳ 0,ab,    0
print(symmetricPower(B, 3)); // zero! (over an exterior algebra!)
↳ 0,0,0,0

```

See also: [Section D.3.1.31 \[exteriorPower\], page 986](#).

### D.3.1.31 exteriorPower

Procedure from library `matrix.lib` (see [Section D.3.1 \[matrix.lib\], page 968](#)).

**Usage:** `exteriorPower(A, k);` A module, k int

**Return:** module: the k-th exterior power of A

**Note:** the chosen bases and most of intermediate data will be shown if `printlevel` is big enough. Last rows will be invisible if zero.

**Example:**



```

LIB "matrix.lib";
ring r = (0),(a, b, c, d, e, f), dp;
r; "base ring:";
↳ // coefficients: QQ
↳ // number of vars : 6
↳ //          block  1 : ordering dp
↳ //          : names  a b c d e f
↳ //          block  2 : ordering C
↳ base ring:
module B = a*gen(1) + c*gen(2) + e*gen(3),
b*gen(1) + d*gen(2) + f*gen(3),
e*gen(1) + f*gen(3);
print(B);
↳ a,b,e,
↳ c,d,0,
↳ e,f,f
print(exteriorPower(B, 2));
↳ df,  cf,  -de+cf,
↳ bf-ef,-e2+af,-be+af,
↳ -de,  -ce,  -bc+ad
print(exteriorPower(B, 3));
↳ -de2-bcf+adf+cef
def g = superCommutative(); setring g; g;
↳ // coefficients: QQ
↳ // number of vars : 6
↳ //          block  1 : ordering dp
↳ //          : names  a b c d e f
↳ //          block  2 : ordering C
↳ // noncommutative relations:
↳ //    ba=-ab
↳ //    ca=-ac
↳ //    da=-ad
↳ //    ea=-ae
↳ //    fa=-af
↳ //    cb=-bc
↳ //    db=-bd
↳ //    eb=-be
↳ //    fb=-bf
↳ //    dc=-cd
↳ //    ec=-ce
↳ //    fc=-cf
↳ //    ed=-de
↳ //    fd=-df
↳ //    fe=-ef
↳ // quotient ring from ideal
↳ _[1]=f2
↳ _[2]=e2
↳ _[3]=d2
↳ _[4]=c2
↳ _[5]=b2
↳ _[6]=a2
module A = a*gen(1), b * gen(1), c*gen(2), d * gen(2);
print(A);

```

```

⇒ a,b,0,0,
⇒ 0,0,c,d
print(exteriorPower(A, 2));
⇒ 0,bd,bc,ad,ac,0
module B = a*gen(1) + c*gen(2) + e*gen(3),
b*gen(1) + d*gen(2) + f*gen(3),
e*gen(1) + f*gen(3);
print(B);
⇒ a,b,e,
⇒ c,d,0,
⇒ e,f,f
print(exteriorPower(B, 2));
⇒ df,    cf, de+cf,
⇒ bf+ef,af, be+af,
⇒ -de,   -ce,bc+ad
print(exteriorPower(B, 3));
⇒ bcf+adf-cef

```

See also: [Section D.3.1.30 \[symmetricPower\]](#), page 985.

## D.3.2 linalg\_lib

**Library:** linalg.lib

**Purpose:** Algorithmic Linear Algebra

**Authors:** Ivor Saynisch (ivs@math.tu-cottbus.de)  
 Mathias Schulze (mschulze@mathematik.uni-kl.de)

**Procedures:**

### D.3.2.1 inverse

Procedure from library linalg.lib (see [Section D.3.2 \[linalg\\_lib\]](#), page 988).

**Usage:** inverse(A [,opt]); A a square matrix, opt integer

**Return:**

a matrix:

- the inverse matrix of A, if A is invertible;
- the 1x1 0-matrix if A is not invertible (in the polynomial ring!).

There are the following options:

- opt=0 or not given: heuristically best option from below
- opt=1 : apply std to (transpose(E,A)), ordering (C,dp).
- opt=2 : apply interred (transpose(E,A)), ordering (C,dp).
- opt=3 : apply lift(A,E), ordering (C,dp).

**Note:** parameters and minpoly are allowed; opt=2 is only correct for matrices with entries in a field

**Example:**

```

LIB "linalg.lib";
ring r=0,(x,y,z),lp;
matrix A[3][3]=
1,4,3,
1,5,7,

```

```

0,4,17;
print(inverse(A));"";
⇒ 57, -56,13,
⇒ -17,17, -4,
⇒ 4, -4, 1
⇒
matrix B[3][3]=
y+1, x+y, y,
z, z+1, z,
y+z+2,x+y+z+2,y+z+1;
print(inverse(B));
⇒ -xz+y+1, -xz-x+y, xz-y,
⇒ z, z+1, -z,
⇒ xz-y-z-2,xz+x-y-z-2,-xz+y+z+1
print(B*inverse(B));
⇒ 1,0,0,
⇒ 0,1,0,
⇒ 0,0,1

```

See also: [Section D.3.2.2 \[inverse\\_B\], page 989](#); [Section D.3.2.3 \[inverse\\_L\], page 990](#).

### D.3.2.2 inverse\_B

Procedure from library `linalg.lib` (see [Section D.3.2 \[linalg.lib\], page 988](#)).

**Usage:** `inverse_B(A)`;  $A$  = square matrix

**Return:** list `Inv` with

- `Inv[1]` = matrix  $I$  and
- `Inv[2]` = poly  $p$

such that  $I*A = \text{unitmat}(n)*p$ ;

**Note:**  $p=1$  if  $1/\det(A)$  is computable and  $p=\det(A)$  if not;  
the computation uses `busadj`.

**Example:**

```

LIB "linalg.lib";
ring r=0,(x,y),lp;
matrix A[3][3]=x,y,1,1,x2,y,x,6,0;
print(A);
⇒ x,y, 1,
⇒ 1,x2,y,
⇒ x,6, 0
list Inv=inverse_B(A);
print(Inv[1]);
⇒ 6y, -6, x2-y2,
⇒ -xy, x, xy-1,
⇒ x3-6,-xy+6x,-x3+y
print(Inv[2]);
⇒ x3-xy2+6xy-6
print(Inv[1]*A);
⇒ x3-xy2+6xy-6,0, 0,
⇒ 0, x3-xy2+6xy-6,0,
⇒ 0, 0, x3-xy2+6xy-6

```

See also: [Section D.3.2.1 \[inverse\], page 988](#); [Section D.3.2.3 \[inverse\\_L\], page 990](#).

### D.3.2.3 inverse\_L

Procedure from library `linalg.lib` (see [Section D.3.2 \[linalg.lib\]](#), page 988).

**Usage:** `inverse_L(A)`;  $A$  = square matrix

**Return:** list `Inv` representing a left inverse of  $A$ , i.e  
 - `Inv[1]` = matrix  $I$  and  
 - `Inv[2]` = poly  $p$   
 such that  $I \cdot A = \text{unitmat}(n) \cdot p$ ;

**Note:**  $p=1$  if  $1/\det(A)$  is computable and  $p=\det(A)$  if not;  
 the computation computes first  $\det(A)$  and then uses lift

**Example:**

```
LIB "linalg.lib";
ring r=0,(x,y),lp;
matrix A[3][3]=x,y,1,1,x2,y,x,6,0;
print(A);
  ↳ x,y, 1,
  ↳ 1,x2,y,
  ↳ x,6, 0
list Inv=inverse_L(A);
print(Inv[1]);
  ↳ -6y, 6, -x2+y2,
  ↳ xy, -x, -xy+1,
  ↳ -x3+6,xy-6x,x3-y
print(Inv[2]);
  ↳ -x3+xy2-6xy+6
print(Inv[1]*A);
  ↳ -x3+xy2-6xy+6,0, 0,
  ↳ 0, -x3+xy2-6xy+6,0,
  ↳ 0, 0, -x3+xy2-6xy+6
```

See also: [Section D.3.2.1 \[inverse\]](#), page 988; [Section D.3.2.2 \[inverse\\_B\]](#), page 989.

### D.3.2.4 sym\_gauss

Procedure from library `linalg.lib` (see [Section D.3.2 \[linalg.lib\]](#), page 988).

**Usage:** `sym_gauss(A)`;  $A$  = symmetric matrix

**Return:** matrix, diagonalisation of  $A$  with symmetric gauss algorithm

**Example:**

```
LIB "linalg.lib";
ring r=0,(x),lp;
matrix A[2][2]=1,4,4,15;
print(A);
  ↳ 1,4,
  ↳ 4,15
print(sym_gauss(A));
  ↳ 1,0,
  ↳ 0,-1
```

### D.3.2.5 orthogonalize

Procedure from library `linalg.lib` (see [Section D.3.2 \[linalg.lib\]](#), page 988).

**Usage:**      `orthogonalize(A)`;  $A$  = matrix of constants

**Return:**     matrix, orthogonal basis of the column space of  $A$

**Example:**

```
LIB "linalg.lib";
ring r=0,(x),lp;
matrix A[4][4]=5,6,12,4,7,3,2,6,12,1,1,2,6,4,2,10;
print(A);
↳ 5, 6,12,4,
↳ 7, 3,2, 6,
↳ 12,1,1, 2,
↳ 6, 4,2, 10
print(orthogonalize(A));
↳ 1,0, 0, 0,
↳ 0,23,0, 0,
↳ 0,0, 21,0,
↳ 0,0, 0, 6
```

### D.3.2.6 diag\_test

Procedure from library `linalg.lib` (see [Section D.3.2 \[linalg.lib\]](#), page 988).

**Usage:**      `diag_test(A)`;  $A$  = const square matrix

**Return:**     int, 1 if  $A$  is diagonalizable,  
                  0 if not  
                  -1 if no statement is possible, since  $A$  does not split.

**Note:**        The test works only for split matrices, i.e if eigenvalues of  $A$  are in the ground field.  
                  Does not work with parameters (uses `factorize,gcd`).

**Example:**

```
LIB "linalg.lib";
ring r=0,(x),dp;
matrix A[4][4]=6,0,0,0,0,0,6,0,0,6,0,0,0,0,6;
print(A);
↳ 6,0,0,0,
↳ 0,0,6,0,
↳ 0,6,0,0,
↳ 0,0,0,6
diag_test(A);
↳ 1
```

### D.3.2.7 busadj

Procedure from library `linalg.lib` (see [Section D.3.2 \[linalg.lib\]](#), page 988).

**Usage:**      `busadj(A)`;  $A$  = square matrix (of size  $n \times n$ )

**Return:**     list  $L$ :

$L[1]$  contains the  $(n+1)$  coefficients of the characteristic polynomial  $X$  of  $A$ , i.e.  
 $X = L[1][1] + \dots + L[1][k] * t^{(k-1)} + \dots + (L[1][n+1]) * t^n$   
 $L[2]$  contains the  $n$   $(n \times n)$ -matrices  $H_k$  which are the coefficients of the busadjoint  $bA = \text{adjoint}(E * t - A)$  of  $A$ , i.e.  
 $bA = (H_{n-1}) * t^{(n-1)} + \dots + H_k * t^k + \dots + H_0, \quad (H_k = L[2][k+1])$

**Example:**

```
LIB "linalg.lib";
ring r = 0, (t,x), lp;
matrix A[2][2] = 1,x2,x,x2+3x;
print(A);
  ↳ 1,x2,
  ↳ x,x2+3x
list L = busadj(A);
poly X = L[1][1]+L[1][2]*t+L[1][3]*t2; X;
  ↳ t2-tx2-3tx-t-x3+x2+3x
matrix bA[2][2] = L[2][1]+L[2][2]*t;
print(bA); //the busadjoint of A;
  ↳ t-x2-3x,x2,
  ↳ x, t-1
print(bA*(t*unitmat(2)-A));
  ↳ t2-tx2-3tx-t-x3+x2+3x,0,
  ↳ 0, t2-tx2-3tx-t-x3+x2+3x
```

**D.3.2.8 charpoly**

Procedure from library `linalg.lib` (see [Section D.3.2 \[linalg.lib\]](#), page 988).

**Usage:** `charpoly(A[,v]);`  $A$  square matrix,  $v$  string, name of a variable

**Return:** `poly`, the characteristic polynomial  $\det(E * v - A)$   
 (default:  $v = \text{name of last variable}$ )

**Note:**  $A$  must be independent of the variable  $v$ . The computation uses `det`. If `printlevel > 0`,  $\det(E * v - A)$  is displayed recursively.

**Example:**

```
LIB "linalg.lib";
ring r=0,(x,t),dp;
matrix A[3][3]=1,x2,x,x2,6,4,x,4,1;
print(A);
  ↳ 1, x2,x,
  ↳ x2,6, 4,
  ↳ x, 4, 1
charpoly(A,"t");
  ↳ -x4t+x4-8x3-x2t+t3+6x2-8t2-3t+10
```

**D.3.2.9 adjoint**

Procedure from library `linalg.lib` (see [Section D.3.2 \[linalg.lib\]](#), page 988).

**Usage:** `adjoint(A);`  $A$  = square matrix

**Return:** adjoint matrix of  $A$ , i.e.  $\text{Adj} * A = \det(A) * E$

**Note:** computation uses `busadj(A)`

**Example:**

```

LIB "linalg.lib";
ring r=0,(t,x),lp;
matrix A[2][2]=1,x2,x,x2+3x;
print(A);
  ↪ 1,x2,
  ↪ x,x2+3x
matrix Adj[2][2]=adjoint(A);
print(Adj); //Adj*A=det(A)*E
  ↪ x2+3x,-x2,
  ↪ -x, 1
print(Adj*A);
  ↪ -x3+x2+3x,0,
  ↪ 0, -x3+x2+3x

```

**D.3.2.10 det\_B**

Procedure from library `linalg.lib` (see [Section D.3.2 \[linalg.lib\], page 988](#)).

**Usage:** `det_B(A)`; A any matrix

**Return:** returns the determinant of A

**Note:** the computation uses the busadj algorithm

**Example:**

```

LIB "linalg.lib";
ring r=0,(x),dp;
matrix A[10][10]=random(2,10,10)+unitmat(10)*x;
print(A);
  ↪ x+2,-1, 2, 0, -1, 1, 1, 2, -1, 1,
  ↪ 2, x-1,-1,2, -1, 1, 1, 2, 1, 0,
  ↪ -2, 2, x, 1, 2, 1, -1, -2, -2, 2,
  ↪ 0, -1, -1,x+2,0, -1,1, 0, -1, 0,
  ↪ 0, 1, -2,0, x+2,1, 1, -2, 2, 1,
  ↪ 2, -2, 0, -2, -1, x, -2, 1, -2, -2,
  ↪ -1, 2, 2, 1, 1, -2,x+1,-1, -2, -2,
  ↪ 1, 1, -1,2, -2, -1,2, x+1,0, 1,
  ↪ -2, 0, 1, 2, 1, -2,-2, 1, x+1,0,
  ↪ 1, 2, 1, 2, 0, 1, -1, 1, -1, x-1
det_B(A);
  ↪ x10+7x9+7x8-39x7-290x6-1777x5-3646x4+3725x3-5511x2-34811x-13241

```

**D.3.2.11 gaussred**

Procedure from library `linalg.lib` (see [Section D.3.2 \[linalg.lib\], page 988](#)).

**Usage:** `gaussred(A)`; A any constant matrix

**Return:** list Z: Z[1]=P , Z[2]=U , Z[3]=S , Z[4]=rank(A)  
gives a row reduced matrix S, a permutation matrix P and a normalized lower triangular matrix U, with  $P*A=U*S$

**Note:** This procedure is designed for teaching purposes mainly. The straight forward implementation in the interpreted library is not very efficient (no standard basis computation).

**Example:**

```

LIB "linalg.lib";
ring r=0,(x),dp;
matrix A[5][4]=1,3,-1,4,2,5,-1,3,1,3,-1,4,0,4,-3,1,-3,1,-5,-2;
print(A);
  ↪ 1, 3,-1,4,
  ↪ 2, 5,-1,3,
  ↪ 1, 3,-1,4,
  ↪ 0, 4,-3,1,
  ↪ -3,1,-5,-2
list Z=gaussred(A); //construct P,U,S s.t. P*A=U*S
print(Z[1]); //P
  ↪ 1,0,0,0,0,
  ↪ 0,1,0,0,0,
  ↪ 0,0,0,0,1,
  ↪ 0,0,0,1,0,
  ↪ 0,0,1,0,0
print(Z[2]); //U
  ↪ 1, 0, 0, 0,0,
  ↪ 2, 1, 0, 0,0,
  ↪ -3,-10,1, 0,0,
  ↪ 0, -4, 1/2,1,0,
  ↪ 1, 0, 0, 0,1
print(Z[3]); //S
  ↪ 1,3, -1,4,
  ↪ 0,-1,1, -5,
  ↪ 0,0, 2, -40,
  ↪ 0,0, 0, 1,
  ↪ 0,0, 0, 0
print(Z[4]); //rank
  ↪ 4
print(Z[1]*A); //P*A
  ↪ 1, 3,-1,4,
  ↪ 2, 5,-1,3,
  ↪ -3,1,-5,-2,
  ↪ 0, 4,-3,1,
  ↪ 1, 3,-1,4
print(Z[2]*Z[3]); //U*S
  ↪ 1, 3,-1,4,
  ↪ 2, 5,-1,3,
  ↪ -3,1,-5,-2,
  ↪ 0, 4,-3,1,
  ↪ 1, 3,-1,4

```

**D.3.2.12 gaussred\_pivot**

Procedure from library `linalg.lib` (see [Section D.3.2 \[linalg.lib\]](#), page 988).

**Usage:** `gaussred_pivot(A)`; A any constant matrix

**Return:** list Z: Z[1]=P , Z[2]=U , Z[3]=S , Z[4]=rank(A)  
 gives a row reduced matrix S, a permutation matrix P and a normalized lower triangular matrix U, with  $P*A=U*S$

**Note:** with row pivoting



**Example:**

```

LIB "linalg.lib";
ring r=0,(x),dp;
matrix A[5][4] = 1, 3,-1,4,
2, 5,-1,3,
1, 3,-1,4,
0, 4,-3,1,
-3,1,-5,-2;
list Z=gaussred_pivot(A); //construct P,U,S s.t. P*A=U*S
print(Z[1]);             //P
⇨ 0,0,0,0,1,
⇨ 0,1,0,0,0,
⇨ 0,0,1,0,0,
⇨ 0,0,0,1,0,
⇨ 1,0,0,0,0
print(Z[2]);             //U
⇨ 1, 0, 0, 0,0,
⇨ -2/3,1, 0, 0,0,
⇨ -1/3,10/17,1, 0,0,
⇨ 0, 12/17,-1/2,1,0,
⇨ -1/3,10/17,1, 0,1
print(Z[3]);             //S
⇨ -3,1, -5, -2,
⇨ 0, 17/3,-13/3,5/3,
⇨ 0, 0, -2/17,40/17,
⇨ 0, 0, 0, 1,
⇨ 0, 0, 0, 0
print(Z[4]);             //rank
⇨ 4
print(Z[1]*A);           //P*A
⇨ -3,1,-5,-2,
⇨ 2, 5,-1,3,
⇨ 1, 3,-1,4,
⇨ 0, 4,-3,1,
⇨ 1, 3,-1,4
print(Z[2]*Z[3]);       //U*S
⇨ -3,1,-5,-2,
⇨ 2, 5,-1,3,
⇨ 1, 3,-1,4,
⇨ 0, 4,-3,1,
⇨ 1, 3,-1,4

```

**D.3.2.13 gauss\_nf**

Procedure from library `linalg.lib` (see [Section D.3.2 \[linalg.lib\]](#), page 988).

**Usage:** `gauss_nf(A)`; A any constant matrix

**Return:** matrix; gauss normal form of A (uses `gaussred`)

**Example:**

```

LIB "linalg.lib";
ring r = 0,(x),dp;
matrix A[4][4] = 1,4,4,7,2,5,5,4,4,1,1,3,0,2,2,7;

```

```

print(gauss_nf(A));
↦ 1,4, 4, 7,
↦ 0,-3,-3,-10,
↦ 0,0, 0, 25,
↦ 0,0, 0, 0

```

### D.3.2.14 mat\_rk

Procedure from library `linalg.lib` (see [Section D.3.2 \[linalg.lib\], page 988](#)).

**Usage:** `mat_rk(A)`; A any constant matrix

**Return:** int, rank of A

**Example:**

```

LIB "linalg.lib";
ring r = 0,(x),dp;
matrix A[4][4] = 1,4,4,7,2,5,5,4,4,1,1,3,0,2,2,7;
mat_rk(A);
↦ 3

```

### D.3.2.15 U\_D\_O

Procedure from library `linalg.lib` (see [Section D.3.2 \[linalg.lib\], page 988](#)).

**Usage:** `U_D_O(A)`; constant invertible matrix A

**Return:** list Z: Z[1]=P , Z[2]=U , Z[3]=D , Z[4]=O  
 gives a permutation matrix P,  
 a normalized lower triangular matrix U ,  
 a diagonal matrix D, and  
 a normalized upper triangular matrix O  
 with  $P*A=U*D*O$

**Note:** Z[1]=-1 means that A is not regular (proc uses gaussred)

**Example:**

```

LIB "linalg.lib";
ring r = 0,(x),dp;
matrix A[5][5] = 10, 4, 0, -9, 8,
-3, 6, -6, -4, 9,
0, 3, -1, -9, -8,
-4,-2, -6, -10,10,
-9, 5, -1, -6, 5;
list Z = U_D_O(A); //construct P,U,D,O s.t. P*A=U*D*O
print(Z[1]); //P
↦ 1,0,0,0,0,
↦ 0,1,0,0,0,
↦ 0,0,1,0,0,
↦ 0,0,0,1,0,
↦ 0,0,0,0,1
print(Z[2]); //U
↦ 1, 0, 0, 0, 0,
↦ -3/10,1, 0, 0, 0,
↦ 0, 5/12, 1, 0, 0,
↦ -2/5, -1/18,-38/9,1, 0,

```

```

⇒ -9/10,43/36,37/9, -1049/2170,1
print(Z[3]); //D
⇒ 10,0, 0, 0, 0,
⇒ 0, 36/5,0, 0, 0,
⇒ 0, 0, 3/2,0, 0,
⇒ 0, 0, 0, -1085/27,0,
⇒ 0, 0, 0, 0, 6871/217
print(Z[4]); //0
⇒ 1,2/5,0, -9/10, 4/5,
⇒ 0,1, -5/6,-67/72, 19/12,
⇒ 0,0, 1, -149/36,-17/2,
⇒ 0,0, 0, 1, 216/217,
⇒ 0,0, 0, 0, 1
print(Z[1]*A); //P*A
⇒ 10,4, 0, -9, 8,
⇒ -3,6, -6,-4, 9,
⇒ 0, 3, -1,-9, -8,
⇒ -4,-2,-6,-10,10,
⇒ -9,5, -1,-6, 5
print(Z[2]*Z[3]*Z[4]); //U*D*0
⇒ 10,4, 0, -9, 8,
⇒ -3,6, -6,-4, 9,
⇒ 0, 3, -1,-9, -8,
⇒ -4,-2,-6,-10,10,
⇒ -9,5, -1,-6, 5

```

### D.3.2.16 pos\_def

Procedure from library `linalg.lib` (see [Section D.3.2 \[linalg.lib\]](#), page 988).

**Usage:** `pos_def(A)`;  $A$  = constant, symmetric square matrix

**Return:** int:  
 1 if  $A$  is positive definit ,  
 0 if not,  
 -1 if unknown

**Example:**

```

LIB "linalg.lib";
ring r = 0,(x),dp;
matrix A[5][5] = 20, 4, 0, -9, 8,
4, 12, -6, -4, 9,
0, -6, -2, -9, -8,
-9, -4, -9, -20, 10,
8, 9, -8, 10, 10;
pos_def(A);
⇒ 0
matrix B[3][3] = 3, 2, 0,
2, 12, 4,
0, 4, 2;
pos_def(B);
⇒ 1

```

### D.3.2.17 hessenberg

Procedure from library `linalg.lib` (see [Section D.3.2 \[linalg.lib\]](#), page 988).

**Usage:**        `hessenberg(M)`; matrix `M`

**Assume:**     `M` constant square matrix

**Return:**      matrix `H`; Hessenberg form of `M`

**Example:**

```
LIB "linalg.lib";
ring R=0,x,dp;
matrix M[3][3]=3,2,1,0,2,1,0,0,3;
print(M);
  ↪ 3,2,1,
  ↪ 0,2,1,
  ↪ 0,0,3
print(hessenberg(M));
  ↪ 3,2,1,
  ↪ 0,2,1,
  ↪ 0,0,3
```

See also: [Section D.3.2.17 \[hessenberg\]](#), page 998; [Section 5.1.155 \[system\]](#), page 275.

### D.3.2.18 eigenvals

Procedure from library `linalg.lib` (see [Section D.3.2 \[linalg.lib\]](#), page 988).

**Usage:**        `eigenvals(M)`; matrix `M`

**Assume:**     eigenvalues of `M` in basefield

**Return:**

```
list l;
  ideal l[1];
    number l[1][i]; i-th eigenvalue of M
  intvec l[2];
    int l[2][i]; multiplicity of i-th eigenvalue of M
```

**Example:**

```
LIB "linalg.lib";
ring R=0,x,dp;
matrix M[3][3]=3,2,1,0,2,1,0,0,3;
print(M);
  ↪ 3,2,1,
  ↪ 0,2,1,
  ↪ 0,0,3
eigenvals(M);
  ↪ [1]:
  ↪   _[1]=2
  ↪   _[2]=3
  ↪ [2]:
  ↪   1,2
```

See also: [Section D.3.2.18 \[eigenvals\]](#), page 998; [Section 5.1.155 \[system\]](#), page 275.

### D.3.2.19 minipoly

Procedure from library `linalg.lib` (see [Section D.3.2 \[linalg.lib\], page 988](#)).

**Usage:** `minipoly(M)`; matrix `M`

**Assume:** eigenvalues of `M` in basefield

**Return:**

list `l`; minimal polynomial of `M`  
 ideal `l[1]`;  
 number `l[1][i]`;  $i$ -th root of minimal polynomial of `M`  
 intvec `l[2]`;  
 int `l[2][i]`; multiplicity of  $i$ -th root of minimal polynomial of `M`

**Example:**

```
LIB "linalg.lib";
ring R=0,x,dp;
matrix M[3][3]=3,2,1,0,2,1,0,0,3;
print(M);
  ↪ 3,2,1,
  ↪ 0,2,1,
  ↪ 0,0,3
minipoly(M);
  ↪ [1]:
  ↪   _[1]=2
  ↪   _[2]=3
  ↪ [2]:
  ↪   1,2
```

### D.3.2.20 spnf

Procedure from library `linalg.lib` (see [Section D.3.2 \[linalg.lib\], page 988](#)).

**Usage:** `spnf(list(a[,m]))`; ideal `a`, intvec `m`

**Assume:** `ncols(a)==size(m)`

**Return:** list `l`:

`l[1]` an ideal, the generators of `a`; sorted and with multiple entries displayed only once  
`l[2]` an intvec, `l[2][i]` provides the multiplicity of `l[1][i]`

**Example:**

```
LIB "linalg.lib";
ring R=0,(x,y),ds;
list sp=list(ideal(-1/2,-3/10,-3/10,-1/10,-1/10,0,1/10,1/10,3/10,3/10,1/2));
spprint(spnf(sp));
  ↪ (-1/2,1),(-3/10,2),(-1/10,2),(0,1),(1/10,2),(3/10,2),(1/2,1)
```

### D.3.2.21 spprint

Procedure from library `linalg.lib` (see [Section D.3.2 \[linalg.lib\], page 988](#)).

**Usage:** `spprint(sp)`; list `sp` (helper routine for `spnf`)

**Return:** string `s`; spectrum `sp`

**Example:**

```

LIB "linalg.lib";
ring R=0,(x,y),ds;
list sp=list(ideal(-1/2,-3/10,-1/10,0,1/10,3/10,1/2),intvec(1,2,2,1,2,2,1));
spprint(sp);
↪ (-1/2,1),(-3/10,2),(-1/10,2),(0,1),(1/10,2),(3/10,2),(1/2,1)

```

See also: [Section D.6.13 \[gmssing\\_lib\], page 1703](#); [Section D.3.2.20 \[spnf\], page 999](#).

### D.3.2.22 jordan

Procedure from library `linalg.lib` (see [Section D.3.2 \[linalg\\_lib\], page 988](#)).

**Usage:** `jordan(M)`; matrix M

**Assume:** eigenvalues of M in basefield

**Return:**

```

list l; Jordan data of M
ideal l[1];
number l[1][i]; eigenvalue of i-th Jordan block of M
intvec l[2];
int l[2][i]; size of i-th Jordan block of M
intvec l[3];
int l[3][i]; multiplicity of i-th Jordan block of M

```

**Example:**

```

LIB "linalg.lib";
ring R=0,x,dp;
matrix M[3][3]=3,2,1,0,2,1,0,0,3;
print(M);
↪ 3,2,1,
↪ 0,2,1,
↪ 0,0,3
jordan(M);
↪ [1]:
↪ _[1]=2
↪ _[2]=3
↪ [2]:
↪ 1,2
↪ [3]:
↪ 1,1

```

### D.3.2.23 jordanbasis

Procedure from library `linalg.lib` (see [Section D.3.2 \[linalg\\_lib\], page 988](#)).

**Usage:** `jordanbasis(M)`; matrix M

**Assume:** eigenvalues of M in basefield

**Return:**

```

list l:
module l[1]; inverse(l[1])*M*l[1] in Jordan normal form
intvec l[2];
int l[2][i]; weight filtration index of l[1][i]

```

**Example:**

```

LIB "linalg.lib";
ring R=0,x,dp;
matrix M[3][3]=3,2,1,0,2,1,0,0,3;
print(M);
↳ 3,2,1,
↳ 0,2,1,
↳ 0,0,3
list l=jordanbasis(M);
print(l[1]);
↳ -2,0,3,
↳ 1, 1,0,
↳ 0, 1,0
print(l[2]);
↳ 0,
↳ 1,
↳ -1
print(inverse(l[1])*M*l[1]);
↳ 2,0,0,
↳ 0,3,0,
↳ 0,1,3

```

### D.3.2.24 jordanmatrix

Procedure from library `linalg.lib` (see [Section D.3.2 \[linalg.lib\], page 988](#)).

**Usage:** `jordanmatrix(list(e,s,m));` ideal `e`, intvec `s`, intvec `m`

**Assume:** `ncols(e)==size(s)==size(m)`

**Return:**

matrix `J`; Jordan matrix with `list(e,s,m)==jordan(J)`

**Example:**

```

LIB "linalg.lib";
ring R=0,x,dp;
ideal e=ideal(2,3);
intvec s=1,2;
intvec m=1,1;
print(jordanmatrix(list(e,s,m)));
↳ 2,0,0,
↳ 0,3,0,
↳ 0,1,3

```

### D.3.2.25 jordannf

Procedure from library `linalg.lib` (see [Section D.3.2 \[linalg.lib\], page 988](#)).

**Usage:** `jordannf(M);` matrix `M`

**Assume:** eigenvalues of `M` in basefield

**Return:** matrix `J`; Jordan normal form of `M`

**Example:**

```

LIB "linalg.lib";
ring R=0,x,dp;

```

```

matrix M[3][3]=3,2,1,0,2,1,0,0,3;
print(M);
  ↦ 3,2,1,
  ↦ 0,2,1,
  ↦ 0,0,3
print(jordannf(M));
  ↦ 2,0,0,
  ↦ 0,3,0,
  ↦ 0,1,3

```

## D.4 Commutative algebra

### D.4.1 absfact.lib

**Library:** absfact.lib

**Purpose:** Absolute factorization for characteristic 0

**Authors:** Wolfram Decker, `decker at math.uni-sb.de`  
 Gregoire Lecerf, `lecerf at math.uvsq.fr`  
 Gerhard Pfister, `pfister at mathematik.uni-kl.de` Martin Lee, `mlee at mathematik.uni-kl.de`

**Overview:** A library for computing the absolute factorization of multivariate polynomials  $f$  with coefficients in a field  $K$  of characteristic zero. Using Trager's idea, the implemented algorithm computes an absolutely irreducible factor by factorizing over some finite extension field  $L$  (which is chosen such that  $V(f)$  has a smooth point with coordinates in  $L$ ). Then a minimal extension field is determined making use of the Rothstein-Trager partial fraction decomposition algorithm. `absFactorizeBCG` uses the algorithm of Bertone, Cheze and Galligo for bivariate polynomials and similar ideas as above to reduce to this case.

**References:**

G. Cheze, G. Lecerf: Lifting and recombination techniques for absolute factorization. *Journal of Complexity*, 23(3):380-420, 2007. C. Bertone, G. Cheze, and A. Galligo: Modular las vegas algorithms for polynomial absolute factorization. *J. Symb. Comput.*, 45(12):1280-1295, December 2010

**Procedures:** See also: [Section 5.1.36 \[factorize\]](#), [page 180](#).

#### D.4.1.1 absFactorize

Procedure from library `absfact.lib` (see [Section D.4.1 \[absfact.lib\]](#), [page 1002](#)).

**Usage:** `absFactorize(p [s]);`  $p$  poly,  $s$  string

**Assume:** coefficient field is the field of rational numbers or a transcendental extension thereof

**Return:** ring  $R$  which is obtained from the current basering by adding a new parameter (if a string  $s$  is given as a second input, the new parameter gets this string as name). The ring  $R$  comes with a list `absolute_factors` with the following entries:

```

absolute_factors[1]: ideal  (the absolute factors)
absolute_factors[2]: intvec (the multiplicities)
absolute_factors[3]: ideal  (the minimal polynomials)
absolute_factors[4]: int    (total number of nontriv. absolute factors)

```



The entry `absolute_factors[1][1]` is a constant, the entry `absolute_factors[3][1]` is the parameter added to the current ring.

Each of the remaining entries `absolute_factors[1][j]` stands for a class of conjugated absolute factors. The corresponding entry `absolute_factors[3][j]` is the minimal polynomial of the field extension over which the factor is minimally defined (its degree is the number of conjugates in the class). If the entry `absolute_factors[3][j]` coincides with `absolute_factors[3][1]`, no field extension was necessary for the  $j$ th (class of) absolute factor(s).

**Note:** All factors are presented denominator- and content-free. The constant factor (first entry) is chosen such that the product of all (!) the (denominator- and content-free) absolute factors of  $p$  equals  $p / \text{absolute\_factors}[1][1]$ .

**Example:**

```
LIB "absfact.lib";
ring R = (0), (x,y), lp;
poly p = (-7*x^2 + 2*x*y^2 + 6*x + y^4 + 14*y^2 + 47)*(5x2+y2)^3*(x-y)^4;
def S = absFactorize(p) ;
↳
↳ // 'absFactorize' created a ring, in which a list absolute_factors (the
↳ // absolute factors) is stored.
↳ // To access the list of absolute factors, type (if the name S was assign\
    ed
↳ // to the return value):
↳ //      setring(S); absolute_factors;
↳
setring(S);
absolute_factors;
↳ [1]:
↳   _[1]=1/21125
↳   _[2]=(-14a+19)*x+13*y2+(-7a+94)
↳   _[3]=5*x+(a)*y
↳   _[4]=x-y
↳ [2]:
↳   1,1,3,4
↳ [3]:
↳   _[1]=(a)
↳   _[2]=(7a2-6a-47)
↳   _[3]=(a2+5)
↳   _[4]=(a)
↳ [4]:
↳   12
```

See also: [Section D.4.26.20 \[absPrimdecGTZ\]](#), page 1251; [Section 5.1.36 \[factorize\]](#), page 180.

### D.4.1.2 absFactorizeBCG

Procedure from library `absfact.lib` (see [Section D.4.1 \[absfact.lib\]](#), page 1002).

**Usage:** `absFactorizeBCG(p [s]);`  $p$  poly,  $s$  string

**Assume:** coefficient field is the field of rational numbers or a transcendental extension thereof

**Return:** ring  $R$  which is obtained from the current basering by adding a new parameter (if a string  $s$  is given as a second input, the new parameter gets this string as name). The ring  $R$  comes with a list `absolute_factors` with the following entries:

```

absolute_factors[1]: ideal  (the absolute factors)
absolute_factors[2]: intvec (the multiplicities)
absolute_factors[3]: ideal  (the minimal polynomials)
absolute_factors[4]: int    (total number of nontriv. absolute factors)

```

The entry `absolute_factors[1][1]` is a constant, the entry `absolute_factors[3][1]` is the parameter added to the current ring.

Each of the remaining entries `absolute_factors[1][j]` stands for a class of conjugated absolute factors. The corresponding entry `absolute_factors[3][j]` is the minimal polynomial of the field extension over which the factor is minimally defined (its degree is the number of conjugates in the class). If the entry `absolute_factors[3][j]` coincides with `absolute_factors[3][1]`, no field extension was necessary for the  $j$ th (class of) absolute factor(s).

**Note:** All factors are presented denominator- and content-free. The constant factor (first entry) is chosen such that the product of all (!) the (denominator- and content-free) absolute factors of  $p$  equals  $p / \text{absolute\_factors}[1][1]$ .

**Example:**

```

LIB "absfact.lib";
ring R = (0), (x,y), lp;
poly p = (-7*x^2 + 2*x*y^2 + 6*x + y^4 + 14*y^2 + 47)*(5x2+y2)^3*(x-y)^4;
def S = absFactorizeBCG(p) ;
⇒
⇒ // 'absFactorizeBCG' created a ring, in which a list absolute_factors (th\
e
⇒ // absolute factors) is stored.
⇒ // To access the list of absolute factors, type (if the name S was assign\
ed
⇒ // to the return value):
⇒ //      setring(S); absolute_factors;
⇒
setring(S);
absolute_factors;
⇒ [1]:
⇒   _[1]=1
⇒   _[2]=-x+y
⇒   _[3]=(-a)*x+y
⇒   _[4]=(2a-13)*x+y2+(a)
⇒ [2]:
⇒   1,4,3,1
⇒ [3]:
⇒   _[1]=(a)
⇒   _[2]=(a)
⇒   _[3]=(a2+5)
⇒   _[4]=(a2-14a+47)
⇒ [4]:
⇒   12

```

See also: [Section D.4.1.1 \[absFactorize\], page 1002](#); [Section D.4.26.20 \[absPrimdecGTZ\], page 1251](#); [Section 5.1.36 \[factorize\], page 180](#).

## D.4.2 algebra\_lib

**Library:** algebra.lib

**Purpose:** Compute with Algebras and Algebra Maps

**Authors:** Gert-Martin Greuel, greuel@mathematik.uni-kl.de,  
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 Gerhard Pfister, pfister@mathematik.uni-kl.de

**Procedures:**

### D.4.2.1 algebra\_containment

Procedure from library `algebra.lib` (see [Section D.4.2 \[algebra.lib\]](#), page 1004).

**Usage:** `algebra_containment(p,A[,k]);` p poly, A ideal, k integer.  
`A = A[1],...,A[m]` generators of subalgebra of the basering

**Return:**

- k=0 (or if k is not given) an integer:
  - 1 : if p is contained in the subalgebra  $K[A[1],\dots,A[m]]$
  - 0 : if p is not contained in  $K[A[1],\dots,A[m]]$
- k=1 : a list, say l, of size 2, l[1] integer, l[2] ring, satisfying
  - l[1]=1 if p is in the subalgebra  $K[A[1],\dots,A[m]]$  and then the ring
  - l[2]: ring, contains poly check =  $h(y(1),\dots,y(m))$  if  $p=h(A[1],\dots,A[m])$
  - l[1]=0 if p is not in the subalgebra  $K[A[1],\dots,A[m]]$  and then
  - l[2] contains the poly check =  $h(x,y(1),\dots,y(m))$  if p satisfies
  - the nonlinear relation  $p = h(x,A[1],\dots,A[m])$  where
  - $x = x(1),\dots,x(n)$  denote the variables of the basering

**Display:** if k=0 and `printlevel >= voice+1` (default) display the polynomial check

**Note:** The proc `inSubring` uses a different algorithm which is sometimes faster.

**Theory:** The ideal of algebraic relations of the algebra generators  $A[1],\dots, A[m]$  is computed introducing new variables  $y(i)$  and the product order with  $x(i) >> y(i)$ .  
 $p$  reduces to a polynomial only in the  $y(i) \iff p$  is contained in the subring generated by the polynomials  $A[1],\dots,A[m]$ .

**Example:**

```
LIB "algebra.lib";
int p = printlevel; printlevel = 1;
ring R = 0,(x,y,z),dp;
ideal A=x2+y2,z2,x4+y4,1,x2z-1y2z,xyz,x3y-1xy3;
poly p1=z;
poly p2=
x10z3-x8y2z3+2x6y4z3-2x4y6z3+x2y8z3-y10z3+x6z4+3x4y2z4+3x2y4z4+y6z4;
algebra_containment(p1,A);
  => // x(3)
  => 0
algebra_containment(p2,A);
  => // y(1)*y(2)*y(5)^2+y(3)*y(5)^3+4*y(1)*y(2)*y(6)^2+4*y(6)^3*y(7)+2*y(2)*y\
    (5)*y(7)^2
  => 1
list L = algebra_containment(p2,A,1);
  =>
  => // 'algebra_containment' created a ring as 2nd element of the list.
  => // The ring contains the polynomial check which defines the algebraic rel\
    ation.
```

```

⇒ // To access to the ring and see check you must give the ring a name,
⇒ // e.g.:
⇒
⇒           def S = l[2]; setring S; check;
⇒
L[1];
⇒ 1
def S = L[2]; setring S;
check;
⇒ y(1)*y(2)*y(5)^2+y(3)*y(5)^3+4*y(1)*y(2)*y(6)^2+4*y(6)^3*y(7)+2*y(2)*y(5)\
    *y(7)^2
printlevel = p;

```

### D.4.2.2 module\_containment

Procedure from library `algebra.lib` (see [Section D.4.2 \[algebra.lib\]](#), page 1004).

**Usage:**     `module_containment(p,P,M[k]);` p poly, P ideal, M ideal, k int  
               P = P[1],...,P[n] generators of a subalgebra of the basering,  
               M = M[1],...,M[m] generators of a module over the subalgebra K[P]

**Assume:**   `ncols(P) = nvars(basering)`, the P[i] are algebraically independent

**Return:**

- k=0 (or if k is not given), an integer:
  - 1   : if p is contained in the module  $\langle M[1], \dots, M[m] \rangle$  over K[P]
  - 0   : if p is not contained in  $\langle M[1], \dots, M[m] \rangle$
- k=1, a list, say l, of size 2, l[1] integer, l[2] ring:
  - l[1]=1 : if p is in  $\langle M[1], \dots, M[m] \rangle$  and then the ring l[2] contains the polynomial  $\text{check} = h(y(1), \dots, y(m), z(1), \dots, z(n))$  if  $p = h(M[1], \dots, M[m], P[1], \dots, P[n])$
  - l[1]=0 : if p is not in  $\langle M[1], \dots, M[m] \rangle$ , then l[2] contains the poly  $\text{check} = h(x, y(1), \dots, y(m), z(1), \dots, z(n))$  if p satisfies the nonlinear relation  $p = h(x, M[1], \dots, M[m], P[1], \dots, P[n])$  where  $x = x(1), \dots, x(n)$  denote the variables of the basering

**Display:**   the polynomial  $h(y(1), \dots, y(m), z(1), \dots, z(n))$  if k=0, resp. a comment how to access the relation check if k=1, provided `printlevel >= voice+1` (default).

**Theory:**     The ideal of algebraic relations of all the generators  $p_1, \dots, p_n, s_1, \dots, s_t$  given by P and S is computed introducing new variables  $y(j), z(i)$  and the product order:  $x^a y^b z^c > x^d y^e z^f$  if  $x^a > x^d$  with respect to the lp ordering or else if  $z^c > z^f$  with respect to the dp ordering or else if  $y^b > y^e$  with respect to the lp ordering again. p reduces to a polynomial only in the  $y(j)$  and  $z(i)$ , linear in the  $z(i) \iff p$  is contained in the module.

**Example:**

```

LIB "algebra.lib";
int p = printlevel; printlevel = 1;
ring R=0,(x,y,z),dp;
ideal P = x2+y2,z2,x4+y4;           //algebra generators
ideal M = 1,x2z-1y2z,xyz,x3y-1xy3; //module generators
poly p1=
x10z3-x8y2z3+2x6y4z3-2x4y6z3+x2y8z3-y10z3+x6z4+3x4y2z4+3x2y4z4+y6z4;
module_containment(p1,P,M);
⇒ // y(2)*z(2)*z(3)^2+z(1)^3*z(2)^2

```

```

⇒ 1
poly p2=z;
list l = module_containment(p2,P,M,1);
⇒
⇒ // 'module_containment' created a ring as 2nd element of the list. The
⇒ // ring contains the polynomial check which defines the algebraic relatio\
    n
⇒ // for p. To access to the ring and see check you must give the ring
⇒ // a name, e.g.:
⇒     def S = l[2]; setring S; check;
⇒
l[1];
⇒ 0
def S = l[2]; setring S; check;
⇒ x(3)
printlevel=p;

```

### D.4.2.3 inSubring

Procedure from library `algebra.lib` (see [Section D.4.2 \[algebra.lib\], page 1004](#)).

**Usage:**     `inSubring(p,i)`; p poly, i ideal

**Return:**

a list l of size 2, l[1] integer, l[2] string  
l[1]=1 if and only if p is in the subring generated by  $i=i[1],\dots,i[k]$ ,  
and then  $l[2] = y(0)-h(y(1),\dots,y(k))$  if  $p = h(i[1],\dots,i[k])$   
l[1]=0 if and only if p is in not the subring generated by i,  
and then  $l[2] = h(y(0),y(1),\dots,y(k))$  where p satisfies the  
nonlinear relation  $h(p,i[1],\dots,i[k])=0$ .

**Note:**     the proc `algebra_containment` tests the same using a different algorithm, which is often faster  
if  $l[1] == 0$  then l[2] may contain more than one relation  $h(y(0),y(1),\dots,y(k))$ , separated by comma

**Example:**

```

LIB "algebra.lib";
ring q=0,(x,y,z,u,v,w),dp;
poly p=xyzu2w-1yzu2w2+u4w2-1xu2vw+u2vw2+xyz-1yzw+2u2w-1xv+vw+2;
ideal I =x-w,u2w+1,yz-v;
inSubring(p,I);
⇒ [1]:
⇒     1
⇒ [2]:
⇒     y(0)-y(1)*y(2)*y(3)-y(2)^2-1

```

### D.4.2.4 algDependent

Procedure from library `algebra.lib` (see [Section D.4.2 \[algebra.lib\], page 1004](#)).

**Usage:**     `algDependent(f[c])`; f ideal (say,  $f = f_1,\dots,f_m$ ), c integer

**Return:**

a list  $l$  of size 2,  $l[1]$  integer,  $l[2]$  ring:  
 -  $l[1] = 1$  if  $f_1, \dots, f_m$  are algebraic dependent, 0 if not  
 -  $l[2]$  is a ring with variables  $x(1), \dots, x(n), y(1), \dots, y(m)$  if the basering has  $n$  variables. It contains the ideal 'ker', depending only on the  $y(i)$  and generating the algebraic relations between the  $f[i]$ , i.e. substituting  $y(i)$  by  $f_i$  yields 0. Of course, ker is nothing but the kernel of the ring map  

$$K[y(1), \dots, y(m)] \longrightarrow \text{basing}, \quad y(i) \rightarrow f_i.$$

**Note:** Three different algorithms are used depending on  $c = 1, 2, 3$ . If  $c$  is not given or  $c=0$ , a heuristically best method is chosen. The basering may be a quotient ring. To access to the ring  $l[2]$  and see ker you must give the ring a name, e.g. `def S=l[2]; setring S; ker;`

**Display:** The above comment is displayed if `printlevel >= 0` (default).

**Example:**

```
LIB "algebra.lib";
int p = printlevel; printlevel = 1;
ring R = 0, (x,y,z,u,v,w), dp;
ideal I = xyzu2w-1yzu2w2+u4w2-1xu2vw+u2vw2+xyz-1yzw+2u2w-1xv+vw+2,
x-w, u2w+1, yz-v;
list l = algDependent(I);
⇒
⇒ // The 2nd element of the list l is a ring with variables x(1),...,x(n),
⇒ // and y(1),...,y(m) if the basering has n variables and if the ideal
⇒ // is f[1],...,f[m]. The ring contains the ideal ker which depends only
⇒ // on the y(i) and generates the relations between the f[i].
⇒ // I.e. substituting y(i) by f[i] yields 0.
⇒ // To access to the ring and see ker you must give the ring a name,
⇒ // e.g.:
⇒
⇒           def S = l[2]; setring S; ker;
⇒
l[1];
⇒ 1
def S = l[2]; setring S;
ker;
⇒ ker[1]=y(2)*y(3)*y(4)+y(3)^2-y(1)+1
printlevel = p;
```

#### D.4.2.5 alg\_kernel

Procedure from library `algebra.lib` (see [Section D.4.2 \[algebra.lib\]](#), page 1004).

**Usage:** `alg_kernel(phi,pr[s,c]);` phi map to basering, pr preimage ring, s string (name of kernel in pr), c integer.

**Return:** a string, the kernel of phi as string.  
 If, moreover, a string  $s$  is given, the algorithm creates, in the preimage ring  $pr$  the kernel of phi with name  $s$ .  
 Three different algorithms are used depending on  $c = 1, 2, 3$ . If  $c$  is not given or  $c=0$ , a heuristically best method is chosen. (algorithm 1 uses the preimage command)

**Note:** Since the kernel of phi lives in  $pr$ , it cannot be returned to the basering. If  $s$  is given, the user has access to it in  $pr$  via  $s$ . The basering may be a quotient ring.

**Example:**

```

LIB "algebra.lib";
ring r = 0,(a,b,c),ds;
ring s = 0,(x,y,z,u,v,w),dp;
ideal I = x-w,u2w+1,yz-v;
map phi = r,I;                // a map from r to s:
alg_kernel(phi,r);            // a,b,c ---> x-w,u2w+1,yz-v
⇨ 0
ring S = 0,(a,b,c),ds;
ring R = 0,(x,y,z),dp;
qring Q = std(x-y);
ideal i = x, y, x2-y3;
map phi = S,i;                // a map to a quotient ring
alg_kernel(phi,S,"ker",3);    // uses algorithm 3
⇨ a-b,b^3-b^2+c
setring S;                    // you have access to kernel in preimage
ker;
⇨ ker[1]=a-b
⇨ ker[2]=c-b2+b3

```

**D.4.2.6 is\_injective**

Procedure from library `algebra.lib` (see [Section D.4.2 \[algebra.lib\]](#), page 1004).

**Usage:** `is_injective(phi,pr[,c,s]);` phi map, pr preimage ring, c int, s string

**Return:**

- 1 (type int) if phi is injective, 0 if not (if s is not given).
- If s is given, return a list l of size 2, l[1] int, l[2] ring:
  - l[1] is 1 if phi is injective, 0 if not
  - l[2] is a ring with variables  $x(1), \dots, x(n), y(1), \dots, y(m)$  if the basering has n variables and the map m components, it contains the ideal 'ker', depending only on the  $y(i)$ , the kernel of the given map

**Note:** Three different algorithms are used depending on  $c = 1, 2, 3$ . If c is not given or  $c=0$ , a heuristically best method is chosen. The basering may be a quotient ring. However, if the preimage ring is a quotient ring, say  $pr = P/I$ , consider phi as a map from P and then the algorithm returns 1 if the kernel of phi is 0 mod I. To access to the ring l[2] and see ker you must give the ring a name, e.g. `def S=l[2]; setring S; ker;`

**Display:** The above comment is displayed if `printlevel >= 0` (default).

**Example:**

```

LIB "algebra.lib";
int p = printlevel;
ring r = 0,(a,b,c),ds;
ring s = 0,(x,y,z,u,v,w),dp;
ideal I = x-w,u2w+1,yz-v;
map phi = r,I;                // a map from r to s:
is_injective(phi,r);          // a,b,c ---> x-w,u2w+1,yz-v
⇨ 1
ring R = 0,(x,y,z),dp;
ideal i = x, y, x2-y3;
map phi = R,i;                // a map from R to itself, z --> x2-y3
list l = is_injective(phi,R,"");

```

```

↳
↳ // The 2nd element of the list is a ring with variables x(1),...,x(n),
↳ // y(1),...,y(m) if the basering has n variables and the map is
↳ // F[1],...,F[m].
↳ // It contains the ideal ker, the kernel of the given map y(i) --> F[i].
↳ // To access to the ring and see ker you must give the ring a name,
↳ // e.g.:
↳     def S = l[2]; setring S; ker;
↳
↳ l[1];
↳ 0
↳ def S = l[2]; setring S;
↳ ker;
↳ ker[1]=y(2)^3-y(1)^2+y(3)

```

### D.4.2.7 is\_surjective

Procedure from library `algebra.lib` (see [Section D.4.2 \[algebra.lib\], page 1004](#)).

**Usage:** `is_surjective(phi)`; phi map to basering, or ideal defining it

**Return:** an integer, 1 if phi is surjective, 0 if not

**Note:** The algorithm returns 1 if and only if all the variables of the basering are contained in the polynomial subalgebra generated by the polynomials defining phi. Hence, it tests surjectivity in the case of a global ordering. If the basering has local or mixed ordering or if the preimage ring is a quotient ring (in which case the map may not be well defined) then the return value 1 needs to be interpreted with care.

**Example:**

```

LIB "algebra.lib";
ring R = 0,(x,y,z),dp;
ideal i = x, y, x2-y3;
map phi = R,i; // a map from R to itself, z->x2-y3
is_surjective(phi);
↳ 0
qring Q = std(ideal(z-x37));
map psi = R, x,y,x2-y3; // the same map to the quotient ring
is_surjective(psi);
↳ 1
ring S = 0,(a,b,c),dp;
map psi = R,ideal(a,a+b,c-a2+b3); // a map from R to S,
is_surjective(psi); // x->a, y->a+b, z->c-a2+b3
↳ 1

```

### D.4.2.8 is\_bijective

Procedure from library `algebra.lib` (see [Section D.4.2 \[algebra.lib\], page 1004](#)).

**Usage:** `is_bijective(phi,pr)`; phi map to basering, pr preimage ring

**Return:** an integer, 1 if phi is bijective, 0 if not

**Note:** The algorithm checks first injectivity and then surjectivity. To interpret this for local/mixed orderings, or for quotient rings type `help is_surjective`; and `help is_injective`;

**Display:** A comment if `printlevel >= voice-1` (default)



**Example:**

```

LIB "algebra.lib";
int p = printlevel; printlevel = 1;
ring R = 0,(x,y,z),dp;
ideal i = x, y, x2-y3;
map phi = R,i; // a map from R to itself, z->x2-y3
is_bijective(phi,R);
⇨ // map not injective
⇨ 0
qring Q = std(z-x2+y3);
is_bijective(ideal(x,y,x2-y3),Q);
⇨ 1
ring S = 0,(a,b,c,d),dp;
map psi = R,ideal(a,a+b,c-a2+b3,0); // a map from R to S,
is_bijective(psi,R); // x->a, y->a+b, z->c-a2+b3
⇨ // map injective, but not surjective
⇨ 0
qring T = std(d,c-a2+b3);
⇨ // ** _ is no standard basis
map chi = Q,a,b,a2-b3; // a map between two quotient rings
is_bijective(chi,Q);
⇨ 1
printlevel = p;

```

**D.4.2.9 noetherNormal**

Procedure from library `algebra.lib` (see [Section D.4.2 \[algebra.lib\]](#), page 1004).

**Usage:** `noetherNormal(id[p]);` id ideal, p integer

**Return:**

a list l of two ideals, say I,J:  
 - I defines a map (coordinate change in the basering), such that:  
 - J is generated by a subset of the variables with  $\text{size}(J) = \dim(\text{id})$   
 if we define `map phi=basing,I;`  
 then  $k[\text{var}(1),\dots,\text{var}(n)]/\text{phi}(\text{id})$  is finite over  $k[J]$ .  
 If p is given,  $0 \leq p \leq 100$ , a sparse coordinate change with p percent  
 of the matrix entries being 0 (default: p=0 i.e. dense)

**Note:** Designed for characteristic 0. It works also in char  $k > 0$  if it terminates, but may result in an infinite loop in small characteristic.

**Example:**

```

LIB "algebra.lib";
ring r=0,(x,y,z),dp;
ideal i= xy,xz;
noetherNormal(i);
⇨ [1]:
⇨ _[1]=x
⇨ _[2]=2x+y
⇨ _[3]=3x+4y+z
⇨ [2]:
⇨ _[1]=y
⇨ _[2]=z

```

### D.4.2.10 mapIsFinite

Procedure from library `algebra.lib` (see [Section D.4.2 \[algebra.lib\], page 1004](#)).

**Usage:** `mapIsFinite(phi,R[,J]);`  $R$  the preimage ring of the map  $\phi: R \rightarrow \text{basering}$   
 $J$  an ideal in the basering,  $J = 0$  if not given

**Return:** 1 if  $R \rightarrow \text{basering}/J$  is finite and 0 else

**Note:**  $R$  may be a quotient ring (this will be ignored since a map  $R/I \rightarrow S/J$  is finite if and only if the induced map  $R \rightarrow S/J$  is finite).

**Example:**

```
LIB "algebra.lib";
ring r = 0,(a,b,c),dp;
ring s = 0,(x,y,z),dp;
ideal i= xy;
map phi= r,(xy)^3+x2+z,y2-1,z3;
mapIsFinite(phi,r,i);
⇒ 1
```

See also: [Section D.4.2.11 \[finitenessTest\], page 1012](#).

### D.4.2.11 finitenessTest

Procedure from library `algebra.lib` (see [Section D.4.2 \[algebra.lib\], page 1004](#)).

**Usage:** `finitenessTest(J[,v]);`  $J$  ideal,  $v$  intvec (say  $v_1, \dots, v_r$  with  $v_i > 0$ )

**Return:**

- a list  $l$  with  $l[1]$  integer,  $l[2]$ ,  $l[3]$ ,  $l[4]$  ideals
- $l[1] = 1$  if  $\text{var}(v_1), \dots, \text{var}(v_r)$  are in  $l[2]$  and 0 else
- $l[2]$  (resp.  $l[3]$ ) contains those variables which occur, (resp. do not occur) as pure power in the leading term of one of the generators of  $J$ ,
- $l[4]$  contains those  $J[i]$  for which the leading term is a pure power of a variable (which is then in  $l[2]$ )
- (default:  $v = [1, 2, \dots, \text{nvars}(\text{basering})]$ )

**Theory:** If  $J$  is a standard basis of an ideal generated by  $x_1 - f_1(y), \dots, x_n - f_n(y)$  with  $y_j$  ordered lexicographically and  $y_j \gg x_i$ , then, if  $y_i$  appears as pure power in the leading term of  $J[k]$ ,  $J[k]$  defines an integral relation for  $y_i$  over the  $y_{(i+1)}, \dots$  and the  $f$ 's. Moreover, in this situation, if  $l[2] = y_1, \dots, y_r$ , then  $K[y_1, \dots, y_r]$  is finite over  $K[f_1, \dots, f_n]$ . If  $J$  contains furthermore polynomials  $h_j(y)$ , then  $K[y_1, \dots, y_r]/\langle h_j \rangle$  is finite over  $K[f_1, \dots, f_n]$ . For a proof cf. Prop. 3.1.5, p. 214. in [G.-M. Greuel, G. Pfister: A SINGULAR Introduction to Commutative Algebra, 2nd Edition, Springer Verlag (2007)]

**Example:**

```
LIB "algebra.lib";
ring s = 0,(x,y,z,a,b,c),(lp(3),dp);
ideal i= a -(xy)^3+x2-z, b -y2-1, c -z3;
ideal j = a -(xy)^3+x2-z, b -y2-1, c -z3, xy;
finitenessTest(std(i),1..3);
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ _[1]=y
```

```

↳   _[2]=z
↳ [3]:
↳   _[1]=x
↳   _[2]=a
↳   _[3]=b
↳   _[4]=c
↳ [4]:
↳   _[1]=z3-c
↳   _[2]=y2-b+1
finitenessTest(std(j),1..3);
↳ [1]:
↳   1
↳ [2]:
↳   _[1]=x
↳   _[2]=y
↳   _[3]=z
↳ [3]:
↳   _[1]=a
↳   _[2]=b
↳   _[3]=c
↳ [4]:
↳   _[1]=z3-c
↳   _[2]=y2-b+1
↳   _[3]=x2-z+a

```

#### D.4.2.12 nonZeroEntry

Procedure from library `algebra.lib` (see [Section D.4.2 \[algebra.lib\], page 1004](#)).

**Usage:** `nonZeroEntry(id)`; `id`=object for which the test '`id[i]!=0`',  $i=1,\dots,N$ ,  $N=\text{size}(\text{id})$  (resp.  $\text{ncols}(\text{id})$  for `id` of type ideal or module) is defined (e.g. ideal, vector, list of polynomials, intvec,...)

**Return:**

- a list, say `l`, with `l[1]` an integer, `l[2]`, `l[3]` integer vectors:
  - `l[1]` number of non-zero entries of `id`
  - `l[2]` intvec of size `l[1]` with `l[2][i]=i` if `id[i] != 0`  
   in case `l[1]!=0` (and `l[2]=0` if `l[1]=0`)
  - `l[3]` intvec with `l[3][i]=1` if `id[i]!=0` and `l[3][i]=0` else

**Note:**

**Example:**

```

LIB "algebra.lib";
ring r = 0,(a,b,c),dp;
poly f = a3c+b3+c2+a;
intvec v = leadexp(f);
nonZeroEntry(v);
↳ [1]:
↳   2
↳ [2]:
↳   1,3
↳ [3]:
↳   1,0,1
intvec w;

```

```

list L = 37,0,f,v,w;
nonZeroEntry(L);
↳ [1]:
↳ 3
↳ [2]:
↳ 1,3,4
↳ [3]:
↳ 1,0,1,1,0

```

### D.4.3 `assprimeszerodim.lib`

**Library:** `assprimeszerodim.lib`

**Purpose:** associated primes of a zero-dimensional ideal

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**Overview:** A library for computing the associated primes and the radical of a zero-dimensional ideal in the polynomial ring over the rational numbers,  $\mathbb{Q}[x_1, \dots, x_n]$ , using modular computations.

**Procedures:** See also: [Section D.4.26 \[primdec.lib\]](#), page 1240.

#### D.4.3.1 `zeroRadical`

Procedure from library `assprimeszerodim.lib` (see [Section D.4.3 \[assprimeszerodim.lib\]](#), page 1014).

**Usage:** `zeroRadical(I[, exactness]);` I ideal, exactness int

**Assume:** I is zero-dimensional in  $\mathbb{Q}[\text{variables}]$

**Return:** the radical of I

**Note:** A final test is applied to the result if `exactness != 0` (default), otherwise no final test is done.

**Example:**

```

LIB "assprimeszerodim.lib";
ring R = 0, (x,y), dp;
ideal I = xy4-2xy2+x, x2-x, y4-2y2+1;
zeroRadical(I);
↳ _[1]=y2-1
↳ _[2]=x2-x

```

#### D.4.3.2 `assPrimes`

Procedure from library `assprimeszerodim.lib` (see [Section D.4.3 \[assprimeszerodim.lib\]](#), page 1014).

**Assume:** I is zero-dimensional over  $\mathbb{Q}[\text{variables}]$

**Return:** a list of the associated primes of I

**Note:** A final test is applied to the result if `exactness != 0` (default), otherwise no final test is done.

**Example:**

```

LIB "assprimeszerodim.lib";
ring R = 0,(a,b,c,d,e,f),dp;
ideal I =
2fb+2ec+d2+a2+a,
2fc+2ed+2ba+b,
2fd+e2+2ca+c+b2,
2fe+2da+d+2cb,
f2+2ea+e+2db+c2,
2fa+f+2eb+2dc;
assPrimes(I);
↳ [1]:
↳ _[1]=cd+be+af+1/2f
↳ _[2]=c2+2bd+2ae+f2+e
↳ _[3]=bc+ad+ef+1/2d
↳ _[4]=b2+2ac+e2+2df+c
↳ _[5]=ab+de+cf+1/2b
↳ _[6]=a2+d2+2ce+2bf+a
↳ _[7]=de2+d2f+2cef+bf2-5/36f
↳ _[8]=d2e+ce2-af2-1/2f2-1/9e
↳ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
↳ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
↳ _[11]=d3-3be2-6aef-f3-3ef-1/4d
↳ _[12]=bd2-2acf-e2f-df2-cf-1/12b
↳ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
↳ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
↳ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
↳ _[16]=be2f+bdf2+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
↳ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
↳ _[18]=acef+1/2cef-1/36af-1/72f
↳ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
↳ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
↳ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
↳ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
↳ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
↳ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
↳ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
↳ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
↳ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
↳ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
↳ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
↳ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
↳ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
↳ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
↳ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
↳ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
↳ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
↳ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
↳ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
↳ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
↳ _[39]=49a-75b-94c+11d+94e+f
↳ [2]:
↳ _[1]=cd+be+af+1/2f

```

```

⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bd2f+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=49a-75b-94c+11d+94e+f-2
⇒ [3]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f

```

```

⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bd2f+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=49a-75b-94c+11d+94e+f+19
⇒ [4]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bd2f+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c

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⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=49a-75b-94c+11d+94e+f+30
⇒ [5]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bdf2+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=49a-75b-94c+11d+94e+f+49
⇒ [6]:

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⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bd2f+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=49a-75b-94c+11d+94e+f+51
⇒ [7]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce

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⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bd2f+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=147a-225b-282c+33d+282e+3f-7
⇒ [8]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bd2f+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef

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⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=147a-225b-282c+33d+282e+3f+1
⇒ [9]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bdf2+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=147a-225b-282c+33d+282e+3f+49

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⇒ [10]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bd2f+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=147a-225b-282c+33d+282e+3f+50
⇒ [11]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b

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⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bdf2+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=147a-225b-282c+33d+282e+3f+56
⇒ [12]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bdf2+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e

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⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=147a-225b-282c+33d+282e+3f+91
⇒ [13]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bdf2+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f

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⇒ _[39]=147a-225b-282c+33d+282e+3f+97
⇒ [14]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bd2f+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=147a-225b-282c+33d+282e+3f+98
⇒ [15]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d

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⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bd2f+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=147a-225b-282c+33d+282e+3f+146
⇒ [16]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bd2f+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f

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⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=147a-225b-282c+33d+282e+3f+154
⇒ [17]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bdf2+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf

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⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=-61386ac-39066ad-57966bd-6840d2-25380ae-36096be-67422ce+28254de+\
9633e2+6498af-14856bf+21486cf-33684df-41736ef-26505f2-3087a+4725b-24771c-\
20226d-18612e+3186f+2032
⇒ [18]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bd2f+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=-61386ac-39066ad-57966bd-6840d2-25380ae-36096be-67422ce+28254de+\
9633e2+6498af-14856bf+21486cf-33684df-41736ef-26505f2-3087a+4725b-24771c-\
20226d-18612e+3186f+9424
⇒ [19]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a

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⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bdf2+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=-61386ac-39066ad-57966bd-6840d2-25380ae-36096be-67422ce+28254de+\
9633e2+6498af-14856bf+21486cf-33684df-41736ef-26505f2+3087a-4725b-36615c-\
18840d-6768e+3312f+5119
⇒ [20]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bdf2+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf

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⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=-61386ac-39066ad-57966bd-6840d2-25380ae-36096be-67422ce+28254de+\
9633e2+6498af-14856bf+21486cf-33684df-41736ef-26505f2+3087a-4725b-36615c-\
18840d-6768e+3312f+12511
⇒ [21]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bdf2+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b

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⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=-81848ac-52088ad-77288bd-9120d2-33840ae-48128be-89896ce+37672de+\
12844e2+8664af-19808bf+28648cf-44912df-55648ef-35340f2-3136a+4800b-34908c\
-26748d-22936e+4268f+6897
⇒ [22]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bdf2+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=-81848ac-52088ad-77288bd-9120d2-33840ae-48128be-89896ce+37672de+\

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12844e2+8664af-19808bf+28648cf-44912df-55648ef-35340f2+3136a-4800b-46940c\
-25340d-10904e+4396f+10033
⇒ [23] :
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bd2f+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=-184158ac-117198ad-173898bd-20520d2-76140ae-108288be-202266ce+84\
762de+28899e2+19494af-44568bf+64458cf-101052df-125208ef-79515f2-7203a+110\
25b-78261c-60216d-51888e+9600f+6733
⇒ [24] :
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e

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⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bdf2+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=--184158ac-117198ad-173898bd-20520d2-76140ae-108288be-202266ce+84\
762de+28899e2+19494af-44568bf+64458cf-101052df-125208ef-79515f2-7203a+110\
25b-78261c-60216d-51888e+9600f+28909
⇒ [25]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bdf2+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e

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⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=-184158ac-117198ad-173898bd-20520d2-76140ae-108288be-202266ce+84\
762de+28899e2+19494af-44568bf+64458cf-101052df-125208ef-79515f2+7203a-110\
25b-105897c-56982d-24252e+9894f+13936
⇒ [26]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bdf2+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b

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⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=-184158ac-117198ad-173898bd-20520d2-76140ae-108288be-202266ce+84\
762de+28899e2+19494af-44568bf+64458cf-101052df-125208ef-79515f2+7203a-110\
25b-105897c-56982d-24252e+9894f+36112
⇒ [27]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bdf2+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=-245544ac-156264ad-231864bd-27360d2-101520ae-144384be-269688ce+1\
13016de+38532e2+25992af-59424bf+85944cf-134736df-166944ef-106020f2-9408a\
14400b-104724c-80244d-68808e+12804f+6403

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⇒ [28]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bd2f+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=-245544ac-156264ad-231864bd-27360d2-101520ae-144384be-269688ce+1\
13016de+38532e2+25992af-59424bf+85944cf-134736df-166944ef-106020f2+9408a-\
14400b-140820c-76020d-32712e+13188f+15811
⇒ [29]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b

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⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bdf2+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=-736632ac-468792ad-695592bd-82080d2-304560ae-433152be-809064ce+3\
39048de+115596e2+77976af-178272bf+257832cf-404208df-500832ef-318060f2-940\
80a+144000b-187836c-255516d-332760e+37068f+52441
⇒ [30]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bdf2+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf

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⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=-736632ac-468792ad-695592bd-82080d2-304560ae-433152be-809064ce+3\
39048de+115596e2+77976af-178272bf+257832cf-404208df-500832ef-318060f2-858\
48a+131400b-203628c-253668d-316968e+37236f+52273
⇒ [31]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bdf2+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f

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⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=-736632ac-468792ad-695592bd-82080d2-304560ae-433152be-809064ce+3\
39048de+115596e2+77976af-178272bf+257832cf-404208df-500832ef-318060f2-376\
32a+57600b-296124c-242844d-224472e+38220f+16297
⇒ [32]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bdf2+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=-736632ac-468792ad-695592bd-82080d2-304560ae-433152be-809064ce+3\
39048de+115596e2+77976af-178272bf+257832cf-404208df-500832ef-318060f2-364\
56a+55800b-298380c-242580d-222216e+38244f+16633
⇒ [33]:
⇒ _[1]=cd+be+af+1/2f

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⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bd2f+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=-736632ac-468792ad-695592bd-82080d2-304560ae-433152be-809064ce+3\
39048de+115596e2+77976af-178272bf+257832cf-404208df-500832ef-318060f2-294\
00a+45000b-311916c-240996d-208680e+38388f+18817
⇒ [34]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b

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⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+ddf2+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=-736632ac-468792ad-695592bd-82080d2-304560ae-433152be-809064ce+3\
39048de+115596e2+77976af-178272bf+257832cf-404208df-500832ef-318060f2-199\
92a+30600b-329964c-238884d-190632e+38580f+65041
⇒ [35]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+ddf2+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e

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⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=-736632ac-468792ad-695592bd-82080d2-304560ae-433152be-809064ce+3\
39048de+115596e2+77976af-178272bf+257832cf-404208df-500832ef-318060f2+199\
92a-30600b-406668c-229908d-113928e+39396f+85033
⇒ [36]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bdf2+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2

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⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=-736632ac-468792ad-695592bd-82080d2-304560ae-433152be-809064ce+3\
39048de+115596e2+77976af-178272bf+257832cf-404208df-500832ef-318060f2+294\
00a-45000b-424716c-227796d-95880e+39588f+48217
⇒ [37]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bdf2+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=-736632ac-468792ad-695592bd-82080d2-304560ae-433152be-809064ce+3\
39048de+115596e2+77976af-178272bf+257832cf-404208df-500832ef-318060f2+364\
56a-55800b-438252c-226212d-82344e+39732f+53089
⇒ [38]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d

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⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bdf2+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=-736632ac-468792ad-695592bd-82080d2-304560ae-433152be-809064ce+3\
39048de+115596e2+77976af-178272bf+257832cf-404208df-500832ef-318060f2+376\
32a-57600b-440508c-225948d-80088e+39756f+53929
⇒ [39] :
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f

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⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bd2f+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e
⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=-736632ac-468792ad-695592bd-82080d2-304560ae-433152be-809064ce+3\
39048de+115596e2+77976af-178272bf+257832cf-404208df-500832ef-318060f2+858\
48a-131400b-533004c-215124d+12408e+40740f+138121
⇒ [40]:
⇒ _[1]=cd+be+af+1/2f
⇒ _[2]=c2+2bd+2ae+f2+e
⇒ _[3]=bc+ad+ef+1/2d
⇒ _[4]=b2+2ac+e2+2df+c
⇒ _[5]=ab+de+cf+1/2b
⇒ _[6]=a2+d2+2ce+2bf+a
⇒ _[7]=de2+d2f+2cef+bf2-5/36f
⇒ _[8]=d2e+ce2-af2-1/2f2-1/9e
⇒ _[9]=bde+ae2+adf+ef2+1/2e2+1/2df+1/18c
⇒ _[10]=ade+acf+e2f+df2+1/2de+1/2cf+1/18b
⇒ _[11]=d3-3be2-6aef-f3-3ef-1/4d
⇒ _[12]=bd2-2acf-e2f-df2-cf-1/12b
⇒ _[13]=ad2+2ace+e3+4def+cf2+1/2d2+ce
⇒ _[14]=e3f+3def2+cf3+1/36be+1/12af+1/24f
⇒ _[15]=ce2f+bef2-1/36ad-1/9ef-1/72d
⇒ _[16]=be2f+bd2f+2aef2+1/2f4+ef2-1/36ac-1/72e2-1/72c
⇒ _[17]=ae2f+adf2+ef3+1/2e2f+1/2df2+1/36de+1/18cf
⇒ _[18]=acef+1/2cef-1/36af-1/72f
⇒ _[19]=e4-6d2f2-12cef2-8bf3+1/18bd+2/9ae+13/12f2+1/9e
⇒ _[20]=ce3-3aef2-1/2f4-3/2ef2-1/36ac-1/8e2-1/9df-1/72c
⇒ _[21]=be3-3adf2-2ef3-3/2df2-5/36cf
⇒ _[22]=ae3-3acf2-2df3+1/2e3-3/2cf2+1/18ce-1/9bf
⇒ _[23]=ace2+d2f2+3cef2+2bf3+1/2ce2-1/18bd-1/9ae-5/18f2-1/18e
⇒ _[24]=e2f3+1/2df4+1/72d2f+1/18cef+5/72bf2-13/2592f
⇒ _[25]=def3+1/2cf4-1/72ce2-1/36bef+1/36af2+1/72f2+1/648e

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⇒ _[26]=cef3+1/2bf4-1/72be2-1/18bdf-1/18aef-1/12f3-1/36ef
⇒ _[27]=bef3+1/2af4+1/4f4-1/72ae2-1/36adf-1/18ef2-1/144e2-1/72df-1/1296c
⇒ _[28]=aef3+1/5f5+1/2ef3+1/36acf+1/36e2f+1/18df2+1/72cf+1/1620b
⇒ _[29]=d2f3+1/2bf4+1/24be2+1/9bdf+1/12aef-1/36f3+1/24ef
⇒ _[30]=bdf3+1/5f5-1/18acf-1/36e2f-1/12df2-1/36cf-1/1080b
⇒ _[31]=adf3+1/2ef4+1/2df3-1/36ace-1/72e3-1/36def+1/36cf2-1/72ce
⇒ _[32]=acf3+1/2df4+1/2cf3-1/72d2f-1/36cef+1/72bf2+5/2592f
⇒ _[33]=ef5+5/36def2+5/36cf3-1/1296be+5/1296af+5/2592f
⇒ _[34]=df5+5/36d2f2+5/36bf3+1/324bd-5/432f2
⇒ _[35]=cf5-5/36bef2+1/1296ad+1/216ef+1/2592d
⇒ _[36]=bf5+5/36bdf2-5/72f4-1/432ac-1/864e2-1/162df-1/864c
⇒ _[37]=af5+1/2f5+5/36adf2+5/36ef3+5/72df2+1/1296de+1/216cf
⇒ _[38]=f7+35/72df4+35/2592d2f+7/288bf2-5/3456f
⇒ _[39]=-736632ac-468792ad-695592bd-82080d2-304560ae-433152be-809064ce+3\
39048de+115596e2+77976af-178272bf+257832cf-404208df-500832ef-318060f2+940\
80a-144000b-548796c-213276d+28200e+40908f+146521

```

#### D.4.4 cisimplicial\_lib

**Library:** cisimplicial.lib

**Purpose:** . Determines if the toric ideal of a simplicial toric variety is a complete intersection

**Authors:** I.Bermejo, ibermejo@ull.es  
I.Garcia-Marco, iggarcia@ull.es

**Overview:** A library for determining if a simplicial toric ideal is a complete intersection with NO NEED of computing explicitly a system of generators of such ideal. The procedures are based on two papers: I. Bermejo, I. Garcia-Marco and J.J. Salazar-Gonzalez: 'An algorithm for checking whether the toric ideal of an affine monomial curve is a complete intersection', J. Symbolic Computation 42 (2007) pags: 971–991 and I.Bermejo and I. Garcia-Marco: 'Complete intersections in simplicial toric varieties', Preprint (2010)

**Procedures:**

##### D.4.4.1 minMult

Procedure from library `cisimplicial.lib` (see [Section D.4.4 \[cisimplicial\\_lib\]](#), page 1046).

**Usage:** minMult (a, b); a integer, b integer vector.

**Return:** an integer k, the minimum positive integer such that  $k \cdot a$  belongs to the semigroup generated by the integers in b.

**Assume:** a is a positive integer, b is a vector of positive integers.

**Example:**

```

LIB "cisimplicial.lib";
int a = 46;
intvec b = 13,17,59;
minMult(a,b);
⇒ 3
"// 3*a = 8*b[1] + 2*b[2]"

```

### D.4.4.2 belongSemigroup

Procedure from library `cisimplicial.lib` (see [Section D.4.4 \[cisimplicial.lib\]](#), page 1046).

**Usage:** `belongSemigroup(v,A[,k]);`  $v$  is an integral vector,  $A$  is an integral matrix,  $n$  is a positive integer.

**Return:** `counters`, a vector with nonnegative entries such that  $A \cdot \text{counters} = v$ . If it does not exist such a vector, it returns 0. If a third parameter  $k$  is introduced, it will only consider the first  $k$  columns of  $A$ .

**Assume:**  $A$  is a matrix with nonnegative entries, nonzero columns,  $v$  is a nonnegative vector and  $\text{nrows}(v) = \text{nrows}(A)$ .

**Example:**

```
LIB "cisimplicial.lib";
intmat A[3][4] = 10,3,2,1,2,1,1,3,5,0,1,2;
print(A);
↪      10      3      2      1
↪      2       1      1      3
↪      5       0      1      2
intvec v = 23,12,10;
belongSemigroup(v,A);
↪ 1,3,1,2
"// A * (1,3,1,2) = v";
↪ // A * (1,3,1,2) = v
belongSemigroup(v,A,3);
↪ 0
"// v is not a combination of the first 3 columns of A";
↪ // v is not a combination of the first 3 columns of A
intvec w = 12,4,1;
belongSemigroup(w,A);
↪ 0
"// w is not a combination of the columns of A";
↪ // w is not a combination of the columns of A
```

### D.4.4.3 oneDimBelongSemigroup

Procedure from library `cisimplicial.lib` (see [Section D.4.4 \[cisimplicial.lib\]](#), page 1046).

**Usage:** `oneDimBelongSemigroup(n,v[,m]);`  $v$  is an integral vector,  $n$  is a positive integer[,  $m$  is a positive integer].

**Return:** `counters`, a vector with nonnegative entries such that  $v \cdot \text{counters} = n$ . If it does not exist such a vector, it returns 0. If a third parameter  $m$  is introduced, it will only consider the first  $m$  entries of  $v$ .

**Assume:**  $v$  is an integral vector with positive entries.

**Example:**

```
LIB "cisimplicial.lib";
int a = 95;
intvec v = 18,51,13;
oneDimBelongSemigroup(a,v);
↪ 1,1,2
"// 95 = 1*18 + 1*25 + 2*13";
```

```

⇒ // 95 = 1*18 + 1*25 + 2*13
oneDimBelongSemigroup(a,v,2);
⇒ 0
"// 95 is not a combination of 18 and 52;";
⇒ // 95 is not a combination of 18 and 52;

```

#### D.4.4.4 cardGroup

Procedure from library `cisimplicial.lib` (see [Section D.4.4 \[cisimplicial.lib\]](#), page 1046).

**Usage:** `cardGroup(A[,n]);` A is a matrix with integral coefficients.

**Return:** It returns a bigint. If we denote by  $ZA$  the group generated by the columns of the matrix  $A$ , then it returns the number of elements of the group of  $Z^m / ZA$ , where  $m$  = number of rows of  $A$ . If a second parameter  $n$  is introduced, it will only consider the first  $n$  columns of  $A$ . It returns 0 if  $Z^m / ZA$  is infinite; this is, when  $\text{rank } ZA < m$ .

**Example:**

```

LIB "cisimplicial.lib";
intmat A[3][5] = 24, 0, 0, 8, 3,
0, 24, 0, 10, 6,
0, 0, 24, 5, 9;
cardGroup(A);
⇒ 72

```

#### D.4.4.5 isCI

Procedure from library `cisimplicial.lib` (see [Section D.4.4 \[cisimplicial.lib\]](#), page 1046).

**Usage:** `isCI(A);` A is an integral matrix

**Return:** 1 if the simplicial toric ideal  $I(A)$  is a complete intersection and 0 otherwise. If `printlevel` > 0 and  $I(A)$  is a complete intersection it also shows a minimal set of generators of  $I(A)$

**Assume:** A is an  $m \times n$  integral matrix with nonnegative entries and for every  $1 \leq i \leq m$ , there exist a column in A whose  $i$ -th coordinate is not null and the rest are 0.

**Example:**

```

LIB "cisimplicial.lib";
intmat A[2][5] = 60,0,140,150,21,0,60,140,150,21;
print(A);
⇒      60      0     140     150     21
⇒      0      60     140     150     21
printlevel = 0;
isCI(A);
⇒ // It is a complete intersection
⇒ 1
printlevel = 1;
isCI(A);
⇒ // Generators of the toric ideal
⇒ toric[1]=-x(1)^7*x(2)^7+x(3)^3
⇒ toric[2]=x(5)^10-x(1)*x(2)*x(4)
⇒ toric[3]=-x(1)^5*x(2)^5+x(4)^2
⇒ // It is a complete intersection
⇒ 1

```

### D.4.5 curveInv\_lib

[illegible]

```

curveDeltaInv(radical(I));
↦ 9
//////////
// Infinite delta invariant //
//////////
ideal J = xyz;
curveDeltaInv(radical(J));
↦ -1

```

See also: [Section D.4.5.2 \[curveConductorMult\]](#), page 1050; [Section D.4.5.3 \[curveDeligneNumber\]](#), page 1051.

### D.4.5.2 curveConductorMult

Procedure from library `curveInv.lib` (see [Section D.4.5 \[curveInv.lib\]](#), page 1049).

**Usage:** `curveConductorMult(I);` I ideal

**Assume:** I is a radical ideal,  $\dim(R/I) = 1$

**Return:** the multiplicity of the conductor

**Note:** the optional parameter can be used if the normalization has already been computed. If a list L contains the output of the procedure `normal` (with options `prim`, `wd` and `using` if the ring has a mixed ordering), apply `curveConductorMult(I,L)`

**Example:**

```

LIB "curveInv.lib";
//////////
// Mutltiplicity of the conductor of curves //
//////////
ring R = 0,(x,y,z),ds;
// Example 1:
ideal I = x2-y4z,z3y2+xy2;
I = std(radical(I));
curveConductorMult(I);
↦ 23
// Example 2:
ideal I = x*(y+z)^3 - y3, x2y2 + z5;
↦ // ** redefining I (ideal I = x*(y+z)^3 - y3, x2y2 + z5;) ./examples/curv\
   eConductorMult.sing:11
I = std(radical(I));
curveConductorMult(I);
↦ 19
kill R;
//////////
// Mutltiplicity of the conductor of Gorenstein curve //
//////////
ring R = 0,(x,y),ds;
ideal I = xy;
// In such a case, the conductor multiplicity c satisfies: c = 2*delta
// Delta invariant:
curveDeltaInv(I);
↦ 1
// Conductor Multiplicity:
curveConductorMult(I);

```



$\mapsto 2$

See also: [Section D.4.23.17 \[normalConductor\]](#), page 1208.

### D.4.5.3 curveDeligneNumber

Procedure from library `curveInv.lib` (see [Section D.4.5 \[curveInv.lib\]](#), page 1049).

**Usage:** `curveDeligneNumber(I)`; I ideal

**Assume:** I is a radical ideal,  $\dim(R/I) = 1$

**Return:** the Deligne number of  $R/I$

**Remarks:** The Deligne number  $e$  satisfies by definition:  $e = 3\delta - m$ . So the algorithm splits the computation into two parts: one part computes the delta invariant, the other part the colength of derivations  $m$ .

**Note:** the optional parameter can be used if the normalization has already been computed. If a list  $L$  contains the output of the procedure `normal` (with options `prim`, `wd` and `using` if the ring has a mixed ordering), apply `curveDeligneNumber(I,L)`

**Example:**

```
LIB "curveInv.lib";
////////////////////////////////////
// Deligne number of curves //
////////////////////////////////////
// Example 1:
ring R = 0,(x,y,z),ds;
ideal I = x2-y4z,z3y2+xy2;
I = std(radical(I));
curveDeligneNumber(I);
 $\mapsto 30$ 
// Example 2:
ring S = 0,(x,y),ds;
ideal I = (x+y)*(x2-y3);
curveDeligneNumber(I);
 $\mapsto 5$ 
// Example 3:
ideal J = (x2-y3)*(x2+y2)*(x-y);
curveDeligneNumber(J);
 $\mapsto 15$ 
// Let us also compute the milnor number of this complete intersection:
milnor(J);
 $\mapsto 17$ 
// We see that the Milnor number is bigger than the Deligne number. Hence, this
// curve cannot be quasi homogeneous. This can also be verified by Saitos criterion:
reduce(J[1],std(jacob(J[1])));
 $\mapsto -1/5y6+19/50y7$ 
```

See also: [Section D.4.5.4 \[curveColengthDerivations\]](#), page 1051; [Section D.4.5.1 \[curveDeltaInv\]](#), page 1049.

### D.4.5.4 curveColengthDerivations

Procedure from library `curveInv.lib` (see [Section D.4.5 \[curveInv.lib\]](#), page 1049).

**Usage:** `curveColengthDerivations(I)`; I ideal

**Assume:** I is a radical ideal

**Return:** the colength of derivations of  $R/I$

**Remarks:** The procedure goes through all branches and computes the colength of derivations there. Then the d-dimension of the minimal primes is computed. After that, everything is summed up.

**Note:** the optional parameter can be used if the normalization has already been computed. If a list L contains the output of the procedure normal (with options prim, wd and using if the ring has a mixed ordering), apply curveColengthDerivations(I,L)

**Example:**

```
LIB "curveInv.lib";
////////////////////////////////////
// colength of derivations of curves //
////////////////////////////////////
// Example 1:
ring R = 0,(x,y,z),ds;
ideal I = x2-y4z,z3y2+xy2;
I = std(radical(I));
curveColengthDerivations(I);
↦ 9
// Example 2:
ring S = 0,(x,y),ds;
ideal I = (x+y)*(x2-y3);
curveColengthDerivations(I);
↦ 4
// Example 3:
ideal J = (x2-y3)*(x2+y2)*(x-y);
curveColengthDerivations(J);
↦ 15
```

## D.4.6 decomp\_lib

**Library:** decomp.lib

**Purpose:** Functional Decomposition of Polynomials

**Author:** Christian Gorzel, University of Muenster  
email: gorzelc@math.uni-muenster.de

**Overview:** This library implements functional uni-multivariate decomposition of multivariate polynomials.

A (multivariate) polynomial  $f$  is a composite if it can be written as

$g \circ h$  where  $g$  is univariate and  $h$  is multivariate, where  $\deg(g), \deg(h) > 1$ .

Uniqueness for monic polynomials is up to linear coordinate change  $g \circ h = g(x/c - d) \circ c(h(x) + d)$ .

If  $f$  is a composite, then `decompose(f)`; returns an ideal  $(g,h)$ ; such that  $\deg(g) < \deg(f)$  is maximal, ( $\deg(h) \geq 2$ ). The polynomial  $h$  is, by the maximality of  $\deg(g)$ , not a composite.

The polynomial  $g$  is univariate in the (first) variable `vvar` of  $f$ , such that  $\deg\_vvar(f)$  is maximal.

`decompose(f,1)`; computes a full decomposition, i.e. if  $f$  is a composite, then an ideal  $(g_1, \dots, g_m, h)$  is returned, where

$g_i$  are univariate and each entry is primitive such that

$$f = g_1 \circ \dots \circ g_m \circ h.$$

If  $f$  is not a composite, for instance if  $\deg(f)$  is prime, then `decompose(f)`; returns  $f$ .

The command `decompose` is the inverse: `compose(decompose(f,1))==f`.

Recall, that Chebyshev polynomials of the first kind commute by composition.

The decomposition algorithms work in the tame case, that is if  $\text{char}(\text{basering})=0$  or  $p:=\text{char}(\text{basering}) > 0$  but  $\deg(g)$  is not divisible by  $p$ . Additionally, it works for monic polynomials over  $Z$  and in some cases for monic polynomials over coefficient rings.

See `is_composite` for examples. (It also works over the reals but there it seems not be numerical stable.)

More information on the univariate resp. multivariate case.

Univariate decomposition is created, with the additional assumption  $\deg(g), \deg(h) > 1$ .

A multivariate polynomial  $f$  is a composite, if  $f$  can be written as

$g \circ h$ , where  $g$  is a univariate polynomial and  $h$

is multivariate. Note, that unlike in the univariate case, the polynomial

$h$  may be of degree 1.

E.g.  $f = (x + y)^2 + 2(x + y) + 1$  is the composite of

$g = x^2 + 2x + 1$  and  $h = x + y$ .

If `nvars(basering)>1`, then, by default, a single-variable multivariate polynomial is not considered to be the same as in the one-variable polynomial ring; it will always be decomposed. That is:

```
> ring r1=0,x,dp;
```

```
> decompose(x3+2x+1);
```

```
x3+2x+1
```

but:

```
> ring r2=0,(x,y),dp;
```

```
> decompose(x3+2x+1);
```

```
_ [1]=x3+2x+1
```

```
_ [2]=x
```

In particular:

```
is_composite(x3+2x+1)==1; in ring r1 but
```

```
is_composite(x3+2x+1)==0; in ring r2.
```

This is justified by interpreting the polynomial decomposition as an affine Stein factorization of the mapping  $f : k^n \rightarrow k, n \geq 2$ .

The behaviour can be changed by the some global variables.

`int DECMETH`; choose von zur Gathen's or Kozen-Landau's method.

`int MINS`; compute  $f = g \circ h$ , such that  $h(0) = 0$ .

`int IMPROVE`; simplify the coefficients of  $g$  and  $h$  if  $f$  is not monic.  
`int DEGONE`; single-variable multivariate are considered uni-variate.

See `decompopts`; for more information.

Additional information is displayed if `printlevel > 0`.

#### References:

- D. Kozen, S. Landau: Polynomial Decomposition Algorithms,  
 J. Symb. Comp. (1989), 7, 445-456.  
 J. von zu Gathen: Functional Decomposition of Polynomials: the Tame Case,  
 J. Symb. Comp. (1990), 9, 281-299.  
 J. von zur Gathen, J. Gerhard: Modern computer algebra,  
 Cambridge University Press, Cambridge, 2003.

#### Procedures: Auxiliary procedures:

##### D.4.6.1 `decompopts`

Procedure from library `decomp.lib` (see [Section D.4.6 \[decomp.lib\]](#), page 1052).

**Usage:** `decompopts()`; or `decompopts("reset")`;

**Return:** nothing

**Note:** in the first case, it shows the setting of the control parameters;  
 in the second case, it kills the user-defined control parameters and  
 resets to the default setting which will then be displayed.

`int DECMETH`; Method for computing the univariate decomposition  
 0 : (default) Kozen-Landau  
 1 : von zur Gathen

`int IMPROVE` Choice of coefficients for the decomposition

$(g_1, \dots, g_l, h)$  of a non-monic polynomials  $f$ .  
 0 : `leadcoef(  $g_1$  ) = leadcoef(  $f$  )` and  $g_2, \dots, g_l, h$  are monic  
 1 : (default), `content(  $g_i$  ) = 1`

`int MINS`

$f = g \circ h, (g_1, \dots, g_m, h)$  of a non-monic polynomials  $f$ .  
 0 : `g(0) = f(0), h(0) = 0` [ueberlegen fuer complete]  
 1 : (default), `g(0)=0, h(0) = f(0)`  
 2 : Tschirnhaus

`int DECORDER`; The order in which the decomposition will be computed  
 0 : minfirst  
 1 : (default) maxfirst

`int DEGONE`; decompose also polynomials built on linear ones  
 0 : (default)  
 1 :

**Example:**

```
LIB "decomp.lib";
decompopts();
⇒
⇒ === Global variables for decomp.lib ===
⇒
⇒ -- DECMETH (int) not defined, implicitly 1
⇒ -- MINS (int) not defined, implicitly 0
⇒ -- IMPROVE (int) not defined, implicitly 1
```

**D.4.6.2 decompose**

Procedure from library `decomp.lib` (see [Section D.4.6 \[decomp.lib\]](#), page 1052).

**Usage:**     `decompose(f); f poly`  
               `decompose(f,1); f poly`

**Return:**    `poly`, the input, if `f` is not a composite  
               `ideal`, if the input is a composite

**Note:**      computes a full decomposition if called by the second variant

**See:**        `compose`

**Example:**

```
LIB "decomp.lib";
ring r2 = 0,(x,y),dp;
decompose(((x3+2y)^6+x3+2y)^4);
⇒ _[1]=x24+4x19+6x14+4x9+x4
⇒ _[2]=x3+2y
// complete decomposition
decompose(((x3+2y)^6+x3+2y)^4,1);
⇒ _[1]=x2
⇒ _[2]=x2
⇒ _[3]=x6+x
⇒ _[4]=x3+2y
// -----
// decompose over the integers
ring rZ = integer,x,dp;
decompose(compose(ideal(x3,x2+2x,x3+2)),1);
⇒ _[1]=x3
⇒ _[2]=x2-1
⇒ _[3]=x3+3
// -----
// prime characteristic
ring r7 = 7,x,dp;
decompose(compose(ideal(x2+x,x7))); // tame case
⇒ _[1]=x2+x
⇒ _[2]=x7
// -----
decompose(compose(ideal(x7+x,x2))); // wild case
⇒ x14+x2
// -----
ring ry = (0,y),x,dp; // y is now a parameter
compose(x2+yx+5,x5-2yx3+x);
```

```

⇒ x10+(-4y)*x8+(4y2+2)*x6+(y)*x5+(-4y)*x4+(-2y2)*x3+x2+(y)*x+5
decompose(_);
⇒ _[1]=1/4*x2+(-y2+20)/4
⇒ _[2]=2*x5+(-4y)*x3+2*x+(y)
// Usage of variable IMPROVE
ideal J = x2+10x, 64x7-112x5+56x3-7x, 4x3-3x;
decompose(compose(J),1);
⇒ _[1]=x2+10*x
⇒ _[2]=64*x7-112*x5+56*x3-7*x
⇒ _[3]=4*x3-3*x
int IMPROVE=0;
exportto(Decomp,IMPROVE);
decompose(compose(J),1);
⇒ _[1]=1099511627776*x2+10485760*x
⇒ _[2]=x7-7/64*x5+7/2048*x3-7/262144*x
⇒ _[3]=x3-3/4*x

```

### D.4.6.3 is\_composite

Procedure from library `decomp.lib` (see [Section D.4.6 \[decomp.lib\]](#), page 1052).

**Usage:**      `is_composite(f); f poly`

**Return:**    `int`  
               1, if `f` is decomposable  
               0, if `f` is not decomposable  
               -1, if `char(basering)>0` and `deg(f)` is divisible by `char(basering)` but no decomposition  
               has been found.

**Note:**      The last case means that it could exist a decomposition  $f=g \circ h$  with  
 $\text{char}(\text{basering}) \mid \text{deg}(g)$ , but this wild case cannot be decided by the algorithm.  
               Some additional information will be displayed when called by the user.

**Example:**

```

LIB "decomp.lib";
ring r0 = 0,x,dp;
is_composite(x4+5x2+6);      // biquadratic polynomial
⇒ 1
is_composite(2x2+x+1);      // prime degree
⇒ The degree is prime.
⇒ 0
// -----
// polynomial ring with several variables
ring R = 0,(x,y),dp;
// -----
// single-variable multivariate polynomials
is_composite(2x+1);
⇒ 1
is_composite(2x2+x+1);
⇒ 1
// -----
// prime characteristic
ring r7 = 7,x,dp;
is_composite(compose(ideal(x2+x,x14)));      // is_composite(x14+x7);
⇒ 1

```

```

is_composite(compose(ideal(x14+x,x2)));      // is_composite(x14+x2);
 $\mapsto$  // -- Warning: wild case, cannot decide whether the polynomial has a
 $\mapsto$  // -- decomposition goh with deg(g) divisible by char(basing) = 7.
 $\mapsto$  -1

```

#### D.4.6.4 chebyshev

Procedure from library `decomp.lib` (see [Section D.4.6 \[decomp.lib\]](#), page 1052).

**Usage:**      `chebyshev(n); n int, n >= 0`  
               `chebyshev(n,c); n int, n >= 0, c number, c!=0`

**Return:**    `poly`, the [monic]  $n$ th Chebyshev polynomial of the first kind.  
               The polynomials are defined in the first variable, say  $x$ , of the basering.

**Note:**       The (generalized) Chebyshev polynomials of the first kind can be defined by the recursion:  $C_0 = c$ ,  $C_1 = x$ ,  $C_n = 2/c \cdot x \cdot C_{n-1} - C_{n-2}$ ,  $n \geq 2, c \neq 0$ .  
               These polynomials commute by composition:  
                $C_m \circ C_n = C_n \circ C_m$ .  
               For  $c=1$ , we obtain the standard (non monic) Chebyshev polynomials  $T_n$  which satisfy  $T_n(x) = \cos(n \cdot \arccos(x))$ .  
               For  $c=2$  (default), we obtain the monic Chebyshev polynomials  $P_n$  which satisfy the relation  $P_n(x + 1/x) = x^n + 1/x^n$ .  
               By default the monic Chebyshev polynomials are returned:  
                $P_n = \text{chebyshev}(n)$  and  $T_n = \text{chebyshev}(n,1)$ .  
               It holds  $P_n(x) = 2 \cdot T_n(x/2)$  and more generally  
                $C_n(c \cdot x) = c \cdot T_n(x)$   
               That is `subst(chebyshev(n,c),var(1),c*var(1))= c*chebyshev(n,1)`.  
               If `char(basing) = 2`, then  
                $C_0 = 1, C_1 = x, C_2 = 1, C_3 = x$ , and so on.

**Example:**

```

LIB "decomp.lib";
ring r = 0,x,lp;
// The monic Chebyshev polynomials
chebyshev(0);
 $\mapsto$  2
chebyshev(1);
 $\mapsto$  x
chebyshev(2);
 $\mapsto$  x2-2
chebyshev(3);
 $\mapsto$  x3-3x
// These polynomials commute
compose(chebyshev(2),chebyshev(6)) ==
compose(chebyshev(6),chebyshev(2));
 $\mapsto$  1
// The standard Chebyshev polynomials
chebyshev(0,1);
 $\mapsto$  1
chebyshev(1,1);
 $\mapsto$  x

```

```

chebyshev(2,1);
↦ 2x2-1
chebyshev(3,1);
↦ 4x3-3x
// -----
// The relation for the various Chebyshev polynomials
5*chebyshev(3,1)==subst(chebyshev(3,5),x,5x);
↦ 1
// -----
// char 2 case
ring r2 = 2,x,dp;
chebyshev(2);
↦ 1
chebyshev(3);
↦ x

```

### D.4.6.5 compose

Procedure from library `decomp.lib` (see [Section D.4.6 \[decomp.lib\], page 1052](#)).

**Usage:**     `compose(f1,...,fn);`  $f_1, \dots, f_n$  poly  
               `compose(I);`  $I$  ideal,

**Assume:**    the ideal consists of  $n = \text{ncols}(I) \geq 1$  entries,  
               where  $I[1], \dots, I[n-1]$  are univariate in the same variable  
               but  $I[n]$  may be multivariate.

**Return:**     poly, the composition  $I[1](I[2](\dots I[n]))$

**Note:**       this procedure is the inverse of `decompose`

**See:**         `decompose`

**Example:**

```

LIB "decomp.lib";
ring r = 0,(x,y),dp;
compose(x3+1,x2,y3+x);
↦ y18+6xy15+15x2y12+20x3y9+15x4y6+6x5y3+x6+1
// or the input as one ideal
compose(ideal(x3+1,x2,x3+y));
↦ x18+6x15y+15x12y2+20x9y3+15x6y4+6x3y5+y6+1

```

### D.4.6.6 makedistinguished

Procedure from library `decomp.lib` (see [Section D.4.6 \[decomp.lib\], page 1052](#)).

**Usage:**     `makedistinguished(f,vvar);`  $f$ ,  $vvar$  poly; where  $vvar$  is a ring variable

**Return:**     (poly, ideal): the transformed polynomial and an ideal defining the map which reverses the transformation.

**Purpose:**     let  $vvar = \text{var}(1)$ . Then  $f$  is transformed by a random linear coordinate change  
 $\text{phi} = (\text{var}(1), \text{var}(2)+c_2*vvar, \dots, \text{var}(n)+c_n*vvar)$   
               such that  $\text{phi}(f) = f \circ \text{phi}$  becomes distinguished with respect to  $vvar$ . That is, the  
               new polynomial contains the monomial  $vvar^d$ , where  $d$  is the degree of  $f$ .  
               If already  $f$  is distinguished w.r.t.  $vvar$ , then  $f$  is left unchanged and the re-  
               transformation is the identity.



**Note 1:** (this proc correctly works independent of the term ordering.) to apply the reverse transformation, either define a map or use substitute (to be loaded from poly.lib).

**Note 2:** If  $p = \text{char}(\text{basering}) > 0$ , then there exist polynomials of degree  $d \geq p$ , e.g.  $(p-1)x^p y + xy^p$ , that cannot be transformed to a vvar-distinguished polynomial. In this case, \*p random trials will be made and the proc may leave with an ERROR message.

**Example:**

```
LIB "decomp.lib";
int randval = system("--random"); // store initial value
system("--random",0815);
ring r = 0,(x,y),dp;
poly g;
map phi;
// -----
// Example 1:
poly f = 3xy4 + 2xy2 + x5y3 + x + y6; // degree 8
// make the polynomial y-distinguished
g, phi = makedistinguished(f,y);
g;
⇒ x5y3+5x4y4+10x3y5+10x2y6+5xy7+y8+y6+3xy4+3y5+2xy2+2y3+x+y
phi;
⇒ phi[1]=x-y
⇒ phi[2]=y
// to reverse the transformation apply the map
f == phi(g);
⇒ 1
// -----
// Example 2:
// The following polynomial is already x-distinguished
f = x6+y4+xy;
g,phi = makedistinguished(f,x);
g; // f is left unchanged
⇒ x6+y4+xy
phi; // the transformation is the identity.
⇒ phi[1]=x
⇒ phi[2]=y
system("--random",randval); // reset random generator
// -----
// Example 3: // polynomials which cannot be transformed
// If p=char(basering)>0, then (p-1)*x^p*y + x*y^p factorizes completely
// in linear factors, since (p-1)*x^p+x equiv 0 on F_p. Hence,
// such polynomials cannot be transformed to a distinguished polynomial.
ring r3 = 3,(x,y),dp;
makedistinguished(2x3y+xy3,y);
⇒ ? it could not be transform to a y-distinguished polynomial.
⇒ ? leaving decomp.lib::makedistinguished (0)
```

#### D.4.7 elim\_lib

**Library:** elim.lib

**Purpose:** Elimination, Saturation and Blowing up

**Procedures:****D.4.7.1 blowup0**

Procedure from library `elim.lib` (see [Section D.4.7 \[elim.lib\]](#), page 1059).

- Usage:** `blowup0(J,C [,W]);` J,C,W ideals  
 C = ideal of center of blowup, J = ideal to be blown up, W = ideal of ambient space
- Assume:** inclusion of ideals : W in J, J in C.  
 If not, the procedure replaces J by J+W and C by C+J+W
- Return:** a ring, say B, containing the ideals C,J,W and the ideals  
 - bR (ideal defining the blown up basering)  
 - aS (ideal of blown up ambient space)  
 - eD (ideal of exceptional divisor)  
 - tT (ideal of total transform)  
 - sT (ideal of strict transform)  
 - bM (ideal of the blowup map from basering to B)  
 such that B/bR is isomorphic to the blowup ring BC.
- Purpose:** compute the projective blowup of the basering in the center C, the exceptional locus, the total and strict transform of J, and the blowup map.  
 The projective blowup is a presentation of the blowup ring  $BC = R[C] = R + t^*C + t^2C^2 + \dots$  (also called Rees ring) of the ideal C in the ring basering R.
- Theory:** If basering =  $K[x_1, \dots, x_n]$  and  $C = \langle f_1, \dots, f_k \rangle$  then let  $B = K[x_1, \dots, x_n, y_1, \dots, y_k]$  and aS the preimage in B of W under the map  $B \rightarrow K[x_1, \dots, x_n, t]$ ,  $x_i \rightarrow x_i$ ,  $y_i \rightarrow t^*f_i$ . aS is homogeneous in the variables  $y_i$  and defines a variety  $Z = V(aS)$  in  $A^n \times P^{(k-1)}$ , the ambient space of the blowup of  $V(W)$ . The projection  $Z \rightarrow A^n$  is an isomorphism outside the preimage of the center  $V(C)$  in  $A^n$  and is called the blowup of the center. The preimage of  $V(C)$  is called the exceptional set, the preimage of  $V(J)$  is called the total transform of  $V(J)$ . The strict transform is the closure of (total transform minus the exceptional set).  
 If  $C = \langle x_1, \dots, x_n \rangle$  then  $aS = \langle y_i * x_j - y_j * x_i \mid i, j = 1, \dots, n \rangle$  and Z is the blowup of  $A^n$  in 0, the exceptional set is  $P^{(k-1)}$ .
- Note:** The procedure creates a new ring with variables  $y(1..k)$  and  $x(1..n)$  where  $n = \text{nvars}(\text{basing})$  and  $k = \text{ncols}(C)$ . The ordering is a block ordering where the x-block has the ordering of the basering and the y-block has ordering dp if C is not homogeneous  
 resp. the weighted ordering  $wp(b_1, \dots, b_k)$  if C is homogeneous with  $\deg(C[i]) = b_i$ .

**Example:**

```
LIB "elim.lib";
ring r = 0, (x,y), dp;
poly f = x2+y3;
ideal C = x,y;           //center of blowup
def B1 = blowup0(f,C);
setring B1;
aS;                       //ideal of blown up ambient space
⇨ aS[1]=x(1)*y(2)-x(2)*y(1)
tT;                       //ideal of total transform of f
⇨ tT[1]=x(1)*y(2)-x(2)*y(1)
⇨ tT[2]=x(2)^3+x(1)^2
```

```

sT;                                //ideal of strict transform of f
⇨ sT[1]=x(2)*y(2)^2+y(1)^2
eD;                                //ideal of exceptional divisor
⇨ eD[1]=x(2)
⇨ eD[2]=x(1)
bM;                                //ideal of blowup map r --> B1
⇨ bM[1]=x(1)
⇨ bM[2]=x(2)
ring R = 0,(x,y,z),ds;
poly f = y2+x3+z5;
ideal C = y2,x,z;
ideal W = z-x;
def B2 = blowup0(f,C,W);
setring B2;
B2;                                //weighted ordering
⇨ // coefficients: QQ
⇨ // number of vars : 6
⇨ //      block   1 : ordering ds
⇨ //                : names    x(1) x(2) x(3)
⇨ //      block   2 : ordering wp
⇨ //                : names    y(1) y(2) y(3)
⇨ //                : weights    2    1    1
⇨ //      block   3 : ordering C
bR;                                //ideal of blown up R
⇨ bR[1]=x(1)*y(1)-x(2)^2*y(2)
⇨ bR[2]=x(3)*y(1)-x(2)^2*y(3)
⇨ bR[3]=-x(1)*y(3)+x(3)*y(2)
aS;                                //ideal of blown up R/W
⇨ aS[1]=x(1)*y(1)-x(2)^2*y(2)
⇨ aS[2]=x(1)*y(3)-x(3)*y(2)
⇨ aS[3]=x(1)-x(3)
⇨ aS[4]=x(3)*y(1)-x(2)^2*y(3)
sT;                                //strict transform of f
⇨ sT[1]=y(1)+x(3)^2*y(3)+x(3)^4*y(3)
eD;                                //ideal of exceptional divisor
⇨ eD[1]=x(1)*y(1)-x(2)^2*y(2)
⇨ eD[2]=x(1)*y(3)-x(3)*y(2)
⇨ eD[3]=x(1)
⇨ eD[4]=x(3)
⇨ eD[5]=x(2)^2*y(3)
⇨ eD[6]=x(2)^2
//Note that the different affine charts are {y(i)=1}

```

See also: [Section D.5.15.1 \[blowUp\], page 1529](#); [Section D.5.15.2 \[blowUp2\], page 1530](#).

### D.4.7.2 elimRing

Procedure from library `elim.lib` (see [Section D.4.7 \[elim.lib\], page 1059](#)).

**Usage:** `elimRing(vars [,w,str]);` vars = product of variables to be eliminated (type poly), w = intvec (specifying weights for all variables), str = string either "a" or "b" (default: w=ringweights, str="a")

**Return:** a list, say L, with R:=L[1] a ring and L[2] an intvec. The ordering in R is an elimination ordering for the variables appearing in vars depending on "a" resp. "b". Let w1

(resp.  $w_2$ ) be the intvec of weights of the variables to be eliminated (resp. not to be eliminated).

The monomial ordering of  $R$  has always 2 blocks, the first block corresponds to the (given) variables to be eliminated.

If  $str = "a"$  the first block is  $a(w_1, 0..0)$  and the second block is  $wp(w)$  resp.  $ws(w)$  if the first variable not to be eliminated is local.

If  $str = "b"$  the 1st block has ordering  $wp(w_1)$  and the 2nd block is  $wp(w_2)$  resp.  $ws(w_2)$  if the first variable not to be eliminated is local.

If the basering is a quotient ring  $P/Q$ , then  $R$  is also a quotient ring with  $Q$  replaced by a standard basis of  $Q$  w.r.t. the new ordering (parameters are not touched).

The intvec  $L[2]$  is the intvec of variable weights (or the given  $w$ ) with weights  $\leq 0$  replaced by 1.

**Purpose:** Prepare a ring for eliminating vars from an ideal/module by computing a standard basis in  $R$  with a fast monomial ordering. This procedure is used by the procedure `elim`.

**Example:**

```
LIB "elim.lib";
ring R = 0,(x,y,z,u,v),(c,lp);
def P = elimRing(yu); P;
-- [1]:
-- // coefficients: QQ
-- // number of vars : 5
-- //      block   1 : ordering a
-- //              : names   y u x z v
-- //              : weights 1 1 0 0 0
-- //      block   2 : ordering dp
-- //              : names   y u x z v
-- //      block   3 : ordering C
-- [2]:
--      1,1,1,1,1
intvec w = 1,1,3,4,5;
elimRing(yu,w);
-- [1]:
-- // coefficients: QQ
-- // number of vars : 5
-- //      block   1 : ordering a
-- //              : names   y u x z v
-- //              : weights 1 4 0 0 0
-- //      block   2 : ordering wp
-- //              : names   y u x z v
-- //              : weights 1 1 3 4 5
-- //      block   3 : ordering C
-- [2]:
--      1,1,3,4,5
ring S = (0,a),(x,y,z,u,v),ws(1,2,3,4,5);
minpoly = a2+1;
qring T = std(ideal(x+y2+v3,(x+v)^2));
def Q = elimRing(yv)[1];
setring Q; Q;
-- // coefficients: QQ[a]/(a2+1)
-- // number of vars : 5
```

```

⇒ //      block    1 : ordering a
⇒ //      : names    y v x z u
⇒ //      : weights  2 5 0 0 0
⇒ //      block    2 : ordering ws
⇒ //      : names    y v x z u
⇒ //      : weights  1 2 3 4 5
⇒ //      block    3 : ordering C
⇒ // quotient ring from ideal
⇒ _[1]=y2+2*yu+u2
⇒ _[2]=v2+y+u3

```

### D.4.7.3 elim

Procedure from library `elim.lib` (see [Section D.4.7 \[elim.lib\]](#), page 1059).

**Usage:** `elim(id,arg[,s]);` `id` ideal/module, `arg` can be either an intvec `v` or a product `p` of variables (type `poly`), `s` a string determining the method which can be "slimgb" or "std" or, additionally, "withWeights".

**Return:** ideal/module obtained from `id` by eliminating either the variables with indices appearing in `v` or the variables appearing in `p`. Does not work in a qring.

**Method:** `elim` uses `elimRing` to create a ring with an elimination ordering for the variables to be eliminated and then applies `std` if "std" is given, or `slimgb` if "slimgb" is given, or a heuristically chosen method.

If the variables in the basering have weights these weights are used in `elimRing`. If a string "withWeights" as (optional) argument is given Singular computes weights for the variables to make the input as homogeneous as possible.

The method is different from that used by `eliminate` and `elim1`; depending on the example, any of these commands can be faster.

**Note:** No special monomial ordering is required, i.e. the ordering can be local or mixed. The result is a SB with respect to the ordering of the second block used by `elimRing`. E.g. if the first var not to be eliminated is global, resp. local, this ordering is `dp`, resp. `ds` (or `wp`, resp. `ws`, with the given weights for these variables). If `printlevel > 0` the ring for which the output is a SB is shown.

**Example:**

```

LIB "elim.lib";
ring r=0,(x,y,u,v,w),dp;
ideal i=x-u,y-u2,w-u3,v-x+y3;
elim(i,3..4);
⇒ _[1]=y2-xw
⇒ _[2]=xy-w
⇒ _[3]=x2-y
elim(i,uv);
⇒ _[1]=y2-xw
⇒ _[2]=xy-w
⇒ _[3]=x2-y
int p = printlevel;
printlevel = 2;
elim(i,uv,"withWeights","slimgb");
⇒ // result is a SB in the following ring:
⇒ // coefficients: QQ
⇒ // number of vars : 5

```

```

⇒ //      block   1 : ordering a
⇒ //      : names   u v x y w
⇒ //      : weights 5 21 0 0 0
⇒ //      block   2 : ordering wp
⇒ //      : names   u v x y w
⇒ //      : weights 5 7 5 21 15
⇒ //      block   3 : ordering C
⇒ _[1]=x3-w
⇒ _[2]=-x2+y
printlevel = p;
ring S = (0,a),(x,y,z,u,v),ws(1,2,3,4,5);
minpoly = a2+1;
ideal i=x-u,y-u2,az-u3,v-x+ay3;
module m=i*gen(1)+i*gen(2);
m=elim(m,xy);
show(m);
⇒ // module, 4 generator(s)
⇒ [0,z+(a)*u3]
⇒ [z+(a)*u3]
⇒ [0,u-v+(-a)*u6]
⇒ [u-v+(-a)*u6]

```

See also: [Section D.4.7.4 \[elim1\]](#), page 1064; [Section 5.1.28 \[eliminate\]](#), page 176.

#### D.4.7.4 elim1

Procedure from library `elim.lib` (see [Section D.4.7 \[elim.lib\]](#), page 1059).

**Usage:** `elim1(id,arg)`; `id` ideal/module, `arg` can be either an intvec `v` or a product `p` of variables (type `poly`)

**Return:** ideal/module obtained from `id` by eliminating either the variables with indices appearing in `v` or the variables appearing in `p`

**Method:** `elim1` calls `eliminate` but in a ring with ordering `dp` (resp. `ls`) if the first var not to be eliminated belongs to a `-p` (resp. `-s`) ordering.

**Note:** no special monomial ordering is required.

**Example:**

```

LIB "elim.lib";
ring r=0,(x,y,t,s,z),dp;
ideal i=x-t,y-t2,z-t3,s-x+y3;
elim1(i,ts);
⇒ _[1]=y2-xz
⇒ _[2]=xy-z
⇒ _[3]=x2-y
module m=i*gen(1)+i*gen(2);
m=elim1(m,3..4); show(m);
⇒ // module, 6 generator(s)
⇒ [y2-xz]
⇒ [0,y2-xz]
⇒ [xy-z]
⇒ [0,xy-z]
⇒ [x2-y]
⇒ [0,x2-y]

```

See also: [Section D.4.7.3 \[elim\]](#), page 1063; [Section 5.1.28 \[eliminate\]](#), page 176.

### D.4.7.5 elim2

Procedure from library `elim.lib` (see [Section D.4.7 \[elim.lib\], page 1059](#)).

**Usage:** `elim2(id,v)`; `id` ideal/module, `v` intvec

**Returns:** ideal/module obtained from `id` by eliminating variables in `v`

**Note:** no special monomial ordering is required, result is a SB with respect to ordering `dp` (resp. `ls`) if the first var not to be eliminated belongs to a `-p` (resp. `-s`) blockordering

**Example:**

```
LIB "elim.lib";
ring r=0,(x,y,u,v,w),dp;
ideal i=x-u,y-u2,w-u3,v-x+y3;
elim2(i,3..4);
↳ _[1]=y2-xw
↳ _[2]=xy-w
↳ _[3]=x2-y
module m=i*gen(1)+i*gen(2);
m=elim2(m,3..4);show(m);
↳ // module, 6 generator(s)
↳ [y2-xw]
↳ [0,y2-xw]
↳ [xy-w]
↳ [0,xy-w]
↳ [x2-y]
↳ [0,x2-y]
```

See also: [Section D.4.7.3 \[elim\], page 1063](#); [Section D.4.7.4 \[elim1\], page 1064](#); [Section 5.1.28 \[eliminate\], page 176](#).

### D.4.7.6 nselect

Procedure from library `elim.lib` (see [Section D.4.7 \[elim.lib\], page 1059](#)).

**Usage:** `nselect(id,v)`; `id` = ideal, module or matrix, `v` = intvec

**Return:** generators (or columns) of `id` not containing the variables with index an entry of `v`

**Example:**

```
LIB "elim.lib";
ring r=0,(x,y,t,s,z),(c,dp);
ideal i=x-y,y-z2,z-t3,s-x+y3;
nselect(i,3);
↳ _[1]=x-y
↳ _[2]=-z2+y
↳ _[3]=y3-x+s
module m=i*(gen(1)+gen(2));
m;
↳ m[1]=[x-y,x-y]
↳ m[2]=[-z2+y,-z2+y]
↳ m[3]=[-t3+z,-t3+z]
↳ m[4]=[y3-x+s,y3-x+s]
nselect(m,3..4);
↳ _[1]=[x-y,x-y]
↳ _[2]=[-z2+y,-z2+y]
```

```
nselect(matrix(m),3..4);
↳ _[1,1]=x-y
↳ _[1,2]=-z2+y
↳ _[2,1]=x-y
↳ _[2,2]=-z2+y
```

See also: [Section D.4.7.9 \[select\]](#), page 1067; [Section D.4.7.10 \[select1\]](#), page 1067.

### D.4.7.7 sat

Procedure from library `elim.lib` (see [Section D.4.7 \[elim.lib\]](#), page 1059).

**Usage:** `sat(id,j)`; `id=ideal/module`, `j=ideal`

**Return:** `ideal/module`  
saturation of `id` with respect to `j` (= `union_(k=1...) of id:j^k`)

**Note:** result is a standard basis in the basering

**Example:**

```
LIB "elim.lib";
ring r      = 2,(x,y,z),dp;
poly F      = x5+y5+(x-y)^2*xyz;
ideal j      = jacob(F);
sat(j,maxideal(1));
↳ _[1]=x3+x2y+xy2+y3
↳ _[2]=y4+x2yz+y3z
↳ _[3]=x2y2+y4
sat(j,maxideal(2));
↳ _[1]=x3+x2y+xy2+y3
↳ _[2]=y4+x2yz+y3z
↳ _[3]=x2y2+y4
```

See also: [Section D.4.14.2 \[modSat\]](#), page 1122; [Section D.4.7.8 \[sat\\_with\\_exp\]](#), page 1066.

### D.4.7.8 sat\_with\_exp

Procedure from library `elim.lib` (see [Section D.4.7 \[elim.lib\]](#), page 1059).

**Usage:** `sat(id,j)`; `id=ideal/module`, `j=ideal`

**Return:** list of an ideal/module [1] and an integer [2]:  
[1] = saturation of `id` with respect to `j` (= `union_(k=1...) of id:j^k`) [2] = saturation exponent (= `min( k | id:j^k = id:j^(k+1) )`)

**Note:** [1] is a standard basis in the basering

**Example:**

```
LIB "elim.lib";
ring r      = 2,(x,y,z),dp;
poly F      = x5+y5+(x-y)^2*xyz;
ideal j      = jacob(F);
sat(j,maxideal(1));
↳ _[1]=x3+x2y+xy2+y3
↳ _[2]=y4+x2yz+y3z
↳ _[3]=x2y2+y4
sat(j,maxideal(2));
↳ _[1]=x3+x2y+xy2+y3
```



```

↳ _[2]=y4+x2yz+y3z
↳ _[3]=x2y2+y4

```

See also: [Section D.4.14.2 \[modSat\]](#), page 1122; [Section D.4.7.7 \[sat\]](#), page 1066.

### D.4.7.9 select

Procedure from library `elim.lib` (see [Section D.4.7 \[elim.lib\]](#), page 1059).

**Usage:** `select(id,n[,m]);` id = ideal/module/matrix, v = intvec

**Return:** generators/columns of id containing all variables with index an entry of v

**Note:** use 'select1' for selecting generators/columns containing at least one of the variables with index an entry of v

**Example:**

```

LIB "elim.lib";
ring r=0,(x,y,t,s,z),(c,dp);
ideal i=x-y,y-z2,z-t3,s-x+y3;
ideal j=select(i,1);
j;
↳ j[1]=x-y
↳ j[2]=y3-x+s
module m=i*(gen(1)+gen(2));
m;
↳ m[1]=[x-y,x-y]
↳ m[2]=[-z2+y,-z2+y]
↳ m[3]=[-t3+z,-t3+z]
↳ m[4]=[y3-x+s,y3-x+s]
select(m,1..2);
↳ _[1]=[x-y,x-y]
↳ _[2]=[y3-x+s,y3-x+s]
select(matrix(m),1..2);
↳ _[1,1]=x-y
↳ _[1,2]=y3-x+s
↳ _[2,1]=x-y
↳ _[2,2]=y3-x+s

```

See also: [Section D.4.7.6 \[nselect\]](#), page 1065; [Section D.4.7.10 \[select1\]](#), page 1067.

### D.4.7.10 select1

Procedure from library `elim.lib` (see [Section D.4.7 \[elim.lib\]](#), page 1059).

**Usage:** `select1(id,v);` id = ideal/module/matrix, v = intvec

**Return:** generators/columns of id containing at least one of the variables with index an entry of v

**Note:** use 'select' for selecting generators/columns containing all variables with index an entry of v

**Example:**

```

LIB "elim.lib";
ring r=0,(x,y,t,s,z),(c,dp);
ideal i=x-y,y-z2,z-t3,s-x+y3;
ideal j=select1(i,1);j;

```

```

↳ j[1]=x-y
↳ j[2]=y3-x+s
module m=i*(gen(1)+gen(2)); m;
↳ m[1]=[x-y,x-y]
↳ m[2]=[-z2+y,-z2+y]
↳ m[3]=[-t3+z,-t3+z]
↳ m[4]=[y3-x+s,y3-x+s]
select1(m,1..2);
↳ _[1]=[x-y,x-y]
↳ _[2]=[-z2+y,-z2+y]
↳ _[3]=[y3-x+s,y3-x+s]
select1(matrix(m),1..2);
↳ _[1,1]=x-y
↳ _[1,2]=-z2+y
↳ _[1,3]=y3-x+s
↳ _[2,1]=x-y
↳ _[2,2]=-z2+y
↳ _[2,3]=y3-x+s

```

See also: [Section D.4.7.6 \[nselect\]](#), page 1065; [Section D.4.7.9 \[select\]](#), page 1067.

## D.4.8 ellipticcovers\_lib

**Library:** ellipticCovers.lib

**Purpose:** Gromov-Witten numbers of elliptic curves

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**Overview:** We implement a formula for computing the number of covers of elliptic curves. It has been obtained by proving mirror symmetry for arbitrary genus by tropical methods in [BBM]. A Feynman graph of genus  $g$  is a trivalent, connected graph of genus  $g$  (with  $2g-2$  vertices and  $3g-3$  edges). The branch type  $b=(b_1,\dots,b_{(3g-3)})$  of a stable map is the multiplicity of the edge  $i$  over a fixed base point.

Given a Feynman graph  $G$  and a branch type  $b$ , we obtain the number  $N_-(G,b)$  of stable maps of branch type  $b$  from a genus  $g$  curve of topological type  $G$  to the elliptic curve by computing a path integral over a rational function. The path integral is computed as a residue.

The sum of  $N_-(G,b)$  over all branch types  $b$  of sum  $d$  gives  $N_-(G,d)*|Aut(G)|$ , with the Gromov-Witten invariant  $N_-(G,d)$  of degree  $d$  stable maps from a genus  $g$  curve of topological type  $G$  to the elliptic curve.

The sum of  $N_-(G,d)$  over all such graphs gives the usual Gromov-Witten invariant  $N_-(g,d)$  of degree  $d$  stable maps from a genus  $g$  curve to the elliptic curve.

The key function computing the numbers  $N_-(G,b)$  and  $N_-(G,d)$  is `gromovWitten`.

**References:**

[BBM] J. Boehm, A. Buchholz, H. Markwig: Tropical mirror symmetry for elliptic curves, arXiv:1309.5893 (2013).

**Types:** graph

**Procedures:**

### D.4.8.1 makeGraph

Procedure from library `ellipticcovers.lib` (see [Section D.4.8 \[ellipticcovers.lib\]](#), page 1068).

**Usage:**      `makeGraph(v,e);` v list, e list

**Assume:**    v is a list of integers, e is a list of two element lists of v.

**Return:**     graph with vertices v and edges e

**Theory:**     Creates a graph from a list of vertices and edges. The vertices can be any type.

**Example:**

```
LIB "ellipticcovers.lib";
ring R=(0,x1,x2,x3,x4),(q1,q2,q3,q4,q5,q6),dp;
graph G = makeGraph(list(1,2,3,4),list(list(1,3),list(1,2),list(1,2),list(2,4),list(3,4)));
G;
↳ [[1, 3], [1, 2], [1, 2], [2, 4], [3, 4], [3, 4]]
↳ Graph with 4 vertices and 6 edges
↳
```

### D.4.8.2 printGraph

Procedure from library `ellipticcovers.lib` (see [Section D.4.8 \[ellipticcovers.lib\]](#), page 1068).

**Usage:**      `printGraph(G);` G graph

**Assume:**    G is a graph.

**Theory:**     This is the print function used by Singular to print a graph.

**Example:**

```
LIB "ellipticcovers.lib";
ring R=(0,x1,x2,x3,x4),(q1,q2,q3,q4,q5,q6),dp;
graph G = makeGraph(list(1,2,3,4),list(list(1,3),list(1,2),list(1,2),list(2,4),list(3,4)));
G;
↳ [[1, 3], [1, 2], [1, 2], [2, 4], [3, 4], [3, 4]]
↳ Graph with 4 vertices and 6 edges
↳
```

### D.4.8.3 propagator

Procedure from library `ellipticcovers.lib` (see [Section D.4.8 \[ellipticcovers.lib\]](#), page 1068).

**Usage:**      `propagator(xy,d);` xy list, d int  
               `propagator(G,b);` G graph, b list

**Assume:**    xy is a list of two numbers x and y in a rational function field, d non-negative integer.  
               G is a Feynman graph, a is a list of integers of length equal to the number of edges of G.

We assume that the coefficient ring has one rational variable for each vertex of G.

**Return:**     number, the propagator associated to the input data.

**Theory:** If  $xy$  and  $d$  are specified, then the function returns  $x^2y^2/(x^2-y^2)^2$  for  $d=0$ , which is associated to an edge with vertices  $x$  and  $y$  not passing above the base point. For  $d>0$  it returns the sum of  $(j^*x^{(4*j)}+j^*y^{(4*j)})/(x^*y)^{(2*j)}$  over all divisors  $j$  of  $d$ , which is associated to an edge with vertices  $x$  and  $y$  passing with multiplicity  $d$  above the base point.

Essentially the variables  $x$  and  $y$  stand for the position of the base points.

In the second way of using this function,  $G$  is a Feynman graph and  $b$  is a branch type over a fixed base point of a cover with source  $G$  and target an elliptic curve. It returns the product of  $\text{propagator}(\text{list}(v[i],w[i]),b[i])$  over all edges  $i$  with multiplicity  $b[i]$  over the base point and vertices  $v[i]$  and  $w[i]$ .

**Example:**

```
LIB "ellipticcovers.lib";
ring R=(0,x1,x2,x3,x4),(q1,q2,q3,q4,q5,q6),dp;
graph G = makeGraph(list(1,2,3,4),list(list(1,3),list(1,2),list(1,2),list(2,4),list(3,4)),list(1,1,1,0,0,0));
propagator(list(x1,x2),0);
⇒ (x1^2*x2^2)/(x1^4-2*x1^2*x2^2+x2^4)
propagator(list(x1,x2),2);
⇒ (2*x1^8+x1^6*x2^2+x1^2*x2^6+2*x2^8)/(x1^4*x2^4)
propagator(G,list(1,1,1,0,0,0));
⇒ (x1^12*x3^2*x4^6+2*x1^8*x2^4*x3^2*x4^6+x1^8*x3^6*x4^6+x1^4*x2^8*x3^2*x4^6\
+2*x1^4*x2^4*x3^6*x4^6+x2^8*x3^6*x4^6)/(x1^6*x2^6*x3^8-4*x1^6*x2^6*x3^6*x\
4^2+6*x1^6*x2^6*x3^4*x4^4-4*x1^6*x2^6*x3^2*x4^6+x1^6*x2^6*x4^8-2*x1^6*x2^2\
4*x3^8*x4^2+8*x1^6*x2^4*x3^6*x4^4-12*x1^6*x2^4*x3^4*x4^6+8*x1^6*x2^4*x3^2\
*x4^8-2*x1^6*x2^4*x4^10+x1^6*x2^2*x3^8*x4^4-4*x1^6*x2^2*x3^6*x4^6+6*x1^6*\
x2^2*x3^4*x4^8-4*x1^6*x2^2*x3^2*x4^10+x1^6*x2^2*x4^12)
```

#### D.4.8.4 computeConstant

Procedure from library `ellipticcovers.lib` (see [Section D.4.8 \[ellipticcovers.lib\]](#), page 1068).

**Usage:** `computeConstant(f,x);`  $f$  number,  $x$  number

**Assume:**  $f$  is a number in a rational function field,  $x$  is a variable of the field.

**Return:** number, the constant coefficient of the Laurent series of  $f$  in the variable  $x$ .

**Theory:** Computes the constant coefficient of the Laurent series by iterative differentiation.

**Example:**

```
LIB "ellipticcovers.lib";
ring R=(0,x1,x2,x3,x4),(q1,q2,q3,q4,q5,q6),dp;
graph G = makeGraph(list(1,2,3,4),list(list(1,3),list(1,2),list(1,2),list(2,4),list(3,4)),list(1,1,1,0,0,0));
number P = propagator(G,list(1,1,1,0,0,0));
computeConstant(P,x2);
⇒ (2*x1^6*x3^2+2*x1^2*x3^6)/(x3^8-4*x3^6*x4^2+6*x3^4*x4^4-4*x3^2*x4^6+x4^8)
```

#### D.4.8.5 evalutateIntegral

Procedure from library `ellipticcovers.lib` (see [Section D.4.8 \[ellipticcovers.lib\]](#), page 1068).

### D.4.8.6 gromovWitten

Procedure from library `ellipticcovers.lib` (see [Section D.4.8 \[ellipticcovers.lib\]](#), page 1068).

**Usage:**     `gromovWitten(P)`; P number  
               `gromovWitten(G,d)`; G graph, d int  
               `gromovWitten(G,b)`; G graph, b list

**Assume:**   P is a propagator, or  
               G is a Feynman graph and d a non-negative integer, or  
               G is a Feynman graph and b is a list of integers of length equal to the number of edges of G  
               We assume that the coefficient ring has one rational variable for each vertex of G.

**Return:**    Gromov-Witten invariant.

**Theory:**    Computes

- the Gromov-Witten invariant of a given propagator P, or
- the invariant  $N_-(G,d) * |\text{Aut}(G)|$  where d is the degree of the covering, or
- the number  $N_-(G,b)$  of coverings with source G and target an elliptic curves with branch type a over a fixed base point (that is, the i-th edge passes over the base point with multiplicity  $b[i]$ ).

**Example:**

```
LIB "ellipticcovers.lib";
ring R=(0,x1,x2,x3,x4),(q1,q2,q3,q4,q5,q6),dp;
graph G = makeGraph(list(1,2,3,4),list(list(1,3),list(1,2),list(1,2),list(2,4),list(
number P = propagator(G,list(0,2,1,0,0,1));
gromovWitten(P);
⇒ 256
gromovWitten(G,list(0,2,1,0,0,1));
⇒ 256
gromovWitten(G,2);
⇒ 32
```

### D.4.8.7 computeGromovWitten

Procedure from library `ellipticcovers.lib` (see [Section D.4.8 \[ellipticcovers.lib\]](#), page 1068).

**Usage:**     `computeGromovWitten(G, d, st, en [, vb] )`; G graph, d int, st int, en int, optional: vb int

**Assume:**    G is a Feynman graph, d a non-negative integer, st specified the start- and en the end partition in the list `pa = partition(d)`. Specifying a positive optional integer vb leads to intermediate printout.  
               We assume that the coefficient ring has one rational variable for each vertex of G.

**Return:** list L, where  $L[i]$  is  $\text{gromovWitten}(G, \text{pa}[i])$  and all others are zero.

**Theory:** This function does essentially the same as the function  $\text{gromovWitten}$ , but is designed for handling complicated examples. Eventually it will also run in parallel.

**Example:**

```
LIB "ellipticcovers.lib";
ring R=(0,x1,x2,x3,x4),(q1,q2,q3,q4,q5,q6),dp;
graph G = makeGraph(list(1,2,3,4),list(list(1,3),list(1,2),list(1,2),list(2,4),list(3,4)),
partitions(6,2);
⇨ [1]:
⇨ 0,0,0,0,0,2
⇨ [2]:
⇨ 0,0,0,0,1,1
⇨ [3]:
⇨ 0,0,0,0,2,0
⇨ [4]:
⇨ 0,0,0,1,0,1
⇨ [5]:
⇨ 0,0,0,1,1,0
⇨ [6]:
⇨ 0,0,0,2,0,0
⇨ [7]:
⇨ 0,0,1,0,0,1
⇨ [8]:
⇨ 0,0,1,0,1,0
⇨ [9]:
⇨ 0,0,1,1,0,0
⇨ [10]:
⇨ 0,0,2,0,0,0
⇨ [11]:
⇨ 0,1,0,0,0,1
⇨ [12]:
⇨ 0,1,0,0,1,0
⇨ [13]:
⇨ 0,1,0,1,0,0
⇨ [14]:
⇨ 0,1,1,0,0,0
⇨ [15]:
⇨ 0,2,0,0,0,0
⇨ [16]:
⇨ 1,0,0,0,0,1
⇨ [17]:
⇨ 1,0,0,0,1,0
⇨ [18]:
⇨ 1,0,0,1,0,0
⇨ [19]:
⇨ 1,0,1,0,0,0
⇨ [20]:
⇨ 1,1,0,0,0,0
⇨ [21]:
⇨ 2,0,0,0,0,0
computeGromovWitten(G,2,3,7);
```

```

↳ [1] :
↳ 0
↳ [2] :
↳ 0
↳ [3] :
↳ 0
↳ [4] :
↳ 0
↳ [5] :
↳ 0
↳ [6] :
↳ 8
↳ [7] :
↳ 0
↳ [8] :
↳ 0
↳ [9] :
↳ 0
↳ [10] :
↳ 0
↳ [11] :
↳ 0
↳ [12] :
↳ 0
↳ [13] :
↳ 0
↳ [14] :
↳ 0
↳ [15] :
↳ 0
↳ [16] :
↳ 0
↳ [17] :
↳ 0
↳ [18] :
↳ 0
↳ [19] :
↳ 0
↳ [20] :
↳ 0
↳ [21] :
↳ 0
computeGromovWitten(G,2,3,7,1);
↳ 21
↳ 3 / 21      0,0,0,0,2,0      0      0      0
↳ 4 / 21      0,0,0,1,0,1      0      0      0
↳ 5 / 21      0,0,0,1,1,0      0      0      0
↳ 6 / 21      0,0,0,2,0,0      8      8      0
↳ 7 / 21      0,0,1,0,0,1      0      8      0
↳ [1] :
↳ 0
↳ [2] :
↳ 0

```

```

    ↦ [3] :
    ↦ 0
    ↦ [4] :
    ↦ 0
    ↦ [5] :
    ↦ 0
    ↦ [6] :
    ↦ 8
    ↦ [7] :
    ↦ 0
    ↦ [8] :
    ↦ 0
    ↦ [9] :
    ↦ 0
    ↦ [10] :
    ↦ 0
    ↦ [11] :
    ↦ 0
    ↦ [12] :
    ↦ 0
    ↦ [13] :
    ↦ 0
    ↦ [14] :
    ↦ 0
    ↦ [15] :
    ↦ 0
    ↦ [16] :
    ↦ 0
    ↦ [17] :
    ↦ 0
    ↦ [18] :
    ↦ 0
    ↦ [19] :
    ↦ 0
    ↦ [20] :
    ↦ 0
    ↦ [21] :
    ↦ 0

```

#### D.4.8.8 partitions

Procedure from library `ellipticcovers.lib` (see [Section D.4.8 \[ellipticcovers.lib\]](#), page 1068).

**Usage:**      `partitions(n,a);` n int, a int

**Assume:**    n and a are positive integers

**Return:**     list of all partitions of a into n summands.

**Theory:**     Computes all partitions of a into n summands.

This may eventually be deleted and become a more efficient kernel function.

**Example:**



```
LIB "ellipticcovers.lib";
partitions(3,7);
⇨ [1]:
⇨ 0,0,7
⇨ [2]:
⇨ 0,1,6
⇨ [3]:
⇨ 0,2,5
⇨ [4]:
⇨ 0,3,4
⇨ [5]:
⇨ 0,4,3
⇨ [6]:
⇨ 0,5,2
⇨ [7]:
⇨ 0,6,1
⇨ [8]:
⇨ 0,7,0
⇨ [9]:
⇨ 1,0,6
⇨ [10]:
⇨ 1,1,5
⇨ [11]:
⇨ 1,2,4
⇨ [12]:
⇨ 1,3,3
⇨ [13]:
⇨ 1,4,2
⇨ [14]:
⇨ 1,5,1
⇨ [15]:
⇨ 1,6,0
⇨ [16]:
⇨ 2,0,5
⇨ [17]:
⇨ 2,1,4
⇨ [18]:
⇨ 2,2,3
⇨ [19]:
⇨ 2,3,2
⇨ [20]:
⇨ 2,4,1
⇨ [21]:
⇨ 2,5,0
⇨ [22]:
⇨ 3,0,4
⇨ [23]:
⇨ 3,1,3
⇨ [24]:
⇨ 3,2,2
⇨ [25]:
⇨ 3,3,1
⇨ [26]:
```

```

⇒      3,4,0
⇒ [27]:
⇒      4,0,3
⇒ [28]:
⇒      4,1,2
⇒ [29]:
⇒      4,2,1
⇒ [30]:
⇒      4,3,0
⇒ [31]:
⇒      5,0,2
⇒ [32]:
⇒      5,1,1
⇒ [33]:
⇒      5,2,0
⇒ [34]:
⇒      6,0,1
⇒ [35]:
⇒      6,1,0
⇒ [36]:
⇒      7,0,0

```

#### D.4.8.9 permute

Procedure from library `ellipticcovers.lib` (see [Section D.4.8 \[ellipticcovers.lib\]](#), page 1068).

**Usage:**     `permute(N)`; N list

**Assume:**    N is a list

**Return:**     list with all permutations of N.

**Theory:**     Computes all permutations of N.

This will eventually be deleted and become a more efficient kernel function.

**Example:**

```

LIB "ellipticcovers.lib";
ring R=(0,x1,x2,x3,x4),(q),dp;
permute(list(x1,x2,x3,x4));
⇒ [1]:
⇒      [1]:
⇒      (x4)
⇒      [2]:
⇒      (x3)
⇒      [3]:
⇒      (x2)
⇒      [4]:
⇒      (x1)
⇒ [2]:
⇒      [1]:
⇒      (x3)
⇒      [2]:
⇒      (x4)

```

```

↳      [3]:
↳ (x2)
↳      [4]:
↳ (x1)
↳ [3]:
↳      [1]:
↳ (x4)
↳      [2]:
↳ (x2)
↳      [3]:
↳ (x3)
↳      [4]:
↳ (x1)
↳ [4]:
↳      [1]:
↳ (x2)
↳      [2]:
↳ (x4)
↳      [3]:
↳ (x3)
↳      [4]:
↳ (x1)
↳ [5]:
↳      [1]:
↳ (x3)
↳      [2]:
↳ (x2)
↳      [3]:
↳ (x4)
↳      [4]:
↳ (x1)
↳ [6]:
↳      [1]:
↳ (x2)
↳      [2]:
↳ (x3)
↳      [3]:
↳ (x4)
↳      [4]:
↳ (x1)
↳ [7]:
↳      [1]:
↳ (x4)
↳      [2]:
↳ (x3)
↳      [3]:
↳ (x1)
↳      [4]:
↳ (x2)
↳ [8]:
↳      [1]:
↳ (x3)
↳      [2]:

```

```

↳ (x4)
↳ [3]:
↳ (x1)
↳ [4]:
↳ (x2)
↳ [9]:
↳ [1]:
↳ (x4)
↳ [2]:
↳ (x1)
↳ [3]:
↳ (x3)
↳ [4]:
↳ (x2)
↳ [10]:
↳ [1]:
↳ (x1)
↳ [2]:
↳ (x4)
↳ [3]:
↳ (x3)
↳ [4]:
↳ (x2)
↳ [11]:
↳ [1]:
↳ (x3)
↳ [2]:
↳ (x1)
↳ [3]:
↳ (x4)
↳ [4]:
↳ (x2)
↳ [12]:
↳ [1]:
↳ (x1)
↳ [2]:
↳ (x3)
↳ [3]:
↳ (x4)
↳ [4]:
↳ (x2)
↳ [13]:
↳ [1]:
↳ (x4)
↳ [2]:
↳ (x2)
↳ [3]:
↳ (x1)
↳ [4]:
↳ (x3)
↳ [14]:
↳ [1]:
↳ (x2)

```

```

↳ [2]:
↳ (x4)
↳ [3]:
↳ (x1)
↳ [4]:
↳ (x3)
↳ [15]:
↳ [1]:
↳ (x4)
↳ [2]:
↳ (x1)
↳ [3]:
↳ (x2)
↳ [4]:
↳ (x3)
↳ [16]:
↳ [1]:
↳ (x1)
↳ [2]:
↳ (x4)
↳ [3]:
↳ (x2)
↳ [4]:
↳ (x3)
↳ [17]:
↳ [1]:
↳ (x2)
↳ [2]:
↳ (x1)
↳ [3]:
↳ (x4)
↳ [4]:
↳ (x3)
↳ [18]:
↳ [1]:
↳ (x1)
↳ [2]:
↳ (x2)
↳ [3]:
↳ (x4)
↳ [4]:
↳ (x3)
↳ [19]:
↳ [1]:
↳ (x3)
↳ [2]:
↳ (x2)
↳ [3]:
↳ (x1)
↳ [4]:
↳ (x4)
↳ [20]:
↳ [1]:

```

```

⇒ (x2)
⇒ [2]:
⇒ (x3)
⇒ [3]:
⇒ (x1)
⇒ [4]:
⇒ (x4)
⇒ [21]:
⇒ [1]:
⇒ (x3)
⇒ [2]:
⇒ (x1)
⇒ [3]:
⇒ (x2)
⇒ [4]:
⇒ (x4)
⇒ [22]:
⇒ [1]:
⇒ (x1)
⇒ [2]:
⇒ (x3)
⇒ [3]:
⇒ (x2)
⇒ [4]:
⇒ (x4)
⇒ [23]:
⇒ [1]:
⇒ (x2)
⇒ [2]:
⇒ (x1)
⇒ [3]:
⇒ (x3)
⇒ [4]:
⇒ (x4)
⇒ [24]:
⇒ [1]:
⇒ (x1)
⇒ [2]:
⇒ (x2)
⇒ [3]:
⇒ (x3)
⇒ [4]:
⇒ (x4)

```

#### D.4.8.10 lsum

Procedure from library `ellipticcovers.lib` (see [Section D.4.8 \[ellipticcovers.lib\]](#), page 1068).

**Usage:** `lsum(L)`; L list

**Assume:** L is a list of things with the binary operator `+` defined.

**Return:** The sum of the elements of L.

**Theory:** Sums the elements of a list.  
Eventually this will be deleted and become a more efficient kernel function.

**Example:**

```
LIB "ellipticcovers.lib";
list L = 1,2,3,4,5;
lsum(L);
↦ 15
```

### D.4.9 fmodstd.lib

**Library:** fmodstd.lib

**Purpose:** Groebner bases of ideals in polynomial rings over rational function fields

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**Overview:** A library for computing a Groebner basis of an ideal in a polynomial ring over an algebraic function field  $Q(T) := Q(t_1, \dots, t_m)$  using modular methods and sparse multivariate rational interpolation, where the  $t_i$  are transcendental over  $Q$ . The idea is as follows: Given an ideal  $I$  in  $Q(T)[X]$ , we map  $I$  to  $J$  via the map sending  $T$  to  $Tz := (t_1z + s_1, \dots, t_mz + s_m)$  for a suitable point  $s$  in  $Q^m \setminus \{(0, \dots, 0)\}$  and for some extra variable  $z$  so that  $J$  is an ideal in  $Q(Tz)[X]$ . For a suitable point  $b$  in  $Z^m \setminus \{(0, \dots, 0)\}$ , we map  $J$  to  $K$  via the map sending  $(T, z)$  to  $(b, z)$ , where  $b := (b_1, \dots, b_m)$  (usually the  $b_i$ 's are distinct primes), so that  $K$  is an ideal in  $Q(z)[X]$ . For such a rational point  $b$ , we compute a Groebner basis  $G_b$  of  $K$  using modular algorithms [1], where prime numbers are replaced by maximal ideals of the form  $\langle z - z_i \rangle$ , and univariate rational interpolation [2, 7]. Note that since  $Q[z]/\langle z - z_i \rangle = Q$  we also use (if required) modular algorithms [1] over  $Q$ . The procedure is repeated for many rational points  $b$  until their number is sufficiently large to recover the correct coefficients in  $Q(T)$ . Once we have these points, we obtain a set of polynomials  $G$  by applying the sparse multivariate rational interpolation algorithm from [4] coefficient-wise to the list of Groebner bases  $G_b$  in  $Q(z)[X]$ , where this algorithm makes use of the following algorithms: univariate polynomial interpolation [2], univariate rational function reconstruction [7], and multivariate polynomial interpolation [3]. The last algorithm uses the well-known Berlekamp/Massey algorithm [5] and its early termination version [6]. The set  $G$  is then a Groebner basis of  $I$  with high probability.

**References:**

- [1] E. A. Arnold: Modular algorithms for computing Groebner bases. J. Symb. Comput. 35, 403-419 (2003).
- [2] R. L. Burden and J. D. Faires: Numerical analysis. 9th ed. (1993). [3] M. Ben-Or and P. Tiwari: A deterministic algorithm for sparse multivariate polynomial interpolation. Proc. of the 20th Annual ACM Symposium on Theory of Computing, 301-309 (1988).
- [4] A. Cuyt and W.-s. Lee: Sparse interpolation of multivariate rational functions. Theor. Comput. Sci. 412, 1445-1456 (2011).
- [5] E. Kaltofen and W.-s. Lee: Early termination in sparse interpolation algorithms. J. Symb. Comput. 36, 365-400 (2003).

- [6] E. Kaltofen, W.-s. Lee and A. A. Lobo: Early termination in Ben-Or/Tiwari sparse interpolation and a hybrid of Zippel's algorithm. Proc. ISSAC (ISSAC '00), 192-201 (2000).
- [7] K. Sara and M. Monagan: Fast Rational Function Reconstruction. Proc. ISSAC (ISSAC '06), 184-190 (2006).

### Procedures:

#### D.4.9.1 fareypoly

Procedure from library `ffmodstd.lib` (see [Section D.4.9 \[ffmodstd.lib\]](#), page 1081).

**Usage:** `fareypoly(f, g[, m]);`  $f$  poly,  $g$  poly,  $m$  int

**Return:** a list  $l$  where  $r/t$  ( $r:=l[1]$ ,  $t:=l[2]$ ) is a univariate rational function such that  $r/t = g \bmod f$ ,  $\gcd(r,t)=\gcd(f,t)=1$  and  $\deg(r) + \deg(t) < \deg(f)$

**Note:** An optional parameter  $m$  can be provided to define the way how  $t$  is normalized. If  $m = 0$  (default), then the leading coefficient of  $t$  is 1. Otherwise, assuming the polynomial  $t$  has a non-zero constant term, the procedure returns the uniquely determined rational function  $r/t$  where the constant term in  $t$  is equal to 1.

If the ground ring has  $n$  variables and  $f$  and  $g$  are in a polynomial ring  $k[\text{var}(i)]$  ( $k$  is a field) for some  $i \leq n$ , then the function  $r/t$  is returned as an element in  $k(\text{var}(i))$ .

In positive characteristic, the condition  $r/t = g \bmod f$  may not be satisfied. The degree  $\deg(f)$  of  $f$  must be higher than the degree  $\deg(g)$  of  $g$ .

### Example:

```
LIB "ffmodstd.lib";
ring rr=23,x,dp;
poly g = 10x5-5x4+3x3+3x2-x-11;
poly f = x6+2x5-9x4+x3-9x2+7x+7;
fareypoly(g,f);
  => [1]:
  => -11x3-7x2-3x+6
  => [2]:
  => x-9
fareypoly(g,f,1);
  => [1]:
  => -9x3+11x2+8x+7
  => [2]:
  => 5x+1
ring R = 0, x,dp;
poly g = (24/1616615)*x6-(732/1616615)*x5+(9558/1616615)*x4-(14187/323323)*x3+
(1148101/1616615)*x2+(4089347/1616615)*x+547356/230945;
poly f = x7-28x6+322x5-1960x4+6769x3-13132x2+13068x-5040;
fareypoly(g,f);
  => [1]:
  => 1/2x3+9/2x2+17/2x+6
  => [2]:
  => x+5/2
fareypoly(g,f,1);
  => [1]:
  => 1/5x3+9/5x2+17/5x+12/5
  => [2]:
  => 2/5x+1
```



```

ring r = (499,a),x,dp;
number N = (-113a4+170a3-29a2+226a+222)/(a7-56a6+114a5+144a4+171a3-64a2+192a);
poly h1 = x4+(-55a5-18a4-141a3+233a2+66a-40)/(a4-28a3+40a2-2a+210)*x3;
poly h2 = (107a6-221a5-68a4-93a3+112a2-54a+216)/(a4-28a3+40a2-2a+210)*x2;
poly h3 = (-53a7+214a6+27a5+12a4+15a3+60a2-167a-83)/(a4-28a3+40a2-2a+210)*x;
poly h4 = (10a6-75a5+47a4+246a3-20a2-217a+196)/(a4-28a3+40a2-2a+210);
poly g = N*(h1+h2+h3+h4);
poly f = x5+(-2a-119)*x4+(a2+237a+3437)*x3+(-118a2-6756a-29401)*x2+
(3319a2+55483a+26082)*x+(-26082a2-26082a);
fareypoly(g,f);
⇒ [1]:
⇒      (a)*x+2
⇒ [2]:
⇒      x2-3*x+(a)

```

See also: [Section 5.1.37 \[farey\]](#), page 181; [Section D.4.9.2 \[polyInterpolation\]](#), page 1083.

### D.4.9.2 polyInterpolation

Procedure from library `ffmodstd.lib` (see [Section D.4.9 \[ffmodstd.lib\]](#), page 1081).

**Usage:** `polyInterpolation(d, e[, n, L]);` d list, e list, n int, L list

**Return:** a list `l_p` where `f:=l_p[1]` is a polynomial of degree at most `size(d)-1` which satisfies the conditions `f(d[i])=e[i]` for all `i`, `l_p[2]` is the product of all `(var(n)-d[i])` for `1 <= i <= size(d)` and `l_p[3]=d`.

**Note:** The procedure applies the Newton interpolation algorithm to the pair `(d,e)` and returns the output w.r.t. the first variable (default) of the ground ring. If an optional parameter `n`, `1<=n<=N` (`N` is the number of variables in the current basering), is given, then the procedure returns the list `l_p` w.r.t. the `n`-th variable. Moreover, if the number of points `(d'[i],e'[i])` is not large enough to obtain the target polynomial, `L = polyInterpolation(d', e', n)` can be provided as an optional parameter to add more interpolation points. The elements in the first list must be distinct.

**Example:**

```

LIB "ffmodstd.lib";
ring rr = 23,(x,y),dp;
list d = 1,2,3,4;
list e = -1,10,3,8;
polyInterpolation(d,e);
⇒ [1]:
⇒      5x3+7x2+x+9
⇒ [2]:
⇒      x4-10x3-11x2-4x+1
⇒ [3]:
⇒      [1]:
⇒          1
⇒      [2]:
⇒          2
⇒      [3]:
⇒          3
⇒      [4]:
⇒          4
polyInterpolation(d,e,2)[1];

```

```

⇒ 5y3+7y2+y+9
list d1 = 5,6;
list e1 = -7,6;
list L = polyInterpolation(d,e);
L = polyInterpolation(d1,e1,1,L); // add points
L;
⇒ [1]:
⇒ 10x5-5x4+3x3+3x2-x-11
⇒ [2]:
⇒ x6+2x5-9x4+x3-9x2+7x+7
⇒ [3]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 2
⇒ [3]:
⇒ 3
⇒ [4]:
⇒ 4
⇒ [5]:
⇒ 5
⇒ [6]:
⇒ 6
ring R = (499,a),x,dp;
list d2 = 2,3a,5;
list e2 = (a-2), (9a2-8a), (a+10);
polyInterpolation(d2,e2);
⇒ [1]:
⇒ x2-3*x+(a)
⇒ [2]:
⇒ x3+(-3a-7)*x2+(21a+10)*x+(-30a)
⇒ [3]:
⇒ [1]:
⇒ 2
⇒ [2]:
⇒ (3a)
⇒ [3]:
⇒ 5

```

### D.4.9.3 modrationalInterpolation

Procedure from library `ffmodstd.lib` (see [Section D.4.9 \[ffmodstd.lib\]](#), page 1081).

**Usage:** `modrationalInterpolation(D, E, vr[, D1, E1]);` D list, E list, vr int, D1 list, E1 list

**Return:** a list L where  $r/t$  ( $r:=L[1]$ ,  $t:=L[2]$ ) is a univariate rational function such that  $r(D[i])/t(D[i]) = E[i]$  (or equivalently  $r/t = g \bmod f$ ,  $\gcd(r,t)=\gcd(f,t)=1$  and  $\deg(r) + \deg(t) < \deg(f)$ )

**Note:** Optional parameters D1 and E1 can be provided to update the existing input, that is, to  $D1+D$  and  $E1+E$ . The rational function  $r/t$  is returned as an element in  $k(\text{var}(vr))$ , where  $k$  is a field. This procedure works only in characteristic zero. The elements in the first list must be distinct.

**Example:**

```

LIB "ffmodstd.lib";
ring rr=0,x,dp;
list D = 2,3,4,5,6,7,8,9,10;
list E = 8/35, 7/123, 22/1027, 4/391, 44/7779, 29/8405, 74/32771, 23/14763, 112/100003;
modrationalInterpolation(D, E, 1);
⇒ [1]:
⇒ x2+x+2
⇒ [2]:
⇒ x5+3
ring R = 0, x, dp;
list D1 = 2,3,4,5,6;
list E1 = 8/35, 7/123, 22/1027, 4/391, 44/7779;
modrationalInterpolation(D1, E1, 1);
⇒ [1]:
⇒ 35185737407/8965053161610x4-659760301271/8965053161610x3+8448757043/16\
419511285x2-7177841332787/4482526580805x+1702354556926/896505316161
⇒ [2]:
⇒ 1
list D = 7,8,9,10;
⇒ // ** redefining D (list D = 7,8,9,10;) ./examples/modrationalInterpolati\
on.sing:10
list E = 29/8405, 74/32771, 23/14763, 112/100003;
modrationalInterpolation(D, E, 1, D1, E1);
⇒ [1]:
⇒ x2+x+2
⇒ [2]:
⇒ x5+3

```

See also: [Section D.4.9.1 \[fareypoly\]](#), page 1082; [Section D.4.9.2 \[polyInterpolation\]](#), page 1083.

#### D.4.9.4 BerlekampMassey

Procedure from library `ffmodstd.lib` (see [Section D.4.9 \[ffmodstd.lib\]](#), page 1081).

**Usage:** BerlekampMassey(L, i[, M]); L list, i int, M list

**Return:** a list Tr where  $f := \text{Tr}[1]$  is the minimal polynomial (w.r.t. the  $i$ -th variable) generated by the sequence  $(L[j])$ ,  $1 \leq j \leq \text{Tr}[2]$ , if the length of the sequence is long enough. In this case, the coefficients  $c_i$  of the polynomial  $f$  satisfy the relation  $-L[j+t] = c_0 L[j] + \dots + c_{t-1} L[j+t-1]$  for all  $j \geq 1$  where  $t = \deg(f)$ .

**Note:** The procedure applies the Berlekamp/Massey algorithm to the sequence  $L[j]$  (elements from the field  $\mathbb{Q}$ ) for  $j > 0$  and returns a polynomial  $f$ . If the polynomial  $f$  splits into linear factors with no multiplicity greater than one, then we say that the length of the sequence  $L$  is long enough. If this polynomial does not split into linear factors, an optional parameter  $M = \text{BerlekampMassey}(L', i)$  can be provided to add more elements to the sequence.

#### References:

- [1] E. Kaltofen and W.-s. Lee: Early termination in sparse interpolation algorithms. *J. Symb. Comput.* 36, 365-400 (2003).
- [2] E. Kaltofen, W.-s. Lee and A. A. Lobo: Early termination in Ben-Or/Tiwari sparse interpolation and a hybrid of Zippel's algorithm. *Proc. ISSAC (ISSAC '00)*, 192-201 (2000).

**Example:**

```

LIB "ffmodstd.lib";
ring rr=0,x,dp;
list L = 150,3204,79272,2245968;
list Tr = BerlekampMassey(L,1);
Tr[1];
⇒ 117288/209x2-10662/209x+1
factorize(Tr[1]); //not linearly factored
⇒ [1]:
⇒   _[1]=1/209
⇒   _[2]=117288x2-10662x+209
⇒ [2]:
⇒   1,1
list L1 = 70411680, 2352815424, 81496927872;
Tr = BerlekampMassey(L1,1,Tr); // increase the length of L by size(L1)
Tr[1];
⇒ x3-66x2+1296x-7776
factorize(Tr[1]); //linearly factored and has distinct roots
⇒ [1]:
⇒   _[1]=1
⇒   _[2]=x-36
⇒   _[3]=x-18
⇒   _[4]=x-12
⇒ [2]:
⇒   1,1,1,1
Tr[2]; //the length of the sequence required to generate Tr[1]
⇒ 6

```

**D.4.9.5 modberlekampMassey**

Procedure from library `ffmodstd.lib` (see [Section D.4.9 \[ffmodstd.lib\]](#), page 1081).

**Usage:** `modberlekampMassey(L [, i]);` L list, i int

**Return:** The minimal polynomial  $f$  (w.r.t. the  $i$ -th variable) generated by the sequence  $(L[j]), j = 1, 2, \dots$ .

**Note:** The procedure first construct polynomials  $f$  and  $g$  of degrees  $\text{size}(L)$  and  $\text{size}(L)-1$ , respectively from the sequence  $L[j]$  (elements from the field  $Q$ ) for  $j > 0$  as described in [1]. It then returns the denominator polynomial  $d$  obtained by applying the SINGULAR command [Section D.4.9.1 \[fareypoly\]](#), page 1082 to the input  $(g,f)$ . If the ground ring has  $n$  variables, the procedure returns  $d$  in a polynomial ring  $k[\text{var}(i)]$  ( $k$  is a field) for some  $i \leq n$ . In this case, an optional parameter  $i$  (default 0) can be provided.

**References:**

- [1] Nadia Ben Atti, Gema M. Diaz-Toca and Henri Lombardi: The Berlekamp-Massey Algorithm Revisited, 2000.

**Example:**

```

LIB "ffmodstd.lib";
ring rr=0, (x,y,z), dp;
list L = 150,3204,79272,2245968, 70411680, 2352815424, 81496927872;
modberlekampMassey(L); // default w.r.t x
⇒ x3-66x2+1296x-7776

```

```
modberlekampMassey(L,3); // returns an output in the ring Q[z]
↦ z3-66z2+1296z-7776
```

See also: [Section D.4.9.4 \[BerlekampMassey\]](#), page 1085.

### D.4.9.6 sparseInterpolation

Procedure from library `ffmodstd.lib` (see [Section D.4.9 \[ffmodstd.lib\]](#), page 1081).

**Usage:** `sparseInterpolation(Br, La, lpr, n[, m]);` Br poly, La list, lpr list, n int, m int

**Return:** a polynomial B in the polynomial ring  $\mathbb{Q}[\text{var}(n+1), \dots, \text{var}(n+\text{size}(\text{lpr}))]$  satisfying the relation  $\text{La}[i] = B(\text{lpr}[1]^i, \dots, \text{lpr}[\text{size}(\text{lpr})]^i)$ .

**Note:** The polynomial Br in  $\mathbb{Q}[\text{var}(n)]$  is the minimal polynomial obtained by applying the SINGULAR command [Section D.4.9.4 \[BerlekampMassey\]](#), page 1085 to the sequence  $(\text{La}[j])$ ,  $1 \leq j \leq \text{size}(\text{La})$ . By default the exponent i starts from 1. However, if the optional parameter  $m \geq 0$  is provided, then it starts from m. The list lpr must be a list of distinct primes.

**Example:**

```
LIB "ffmodstd.lib";
ring rr=0,(x,y),dp;
list lpr = 2,3; // assign 2 for x and 3 for y
list La = 150,3204,79272,2245968,70411680, 2352815424, 81496927872;
// La[i] = number(subst(f,y,lpr[1]^i,z,lpr[2]^i)); for f = x2y2+2x2y+5xy2 and i=1,..
poly Br = BerlekampMassey(La,1)[1];
Br;
↦ x3-66x2+1296x-7776
sparseInterpolation(Br,La,lpr,0); // reconstruct f default
↦ x2y2+2x2y+5xy2
La = 97,275,793,2315,6817;
// La[i] = number(subst(g,y,lpr[1]^i,z,lpr[2]^i)); for g = x+y and i=4,...,8
Br = BerlekampMassey(La,1)[1];
Br;
↦ x2-5x+6
sparseInterpolation(Br,La,lpr,0,4);
↦ x+y
```

See also: [Section D.4.9.4 \[BerlekampMassey\]](#), page 1085.

### D.4.9.7 ffmodStd

Procedure from library `ffmodstd.lib` (see [Section D.4.9 \[ffmodstd.lib\]](#), page 1081).

**Usage:** `ffmodStd(I [, d]);` I ideal, d int

**Return:** Groebner basis of I over an algebraic function field

**Note:** An optional parameter  $d > 0$ , a positive integer, can be provided for the procedure. It refers to a number of evaluation points to used.

**Example:**

```
LIB "ffmodstd.lib";
ring Ra=(0,a),(x,y,z),dp;
ideal I = (a^2+2)*x^2*y+a*y*z^2, x*z^2+(a+1)*x^2-a*y^2;
ffmodStd(I);
↦ _[1]=xz2+(a+1)*x2+(-a)*y2
```

```

⇒ _[2]=(a2+2)*x2y+(a)*yz2
⇒ _[3]=(a2+2)*yz4+(a4+4a2+4)*xy3+(a4+a3+2a2+2a)*y3+(a3+2a2+a)*yz2
ideal J = x^2*y+y*z^2, x*z^2+x^2-y^2;
ffmodStd(J);
⇒ _[1]=xz2+x2-y2
⇒ _[2]=x2y+yz2
⇒ _[3]=yz4+xy3+y3+yz2
ring R1=(0,a,b),(x,y,z),dp;
ideal I = x^2*y^3*z+2*a*x*y*z^2+7*y^3,
x^2*y^4*z+(a-7b)*x^2*y*z^2-x*y^2*z^2+2*x^2*y*z-12*x+by,
(a2+b-2)*y^5*z+(a+5b)*x^2*y^2*z-b*x*y^3*z-x*y^3+y^4+2*a2*y^2*z,
a*x^2*y^2*z-x*y^3*z+3a*x*y*z^3+(-a+4)*y^3*z^2+4*z^2-bx;
ffmodStd(I);
⇒ _[1]=12*x+(-b)*y
⇒ _[2]=48*z2+(-b2)*y
⇒ _[3]=y2z
⇒ _[4]=y3

```

See also: [Section D.4.20.2 \[nfmodStd\]](#), page 1177.

## D.4.10 grwalk\_lib

**Library:** grwalk.lib

**Purpose:** Groebner Walk Conversion Algorithms

**Author:** I Made Sulandra

**Procedures:** See also: [Section D.15.17 \[rwalk\\_lib\]](#), page 2537; [Section D.15.22 \[swalk\\_lib\]](#), page 2553.

### D.4.10.1 fwalk

Procedure from library `grwalk.lib` (see [Section D.4.10 \[grwalk\\_lib\]](#), page 1088).

**Syntax:** `fwalk(ideal i);`  
`fwalk(ideal i, intvec v, intvec w);`

**Type:** ideal

**Purpose:** compute the standard basis of the ideal w.r.t. the lexicographical ordering or a weighted-lex ordering, calculated via the fractal walk algorithm.

**Example:**

```

LIB "grwalk.lib";
ring r = 32003,(z,y,x), lp;
ideal I = y3+xyz+y2z+xz3, 3+xy+x2y+y2z;
fwalk(I);
⇒ _[1]=y9-y7x2-y7x-y6x3-y6x2-3y6-3y5x-y3x7-3y3x6-3y3x5-y3x4-9y2x5-18y2x4-9y\
2x3-27yx3-27yx2-27x
⇒ _[2]=zx+8297y8x2+8297y8x+3556y7-8297y6x4+15409y6x3-8297y6x2-8297y5x5+1540\
9y5x4-8297y5x3+3556y5x2+3556y5x+3556y4x3+3556y4x2-10668y4-10668y3x-8297y2\
x9-1185y2x8+14224y2x7-1185y2x6-8297y2x5-14223yx7-10666yx6-10666yx5-14223y\
x4+x5+2x4+x3
⇒ _[3]=zy2+yx2+yx+3

```

See also: [Section D.4.10.3 \[awalk1\]](#), page 1089; [Section D.4.10.4 \[awalk2\]](#), page 1090; [Section 5.1.53 \[groebner\]](#), page 191; [Section D.4.10.6 \[gwalk\]](#), page 1091; [Section D.4.10.5 \[pwalk\]](#), page 1090; [Section 5.1.151 \[std\]](#), page 271; [Section 5.1.152 \[stdfglm\]](#), page 272; [Section D.4.10.2 \[twalk\]](#), page 1089.

### D.4.10.2 twalk

Procedure from library `grwalk.lib` (see [Section D.4.10 \[grwalk.lib\]](#), page 1088).

**Syntax:**     `twalk(ideal i);`  
               `twalk(ideal i, intvec v, intvec w);`

**Type:**       ideal

**Purpose:**     compute the standard basis of the ideal w.r.t.  
               the ordering "`(a(w),lp)`" or "`(a(1,0,...,0),lp)`",  
               calculated via the Tran algorithm.

**Example:**

```
LIB "grwalk.lib";
ring r = 32003,(z,y,x), lp;
ideal I = y3+xyz+y2z+xz3, 3+xy+x2y+y2z;
twalk(I);
↦ _[1]=y9-y7x2-y7x-y6x3-y6x2-3y6-3y5x-y3x7-3y3x6-3y3x5-y3x4-9y2x5-18y2x4-9y\
2x3-27yx3-27yx2-27x
↦ _[2]=zx+8297y8x2+8297y8x+3556y7-8297y6x4+15409y6x3-8297y6x2-8297y5x5+1540\
9y5x4-8297y5x3+3556y5x2+3556y5x+3556y4x3+3556y4x2-10668y4-10668y3x-8297y2\
x9-1185y2x8+14224y2x7-1185y2x6-8297y2x5-14223yx7-10666yx6-10666yx5-14223y\
x4+x5+2x4+x3
↦ _[3]=zy2+yx2+yx+3
```

See also: [Section D.4.10.3 \[awalk1\]](#), page 1089; [Section D.4.10.4 \[awalk2\]](#), page 1090; [Section D.4.10.1 \[fwalk\]](#), page 1088; [Section 5.1.53 \[groebner\]](#), page 191; [Section D.4.10.6 \[gwalk\]](#), page 1091; [Section D.4.10.5 \[pwalk\]](#), page 1090; [Section 5.1.151 \[std\]](#), page 271; [Section 5.1.152 \[stdfglm\]](#), page 272.

### D.4.10.3 awalk1

Procedure from library `grwalk.lib` (see [Section D.4.10 \[grwalk.lib\]](#), page 1088).

**Syntax:**     `awalk1(ideal i);`  
               `awalk1(ideal i, int n);`  
               `awalk1(ideal i, int n, intvec v, intvec w);`  
               `awalk1(ideal i, intvec v, intvec w);`

**Type:**       ideal

**Purpose:**     compute the standard basis of the ideal, calculated via the first alternative algorithm  
               from an ordering  
               "`(a(v),lp)`", "`dp`" or "`Dp`" to the ordering  
               "`(a(w),lp)`" or "`(a(1,0,...,0),lp)`"  
               with a perturbation degree `n` for the weight vector `w`.

**Example:**

```
LIB "grwalk.lib";
ring r = 32003,(z,y,x), lp;
ideal I = y3+xyz+y2z+xz3, 3+xy+x2y+y2z;
awalk1(I,3);
↦ _[1]=y9-y7x2-y7x-y6x3-y6x2-3y6-3y5x-y3x7-3y3x6-3y3x5-y3x4-9y2x5-18y2x4-9y\
2x3-27yx3-27yx2-27x
↦ _[2]=zx+8297y8x2+8297y8x+3556y7-8297y6x4+15409y6x3-8297y6x2-8297y5x5+1540\
9y5x4-8297y5x3+3556y5x2+3556y5x+3556y4x3+3556y4x2-10668y4-10668y3x-8297y2\
```

```

x9-1185y2x8+14224y2x7-1185y2x6-8297y2x5-14223yx7-10666yx6-10666yx5-14223y\
x4+x5+2x4+x3
↦ _[3]=zy2+yx2+yx+3

```

See also: [Section D.4.10.4 \[awalk2\]](#), page 1090; [Section D.4.10.1 \[fwalk\]](#), page 1088; [Section 5.1.53 \[groebner\]](#), page 191; [Section D.4.10.6 \[gwalk\]](#), page 1091; [Section D.4.10.5 \[pwalk\]](#), page 1090; [Section 5.1.151 \[std\]](#), page 271; [Section 5.1.152 \[stdfglm\]](#), page 272; [Section D.4.10.2 \[twalk\]](#), page 1089.

#### D.4.10.4 awalk2

Procedure from library `grwalk.lib` (see [Section D.4.10 \[grwalk.lib\]](#), page 1088).

**Syntax:**     `awalk2(ideal i);`  
               `awalk2(ideal i, intvec v, intvec w);`

**Type:**       ideal

**Purpose:**     compute the standard basis of the ideal, calculated via the second alternative algorithm from the ordering  
               `"(a(v),lp)"`, `"dp"` or `"Dp"`  
               to the ordering `"(a(w),lp)"` or `"(a(1,0,...,0),lp)"`.

**Example:**

```

LIB "grwalk.lib";
ring r = 32003,(z,y,x), lp;
ideal I = y3+xyz+y2z+xz3, 3+xy+x2y+y2z;
awalk2(I);
↦ _[1]=y9-y7x2-y7x-y6x3-y6x2-3y6-3y5x-y3x7-3y3x6-3y3x5-y3x4-9y2x5-18y2x4-9y\
2x3-27yx3-27yx2-27x
↦ _[2]=zx+8297y8x2+8297y8x+3556y7-8297y6x4+15409y6x3-8297y6x2-8297y5x5+1540\
9y5x4-8297y5x3+3556y5x2+3556y5x+3556y4x3+3556y4x2-10668y4-10668y3x-8297y2\
x9-1185y2x8+14224y2x7-1185y2x6-8297y2x5-14223yx7-10666yx6-10666yx5-14223y\
x4+x5+2x4+x3
↦ _[3]=zy2+yx2+yx+3

```

See also: [Section D.4.10.3 \[awalk1\]](#), page 1089; [Section D.4.10.1 \[fwalk\]](#), page 1088; [Section 5.1.53 \[groebner\]](#), page 191; [Section D.4.10.6 \[gwalk\]](#), page 1091; [Section D.4.10.5 \[pwalk\]](#), page 1090; [Section 5.1.151 \[std\]](#), page 271; [Section 5.1.152 \[stdfglm\]](#), page 272; [Section D.4.10.2 \[twalk\]](#), page 1089.

#### D.4.10.5 pwalk

Procedure from library `grwalk.lib` (see [Section D.4.10 \[grwalk.lib\]](#), page 1088).

**Syntax:**     `pwalk(int d, ideal i, int n1, int n2);`  
               `pwalk(int d, ideal i, int n1, int n2, intvec v, intvec w);`

**Type:**       ideal

**Purpose:**     compute the standard basis of the ideal, calculated via the perturbation walk algorithm from the ordering  
               `"(a(v),lp)"`, `"dp"` or `"Dp"`  
               to the ordering `"(a(w),lp)"` or `"(a(1,0,...,0),lp)"`  
               with a perturbation degree `n`, `m` for `v` and `w`, resp.

**Example:**

```

LIB "grwalk.lib";
ring r = 32003,(z,y,x), lp;

```



```

ideal I = y3+xyz+y2z+xz3, 3+xy+x2y+y2z;
pwalk(I,2,2);
↦ _[1]=zx+8297y8x2+8297y8x+3556y7-8297y6x4+15409y6x3-8297y6x2-8297y5x5+1540\
9y5x4-8297y5x3+3556y5x2+3556y5x+3556y4x3+3556y4x2-10668y4-10668y3x-8297y2\
x9-1185y2x8+14224y2x7-1185y2x6-8297y2x5-14223yx7-10666yx6-10666yx5-14223y\
x4+x5+2x4+x3
↦ _[2]=y9-y7x2-y7x-y6x3-y6x2-3y6-3y5x-y3x7-3y3x6-3y3x5-y3x4-9y2x5-18y2x4-9y\
2x3-27yx3-27yx2-27x
↦ _[3]=zy2+yx2+yx+3

```

See also: [Section D.4.10.3 \[awalk1\]](#), page 1089; [Section D.4.10.4 \[awalk2\]](#), page 1090; [Section D.4.10.1 \[fwalk\]](#), page 1088; [Section 5.1.53 \[groebner\]](#), page 191; [Section D.4.10.6 \[gwalk\]](#), page 1091; [Section 5.1.151 \[std\]](#), page 271; [Section 5.1.152 \[stdfglm\]](#), page 272; [Section D.4.10.2 \[twalk\]](#), page 1089.

### D.4.10.6 gwalk

Procedure from library `grwalk.lib` (see [Section D.4.10 \[grwalk.lib\]](#), page 1088).

**Syntax:** `gwalk(ideal i);`  
`gwalk(ideal i, intvec v, intvec w);`

**Type:** ideal

**Purpose:** compute the standard basis of the ideal, calculated via the improved Groebner walk algorithm from the ordering " $(a(v),lp)$ ", " $dp$ " or " $Dp$ " to the ordering " $(a(w),lp)$ " or " $(a(1,0,...,0),lp)$ ".

**Example:**

```

LIB "grwalk.lib";
/** compute a Groebner basis of I w.r.t. lp.
ring r = 32003,(z,y,x), lp;
ideal I = zy2+yx2+yx+3,
z3x+y3+zyx-yx2-yx-3,
z2yx3-y5+z2yx2+y3x2+y2x3+y3x+y2x2+3z2x+3y2+3yx,
zyx5+y6-y4x2-y3x3+2zyx4-y4x-y3x2+zyx3-3z2yx+3zx3-3y3-3y2x+3zx2,
yx7-y7+y5x2+y4x3+3yx6+y5x+y4x2+3yx5-6zyx3+yx4+3x5+3y4+3y3x-6zyx2+6x4+3x3-9zx;
gwalk(I);
↦ _[1]=y9-y7x2-y7x-y6x3-y6x2-3y6-3y5x-y3x7-3y3x6-3y3x5-y3x4-9y2x5-18y2x4-9y\
2x3-27yx3-27yx2-27x
↦ _[2]=zx+8297y8x2+8297y8x+3556y7-8297y6x4+15409y6x3-8297y6x2-8297y5x5+1540\
9y5x4-8297y5x3+3556y5x2+3556y5x+3556y4x3+3556y4x2-10668y4-10668y3x-8297y2\
x9-1185y2x8+14224y2x7-1185y2x6-8297y2x5-14223yx7-10666yx6-10666yx5-14223y\
x4+x5+2x4+x3
↦ _[3]=zy2+yx2+yx+3

```

See also: [Section D.4.10.3 \[awalk1\]](#), page 1089; [Section D.4.10.4 \[awalk2\]](#), page 1090; [Section D.4.10.1 \[fwalk\]](#), page 1088; [Section 5.1.53 \[groebner\]](#), page 191; [Section D.4.10.5 \[pwalk\]](#), page 1090; [Section 5.1.151 \[std\]](#), page 271; [Section 5.1.152 \[stdfglm\]](#), page 272; [Section D.4.10.2 \[twalk\]](#), page 1089.

### D.4.11 homolog.lib

**Library:** `homolog.lib`

**Purpose:** Procedures for Homological Algebra

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**Procedures:**

#### D.4.11.1 canonMap

Procedure from library `homolog.lib` (see [Section D.4.11 \[homolog.lib\]](#), page 1091).

**Usage:** `canonMap(id)`; `id`= ideal/module,

**Return:** a list `L`, the kernel in two different representations and  
 the cokernel of the canonical map  
 $M \rightarrow \text{Ext}^c_R(\text{Ext}^c_R(M, R), R)$  given by presentations  
 Here `M` is the `R`-module (`R`=basering) given by the presentation  
 defined by `id`, i.e.  $M=R/\text{id}$  resp.  $M=R^n/\text{id}$   
`c` is the codimension of `M`  
`L[1]` is the preimage of the kernel in `R` resp.  $R^n$   
`L[2]` is a presentation of the kernel  
`L[3]` is a presentation of the cokernel

**Example:**

```
LIB "homolog.lib";
ring s=0,(x,y),dp;
ideal i = x,y;
canonMap(i);
↳ [1]:
↳ _[1]=y*gen(1)
↳ _[2]=x*gen(1)
↳ [2]:
↳ _[1]=0
↳ [3]:
↳ _[1]=0
ring R = 0,(x,y,z,w),dp;
ideal I1 = x,y;
ideal I2 = z,w;
ideal I = intersect(I1,I2);
canonMap(I);
↳ [1]:
↳ _[1]=yw*gen(1)
↳ _[2]=xw*gen(1)
↳ _[3]=yz*gen(1)
↳ _[4]=xz*gen(1)
↳ [2]:
↳ _[1]=0
↳ [3]:
↳ _[1]=-w*gen(1)
↳ _[2]=-z*gen(1)
↳ _[3]=-y*gen(1)
↳ _[4]=-x*gen(1)
module M = syz(I);
canonMap(M);
↳ [1]:
```

```

↳   _[1]=z*gen(1)-w*gen(3)
↳   _[2]=z*gen(2)-w*gen(4)
↳   _[3]=x*gen(1)-y*gen(2)
↳   _[4]=x*gen(3)-y*gen(4)
↳ [2]:
↳   _[1]=0
↳ [3]:
↳   _[1]=yw*gen(1)
↳   _[2]=xw*gen(1)
↳   _[3]=yz*gen(1)
↳   _[4]=xz*gen(1)
ring S = 0,(x,y,z,t),Wp(3,4,5,1);
ideal I = x-t3,y-t4,z-t5;
ideal J = eliminate(I,t);
ring T = 0,(x,y,z),Wp(3,4,5);
ideal p = imap(S,J);
ideal p2 = p^2;
canonMap(p2);
↳ [1]:
↳   _[1]=x5*gen(1)-3x2yz*gen(1)+xy3*gen(1)+z3*gen(1)
↳   _[2]=x2z2*gen(1)-2xy2z*gen(1)+y4*gen(1)
↳   _[3]=x4z*gen(1)-x3y2*gen(1)-xyz2*gen(1)+y3z*gen(1)
↳   _[4]=x3yz*gen(1)-x2y3*gen(1)-xz3*gen(1)+y2z2*gen(1)
↳   _[5]=x4y2*gen(1)-4xy3z*gen(1)+2y5*gen(1)+z4*gen(1)
↳ [2]:
↳   _[1]=x*gen(1)
↳   _[2]=y*gen(1)
↳   _[3]=z*gen(1)
↳ [3]:
↳   _[1]=0

```

#### D.4.11.2 cup

Procedure from library `homolog.lib` (see [Section D.4.11 \[homolog.lib\]](#), page 1091).

**Usage:** `cup(M,[,any,any]);` M=module

**Compute:**  $\text{cup-product } \text{Ext}^1(M',M') \times \text{Ext}^1(M',M') \longrightarrow \text{Ext}^2(M',M')$ , where  $M' := R^m/M$ , if M in  $R^m$ , R basering (i.e.  $M' := \text{coker}(\text{matrix}(M))$ ).

If called with  $\geq 2$  arguments: compute symmetrized cup-product

**Assume:** all Ext's are finite dimensional

**Return:**

- if called with 1 argument: matrix, the columns of the output present the coordinates of  $b_i \& b_j$  with respect to a kbase of  $\text{Ext}^2$ , where  $b_1, b_2, \dots$  is a kbase of  $\text{Ext}^1$  and  $\&$  denotes cup product;
- if called with 2 arguments: matrix, the columns of the output present the coordinates of  $(1/2)(b_i \& b_j + b_j \& b_i)$  with respect to a kbase of  $\text{Ext}^2$ ;
- if called with 3 arguments: list,

$L[1]$  = matrix see above (symmetric case, for  $\geq 2$  arguments)

$L[2]$  = matrix of kbase of  $\text{Ext}^1$

$L[3]$  = matrix of kbase of  $\text{Ext}^2$

**Note:** `printlevel`  $\geq 1$ ; shows what is going on.

`printlevel`  $\geq 2$ ; shows result in another representation.

For computing cupproduct of M itself, apply `proc` to `syz(M)`!

**Example:**

```

LIB "homolog.lib";
int p      = printlevel;
ring rr    = 32003,(x,y,z),(dp,C);
ideal I    = x4+y3+z2;
qring o    = std(I);
module M   = [x,y,0,z],[y2,-x3,z,0],[z,0,-y,-x3],[0,z,x,-y2];
print(cup(M));
↳ 1,0, 0, 0,0,0,0,0,0,0, 0,0,0,0,0,0,
↳ 0,-1,0, 0,1,0,0,0,0,0, 0,0,0,0,0,0,
↳ 0,0, -1,0,0,0,0,0,1,0, 0,0,0,0,0,0,
↳ 0,0, 0, 1,0,0,1,0,0,-1,0,0,1,0,0,0
print(cup(M,1));
↳ 1,0,0,0,0,0,0,0,0,0,
↳ 0,0,0,0,0,0,0,0,0,0,
↳ 0,0,0,0,0,0,0,0,0,0,
↳ 0,0,0,1,0,0,0,0,0,0
// 2nd EXAMPLE (shows what is going on)
printlevel = 3;
ring r      = 0,(x,y),(dp,C);
ideal i     = x2-y3;
qring q     = std(i);
module M    = [-x,y],[-y2,x];
print(cup(M));
↳ // vdim (Ext^1) = 2
↳ // kbase of Ext^1(M,M)
↳ // - the columns present the kbase elements in Hom(F(1),F(0))
↳ // - F(*) a free resolution of M
↳ -1,0,
↳ 0, y,
↳ 0, 1,
↳ -1,0
↳ // lift kbase of Ext^1:
↳ // - the columns present liftings of kbase elements into Hom(F(2),F(1))
↳ // - F(*) a free resolution of M
↳ 1,0,
↳ 0,y,
↳ 0,1,
↳ 1,0
↳ // vdim (Ext^2) = 2
↳ // kbase of Ext^2(M,M)
↳ // - the columns present the kbase elements in Hom(F(2),F(0))
↳ // - F(*) is a free resolution of M
↳ -1,0,
↳ 0, y,
↳ 0, 1,
↳ -1,0
↳ // matrix of cup-products (in Ext^2)
↳ -1,0, 0,y,
↳ 0, -y,y,0,
↳ 0, -1,1,0,
↳ -1,0, 0,y
↳ ///// end level 2 /////

```

```

⇒ // the associated matrices of the bilinear mapping 'cup'
⇒ // corresponding to the kbase elements of Ext^2(M,M) are shown,
⇒ // i.e. the rows of the final matrix are written as matrix of
⇒ // a bilinear form on Ext^1 x Ext^1
⇒ //-----component 1:
⇒ 1,0,
⇒ 0,0
⇒ //-----component 2:
⇒ 0,-1,
⇒ 1,0
⇒ ////////// end level 3 //////////
⇒ 1,0, 0,0,
⇒ 0,-1,1,0
printlevel = p;

```

### D.4.11.3 cupproduct

Procedure from library `homolog.lib` (see [Section D.4.11 \[homolog-lib\]](#), page 1091).

**Usage:** `cupproduct(M,N,P,p,q[,any]);`  $M,N,P$  modules,  $p,q$  integers

**Compute:**  $\text{cup-product } \text{Ext}^p(M',N') \times \text{Ext}^q(N',P') \rightarrow \text{Ext}^{p+q}(M',P')$ , where  $M' := R^m/M$ , if  $M$  in  $R^m$ ,  $R$  basering (i.e.  $M' := \text{coker}(\text{matrix}(M))$ )

**Assume:** all  $\text{Ext}$ 's are of finite dimension

**Return:** - if called with 5 arguments: matrix of the associated linear map  $\text{Ext}^p$  (tensor)  $\text{Ext}^q \rightarrow \text{Ext}^{p+q}$ , i.e. the columns of `<matrix>` present the coordinates of the cup products ( $b_i$  &  $c_j$ ) with respect to a kbase of  $\text{Ext}^{p+q}$  ( $b_i$  resp.  $c_j$  are the chosen bases of  $\text{Ext}^p$ , resp.  $\text{Ext}^q$ ).  
 - if called with 6 arguments: list  $L$ ,  
 $L[1]$  = matrix (see above)  
 $L[2]$  = matrix of kbase of  $\text{Ext}^p(M',N')$   
 $L[3]$  = matrix of kbase of  $\text{Ext}^q(N',P')$   
 $L[4]$  = matrix of kbase of  $\text{Ext}^{p+q}(N',P')$

**Note:** `printlevel >=1`; shows what is going on.  
`printlevel >=2`; shows the result in another representation.  
 For computing the cupproduct of  $M,N$  itself, apply `proc` to `syz(M)`, `syz(N)`!

**Example:**

```

LIB "homolog.lib";
int p      = printlevel;
ring rr    = 32003,(x,y,z),(dp,C);
ideal I    = x4+y3+z2;
qring o    = std(I);
module M   = [x,y,0,z],[y2,-x3,z,0],[z,0,-y,-x3],[0,z,x,-y2];
print(cupproduct(M,M,M,1,3));
⇒ 1,0, 0, 0,0,0,0,0,0, 0,0,0,0,0,0,
⇒ 0,-1,0, 0,1,0,0,0,0, 0,0,0,0,0,0,
⇒ 0,0, -1,0,0,0,0,0,1,0, 0,0,0,0,0,0,
⇒ 0,0, 0, 1,0,0,1,0,0,-1,0,0,1,0,0,0
printlevel = 3;
list l     = (cupproduct(M,M,M,1,3,"any"));
⇒ // vdim Ext(M,N) = 4

```

```

⇒ // kbase of Extp(M,N)
⇒ // - the columns present the kbase elements in Hom(F(p),G(0))
⇒ // - F(*),G(*) are free resolutions of M and N
⇒ 0, 0, 1, 0,
⇒ 0, y, 0, 0,
⇒ 1, 0, 0, 0,
⇒ 0, 0, 0, y,
⇒ 0, -1,0, 0,
⇒ 0, 0, x2,0,
⇒ 0, 0, 0, -x2,
⇒ 1, 0, 0, 0,
⇒ 0, 0, 0, -1,
⇒ -1,0, 0, 0,
⇒ 0, 1, 0, 0,
⇒ 0, 0, 1, 0,
⇒ -1,0, 0, 0,
⇒ 0, 0, 0, x2y,
⇒ 0, 0, x2,0,
⇒ 0, -y,0, 0
⇒ // vdim Ext(N,P) = 4
⇒ // kbase of Ext(N,P):
⇒ 0, 0, 1, 0,
⇒ 0, 0, 0, y,
⇒ 1, 0, 0, 0,
⇒ 0, -y,0, 0,
⇒ 0, -1,0, 0,
⇒ 1, 0, 0, 0,
⇒ 0, 0, 0, -x2,
⇒ 0, 0, -x2,0,
⇒ 0, 0, 0, -1,
⇒ 0, 0, 1, 0,
⇒ 0, 1, 0, 0,
⇒ 1, 0, 0, 0,
⇒ -1,0, 0, 0,
⇒ 0, -y,0, 0,
⇒ 0, 0, x2, 0,
⇒ 0, 0, 0, -x2y
⇒ // kbase of Extq(N,P)
⇒ // - the columns present the kbase elements in Hom(G(q),H(0))
⇒ // - G(*),H(*) are free resolutions of N and P
⇒ 0, 0, 1, 0,
⇒ 0, 0, 0, y,
⇒ 1, 0, 0, 0,
⇒ 0, -y,0, 0,
⇒ 0, -1,0, 0,
⇒ 1, 0, 0, 0,
⇒ 0, 0, 0, -x2,
⇒ 0, 0, -x2,0,
⇒ 0, 0, 0, -1,
⇒ 0, 0, 1, 0,
⇒ 0, 1, 0, 0,
⇒ 1, 0, 0, 0,
⇒ -1,0, 0, 0,

```

```

⇒ 0, -y, 0, 0,
⇒ 0, 0, x2, 0,
⇒ 0, 0, 0, -x2y
⇒ // vdim Ext(M,P) = 4
⇒ // kbase of Extp+q(M,P)
⇒ // - the columns present the kbase elements in Hom(F(p+q),H(0))
⇒ // - F(*),H(*) are free resolutions of M and P
⇒ 0, 0, 1, 0,
⇒ 0, 0, 0, y,
⇒ 1, 0, 0, 0,
⇒ 0, -y, 0, 0,
⇒ 0, -1, 0, 0,
⇒ 1, 0, 0, 0,
⇒ 0, 0, 0, -x2,
⇒ 0, 0, -x2, 0,
⇒ 0, 0, 0, -1,
⇒ 0, 0, 1, 0,
⇒ 0, 1, 0, 0,
⇒ 1, 0, 0, 0,
⇒ -1, 0, 0, 0,
⇒ 0, -y, 0, 0,
⇒ 0, 0, x2, 0,
⇒ 0, 0, 0, -x2y
⇒ // lifting of kbase of Extp(M,N)
⇒ // - the columns present liftings of kbase elements in Hom(F(p+q),G(q))
⇒ 1, 0, 0, 0,
⇒ 0, -y, 0, 0,
⇒ 0, 0, x2, 0,
⇒ 0, 0, 0, x2y,
⇒ 0, 1, 0, 0,
⇒ 1, 0, 0, 0,
⇒ 0, 0, 0, -x2,
⇒ 0, 0, x2, 0,
⇒ 0, 0, -1, 0,
⇒ 0, 0, 0, y,
⇒ 1, 0, 0, 0,
⇒ 0, y, 0, 0,
⇒ 0, 0, 0, -1,
⇒ 0, 0, -1, 0,
⇒ 0, -1, 0, 0,
⇒ 1, 0, 0, 0
⇒ // matrix of cup-products (in Extp+q)
⇒ 0, 0, -1, 0, 0, 0, 0, y, 1, 0, 0, 0, 0, 0, y, 0, 0,
⇒ 0, 0, 0, y, 0, 0, y, 0, 0, -y, 0, 0, y, 0, 0, 0, 0,
⇒ 1, 0, 0, 0, 0, y, 0, 0, 0, 0, x2, 0, 0, 0, 0, -x2y,
⇒ 0, y, 0, 0, -y, 0, 0, 0, 0, 0, 0, x2y, 0, 0, x2y, 0,
⇒ 0, 1, 0, 0, -1, 0, 0, 0, 0, 0, 0, x2, 0, 0, x2, 0,
⇒ 1, 0, 0, 0, 0, y, 0, 0, 0, 0, x2, 0, 0, 0, 0, -x2y,
⇒ 0, 0, 0, -x2, 0, 0, -x2, 0, 0, x2, 0, 0, -x2, 0, 0, 0,
⇒ 0, 0, x2, 0, 0, 0, 0, -x2y, -x2, 0, 0, 0, 0, -x2y, 0, 0,
⇒ 0, 0, 0, -1, 0, 0, -1, 0, 0, 1, 0, 0, -1, 0, 0, 0,
⇒ 0, 0, -1, 0, 0, 0, 0, y, 1, 0, 0, 0, 0, y, 0, 0,
⇒ 0, -1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, -x2, 0, 0, -x2, 0,

```

```

⇒ 1, 0, 0, 0, 0, y, 0, 0, 0, 0, x2, 0, 0, 0, 0, -x2y,
⇒ -1, 0, 0, 0, 0, -y, 0, 0, 0, 0, -x2, 0, 0, 0, 0, x2y,
⇒ 0, y, 0, 0, -y, 0, 0, 0, 0, 0, 0, x2y, 0, 0, x2y, 0,
⇒ 0, 0, -x2, 0, 0, 0, 0, x2y, x2, 0, 0, 0, 0, 0, x2y, 0, 0,
⇒ 0, 0, 0, -x2y, 0, 0, -x2y, 0, 0, x2y, 0, 0, -x2y, 0, 0, 0
⇒ ##### end level 2 #####
⇒ // the associated matrices of the bilinear mapping 'cup'
⇒ // corresponding to the kbase elements of Extp+q(M,P) are shown,
⇒ // i.e. the rows of the final matrix are written as matrix of
⇒ // a bilinear form on Extp x Extq
⇒ //----component 1:
⇒ 1,0,0,0,
⇒ 0,0,0,0,
⇒ 0,0,0,0,
⇒ 0,0,0,0,
⇒ 0,0,0,0,
⇒ //----component 2:
⇒ 0,-1,0,0,
⇒ 1,0, 0,0,
⇒ 0,0, 0,0,
⇒ 0,0, 0,0
⇒ //----component 3:
⇒ 0,0,-1,0,
⇒ 0,0,0, 0,
⇒ 1,0,0, 0,
⇒ 0,0,0, 0
⇒ //----component 4:
⇒ 0,0, 0,1,
⇒ 0,0, 1,0,
⇒ 0,-1,0,0,
⇒ 1,0, 0,0
⇒ ##### end level 3 #####
show(l[1]);show(l[2]);
⇒ // matrix, 4x16
⇒ 1,0, 0, 0,0,0,0,0,0,0, 0,0,0,0,0,0,0,
⇒ 0,-1,0, 0,1,0,0,0,0,0,0, 0,0,0,0,0,0,0,
⇒ 0,0, -1,0,0,0,0,0,0,1,0, 0,0,0,0,0,0,0,
⇒ 0,0, 0, 1,0,0,1,0,0,-1,0,0,1,0,0,0,0
⇒ // matrix, 16x4
⇒ 0, 0, 1, 0,
⇒ 0, y, 0, 0,
⇒ 1, 0, 0, 0,
⇒ 0, 0, 0, y,
⇒ 0, -1,0, 0,
⇒ 0, 0, x2,0,
⇒ 0, 0, 0, -x2,
⇒ 1, 0, 0, 0,
⇒ 0, 0, 0, -1,
⇒ -1,0, 0, 0,
⇒ 0, 1, 0, 0,
⇒ 0, 0, 1, 0,
⇒ -1,0, 0, 0,
⇒ 0, 0, 0, x2y,
⇒ 0, 0, x2,0,

```



```

    ↦ 0, -y, 0, 0
    printlevel = p;

```

#### D.4.11.4 depth

Procedure from library `homolog.lib` (see [Section D.4.11 \[homolog-lib\], page 1091](#)).

**Usage:** `depth(M,[I]);` M module, I ideal

**Return:** int,  
 - if called with 1 argument: the depth of  $M'=\text{coker}(M)$  w.r.t. the ideal generated by the variables in the basering  
 (the maximal ideal, if the ring is local)  
 - if called with 2 arguments: the depth of  $M'=\text{coker}(M)$  w.r.t. the ideal I.

**Note:** Not checked: if  $I \cdot M' = M'$ , depth is infinity.

**Note:** procedure makes use of KoszulHomology.

**Example:**

```

LIB "homolog.lib";
ring R=0,(x,y,z),dp;
ideal I=x^2,xy,yz;
module M=0;
depth(M,I); // depth(<x^2,xy,yz>,Q[x,y,z])
↦ 2
M=[1];
depth(M); // depth(0)
↦ 3
ring r=0,(x,y,z),ds; // local ring
matrix M[2][2]=x,xy,1+yz,0;
print(M);
↦ x, xy,
↦ 1+yz,0
depth(M); // depth(maxideal,coker(M))
↦ 2
ideal I=x;
depth(M,I); // depth(<x>,coker(M))
↦ 0
I=x+z;
depth(M,I); // depth(<x+z>,coker(M))
↦ 1

```

#### D.4.11.5 Ext\_R

Procedure from library `homolog.lib` (see [Section D.4.11 \[homolog-lib\], page 1091](#)).

**Usage:** `Ext_R(v,M[,p]);` v int resp. intvec, M module, p int

**Compute:** A presentation of  $\text{Ext}^k(M',R)$ ; for  $k=v[1],v[2],\dots$ ,  $M'=\text{coker}(M)$ . Let

$$0 \leftarrow M' \leftarrow F_0 \leftarrow M \leftarrow F_1 \leftarrow F_2 \leftarrow \dots$$

be a free resolution of  $M'$ . If

$$0 \rightarrow F_0^* \xrightarrow{-A_1} F_1^* \xrightarrow{-A_2} F_2^* \xrightarrow{-A_3} \dots$$

is the dual sequence,  $F_i^* = \text{Hom}(F_i, R)$ , then  $\text{Ext}^k = \ker(A_{k+1})/\text{im}(A_k)$  is presented as in the following exact sequences:

$$\begin{aligned} R^p &\xrightarrow{\text{syz}(A_{k+1})} F_k^* \xrightarrow{A_{k+1}} F_{k+1}^* , \\ R^q &\xrightarrow{\text{Ext}^k} R^p \xrightarrow{\text{syz}(A_{k+1})} F_k^* / \text{im}(A_k) . \end{aligned}$$

Hence,  $\text{Ext}^k = \text{modulo}(\text{syz}(A_{k+1}), A_k)$  presents  $\text{Ext}^k(M', R)$ .

**Return:** - module  $\text{Ext}$ , a presentation of  $\text{Ext}^k(M', R)$  if  $v$  is of type `int`  
 - a list of  $\text{Ext}^k$  ( $k=v[1], v[2], \dots$ ) if  $v$  is of type `intvec`.  
 - In case of a third argument of type `int` return a list  $l$ :

$l[1]$  = module  $\text{Ext}^k$  resp. list of  $\text{Ext}^k$   
 $l[2]$  = SB of  $\text{Ext}^k$  resp. list of SB of  $\text{Ext}^k$   
 $l[3]$  = matrix resp. list of matrices, each representing a kbase of  $\text{Ext}^k$   
 (if finite dimensional)

**Display:** `printlevel >= 0`: (affine) dimension of  $\text{Ext}^k$  for each  $k$  (default) `printlevel >= 1`:  $A_k$ ,  $A_{k+1}$  and kbase of  $\text{Ext}^k$  in  $F_k^*$

**Note:** In order to compute  $\text{Ext}^k(M, R)$  use the command `Ext_R(k, syz(M))`;  
 By default, the procedure uses the `mres` command. If called with the additional parameter `"sres"`, the `sres` command is used instead.  
 If the attribute `"isHomog"` has been set for the input module, it is also set for the returned module (accordingly).

**Example:**

```
LIB "homolog.lib";
int p      = printlevel;
printlevel = 1;
ring r      = 0, (x,y,z), dp;
ideal i     = x2y, y2z, z3x;
module E    = Ext_R(1, i);    //computes Ext^1(r/i, r)
⇒ // Computing Ext^1:
⇒ // Let 0 ← coker(M) ← F0 ← F1 ← F2 ← ... be a resolution of M,
⇒ // then F1* → F2* is given by:
⇒ x2, -yz, 0,
⇒ 0, z3, -xy,
⇒ xz2, 0, -y2
⇒ // and F0* → F1* is given by:
⇒ y2z,
⇒ x2y,
⇒ xz3
⇒
⇒ // dimension of Ext^1: -1
⇒
is_zero(E);
⇒ 1
qring R     = std(x2+yz);
intvec v    = 0, 2;
printlevel = 2;           //shows what is going on
ideal i     = x, y, z;     //computes Ext^i(r/(x,y,z), r/(x2+yz)), i=0, 2
list L      = Ext_R(v, i, 1); //over the qring R=r/(x2+yz), std and kbase
⇒ // Computing Ext^0:
⇒ // Let 0 ← coker(M) ← F0 ← F1 ← F2 ← ... be a resolution of M,
⇒ // then F0* → F1* is given by:
⇒ z,
⇒ y,
⇒ x
```

```

⇒ // and  $F_1 \rightarrow F_0$  is given by:
⇒ 0
⇒
⇒ // dimension of  $\text{Ext}^0$ : -1
⇒
⇒ // columns of matrix are kbase of  $\text{Ext}^0$  in  $F_0$ :
⇒ 0
⇒
⇒ // Computing  $\text{Ext}^2$ :
⇒ // Let  $0 \leftarrow \text{coker}(M) \leftarrow F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow \dots$  be a resolution of  $M$ ,
⇒ // then  $F_2 \rightarrow F_3$  is given by:
⇒ x,-y,z, 0,
⇒ z,x, 0, z,
⇒ 0,0, x, y,
⇒ 0,0, -z,x
⇒ // and  $F_1 \rightarrow F_2$  is given by:
⇒ y,-z,0,
⇒ x,0, -z,
⇒ 0,x, -y,
⇒ 0,z, x
⇒
⇒ // dimension of  $\text{Ext}^2$ : 0
⇒ // vdim of  $\text{Ext}^2$ : 1
⇒
⇒ // columns of matrix are kbase of  $\text{Ext}^2$  in  $F_2$ :
⇒ x,
⇒ -z,
⇒ 0,
⇒ 0
⇒
printlevel = p;

```

#### D.4.11.6 Ext

Procedure from library `homolog.lib` (see [Section D.4.11 \[homolog.lib\]](#), page 1091).

**Usage:** `Ext(v,M,N[,any]);`  $v$  int resp. intvec,  $M, N$  modules

**Compute:** A presentation of  $\text{Ext}^k(M', N')$ ; for  $k=v[1], v[2], \dots$  where  $M' = \text{coker}(M)$  and  $N' = \text{coker}(N)$ . Let

$$\begin{aligned} 0 &\leftarrow M' \leftarrow F_0 \leftarrow M \leftarrow F_1 \leftarrow F_2 \leftarrow \dots, \\ 0 &\leftarrow N' \leftarrow G_0 \leftarrow N \leftarrow G_1 \end{aligned}$$

be a free resolution of  $M'$ , resp. a presentation of  $N'$ . Consider the commutative diagram

$$\begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & |^{\wedge} & & |^{\wedge} & & |^{\wedge} & \\ \rightarrow & \text{Hom}(F_{k-1}, N') & \xrightarrow{-A_k} & \text{Hom}(F_k, N') & \xrightarrow{-A_{k+1}} & \text{Hom}(F_{k+1}, N') & \\ & |^{\wedge} & & |^{\wedge} & & |^{\wedge} & \\ \rightarrow & \text{Hom}(F_{k-1}, G_0) & \xrightarrow{-A_k} & \text{Hom}(F_k, G_0) & \xrightarrow{-A_{k+1}} & \text{Hom}(F_{k+1}, G_0) & \\ & & & |^{\wedge} & & |^{\wedge} & \\ & & & |^C & & |^B & \\ & & & \text{Hom}(F_k, G_1) & \xrightarrow{\quad\quad\quad} & \text{Hom}(F_{k+1}, G_1) & \end{array}$$

( $A_k, A_{k+1}$  induced by  $M$  and  $B, C$  induced by  $N$ ).

Let  $K = \text{modulo}(A_{k+1}, B)$ ,  $J = \text{module}(A_k) + \text{module}(C)$  and  $\text{Ext} = \text{modulo}(K, J)$ , then we have exact sequences

$$R^p \xrightarrow{-K} \text{Hom}(F_k, G_0) \xrightarrow{-A_{k+1}} \text{Hom}(F_{k+1}, G_0) / \text{im}(B),$$

$$R^q \xrightarrow{-\text{Ext}} R^p \xrightarrow{-K} \text{Hom}(F_k, G_0) / (\text{im}(A_k) + \text{im}(C)).$$

Hence,  $\text{Ext}$  presents  $\text{Ext}^k(M', N')$ .

**Return:** - module  $\text{Ext}$ , a presentation of  $\text{Ext}^k(M', N')$  if  $v$  is of type `int`  
 - a list of  $\text{Ext}^k$  ( $k=v[1], v[2], \dots$ ) if  $v$  is of type `intvec`.  
 - In case of a third argument of any type return a list  $l$ :

$l[1]$  = module  $\text{Ext}$ /list of  $\text{Ext}^k$   
 $l[2]$  = SB of  $\text{Ext}$ /list of SB of  $\text{Ext}^k$   
 $l[3]$  = matrix/list of matrices, each representing a kbase of  $\text{Ext}^k$   
 (if finite dimensional)

**Display:** `printlevel >= 0`: dimension, `vdim` of  $\text{Ext}^k$  for each  $k$  (default).  
`printlevel >= 1`: matrices  $A_k$ ,  $A_{k+1}$  and kbase of  $\text{Ext}^k$  in  $\text{Hom}(F_k, G_0)$  (if finite dimensional)

**Note:** In order to compute  $\text{Ext}^k(M, N)$  use the command  $\text{Ext}(k, \text{syz}(M), \text{syz}(N))$ ; or: list  $P = \text{mres}(M, 2)$ ; list  $Q = \text{mres}(N, 2)$ ;  $\text{Ext}(k, P[2], Q[2])$ ;

**Example:**

```
LIB "homolog.lib";
int p      = printlevel;
printlevel = 1;
ring r      = 0, (x, y), dp;
ideal i     = x2-y3;
ideal j     = x2-y5;
list E      = Ext(0..2, i, j); // Ext^k(r/i, r/j) for k=0,1,2 over r
⇒ // Computing Ext^0 (help Ext; gives an explanation):
⇒ // Let 0<--coker(M)<--F0<--F1<--F2<--... be a resolution of coker(M),
⇒ // and 0<--coker(N)<--G0<--G1 a presentation of coker(N),
⇒ // then Hom(F0, G0)-->Hom(F1, G0) is given by:
⇒ y3-x2
⇒ // and Hom(F-1, G0) + Hom(F0, G1)-->Hom(F0, G0) is given by:
⇒ 0, -y5+x2
⇒
⇒ // dimension of Ext^0:  -1
⇒
⇒ // Computing Ext^1 (help Ext; gives an explanation):
⇒ // Let 0<--coker(M)<--F0<--F1<--F2<--... be a resolution of coker(M),
⇒ // and 0<--coker(N)<--G0<--G1 a presentation of coker(N),
⇒ // then Hom(F1, G0)-->Hom(F2, G0) is given by:
⇒ 0
⇒ // and Hom(F0, G0) + Hom(F1, G1)-->Hom(F1, G0) is given by:
⇒ y3-x2, -y5+x2
⇒
⇒ // dimension of Ext^1:  0
⇒ // vdim of Ext^1:      10
⇒
⇒ // Computing Ext^2 (help Ext; gives an explanation):
⇒ // Let 0<--coker(M)<--F0<--F1<--F2<--... be a resolution of coker(M),
⇒ // and 0<--coker(N)<--G0<--G1 a presentation of coker(N),
```

```

⇒ // then Hom(F2,G0)→Hom(F3,G0) is given by:
⇒ 1
⇒ // and Hom(F1,G0) + Hom(F2,G1)→Hom(F2,G0) is given by:
⇒ 0,-y5+x2
⇒
⇒ // dimension of Ext^2: -1
⇒
qring R      = std(i);
ideal j      = fetch(r,j);
module M     = [-x,y],[-y2,x];
printlevel = 2;
module E1    = Ext(1,M,j);      // Ext^1(R^2/M,R/j) over R=r/i
⇒ // Computing Ext^1 (help Ext; gives an explanation):
⇒ // Let 0←coker(M)←F0←F1←F2←... be a resolution of coker(M),
⇒ // and 0←coker(N)←G0←G1 a presentation of coker(N),
⇒ // then Hom(F1,G0)→Hom(F2,G0) is given by:
⇒ x, -y,
⇒ y2,-x
⇒ // and Hom(F0,G0) + Hom(F1,G1)→Hom(F1,G0) is given by:
⇒ x, -y,-y5+x2,0,
⇒ y2,-x,0,      -y5+x2
⇒
⇒ // dimension of Ext^1: -1
⇒
list l       = Ext(4,M,M,1);    // Ext^4(R^2/M,R^2/M) over R=r/i
⇒ // Computing Ext^4 (help Ext; gives an explanation):
⇒ // Let 0←coker(M)←F0←F1←F2←... be a resolution of coker(M),
⇒ // and 0←coker(N)←G0←G1 a presentation of coker(N),
⇒ // then Hom(F4,G0)→Hom(F5,G0) is given by:
⇒ x, -y,0, 0,
⇒ y2,-x,0, 0,
⇒ 0, 0, x, -y,
⇒ 0, 0, y2,-x
⇒ // and Hom(F3,G0) + Hom(F4,G1)→Hom(F4,G0) is given by:
⇒ x, -y,0, 0, -x,0, -y2,0,
⇒ y2,-x,0, 0, 0, -x,0, -y2,
⇒ 0, 0, x, -y,y, 0, x, 0,
⇒ 0, 0, y2,-x,0, y, 0, x
⇒
⇒ // dimension of Ext^4: 0
⇒ // vdim of Ext^4: 2
⇒
⇒ // columns of matrix are kbase of Ext^4 in Hom(F4,G0)
⇒ 1,0,
⇒ 0,y,
⇒ 0,1,
⇒ 1,0
⇒
⇒ // element 1 of kbase of Ext^4 in Hom(F4,G0)
⇒ // as matrix: F4→G0
⇒ 1,0,
⇒ 0,1
⇒ // element 2 of kbase of Ext^4 in Hom(F4,G0)

```

```

⇒ // as matrix: F4-->G0
⇒ 0,y,
⇒ 1,0
⇒
printlevel = p;

```

#### D.4.11.7 fitting

Procedure from library `homolog.lib` (see [Section D.4.11 \[homolog.lib\], page 1091](#)).

**Usage:**     `fitting (M,n)`; M module, n int

**Return:**    ideal, (standard basis of) n-th Fitting ideal of  $M'=\text{coker}(M)$ .

**Example:**

```

LIB "homolog.lib";
ring R=0,x(0..4),dp;
matrix M[2][4]=x(0),x(1),x(2),x(3),x(1),x(2),x(3),x(4);
print(M);
⇒ x(0),x(1),x(2),x(3),
⇒ x(1),x(2),x(3),x(4)
fitting(M,-1);
⇒ _[1]=0
fitting(M,0);
⇒ _[1]=x(3)^2-x(2)*x(4)
⇒ _[2]=x(2)*x(3)-x(1)*x(4)
⇒ _[3]=x(1)*x(3)-x(0)*x(4)
⇒ _[4]=x(2)^2-x(0)*x(4)
⇒ _[5]=x(1)*x(2)-x(0)*x(3)
⇒ _[6]=x(1)^2-x(0)*x(2)
fitting(M,1);
⇒ _[1]=x(4)
⇒ _[2]=x(3)
⇒ _[3]=x(2)
⇒ _[4]=x(1)
⇒ _[5]=x(0)
fitting(M,2);
⇒ _[1]=1

```

#### D.4.11.8 flatteningStrat

Procedure from library `homolog.lib` (see [Section D.4.11 \[homolog.lib\], page 1091](#)).

**Usage:**     `flatteningStrat(M)`; M module

**Return:**    list of ideals.

The list entries  $L[1], \dots, L[r]$  describe the flattening stratification of  $M'=\text{coker}(M)$ : setting  $L[0]=0$ ,  $L[r+1]=1$ , the flattening stratification is given by the open sets  $\text{Spec}(A/V(L[i-1])) \setminus V(L[i])$ ,  $i=1, \dots, r+1$  ( $A = \text{basering}$ ).

**Note:**     for more information see the book 'A Singular Introduction to Commutative Algebra' (by Greuel/Pfister, Springer 2002).

**Example:**

```

LIB "homolog.lib";
ring A = 0,x(0..4),dp;

```

```

// presentation matrix:
matrix M[2][4] = x(0),x(1),x(2),x(3),x(1),x(2),x(3),x(4);
list L = flatteningStrat(M);
L;
⇒ [1]:
⇒   _[1]=x(3)^2-x(2)*x(4)
⇒   _[2]=x(2)*x(3)-x(1)*x(4)
⇒   _[3]=x(1)*x(3)-x(0)*x(4)
⇒   _[4]=x(2)^2-x(0)*x(4)
⇒   _[5]=x(1)*x(2)-x(0)*x(3)
⇒   _[6]=x(1)^2-x(0)*x(2)
⇒ [2]:
⇒   _[1]=x(4)
⇒   _[2]=x(3)
⇒   _[3]=x(2)
⇒   _[4]=x(1)
⇒   _[5]=x(0)

```

#### D.4.11.9 Hom

Procedure from library `homolog.lib` (see [Section D.4.11 \[homolog-lib\]](#), page 1091).

**Usage:** `Hom(M,N,[any]);` M,N=modules

**Compute:** A presentation of  $\text{Hom}(M',N')$ ,  $M'=\text{coker}(M)$ ,  $N'=\text{coker}(N)$  as follows: let

$$F_1 \xrightarrow{-M-} F_0 \xrightarrow{->} M' \xrightarrow{->} 0, \quad G_1 \xrightarrow{-N-} G_0 \xrightarrow{->} N' \xrightarrow{->} 0$$

be presentations of  $M'$  and  $N'$ . Consider

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & |^{\wedge} & & |^{\wedge} \\
 0 \xrightarrow{->} \text{Hom}(M',N') & \xrightarrow{->} & \text{Hom}(F_0,N') & \xrightarrow{->} & \text{Hom}(F_1,N') \\
 & & |^{\wedge} & & |^{\wedge} \\
 \text{(A: induced by M)} & & \text{Hom}(F_0,G_0) \xrightarrow{-A-} & & \text{Hom}(F_1,G_0) \\
 & & |^{\wedge} & & |^{\wedge} \\
 \text{(B,C:induced by N)} & & |^{\wedge} & & |^{\wedge} \\
 & & |^{\wedge} & & |^{\wedge} \\
 & & \text{Hom}(F_0,G_1) \xrightarrow{->} & & \text{Hom}(F_1,G_1)
 \end{array}$$

Let  $D=\text{modulo}(A,B)$  and  $\text{Hom}=\text{modulo}(D,C)$ , then we have exact sequences

$$R^p \xrightarrow{-D-} \text{Hom}(F_0,G_0) \xrightarrow{-A-} \text{Hom}(F_1,G_0)/\text{im}(B),$$

$$R^q \xrightarrow{-\text{Hom}-} R^p \xrightarrow{-D-} \text{Hom}(F_0,G_0)/\text{im}(C) \xrightarrow{-A-} \text{Hom}(F_1,G_0)/\text{im}(B).$$

Hence  $\text{Hom}$  presents  $\text{Hom}(M',N')$

**Return:** module  $\text{Hom}$ , a presentation of  $\text{Hom}(M',N')$ , resp., in case of 3 arguments, a list  $l$  (of size  $\leq 3$ ):

- $l[1] = \text{Hom}$
- $l[2] = \text{SB of Hom}$
- $l[3] = \text{kbase of coker(Hom) (if finite dimensional, not 0),}$   
represented by elements in  $\text{Hom}(F_0,G_0)$  via mapping  $D$

**Display:** `printlevel >=0:` (affine) dimension of  $\text{Hom}$  (default)  
`printlevel >=1:`  $D$  and  $C$  and kbase of  $\text{coker}(\text{Hom})$  in  $\text{Hom}(F_0,G_0)$   
`printlevel >=2:` elements of kbase of  $\text{coker}(\text{Hom})$  as matrix  $:F_0 \rightarrow G_0$

**Note:** DISPLAY is as described only for a direct call of 'Hom'. Calling 'Hom' from another proc has the same effect as decreasing printlevel by 1.

**Example:**

```
LIB "homolog.lib";
int p      = printlevel;
printlevel= 1; //in 'example proc' printlevel has to be increased by 1
ring r      = 0,(x,y),dp;
ideal i     = x2-y3,xy;
qring q     = std(i);
ideal i     = fetch(r,i);
module M    = [-x,y],[-y2,x],[x3];
module H    = Hom(M,i);
⇒ // dimension of Hom: 0
⇒ // vdim of Hom:      5
⇒
⇒ // given F1 --M-> F0 -->M'--> 0 and G1 --N-> G0 -->N'--> 0,
⇒ // show D = ker( Hom(F0,G0) --> Hom(F1,G0)/im(Hom(F1,G1)->Hom(F1,G0)) )
⇒ y,x, 0,
⇒ x,y2,x2
⇒ // show C = im ( Hom(F0,G1) --> Hom(F0,G0) )
⇒ -y3+x2,0, xy,0,
⇒ 0, -y3+x2,0, xy
⇒
print(H);
⇒ 0, x, 0,y2,0,
⇒ y, 0, 0,-x,x2,
⇒ -1,-1,x,0, 0
printlevel= 2;
list L      = Hom(M,i,1);"";
⇒ // dimension of Hom: 0
⇒ // vdim of Hom:      5
⇒
⇒ // given F1 --M-> F0 -->M'--> 0 and G1 --N-> G0 -->N'--> 0,
⇒ // show D = ker( Hom(F0,G0) --> Hom(F1,G0)/im(Hom(F1,G1)->Hom(F1,G0)) )
⇒ y,x, 0,
⇒ x,y2,x2
⇒ // show C = im ( Hom(F0,G1) --> Hom(F0,G0) )
⇒ -y3+x2,0, xy,0,
⇒ 0, -y3+x2,0, xy
⇒
⇒ // element 1 of kbase of Hom in Hom(F0,G0) as matrix: F0-->G0:
⇒ y2,xy
⇒ // element 2 of kbase of Hom in Hom(F0,G0) as matrix: F0-->G0:
⇒ y,x
⇒ // element 3 of kbase of Hom in Hom(F0,G0) as matrix: F0-->G0:
⇒ x2,xy2
⇒ // element 4 of kbase of Hom in Hom(F0,G0) as matrix: F0-->G0:
⇒ x,y2
⇒ // element 5 of kbase of Hom in Hom(F0,G0) as matrix: F0-->G0:
⇒ 0,x2
⇒
printlevel=1;
```



```

ring s      = 3,(x,y,z),(c,dp);
ideal i     = jacob(ideal(x2+y5+z4));
qring rq=std(i);
matrix M[2][2]=xy,x3,5y,4z,x2;
matrix N[3][2]=x2,x,y3,3xz,x2z,z;
print(M);
↳ xy,x3,
↳ -y,z
print(N);
↳ x2, x,
↳ y3, 0,
↳ x2z,z
list l=Hom(M,N,1);
↳ // dimension of Hom: 0
↳ // vdim of Hom: 16
↳
↳ // given F1 --M-> F0 -->M'--> 0 and G1 --N-> G0 -->N'--> 0,
↳ // show D = ker( Hom(F0,G0) --> Hom(F1,G0)/im(Hom(F1,G1)->Hom(F1,G0)) )
↳ 0,0, 0,0, 0, 0,0, 1,
↳ 0,0, 0,0, 0, 0,y3z2,0,
↳ 0,0, 0,0, 0, 1,0, 0,
↳ 0,0, 0,y3,y2z2,0,0, 0,
↳ 0,0, 1,0, 0, 0,0, 0,
↳ z,y3,0,0, 0, 0,0, 0
↳ // show C = im ( Hom(F0,G1) --> Hom(F0,G0) )
↳ x2, 0, x,0,
↳ 0, x2, 0,x,
↳ y3, 0, 0,0,
↳ 0, y3, 0,0,
↳ x2z,0, z,0,
↳ 0, x2z,0,z
↳
↳ // columns of matrix are kbase of Hom in Hom(F0,G0)
↳ 0, 0, 0, 0,0,0, 0, 0, 0, 0, 0, 0,0, 0,0,0,
↳ 0, 0, 0, 0,0,0, 0, 0, 0, 0, 0, 0,0, 0,0,y3z2,
↳ 0, 0, 0, 0,0,0, y2z2,yz2,z2,y2z,yz,z,y2,y,1,0,
↳ 0, 0, 0, 0,0,y2z2,0, 0, 0, 0, 0, 0,0, 0,0,0,
↳ 0, y3,y2,y,1,0, 0, 0, 0, 0, 0, 0,0, 0,0,0,
↳ y3,0, 0, 0,0,0, 0, 0, 0, 0, 0, 0,0, 0,0,0
printlevel = p;

```

#### D.4.11.10 homology

Procedure from library `homolog.lib` (see [Section D.4.11 \[homolog.lib\]](#), page 1091).

**Usage:** `homology(A,B,M,N);`

**Compute:** Let  $M$  and  $N$  be submodules of  $R^m$  and  $R^n$  presenting  $M'=R^m/M$ ,  $N'=R^n/N$  ( $R$ =basering) and let  $A,B$  matrices inducing maps

$$R^k \xrightarrow{A} R^m \xrightarrow{B} R^n.$$

Compute a presentation of the module

$$\ker(B)/\operatorname{im}(A) := \ker(M'/\operatorname{im}(A) \xrightarrow{B} N'/\operatorname{im}(BM)+\operatorname{im}(BA)).$$

If  $B$  induces a map  $M' \rightarrow N'$  (i.e  $BM=0$ ) and if  $\operatorname{im}(A)$  is contained in  $\ker(B)$  (that is,  $BA=0$ ) then  $\ker(B)/\operatorname{im}(A)$  is the homology of the complex

$$R^k \xrightarrow{A} M' \xrightarrow{B} N'.$$

**Return:** module H, a presentation of  $\ker(B)/\text{im}(A)$ .

**Note:** homology returns a free module of rank m if  $\ker(B)=\text{im}(A)$ .

**Example:**

```
LIB "homolog.lib";
ring r;
ideal id=maxideal(4); // GB
qring qr=id;
module N=maxideal(3)*freemodule(2);
module M=maxideal(2)*freemodule(2);
module B=[2x,0],[x,y],[z2,y];
module A=M;
module H=homology(A,B,M,N);
H=std(H);
// dimension of homology:
dim(H);
↪ 0
// vector space dimension:
vdim(H);
↪ 19
ring s=0,x,ds;
qring qs=std(x4);
module A=[x];
module B=A;
module M=[x3];
module N=M;
homology(A,B,M,N);
↪ _[1]=gen(1)
```

#### D.4.11.11 isCM

Procedure from library `homolog.lib` (see [Section D.4.11 \[homolog.lib\], page 1091](#)).

**Usage:** `isCM(M)`; M module

**Return:** 1 if  $M'=\text{coker}(M)$  is Cohen-Macaulay;  
0 if this is not the case.

**Assume:** basering is local.

**Example:**

```
LIB "homolog.lib";
ring R=0,(x,y,z),ds; // local ring R = Q[x,y,z]_<x,y,z>
module M=xz,yz,z2;
isCM(M); // test if R/<xz,yz,z2> is Cohen-Macaulay
↪ 0
M=x2+y2,z7; // test if R/<x2+y2,z7> is Cohen-Macaulay
isCM(M);
↪ 1
```

#### D.4.11.12 isFlat

Procedure from library `homolog.lib` (see [Section D.4.11 \[homolog.lib\], page 1091](#)).

**Usage:** `isFlat(M)`;  $M$  module

**Return:** 1 if  $M' = \text{coker}(M)$  is flat;  
0 if this is not the case.

**Example:**

```
LIB "homolog.lib";
ring A = 0, (x,y), dp;
matrix M[3][3] = x-1,y,x,x,x+1,y,x2,xy+x+1,x2+y;
print(M);
  ↪ x-1,y,      x,
  ↪ x,  x+1,    y,
  ↪ x2, xy+x+1,x2+y
isFlat(M);           // coker(M) is not flat over A=Q[x,y]
  ↪ 0
qring B = std(x2+x-y); // the ring B=Q[x,y]/<x2+x-y>
matrix M = fetch(A,M);
isFlat(M);           // coker(M) is flat over B
  ↪ 1
setring A;
qring C = std(x2+x+y); // the ring C=Q[x,y]/<x2+x+y>
matrix M = fetch(A,M);
isFlat(M);           // coker(M) is not flat over C
  ↪ 0
```

#### D.4.11.13 isLocallyFree

Procedure from library `homolog.lib` (see [Section D.4.11 \[homolog-lib\]](#), page 1091).

**Usage:** `isLocallyFree(M,r)`;  $M$  module,  $r$  int

**Return:** 1 if  $M' = \text{coker}(M)$  is locally free of constant rank  $r$ ;  
0 if this is not the case.

**Example:**

```
LIB "homolog.lib";
ring R=0,(x,y,z),dp;
matrix M[2][3]; // the presentation matrix
M=x-1,y-1,z,y-1,x-2,x;
ideal I=fitting(M,0); // 0-th Fitting ideal of coker(M)
qring Q=I;
matrix M=fetch(R,M);
isLocallyFree(M,1); // as R/I-module, coker(M) is locally free of rk 1
  ↪ 1
isLocallyFree(M,0);
  ↪ 0
```

#### D.4.11.14 isReg

Procedure from library `homolog.lib` (see [Section D.4.11 \[homolog-lib\]](#), page 1091).

**Usage:** `isReg(I,M)`;  $I$  ideal,  $M$  module

**Return:** 1 if given (ordered) list of generators for  $I$  is  $\text{coker}(M)$ -sequence;  
0 if this is not the case.

**Example:**

```

LIB "homolog.lib";
ring R = 0,(x,y,z),dp;
ideal I = x*(y-1),y,z*(y-1);
isReg(I,0);           // given list of generators is Q[x,y,z]-sequence
⇨ 1
I = x*(y-1),z*(y-1),y; // change sorting of generators
isReg(I,0);
⇨ 0
ring r = 0,(x,y,z),ds; // local ring
ideal I=fetch(R,I);
isReg(I,0);           // result independent of sorting of generators
⇨ 1

```

#### D.4.11.15 hom\_kernel

Procedure from library `homolog.lib` (see [Section D.4.11 \[homolog.lib\]](#), page 1091).

**Usage:** `hom_kernel(A,M,N);`

**Compute:** Let  $M$  and  $N$  be submodules of  $R^m$  and  $R^n$ , presenting  $M'=R^m/M$ ,  $N'=R^n/N$  ( $R$ =basering), and let  $A:R^m \rightarrow R^n$  be a matrix inducing a map  $A':M' \rightarrow N'$ . Then `ker(A,M,N)`; computes a presentation  $K$  of  $\ker(A')$  as in the commutative diagram:

$$\begin{array}{ccccc}
 \ker(A') & \dashrightarrow & M' & \xrightarrow{A'} & N' \\
 | & & | & & | \\
 | & & | & & | \\
 R^r & \dashrightarrow & R^m & \xrightarrow{A} & R^n \\
 | & & | & & | \\
 |K & & |M & & |N \\
 | & & | & & | \\
 R^s & \dashrightarrow & R^p & \dashrightarrow & R^q
 \end{array}$$

**Return:** module  $K$ , a presentation of  $\ker(A':\text{coker}(M) \rightarrow \text{coker}(N))$ .

**Example:**

```

LIB "homolog.lib";
ring r;
module N=[2x,x],[0,y];
module M=maxideal(1)*freemodule(2);
matrix A[2][3]=2x,0,x,y,z2,y;
module K=hom_kernel(A,M,N);
// dimension of kernel:
dim(std(K));
⇨ 3
// vector space dimension of kernel:
vdim(std(K));
⇨ -1
print(K);
⇨ 0,0,0,
⇨ 1,0,0,
⇨ 0,1,0,
⇨ 0,0,1

```

#### D.4.11.16 kohom

Procedure from library `homolog.lib` (see [Section D.4.11 \[homolog.lib\]](#), page 1091).

**Usage:** kohom(A,k); A=matrix, k=integer

**Return:** matrix  $\text{Hom}(R^k, A)$ , i.e. let A be a matrix defining a map  $F1 \rightarrow F2$  of free R-modules, then the matrix of  $\text{Hom}(R^k, F1) \rightarrow \text{Hom}(R^k, F2)$  is computed (R=basing).

**Example:**

```
LIB "homolog.lib";
ring r;
matrix n[2][3]=x,y,5,z,77,33;
print(kohom(n,3));
↪ x,0,0,y, 0, 0, 5, 0, 0,
↪ 0,x,0,0, y, 0, 0, 5, 0,
↪ 0,0,x,0, 0, y, 0, 0, 5,
↪ z,0,0,77,0, 0, 33,0, 0,
↪ 0,z,0,0, 77,0, 0, 33,0,
↪ 0,0,z,0, 0, 77,0, 0, 33
```

#### D.4.11.17 kontrahom

Procedure from library `homolog.lib` (see [Section D.4.11 \[homolog.lib\], page 1091](#)).

**Usage:** kontrahom(A,k); A=matrix, k=integer

**Return:** matrix  $\text{Hom}(A, R^k)$ , i.e. let A be a matrix defining a map  $F1 \rightarrow F2$  of free R-modules, then the matrix of  $\text{Hom}(F2, R^k) \rightarrow \text{Hom}(F1, R^k)$  is computed (R=basing).

**Example:**

```
LIB "homolog.lib";
ring r;
matrix n[2][3]=x,y,5,z,77,33;
print(kontrahom(n,3));
↪ x,z, 0,0, 0,0,
↪ y,77,0,0, 0,0,
↪ 5,33,0,0, 0,0,
↪ 0,0, x,z, 0,0,
↪ 0,0, y,77,0,0,
↪ 0,0, 5,33,0,0,
↪ 0,0, 0,0, x,z,
↪ 0,0, 0,0, y,77,
↪ 0,0, 0,0, 5,33
```

#### D.4.11.18 KoszulHomology

Procedure from library `homolog.lib` (see [Section D.4.11 \[homolog.lib\], page 1091](#)).

**Compute:** A presentation of the p-th Koszul homology module  $H_p(f_1, \dots, f_k; M')$ , where  $M' = \text{coker}(M)$  and  $f_1, \dots, f_k$  are the given (ordered list of non-zero) generators of the ideal I.

The computed presentation is minimized via `prune`.

In particular, if  $H_p(f_1, \dots, f_k; M') = 0$  then the return value is 0.

**Return:** module H, s.th.  $\text{coker}(H) = H_p(f_1, \dots, f_k; M')$ .

**Note:** size of input ideal has to be  $\leq 20$ .

**Example:**

```

LIB "homolog.lib";
ring R=0,x(1..3),dp;
ideal x=maxideal(1);
module M=0;
KoszulHomology(x,M,0); // H_0(x,R), x=(x_1,x_2,x_3)
↳ _[1]=x(3)*gen(1)
↳ _[2]=x(2)*gen(1)
↳ _[3]=x(1)*gen(1)
KoszulHomology(x,M,1); // H_1(x,R), x=(x_1,x_2,x_3)
↳ _[1]=0
qring S=std(x(1)*x(2));
module M=0;
ideal x=maxideal(1);
KoszulHomology(x,M,1);
↳ _[1]=-x(3)*gen(1)
↳ _[2]=-x(2)*gen(1)
↳ _[3]=-x(1)*gen(1)
KoszulHomology(x,M,2);
↳ _[1]=0

```

#### D.4.11.19 tensorMod

Procedure from library `homolog.lib` (see [Section D.4.11 \[homolog-lib\]](#), page 1091).

**Usage:**      `tensorMod(M,N)`; M,N modules

**Compute:**    presentation matrix A of the tensor product T of the modules  $M'=\text{coker}(M)$ ,  $N'=\text{coker}(N)$ : if `matrix(M)` defines a map  $M: R^r \rightarrow R^s$  and `matrix(N)` defines a map  $N: R^p \rightarrow R^q$ , then A defines a presentation

$$R^{(sp+rq)} \xrightarrow{-A} R^{(sq)} \twoheadrightarrow T \twoheadrightarrow 0 .$$

**Return:**      matrix A satisfying  $\text{coker}(A) = \text{tensorprod}(\text{coker}(M), \text{coker}(N))$  .

**Example:**

```

LIB "homolog.lib";
ring A=0,(x,y,z),dp;
matrix M[3][3]=1,2,3,4,5,6,7,8,9;
matrix N[2][2]=x,y,0,z;
print(M);
↳ 1,2,3,
↳ 4,5,6,
↳ 7,8,9
print(N);
↳ x,y,
↳ 0,z
print(tensorMod(M,N));
↳ x,y,0,0,0,0,1,0,2,0,3,0,
↳ 0,z,0,0,0,0,0,1,0,2,0,3,
↳ 0,0,x,y,0,0,4,0,5,0,6,0,
↳ 0,0,0,z,0,0,0,4,0,5,0,6,
↳ 0,0,0,0,x,y,7,0,8,0,9,0,
↳ 0,0,0,0,0,z,0,7,0,8,0,9

```

### D.4.11.20 Tor

Procedure from library `homolog.lib` (see [Section D.4.11 \[homolog.lib\]](#), page 1091).

**Compute:** a presentation of  $\text{Tor}_k(M', N')$ , for  $k=v[1], v[2], \dots$ , where  $M'=\text{coker}(M)$  and  $N'=\text{coker}(N)$ : let

```
0 <-- M' <-- G0 <-M-- G1
0 <-- N' <-- F0 <-N- F1 <-- F2 <-- ...
```

be a presentation of  $M'$ , resp. a free resolution of  $N'$ , and consider the commutative diagram

$$\begin{array}{ccccc}
 0 & & 0 & & 0 \\
 |^{\wedge} & & |^{\wedge} & & |^{\wedge} \\
 \text{Tensor}(M', F_{k+1}) & \xrightarrow{-A_{k+1}} & \text{Tensor}(M', F_k) & \xrightarrow{-A_k} & \text{Tensor}(M', F_{k-1}) \\
 |^{\wedge} & & |^{\wedge} & & |^{\wedge} \\
 \text{Tensor}(G_0, F_{k+1}) & \xrightarrow{-A_{k+1}} & \text{Tensor}(G_0, F_k) & \xrightarrow{-A_k} & \text{Tensor}(G_0, F_{k-1}) \\
 & & |^{\wedge} & & |^{\wedge} \\
 & & |C & & |B \\
 & & \text{Tensor}(G_1, F_k) & \xrightarrow{\quad} & \text{Tensor}(G_1, F_{k-1})
 \end{array}$$

( $A_k, A_{k+1}$  induced by  $N$  and  $B, C$  induced by  $M$ ).

Let  $K=\text{modulo}(A_k, B)$ ,  $J=\text{module}(C)+\text{module}(A_{k+1})$  and  $\text{Tor}=\text{modulo}(K, J)$ , then we have exact sequences

$$R^p \xrightarrow{-K} \text{Tensor}(G_0, F_k) \xrightarrow{-A_k} \text{Tensor}(G_0, F_{k-1}) / \text{im}(B),$$

$$R^q \xrightarrow{-\text{Tor}} R^p \xrightarrow{-K} \text{Tensor}(G_0, F_k) / (\text{im}(C) + \text{im}(A_{k+1})).$$

Hence,  $\text{Tor}$  presents  $\text{Tor}_k(M', N')$ .

**Return:**

- if  $v$  is of type `int`: module  $\text{Tor}$ , a presentation of  $\text{Tor}_k(M', N')$ ;
- if  $v$  is of type `intvec`: a list of  $\text{Tor}_k(M', N')$  ( $k=v[1], v[2], \dots$ );
- in case of a third argument of any type: list  $l$  with
  - $l[1] = \text{module Tor}/\text{list of Tor}_k(M', N')$ ,
  - $l[2] = \text{SB of Tor}/\text{list of SB of Tor}_k(M', N')$ ,
  - $l[3] = \text{matrix}/\text{list of matrices, each representing a kbase of Tor}_k(M', N')$   
(if finite dimensional), or 0.

**Display:** `printlevel >=0`: (affine) dimension of  $\text{Tor}_k$  for each  $k$  (default).  
`printlevel >=1`: matrices  $A_k, A_{k+1}$  and kbase of  $\text{Tor}_k$  in  $\text{Tensor}(G_0, F_k)$  (if finite dimensional).

**Note:** In order to compute  $\text{Tor}_k(M, N)$  use the command `Tor(k, syz(M), syz(N))`; or: list  $P=\text{mres}(M, 2)$ ; list  $Q=\text{mres}(N, 2)$ ; `Tor(k, P[2], Q[2])`;

**Example:**

```
LIB "homolog.lib";
int p      = printlevel;
printlevel = 1;
ring r      = 0, (x, y), dp;
ideal i     = x^2, y;
ideal j     = x;
list E      = Tor(0..2, i, j); // Tor_k(r/i, r/j) for k=0,1,2 over r
⇨ // dimension of Tor_0: 0
⇨ // vdim of Tor_0: 1
⇨
```

```

⇒ // Computing Tor_1 (help Tor; gives an explanation):
⇒ // Let 0 <- coker(M) <- G0 <-M- G1 be the present. of coker(M),
⇒ // and 0 <- coker(N) <- F0 <-N- F1 <- F2 <- ... a resolution of
⇒ // coker(N), then Tensor(G0,F1)-->Tensor(G0,F0) is given by:
⇒ x
⇒ // and Tensor(G0,F2) + Tensor(G1,F1)-->Tensor(G0,F1) is given by:
⇒ 0,x2,y
⇒
⇒ // dimension of Tor_1: 0
⇒ // vdim of Tor_1: 1
⇒
⇒ // Computing Tor_2 (help Tor; gives an explanation):
⇒ // Let 0 <- coker(M) <- G0 <-M- G1 be the present. of coker(M),
⇒ // and 0 <- coker(N) <- F0 <-N- F1 <- F2 <- ... a resolution of
⇒ // coker(N), then Tensor(G0,F2)-->Tensor(G0,F1) is given by:
⇒ 0
⇒ // and Tensor(G0,F3) + Tensor(G1,F2)-->Tensor(G0,F2) is given by:
⇒ 1,x2,y
⇒
⇒ // dimension of Tor_2: -1
⇒
qring R = std(i);
ideal j = fetch(r,j);
module M = [x,0],[0,x];
printlevel = 2;
module E1 = Tor(1,M,j); // Tor_1(R^2/M,R/j) over R=r/i
⇒ // Computing Tor_1 (help Tor; gives an explanation):
⇒ // Let 0 <- coker(M) <- G0 <-M- G1 be the present. of coker(M),
⇒ // and 0 <- coker(N) <- F0 <-N- F1 <- F2 <- ... a resolution of
⇒ // coker(N), then Tensor(G0,F1)-->Tensor(G0,F0) is given by:
⇒ x,0,
⇒ 0,x
⇒ // and Tensor(G0,F2) + Tensor(G1,F1)-->Tensor(G0,F1) is given by:
⇒ x,0,x,0,
⇒ 0,x,0,x
⇒
⇒ // dimension of Tor_1: 0
⇒ // vdim of Tor_1: 2
⇒
list l = Tor(3,M,M,1); // Tor_3(R^2/M,R^2/M) over R=r/i
⇒ // Computing Tor_3 (help Tor; gives an explanation):
⇒ // Let 0 <- coker(M) <- G0 <-M- G1 be the present. of coker(M),
⇒ // and 0 <- coker(N) <- F0 <-N- F1 <- F2 <- ... a resolution of
⇒ // coker(N), then Tensor(G0,F3)-->Tensor(G0,F2) is given by:
⇒ x,0,0,0,
⇒ 0,x,0,0,
⇒ 0,0,x,0,
⇒ 0,0,0,x
⇒ // and Tensor(G0,F4) + Tensor(G1,F3)-->Tensor(G0,F3) is given by:
⇒ x,0,0,0,x,0,0,0,
⇒ 0,x,0,0,0,x,0,0,
⇒ 0,0,x,0,0,0,x,0,
⇒ 0,0,0,x,0,0,0,x

```



```

↳
↳ // dimension of Tor_3:  0
↳ // vdim of Tor_3:      4
↳
↳ // columns of matrix are kbase of Tor_3 in Tensor(G0,F3)
↳ 1,0,0,0,
↳ 0,1,0,0,
↳ 0,0,1,0,
↳ 0,0,0,1
↳
printlevel = p;

```

### D.4.12 integralbasis.lib

**Library:** integralbasis.lib

**Purpose:** Integral basis in algebraic function fields

**Authors:** J. Boehm, boehm at mathematik.uni-kl.de  
 W. Decker, decker at mathematik.uni-kl.de  
 S. Laplagne, slaplagn at dm.uba.ar  
 G. Pfister, pfister at mathematik.uni-kl.de

**Overview:** Given an irreducible polynomial  $f$  in two variables defining a plane curve, this library implements algorithms for computing an integral basis of the integral closure of the affine coordinate ring in the algebraic function field.

**Procedures:**

#### D.4.12.1 integralBasis

Procedure from library `integralbasis.lib` (see [Section D.4.12 \[integralbasis.lib\], page 1115](#)).

**Usage:** `integralBasis(f, intVar)`;  $f$  irreducible polynomial in two variables, `intVar` integer indicating that the `intVar`-th variable of the ring is the integral element.

The base ring must be a ring in two variables, and the polynomial  $f$  must be monic as polynomial in the `intVar`-th variable.

Optional parameters in list `choose` (can be entered in any order):

Algorithm:

- "normal" -> the integral basis is computed using the general normalization algorithm.
- "hensel" -> the integral bases is computed using an algorithm based on Puiseux expansions and Hensel lifting. (only available for polynomials with rational coefficients; default option in that case)

Options for normal algorithm:

- "global" -> computes the normalization of  $R / \langle f \rangle$  and puts the results in integral basis shape.
- "local" -> computes the normalization at each component of the singular locus of  $R/\langle f \rangle$  and puts everything together. (Default option for normal algorithm.)

Options for hensel algorithm:

- "opti" -> The integral basis for the branches are merged using a combinatorial approach. (Default option.)
- "noOpti" -> the integral basis for the different branches are merged using the Chinese remainder theorem. This is usually slower when the number of branches is large.

- "local" -> computes the normalization at each component of the singular locus of  $R/\langle f \rangle$  and puts everything together. (Default option for normal algorithm.)

Other options:

- "modular" -> uses modular algorithms for computing Groebner bases, radicals and decompositions whenever possible. Can be used together with any of the other options. The ground field must have characteristic 0. (Default option for ground fields of characteristic 0.)

- "nonModular" -> do not use modular algorithms. (Default option for ground fields of positive characteristic.)

- "atOrigin" -> will compute the local contribution to the integral basis at the origin only (naturally, this contribution is only relevant if the curve defined by  $f$  has a singularity at the origin).

- "isIrred" -> assumes that the input polynomial  $f$  is irreducible, and therefore will not check this. If this option is given but  $f$  is not irreducible, the output might be wrong.

- list("inputJ", ideal inputJ) -> takes as initial test ideal the ideal inputJ. This option is only for use in other procedures. Using this option, the result might not be the integral basis. (When this option is given, the global option will be used.)

- list("inputC", ideal inputC) -> takes as initial conductor the ideal inputC. This option is only for use in other procedures. Using this option, the result might not be the integral basis. (When this option is given, the global option will be used.)

- "locBasis" -> when computing the integral basis at a prime or primary component, it computes a local basis, that is, a basis that is integral only over the ring localized at the component. This option is only valid when "atOrigin" is chosen or an initial test ideal or conductor is given.

**Return:** a list, say  $l$ , of size 2.

$l[1]$  is an ideal  $I$  and  $l[2]$  is a polynomial  $D$  such that the integral basis is  $b_0 = I[1] / D$ ,  $b_1 = I[2] / D$ , ...,  $b_{n-1} = I[n] / D$ .

That is, the integral closure of  $k[x]$  in the algebraic function field  $k(x,y)$  is

$k[x] b_0 + k[x] b_1 + \dots + k[x] b_{n-1}$ ,

where we assume that  $x$  is the transcendental variable,  $y$  is the integral element (indicated by `intVar`),  $f$  gives the integral equation and  $n$  is the degree of  $f$  as a polynomial in  $y$ .

**Theory:** We compute the integral basis of the integral closure of  $k[x]$  in  $k(x,y)$ . When option "normal" is selected, the normalization of the affine ring  $k[x,y]/\langle f \rangle$  is computed using procedure `normal` from `normal.lib`, which implements a general algorithm for normalization of rings by G. Greuel, S. Laplagne and F. Seelisch, and the  $k[x,y]$ -module generators are converted into a  $k[x]$ -basis.

When option "Hensel" is selected, the algorithm by J. Boehm, W. Decker, S. Laplagne and G. Pfister is used.

**Example:**

```
LIB "integralbasis.lib";
printlevel = printlevel+1;
ring s = 0,(x,y),dp;
poly f = y5-y4x+4y2x2-x4;
list l = integralBasis(f, 2);
↪ Computing the integral basis...
```

```

⇒ DBG - optimize = 1
⇒ --Computing the associated primes of the singular locus...
⇒ (Using non-modular algorithm.)
⇒ --Computing the integral basis at each component...
⇒ ----Computing the integral basis of component
⇒ 1
⇒ ----Component:
⇒ compo[1]=y
⇒ compo[2]=x
⇒ Integral basis computation finished.
1;
⇒ [1]:
⇒ _[1]=x3
⇒ _[2]=21/524288x7-5/16384x6-1/128x5-1/2x4+x3y
⇒ _[3]=-441/274877906944x10+777/4294967296x9-9/268435456x8-5/2048x6+1/25\
6x5y-1/64x5+1/64x4y-1/4x4+x2y2
⇒ _[4]=-441/274877906944x9y+777/4294967296x8y-9/268435456x7y-5/2048x5y+1\
/256x4y2-1/64x4y+1/64x3y2-1/4x3y+xy3
⇒ _[5]=567/2147483648x8y-21/134217728x7y2-63/67108864x8-51/67108864x7y+1\
9/33554432x6y2-3/1048576x7-107/2097152x6y-5/1048576x5y2+21/524288x4y3+55/\
131072x6-237/65536x5y+147/16384x4y2-69/16384x3y3+51/4096x5-21/512x4y+3/64\
x3y2-3/128x2y3+3/32x4-3/8x3y+3/4x2y2-3/2xy3+y4-2x3+4x2y
⇒ [2]:
⇒ x3
// The integral basis of the integral closure of Q[x] in Q(x,y) consists
// of the elements of l[1] divided by the polynomial l[2].
printlevel = printlevel-1;

```

See also: [Section D.4.23.1 \[normal\]](#), page 1184.

### D.4.12.2 polyDK

Procedure from library `integralbasis.lib` (see [Section D.4.12 \[integralbasis.lib\]](#), page 1115).

**Usage:** `polyDK(d,k,#);` d integer, k<=d integer.

**Return:** polynomial of degree d in the second variable of the ring with an ordinary multiple point at the origin of order k.

**Example:**

```

LIB "integralbasis.lib";
ring R = 0, (x,y), dp;
int k = 3;
int d = 6;
// Polynomial of degree 6 in y with an ordinary multiple point
// at the origin of order k.
poly f = polyDK(d, k, 1231);
f;
⇒ -1/3x6-4/3x5y+x4y2+1/3x3y3-x2y4-4/3xy5+y6+1/3x5-x4y+x3y2+1/3x2y3-1/3xy4+4\
/3y5+x4+4/3x3y-2/3x2y2+2/3xy3-y4+x3+2/3xy2+4/3y3
// The integral basis of R / <f>
list l = integralBasis(f, 2, "atOrigin");
l;
⇒ [1]:
⇒ _[1]=x2

```

```

⇒  _[2]=x2y
⇒  _[3]=x2y2
⇒  _[4]=-1413399/32768x8y2-1016807/12288x8y+19063/2048x7y2+1386535/32768x\
    8+15505/1536x7y-657/1024x6y2-151351/6144x7+2399/384x6y-65/32x5y2+36083/30\
    72x6-155/24x5y+15/16x4y2-191/96x5+7/6x4y-11/6x3y2+x2y3-5/48x4-x3y+4/3x2y2\
    +7/6x3-x2y+4/3x2
⇒  _[5]=-1413399/32768x7y3-1016807/12288x7y2+19063/2048x6y3+1386535/32768\
    x7y+15505/1536x6y2-657/1024x5y3-151351/6144x6y+2399/384x5y2-65/32x4y3+360\
    83/3072x5y-155/24x4y2+15/16x3y3-191/96x4y+7/6x3y2-11/6x2y3+xy4-5/48x3y-x2\
    y2+4/3xy3+7/6x2y-xy2+4/3xy
⇒  _[6]=-1413399/32768x6y4-1016807/12288x6y3+19063/2048x5y4+1386535/32768\
    x6y2+15505/1536x5y3-657/1024x4y4-151351/6144x5y2+2399/384x4y3-65/32x3y4+3\
    6083/3072x4y2-155/24x3y3+15/16x2y4-191/96x3y2+7/6x2y3-11/6xy4+y5-5/48x2y2\
    -xy3+4/3y4+7/6xy2-y3+4/3y2
⇒  [2]:
⇒  x2

```

### D.4.12.3 monic

Procedure from library `integralbasis.lib` (see [Section D.4.12 \[integralbasis.lib\], page 1115](#)).

**Usage:** `monic(f)`;  $f$  polynomial in two variables whose lead.

**Return:** a multiple of  $f$  monic as polynomial in the second variables, polynomial of degree  $d$  in the second variable of the ring with an ordinary multiple point at the origin of order  $k$ .

**Assume:**  $f$  is a bivariate polynomial whose leading coefficients as polynomial in the second variable is a unit.

**Example:**

```

LIB "integralbasis.lib";
ring R = 0, (x,y), dp;
poly f = -3y3 + 3x2y + y2 + 1;
monic(f);
⇒ -x2y+y3-1/3y2-1/3

```

### D.4.12.4 henselGlobal

Procedure from library `integralbasis.lib` (see [Section D.4.12 \[integralbasis.lib\], page 1115](#)).

**Usage:** `henselGlobal(f,g,h,order)`;  $h$  is polynomial in  $x$  and  $y$ .  $f$  and  $g$  are polynomials in  $y$  such that  $f(y)*g(y) = h(0,y)$  and  $\langle f, g \rangle = 1$ .

**Return:** polynomials  $f1$  and  $g1$  such that  
 1)  $h = f1*g1$  up to the required order in  $x$ .  
 2)  $f1(0,y) = f, g1(0,y) = g$

**Example:**

```

LIB "integralbasis.lib";
ring R = 0, (x,y), dp;
// Polynomial of degree 6 in y with an ordinary multiple point
// at the origin of order k.
poly h = (y2 + 3xy + x3 + x4)*(y3 + 2x + 1);
poly f = y2;
poly g = y3 + 1;
henselGlobal(f, g, h, 3);

```

```

⇒ [1] :
⇒      x3+3xy+y2
⇒ [2] :
⇒      y3+2x+1

```

### D.4.13 intprog\_lib

**Library:** intprog.lib

**Purpose:** Integer Programming with Groebner Basis Methods

**Author:** Christine Theis, email: ctheis@math.uni-sb.de

**Procedures:**

#### D.4.13.1 solve\_IP

Procedure from library `intprog.lib` (see [Section D.4.13 \[intprog\\_lib\]](#), page 1119).

**Usage:** solve\_IP(A,bx,c,alg); A intmat, bx intvec, c intvec, alg string.  
 solve\_IP(A,bx,c,alg); A intmat, bx list of intvec, c intvec, alg string.  
 solve\_IP(A,bx,c,alg,prsv); A intmat, bx intvec, c intvec, alg string, prsv intvec.  
 solve\_IP(A,bx,c,alg,prsv); A intmat, bx list of intvec, c intvec, alg string, prsv intvec.

**Return:** same type as bx: solution of the associated integer programming problem(s) as explained in [Section C.6 \[Toric ideals and integer programming\]](#), page 782.

**Note:** This procedure returns the solution(s) of the given IP-problem(s) or the message ‘not solvable’.

One may call the procedure with several different algorithms:

- the algorithm of Conti/Traverso (ct),
- the positive variant of the algorithm of Conti/Traverso (pct),
- the algorithm of Conti/Traverso using elimination (ect),
- the algorithm of Pottier (pt),
- an algorithm of Bigatti/La Scala/Robbiano (blr),
- the algorithm of Hosten/Sturmfels (hs),
- the algorithm of DiBiase/Urbanke (du).

The argument ‘alg’ should be the abbreviation for an algorithm as above: ct, pct, ect, pt, blr, hs or du.

‘ct’ allows computation of an optimal solution of the IP-problem directly from the right-hand vector b.

The same is true for its ‘positive’ variant ‘pct’ which may only be applied if A and b have nonnegative entries.

All other algorithms need initial solutions of the IP-problem.

If ‘alg’ is chosen to be ‘ct’ or ‘pct’, bx is read as the right hand vector b of the system  $Ax=b$ . b should then be an intvec of size m where m is the number of rows of A.

Furthermore, bx and A should be nonnegative if ‘pct’ is used. If ‘alg’ is chosen to be ‘ect’, ‘pt’, ‘blr’, ‘hs’ or ‘du’, bx is read as an initial solution x of the system  $Ax=b$ . bx should then be a nonnegative intvec of size n where n is the number of columns of A.

If ‘alg’ is chosen to be ‘blr’ or ‘hs’, the algorithm needs a vector with positive coefficients in the row space of A.

If no row of A contains only positive entries, one has to use the versions of solve\_IP which take such a vector prsv as an argument.

`solve_IP` may also be called with a list `bx` of `intvecs` instead of a single `intvec`.

**Example:**

```
LIB "intprog.lib";
// 1. call with single right-hand vector
intmat A[2][3]=1,1,0,0,1,1;
intvec b1=1,1;
intvec c=2,2,1;
intvec solution_vector=solve_IP(A,b1,c,"pct");
solution_vector;""
⇒ 0,1,0
⇒
// 2. call with list of right-hand vectors
intvec b2=-1,1;
list l=b1,b2;
l;
⇒ [1]:
⇒ 1,1
⇒ [2]:
⇒ -1,1
list solution_list=solve_IP(A,l,c,"ct");
solution_list;""
⇒ [1]:
⇒ 0,1,0
⇒ [2]:
⇒ not solvable
⇒
// 3. call with single initial solution vector
A=2,1,-1,-1,1,2;
b1=3,4,5;
solve_IP(A,b1,c,"du");""
⇒ 0,7,2
⇒
// 4. call with single initial solution vector
// and algorithm needing a positive row space vector
solution_vector=solve_IP(A,b1,c,"hs");""
⇒ ERROR: The chosen algorithm needs a positive vector in the row space of t\
he matrix.
⇒ 0
⇒
// 5. call with single initial solution vector
// and positive row space vector
intvec prsv=1,2,1;
solution_vector=solve_IP(A,b1,c,"hs",prsv);
solution_vector;""
⇒ 0,7,2
⇒
// 6. call with list of initial solution vectors
// and positive row space vector
b2=7,8,0;
l=b1,b2;
l;
⇒ [1]:
```

```

⇒      3,4,5
⇒ [2]:
⇒      7,8,0
solution_list=solve_IP(A,l,c,"blr",prsv);
solution_list;
⇒ [1]:
⇒      0,7,2
⇒ [2]:
⇒      7,8,0

```

See also: [Section C.6.4 \[Integer programming\]](#), page 784; [Section D.4.13 \[intprog.lib\]](#), page 1119; [Section D.4.35 \[toric.lib\]](#), page 1388.

## D.4.14 moddiq.lib

**Library:** moddiq.lib

**Purpose:** Double ideal quotient using modular methods

**Authors:** Y. Ishihara yishihara@rikkyo.ac.jp

**Overview:** A library for computing ideal quotient and saturation in the polynomial ring over the rational numbers using modular methods.

**References:**

M. Noro, K. Yokoyama: Usage of Modular Techniques for Efficient Computation of Ideal Operations. Math.Comput.Sci. 12: 1, 1-32. (2017).

**Procedures:**

### D.4.14.1 modQuotient

Procedure from library moddiq.lib (see [Section D.4.14 \[moddiq.lib\]](#), page 1121).

**Usage:** modQuotient(I,J); I,J ideal

**Return:** a standard basis of (I:J)

**Note:** The procedure computes a standard basis of (I:J) (over the rational numbers) by using modular methods.

**Example:**

```

LIB "moddiq.lib";
ring r=0,x(1..6),dp;
ideal i=cyclic(6);
ideal j=-15*var(5)+16*var(6)^3-60*var(6)^2+225*var(6)-4,2*var(5)^2-7*var(5)+2*var(6);
modQuotient(i,modQuotient(i,j));
⇒ _[1]=x(4)+1/4*x(5)+1/4*x(6)
⇒ _[2]=x(3)+1/4*x(5)+1/4*x(6)
⇒ _[3]=x(2)+1/4*x(5)+1/4*x(6)
⇒ _[4]=x(1)+1/4*x(5)+1/4*x(6)
⇒ _[5]=x(5)*x(6)-1/4*x(5)-1/4*x(6)+1
⇒ _[6]=x(5)^2+x(6)^2-7/2*x(5)-7/2*x(6)+14
⇒ _[7]=x(6)^3-15/4*x(6)^2-15/16*x(5)+225/16*x(6)-1/4
ideal id2=x(1)^2+x(1)*x(2)*x(3),x(2)^2-x(3)^3*x(2),x(3)^3+x(2)^5*x(1)*x(3);
quotient(id2,maxideal(3));
⇒ _[1]=x(1)*x(2)*x(3)+x(1)^2

```

```

⇒ _[2]=x(2)*x(3)^3-x(2)^2
⇒ _[3]=x(1)*x(3)^3+x(2)^2*x(3)
⇒ _[4]=x(1)^2*x(3)^2+x(1)*x(2)^2
⇒ _[5]=x(2)^3*x(3)+x(1)*x(2)^2
⇒ _[6]=x(2)^4+x(1)^3
⇒ _[7]=x(1)*x(2)^3-x(1)^3*x(3)
⇒ _[8]=x(1)^5+x(3)^3
⇒ _[9]=x(3)^6-x(2)^2
⇒ _[10]=x(1)^4*x(2)^2-x(3)^5

```

See also: [Section D.2.6.1 \[modular\]](#), page 884; [Section 5.1.127 \[quotient\]](#), page 248.

### D.4.14.2 modSat

Procedure from library `moddiq.lib` (see [Section D.4.14 \[moddiq.lib\]](#), page 1121).

**Usage:** `modSat(I,J)`; I,J ideal

**Return:** a standard basis of  $(I:J^\infty)$

**Note:** The procedure computes a standard basis of  $(I:J^\infty)$  (over the rational numbers) by using modular methods.

**Example:**

```

LIB "moddiq.lib";
ring r=0,x(1..6),dp;
ideal i=cyclic(6);
ideal j=-15*var(5)+16*var(6)^3-60*var(6)^2+225*var(6)-4,2*var(5)^2-7*var(5)+2*var(6);
modSat(i,modSat(i,j)[1])[1];
⇒ _[1]=x(4)+1/4*x(5)+1/4*x(6)
⇒ _[2]=x(3)+1/4*x(5)+1/4*x(6)
⇒ _[3]=x(2)+1/4*x(5)+1/4*x(6)
⇒ _[4]=x(1)+1/4*x(5)+1/4*x(6)
⇒ _[5]=x(5)*x(6)-1/4*x(5)-1/4*x(6)+1
⇒ _[6]=x(5)^2+x(6)^2-7/2*x(5)-7/2*x(6)+14
⇒ _[7]=x(6)^3-15/4*x(6)^2-15/16*x(5)+225/16*x(6)-1/4
poly F      = x(1)^5+x(2)^5+(x(1)-x(2))^2*x(1)*x(2)*x(3);
ideal J      = jacob(F);
modSat(J,maxideal(1));
⇒ [1]:
⇒ _[1]=x(2)^4+1/5*x(1)^3*x(3)-4/5*x(1)^2*x(2)*x(3)+3/5*x(1)*x(2)^2*x(3)
⇒ _[2]=x(1)^3*x(2)-2*x(1)^2*x(2)^2+x(1)*x(2)^3
⇒ _[3]=x(1)^4+3/5*x(1)^2*x(2)*x(3)-4/5*x(1)*x(2)^2*x(3)+1/5*x(2)^3*x(3)
⇒ _[4]=x(1)^2*x(2)^3-7/15*x(1)^2*x(2)^2*x(3)+4/15*x(1)*x(2)^3*x(3)-1/75*\
x(1)^3*x(3)^2-2/75*x(1)^2*x(2)*x(3)^2+1/15*x(1)*x(2)^2*x(3)^2-2/75*x(2)^3*\
*x(3)^2
⇒ _[5]=x(1)*x(2)^3*x(3)^2+29/220*x(1)^3*x(3)^3-71/220*x(1)^2*x(2)*x(3)^3\
+27/220*x(1)*x(2)^2*x(3)^3+3/44*x(2)^3*x(3)^3
⇒ _[6]=x(1)^2*x(2)^2*x(3)^2-44/41*x(1)*x(2)^3*x(3)^2-17/410*x(1)^3*x(3)^\
3+101/410*x(1)^2*x(2)*x(3)^3-19/82*x(1)*x(2)^2*x(3)^3+11/410*x(2)^3*x(3)^\
3
⇒ _[7]=x(1)^3*x(3)^4-7*x(1)^2*x(2)*x(3)^4+7*x(1)*x(2)^2*x(3)^4-x(2)^3*x(\
3)^4
⇒ [2]:
⇒ 0

```

See also: [Section D.2.6.1 \[modular\]](#), page 884; [Section D.4.7.8 \[sat\\_with\\_exp\]](#), page 1066.



### D.4.15 modules\_lib

**Library:** modules.lib

**Purpose:** Modules

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**Overview:** This library is used for the computation of graded free resolutions with an own graduation of the monomials. For these Resolution is a new class of modules needed. These modules, can be computed via the image, kernel, cokernel of a matrix or the subquotient of two matrices. The used matrices also have a free module as source and target, with graded generators if the matrix is homogeneous. A matrix of this new form is created by a normal matrix, source, target and the graduation, if the matrix is homogeneous, are done automatically. With this matrices it is then possible to compute the new class of modules.

This library also offers the opportunity to create R-module-homomorphisms between two modules. For these homomorphisms the kernel can be computed and will be returned as a module of the new class.

This is experimental work in progress!!!

**Types:** Matrix the class of matrices with source and target in form of free modules FreeModule  
 free modules represented with the ring and degree Resolution class of graded resolutions  
 Module modules represented by either the image, coker, kernel of a matrix or the subquotient of two matrices Vector element of a Module  
 Ideal same as ideal, but with its own basering saved, used to compute resolutions  
 Homomorphism class of R-module-homomorphisms

**Procedures:**

#### D.4.15.1 id

Procedure from library modules.lib (see [Section D.4.15 \[modules\\_lib\], page 1123](#)).

**Usage:** id(n); n integer

**Return:** returns the n x n identity matrix, with nongraded free modules for source and target

**Example:**

```
LIB "modules.lib";
ring r;
int n=4;
id(n);
↪      {0} {0} {0} {0}
↪ {0}    1  0  0  0
↪ {0}    0  1  0  0
↪ {0}    0  0  1  0
↪ {0}    0  0  0  1
↪
```

### D.4.15.2 zero

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Usage:**      `zero(n)`; `n` and `m` integer

**Return:**    returns the `n` x `m` zero matrix, with nongraded free modules for source and target

**Example:**

```
LIB "modules.lib";
ring r;
int n=4;
int m=3;
zero(n,m);
↪      {0} {0} {0}
↪ {0}   0   0   0
↪ {0}   0   0   0
↪ {0}   0   0   0
↪ {0}   0   0   0
↪
```

### D.4.15.3 freeModule

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Usage:**      `freeModule(r,n,l)`; `r` ring, `n` integer, `l` list

**Return:**    a free Module over the ring `r`, with rank `n`, and degrees `l` for the generators

**Note:**      -1 for no graduation and 0 to set every degree to 0

**Example:**

```
LIB "modules.lib";
ring r;
int n=3;
list l=1,2,3;
freeModule(r,n,l);
↪ 3
↪ r
↪ free Module
↪ Degrees of the generators: {1} {2} {3}
↪
```

### D.4.15.4 makeMatrix

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Usage:**      `makeMatrix(m)`, `m` matrix

RETURN Matrix, with graded source and target if the matrix is homogeneous

**Example:**

```
LIB "modules.lib";
ring r;
matrix m[2][2]=x,y3,z,xz;
m;
↪ m[1,1]=x
↪ m[1,2]=y3
```

```

    ↪ m[2,1]=z
    ↪ m[2,2]=xz
    Matrix M=m;

```

#### D.4.15.5 makeIdeal

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Usage:**      `makeIdeal(i)` or `Ideal I=i; i ideal`

**Return:**     Ideal with saved basering

**Example:**

```

    LIB "modules.lib";
    ring r;
    ideal i=x,y,z+x;
    i;
    ↪ i[1]=x
    ↪ i[2]=y
    ↪ i[3]=x+z
    Ideal I=i;

```

#### D.4.15.6 Target

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Usage:**      `Target(M); M Matrix`

**Return:**     FreeModule, target of the Matrix

**Example:**

```

    LIB "modules.lib";
    ring r;
    matrix m[2][2]=x,y3,z,xz;
    Matrix M=m;
    M;
    ↪ x,y3,
    ↪ z,xz
    ↪
    Target(M);
    ↪ 2
    ↪ r
    ↪ free Module
    ↪

```

#### D.4.15.7 Source

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Usage:**      `Source(M); M Matrix`

**Return:**     FreeModule, source of the Matrix

**Example:**

```

    LIB "modules.lib";
    ring r;
    matrix m[2][2]=x,y3,z,xz;

```

```

Matrix M=m;
M;
↳ x,y3,
↳ z,xz
↳
Source(M);
↳ 2
↳ r
↳ free Module
↳

```

#### D.4.15.8 printMatrix

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Usage:**     `printMatrix(M)`; or `M`; `M` Matrix

**Return:**    nothing, prints the matrix with degrees of the generators from target and source

**Example:**

```

LIB "modules.lib";
ring r;
matrix m[2][2]=x,y3,z,xz;
Matrix M=m;
M;
↳ x,y3,
↳ z,xz
↳

```

#### D.4.15.9 printFreeModule

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Usage:**     `printFreeModule(M)`; or `M`; `M` FreeModule

**Return:**    nothing, prints a free Module

**Example:**

```

LIB "modules.lib";
ring r;
int n=3;
list l=1,2,3;
freeModule(r,n,l);
↳ 3
↳ r
↳ free Module
↳ Degrees of the generators: {1} {2} {3}
↳

```

#### D.4.15.10 printResolution

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Usage:**     `printResolution(R)`; or `R`; `R` Resolution

**Return:**    nothing, prints the resolution

**Example:**

```

LIB "modules.lib";
ring r;
matrix m[1][3]=x,y,z;
Matrix M=m;
Module N=coker(M);
N;
⇒ cokernel | x y z |
⇒
⇒
Resolution R = Res(N);
R;
⇒ 1      3      3      1
⇒ r <--  r <--  r <--  r
⇒
⇒ 0      1      2      3
⇒ resolution not minimized yet
⇒
⇒

```

**D.4.15.11 printModule**

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Return:** nothing, prints the Module

**Example:**

```

LIB "modules.lib";
ring r;
matrix m[2][2]=x,y2,z,xz;
Matrix M=m;
M;
⇒ {1} {2}
⇒ {0} x y2
⇒ {0} z xz
⇒
Module N=image(M);
N;
⇒ image | x y2 |
⇒      | z xz |
⇒
⇒

```

**D.4.15.12 printHom**

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Usage:** `printHom(M); f` a Homomorphism

**Return:** nothing, prints `f`

**Example:**

```

LIB "modules.lib";
ring R=0,(x,y),(lp,c);
Matrix M=id(2);

```

```

Module src=image(M);
matrix rules[2][2]=x,y,x2,y2;
Module tar=coker(M);
src;
⇒ R^2
⇒
⇒
tar;
⇒ cokernel | 1 0 |
⇒          | 0 1 |
⇒
⇒
rules;
⇒ rules[1,1]=x
⇒ rules[1,2]=y
⇒ rules[2,1]=x2
⇒ rules[2,2]=y2
Homomorphism f=homomorphism(rules,src,tar);
f;
⇒ | x  y  |
⇒ | x2 y2 |
⇒
⇒ cokernel | 1 0 | <--- R^2
⇒          | 0 1 |
⇒

```

#### D.4.15.13 mRes

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\]](#), page 1123).

**Usage:** `mRes(M,n)`; `M` Module or Ideal, `n` integer

**Return:** Resolution, minimized resolution with graded modules

**Note:** `n` is optional, if `n` is positive only that many steps will be computed use `R.dd[i]`; to return the different modules as image of matrices, `R` Resolution `i` integer **EXAMPLE.** example `mRes`, shows an example

**Example:**

```

LIB "modules.lib";
ring r;
matrix m[1][3]=x,y,z;
Matrix M=m;
Module N=coker(M);
N;
⇒ cokernel | x y z |
⇒
⇒
Resolution R = mRes(N);
R;
⇒ 1      3      3      1
⇒ r <--  r <--  r <--  r
⇒
⇒ 0      1      2      3
⇒

```

```

↳
R.dd[2];
↳      {2}  {2}  {2}
↳ {1}    y   x   0
↳ {1}   -z   0   x
↳ {1}    0  -z  -y
↳

```

#### D.4.15.14 sRes

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Usage:** `sRes(M,n)`; M Module or Ideal, n integer

**Return:** Resolution, with graded modules, computed with Schreyer's method using the function `sres`

**Note:** n is optional, if n is positive only that many steps will be computed use `R.dd[i]`; to return the different modules as image of matrices, `R` Resolution i integer **EXAMPLE.** example `sRes`, shows an example

**Example:**

```

LIB "modules.lib";
ring r;
matrix m[1][3]=x,y,z;
Matrix M=m;
Module N=coker(M);
N;
↳ cokernel | x y z |
↳
↳
Resolution R = sRes(N);
R;
↳ 1      3      3      1
↳ r <--  r <--  r <--  r
↳
↳ 0      1      2      3
↳ resolution not minimized yet
↳
↳
R.dd[2];
↳      {2}  {2}  {2}
↳ {1}   -y  -x   0
↳ {1}    z   0  -x
↳ {1}    0   z   y
↳

```

#### D.4.15.15 Res

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Usage:** `Res(M,n)`; M Module or Ideal, n integer

**Return:** Resolution, resolution with graded modules

**Note:** `n` is optional, if `n` is positive only that many steps will be computed use `R.dd[i]`; to return the different modules as image of matrices, `R` Resolution `i` integer **EXAMPLE.** example `Res`, shows an example

**Example:**

```
LIB "modules.lib";
ring r;
matrix m[1][3]=x,y,z;
Matrix M=m;
Module N=coker(M);
N;
⇒ cokernel | x y z |
⇒
⇒
Resolution R = Res(N);
R;
⇒ 1      3      3      1
⇒ r <--  r <--  r <--  r
⇒
⇒ 0      1      2      3
⇒ resolution not minimized yet
⇒
⇒
R.dd[2];
⇒      {2}  {2}  {2}
⇒ {1}  -y  -x   0
⇒ {1}   z   0  -x
⇒ {1}   0   z   y
⇒
```

#### D.4.15.16 Betti

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\]](#), page 1123).

**Usage:** `Betti(reso)`; `reso` Resolution

**Return:** `intmat`, Bettimatrix of the resolution

**Note:** for a clear overview use `printBetti`

**Example:**

```
LIB "modules.lib";
ring r;
matrix m[1][3]=x,y,z;
Matrix M=m;
Module N=coker(M);
Resolution R=mRes(N);
R;
⇒ 1      3      3      1
⇒ r <--  r <--  r <--  r
⇒
⇒ 0      1      2      3
⇒
⇒
Betti(R);
⇒ 1,3,3,1
```



### D.4.15.17 printBetti

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Usage:**     `printBetti(resi)`, `resi` Resolution

**Return:**    nothing, prints the Bettimatrix of the Resolution

**Example:**

```
LIB "modules.lib";
ring r;
matrix m[1][3]=x,y,z;
Matrix M=m;
Module N=coker(M);
Resolution R=mRes(N);
R;
⇒ 1      3      3      1
⇒ r <--  r <--  r <--  r
⇒
⇒ 0      1      2      3
⇒
⇒
Betti(R);
⇒ 1,3,3,1
```

### D.4.15.18 SetDeg

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Usage:**     `SetDeg(l)`, `l` list or intvec

**Return:**    nothing, saves own degrees for the different variables of the basering

**Note:**     should be used after declaration of the ring and shouldn't be changed afterwards

**Example:**

```
LIB "modules.lib";
ring r;
Deg(x);
⇒ 1
list l=2,2,2;
SetDeg(l);
Deg(x);
⇒ 2
```

### D.4.15.19 Deg

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Usage:**     same as `deg`

**Note:**     only needed if custom degrees are set with `SetDeg`

**Example:**

```
LIB "modules.lib";
ring r;
Deg(x);
```

```

↳ 1
list l=2,2,2;
SetDeg(l);
Deg(x);
↳ 2

```

#### D.4.15.20 Degree

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\]](#), page 1123).

**Usage:**      `Degree(M);` M FreeModule

**Return:**     list, degrees of the generators from the module, if they are graded

**Example:**

```

LIB "modules.lib";
ring r;
matrix m[2][2]=x,y3,z,xz;
Matrix Ma=m;
FreeModule M=Source(Ma);
M;
↳ 2
↳ r
↳ free Module
↳
Degree(M);
↳ The module isn't graded

```

#### D.4.15.21 Degrees

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\]](#), page 1123).

**Usage:**      `Degrees(M);` M a Module

**Return:**     list, grading of the Module

**Example:**

```

LIB "modules.lib";
ring r;
matrix ma[2][2]=x,y,x,y;
Matrix m=ma;
Module M=image(m);
M;
↳ image | x y |
↳       | x y |
↳
↳
Degrees(M);
↳ [1]:
↳ 1
↳ [2]:
↳ 1

```

### D.4.15.22 subquotient

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Usage:** `subquotient(gens,rels);` gens and rels Matrices

**Return:** Module, the subquotient of two entered Matrices

**Example:**

```
LIB "modules.lib";
ring r;
matrix m[2][2]=x,y2,z,xz;
Matrix M=m;
matrix n[2][3]=z2,xyz,x2y2,xy,x3,y4;
Matrix N=n;
M;
↪      {1}  {2}
↪ {0}    x   y2
↪ {0}    z   xz
↪
N;
↪      {2}  {3}  {4}
↪ {0}    z2  xyz  x2y2
↪ {0}    xy  x3   y4
↪
subquotient(M,N);
↪ subquotient (| x y2 |, | z2 xyz x2y2 |)
↪               | z xz | | xy x3 y4 |
↪
↪
```

### D.4.15.23 coker

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Usage:** `coker(M)`, M a Matrix

**Return:** Module, the coker of M

**Example:**

```
LIB "modules.lib";
ring r;
matrix m[2][2]=x,y2,z,xz;
Matrix M=m;
M;
↪      {1}  {2}
↪ {0}    x   y2
↪ {0}    z   xz
↪
coker(M);
↪ cokernel | x y2 |
↪           | z xz |
↪
↪
```

**D.4.15.24 image**

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Usage:** `image(M)`; M a Matrix

**Return:** Module, the image of M

**Example:**

```
LIB "modules.lib";
ring r;
matrix m[2][2]=x,y2,z,xz;
Matrix M=m;
M;
↪      {1}  {2}
↪ {0}    x   y2
↪ {0}    z   xz
↪
image(M);
↪ image | x y2 |
↪      | z xz |
↪
↪
```

**D.4.15.25 Ker**

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Usage:** `Ker(M)`; M a Matrix

**Return:** Module, image of a Matrix N with `image(N)=Ker(M)`

**Example:**

```
LIB "modules.lib";
ring r;
matrix m[1][3]=x,y2,z3;
Matrix M=m;
M;
↪      {1}  {2}  {3}
↪ {0}    x   y2   z3
↪
Ker(M);
↪ image | y2 z3 0 |
↪      | -x 0  z3 |
↪      | 0  -x -y2 |
↪
↪
```

**D.4.15.26 compareModules**

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Usage:** `compareModules(M,N)`; or `M==N`; compares two Modules up to isomorphism

**Return:** 1 or 0, if the are isomorphic or aren't

**Example:**

```

LIB "modules.lib";
ring r;
matrix ma[2][2]=x,y,x,y;
Matrix m=ma;
Module M=image(m);
matrix na[2][1]=-y,x;
Matrix n=na;
M;
 $\mapsto$  image | x y |
 $\mapsto$          | x y |
 $\mapsto$ 
 $\mapsto$ 
Module N=image(n);
N;
 $\mapsto$  image | -y |
 $\mapsto$          | x  |
 $\mapsto$ 
 $\mapsto$ 
N==M;
 $\mapsto$  0
N=coker(n);
N;
 $\mapsto$  cokernel | -y |
 $\mapsto$            | x  |
 $\mapsto$ 
 $\mapsto$ 
N==M;
 $\mapsto$  0

```

#### D.4.15.27 addModules

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\]](#), page 1123).

**Usage:**      `addModules(M,N);` or `M+N`; M and N Modules

**Return:**    Module, sum of the two Modules

**Example:**

```

LIB "modules.lib";
ring r;
matrix ma[2][2]=x,y,x2,y2;
Matrix m=ma;
Module M=image(m);
matrix na[2][2]=xy,x2,y2,x;
Matrix n=na;
Module N=image(na);
M;
 $\mapsto$  image | x y |
 $\mapsto$          | x2 y2 |
 $\mapsto$ 
 $\mapsto$ 
N;
 $\mapsto$  image | xy x2 |
 $\mapsto$          | y2 x  |

```

```

↳
↳
N+M;
↳ image | xy x2 x y |
↳       | y2 x  x2 y2 |
↳
↳
↳

```

#### D.4.15.28 homomorphism

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Usage:**      `homomorphism(rules,sources,targets);` rules a matrix, sources and targets are Modules  
**Return:** Homomorphism

**Example:**

```

LIB "modules.lib";
ring R=0,(x,y),(lp,c);
Matrix M=id(2);
Module src=image(M);
matrix rules[2][2]=x,y,x2,y2;
Module tar=coker(M);
src;
↳ R^2
↳
↳
tar;
↳ cokernel | 1 0 |
↳          | 0 1 |
↳
↳
rules;
↳ rules[1,1]=x
↳ rules[1,2]=y
↳ rules[2,1]=x2
↳ rules[2,2]=y2
homomorphism(rules,src,tar);
↳ | x  y |
↳ | x2 y2 |
↳
↳ cokernel | 1 0 | <--- R^2
↳          | 0 1 |
↳

```

#### D.4.15.29 target

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Usage:**      `target(f);` f Homomorphism

**Return:**      Module, the target of f

**Example:**

```

LIB "modules.lib";
ring R=0,(x,y),(lp,c);

```

```

Matrix M=id(2);
Module src=image(M);
matrix rules[2][2]=x,y,x2,y2;
Module tar=coker(M);
src;
↳ R^2
↳
↳
tar;
↳ cokernel | 1 0 |
↳          | 0 1 |
↳
↳
rules;
↳ rules[1,1]=x
↳ rules[1,2]=y
↳ rules[2,1]=x2
↳ rules[2,2]=y2
Homomorphism f=homomorphism(rules,src,tar);
f;
↳ | x  y  |
↳ | x2 y2 |
↳
↳ cokernel | 1 0 | <--- R^2
↳          | 0 1 |
↳
target(f);
↳ cokernel | 1 0 |
↳          | 0 1 |
↳
↳

```

#### D.4.15.30 source

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\]](#), page 1123).

**Usage:**      `source(f);` `f` Homomorphism

**Return:**    Module, the source of `f`

**Example:**

```

LIB "modules.lib";
ring R=0,(x,y),(lp,c);
Matrix M=id(2);
Module src=image(M);
matrix rules[2][2]=x,y,x2,y2;
Module tar=coker(M);
src;
↳ R^2
↳
↳
tar;
↳ cokernel | 1 0 |
↳          | 0 1 |

```

```

↳
↳
rules;
↳ rules[1,1]=x
↳ rules[1,2]=y
↳ rules[2,1]=x2
↳ rules[2,2]=y2
Homomorphism f=homomorphism(rules,src,tar);
f;
↳ | x  y  |
↳ | x2 y2 |
↳
↳ cokernel | 1 0 | <--- R^2
↳          | 0 1 |
↳
source(f);
↳ R^2
↳
↳

```

#### D.4.15.31 compareMatrix

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Usage:**      `compareMatrix(M,N)`; or `M==N`; compares two Matrices

**Return:**     1 or 0, if the are the same or aren't

**Example:**

```

LIB "modules.lib";
ring r;
matrix ma[2][2]=x,y,x,y;
Matrix M=ma;
matrix na[2][1]=-y,x;
Matrix N=na;
M;
↳      {1} {1}
↳ {0}   x  y
↳ {0}   x  y
↳
N;
↳      {1}
↳ {0}   -y
↳ {0}   x
↳
N==M;
↳ 0
M==M;
↳ 1

```

#### D.4.15.32 freeModule2Module

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Usage:**      `freeModule2Module(F)`; F FreeModule



**Return:** returns F as a Module

**Example:**

```
LIB "modules.lib";
ring r;
list L = 1,1,1;
FreeModule F = freeModule(r,3,L);
freeModule2Module(F);
↪ R^3
↪
↪
```

### D.4.15.33 makeVector

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Return:** a Vector V, element of M with entries v

**Example:**

```
LIB "modules.lib";
ring r;
Module M = image(id(3));
makeVector([x,y,z],M);
↪ | x |
↪ | y |
↪ | z |
↪
↪
```

### D.4.15.34 netVector

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Usage:** netVector(V); V Vector

**Return:** pretty print for Vector

**Example:**

```
LIB "modules.lib";
ring r;
Module M = image(id(3));
Vector V = makeVector([x,y,z],M);
netVector(V);
↪ | x |
↪ | y |
↪ | z |
↪
```

### D.4.15.35 netMatrix

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Return:** nothing, prints M

**Example:**

```

LIB "modules.lib";
ring r;
matrix m[2][2]=x,y2,z,xz;
Matrix M=m;
netMatrix(M);
↪ | x y2 |
↪ | z xz |
↪

```

### D.4.15.36 presentation

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\]](#), page 1123).

**Return:** Subquotient M converted to `coker(C)`

**Example:**

```

LIB "modules.lib";
ring R = 0,(x,y),dp;
matrix a[1][2] = x,y;
Matrix A = a;
matrix b[1][2] = x2,y2;
Matrix B = b;
Module M = subquotient(A,B);
presentation(M);
↪ cokernel | 0 x -y |
↪          | y 0 x  |
↪
↪

```

### D.4.15.37 tensorMatrix

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\]](#), page 1123).

**Return:** Tensorprodukt of A,B

**Example:**

```

LIB "modules.lib";
ring r;
matrix m[2][2]=x,y2,z,xz;
matrix n[2][2]=1,2,3,4;
Matrix M = m;
Matrix N = n;
tensorMatrix(M,N);
↪      {1}  {1}  {2}  {2}
↪ {0}    x   2x   y2  2y2
↪ {0}   3x   4x  3y2  4y2
↪ {0}    z   2z   xz  2xz
↪ {0}   3z   4z  3xz  4xz
↪

```

### D.4.15.38 tensorModule

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\]](#), page 1123).

**Return:** Tensorprodukt of M,N

**Example:**

```

LIB "modules.lib";
ring R = 0,(x,y,z),dp;
matrix a[1][2] = x,y;
Matrix A = a;
matrix b[1][2] = x2,y2;
Matrix B = b;
Module M = subquotient(A,B);
M;
↳ subquotient (| x y |, | x2 y2 |)
↳
↳
matrix c[2][2]=x,y2,z,xz;
Matrix C=c;
matrix d[2][3]=z2,xyz,x2y2,xy,x3,y4;
Matrix D=d;
Module N = subquotient(C,D);
N;
↳ subquotient (| x y2 |, | z2 xyz x2y2 |)
↳
| z xz | | xy x3 y4 |
↳
↳
tensorModule(M,N);
↳ cokernel | 0 x -y 0 0 0 -xyz+y2 0 x2z-xyz 0 y4-xyz2-x2z+xyz+\
y2z-y2 0 -xy2z-y3z+xy2+y3+x2z-xyz+xz2+y2 0 \
-y3z+xz2 0 xy2z2-xy2z-y3z+y3 0 \
y4z-y4-x2z2+xyz2+x2z-xyz-y2z+y2 0 xy3z-xy \
3+xy2z+y3z-xy2-y3-x2z+xyz-xz2-y2 0 \
0 0 x2y2z-x2y2+xy2z+y3z-xy2-y3-x2z+xyz-xz2-y2 0 \
|
↳ | y 0 x 0 0 0 0 -xyz+y2 0 x2z-xyz 0 \
y4-xyz2-x2z+xyz+y2z-y2 0 -xy2z-y3z+x \
y2+y3+x2z-xyz+xz2+y2 0 -y3z+xz2 0 xy2z2-xy2z-y3z+y \
3 0 y4z-y4-x2z2+xyz2+x2z-xyz-y2z+y2 0 \
xy3z-xy3+xy2z+y3z-xy2-y3-x2z+xyz-xz2-y2 \
0 0 0 x2y2z-x2y2+xy2z \
+y3z-xy2-y3-x2z+xyz-xz2-y2 |
↳ | 0 0 0 0 x -y z2-x 0 0 0 -xy2+z3-z2+x \
0 -y2z+yz2-x2-xy-xz+z2-x 0 \
0 0 -yz3+x2z+yz2-xy 0 \
xy2-z3+z2-x 0 -y2z2+x \
2y+y2z-yz2+x2+xy+xz-z2+x 0 \
xyz-y2z 0 -xyz2+x3+y2z-yz2+x2+xy+xz-z2+x 0 \
|
↳ | 0 0 0 y 0 x 0 z2-x 0 0 0 \
-xy2+z3-z2+x 0 -y2z+yz2-x2 \
-xy-xz+z2-x 0 0 -yz3+x2z+yz2-xy \
0 xy2-z3+z2-x 0 \
-y2z2+x2y+y2z-yz2+x2+xy+xz-z2+x \
0 xyz-y2z 0 -xyz2+x3+y2z-yz \
2+x2+xy+xz-z2+x |
↳
↳

```

### D.4.15.39 tensorModFreemod

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Return:** Tensorprodukt of M,F

**Example:**

```
LIB "modules.lib";
ring R = 0,(x,y,z),dp;
matrix a[1][2] = x,y;
Matrix A = a;
matrix b[1][2] = x2,y2;
Matrix B = b;
Module M = subquotient(A,B);
M;
↳ subquotient (| x y |, | x2 y2 |)
↳
↳
FreeModule F = freeModule(R,3,0);
F;
↳ 3
↳ R
↳ free Module
↳ Degrees of the generators: {0} {0} {0}
↳
tensorModFreemod(M,F);
↳ cokernel | 0 0 0 x 0 0 -y 0 0 |
↳          | 0 0 0 0 x 0 0 -y 0 |
↳          | 0 0 0 0 0 x 0 0 -y |
↳          | y 0 0 0 0 0 x 0 0 |
↳          | 0 y 0 0 0 0 0 x 0 |
↳          | 0 0 y 0 0 0 0 0 x |
↳
↳
```

### D.4.15.40 tensorFreemodMod

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

### D.4.15.41 tensorFreeModule

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Return:** Tensorprodukt of M,N

**Example:**

```
LIB "modules.lib";
ring R = 0,(x,y,z),dp;
FreeModule F = freeModule(R,3,0);
F;
↳ 3
↳ R
↳ free Module
↳ Degrees of the generators: {0} {0} {0}
↳
```

```

tensorFreeModule(F,F);
↳ 9
↳ R
↳ free Module
↳

```

#### D.4.15.42 tensorProduct

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\]](#), page 1123).

#### D.4.15.43 pruneModule

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\]](#), page 1123).

**Return:** M in a simplified presentation

**Example:**

```

LIB "modules.lib";
ring R = 0,(x,y,z),dp;
matrix a[2][3] = -x,-y^2,x^3,y,x,0;
matrix b[1][2] = x^2-y^3,xy;
Matrix A = a;
Matrix B = b;
Module M = coker(A);
Module N = coker(B);
Module H = hom(M,N);
H;
↳ subquotient (| 0 -y -y 0 -x |, | -y3+x2 xy 0 0 |)
↳ | xy -x -x y3 -y2 | | 0 0 -y3+x2 xy |
↳
↳
pruneModule(H);
↳ cokernel | 0 y2 x2 x |
↳ | x 0 0 y3 |
↳
↳

```

#### D.4.15.44 hom

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\]](#), page 1123).

**Return:** calculates  $\text{Hom}(M,N)$  as a subquotient and yields an interpretation for the elements

**Example:**

```

LIB "modules.lib";
"Example:";
↳ Example:
ring R = 0,(x,y,z),dp;
matrix a[2][3] = -x,-y^2,x^3,y,x,0;
matrix b[1][2] = x^2-y^3,xy;
Matrix A = a;
Matrix B = b;
Module M = coker(A);
Module N = coker(B);
hom(M,N);

```

```

⇒ subquotient (| 0  -y -y 0  -x  |, | -y3+x2 xy 0      0  |)
⇒              | xy -x -x y3 -y2 | | 0      0  -y3+x2 xy |
⇒
⇒

```

#### D.4.15.45 kerHom

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\]](#), page 1123).

**Return:** returns the kernel of the given homomorphism

**Example:**

```

LIB "modules.lib";
ring R=0,(x,y),(lp,c);
Matrix M=id(2);
Module src=image(M);
matrix rules[2][2]=x,y,xy,y2;
Module tar=coker(M);
Homomorphism f=homomorphism(rules,src,tar);
f;
⇒ | x  y  |
⇒ | xy y2 |
⇒
⇒ cokernel | 1 0 | <--- R^2
⇒          | 0 1 |
⇒
kerHom(f);
⇒ image | 0  -1 |
⇒       | -1 0  |
⇒
⇒

```

#### D.4.15.46 interpret

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\]](#), page 1123).

**Return:** interpretation of  $V$  into some space that is stored in the interpretationlist of  $V$ .space

**Example:**

```

LIB "modules.lib";
"example 1:";
⇒ example 1:
ring R = 0,(x,y),dp;
matrix a[1][2] = x,y;
Matrix A = a;
matrix b[1][2] = x2,y2;
Matrix B = b;
Module M = subquotient(A,B);
Module C = presentation(M);
Vector V = [x2,y4],C;
interpret(V);
⇒ | y5+x3 |
⇒
⇒
"example 2:";

```

```

↳ example 2:
ring S;
matrix gens[2][3] = x2+y-3z4,y+xy,xyz+4,3+z2x,z3-3x+3,2+x+y+z7;
vector v = 2x*[gens[1..2,1]] + (y-z2)*[gens[1..2,2]] + 5*[gens[1..2,3]];
Matrix Gens = gens;
M = subquotient(Gens,zero(2,3));
M.interpretation = list(list(1,1,1),list());
V = v,M;
interpret(V);
↳ -z2+2x+y+5

```

#### D.4.15.47 interpretInv

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Return:** interpretation of V into some Module N (inverse to `interpret`)

**Example:**

```

LIB "modules.lib";
ring R;
matrix gens[2][3] = x2,xy,4,z2x,3x+3,z;
vector v = 2x*[gens[1..2,1]] + (y-z2)*[gens[1..2,2]] + 5*[gens[1..2,3]];
Matrix Gens = gens;
Module S = subquotient(Gens,zero(2,3));
Module N = coker(id(3));
matrix E = N.generators.hom;
Vector E1 = [1,0,0],N;
Vector E2 = [0,1,0],N;
Vector E3 = [0,0,1],N;
S.interpretation = list(list(E1,E2,E3),list([gens[1..2,1]], [gens[1..2,2]], [gens[1..2,3]]));
Vector V = v,S;
Vector W = interpret(V),N;
net(V);
↳ | -xyz2+2x3+xy2+20 |
↳ | 2x2z2-3xz2+3xy-3z2+3y+5z |
↳
Vector Vnew = interpretInv(W,S);
net(Vnew);
↳ | -xyz2+2x3+xy2+20 |
↳ | 2x2z2-3xz2+3xy-3z2+3y+5z |
↳
V==Vnew;
↳ 1

```

#### D.4.15.48 reduceIntChain

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Return:** Module C with minimized (or # steps) interpretation list

**Example:**

```

LIB "modules.lib";
ring R;
matrix a[5][4];
Module M = coker(a+1);

```

```
Module N = simplePrune(simplePrune(simplePrune(simplePrune(M))));
//reduceIntChain(N);
```

#### D.4.15.49 interpretElem

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Return:** interpretation of a Vector with # steps or until can't interpret further

**Example:**

```
LIB "modules.lib";
ring R;
matrix a[5][4];
Module M = coker(a+1);
Module N = simplePrune(simplePrune(simplePrune(simplePrune(M))));
Vector V = [x+y],N;
interpretElem(V,3);
⇒ | 0 |
⇒ | 0 |
⇒ | 0 |
⇒ | x+y |
⇒
⇒
```

#### D.4.15.50 interpretList

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Return:** interpretation of Elements in some abstract structure defined by the user or into a Module

**Example:**

```
LIB "modules.lib";
ring R;
matrix a[5][4];
Module M = coker(a+1);
Module N = simplePrune(simplePrune(simplePrune(simplePrune(M))));
Vector V = [x+y],N;
Vector W = [x2+y2+3*z2],N;
Vector U = [x+2y+27z],N;
list L = U,V,W;
//interpretList(L,3);
```

#### D.4.15.51 compareVectors

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\], page 1123](#)).

**Usage:** `compareVector(V1,V2);` Vector V1,V2

**Return:** compares the given Vectors up to equivalence

**Example:**

```
LIB "modules.lib";
ring r;
matrix m[2][1] = x,-y;
Module M = subquotient(id(2),m);
```



```

Vector V = [x,y],M;
Vector W = [0,2y],M;
Vector U = [x,y2],M;
compareVectors(V,W);
↦ 1
compareVectors(U,V);
↦ 0

```

#### D.4.15.52 simplePrune

Procedure from library `modules.lib` (see [Section D.4.15 \[modules.lib\]](#), page 1123).

**Return:** Simplified Module with reduced dimension

**Example:**

```

LIB "modules.lib";
ring R;
matrix a[5][4];
Module M = coker(a+1);
Module N = simplePrune(M);

```

#### D.4.16 modstd.lib

**Library:** `modstd.lib`

**Purpose:** Groebner bases of ideals/modules using modular methods

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**Overview:** A library for computing Groebner bases of ideals/modules in the polynomial ring over the rational numbers using modular methods.

**References:**

E. A. Arnold: Modular algorithms for computing Groebner bases. *J. Symb. Comp.* 35, 403-419 (2003).  
 N. Idrees, G. Pfister, S. Steidel: Parallelization of Modular Algorithms. *J. Symb. Comp.* 46, 672-684 (2011).

**Procedures:**

##### D.4.16.1 modStd

Procedure from library `modstd.lib` (see [Section D.4.16 \[modstd.lib\]](#), page 1147).

**Usage:** `modStd(I[, exactness]);` I ideal/module, exactness int

**Return:** a standard basis of I

**Note:** The procedure computes a standard basis of I (over the rational numbers) by using modular methods.

An optional parameter 'exactness' can be provided. If `exactness = 1`(default), the procedure computes a standard basis of I for sure; if `exactness = 0`, it computes a standard basis of I with high probability.

**Example:**

```

LIB "modstd.lib";
ring R1 = 0, (x,y,z,t), dp;
ideal I = 3x3+x2+1, 11y5+y3+2, 5z4+z2+4;
ideal J = modStd(I);
J;
↪ J[1]=x3+1/3x2+1/3
↪ J[2]=z4+1/5z2+4/5
↪ J[3]=y5+1/11y3+2/11
I = homog(I, t);
J = modStd(I);
J;
↪ J[1]=x3+1/3x2t+1/3t3
↪ J[2]=z4+1/5z2t2+4/5t4
↪ J[3]=y5+1/11y3t2+2/11t5
ring R2 = 0, (x,y,z), ds;
ideal I = jacob(x5+y6+z7+xyz);
ideal J = modStd(I, 0);
J;
↪ J[1]=xy+7z6
↪ J[2]=xz+6y5
↪ J[3]=yz+5x4
↪ J[4]=x5-7/5z7
↪ J[5]=y6-7/6z7
↪ J[6]=z8
ring R3 = 0, x(1..4), lp;
ideal I = cyclic(4);
ideal J1 = modStd(I, 1); // default
ideal J2 = modStd(I, 0);
size(reduce(J1, J2));
↪ 0
size(reduce(J2, J1));
↪ 0

```

See also: [Section D.2.6.1 \[modular\]](#), page 884.

**D.4.16.2 modGB**

Procedure from library `modstd.lib` (see [Section D.4.16 \[modstd.lib\]](#), page 1147).

**Usage:** `modGB(method, I[, exactness]);` I ideal/module, exactness int method can be: std, slimgb, sba

**Return:** a standard basis of I

**Note:** The procedure computes a standard basis of I (over the rational numbers) by using modular methods.

An optional parameter 'exactness' can be provided. If `exactness = 1`(default), the procedure computes a standard basis of I for sure; if `exactness = 0`, it computes a standard basis of I with high probability.

**Example:**

```

LIB "modstd.lib";
ring R1 = 0, (x,y,z,t), dp;
ideal I = 3x3+x2+1, 11y5+y3+2, 5z4+z2+4;

```

```

ideal J = modGB("slimgb",I);
J;
⇒ J[1]=x3+1/3x2+1/3
⇒ J[2]=z4+1/5z2+4/5
⇒ J[3]=y5+1/11y3+2/11
I = homog(I, t);
J = modGB("slimgb",I);
J;
⇒ J[1]=x3+1/3x2t+1/3t3
⇒ J[2]=z4+1/5z2t2+4/5t4
⇒ J[3]=y5+1/11y3t2+2/11t5
ring R3 = 0, x(1..4), lp;
ideal I = cyclic(4);
ideal J1 = modGB("slimgb",I, 1); // default
ideal J2 = modGB("slimgb",I, 0);
size(reduce(J1, J2));
⇒ 0
size(reduce(J2, J1));
⇒ 0

```

See also: [Section D.2.6.1 \[modular\]](#), page 884.

### D.4.16.3 modSyz

Procedure from library `modstd.lib` (see [Section D.4.16 \[modstd.lib\]](#), page 1147).

**Usage:** `modSyz(I);` I ideal/module

**Return:** a generating set of syzygies of I

**Note:** The procedure computes a the syzygy module of I (over the rational numbers) by using modular methods with high probability. The property of being a syzygy is tested.

**Example:**

```

LIB "modstd.lib";
ring R1 = 0, (x,y,z,t), dp;
ideal I = 3x3+x2+1, 11y5+y3+2, 5z4+z2+4;
modSyz(I);
⇒ _[1]=z4*gen(1)-3/5x3*gen(3)-1/5x2*gen(3)+1/5z2*gen(1)-1/5*gen(3)+4/5*gen(\
1)
⇒ _[2]=y5*gen(1)-3/11x3*gen(2)+1/11y3*gen(1)-1/11x2*gen(2)-1/11*gen(2)+2/11\
*gen(1)
⇒ _[3]=y5*gen(3)-5/11z4*gen(2)+1/11y3*gen(3)-1/11z2*gen(2)+2/11*gen(3)-4/11\
*gen(2)
simplify(syz(I),1);
⇒ _[1]=z4*gen(1)-3/5x3*gen(3)-1/5x2*gen(3)+1/5z2*gen(1)-1/5*gen(3)+4/5*gen(\
1)
⇒ _[2]=y5*gen(1)-3/11x3*gen(2)+1/11y3*gen(1)-1/11x2*gen(2)-1/11*gen(2)+2/11\
*gen(1)
⇒ _[3]=y5*gen(3)-5/11z4*gen(2)+1/11y3*gen(3)-1/11z2*gen(2)+2/11*gen(3)-4/11\
*gen(2)

```

See also: [Section D.2.6.1 \[modular\]](#), page 884.

### D.4.16.4 modIntersect

Procedure from library `modstd.lib` (see [Section D.4.16 \[modstd.lib\]](#), page 1147).

**Usage:** `modIntersect(I,J);` I,J ideal/module  
**Return:** a generating set of the intersection of I and J  
**Note:** The procedure computes a the intersection of I and J  
 (over the rational numbers) by using modular methods  
 with high probability.  
 No additional tests are performed.

**Example:**

```
LIB "modstd.lib";
ring R1 = 0, (x,y,z,t), dp;
ideal I = 3x3+x2+1, 11y5+y3+2, 5z4+z2+4;
ideal J = maxideal(2);
modIntersect(I,J);
↪ _[1]=z4-12/5x3-4/5x2+1/5z2
↪ _[2]=x3t2+1/3x2t2+1/3t2
↪ _[3]=x3zt+1/3x2zt+1/3zt
↪ _[4]=x3yt+1/3x2yt+1/3yt
↪ _[5]=x4t+1/3x3t+1/3xt
↪ _[6]=x3z2+1/3x2z2+1/3z2
↪ _[7]=x3yz+1/3x2yz+1/3yz
↪ _[8]=x4z+1/3x3z+1/3xz
↪ _[9]=y5-6/11x3+1/11y3-2/11x2
↪ _[10]=x3y2+1/3x2y2+1/3y2
↪ _[11]=x4y+1/3x3y+1/3xy
↪ _[12]=x5+1/3x4+1/3x2
simplify(intersect(I,J),1);
↪ _[1]=z4-12/5x3-4/5x2+1/5z2
↪ _[2]=x3t2+1/3x2t2+1/3t2
↪ _[3]=x3zt+1/3x2zt+1/3zt
↪ _[4]=x3yt+1/3x2yt+1/3yt
↪ _[5]=x4t+1/3x3t+1/3xt
↪ _[6]=x3z2+1/3x2z2+1/3z2
↪ _[7]=x3yz+1/3x2yz+1/3yz
↪ _[8]=x4z+1/3x3z+1/3xz
↪ _[9]=y5-6/11x3+1/11y3-2/11x2
↪ _[10]=x3y2+1/3x2y2+1/3y2
↪ _[11]=x4y+1/3x3y+1/3xy
↪ _[12]=x5+1/3x4+1/3x2
```

See also: [Section D.2.6.1 \[modular\]](#), page 884.

**D.4.17 monomialideal\_lib**

**Library:** `monomialideal.lib`  
**Purpose:** Primary and irreducible decompositions of monomial ideals  
**Authors:** I.Bermejo, [ibermejo@ull.es](mailto:ibermejo@ull.es)  
 E.Garcia-Llorente, [evgarcia@ull.es](mailto:evgarcia@ull.es)  
 Ph.Gimenez, [pgimenez@agt.uva.es](mailto:pgimenez@agt.uva.es)  
**Overview:** A library for computing a primary and the irreducible decompositions of a monomial ideal using several methods.  
 In this library we also take advantage of the fact that the ideal is monomial to make some computations that are Grobner free in this case (radical, intersection, quotient...).

**Procedures:****D.4.17.1 isMonomial**

Procedure from library `monomialideal.lib` (see [Section D.4.17 \[monomialideal.lib\], page 1150](#)).

**Usage:** `isMonomial (I)`; I ideal.

**Return:** 1, if I is monomial ideal; 0, otherwise.

**Assume:** I is an ideal of the basering.

**Example:**

```
LIB "monomialideal.lib";
ring R = 0,(w,x,y,z,t),lp;
ideal I = w^3*x*y, w^2*x^2*y^2*z^2 - y^3*z+x^4*z^4*t^4, w*x*y*z*t - w*x^6*y^5*z^4, x^2*y^3;
isMonomial(I);
↳ 1
ideal J = w^3*x*y + x^3*y^9*t^3, w^2*x^2*y^2*z^2 - y^3*z, w*x*y*z*t - w*x^6*y^5*z^4, x^2*y^3;
isMonomial(J);
↳ 0
```

**D.4.17.2 minbaseMon**

Procedure from library `monomialideal.lib` (see [Section D.4.17 \[monomialideal.lib\], page 1150](#)).

**Usage:** `minbaseMon (I)`; I ideal.

**Return:** an ideal, the minimal monomial generators of I.  
(-1 if the generators of I are not monomials)

**Assume:** I is an ideal generated by a list of monomials of the basering.

**Example:**

```
LIB "monomialideal.lib";
ring R = 0,(w,x,y,z,t),lp;
ideal I = w^3*x*y, w^2*x^2*y^2*z^2, y^3*z,x^4*z^4*t^4, w*x*y*z*t,w*x^6*y^5*z^4, x^2*y^3;
minbaseMon(I);
↳ _[1]=w3xy
↳ _[2]=y3z
↳ _[3]=wxyz
↳ _[4]=x2z4t3
↳ _[5]=x2y2z2
```

**D.4.17.3 gcdMon**

Procedure from library `monomialideal.lib` (see [Section D.4.17 \[monomialideal.lib\], page 1150](#)).

**Usage:** `gcdMon (f,g)`; f,g polynomials.

**Return:** a monomial, the greatest common divisor of f and g.

**Assume:** f and g are monomials of the basering.

**Example:**

```
LIB "monomialideal.lib";
ring R = 0,(x,y,z,t),dp;
poly f = x^3*z^5*t^2;
poly g = y^6*z^3*t^3;
gcdMon(f,g);
↳ z3t2
```

#### D.4.17.4 lcmMon

Procedure from library `monomialideal.lib` (see [Section D.4.17 \[monomialideal.lib\]](#), page 1150).

**Usage:** `lcmMon (f,g)`;  $f, g$  polynomials.

**Return:** a monomial, the least common multiple of  $f$  and  $g$ .

**Assume:**  $f, g$  are monomials of the basering.

**Example:**

```
LIB "monomialideal.lib";
ring R = 0, (x,y,z,t), dp;
poly f = x^3*z^5*t^2;
poly g = y^6*z^3*t^3;
lcmMon(f,g);
↪ x3y6z5t3
```

#### D.4.17.5 membershipMon

Procedure from library `monomialideal.lib` (see [Section D.4.17 \[monomialideal.lib\]](#), page 1150).

**Usage:** `membershipMon(f,I)`;  $f$  polynomial,  $I$  ideal.

**Return:** 1, if  $f$  lies in  $I$ ; 0 otherwise.  
(-1 if  $I$  and  $f$  are nonzero and  $I$  is not a monomial ideal)

**Assume:**  $I$  is a monomial ideal of the basering.

**Example:**

```
LIB "monomialideal.lib";
ring R = 0, (w,x,y,z,t), lp;
ideal I = w*x, x^2, y*z*t, y^5*t;
poly f = 3*x^2*y + 6*t^5*z*y^6 - 4*x^2 + 8*w*x^5*y^6 - 10*y^10*t^10;
membershipMon(f,I);
↪ 1
poly g = 3*w^2*t^3 - 4*y^3*z*t^3 - 2*x^2*y^5*t + 4*x*y^3;
membershipMon(g,I);
↪ 0
```

#### D.4.17.6 intersectMon

Procedure from library `monomialideal.lib` (see [Section D.4.17 \[monomialideal.lib\]](#), page 1150).

**Usage:** `intersectMon (I,J)`;  $I, J$  ideals.

**Return:** an ideal, the intersection of  $I$  and  $J$ .  
(it returns -1 if  $I$  or  $J$  are not monomial ideals)

**Assume:**  $I, J$  are monomial ideals of the basering.

**Note:** the minimal monomial generating set is returned.

**Example:**

```
LIB "monomialideal.lib";
ring R = 0, (w,x,y,z,t), lp;
ideal I = w^3*x*y, w*x*y*z*t, x^2*y^2*z^2, x^2*z^4*t^3, y^3*z;
ideal J = w*x, x^2, y*z*t, y^5*t;
intersectMon (I,J);
```

```

↳ _[1]=y3zt
↳ _[2]=wxyz
↳ _[3]=w3xy
↳ _[4]=x2y2z2
↳ _[5]=x2z4t3
↳ _[6]=x2y3z
↳ _[7]=wxy3z

```

#### D.4.17.7 quotientMon

Procedure from library `monomialideal.lib` (see [Section D.4.17 \[monomialideal.lib\]](#), page 1150).

**Usage:** `quotientMon (I,J)`; I,J ideals.

**Return:** an ideal, the quotient I:J.  
(returns -1 if I or J is not monomial)

**Assume:** I,J are monomial ideals of the basering.

**Note:** the minimal monomial generating set is returned.

**Example:**

```

LIB "monomialideal.lib";
ring R = 0,(w,x,y,z,t),lp;
ideal I = w^3*x*y,w*x*y*z*t,x^2*y^2*z^2,x^2*z^4*t^3,y^3*z;
ideal J = w*x, x^2, y*z*t, y^5*t;
quotientMon (I,J);
↳ _[1]=y2z2t
↳ _[2]=y3z
↳ _[3]=xy2z2
↳ _[4]=x2z4t3
↳ _[5]=wy2zt
↳ _[6]=wxz4t3
↳ _[7]=wxyz
↳ _[8]=w2y2z2
↳ _[9]=w3y2z
↳ _[10]=w3xy

```

#### D.4.17.8 radicalMon

Procedure from library `monomialideal.lib` (see [Section D.4.17 \[monomialideal.lib\]](#), page 1150).

**Usage:** `radicalMon(I)`; I ideal

**Return:** an ideal, the radical ideal of the ideal I.  
(returns -1 if I is not a monomial ideal)

**Assume:** I is a monomial ideal of the basering.

**Note:** the minimal monomial generating set is returned.

**Example:**

```

LIB "monomialideal.lib";
ring R = 0,(w,x,y,z,t),lp;
ideal I = w^3*x*y,w*x*y*z*t,x^2*y^2*z^2,x^2*z^4*t^3,y^3*z;
radicalMon(I);
↳ _[1]=yz
↳ _[2]=xzt
↳ _[3]=wxy

```

### D.4.17.9 isprimeMon

Procedure from library `monomialideal.lib` (see [Section D.4.17 \[monomialideal.lib\]](#), page 1150).

**Usage:**      `isprimeMon (I)`; I ideal  
**Return:**      1, if I is prime; 0, otherwise.  
                  (returns -1 if I is not a monomial ideal)  
**Assume:**    I is a monomial ideal of the basering.

**Example:**

```
LIB "monomialideal.lib";
ring R = 0,(w,x,y,z,t),lp;
ideal I = w,y,t;
isprimeMon (I);
↪ 1
ideal J = w,y,t,x*z;
isprimeMon (J);
↪ 0
```

### D.4.17.10 isprimaryMon

Procedure from library `monomialideal.lib` (see [Section D.4.17 \[monomialideal.lib\]](#), page 1150).

**Usage:**      `isprimaryMon (I)`; I ideal  
**Return:**      1, if I is primary; 0, otherwise.  
                  (returns -1 if I is not a monomial ideal)  
**Assume:**    I is a monomial ideal of the basering.

**Example:**

```
LIB "monomialideal.lib";
ring R = 0,(w,x,y,z,t),lp;
ideal I = w^4,x^3,z^2,t^5,x*t,w*x^2*z;
isprimaryMon (I);
↪ 1
ideal J = w^4,x^3,z^2,t^5,x*t,w*x^2*z,y^3*t^3;
isprimaryMon (J);
↪ 0
```

### D.4.17.11 isirreducibleMon

Procedure from library `monomialideal.lib` (see [Section D.4.17 \[monomialideal.lib\]](#), page 1150).

**Usage:**      `isirreducibleMon(I)`; I ideal  
**Return:**      1, if I is irreducible; 0, otherwise.  
                  (return -1 if I is not a monomial ideal)  
**Assume:**    I is a monomial ideal of the basering.

**Example:**

```
LIB "monomialideal.lib";
ring R = 0,(w,x,y,z,t),lp;
ideal I = w^4,x^3,z^2,t^5;
isirreducibleMon (I);
```



```

⇒ 1
ideal J = w^4*x,x^3,z^2,t^5;
isirreducibleMon (J);
⇒ 0

```

#### D.4.17.12 isartinianMon

Procedure from library `monomialideal.lib` (see [Section D.4.17 \[monomialideal.lib\]](#), page 1150).

**Usage:** `isartinianMon(I);` I ideal.

**Return:** 1, if ideal is artinian; 0, otherwise.  
(return -1 if ideal I is not a monomial ideal).

**Assume:** I is a monomial ideal of the basering.

**Example:**

```

LIB "monomialideal.lib";
ring R = 0,(w,x,y,z,t),lp;
ideal I = w^4,x^3,y^4,z^2,t^6,w^2*x^2*y,w*z*t^4,x^2*y^3,z^2*t^5;
isartinianMon (I);
⇒ 1
ideal J = w^4,x^3,y^4,z^2,w^2*x^2*y,w*z*t^4,x^2*y^3,z^2*t^5;
isartinianMon (J);
⇒ 0

```

#### D.4.17.13 isgenericMon

Procedure from library `monomialideal.lib` (see [Section D.4.17 \[monomialideal.lib\]](#), page 1150).

**Usage:** `isgenericMon(I);` I ideal.

**Return:** 1, if ideal is generic; 0, otherwise.  
(return -1 if ideal I is not a monomial ideal)

**Assume:** I is a monomial ideal of the basering.

**Example:**

```

LIB "monomialideal.lib";
ring R = 0,(w,x,y,z,t),lp;
ideal I = w^4,x^3,y^4,z^2,w^2*x^2*y,w*z*t^4,x*y^3,z*t^5;
isgenericMon (I);
⇒ 0
ideal J = w^4,x^3,y^4,z^3,w^2*x^2*y,w*z*t^4,x*y^3,z^2*t^5;
isgenericMon (J);
⇒ 1

```

#### D.4.17.14 dimMon

Procedure from library `monomialideal.lib` (see [Section D.4.17 \[monomialideal.lib\]](#), page 1150).

**Usage:** `dimMon (I);` I ideal

**Return:** an integer, the dimension of the affine variety defined by the ideal I.  
(returns -1 if I is not a monomial ideal)

**Assume:** I is a monomial ideal of the basering.

**Example:**

```

LIB "monomialideal.lib";
ring R = 0,(w,x,y,z,t),lp;
ideal I = w^3*x*y,w*x*y*z*t,x^2*y^2*z^2,x^2*z^4*t^3,y^3*z;
dimMon (I);
⇒ 3
ideal J = w^4,x^3,y^4,z^2,t^6,w^2*x^2*y,w*z*t^4,x^2*y^3,z*t^5;
dimMon (J);
⇒ 0

```

**D.4.17.15 irreddecMon**

Procedure from library `monomialideal.lib` (see [Section D.4.17 \[monomialideal.lib\]](#), page 1150).

**Usage:** `irreddecMon (I[,alg]);` I ideal, alg string.

**Return:** list, the irreducible components of the monomial ideal I. (returns -1 if I is not a monomial ideal).

**Assume:** I is a monomial ideal of the basering  $k[x(1)..x(n)]$ .

**Note:** This procedure returns the irreducible decomposition of I. One may call the procedure with different algorithms using the optional argument 'alg':

- the direct method following Vasconcelos' book (`alg=vas`) - via the Alexander dual and using double dual (`alg=add`), - via the Alexander dual and quotients following E. Miller (`alg=ad`),
- the formula of irreducible components (`alg=for`),
- via the Scarf complex following Milowski (`alg=mil`),
- using the label algorithm of Roune (`alg=lr`),
- using the algorithm of Gao-Zhu (`alg=gz`).
- using the slice algorithm of Roune (`alg=sr`).

**Example:**

```

LIB "monomialideal.lib";
ring R = 0,(w,x,y,z),Dp;
ideal I = w^3*x*y,w*x*y*z,x^2*y^2*z^2,x^2*z^4,y^3*z;
// Vasconcelos
irreddecMon (I,"vas");
⇒ [1]:
⇒   _[1]=y
⇒   _[2]=x2
⇒ [2]:
⇒   _[1]=w
⇒   _[2]=z2
⇒   _[3]=y3
⇒ [3]:
⇒   _[1]=y
⇒   _[2]=z4
⇒ [4]:
⇒   _[1]=w
⇒   _[2]=x2
⇒   _[3]=y3
⇒ [5]:
⇒   _[1]=w
⇒   _[2]=y2

```

```

    ↪      _[3]=z4
    ↪ [6]:
    ↪      _[1]=z
    ↪      _[2]=w3
    ↪ [7]:
    ↪      _[1]=z
    ↪      _[2]=x
    ↪ [8]:
    ↪      _[1]=x
    ↪      _[2]=y3
    // Alexander Dual
    irreddecMon (I,"ad");
    ↪ [1]:
    ↪      _[1]=w
    ↪      _[2]=y3
    ↪      _[3]=z2
    ↪ [2]:
    ↪      _[1]=w
    ↪      _[2]=y2
    ↪      _[3]=z4
    ↪ [3]:
    ↪      _[1]=x
    ↪      _[2]=z
    ↪ [4]:
    ↪      _[1]=w
    ↪      _[2]=x2
    ↪      _[3]=y3
    ↪ [5]:
    ↪      _[1]=w3
    ↪      _[2]=z
    ↪ [6]:
    ↪      _[1]=x2
    ↪      _[2]=y
    ↪ [7]:
    ↪      _[1]=y
    ↪      _[2]=z4
    ↪ [8]:
    ↪      _[1]=x
    ↪      _[2]=y3
    // Scarf Complex
    irreddecMon (I,"mil");
    ↪ [1]:
    ↪      _[1]=y
    ↪      _[2]=z4
    ↪ [2]:
    ↪      _[1]=w3
    ↪      _[2]=z
    ↪ [3]:
    ↪      _[1]=w
    ↪      _[2]=y3
    ↪      _[3]=z2
    ↪ [4]:
    ↪      _[1]=w

```

```

⇒      _[2]=y2
⇒      _[3]=z4
⇒ [5] :
⇒      _[1]=w
⇒      _[2]=x2
⇒      _[3]=y3
⇒ [6] :
⇒      _[1]=x
⇒      _[2]=y3
⇒ [7] :
⇒      _[1]=x2
⇒      _[2]=y
⇒ [8] :
⇒      _[1]=x
⇒      _[2]=z
// slice algorithm
irreddecMon(I,"sr");
⇒ [1] :
⇒      _[1]=y
⇒      _[2]=z4
⇒ [2] :
⇒      _[1]=x2
⇒      _[2]=y
⇒ [3] :
⇒      _[1]=x
⇒      _[2]=z
⇒ [4] :
⇒      _[1]=x
⇒      _[2]=y3
⇒ [5] :
⇒      _[1]=w3
⇒      _[2]=z
⇒ [6] :
⇒      _[1]=w
⇒      _[2]=y3
⇒      _[3]=z2
⇒ [7] :
⇒      _[1]=w
⇒      _[2]=y2
⇒      _[3]=z4
⇒ [8] :
⇒      _[1]=w
⇒      _[2]=x2
⇒      _[3]=y3

```

#### D.4.17.16 primdecMon

Procedure from library `monomialideal.lib` (see [Section D.4.17 \[monomialideal.lib\]](#), page 1150).

**Usage:**     `primdecMon (I[,alg]);` I ideal, alg string

**Return:**    list, the components in a minimal primary decomposition of I. (returns -1 if I is not a monomial ideal).

**Assume:**    I is a monomial ideal of the basering  $k[x(1)..x(n)]$ .

**Note:** This procedure returns a minimal primary decomposition of  $I$ . One may call the procedure with different algorithms using the optional argument 'alg':

- the direct method for a primary decomposition following Vasconcelos' book (alg=vp),
- from the irreducible decomposition obtained via the direct method following Vasconcelos' book (alg=vi),
- from the irreducible decomposition obtained via the Alexander dual and using double dual (alg=add),
- from the irreducible decomposition obtained via the Alexander dual and quotients following E. Miller (alg=ad),
- from the irreducible decomposition obtained via ..... (alg=for),
- from the irreducible decomposition obtained via the Scarf complex following Milowski (alg=mil),
- from the irreducible decomposition obtained using the label algorithm of Roune (alg=lr),
- from the irreducible decomposition obtained using the algorithm of Gao-Zhu (alg=gz),
- from the irreducible decomposition obtained using the slice algorithm of Roune (alg=sr).

**Example:**

```
LIB "monomialideal.lib";
ring R = 0,(w,x,y,z),Dp;
ideal I = w^3*x*y,w*x*y*z,x^2*y^2*z^2,x^2*z^4,y^3*z;
// Vasconcelos para primaria
primdecMon(I,"vp");
⇒ [1]:
⇒ _[1]=x2
⇒ _[2]=y3
⇒ _[3]=wxy
⇒ _[4]=w3
⇒ [2]:
⇒ _[1]=xy
⇒ _[2]=x2
⇒ _[3]=y3
⇒ [3]:
⇒ _[1]=z
⇒ _[2]=x
⇒ [4]:
⇒ _[1]=y3
⇒ _[2]=wyz
⇒ _[3]=w3
⇒ _[4]=z4
⇒ _[5]=y2z2
⇒ [5]:
⇒ _[1]=y
⇒ _[2]=z4
⇒ [6]:
⇒ _[1]=z
⇒ _[2]=w3
// Alexander dual
```

```

primdecMon(I,"add");
↳ [1]:
↳   _[1]=y
↳   _[2]=z4
↳ [2]:
↳   _[1]=xy
↳   _[2]=x2
↳   _[3]=y3
↳ [3]:
↳   _[1]=w3
↳   _[2]=z
↳ [4]:
↳   _[1]=w
↳   _[2]=x2
↳   _[3]=y3
↳ [5]:
↳   _[1]=x
↳   _[2]=z
↳ [6]:
↳   _[1]=w
↳   _[2]=y3
↳   _[3]=z4
↳   _[4]=y2z2
// label algorithm
primdecMon(I,"lr");
↳ [1]:
↳   _[1]=w
↳   _[2]=x2
↳   _[3]=y3
↳ [2]:
↳   _[1]=w
↳   _[2]=y3
↳   _[3]=z4
↳   _[4]=y2z2
↳ [3]:
↳   _[1]=w3
↳   _[2]=z
↳ [4]:
↳   _[1]=x
↳   _[2]=z
↳ [5]:
↳   _[1]=xy
↳   _[2]=x2
↳   _[3]=y3
↳ [6]:
↳   _[1]=y
↳   _[2]=z4
//slice algorithm
primdecMon(I,"sr");
↳ [1]:
↳   _[1]=w
↳   _[2]=x2
↳   _[3]=y3

```

```

⇒ [2] :
⇒   _[1]=w
⇒   _[2]=y3
⇒   _[3]=z4
⇒   _[4]=y2z2
⇒ [3] :
⇒   _[1]=w3
⇒   _[2]=z
⇒ [4] :
⇒   _[1]=x
⇒   _[2]=z
⇒ [5] :
⇒   _[1]=xy
⇒   _[2]=x2
⇒   _[3]=y3
⇒ [6] :
⇒   _[1]=y
⇒   _[2]=z4

```

#### D.4.18 mprimdec\_lib

**Library:** mprimdec.lib

**Purpose:** procedures for primary decomposition of modules

**Authors:** Alexander Dreyer, dreyer@mathematik.uni-kl.de; adreyer@web.de

**Overview:** Algorithms for primary decomposition for modules based on the algorithms of Gianni, Trager and Zacharias and Shimoyama and Yokoyama (generalization of the latter suggested by Hans-Gert Graebe, Leipzig ) using elements of primdec.lib

**Remark:** These procedures are implemented to be used in characteristic 0. They also work in positive characteristic  $\gg 0$ . In small characteristic and for algebraic extensions, the procedures via Gianni, Trager, Zacharias may not terminate.

**Procedures:**

##### D.4.18.1 separator

Procedure from library `mprimdec.lib` (see [Section D.4.18 \[mprimdec.lib\]](#), page 1161).

**Usage:** separator(l); list l of prime ideals

**Return:** list sepList;  
a list of separators of the prime ideals in l,  
i.e. polynomials  $p_{ij}$ , s.th.  $p_{ij}$  is in  $l[j]$ ,  
for all  $l[j]$  not contained in  $l[i]$   
but  $p_{ij}$  is not in  $l[i]$

**Example:**

```

LIB "mprimdec.lib";
ring r=0,(x,y,z),dp;
ideal i=(x2y,xz2,y2z,z3);

```

```

list l=minAssGTZ(i);
list sepL=separator(l);
sepL;
↳ [1]:
↳      x
↳ [2]:
↳      y

```

### D.4.18.2 PrimdecA

Procedure from library `mprimdec.lib` (see [Section D.4.18 \[mprimdec.lib\]](#), page 1161).

**Usage:**     PrimdecA (N[, i]); module N, int i

**Return:**    list l  
               a (not necessarily minimal) primary decomposition of N computed by a generalized version of the algorithm of Shimoyama/Yokoyama,  
               if  $i \neq 0$  is given, the factorizing Groebner is used to compute the isolated primes

**Example:**

```

LIB "mprimdec.lib";
ring r=0,(x,y,z),dp;
module N=x*gen(1)+ y*gen(2),
x*gen(1)-x2*gen(2);
list l=PrimdecA(N);
l;
↳ [1]:
↳      [1]:
↳      _[1]=x*gen(1)+y*gen(2)
↳      _[2]=x*gen(2)-gen(1)
↳      [2]:
↳      _[1]=x2+y
↳ [2]:
↳      [1]:
↳      _[1]=gen(2)
↳      _[2]=x*gen(1)
↳      [2]:
↳      _[1]=x
↳ [3]:
↳      [1]:
↳      _[1]=y*gen(1)
↳      _[2]=y*gen(2)
↳      _[3]=x*gen(1)
↳      _[4]=x*gen(2)
↳      [2]:
↳      _[1]=y
↳      _[2]=x

```

### D.4.18.3 PrimdecB

Procedure from library `mprimdec.lib` (see [Section D.4.18 \[mprimdec.lib\]](#), page 1161).

**Usage:**     PrimdecB (N, p); pseudo-primary module N, isolated prime ideal p

**Return:**    list l  
               a (not necessarily minimal) primary decomposition of N



**Example:**

```

LIB "mprimdec.lib";
ring r=0,(x,y,z),dp;
module N=y*gen(1),y2*gen(2),yz*gen(2),yx*gen(2);
ideal p=y;
list l=PrimdecB(N,p);
l;
⇒ [1]:
⇒ [1]:
⇒ _[1]=y*gen(1)
⇒ _[2]=y*gen(2)
⇒ [2]:
⇒ _[1]=y
⇒ [2]:
⇒ [1]:
⇒ _[1]=y*gen(1)
⇒ _[2]=y*gen(2)
⇒ _[3]=x*gen(1)
⇒ _[4]=x*gen(2)
⇒ [2]:
⇒ _[1]=y
⇒ _[2]=x
⇒ [3]:
⇒ [1]:
⇒ _[1]=z*gen(1)
⇒ _[2]=z*gen(2)
⇒ _[3]=y*gen(1)
⇒ _[4]=x*gen(1)
⇒ _[5]=x*gen(2)
⇒ _[6]=y2*gen(2)
⇒ [2]:
⇒ _[1]=z
⇒ _[2]=y
⇒ _[3]=x

```

**D.4.18.4 modDec**

Procedure from library `mprimdec.lib` (see [Section D.4.18 \[mprimdec.lib\]](#), page 1161).

**Usage:** modDec (N[, i]); module N, int i

**Return:** list l  
 a minimal primary decomposition of N  
 computed by an generalized version of the algorithm of Shimoyama/Yokoyama,  
 if i=1 is given, the factorizing Groebner basis algorithm is used internally.

**Example:**

```

LIB "mprimdec.lib";
ring r=0,(x,y,z),dp;
module N=x*gen(1)+ y*gen(2),
x*gen(1)-x2*gen(2);
list l=modDec(N);
l;
⇒ [1]:

```

```

⇒      [1]:
⇒      _[1]=x*gen(1)+y*gen(2)
⇒      _[2]=x*gen(2)-gen(1)
⇒      [2]:
⇒      _[1]=x2+y
⇒ [2]:
⇒      [1]:
⇒      _[1]=gen(2)
⇒      _[2]=x*gen(1)
⇒      [2]:
⇒      _[1]=x

```

#### D.4.18.5 zeroMod

Procedure from library `mprimdec.lib` (see [Section D.4.18 \[mprimdec.lib\], page 1161](#)).

**Usage:**      `zeroMod (N[, check]);` zero-dimensional module `N[, module check]`

**Return:**     `list l`  
               the minimal primary decomposition of a zero-dimensional module `N`, computed by a generalized version of the algorithm of Gianni, Trager and Zacharias

**Note:**        if the parameter `check` is given, only components not containing `check` are computed

**Example:**

```

LIB "mprimdec.lib";
ring r=0,z,dp;
module N=z*gen(1),(z-1)*gen(2),(z+1)*gen(3);
list l=zeroMod(N);
⇒ 2
l;
⇒ [1]:
⇒      [1]:
⇒      _[1]=gen(2)
⇒      _[2]=gen(3)
⇒      _[3]=z*gen(1)
⇒      [2]:
⇒      _[1]=z
⇒ [2]:
⇒      [1]:
⇒      _[1]=gen(1)
⇒      _[2]=gen(3)
⇒      _[3]=z*gen(2)-gen(2)
⇒      [2]:
⇒      _[1]=z-1
⇒ [3]:
⇒      [1]:
⇒      _[1]=gen(1)
⇒      _[2]=gen(2)
⇒      _[3]=z*gen(3)+gen(3)
⇒      [2]:
⇒      _[1]=z+1

```

#### D.4.18.6 GTZmod

Procedure from library `mprimdec.lib` (see [Section D.4.18 \[mprimdec.lib\], page 1161](#)).

**Usage:** GTZmod (N[, check]); module N[, module check]

**Return:** list l  
the minimal primary decomposition of the module N,  
computed by a generalized version of the algorithm of Gianni, Trager and Zacharias

**Note:** if the parameter check is given, only components not containing check are computed

**Example:**

```
LIB "mprimdec.lib";
ring r=0,(x,y,z),dp;
module N=x*gen(1)+ y*gen(2),
x*gen(1)-x2*gen(2);
list l=GTZmod(N);
⇒ 2
l;
⇒ [1]:
⇒ [1]:
⇒ _[1]=gen(2)
⇒ _[2]=x*gen(1)
⇒ [2]:
⇒ _[1]=x
⇒ [2]:
⇒ [1]:
⇒ _[1]=x*gen(1)+y*gen(2)
⇒ _[2]=x*gen(2)-gen(1)
⇒ [2]:
⇒ _[1]=x2+y
```

#### D.4.18.7 declvar

Procedure from library `mprimdec.lib` (see [Section D.4.18 \[mprimdec.lib\]](#), page 1161).

**Usage:** declvar (N); zero-dimensional module N[, module check]

**Return:** list l  
the minimal primary decomposition of a submodule N of  $R^s$  if  $\text{nvars}(R)=1$

**Note:** if the parameter check is given, only components not containing check are computed

**Example:**

```
LIB "mprimdec.lib";
ring r=0,z,dp;
module N=z*gen(1),(z-1)*gen(2),(z+1)*gen(3);
list l=declvar(N);
l;
⇒ [1]:
⇒ [1]:
⇒ _[1]=gen(2)
⇒ _[2]=gen(3)
⇒ _[3]=z*gen(1)
⇒ [2]:
⇒ _[1]=z
⇒ [2]:
⇒ [1]:
⇒ _[1]=gen(1)
```

```

↳      _[2]=gen(3)
↳      _[3]=z*gen(2)-gen(2)
↳      [2]:
↳      _[1]=z-1
↳      [3]:
↳      [1]:
↳      _[1]=gen(1)
↳      _[2]=gen(2)
↳      _[3]=z*gen(3)+gen(3)
↳      [2]:
↳      _[1]=z+1

```

#### D.4.18.8 annil

Procedure from library `mprimdec.lib` (see [Section D.4.18 \[mprimdec.lib\]](#), page 1161).

**Usage:**      `annil(N); modul N`

**Return:**    `ideal ann=std(quotient(N, freemodule(nrows(N))));`  
the annihilator of  $M/N$  in the basering

**Note:**      `ann` is a std basis in the basering

**Example:**

```

LIB "mprimdec.lib";
ring r=0,(x,y,z),dp;
module N=x*gen(1), y*gen(2);
ideal ann=annil(N);
ann;
↳ ann[1]=xy

```

#### D.4.18.9 splitting

Procedure from library `mprimdec.lib` (see [Section D.4.18 \[mprimdec.lib\]](#), page 1161).

**Usage:**      `splitting(N[,check[, ann]]); modul N, module check, ideal ann`

**Return:**    `(l, check)` list `l`, module `check`  
the elements of `l` consists of quadruples, where  
`[1]` is of type module, `[2]`, `[3]` and `[4]` are of type ideal, s.th. the intersection of the  
modules is equal to the zero-dimensional module  $N$ , furthermore `l[j][3]=annil(l[j][1])`  
and `l[j][4]` contains internal ideal data;  
if `l[j][2]!=0` then the module `l[j][1]` is primary  
with associated prime `l[j][2]`, and `check=intersect(check, l[j][1])` is computed

**Note:**      if the parameter `check` is given, only components not containing `check` are computed;  
if `ann` is given, `ann` is used instead of `annil(N)`

**Example:**

```

LIB "mprimdec.lib";
ring r=0,z,lp;
module N=z*gen(1), (z+1)*gen(2);
N=std(N);
list l; module check;
(l, check)=splitting(N);
l;

```

```

⇒ [1]:
⇒ [1]:
⇒ [1]=gen(2)
⇒ [2]=z*gen(1)
⇒ [2]:
⇒ [1]=z
⇒ [3]:
⇒ [1]=z
⇒ [4]:
⇒ [1]=z
⇒ [2]:
⇒ [1]:
⇒ [1]=gen(1)
⇒ [2]=z*gen(2)+gen(2)
⇒ [2]:
⇒ [1]=z+1
⇒ [3]:
⇒ [1]=z+1
⇒ [4]:
⇒ [1]=z+1
check;
⇒ check[1]=z*gen(2)+gen(2)
⇒ check[2]=z*gen(1)

```

#### D.4.18.10 primTest

Procedure from library `mprimdec.lib` (see [Section D.4.18 \[mprimdec.lib\]](#), page 1161).

**Usage:** `primTest(i[, p]);` a zero-dimensional ideal `i`, irreducible poly `p` in `i`

**Return:** if `i` is neither prime nor homogeneous then `ideal(0)` is returned, otherwise `radical(i)`

**Example:**

```

LIB "mprimdec.lib";
ring r=0,(x,y,z),lp;
ideal i=x+1,y-1,z;
i=std(i);
ideal primId=primTest(i,z);
primId;
⇒ primId[1]=z
⇒ primId[2]=y-1
⇒ primId[3]=x+1
i=x,z2,yz,y2;
i=std(i);
primId=primTest(i);
primId;
⇒ primId[1]=x
⇒ primId[2]=y
⇒ primId[3]=z

```

#### D.4.18.11 preComp

Procedure from library `mprimdec.lib` (see [Section D.4.18 \[mprimdec.lib\]](#), page 1161).

**Usage:** `preComp(N,check[, ann]);` modul `N`, module `check`, ideal `ann`

- Return:** (l, check) list l, module check  
the elements of l consists of a triple with  
[1] of type module [2] and [3] of type ideal  
s.th. the intersection of the modules is equal to the zero-dimensional module N, furthermore  $l[j][3] = \text{annil}(l[j][1])$  if  $l[j][2] \neq 0$  then the module  $l[j][1]$  is primary with associated prime  $l[j][2]$ ,  
and  $\text{check} = \text{intersect}(\text{check}, l[j][1])$  is computed
- Note:** only components not containing check are computed;  
if ann is given, ann is used instead of  $\text{annil}(N)$

**Example:**

```
LIB "mprimdec.lib";
ring r=0,z,lp;
module N=z*gen(1), (z+1)*gen(2);
N=std(N);
list l; module check;
(l, check)=preComp(N, freemodule(2));
l;
⇒ [1]:
⇒ [1]:
⇒ _[1]=z*gen(1)
⇒ _[2]=gen(2)
⇒ [2]:
⇒ _[1]=z
⇒ [3]:
⇒ _[1]=z
⇒ [2]:
⇒ [1]:
⇒ _[1]=gen(1)
⇒ _[2]=z*gen(2)+gen(2)
⇒ [2]:
⇒ _[1]=z+1
⇒ [3]:
⇒ _[1]=z+1
check;
⇒ check[1]=z*gen(1)
⇒ check[2]=z*gen(2)+gen(2)
```

**D.4.18.12 indSet**

Procedure from library `mprimdec.lib` (see [Section D.4.18 \[mprimdec.lib\]](#), page 1161).

- Usage:** `indSet(i); i ideal`
- Return:** list with two entries  
both are lists of new varstrings with the dependent variables, the independent set, the ordstring with the corresp. block ordering, and the integer where the independent set starts in the varstring
- Note:** the first entry gives the strings for all maximal independent sets the second gives the strings for the independent sets, which cannot be enhanced

**Example:**

```

LIB "mprimdec.lib";
ring s1=(0,x,y),(a,b,c,d,e,f,g),lp;
ideal i=ea-fbg,fa+be,ec-fdg,fc+de;
i=std(i);
list l=indSet(i);
l;
⇒ [1]:
⇒   [1]:
⇒     [1]:
⇒       e,f
⇒     [2]:
⇒       a,b,c,d,g
⇒     [3]:
⇒       (C,dp(2),dp)
⇒     [4]:
⇒       5
⇒ [2]:
⇒   [1]:
⇒     [1]:
⇒       a,b,c,d
⇒     [2]:
⇒       e,f,g
⇒     [3]:
⇒       (C,dp(4),dp)
⇒     [4]:
⇒       3
⇒   [2]:
⇒     [1]:
⇒       a,c,e
⇒     [2]:
⇒       b,d,f,g
⇒     [3]:
⇒       (C,dp(3),dp)
⇒     [4]:
⇒       4

```

#### D.4.18.13 GTZopt

Procedure from library `mprimdec.lib` (see [Section D.4.18 \[mprimdec.lib\]](#), page 1161).

**Usage:** GTZopt (N[, check]); module N[, module check]

**Return:** list l  
the minimal primary decomposition of the module N,  
computed by a generalized and optimized version of  
the algorithm of Gianni, Trager and Zacharias

**Note:** if the parameter check is given, only components  
not containing check are computed

**Example:**

```

LIB "mprimdec.lib";
ring r=0,(x,y,z),dp;
module N=x*gen(1)+ y*gen(2),
x*gen(1)-x2*gen(2);

```

```

list l=GTZopt(N);
l;
⇒ [1]:
⇒   [1]:
⇒     _[1]=gen(2)
⇒     _[2]=x*gen(1)
⇒   [2]:
⇒     _[1]=x
⇒ [2]:
⇒   [1]:
⇒     _[1]=x*gen(1)+y*gen(2)
⇒     _[2]=x*gen(2)-gen(1)
⇒   [2]:
⇒     _[1]=x2+y

```

#### D.4.18.14 zeroOpt

Procedure from library `mprimdec.lib` (see [Section D.4.18 \[mprimdec.lib\]](#), page 1161).

**Usage:**      `zeroOpt (N[, check]);` zero-dimensional module `N`[, module `check`]

**Return:**    `list l`  
               the minimal primary decomposition of a zero-dimensional module `N`, computed by a generalized and optimized version of the algorithm of Gianni, Trager and Zacharias

**Note:**       if the parameter `check` is given, only components  
               not containing `check` are computed

**Example:**

```

LIB "mprimdec.lib";
ring r=0,z,dp;
module N=z*gen(1),(z-1)*gen(2),(z+1)*gen(3);
list l=zeroOpt(N);
l;
⇒ [1]:
⇒   [1]:
⇒     _[1]=z*gen(1)
⇒     _[2]=gen(2)
⇒     _[3]=gen(3)
⇒   [2]:
⇒     _[1]=z
⇒ [2]:
⇒   [1]:
⇒     _[1]=gen(1)
⇒     _[2]=z*gen(2)-gen(2)
⇒     _[3]=gen(3)
⇒   [2]:
⇒     _[1]=z-1
⇒ [3]:
⇒   [1]:
⇒     _[1]=gen(1)
⇒     _[2]=gen(2)
⇒     _[3]=z*gen(3)+gen(3)
⇒   [2]:
⇒     _[1]=z+1

```



### D.4.19 mregular.lib

**Library:** mregular.lib

**Purpose:** Castelnuovo-Mumford regularity of homogeneous ideals

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**Overview:** A library for computing the Castelnuovo-Mumford regularity of a homogeneous ideal that DOES NOT require the computation of a minimal graded free resolution of the ideal.

It also determines  $\text{depth}(\text{basing}/\text{ideal})$  and  $\text{satiety}(\text{ideal})$ . The procedures are based on 3 papers by Isabel Bermejo and Philippe Gimenez: 'On Castelnuovo-Mumford regularity of projective curves' Proc.Amer.Math.Soc. 128(5) (2000), 'Computing the Castelnuovo-Mumford regularity of some subschemes of  $P^n$  using quotients of monomial ideals', Proceedings of MEGA-2000, J. Pure Appl. Algebra 164 (2001), and 'Saturation and Castelnuovo-Mumford regularity', Preprint (2004).

**Procedures:**

#### D.4.19.1 regIdeal

Procedure from library mregular.lib (see [Section D.4.19 \[mregular.lib\]](#), page 1171).

**Usage:** regIdeal (i[e]); i ideal, e integer

**Return:** an integer, the Castelnuovo-Mumford regularity of i.  
 (returns -1 if i is not homogeneous)

**Assume:** i is a homogeneous ideal of the basering  $S=K[x(0)..x(n)]$ . e=0: (default)  
 If K is an infinite field, makes random changes of coordinates. If K is a finite field, works over a transcendental extension. e=1: Makes random changes of coordinates even when K is finite. It works if it terminates, but may result in an infinite loop. After 30 loops, a warning message is displayed and -1 is returned.

**Note:** If  $\text{printlevel} > 0$  (default = 0), additional info is displayed:  $\dim(S/i)$ ,  $\text{depth}(S/i)$  and  $\text{end}(H^{\text{depth}(S/i)}(S/i))$  are computed, and an upper bound for the a-invariant of  $S/i$  is given. The algorithm also determines whether the regularity is attained or not at the last step of a minimal graded free resolution of i, and if the answer is positive, the regularity of the Hilbert function of  $S/i$  is given.

**Example:**

```
LIB "mregular.lib";
ring r=0,(x,y,z,t,w),dp;
ideal i=y2t,x2y-x2z+yt2,x2y2,xyztw,x3z2,y5+xz3w-x2zw2,x7-yt2w4;
regIdeal(i);
  ↳ 10
regIdeal(lead(std(i)));
  ↳ 13
// Additional information is displayed if you change printlevel (=1);
```

### D.4.19.2 depthIdeal

Procedure from library `mregular.lib` (see [Section D.4.19 \[mregular.lib\]](#), page 1171).

**Usage:** `depthIdeal (i[,e]);`  $i$  ideal,  $e$  integer

**Return:** an integer, the depth of  $S/i$  where  $S=K[x(0)..x(n)]$  is the basering. (returns -1 if  $i$  is not homogeneous or if  $i=(1)$ )

**Assume:**  $i$  is a proper homogeneous ideal.

$e=0$ : (default)

If  $K$  is an infinite field, makes random changes of coordinates. If  $K$  is a finite field, works over a transcendental extension.  $e=1$ : Makes random changes of coordinates even when  $K$  is finite. It works if it terminates, but may result in an infinite loop. After 30 loops, a warning message is displayed and -1 is returned.

**Note:** If `printlevel > 0` (default = 0),  $\dim(S/i)$  is also displayed.

**Example:**

```
LIB "mregular.lib";
ring r=0,(x,y,z,t,w),dp;
ideal i=y2t,x2y-x2z+yt2,x2y2,xyztw,x3z2,y5+xz3w-x2zw2,x7-yt2w4;
depthIdeal(i);
  ↪ 1
depthIdeal(lead(std(i)));
  ↪ 0
// Additional information is displayed if you change printlevel (=1);
```

### D.4.19.3 satiety

Procedure from library `mregular.lib` (see [Section D.4.19 \[mregular.lib\]](#), page 1171).

**Usage:** `satiety (i[,e]);`  $i$  ideal,  $e$  integer

**Return:** an integer, the satiety of  $i$ .  
(returns -1 if  $i$  is not homogeneous)

**Assume:**  $i$  is a homogeneous ideal of the basering  $S=K[x(0)..x(n)]$ .  $e=0$ : (default)

The satiety is computed determining the fresh elements in the socle of  $i$ . It works over arbitrary fields.

$e=1$ : Makes random changes of coordinates to find a monomial ideal with same satiety. It works over infinite fields only. If  $K$  is finite, it works if it terminates, but may result in an infinite loop. After 30 loops, a warning message is displayed and -1 is returned.

**Theory:** The satiety, or saturation index, of a homogeneous ideal  $i$  is the least integer  $s$  such that, for all  $d \geq s$ , the degree  $d$  part of the ideals  $i$  and  $\text{isat}=\text{sat}(i, \text{maxideal}(1))$  coincide.

**Note:** If `printlevel > 0` (default = 0),  $\dim(S/i)$  is also displayed.

**Example:**

```
LIB "mregular.lib";
ring r=0,(x,y,z,t,w),dp;
ideal i=y2t,x2y-x2z+yt2,x2y2,xyztw,x3z2,y5+xz3w-x2zw2,x7-yt2w4;
satiety(i);
  ↪ 0
ideal I=lead(std(i));
satiety(I); // First method: direct computation
```

```

    ↪ 12
    satiety(I,1); // Second method: doing changes of coordinates
    ↪ 12
    // Additional information is displayed if you change printlevel (=1);

```

#### D.4.19.4 regMonCurve

Procedure from library `mregular.lib` (see [Section D.4.19 \[mregular.lib\]](#), page 1171).

- Usage:** `regMonCurve (a0,...,an)` ;  $a_i$  integers with  $a_0=0 < a_1 < \dots < a_n=:d$
- Return:** an integer, the Castelnuovo-Mumford regularity of the projective monomial curve  $C$  in  $P_n(K)$  parametrically defined by  $x(0) = t^d$ ,  $x(1) = s^{a_1}t^{(d-a_1)}$ , ...,  $x(n) = s^d$  where  $K$  is the field of complex numbers.  
(returns -1 if  $a_0=0 < a_1 < \dots < a_n$  is not satisfied)
- Assume:**  $a_0=0 < a_1 < \dots < a_n$  are integers.
- Notes:**
1. The defining ideal of the curve  $C$ ,  $I$  in  $S=K[x(0),\dots,x(n)]$ , is determined by elimination.
  2. The procedure `regIdeal` has been improved in this case since one knows beforehand that the monomial ideal  $J=\text{lead}(\text{std}(I))$  is of nested type if the monomial ordering is `dp`, and that  $\text{reg}(C)=\text{reg}(J)$  (see preprint 'Saturation and Castelnuovo-Mumford regularity' by Bermejo-Gimenez, 2004).
  3. If `printlevel > 0` (default = 0) additional info is displayed: - It says whether  $C$  is arithmetically Cohen-Macaulay or not. - If  $C$  is not arith. Cohen-Macaulay,  $\text{end}(H^1(S/I))$  is computed and an upper bound for the  $a$ -invariant of  $S/I$  is given. - It also determines one step of the minimal graded free resolution (m.g.f.r.) of  $I$  where the regularity is attained and gives the value of the regularity of the Hilbert function of  $S/I$  when  $\text{reg}(I)$  is attained at the last step of a m.g.f.r.

**Example:**

```

LIB "mregular.lib";
// The 1st example is the twisted cubic:
regMonCurve(0,1,2,3);
↪ 2
// The 2nd. example is the non arithm. Cohen-Macaulay monomial curve in P4
// parametrized by: x(0)-s6,x(1)-s5t,x(2)-s3t3,x(3)-st5,x(4)-t6:
regMonCurve(0,1,3,5,6);
↪ 3
// Additional information is displayed if you change printlevel (=1);

```

#### D.4.19.5 NoetherPosition

Procedure from library `mregular.lib` (see [Section D.4.19 \[mregular.lib\]](#), page 1171).

- Usage:** `NoetherPosition (i)`;  $i$  ideal
- Return:** ideal such that, for the homogeneous linear transformation  $\text{map } \phi=S, \text{NoetherPosition}(i)$ ; one has that  $K[x(n-d+1),\dots,x(n)]$  is a Noether normalization of  $S/\phi(i)$  where  $S=K[x(0),\dots,x(n)]$  is the basering and  $d=\dim(S/i)$ . (returns -1 if  $i = (0)$  or  $(1)$ ).
- Assume:** The field  $K$  is infinite and  $i$  is a nonzero proper ideal.

- Note:**
1. It works also if  $K$  is a finite field if it terminates, but may result in an infinite loop. If the procedure enters more than 30 loops, -1 is returned and a warning message is displayed.
  2. If `printlevel > 0` (default = 0), additional info is displayed:  $\dim(S/i)$  and  $K[x(n-d+1), \dots, x(n)]$  are given.

**Example:**

```
LIB "mregular.lib";
ring r=0,(x,y,z,t,u),dp;
ideal i1=y,z,t,u; ideal i2=x,z,t,u; ideal i3=x,y,t,u; ideal i4=x,y,z,u;
ideal i5=x,y,z,t; ideal i=intersect(i1,i2,i3,i4,i5);
map phi=r,NoetherPosition(i);
phi;
  ↳ phi[1]=x
  ↳ phi[2]=y
  ↳ phi[3]=z
  ↳ phi[4]=t
  ↳ phi[5]=53x+27y-75z+45t+u
ring r5=5,(x,y,z,t,u),dp;
ideal i=imap(r,i);
map phi=r5,NoetherPosition(i);
phi;
  ↳ phi[1]=x
  ↳ phi[2]=y
  ↳ phi[3]=z
  ↳ phi[4]=t
  ↳ phi[5]=x-y+z-t+u
// Additional information is displayed if you change printlevel (=1);
```

**D.4.19.6 is\_NP**

Procedure from library `mregular.lib` (see [Section D.4.19 \[mregular.lib\]](#), page 1171).

**Usage:** `is_NP (i); i ideal`

**Return:** 1 if  $K[x(n-d+1), \dots, x(n)]$  is a Noether normalization of  $S/i$  where  $S=K[x(0), \dots, x(n)]$  is the basering, and  $d=\dim(S/i)$ , 0 otherwise.  
(returns -1 if  $i=(0)$  or  $i=(1)$ ).

**Assume:**  $i$  is a nonzero proper homogeneous ideal.

- Note:**
1. If  $i$  is not homogeneous and  $\text{is\_NP}(i)=1$  then  $K[x(n-d+1), \dots, x(n)]$  is a Noether normalization of  $S/i$ . The converse may be wrong if the ideal is not homogeneous.
  2. `is_NP` is used in the procedures `regIdeal`, `depthIdeal`, `satiety`, and `NoetherPosition`.

**Example:**

```
LIB "mregular.lib";
ring r=0,(x,y,z,t,u),dp;
ideal i1=y,z,t,u; ideal i2=x,z,t,u; ideal i3=x,y,t,u; ideal i4=x,y,z,u;
ideal i5=x,y,z,t; ideal i=intersect(i1,i2,i3,i4,i5);
is_NP(i);
  ↳ 0
ideal ch=x,y,z,t,x+y+z+t+u;
map phi=ch;
is_NP(phi(i));
  ↳ 1
```

### D.4.19.7 is\_nested

Procedure from library `mregular.lib` (see [Section D.4.19 \[mregular.lib\]](#), page 1171).

**Usage:** `is_nested (i);`  $i$  monomial ideal

**Return:** 1 if  $i$  is of nested type, 0 otherwise.  
(returns -1 if  $i=(0)$  or  $i=(1)$ ).

**Assume:**  $i$  is a nonzero proper monomial ideal.

**Notes:** 1. The ideal must be monomial, otherwise the result has no meaning (so check this before using this procedure).  
2. `is_nested` is used in procedures `depthIdeal`, `regIdeal` and `satiety`.  
3. When  $i$  is a monomial ideal of nested type of  $S=K[x(0)..x(n)]$ , the  $a$ -invariant of  $S/i$  coincides with the upper bound obtained using the procedure `regIdeal` with `printlevel > 0`.

**Theory:** A monomial ideal is of nested type if its associated primes are all of the form  $(x(0), \dots, x(i))$  for some  $i \leq n$ .  
(see definition and effective criterion to check this property in the preprint 'Saturation and Castelnuovo-Mumford regularity' by Bermejo-Gimenez, 2004).

**Example:**

```
LIB "mregular.lib";
ring s=0,(x,y,z,t),dp;
ideal i1=x2,y3; ideal i2=x3,y2,z2; ideal i3=x3,y2,t2;
ideal i=intersect(i1,i2,i3);
is_nested(i);
  ↦ 0
ideal ch=x,y,z,z+t;
map phi=ch;
ideal I=lead(std(phi(i)));
is_nested(I);
  ↦ 1
```

### D.4.20 nfmodstd.lib

**Library:** `nfmodstd.lib`

**Purpose:** Groebner bases of ideals in polynomial rings over algebraic number fields

**Authors:** D.K. Boku [boku@mathematik.uni-kl.de](mailto:boku@mathematik.uni-kl.de)  
W. Decker [decker@mathematik.uni-kl.de](mailto:decker@mathematik.uni-kl.de)  
C. Fieker [fieler@mathematik.uni-kl.de](mailto:fieler@mathematik.uni-kl.de)

**Overview:** A library for computing the Groebner basis of an ideal in the polynomial ring over an algebraic number field  $Q(t)$  using the modular methods, where  $t$  is algebraic over the field of rational numbers  $Q$ . For the case  $Q(t) = Q$ , the procedure is inspired by Arnold [1]. This idea is then extended to the case  $t$  not in  $Q$  using factorization as follows:  
Let  $f$  be the minimal polynomial of  $t$ .  
For  $I, I'$  ideals in  $Q(t)[X]$ ,  $Q[X,t]/\langle f \rangle$  respectively, we map  $I$  to  $I'$  via the map sending  $t$  to  $t + \langle f \rangle$ . We first choose a prime  $p$  such that  $f$  has at least two factors in characteristic  $p$  and add each factor  $f_i$  to  $I'$  to obtain the ideal  $J'_i = I' + \langle f_i \rangle$ . We then compute a standard basis  $G'_i$  of  $J'_i$  for each  $i$  and combine the  $G'_i$  to  $G_p$  (a standard basis of  $I'_p$ ) using chinese remaindering for polynomials. The procedure is repeated for

many primes  $p$ , where we compute the  $G_p$  in parallel until the number of primes is sufficiently large to recover the correct standard basis  $G'$  of  $I'$ . Finally, by mapping  $G'$  back to  $Q(t)[X]$ , a standard basis  $G$  of  $I$  is obtained.

The procedure also works if the input is a module. For this, we consider the rings  $A = Q(t)[X]$  and  $A' = (Q[t]/\langle f \rangle)[X]$ . For submodules  $I, I'$  in  $A^m, A'^m$ , respectively, we map  $I$  to  $I'$  via the map sending  $t$  to  $t + \langle f \rangle$ . As above, we first choose a prime  $p$  such that  $f$  has at least two factors in characteristic  $p$ . For each factor  $f_{i,p}$  of  $f_p := (f \bmod p)$ , we set  $I'_{i,p} := (I'_p \bmod f_{i,p})$ . We then compute a standard basis  $G'_i$  of  $I'_{i,p}$  over  $F_p[t]/\langle f_{i,p} \rangle$  for each  $i$  and combine the  $G'_i$  to  $G_p$  (a standard basis of  $I'_p$ ) using chinese remaindering for polynomials. The procedure is repeated for many primes  $p$  as described above and we finally obtain a standard basis of  $I$ .

#### References:

- [1] E. A. Arnold: Modular algorithms for computing Groebner bases. J. Symb. Comp. 35, 403-419 (2003).

#### Procedures:

### D.4.20.1 chinrempoly

Procedure from library `nfmodstd.lib` (see [Section D.4.20 \[nfmodstd.lib\]](#), page 1175).

**Usage:** `chinrempoly(l, m);`  $l$  list,  $m$  list

**Return:** a polynomial (resp. ideal/module) which is congruent to  $l[i]$  modulo  $m[i]$  for all  $i$

**Note:** The procedure applies chinese remaindering to the first argument w.r.t. the moduli given in the second. The elements in the first list must be of the same type which can be polynomial, ideal, or module. The moduli must be of type polynomial. The elements in the second list must be distinct and co-prime.

#### Example:

```
LIB "nfmodstd.lib";
ring rr=97,x,dp;
poly f=x^7-7*x + 3;
ideal J=factorize(f,1);
J;
↪ J[1]=x+37
↪ J[2]=x3+9x2+20x-20
↪ J[3]=x3-46x2+17x-8
list m=J[1..ncols(J)];
list l= x^2+2*x+3, x^2+5, x^2+7;
ideal I=chinrempoly(l,m);
I;
↪ I[1]=-44x6-36x5-45x4+12x3-36x2+25x-32
ring s=0,x,dp;
list m= x^2+2*x+3, x^3+5, x^4+x^3+7;
list l=x^3 + 2, x^4 + 7, x^5 + 11;
ideal I=chinrempoly(l,m);
I;
↪ I[1]=18113/107610x8+5826/17935x7-5257/107610x6+3975/7174x5+246151/107610x\
4+131573/53805x3-910/633x2-36239/21522x+146695/7174
int p=prime(536546513);
ring r = p, (x,y,a), (dp(2),dp(1));
poly minpolynomial = a^2+1;
```

```

ideal kf=factorize(minpolynomial,1); //return factors without multiplicity
kf;
⇨ kf[1]=a+222052315
⇨ kf[2]=a-222052315
ideal k=(a+1)*x2+y, 3x-ay+ a+2;
option(redSB);
ideal k1=k,kf[1];
ideal k2 =k,kf[2];
k1=std(k1);
k2=std(k2);
list l=k1,k2;
list m=kf[1..ncols(kf)];
ideal I=chinrempoly(l,m);
I=simplify(I,2);
I;
⇨ I[1]=x-178848838ya+178848838a-178848837
⇨ I[2]=y2-268273248ya+268273250y-4a-3
l = module(k1[2..ncols(k1)]), module(k2[2..ncols(k2)]);
module M = chinrempoly(l,m);
M;
⇨ M[1]=x*gen(1)-178848838ya*gen(1)+178848838a*gen(1)-178848837*gen(1)
⇨ M[2]=y2*gen(1)-268273248ya*gen(1)+268273250y*gen(1)-4a*gen(1)-3*gen(1)

```

See also: [Section 5.1.8 \[chinrem\]](#), [page 162](#).

#### D.4.20.2 nfmodStd

Procedure from library `nfmodstd.lib` (see [Section D.4.20 \[nfmodstd.lib\]](#), [page 1175](#)).

**Usage:** `nfmodStd(I, #)`; I ideal or module, # optional parameters

**Return:** standard basis of I over algebraic number field

**Note:** The procedure passes to [Section D.4.16.1 \[modStd\]](#), [page 1147](#) if the ground field has no parameter. In this case, the optional parameters # (if given) are directly passed to [Section D.4.16.1 \[modStd\]](#), [page 1147](#).

**Example:**

```

LIB "nfmodstd.lib";
ring r1 = (0,a),(x,y),dp;
minpoly = a2+1;
ideal k = (a/2+1)*x2+2/3y, 3*x-a*y+ a/7+2;
ideal I = nfmodStd(k);
I;
⇨ I[1]=x+(-1/3a)*y+(1/21a+2/3)
⇨ I[2]=y2+(32/5a-178/35)*y+(-4/7a-195/49)
ring rm = (0,a),(x,y),(c,dp);
minpoly = a3+2a+7;
module M = [(a/2+1)*x2+2/3y, 3*x-a*y+ a/7+2], [ax, y];
M = nfmodStd(M);
M;
⇨ M[1]=[0,x2y+(12/5a2-24/5a+42/5)*x2+(8/5a2-6/5a+28/5)*xy+(4/15a2-8/15a+8/5\
) *y2+(48/35a2-106/35a+24/5)*x]
⇨ M[2]=[y, (3/14a2-9/28)*xy+9/2*x+(-3/2a)*y+(3/14a+3)]
⇨ M[3]=[x, (-1/7a2-2/7)*y]

```

```

ring r2 = (0,a),(x,y,z),dp;
minpoly = a^3 +2;
ideal k = (a^2+a/2)*x^2+(a^2 -2/3*a)*yz, (3*a^2+1)*zx-(a+4/7)*y+ a+2/5;
ideal IJ = nfmodStd(k);
IJ;
⇒ IJ[1]=xz+(138/763a2+65/763a-46/763)*y+(-96/545a2-31/545a+32/545)
⇒ IJ[2]=x2+(28/45a2-14/45a+52/45)*yz
⇒ IJ[3]=yz2+(-3354/23653a2-6390/23653a-7683/47306)*xy+(993/6758a2+4104/1689\
5a+4449/33790)*x
ring r3 = 0, (x,y), dp; // ring without parameter
ideal I = x2 + y, xy - 7y + 2x;
ideal J1 = nfmodStd(I);
J1;
⇒ J1[1]=y2-14x+51y
⇒ J1[2]=xy+2x-7y
⇒ J1[3]=x2+y
module J2 = nfmodStd(module(I));
J2;
⇒ J2[1]=y2*gen(1)-14x*gen(1)+51y*gen(1)
⇒ J2[2]=xy*gen(1)+2x*gen(1)-7y*gen(1)
⇒ J2[3]=x2*gen(1)+y*gen(1)
ring r4 = 0, (x,y), (c,dp);
module I = [x2, x-y], [xy,0], [0,-7y + 2x];
I=nfmodStd(I);
I;
⇒ I[1]=[0,x-7/2y]
⇒ I[2]=[0,y2]
⇒ I[3]=[xy]
⇒ I[4]=[x2,5/2y]

```

See also: [Section D.4.16.1 \[modStd\]](#), page 1147.

## D.4.21 nfmodsyz\_lib

**Library:** nfmodsyz.lib

**Purpose:** Syzygy modules of submodules of free modules over algebraic number fields

**Authors:** D.K. Boku boku@mathematik.uni-kl.de  
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**Overview:** A library for computing the syzygy module of a given submodule  $I$  in a polynomial ring over an algebraic number field  $\mathbb{Q}(t)$ , where  $t$  is an algebraic number, using modular methods. For the case  $\mathbb{Q}(t)=\mathbb{Q}$ , that is, where  $t$  is an element of  $\mathbb{Q}$ , we compute, following [1], the syzygy module of  $I$  as follows: For a submodule  $I$  of  $A^m$  with  $A = \mathbb{Q}[X]$ , we first choose a sufficiently large set of primes  $P$  and compute the reduced Groebner basis of the syzygy module of  $I_p$ , for each  $p$  in  $P$ , in parallel. We then use the Chinese remainder algorithm and rational reconstruction to obtain the syzygy module of  $I$  over  $\mathbb{Q}$ . For the case where  $t$  is not in  $\mathbb{Q}$ , we compute, following [2], the syzygy module of  $I$  as follows:

Let  $f$  be the minimal polynomial of  $t$ . For a submodule  $I$  in  $A^m$  with  $A = \mathbb{Q}(t)[X]$ , we map  $I$  to a submodule  $I'$  in  $A^m$  with  $A = (\mathbb{Q}[t]/\langle f \rangle)[X]$  via the map sending  $t$  to  $t + \langle f \rangle$ . We first choose a prime  $p$  such that  $f$  has at least two factors in characteristic



$p$ . For each factor  $f_{\{i,p\}}$  of  $f_p := (f \bmod p)$ , we set  $I'_{\{i,p\}} := (I'_p \bmod f_{\{i,p\}})$ . We then compute the reduced Groebner bases  $G'_i$  of the syzygy modules of  $I'_{\{i,p\}}$  over  $F_p[t]/\langle f_{\{i,p\}} \rangle$  and combine the  $G'_i$  to  $G_p$  (the syzygy module of  $I'_p$ ) using chinese remaindering for polynomials. As described in [2], the procedure is repeated for many primes  $p$ , where we compute the  $G_p$  in parallel until the number of primes is sufficiently large to recover the correct generating set for the syzygy module  $G'$  of  $I'$  which is, considered over  $Q(t)$ , also a generating set for the syzygy module of  $I$ .

#### References:

- [1] E. A. Arnold: Modular algorithms for computing Groebner bases. J. Symb. Comp. 35, 403-419 (2003).
- [2] D. Boku, W. Decker, C. Fieker, and A. Steenpass. Groebner bases over algebraic number fields. In: Proceedings of the 2015 International Workshop on Parallel Symb. Comp. PASCO'15, pages 16-24 (2015).

#### Procedures:

### D.4.21.1 nfmodSyz

Procedure from library `nfmodsyz.lib` (see [Section D.4.21 \[nfmodsyz.lib\]](#), page 1178).

**Usage:** `nfmodSyz(I)`;  $I$  ideal or module

**Return:** syzygy module of  $I$  over an algebraic number field

#### Example:

```
LIB "nfmodsyz.lib";
ring r1 =(0,a),(x,y),(c,dp);
minpoly = (a^3+2a+7);
module M1 = [(a/2+1)*y, 3*x-a*y],
[y-x,y2],
[x2-xy, ax-y];
nfmodSyz(M1);
⇨ _[1]=[x2y2-xy3+(a)*x2+(-a-1)*xy+y2,-3*x3+(a+3)*x2y+(-a)*xy2+(1/2a2+a)*xy+
(-1/2a-1)*y2,(-1/2a-1)*y3-3*x2+(a+3)*xy+(-a)*y2]
ring r2 = (0,a),(x,y,z),(dp,c);
minpoly = (a3+a+1);
module M2 = [x2z+x+(-a)*y,z2+(a+2)*x],
[y2+(a)*z+(a),(a+3)*z3+(-a)*x2],
[-xz+(a+3)*yz,xy+(a2)*z];
nfmodSyz(M2);
⇨ _[1]=x2z4*gen(3)+(3/29a2-9/29a+1/29)*x4z*gen(3)+(-1/29a2+3/29a-10/29)*x3y\
z*gen(2)+xz4*gen(1)+(-a2-3)*yz4*gen(1)+(1/29a2-3/29a+10/29)*xy3*gen(1)+(3\
/29a2-9/29a+1/29)*x3z*gen(1)+(-7/29a2+21/29a-12/29)*x2yz*gen(1)+(-9/29a2-\
2/29a-3/29)*x2z2*gen(2)+(-1/29a2+3/29a-10/29)*y2z2*gen(3)+(-1/29a2+3/29a-\
10/29)*xz3*gen(2)+xz3*gen(3)+(12/29a2-7/29a+33/29)*yz3*gen(2)+(-a)*yz3*ge\
n(3)+(3/29a2-9/29a+1/29)*x3*gen(3)+(-1/29a2+3/29a-10/29)*x2y*gen(2)+(9/29\
a2+2/29a+3/29)*x2y*gen(3)+(-3/29a2+9/29a-1/29)*xy2*gen(2)+(1/29a2-3/29a-1\
9/29)*xy2*gen(3)+(1/29a2-3/29a-19/29)*x2z*gen(2)+(-3/29a2+9/29a-1/29)*xyz\
*gen(1)+(17/29a2+7/29a+54/29)*xyz*gen(2)+(9/29a2+2/29a+3/29)*y2z*gen(1)+(\
3/29a2-9/29a+1/29)*z3*gen(3)+(-3/29a2+9/29a-1/29)*xy*gen(1)+(-9/29a2-2/29\
a-3/29)*xz*gen(2)+(-3/29a2-20/29a-1/29)*xz*gen(3)+(2/29a2-6/29a-9/29)*yz*\
gen(2)+(2/29a2-6/29a-9/29)*z2*gen(1)+(3/29a2-9/29a+1/29)*z2*gen(3)+(-3/29\
a2-20/29a-1/29)*x*gen(3)+(2/29a2-6/29a-9/29)*z*gen(1)
ring r3=0,(x,y),dp; // ring without parameter
```

```

module M3 = [x2 + y, xy], [-7y, 2x], [x2-y, 0];
nfmodSyz(M3);
⇨ _[1]=x2y*gen(2)+2x2*gen(3)-2x2*gen(1)+7y2*gen(3)-y2*gen(2)+2y*gen(3)+2y*g\
    en(1)
ring r4=0,(x,y),(c,dp); // ring without parameter
module M4 = [xy, x-y],
[x2 + y, 5y],
[- 7y, 2x],
[x2-y, 0];
nfmodSyz(M4);
⇨ _[1]=[0,x3-xy,-5/2x2y+5/2y2,-x3-xy-35/2y2]
⇨ _[2]=[x+35/4y,-1/2x2-7/4x+7/4y,5/4xy-1/2x+1/2y,1/2x2+7/4x-7/4y]
⇨ _[3]=[y2-16/1225y,-2/35x2y+156/6125x2-53/245xy+1/5y2-16/6125y,1/7xy2-78/1\
    225xy+2/49y2+8/1225y,2/35x2y-156/6125x2+53/245xy-3/35y2-296/6125y]

```

See also: [Section 5.1.156 \[syz\]](#), [page 280](#).

## D.4.22 noether\_lib

**Library:** noether.lib

**Purpose:** Noether normalization of an ideal (not necessary homogeneous)

**Authors:** A. Hashemi, Amir.Hashemi@lip6.fr

**Overview:** A library for computing the Noether normalization of an ideal that DOES NOT require the computation of the dimension of the ideal. It checks whether an ideal is in Noether position. A modular version of these algorithms is also provided.

The procedures are based on a paper of Amir Hashemi 'Efficient Algorithms for Computing Noether Normalization' (presented in ASCM 2007)

This library computes also Castelnuovo-Mumford regularity and satiety of an ideal. A modular version of these algorithms is also provided. The procedures are based on a paper of Amir Hashemi 'Computation of Castelnuovo-Mumford regularity and satiety' (preprint 2008)

**Procedures:**

### D.4.22.1 NPos\_test

Procedure from library `noether.lib` (see [Section D.4.22 \[noether.lib\]](#), [page 1180](#)).

**Usage:** NPos\_test (I); I monomial ideal

**Return:** A list whose first element is 1, if i is in Noether position, 0 otherwise. The second element of this list is a list of variables ordered such that those variables are listed first, of which a power belongs to the initial ideal of i. If i is in Noether position, the method returns furthermore the dimension of i.

**Assume:** i is a nonzero monomial ideal.

**Example:**

```

LIB "noether.lib";
ring r=0,(X,Y,a,b),dp;
poly f=X^8+a*Y^4-Y;
poly g=Y^8+b*X^4-X;
poly h=diff(f,X)*diff(g,Y)-diff(f,Y)*diff(g,X);

```

```

ideal i=f,g,h;
NPos_test(i);
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ [1]:
⇒ b
⇒ [2]:
⇒ a
⇒ [3]:
⇒ Y
⇒ [4]:
⇒ X

```

#### D.4.22.2 modNpos\_test

Procedure from library `noether.lib` (see [Section D.4.22 \[noether.lib\]](#), page 1180).

**Usage:** `modNpos_test(i)`; `i` an ideal

**Return:** 1 if `i` is in Noether position 0 otherwise.

**Note:** This test is a probabilistic test, and it computes the initial of the ideal modulo the prime number 2147483647 (the biggest prime less than  $2^{31}$ ).

**Example:**

```

LIB "noether.lib";
ring r=0,(X,Y,a,b),dp;
poly f=X^8+a*Y^4-Y;
poly g=Y^8+b*X^4-X;
poly h=diff(f,X)*diff(g,Y)-diff(f,Y)*diff(g,X);
ideal i=f,g,h;
modNpos_test(i);
⇒ // WARNING:
⇒ // The procedure is probabilistic and it computes the initial of the ide\
    al modulo the prime number 2147483647
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ [1]:
⇒ X
⇒ [2]:
⇒ Y
⇒ [3]:
⇒ b
⇒ [4]:
⇒ a

```

#### D.4.22.3 NPos

Procedure from library `noether.lib` (see [Section D.4.22 \[noether.lib\]](#), page 1180).

**Usage:** `NPos(i)`; `i` ideal

**Return:** A linear map `phi` such that `phi(i)` is in Noether position

**Example:**

```
LIB "noether.lib";
ring r=0,(X,Y,a,b),dp;
poly f=X^8+a*Y^4-Y;
poly g=Y^8+b*X^4-X;
poly h=diff(f,X)*diff(g,Y)-diff(f,Y)*diff(g,X);
ideal i=f,g,h;
NPos(i);
⇒ The dimension of the ideal is:
⇒ 1
⇒ _[1]=X
⇒ _[2]=Y
⇒ _[3]=a
⇒ _[4]=53X+27Y-75a+b and the time of this computation is: 645 /10 sec.
```

**D.4.22.4 modNPos**

Procedure from library `noether.lib` (see [Section D.4.22 \[noether.lib\]](#), page 1180).

**Usage:** `modNPos(i); i ideal`

**Return:** A linear map  $\phi$  such that  $\phi(i)$  is in Noether position

**Note:** It uses the procedure `modNpos_test` to test Noether position.

**Example:**

```
LIB "noether.lib";
ring r=0,(X,Y,a,b),dp;
poly f=X^8+a*Y^4-Y;
poly g=Y^8+b*X^4-X;
poly h=diff(f,X)*diff(g,Y)-diff(f,Y)*diff(g,X);
ideal i=f,g,h;
modNPos(i);
⇒ // WARNING:
⇒ // The procedure is probabilistic and it computes the initial of the ide\
  al modulo the prime number 2147483647
⇒ // WARNING:
⇒ // The procedure is probabilistic and it computes the initial of the ide\
  al modulo the prime number 2147483647
⇒ // WARNING:
⇒ // The procedure is probabilistic and it computes the initial of the ide\
  al modulo the prime number 2147483647
⇒ The dimension of the ideal is:
⇒ 1
⇒ _[1]=X
⇒ _[2]=Y
⇒ _[3]=a
⇒ _[4]=53X+27Y-75a+b and the time of this computation is: 12 /10 sec.
```

**D.4.22.5 nsatiety**

Procedure from library `noether.lib` (see [Section D.4.22 \[noether.lib\]](#), page 1180).

**Usage:** `nsatiety (i); i ideal,`

- Return:** an integer, the satiety of  $i$ .  
(returns -1 if  $i$  is not homogeneous)
- Assume:**  $i$  is a homogeneous ideal of the basering  $R=K[x(0)..x(n)]$ .
- Theory:** The satiety, or saturation index, of a homogeneous ideal  $i$  is the least integer  $s$  such that, for all  $d \geq s$ , the degree  $d$  part of the ideals  $i$  and  $\text{isat}=\text{sat}(i, \text{maxideal}(1))$  coincide.

**Example:**

```
LIB "noether.lib";
ring r=0,(t,a,b,c,d),dp;
ideal i=b4-a3d, ab3-a3c, bc4-ac3d-bcd3+ad4, c6-bc3d2-c3d3+bd5, ac5-b2c3d-ac2d3+b2d4,
nsatiety(i);
↪ sat(i)=0 and the time of this computation: 2/100sec.
```

**D.4.22.6 modsatiety**

Procedure from library `noether.lib` (see [Section D.4.22 \[noether.lib\]](#), page 1180).

- Usage:** `modsatiety(i);`  $i$  ideal,
- Return:** an integer, the satiety of  $i$ .  
(returns -1 if  $i$  is not homogeneous)
- Assume:**  $i$  is a homogeneous ideal of the basering  $R=K[x(0)..x(n)]$ .
- Theory:** The satiety, or saturation index, of a homogeneous ideal  $i$  is the least integer  $s$  such that, for all  $d \geq s$ , the degree  $d$  part of the ideals  $i$  and  $\text{isat}=\text{sat}(i, \text{maxideal}(1))$  coincide.
- Note:** This is a probabilistic procedure, and it computes the initial of the ideal modulo the prime number 2147483647 (the biggest prime less than  $2^{31}$ ).

**Example:**

```
LIB "noether.lib";
ring r=0,(t,a,b,c,d),dp;
ideal i=b4-a3d, ab3-a3c, bc4-ac3d-bcd3+ad4, c6-bc3d2-c3d3+bd5, ac5-b2c3d-ac2d3+b2d4,
modsatiety(i);
↪ // WARNING: The characteristic of base field must be zero.
↪ // The procedure is probabilistic and it computes the
↪ //initial ideals modulo the prime number 2147483647.
↪ msat(i)=0 and the time of this computation: 2/100sec.
```

**D.4.22.7 regCM**

Procedure from library `noether.lib` (see [Section D.4.22 \[noether.lib\]](#), page 1180).

- Usage:** `regCM(i);`  $i$  ideal
- Return:** the Castelnuovo-Mumford regularity of  $i$ .  
(returns -1 if  $i$  is not homogeneous)
- Assume:**  $i$  is a homogeneous ideal.

**Example:**

```
LIB "noether.lib";
ring r=0,(t,a,b,c,d),dp;
ideal i=b4-a3d, ab3-a3c, bc4-ac3d-bcd3+ad4, c6-bc3d2-c3d3+bd5, ac5-b2c3d-ac2d3+b2d4,
regCM(i);
↪ reg(i)=7 and the time of this computation: 4 sec./100
```

### D.4.22.8 modregCM

Procedure from library `noether.lib` (see [Section D.4.22 \[noether.lib\]](#), page 1180).

**Usage:** `modregCM(i); i ideal`

**Return:** an integer, the Castelnuovo-Mumford regularity of `i`.  
(returns -1 if `i` is not homogeneous)

**Assume:** `i` is a homogeneous ideal and the characteristic of base field is zero..

**Note:** This is a probabilistic procedure, and it computes the initial of the ideal modulo the prime number 2147483647 (the biggest prime less than  $2^{31}$ ).

**Example:**

```
LIB "noether.lib";
ring r=0,(t,a,b,c,d),dp;
ideal i=b4-a3d, ab3-a3c, bc4-ac3d-bcd3+ad4, c6-bc3d2-c3d3+bd5, ac5-b2c3d-ac2d3+b2d4,
modregCM(i);
⇒ // WARNING: The characteristic of base field must be zero.
⇒ // This procedure is probabilistic and it computes the initial
⇒ //ideals modulo the prime number 2147483647
⇒ // ** redefining zz ( list l11;int zz;) noether.lib::modregCM:937
⇒ mreg(i)=7 and the time of this computation: 4sec./100
```

### D.4.23 normal\_lib

**Library:** `normal.lib`

**Purpose:** Normalization of Affine Rings

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**Procedures:** See also: [Section D.4.12 \[integralbasis.lib\]](#), page 1115.

#### D.4.23.1 normal

Procedure from library `normal.lib` (see [Section D.4.23 \[normal.lib\]](#), page 1184).

**Usage:** `normal(id [,choose]); id = radical ideal, choose = list of options.`

Optional parameters in list `choose` (can be entered in any order):

Decomposition:

- "equidim" -> computes first an equidimensional decomposition of the input ideal, and then the normalization of each component (default).
- "prim" -> computes first the minimal associated primes of the input ideal, and then the normalization of each prime. (When the input ideal is not prime and the minimal associated primes are easy to compute, this method is usually faster than "equidim".)
- "noDeco" -> no preliminary decomposition is done. If the ideal is not equidimensional radical, output might be wrong.

- "isPrim" -> assumes that the ideal is prime. If this assumption does not hold, the output might be wrong.

- "noFac" -> factorization is avoided in the computation of the minimal associated primes;

Other:

- "useRing" -> uses the original ring ordering.

If this option is set and if the ring ordering is not global, normal will change to a global ordering only for computing radicals and prime or equidimensional decompositions.

If this option is not set, normal changes to dp ordering and performs all computations with respect to this ordering.

- "withDelta" (or "wd") -> returns also the delta invariants. Not valid for options "locNormal" and "modNormal".

If the optional parameter choose is not given or empty, only "equidim" but no other option is used.

- list("inputJ", ideal inputJ) -> takes as initial test ideal the ideal inputJ. This option is only for use in other procedures. Using this option, the result might not be the normalization.

(Option only valid for global algorithm.)

- list("inputC", ideal inputC) -> takes as initial conductor the ideal inputC. This option is only for use in other procedures. Using this option, the result might not be the normalization.

(Option only valid for global algorithm.)

Options used for computing integral basis (over rings of two variables):

- "normalCheck" -> checks only if the ring is normal. The output is 1 if the ring is normal and 0 if not.

- "var1" -> uses a polynomial in the first variable as universal denominator.

- "var2" -> uses a polynomial in the second variable as universal denominator.

- "GBRadical" -> it computes a Groebner basis of the radical ideal computed for the test ideal.

- "noGBRadical" -> it does not compute a Groebner basis of the radical ideal computed for the test ideal.

- "locNormal" -> uses locNormal procedure to compute the normalization, which implements a local-to-global algorithm (for this option it is required that the ideal I is prime).

- "modular" -> combined with "locNormal" option, it will use modNormal algorithm for the local computations.

- "modNormal" -> uses modNormal procedure to compute the normalization, which implements a modular algorithm (for this option it is required that the ideal I is prime).

- list("nPrimes", int n) -> parameter for modNormal. We consider n primes at a time until the result lifted to the rationals is correct modulo one additional prime. (default: n = 10)

- "noVerification" -> if the modular approach is used (using modNormal option or modular option in locNormal), the result will not be verified.

If the optional parameter choose is not given or empty, only "equidim" but no other option is used.

- "withRing" -> the ring structure of the normalization is computed. The number of variables in the new ring is reduced as much as possible. This is the default option, except when using modNormal or locNormal options.

**Assume:** The ideal must be radical, for non-radical ideals the output may be wrong (`id=radical(id)`; makes `id` radical). However, when using the "prim" option the minimal associated primes of `id` are computed first and hence `normal` computes the normalization of the radical of `id`.

**Note:** "isPrim" should only be used if `id` is known to be prime.

**Return:** If "withRing" option is set, a list, say `nor`, of size 2 (resp. 3 with option "withDelta"). Let `R` denote the basering and `id` the input ideal.

- \* `nor[1]` is a list of `r` rings, where `r` is the number of associated primes `Pi` with option "prim" (resp.  $\geq$  no of equidimensional components `Pi` with option "equidim"). Each ring `Ri := nor[1][i]`,  $i=1..r$ , contains two ideals with given names `norid` and `normap` such that:
  - `Ri/norid` is the normalization of the  $i$ -th component, i.e. the integral closure of `R/Pi` in its field of fractions (as affine ring);
  - `normap` gives the normalization map from `R/id` to `Ri/norid` for each  $i$ .
  - the direct sum of the rings `Ri/norid`,  $i=1..r$ , is the normalization of `R/id` as affine algebra;
- \* `nor[2]` is a list of size `r` with information on the normalization of the  $i$ -th component as module over the basering `R`:
  - `nor[2][i]` is an ideal, say `U`, in `R` such that the integral closure of `basing/Pi` is generated as module over `R` by  $1/c * U$ , with `c` the last element `U[size(U)]` of `U`.
- \* `nor[3]` (if option "withDelta" is set) is a list of an intvec of size `r`, the delta invariants of the `r` components, and an integer, the total delta invariant of `basing/id` (-1 means infinite, and 0 that `R/Pi` resp. `R/id` is normal).

If "withRing" option is not set, a list of an ideal `U` and a polynomial `d` such that `U/d` is the normalization

**Theory:** We use here a general algorithm described in [G.-M.Greuel, S.Laplagne, F.Seelisch: Normalization of Rings (2009)].

The procedure computes the `R`-module structure, the algebra structure and the delta invariant of the normalization of `R/id`:

The normalization of `R/id` is the integral closure of `R/id` in its total ring of fractions. It is a finitely generated `R`-module and `nor[2]` computes `R`-module generators of it. More precisely: If `U:=nor[2][i]` and `c:=U[size(U)]`, then `c` is a non-zero divisor and `U/c` is an `R`-module in the total ring of fractions, the integral closure of `R/Pi`. Since `U[size(U)]/c` is equal to 1, `R/Pi` resp. `R/id` is contained in the integral closure.

The normalization is also an affine algebra over the ground field and `nor[1]` presents it as such. For geometric considerations `nor[1]` is relevant since the variety of the ideal `norid` in `Ri` is the normalization of the variety of the ideal `Pi` in `R`.

The delta invariant of a reduced ring `A` is  $\dim_K(\text{normalization}(A)/A)$ . For `A=K[x1,...,xn]/id` we call this number also the delta invariant of `id`. `nor[3]` returns the delta invariants of the components `Pi` and of `id`.

**Note:** To use the  $i$ -th ring type e.g.: `def R=nor[1][i]; setring R;`

Increasing/decreasing `printlevel` displays more/less comments (default: `printlevel=0`).

Implementation works also for local rings.

Not implemented for quotient rings.

If the input ideal `id` is weighted homogeneous a weighted ordering may be used together with the `useRing`-option (`qhweight(id)`; computes weights).

**Example:**



```

LIB "normal.lib";
printlevel = printlevel+1;
ring s = 0,(x,y),dp;
ideal i = (x2-y3)*(x2+y2)*x;
list nor = normal(i, "withDelta", "prim");
⇒ // Computing the minimal associated primes...
⇒ [1]:
⇒   _[1]=-y3+x2
⇒ [2]:
⇒   _[1]=x2+y2
⇒ [3]:
⇒   _[1]=x
⇒
⇒ // number of components is 3
⇒
⇒ // start computation of component 1
⇒   -----
⇒ Computing the jacobian ideal...
⇒
⇒ The universal denominator is  x
⇒ The original singular locus is
⇒ _[1]=x
⇒ _[2]=y2
⇒
⇒ The radical of the original singular locus is
⇒ J[1]=x
⇒ J[2]=y
⇒ The non zero divisor is  y
⇒
⇒ Preliminar step begins.
⇒ Computing the quotient (DJ : J)...
⇒ In this step, we have the ring 1/c * U, with c = y
⇒ and U =
⇒ U[1]=y
⇒ U[2]=x
⇒
⇒ Step 1  begins.
⇒ Computing the test ideal...
⇒ Computing the quotient (c*D*cJ : cJ)...
⇒ The ring in the previous step was already normal.
⇒
⇒ // start computation of component 2
⇒   -----
⇒ Computing the jacobian ideal...
⇒
⇒ The universal denominator is  y
⇒ The original singular locus is
⇒ _[1]=y
⇒ _[2]=x
⇒
⇒ The radical of the original singular locus is
⇒ J[1]=x
⇒ J[2]=y

```

```

⇒ The non zero divisor is y
⇒
⇒ Preliminar step begins.
⇒ Computing the quotient (DJ : J)...
⇒ In this step, we have the ring  $1/c * U$ , with  $c = y$ 
⇒ and  $U =$ 
⇒  $U[1]=y$ 
⇒  $U[2]=x$ 
⇒
⇒ Step 1 begins.
⇒ Computing the test ideal...
⇒ Computing the quotient  $(c*D*cJ : cJ)$ ...
⇒ The ring in the previous step was already normal.
⇒
⇒ // start computation of component 3
⇒ -----
⇒ Computing the jacobian ideal...
⇒ // Sum of delta for all components: 2
⇒ // Computing the sum of the intersection multiplicities of the components\
...
⇒ // Intersection multiplicity is : 11
⇒
⇒ // 'normal' created a list, say nor, of three elements.
⇒ // To see the list type
⇒     nor;
⇒
⇒ // * nor[1] is a list of 3 ring(s).
⇒ // To access the i-th ring nor[1][i], give it a name, say Ri, and type
⇒     def R1 = nor[1][1]; setring R1; norid; normap;
⇒ // For the other rings type first (if R is the name of your base ring)
⇒     setring R;
⇒ // and then continue as for R1.
⇒ // Ri/norid is the affine algebra of the normalization of  $R/P_i$  where
⇒ //  $P_i$  is the i-th component of a decomposition of the input ideal id
⇒ // and normap the normalization map from R to Ri/norid.
⇒
⇒ // * nor[2] is a list of 3 ideal(s). Let ci be the last generator
⇒ // of the ideal nor[2][i]. Then the integral closure of  $R/P_i$  is
⇒ // generated as R-submodule of the total ring of fractions by
⇒ //  $1/c_i * nor[2][i]$ .
⇒
⇒ // * nor[3] is a list of an intvec of size 3 the delta invariants
⇒ // of the components, and an integer, the total delta invariant
⇒ // of R/id (-1 means infinite, and 0 that  $R/P_i$  resp. R/id is normal).
nor;
⇒ [1]:
⇒     [1]:
⇒         // coefficients: QQ
⇒ // number of vars : 3
⇒ //         block 1 : ordering dp
⇒ //         : names T(1)
⇒ //         block 2 : ordering dp
⇒ //         : names x y

```

```

⇒ //      block   3 : ordering C
⇒      [2]:
⇒      // coefficients: QQ
⇒ // number of vars : 3
⇒ //      block   1 : ordering dp
⇒ //      : names   T(1)
⇒ //      block   2 : ordering dp
⇒ //      : names   x y
⇒ //      block   3 : ordering C
⇒      [3]:
⇒      // coefficients: QQ
⇒ // number of vars : 2
⇒ //      block   1 : ordering dp
⇒ //      : names   x y
⇒ //      block   2 : ordering C
⇒ [2]:
⇒      [1]:
⇒      _[1]=x
⇒      _[2]=y
⇒      [2]:
⇒      _[1]=x
⇒      _[2]=y
⇒      [3]:
⇒      _[1]=1
⇒ [3]:
⇒      [1]:
⇒      1,1,0
⇒      [2]:
⇒      13
// 2 branches have delta = 1, and 1 branch has delta = 0
// the total delta invariant is 13
def R2 = nor[1][2]; setring R2;
norid; normap;
⇒ norid[1]=-T(1)*y+x
⇒ norid[2]=T(1)*x+y
⇒ norid[3]=T(1)^2+1
⇒ norid[4]=x^2+y^2
⇒ normap[1]=x
⇒ normap[2]=y
printlevel = printlevel-1;
ring r = 2,(x,y,z),dp;
ideal i = z3-xy4;
list nor = normal(i, "withDelta", "prim"); nor;
⇒
⇒ // 'normal' created a list, say nor, of three elements.
⇒ // To see the list type
⇒      nor;
⇒
⇒ // * nor[1] is a list of 1 ring(s).
⇒ // To access the i-th ring nor[1][i], give it a name, say Ri, and type
⇒      def R1 = nor[1][1]; setring R1; norid; normap;
⇒ // For the other rings type first (if R is the name of your base ring)
⇒      setring R;

```

```

⇒ // and then continue as for R1.
⇒ // Ri/norid is the affine algebra of the normalization of R/P_i where
⇒ // P_i is the i-th component of a decomposition of the input ideal id
⇒ // and normap the normalization map from R to Ri/norid.
⇒
⇒ // * nor[2] is a list of 1 ideal(s). Let ci be the last generator
⇒ // of the ideal nor[2][i]. Then the integral closure of R/P_i is
⇒ // generated as R-submodule of the total ring of fractions by
⇒ // 1/ci * nor[2][i].
⇒
⇒ // * nor[3] is a list of an intvec of size 1 the delta invariants
⇒ // of the components, and an integer, the total delta invariant
⇒ // of R/id (-1 means infinite, and 0 that R/P_i resp. R/id is normal).
⇒ [1]:
⇒   [1]:
⇒     // coefficients: ZZ/2
⇒ // number of vars : 5
⇒ //      block  1 : ordering dp
⇒ //              : names    T(1) T(2)
⇒ //      block  2 : ordering dp
⇒ //              : names    x y z
⇒ //      block  3 : ordering C
⇒ [2]:
⇒   [1]:
⇒     _[1]=xy2z
⇒     _[2]=xy3
⇒     _[3]=z2
⇒ [3]:
⇒   [1]:
⇒     -1
⇒   [2]:
⇒     -1
⇒ // the delta invariant is infinite
⇒ // xy2z/z2 and xy3/z2 generate the integral closure of r/i as r/i-module
⇒ // in its quotient field Quot(r/i)
⇒ // the normalization as affine algebra over the ground field:
def R = nor[1][1]; setring R;
norid; normap;
⇒ norid[1]=T(1)*y+T(2)*z
⇒ norid[2]=T(2)*y+z
⇒ norid[3]=T(1)*z+x*y^2
⇒ norid[4]=T(1)^2+x*z
⇒ norid[5]=T(1)*T(2)+x*y
⇒ norid[6]=T(2)^2+T(1)
⇒ norid[7]=x*y^4+z^3
⇒ normap[1]=x
⇒ normap[2]=y
⇒ normap[3]=z

```

See also: [Section D.4.23.5 \[normalC\]](#), page 1195; [Section D.4.23.4 \[normalP\]](#), page 1192.

### D.4.23.2 locNormal

Procedure from library `normal.lib` (see [Section D.4.23 \[normal\\_lib\]](#), page 1184).

- Usage:** `locNormal(I [,options]);`  $I$  = prime ideal, options = list of options.  
Optional parameters in list options (can be entered in any order):  
modular: use a modular approach for the local computations. The number of primes is increased one at a time, starting with 2 primes, until the result stabilizes.  
noVerification: if the modular approach is used, the result will not be verified.
- Assume:**  $I$  is a prime ideal (the algorithm will also work for radical ideals as long as the normal command does not detect that the ideal under consideration is not prime).
- Return:** a list of an ideal  $U$  and a universal denominator  $d$  such that  $U/d$  is the normalization.
- Remarks:** We use the local-to-global algorithm given in [1] to compute the normalization of  $A = R/I$ , where  $R$  is the basering.  
The idea is to stratify the singular locus of  $A$ , apply the normalization algorithm given in [2] locally at each stratum, and put the local results together.  
If the option modular is given, the result is returned as a probabilistic result or verified, depending on whether the option noVerification is used or not.  
The normalization of  $A$  is represented as an  $R$ -module by returning a list of  $U$  and  $d$ , where  $U$  is an ideal of  $A$  and  $d$  is an element of  $A$  such that  $U/d$  is the normalization of  $A$ . In fact,  $U$  and  $d$  are returned as an ideal and a polynomial of the base ring  $R$ .

**References:**

- [1] Janko Boehm, Wolfram Decker, Santiago Laplagne, Gerhard Pfister, Stefan Steidel, Andreas Steenpass: Parallel algorithms for normalization, <http://arxiv.org/abs/1110.4299>, 2011.  
[2] Gert-Martin Greuel, Santiago Laplagne, Frank Seelisch: Normalization of Rings, Journal of Symbolic Computation 9 (2010), p. 887-901

**Example:**

```
LIB "normal.lib";
ring R = 0,(x,y,z),dp;
int k = 4;
poly f = (x^(k+1)+y^(k+1)+z^(k+1))^2-4*(x^(k+1)*y^(k+1)+y^(k+1)*z^(k+1)+z^(k+1)*x^(k+1));
f = subst(f,z,3x-2y+1);
ring S = 0,(x,y),dp;
poly f = imap(R,f);
ideal i = f;
list L = locNormal(i);
```

See also: [Section D.4.23 \[normal.lib\]](#), page 1184.

**D.4.23.3 modNormal**

Procedure from library `normal.lib` (see [Section D.4.23 \[normal.lib\]](#), page 1184).

- Usage:** `modNormal(I, n [,options]);`  $I$  = prime ideal,  $n$  = positive integer, options = list of options.  
Optional parameters in list options (can be entered in any order):  
noVerification: do not verify the result.  
printTimings: print timings.  
int ncores: number of cores to be used (default = 1).
- Assume:**  $I$  is a prime ideal (the algorithm will also work for radical ideals as long as the normal command does not detect that the ideal under consideration is not prime).
- Return:** a list of an ideal  $U$  and a universal denominator  $d$  such that  $U/d$  is the normalization.

**Remarks:** We use the algorithm given in [1] to compute the normalization of  $A = R/I$  where  $R$  is the basering. We apply the algorithm for  $n$  primes at a time until the result lifted to the rationals is correct modulo one additional prime. Depending on whether the option `noVerification` is used or not, the result is returned as a probabilistic result or verified over the rationals.

The normalization of  $A$  is represented as an  $R$ -module by returning a list of  $U$  and  $d$ , where  $U$  is an ideal of  $A$  and  $d$  is an element of  $A$  such that  $U/d$  is the normalization of  $A$ . In fact,  $U$  and  $d$  are returned as an ideal and a polynomial of the base ring  $R$ .

**Example:**

```
LIB "normal.lib";
ring R = 0,(x,y,z),dp;
int k = 4;
poly f = (x^(k+1)+y^(k+1)+z^(k+1))^2-4*(x^(k+1)*y^(k+1)+y^(k+1)*z^(k+1)+z^(k+1)*x^(k+1));
f = subst(f,z,3x-2y+1);
ring S = 0,(x,y),dp;
poly f = imap(R,f);
ideal i = f;
list L = modNormal(i,1,"noVerification");
```

See also: [Section D.4.12 \[integralbasis.lib\], page 1115](#).

#### D.4.23.4 normalP

Procedure from library `normal.lib` (see [Section D.4.23 \[normal.lib\], page 1184](#)).

**Usage:** `normalP(id [,choose]);` `id` = radical ideal, `choose` = optional list of strings.  
Optional parameters in list `choose` (can be entered in any order):  
"withRing", "isPrim", "noFac", "noRed", where  
- "noFac" -> factorization is avoided during the computation of the minimal associated primes.  
- "isPrim" -> assumes that the ideal is prime. If the assumption does not hold, output might be wrong.  
- "withRing" -> the ring structure of the normalization is computed. The number of variables in the new ring is reduced as much as possible.  
- "noRed" -> when computing the ring structure, no reduction on the number of variables is done, it creates one new variable for every new module generator of the integral closure in the quotient field.

**Assume:** The characteristic of the ground field must be positive. If the option "isPrim" is not set, the minimal associated primes of `id` are computed first and hence `normalP` computes the normalization of the radical of `id`. If option "isPrim" is set, the ideal must be a prime ideal otherwise the result may be wrong.

**Return:** a list, say 'nor' of size 2 (resp. 3 if "withRing" is set).  
\*\* If option "withRing" is not set:  
Only the module structure is computed:  
\* `nor[1]` is a list of ideals  $I_i$ ,  $i=1..r$ , in the basering  $R$  where  $r$  is the number of minimal associated prime ideals  $P_i$  of the input ideal `id`, describing the module structure:  
If  $I_i$  is given by polynomials  $g_{i,1}, \dots, g_{i,k}$  in  $R$ , then  $c := g_{i,k}$  is non-zero in the ring  $R/P_i$  and  $g_{i,1}/c, \dots, g_{i,k}/c$  generate the integral closure of  $R/P_i$  as  $R$ -module in the quotient field of  $R/P_i$ .  
\* `nor[2]` shows the delta invariants: it is a list of an intvec of size  $r$ , the delta invariants

of the  $r$  components, and an integer, the total delta invariant of  $R/\text{id}$  (-1 means infinite, and 0 that  $R/P_i$  resp.  $R/\text{id}$  is normal).

**\*\*** If option "withRing" is set:

The ring structure is also computed, and in this case:

\*  $\text{nor}[1]$  is a list of  $r$  rings.

Each ring  $R_i = \text{nor}[1][i]$ ,  $i=1..r$ , contains two ideals with given names **norid** and **normap** such that

-  $R_i/\text{norid}$  is the normalization of  $R/P_i$ , i.e. isomorphic as  $K$ -algebra ( $K$  the ground field) to the integral closure of  $R/P_i$  in the field of fractions of  $R/P_i$ ;

- the direct sum of the rings  $R_i/\text{norid}$  is the normalization of  $R/\text{id}$ ;

- **normap** gives the normalization map from  $R$  to  $R_i/\text{norid}$ .

\*  $\text{nor}[2]$  gives the module generators of the normalization of  $R/P_i$ , it is the same as  $\text{nor}[1]$  if "withRing" is not set.

\*  $\text{nor}[3]$  shows the delta invariants, it is the same as  $\text{nor}[2]$  if "withRing" is not set.

**Theory:** `normalP` uses the Leonard-Pellikaan-Singh-Swanson algorithm (using the Frobenius) cf. [A. K. Singh, I. Swanson: An algorithm for computing the integral closure, arXiv:0901.0871].

The delta invariant of a reduced ring  $A$  is  $\dim_K(\text{normalization}(A)/A)$ . For  $A=K[x_1, \dots, x_n]/\text{id}$  we call this number also the delta invariant of  $\text{id}$ . The procedure returns the delta invariants of the components  $P_i$  and of  $\text{id}$ .

**Note:** To use the  $i$ -th ring type: `def R=nor[1][i]; setring R;`

Increasing/decreasing `printlevel` displays more/less comments (default: `printlevel = 0`).

Not implemented for local or mixed orderings or quotient rings. For local or mixed orderings use `proc 'normal'`.

If the input ideal  $\text{id}$  is weighted homogeneous a weighted ordering may be used (`qh-weight(id)`; computes weights).

Works only in characteristic  $p > 0$ ; use `proc normal` in char 0.

**Example:**

```
LIB "normal.lib";
ring r = 11,(x,y,z),wp(2,1,2);
ideal i = x*(z3 - xy4 + x2);
list nor= normalP(i); nor;
⇒
⇒ // 'normalP' computed a list, say nor, of two lists:
⇒ // To see the result, type
⇒     nor;
⇒
⇒ // * nor[1] is a list of 2 ideal(s), where each ideal nor[1][i] consists
⇒ // of elements g1..gk of the basering R such that gj/gk generate the inte\
    gral
⇒ // closure of R/P_i (where P_i is a min. associated prime of the input id\
    eal)
⇒ // as R-module in the quotient field of R/P_i;
⇒
⇒ // * nor[2] shows the delta-invariant of each component and of the input \
    ideal
⇒ // (-1 means infinite, and 0 that R/P_i is normal).
⇒ [1]:
⇒     [1]:
```

```

⇒      _[1]=1
⇒      [2]:
⇒      _[1]=1
⇒      [2]:
⇒      [1]:
⇒      0,0
⇒      [2]:
⇒      -1
//the result says that both components of i are normal, but i itself
//has infinite delta
ring s = 2,(x,y),dp;
ideal i = y*((x-y^2)^2 - x^3);
list nor = normalP(i,"withRing"); nor;
⇒
⇒ // 'normalP' created a list, say nor, of three lists:
⇒ // To see the result, type
⇒      nor;
⇒
⇒ // * nor[1] is a list of 2 ring(s):
⇒ // To access the i-th ring nor[1][i] give it a name, say Ri, and type e.g\
.
⇒      def R1 = nor[1][1]; setring R1; norid; normap;
⇒ // for the other rings type first setring R; (if R is the name of your
⇒ // original basering) and then continue as for R1;
⇒ // Ri/norid is the affine algebra of the normalization of the i-th
⇒ // component R/P_i (where P_i is a min. associated prime of the input ide\
al)
⇒ // and normap the normalization map from R to Ri/norid;
⇒
⇒ // * nor[2] is a list of 2 ideal(s), each ideal nor[2][i] consists of
⇒ // elements g1..gk of r such that the gj/gk generate the integral
⇒ // closure of R/P_i as R-module in the quotient field of R/P_i.
⇒
⇒ // * nor[3] shows the delta-invariant of each component and of the input
⇒ // ideal (-1 means infinite, and 0 that r/P_i is normal).
⇒ [1]:
⇒      [1]:
⇒      // coefficients: ZZ/2
⇒ // number of vars : 1
⇒ //      block 1 : ordering dp
⇒ //      : names x
⇒ //      block 2 : ordering C
⇒      [2]:
⇒      // coefficients: ZZ/2
⇒ // number of vars : 2
⇒ //      block 1 : ordering dp
⇒ //      : names T(1)
⇒ //      block 2 : ordering dp
⇒ //      : names y
⇒ //      block 3 : ordering C
⇒ [2]:
⇒      [1]:
⇒      _[1]=1

```



```

⇒      [2]:
⇒      _[1]=y3+xy
⇒      _[2]=x2
⇒ [3]:
⇒      [1]:
⇒      0,3
⇒      [2]:
⇒      6
def R2 = nor[1][2]; setring R2;
norid; normap;
⇒ norid[1]=T(1)^4+T(1)^3*y+T(1)^2+y^2
⇒ norid[2]=T(1)^3+T(1)+y
⇒ norid[3]=T(1)^6+T(1)^5*y+T(1)^4*y^2+T(1)^3*y^3+T(1)^2+T(1)*y+y^4
⇒ normap[1]=T(1)^2+T(1)*y+1
⇒ normap[2]=y

```

See also: [Section D.4.23.1 \[normal\], page 1184](#); [Section D.4.23.5 \[normalC\], page 1195](#).

### D.4.23.5 normalC

Procedure from library `normal.lib` (see [Section D.4.23 \[normal\\_lib\], page 1184](#)).

**Usage:**     `normalC(id [,choose])`; `id` = radical ideal, `choose` = optional list of string.  
Optional parameters in list `choose` (can be entered in any order):  
Decomposition:  
- "equidim" -> computes first an equidimensional decomposition, and then the normalization of each component (default).  
- "prim" -> computes first the minimal associated primes, and then the normalization of each prime.  
- "noDeco" -> no preliminary decomposition is done. If the ideal is not equidimensional radical, output might be wrong.  
- "isPrim" -> assumes that the ideal is prime. If the assumption does not hold, output might be wrong.  
- "noFac" -> factorization is avoided in the computation of the minimal associated primes;  
Other:  
- "withGens" -> the minimal associated primes  $P_i$  of `id` are computed and for each  $P_i$ , algebra generators of the integral closure of  $\text{basering}/P_i$  are computed as elements of its quotient field;  
If `choose` is not given or empty, the default options are used.

**Assume:**     The ideal must be radical, for non-radical ideals the output may be wrong (`id=radical(id)`; makes `id` radical). However, if option "prim" is set the minimal associated primes are computed first and hence `normalC` computes the normalization of the radical of `id`. "isPrim" should only be used if `id` is known to be irreducible.

**Return:**     a list, say `nor`, of size 2 (resp. 3 if option "withGens" is set).  
\* `nor[1]` is always a of  $r$  rings, where  $r$  is the number of associated primes with option "prim" (resp.  $\geq$  no of equidimensional components with option "equidim").  
Each ring  $R_i = \text{nor}[1][i]$ ,  $i = 1..r$ , contains two ideals with given names `norid` and `normap` such that  
-  $R_i/\text{norid}$  is the normalization of the  $i$ -th component, i.e. the integral closure in its field of fractions as affine ring, i.e.  $R_i$  is given in the form  $K[X(1..p), T(1..q)]$ , where  $K$

is the ground field; - normap gives the normalization map from basering/id to  $R_i/\text{norid}$  for each  $i$  (the  $j$ -th element of normap is mapped to the  $j$ -th variable of  $R$ ).

- the direct sum of the rings  $R_i/\text{norid}$  is the normalization of basering/id;

\*\* If option "withGens" is not set:

\* nor[2] shows the delta invariants: nor[2] is a list of an intvec of size  $r$ , the delta invariants of the  $r$  components, and an integer, the delta invariant of basering/id. (-1 means infinite, 0 that basering/ $P_i$  resp. basering/input is normal, -2 means that delta resp. delta of one of the components is not computed (which may happen if "equidim" is given).

\*\* If option "withGens" is set:

\* nor[2] is a list of ideals  $I_i = \text{nor}[2][i]$ ,  $i = 1..r$ , in the basering, generating the integral closure of basering/ $P_i$  in its quotient field as  $K$ -algebra ( $K$  the ground field):

If  $I_i$  is given by polynomials  $g_1, \dots, g_k$ , then  $c = g_k$  is a non-zero divisor and the  $j$ -th variables of the ring  $R_i$  satisfies  $\text{var}(j) = g_j/c$ ,  $j = 1..k-1$ , as element in the quotient field of basering/ $P_i$ . The  $g_j/g_{k+1}$  are  $K$ -algebra generators of the integral closure of basering/ $P_i$ .

\* nor[3] shows the delta invariant as above.

**Theory:** We use the Grauert-Remmert-de Jong algorithm [c.f. G.-M. Greuel, G. Pfister: A SINGULAR Introduction to Commutative Algebra, 2nd Edition. Springer Verlag (2007)].

The procedure computes the algebra structure and the delta invariant of the normalization of  $R/\text{id}$ :

The normalization is an affine algebra over the ground field  $K$  and nor[1] presents it as such:  $R_i = K[X(1..p), T(1..q)]$  and  $R_i/\text{norid}$  is the integral closure of  $R/P_i$ ; if option "withGens" is set the  $X(j)$  and  $T(j)$  are expressed as quotients in the total ring of fractions. Note that the  $X(j)$  and  $T(j)$  generate the integral closure as  $K$ -algebra, but not necessarily as  $R$ -module (since relations of the form  $X(1) = T(1) \cdot T(2)$  may have been eliminated). Geometrically the algebra structure is relevant since the variety of the ideal norid in  $R_i$  is the normalization of the variety of the ideal  $P_i$  in  $R$ .

The delta invariant of a reduced ring  $A$  is  $\dim_K(\text{normalization}(A)/A)$ . For  $A = K[x_1, \dots, x_n]/\text{id}$  we call this number also the delta invariant of id. nor[3] returns the delta invariants of the components  $P_i$  and of id.

**Note:** To use the  $i$ -th ring type: `def R=nor[1][i]; setring R;`

Increasing/decreasing printlevel displays more/less comments (default: printlevel=0).

Not implemented for local or mixed orderings or quotient rings. For local or mixed orderings use proc 'normal'.

If the input ideal id is weighted homogeneous a weighted ordering may be used (qh-weight(id); computes weights).

**Example:**

```
LIB "normal.lib";
printlevel = printlevel+1;
ring s = 0, (x,y), dp;
ideal i = (x2-y3)*(x2+y2)*x;
list nor = normalC(i);
⇒ // We use method 'prim'
⇒
⇒ // number of irreducible components: 3
⇒
⇒ // computing the normalization of component 1
```

```

⇒ -----
⇒ // delta of component 1
⇒ 1
⇒
⇒ // computing the normalization of component 2
⇒ -----
⇒ // delta of component 2
⇒ 1
⇒
⇒ // computing the normalization of component 3
⇒ -----
⇒ // delta of component 3
⇒ 0
⇒ // Sum of delta for all components
⇒ 2
⇒ // Compute intersection multiplicities of the components
⇒
⇒ // 'normalC' created a list, say nor, of two lists:
⇒ // To see the result, type
⇒     nor;
⇒
⇒ // * nor[1] is a list of 3 ring(s).
⇒ // To access the i-th ring nor[1][i] give it a name, say Ri, and type e.g\
⇒
⇒     def R1 = nor[1][1]; setring R1; norid; normap;
⇒ // and similar for the other rings nor[1][i];
⇒ // Ri/norid is the affine algebra of the normalization of r/P_i (where P\
⇒ _i
⇒ // is an associated prime or an equidimensional part of the input ideal i\
⇒ d)
⇒ // and normap the normalization map from the basering to Ri/norid;
⇒
⇒ // * nor[2] shows the delta-invariant of each component and of id
⇒ // (-1 means infinite, 0 that r/P_i resp. r/id is normal, and -2 that del\
⇒ ta
⇒ // of a component was not computed).
nor;
⇒ [1]:
⇒     [1]:
⇒         // coefficients: QQ
⇒ // number of vars : 1
⇒ //         block 1 : ordering dp
⇒ //             : names   T(1)
⇒ //         block 2 : ordering C
⇒     [2]:
⇒         // coefficients: QQ
⇒ // number of vars : 2
⇒ //         block 1 : ordering dp
⇒ //             : names   T(1) T(2)
⇒ //         block 2 : ordering C
⇒     [3]:
⇒         // coefficients: QQ
⇒ // number of vars : 1

```

```

⇒ //          block   1 : ordering dp
⇒ //                      : names   T(1)
⇒ //          block   2 : ordering C
⇒ [2]:
⇒   [1]:
⇒       0,1,1
⇒   [2]:
⇒       13
// 2 branches have delta = 1, and 1 branch has delta = 0
// the total delta invariant is 13
def R2 = nor[1][2]; setring R2;
norid; normap;
⇒ norid[1]=T(2)^2+1
⇒ normap[1]=-T(1)*T(2)
⇒ normap[2]=T(1)
ring r = 2,(x,y,z),dp;
ideal i = z3-xy4;
nor = normalC(i); nor;
⇒ // We use method 'equidim'
⇒
⇒ // number of equidimensional components: 1
⇒
⇒ // number of components after application of facstd: 1
⇒
⇒ // computing the normalization of component 1
⇒ -----
⇒ // delta of component 1
⇒ -1
⇒ // Sum of delta for all components
⇒ -1
⇒
⇒ // 'normalC' created a list, say nor, of two lists:
⇒ // To see the result, type
⇒     nor;
⇒
⇒ // * nor[1] is a list of 1 ring(s).
⇒ // To access the i-th ring nor[1][i] give it a name, say Ri, and type e.g\
.
⇒     def R1 = nor[1][1]; setring R1; norid; normap;
⇒ // and similar for the other rings nor[1][i];
⇒ // Ri/norid is the affine algebra of the normalization of r/P_i (where P\
_i
⇒ // is an associated prime or an equidimensional part of the input ideal i\
d)
⇒ // and normap the normalization map from the basering to Ri/norid;
⇒
⇒ // * nor[2] shows the delta-invariant of each component and of id
⇒ // (-1 means infinite, 0 that r/P_i resp. r/id is normal, and -2 that del\
ta
⇒ // of a component was not computed).
⇒ [1]:
⇒   [1]:
⇒       // coefficients: ZZ/2

```

```

⇒ // number of vars : 3
⇒ //      block   1 : ordering dp
⇒ //      : names   T(1) T(2) T(3)
⇒ //      block   2 : ordering C
⇒ [2]:
⇒   [1]:
⇒   -1
⇒   [2]:
⇒   -1
// the delta invariant is infinite
// xy2z/z2 and xy3/z2 generate the integral closure of r/i as r/i-module
// in its quotient field Quot(r/i)
// the normalization as affine algebra over the ground field:
def R = nor[1][1]; setring R;
norid; normap;
⇒ norid[1]=T(3)^3+T(1)*T(2)
⇒ normap[1]=T(1)
⇒ normap[2]=T(2)
⇒ normap[3]=T(2)*T(3)
setring r;
nor = normalC(i, "withGens", "prim");    // a different algorithm
⇒
⇒ // We use method 'withGens'
⇒ // Computing minimal associated primes...
⇒ // number of irreducible components is 1
⇒
⇒ pause>
⇒ // Computing normalization of component 1
⇒ -----
⇒
⇒ // 'normalC' created a list, say nor, of three lists:
⇒ // To see the list type
⇒     nor;
⇒
⇒ // * nor[1] is a list of 1 ring(s)
⇒ // To access the i-th ring nor[1][i] give it a name, say Ri, and type e.g\
.
⇒     def R1 = nor[1][1]; setring R1; norid; normap;
⇒ // For the other rings type first (if R is the name of your original base\
ring)
⇒     setring R;
⇒ // and then continue as for R1.
⇒ // Ri/norid is the affine algebra of the normalization of the i-th
⇒ // component R/P_i (where P_i is an associated prime of the input ideal i\
d)
⇒ // and normap the normalization map from R to Ri/norid.
⇒
⇒ // * nor[2] is a list of 1 ideal(s), each ideal nor[2][i] consists of
⇒ // elements g1..gk of R such that the gj/gk generate the integral
⇒ // closure of R/P_i as sub-algebra in the quotient field of R/P_i, with
⇒ // gj/gk being mapped by normap to the j-th variable of Ri;
⇒
⇒ // * nor[3] shows the delta-invariant of each component and of id

```

```

⇒ // (-1 means infinite, and 0 that R/P_i resp. R/id is normal).
nor;
⇒ [1]:
⇒   [1]:
⇒   // coefficients: ZZ/2
⇒ // number of vars : 6
⇒ //      block  1 : ordering dp
⇒ //      : names  X(1) X(2) X(3) X(4) T(2) T(3)
⇒ //      block  2 : ordering C
⇒ [2]:
⇒   [1]:
⇒   _[1]=x2y3
⇒   _[2]=z3
⇒   _[3]=xy3z
⇒   _[4]=xy2z2
⇒   _[5]=xyz2
⇒   _[6]=xy2z
⇒   _[7]=xy3
⇒ [3]:
⇒   [1]:
⇒   -1
⇒   [2]:
⇒   -1

```

See also: [Section D.4.23.1 \[normal\]](#), page 1184; [Section D.4.23.4 \[normalP\]](#), page 1192.

### D.4.23.6 HomJJ

Procedure from library `normal.lib` (see [Section D.4.23 \[normal.lib\]](#), page 1184).

**Usage:** HomJJ (Li); Li = list: ideal SBid, ideal id, ideal J, poly p

**Assume:** R = P/id, P = basering, a polynomial ring, id an ideal of P,  
 SBid = standard basis of id,  
 J = ideal of P containing the polynomial p,  
 p = nonzero divisor of R

**Compute:** Endomorphism ring  $\text{End}_R(J) = \text{Hom}_R(J, J)$  with its ring structure as affine ring, together with the map  $R \rightarrow \text{Hom}_R(J, J)$  of affine rings, where R is the quotient ring of P modulo the standard basis SBid.

**Return:** a list l of three objects

```

l[1] : a polynomial ring, containing two ideals, 'endid' and 'endphi'
      such that l[1]/endid = Hom_R(J,J) and
      endphi describes the canonical map R -> Hom_R(J,J)
l[2] : an integer which is 1 if phi is an isomorphism, 0 if not
l[3] : an integer, = dim_K(Hom_R(J,J)/R) (the contribution to delta)
      if the dimension is finite, -1 otherwise

```

**Note:** printlevel >=1: display comments (default: printlevel=0)

**Example:**

```

LIB "normal.lib";
ring r = 0, (x,y), wp(2,3);
ideal id = y^2-x^3;
ideal J = x,y;

```

```

poly p    = x;
list Li   = std(id),id,J,p;
list L    = HomJJ(Li);
def end = L[1];    // defines ring L[1], containing ideals endid, endphi
setring end;      // makes end the basering
end;
⇒ // coefficients: QQ
⇒ // number of vars : 1
⇒ //          block 1 : ordering dp
⇒ //          : names  T(1)
⇒ //          block 2 : ordering C
endid;           // end/endid is isomorphic to End(r/id) as ring
⇒ endid[1]=0
map psi = r,endphi; // defines the canonical map r/id -> End(r/id)
psi;
⇒ psi[1]=T(1)^2
⇒ psi[2]=T(1)^3
L[3];            // contribution to delta
⇒ 1

```

### D.4.23.7 genus

Procedure from library `normal.lib` (see [Section D.4.23 \[normal.lib\], page 1184](#)).

**Return:** an integer, the geometric genus  $p_g = p_a - \delta$  of the projective curve defined by  $i$ , where  $p_a$  is the arithmetic genus.

**Note:** genus always treats projective curves and takes projective closure if input is affine 1-dim variety.  $\delta$  is the sum of all local  $\delta$ -invariants of the singularities, i.e.  $\dim(R'/R)$ ,  $R'$  the normalization of the local ring  $R$  of the singularity.  
`genus(I,"nor")` uses the normalization to compute  $\delta$ . Usually `genus(I,"nor")` is slower than `genus(I)` but sometimes not.  
`genus(I,"pri")` starts with a primary decomposition.

**Example:**

```

LIB "normal.lib";
ring r=0,(x,y),dp;
ideal i=y^9 - x^2*(x - 1)^9;
genus(i);
⇒ 0
ring r7=7,(x,y),dp;
ideal i=y^9 - x^2*(x - 1)^9;
genus(i);
⇒ 0

```

### D.4.23.8 primeClosure

Procedure from library `normal.lib` (see [Section D.4.23 \[normal.lib\], page 1184](#)).

**Usage:** `primeClosure(L [,c]);`  $L$  a list of a ring containing a prime ideal  $\ker$ ,  $c$  an optional integer

**Return:** a list  $L$  (of size  $n+1$ ) consisting of rings  $L[1], \dots, L[n]$  such that -  $L[1]$  is a copy of (not a reference to!) the input ring  $L[1]$  - all rings  $L[i]$  contain ideals  $\ker$ ,  $L[2], \dots, L[n]$  contain ideals  $\phi$  such that  
 $L[1]/\ker \rightarrow \dots \rightarrow L[n]/\ker$

are injections given by the corresponding ideals  $\phi_i$ , and  $L[n]/\ker$  is the integral closure of  $L[1]/\ker$  in its quotient field. - all rings  $L[i]$  contain a polynomial  $\text{nzd}$  such that elements of  $L[i]/\ker$  are quotients of elements of  $L[i-1]/\ker$  with denominator  $\text{nzd}$  via the injection  $\phi_i$ .

$L[n+1]$  is the delta invariant

**Note:**

- $L$  is constructed by recursive calls of `primeClosure` itself. -  $c$  determines the choice of  $\text{nzd}$ :
- $c$  not given or equal to 0: first generator of the ideal  $SL$ , the singular locus of  $\text{Spec}(L[i]/\ker)$
- $c > 0$ : the generator of  $SL$  with least number of monomials.

**Example:**

```
LIB "normal.lib";
ring R=0,(x,y),dp;
ideal I=x4,y4;
def K=ReesAlgebra(I)[1];          // K contains ker such that K/ker=R[It]
list L=primeClosure(K);
def R(1)=L[1];                    // L[4] contains ker, L[4]/ker is the
def R(4)=L[4];                    // integral closure of L[1]/ker
setring R(1);
R(1);
⇒ // coefficients: QQ
⇒ // number of vars : 4
⇒ //          block 1 : ordering dp
⇒ //          : names   x y U(1) U(2)
⇒ //          block 2 : ordering C
ker;
⇒ ker[1]=y^4*U(1)-x^4*U(2)
setring R(4);
R(4);
⇒ // coefficients: QQ
⇒ // number of vars : 10
⇒ //          block 1 : ordering a
⇒ //          : names   X(1) X(2) X(3) X(4) X(5) X(6) X(7) T(2) T(\
3) T(4)
⇒ //          : weights   1    1    1    1    3    2    2    1    \
1    1
⇒ //          block 2 : ordering dp
⇒ //          : names   X(1) X(2) X(3) X(4) X(5) X(6) X(7) T(2) T(\
3) T(4)
⇒ //          block 3 : ordering C
ker;
⇒ ker[1]=X(2)*X(7)-X(5)
⇒ ker[2]=X(1)*X(6)-X(5)
⇒ ker[3]=X(1)^2*X(4)-X(2)*X(6)
⇒ ker[4]=X(2)^2*X(3)-X(1)*X(7)
⇒ ker[5]=X(1)*X(4)*X(7)-X(6)^2
⇒ ker[6]=X(1)*X(2)*X(3)*X(4)-X(6)*X(7)
⇒ ker[7]=X(2)*X(3)*X(6)-X(7)^2
⇒ ker[8]=X(1)*T(3)-X(2)*T(4)
⇒ ker[9]=-X(2)*T(2)+X(1)*T(4)
⇒ ker[10]=-X(2)*T(2)+X(7)
```



```

⇒ ker[11]=-X(2)*T(4)+X(6)
⇒ ker[12]=X(6)*T(2)-X(7)*T(4)
⇒ ker[13]=X(2)*X(4)*T(2)-X(6)*T(3)
⇒ ker[14]=X(1)*X(4)-X(2)*T(3)
⇒ ker[15]=X(1)*X(4)*T(2)-X(6)*T(4)
⇒ ker[16]=X(2)*X(3)-X(1)*T(2)
⇒ ker[17]=-X(2)^2*T(2)+X(5)
⇒ ker[18]=X(3)*X(6)-X(7)*T(2)
⇒ ker[19]=X(2)*X(3)*X(4)-X(7)*T(3)
⇒ ker[20]=X(1)*X(3)*X(4)-X(7)*T(4)
⇒ ker[21]=X(2)^2*X(6)*T(4)-X(2)*X(5)*T(3)
⇒ ker[22]=T(2)^2-X(3)*T(4)
⇒ ker[23]=-X(3)*X(4)+T(2)*T(3)
⇒ ker[24]=T(3)^2-X(4)*T(4)
⇒ ker[25]=-X(3)*T(3)+T(2)*T(4)
⇒ ker[26]=-X(4)*T(2)+T(3)*T(4)
⇒ ker[27]=-X(3)*X(4)+T(4)^2

```

#### D.4.23.9 closureFrac

Procedure from library `normal.lib` (see [Section D.4.23 \[normal.lib\]](#), page 1184).

**Usage:**      `closureFrac (L)`;  $L$  a list of size  $n+1$  as in the result of `primeClosure`,  $L[n]$  contains an additional polynomial  $f$

**Create:**     a list fraction of two elements of  $L[1]$ , such that  
 $f = \text{fraction}[1]/\text{fraction}[2]$  via the injections  $\text{phi } L[i] \rightarrow L[i+1]$ .

**Example:**

```

LIB "normal.lib";
ring R=0,(x,y),dp;
ideal ker=x2+y2;
export ker;
list L=primeClosure(R);           // We normalize R/ker
for (int i=1;i<=size(L);i++) { def R(i)=L[i]; }
setring R(2);
kill R;
phi;                               // The map R(1)-->R(2)
⇒ phi[1]=X(1)
⇒ phi[2]=X(2)
poly f=T(2);                       // We will get a representation of f
export f;
L[2]=R(2);
closureFrac(L);
setring R(1);
kill R(2);
fraction;                          // f=fraction[1]/fraction[2] via phi
⇒ [1]:
⇒ x
⇒ [2]:
⇒ y
kill R(1);

```

### D.4.23.10 iMult

Procedure from library `normal.lib` (see [Section D.4.23 \[normal.lib\], page 1184](#)).

**Usage:** `iMult(L)`;  $L$  a list of ideals

**Return:** `int`, the intersection multiplicity of the ideals of  $L$ ; if `iMult(L)` is infinite, -1 is returned.

**Theory:** If  $r = \text{size}(L) = 2$  then  $\text{iMult}(L) = \text{vdim}(\text{std}(L[1] + L[2]))$  and in general  $\text{iMult}(L) = \sum \{ \text{iMult}(L[j], L_j) \mid j = 1..r-1 \}$  with  $L_j$  the intersection of  $L[j+1], \dots, L[r]$ . If  $I$  is the intersection of all ideals in  $L$  then we have  $\text{delta}(I) = \text{delta}(L[1]) + \dots + \text{delta}(L[r]) + \text{iMult}(L)$  where  $\text{delta}(I) = \text{vdim}(\text{normalisation}(R/I)/(R/I))$ ,  $R$  the basering.

**Example:**

```
LIB "normal.lib";
ring s = 23, (x,y), dp;
list L = (x-y), (x3+y2);
iMult(L);
  ↪ 3
L = (x-y), (x3+y2), (x3-y4);
iMult(L);
  ↪ 19
```

### D.4.23.11 deltaLoc

Procedure from library `normal.lib` (see [Section D.4.23 \[normal.lib\], page 1184](#)).

**Usage:** `deltaLoc(f,J)`;  $f$  poly,  $J$  ideal

**Assume:**  $f$  is reduced bivariate polynomial; basering has exactly two variables;  $J$  is irreducible prime component of the singular locus of  $f$  (e.g., one entry of the output of `minAssGTZ(I)`;  $I = \langle f, \text{jacob}(f) \rangle$ ).

**Return:** list  $L$ :

$L[1]$ ; `int`: the sum of (local) delta invariants of  $f$  at the (conjugated) singular points given by  $J$ .

$L[2]$ ; `int`: the sum of (local) Tjurina numbers of  $f$  at the (conjugated) singular points given by  $J$ .

$L[3]$ ; `int`: the sum of (local) number of branches of  $f$  at the (conjugated) singular points given by  $J$ .

**Note:** procedure makes use of `execute`; increasing `printlevel` displays more comments (default: `printlevel=0`).

**Example:**

```
LIB "normal.lib";
ring r=0, (x,y), dp;
poly f=(x2+y^2-1)^3 +27x2y2;
ideal I=f,jacob(f);
I=std(I);
list qr=minAssGTZ(I);
size(qr);
  ↪ 6
// each component of the singular locus either describes a cusp or a pair
// of conjugated nodes:
```

```

deltaLoc(f,qr[1]);
↳ [1]:
↳ 1
↳ [2]:
↳ 2
↳ [3]:
↳ 1
deltaLoc(f,qr[2]);
↳ [1]:
↳ 1
↳ [2]:
↳ 2
↳ [3]:
↳ 1
deltaLoc(f,qr[3]);
↳ [1]:
↳ 1
↳ [2]:
↳ 2
↳ [3]:
↳ 1
deltaLoc(f,qr[4]);
↳ [1]:
↳ 1
↳ [2]:
↳ 2
↳ [3]:
↳ 1
deltaLoc(f,qr[5]);
↳ [1]:
↳ 2
↳ [2]:
↳ 2
↳ [3]:
↳ 4
deltaLoc(f,qr[6]);
↳ [1]:
↳ 2
↳ [2]:
↳ 2
↳ [3]:
↳ 4

```

See also: [Section D.6.15.12 \[delta\], page 1731](#); [Section D.6.20.15 \[tjurina\], page 1764](#).

#### D.4.23.12 locAtZero

Procedure from library `normal.lib` (see [Section D.4.23 \[normal\\_lib\], page 1184](#)).

**Usage:**      `locAtZero(I)`;  $I$  = ideal

**Return:**    `int`, 1 if  $I$  has only one point which is located at zero, 0 otherwise

**Assume:**     $I$  is given as a standard bases in the basering

**Note:**      only useful in affine rings, in local rings `vdim` does the check

**Example:**

```

LIB "normal.lib";
ring r = 0,(x,y,z),dp;
poly f = z5+y4+x3+xyz;
ideal i = jacob(f),f;
i=std(i);
locAtZero(i);
↪ 1
i= std(i*ideal(x-1,y,z));
locAtZero(i);
↪ 0

```

**D.4.23.13 norTest**

Procedure from library `normal.lib` (see [Section D.4.23 \[normal.lib\]](#), page 1184).

**Assume:** `nor` is the output of `normal(i)` (any options) or `normalP(i,"withRing")` or `normalC(i)` (any options).  
In particular, the ring `nor[1][1]` contains the ideal `norid` and the map `normap: basering/i → nor[1][1]/norid`.

**Return:** an intvec `v` such that:

$v[1] = 1$  if the normap is injective and 0 otherwise  
 $v[2] = 1$  if the normap is finite and 0 otherwise  
 $v[3] = 1$  if `nor[1][1]/norid` is normal and 0 otherwise

If `n=1` (resp `n=2`) only `v[1]` (resp. `v[2]`) is computed and returned

**Theory:** The procedure can be used to test whether the computation of the normalization was correct: `basering/i → nor[1][1]/norid` is the normalization of `basering/i` if and only if `v=1,1,0`.

**Note:** For big examples it can be hard to fully test correctness; the partial test `norTest(i,nor,2)` is usually fast

**Example:**

```

LIB "normal.lib";
int prl = printlevel;
printlevel = -1;
ring r = 0,(x,y),dp;
ideal i = (x-y^2)^2 - y*x^3;
list nor = normal(i);
norTest(i,nor);                                //1,1,1 means that normal was correct
↪ 1
↪ 1
↪ 1,1,1
nor = normalC(i);
norTest(i,nor);                                //1,1,1 means that normal was correct
↪ 1
↪ 1
↪ 1,1,1
ring s = 2,(x,y),dp;
ideal i = (x-y^2)^2 - y*x^3;
nor = normalP(i,"withRing");
norTest(i,nor);                                //1,1,1 means that normalP was correct

```

```

    ↦ 1
    ↦ 1
    ↦ 1,1,1
    printlevel = prl;

```

#### D.4.23.14 getSmallest

Procedure from library `normal.lib` (see [Section D.4.23 \[normal.lib\], page 1184](#)).

**Usage:** `getSmallest(J)`; J is an ideal.

**Return:** the generator of J of smallest degree. If there are more than one, it chooses the one with smallest number of monomials.

**Note:** It looks only at the generator of J, not at all the polynomials in the ideal. It is intended mainly to compute a good universal denominator in the normalization algorithms.

**Example:**

```

LIB "normal.lib";
printlevel = printlevel+1;
ring s = 0,(x,y),dp;
ideal J = x3-y, y5, x2-y2+1;
getSmallest(J);
↦ x2-y2+1
printlevel = printlevel-1;

```

#### D.4.23.15 getOneVar

Procedure from library `normal.lib` (see [Section D.4.23 \[normal.lib\], page 1184](#)).

**Usage:** `getOneVar(J, vari)`; J is a 0-dimensional ideal, vari is an integer.

**Return:** a polynomial of J in the variable indicated by vari of smallest degree.

**Note:** Works only over rings of two variables. It is intended mainly as an auxiliary procedure for computing integral bases.

**Example:**

```

LIB "normal.lib";
printlevel = printlevel+1;
ring s = 0,(x,y),dp;
ideal J = x3-y, y3;
getOneVar(J, 1);
↦ x9
printlevel = printlevel-1;

```

#### D.4.23.16 changeDenominator

Procedure from library `normal.lib` (see [Section D.4.23 \[normal.lib\], page 1184](#)).

**Usage:** `changeDenominator(U1, c1, c2, I)`; U1 and I ideals, c1 and c2 polynomials.

**Return:** an ideal  $U_2$  such that the  $A$ -modules  $1/c_1 * U_1$  and  $1/c_2 * U_2$  are equal, where  $A = R/I$  and  $R$  is the basering.

**Note:** It assumes that such  $U_2$  exists. It is intended mainly as an auxiliary procedure in the normalization algorithms.

**Example:**

```
LIB "normal.lib";
ring s = 0,(x,y),dp;
ideal I = y5-y4x+4y2x2-x4;
ideal U1 = normal(I)[2][1];
⇒
⇒ // 'normal' created a list, say nor, of two elements.
⇒ // To see the list type
⇒     nor;
⇒
⇒ // * nor[1] is a list of 1 ring(s).
⇒ // To access the i-th ring nor[1][i], give it a name, say Ri, and type
⇒     def R1 = nor[1][1]; setring R1; norid; normap;
⇒ // For the other rings type first (if R is the name of your base ring)
⇒     setring R;
⇒ // and then continue as for R1.
⇒ // Ri/norid is the affine algebra of the normalization of R/P_i where
⇒ // P_i is the i-th component of a decomposition of the input ideal id
⇒ // and normap the normalization map from R to Ri/norid.
⇒
⇒ // * nor[2] is a list of 1 ideal(s). Let ci be the last generator
⇒ // of the ideal nor[2][i]. Then the integral closure of R/P_i is
⇒ // generated as R-submodule of the total ring of fractions by
⇒ // 1/ci * nor[2][i].
poly c1 = U1[4];
U1;c1;
⇒ U1[1]=xy2
⇒ U1[2]=x2y
⇒ U1[3]=x3
⇒ U1[4]=y3
⇒ y3
⇒ // 1/c1 * U1 is the normalization of I.
ideal U2 = changeDenominator(U1, c1, x3, I);
U2;
⇒ U2[1]=-xy3+y4+4x2y
⇒ U2[2]=-x2y2+xy3+4x3
⇒ U2[3]=-x3y+x2y2-4xy3+4y4+16x2y
⇒ U2[4]=x3
⇒ // 1/x3 * U2 is also the normalization of I, but with a different denominator.
```

### D.4.23.17 normalConductor

Procedure from library `normal.lib` (see [Section D.4.23 \[normal.lib\]](#), page 1184).

**Usage:** `normalConductor(I);`  $I$  ideal

**Assume:**  $I$  is a radical ideal

**Return:** the conductor of  $R/I$  as ideal in  $R$

**Remarks:** The procedure makes use of the minimal primes and the generators of the normalization given by the normalization algorithm.

**Note:** the optional parameter can be used if the normalization has already been computed. If a list  $L$  contains the output of the procedure `normal` (with options `prim`, `wd` and `using` if the ring has a mixed ordering), apply `normalConductor(I,L)`

**Example:**

```
LIB "normal.lib";
////////////////////////////////////
// Computation of small conductor ideals //
////////////////////////////////////
ring R = 0,(x,y,z),ds;
ideal I = x2y2 - z;
normalConductor(I);
⇒ _[1]=1
// The conductor is the whole ring - so the ring is normal
// We can also see this using the delta invariant:
curveDeltaInv(I);
⇒ 0
ring S = 0,(a,b,c),dp;
ideal J = abc;
normalConductor(J);
⇒ _[1]=bc
⇒ _[2]=ac
⇒ _[3]=ab
// The conductor is not the whole ring - so it is not normal
// We can also see this using the delta invariant, which is even infinite
curveDeltaInv(J);
⇒ -1
kill R,S;
////////////////////////////////////
// Computation of a bigger example //
////////////////////////////////////
ring R = 0,(x,y,z,t),ds;
ideal I = xyz - yzt, x2y3 - z2t4;
I = std(radical(I));
// Ideal I
I;
⇒ I[1]=x2y-xyt
⇒ I[2]=xyz-yzt
⇒ I[3]=xzt-zt2
⇒ I[4]=xy3-z2t3
⇒ I[5]=y3zt-z3t3
// Conductor
normalConductor(I);
⇒ _[1]=xy
⇒ _[2]=xz-zt
⇒ _[3]=yt
⇒ _[4]=x2t-xt2
⇒ _[5]=zt2
⇒ _[6]=y3z
```

See also: [Section D.4.5.2 \[curveConductorMult\]](#), page 1050.

### D.4.23.18 isNormal

Procedure from library `normal.lib` (see [Section D.4.23 \[normal.lib\]](#), page 1184).

**Usage:** `isNormal(I)`;  $I$  ideal.

**Return:** 1 if  $R/I$  is normal and 0 if  $R/I$  is not normal, where  $R$  is the basering.

**Example:**

```
LIB "normal.lib";
ring R = 0, (x,y), dp;
ideal I1 = x2 - y3;
isNormal(I1);
↳ 0
ideal I2 = x - y3;
isNormal(I2);
↳ 1
```

### D.4.24 normaliz.lib

**Library:** `normaliz.lib`

**Purpose:** Provides an interface for the use of Normaliz 3.10.0 or newer within SINGULAR.

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**Overview:** The library `normaliz.lib` provides an interface for the use of Normaliz 3.10.0 or newer within SINGULAR. The exchange of data is via files. In addition to the top level functions that aim at objects of type ideal or ring, several other auxiliary functions allow the user to apply Normaliz to data of type intmat. Options such as computationn goals or algorithmic variants can be activated. To some extent, SINGULAR can therefore be used as an environment for interactive access to Normaliz. Please see the `Normaliz.pdf` (included in the Normaliz distribution) for a more extensive documentation of Normaliz.

Normaliz allows the use of a grading. In the Singular functions that access Normaliz the parameter grading is an intvec that assigns a (not necessarily positive) degree to every variable of the ambient polynomial ring. But it must give positive degrees to the generators given to the function.

Singular and Normaliz exchange data via files. The input files use the Normaliz 3 syntax. These files are automatically created and erased behind the scenes. As long as one wants to use only the ring-theoretic functions there is no need for file management. Note that the numerical invariants computed by Normaliz can be accessed in this "automatic file mode".

However, if Singular is used as a frontend for Normaliz or the user wants to inspect data not automatically returned to Singular, then an explicit filename and a path can be specified for the exchange of data. Moreover, the library provides functions for access to these files. Deletion of the files is left to the user. (Not all output files of Normaliz can be read by this library.)

Use of this library requires the program Normaliz  $\geq 3.10.0$  to be installed. You can download it from <https://github.com/Normaliz/Normaliz/releases>. Please make sure that the executable is in the search path or use `setNmzExecPath` ([Section D.4.24.23 \[setNmzExecPath\]](#), page 1228).



**Procedures:****D.4.24.1 intclToricRing**

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:**     `intclToricRing(ideal I);`  
               `intclToricRing(ideal I, intvec grading);`

**Return:**    The toric ring  $S$  is the subalgebra of the basering generated by the leading monomials of the elements of  $I$  (considered as a list of polynomials). The function computes the integral closure  $T$  of  $S$  in the basering and returns an ideal listing the algebra generators of  $T$  over the coefficient field.  
 The function returns the input ideal  $I$  if an option blocking the computation of Hilbert bases has been activated. However, in this case some numerical invariants are computed, and some other data may be contained in files that you can read into Singular (see [Section D.4.24.15 \[showNuminvs\]](#), page 1222, [Section D.4.24.16 \[exportNuminvs\]](#), page 1223).

**Note:**     A mathematical remark: the toric ring depends on the list of monomials given, and not only on the ideal they generate!

**Example:**

```
LIB "normaliz.lib";
ring R=37,(x,y,t),dp;
ideal I=x3,x2y,y3;
intclToricRing(I);
↪ _[1]=y
↪ _[2]=x
showNuminvs();
↪ hilbert_basis_elements : 2
↪ number_extreme_rays : 2
↪ dim_max_subspace : 0
↪ embedding_dim : 3
↪ rank : 2
↪ external_index : 1
↪ internal_index : 3
↪ number_support_hyperplanes : 2
↪ size_triangulation : 1
↪ sum_dets : 1
↪ integrally_closed : 0
↪ inhomogeneous : 0
↪ graded : 1
↪ degree_1_elements : 2
↪ grading : 1,1,0
↪ grading_denom : 1
↪ multiplicity : 1
↪ multiplicity_denom : 1
↪ hilbert_series_num : 1
↪ hilbert_series_denom : 1,1
↪ class_group : 0
//now the same example with another grading
intvec grading = 2,3,1;
```

```

intclToricRing(I,grading);
↳ _[1]=x
↳ _[2]=y
showNuminvs();
↳ hilbert_basis_elements : 2
↳ number_extreme_rays : 2
↳ dim_max_subspace : 0
↳ embedding_dim : 3
↳ rank : 2
↳ external_index : 1
↳ internal_index : 3
↳ number_support_hyperplanes : 2
↳ size_triangulation : 1
↳ sum_dets : 1
↳ integrally_closed : 0
↳ inhomogeneous : 0
↳ graded : 1
↳ degree_1_elements : 0
↳ grading : 2,3,1
↳ grading_denom : 1
↳ multiplicity : 1
↳ multiplicity_denom : 6
↳ hilbert_series_num : 1,-1,1
↳ hilbert_series_denom : 1,6
↳ class_group : 0

```

See also: [Section D.4.24.5 \[ehrhartRing\]](#), page 1214; [Section D.4.24.6 \[intclMonIdeal\]](#), page 1215; [Section D.4.24.2 \[normalToricRing\]](#), page 1212.

### D.4.24.2 normalToricRing

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:**     `normalToricRing(ideal I);`  
               `normalToricRing(ideal I, intvec grading);`

**Return:**    The toric ring  $S$  is the subalgebra of the basering generated by the leading monomials of the elements of  $I$  (considered as a list of polynomials). The function computes the normalisation  $T$  of  $S$  and returns an ideal listing the algebra generators of  $T$  over the coefficient field.

The function returns the input ideal  $I$  if one of the options blocking the computation of Hilbert bases has been activated. However, in this case some numerical invariants are computed, and some other data may be contained in files that you can read into Singular (see [Section D.4.24.15 \[showNuminvs\]](#), page 1222, [Section D.4.24.16 \[exportNuminvs\]](#), page 1223).

**Note:**       A mathematical remark: the toric ring depends on the list of monomials given, and not only on the ideal they generate!

**Example:**

```

LIB "normaliz.lib";
ring R = 37,(x,y,t),dp;
ideal I = x3,x2y,y3;
normalToricRing(I);
↳ _[1]=y3

```

```

↳ _[2]=xy2
↳ _[3]=x2y
↳ _[4]=x3

```

See also: [Section D.4.24.5 \[ehrhartRing\]](#), page 1214; [Section D.4.24.6 \[intclMonIdeal\]](#), page 1215; [Section D.4.24.1 \[intclToricRing\]](#), page 1211; [Section D.4.24.3 \[normalToricRingFromBinomials\]](#), page 1213; [Section D.4.24.4 \[toricRingFromBinomials\]](#), page 1214.

### D.4.24.3 normalToricRingFromBinomials

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:**      `normalToricRingFromBinomials(ideal I);`  
               `normalToricRingFromBinomials(ideal I, intvec grading);`

**Return:**     The ideal  $I$  is generated by binomials of type  $X^a - X^b$  (multiindex notation) in the surrounding polynomial ring  $K[X] = K[X_1, \dots, X_n]$ . The binomials represent a congruence on the monoid  $Z^n$  with residue monoid  $M$ . Let  $N$  be the image of  $M$  in  $\text{gp}(M)/\text{torsion}$ . Then  $N$  is universal in the sense that every homomorphism from  $M$  to an affine monoid factors through  $N$ . If  $I$  is a prime ideal, then  $K[N] = K[X]/I$ . In general,  $K[N] = K[X]/P$  where  $P$  is the unique minimal prime ideal of  $I$  generated by binomials of type  $X^a - X^b$ .

The function computes the normalization of  $K[N]$  and returns a newly created polynomial ring of the same Krull dimension, whose variables are  $x(1), \dots, x(n-r)$ , where  $r$  is the rank of the matrix with rows  $a-b$ . (In general there is no canonical choice for such an embedding.) Inside this polynomial ring there is an ideal  $I$  which lists the algebra generators of the normalization of  $K[N]$ .

The function returns the input ideal  $I$  if an option blocking the computation of Hilbert bases has been activated.

However, in this case some numerical invariants are computed, and some other data may be contained in files that you can read into Singular (see [Section D.4.24.15 \[showNuminvs\]](#), page 1222, [Section D.4.24.16 \[exportNuminvs\]](#), page 1223).

#### Example:

```

LIB "normaliz.lib";
ring R = 37,(u,v,w,x,y,z),dp;
ideal I = u2v-xyz, ux2-wyz, uvw-y2z;
def S = normalToricRingFromBinomials(I);
setring S;
I;
↳ I[1]=x(3)
↳ I[2]=x(1)
↳ I[3]=x(2)*x(3)^3
↳ I[4]=x(1)*x(2)*x(3)^2
↳ I[5]=x(1)^2*x(2)*x(3)
↳ I[6]=x(1)^3*x(2)
↳ I[7]=x(1)*x(2)^2*x(3)^4
↳ I[8]=x(1)^2*x(2)^2*x(3)^3
↳ I[9]=x(1)^2*x(2)^3*x(3)^5

```

See also: [Section D.4.24.5 \[ehrhartRing\]](#), page 1214; [Section D.4.24.6 \[intclMonIdeal\]](#), page 1215; [Section D.4.24.1 \[intclToricRing\]](#), page 1211; [Section D.4.24.2 \[normalToricRing\]](#), page 1212.

#### D.4.24.4 toricRingFromBinomials

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:** `toricRingFromBinomials(ideal I);`  
`toricRingFromBinomials(ideal I, intvec grading);`

**Return:** The ideal  $I$  is generated by binomials of type  $X^a - X^b$  (multiindex notation) in the surrounding polynomial ring  $K[X] = K[X_1, \dots, X_n]$ . The binomials represent a congruence on the monoid  $\mathbb{Z}^n$  with residue monoid  $M$ . Let  $N$  be the image of  $M$  in  $\text{gp}(M)/\text{torsion}$ . Then  $N$  is universal in the sense that every homomorphism from  $M$  to an affine monoid factors through  $N$ . If  $I$  is a prime ideal, then  $K[N] = K[X]/I$ . In general,  $K[N] = K[X]/P$  where  $P$  is the unique minimal prime ideal of  $I$  generated by binomials of type  $X^a - X^b$ .

The function computes  $K[N]$  and returns a newly created polynomial ring of the same Krull dimension, whose variables are  $x(1), \dots, x(n-r)$ , where  $r$  is the rank of the matrix with rows  $a - b$ . (In general there is no canonical choice for such an embedding.)

The function returns the input ideal  $I$  if an option blocking the computation of Hilbert bases has been activated.

However, in this case some numerical invariants are computed, and some other data may be contained in files that you can read into Singular (see [Section D.4.24.15 \[showN-uminvs\]](#), page 1222, [Section D.4.24.16 \[exportNuminvs\]](#), page 1223).

**Example:**

```
LIB "normaliz.lib";
ring R = 37,(u,v,w,x,y,z),dp;
ideal I = u2v-xyz, ux2-wyz, uvw-y2z;
def S = toricRingFromBinomials(I);
setring S;
I;
⇒ I[1]=x(3)
⇒ I[2]=x(1)
⇒ I[3]=x(2)*x(3)^3
⇒ I[4]=x(1)*x(2)*x(3)^2
⇒ I[5]=x(1)^3*x(2)
⇒ I[6]=x(1)^2*x(2)^3*x(3)^5
```

See also: [Section D.4.24.3 \[normalToricRingFromBinomials\]](#), page 1213; [Section D.4.24.4 \[toricRingFromBinomials\]](#), page 1214.

#### D.4.24.5 ehrhartRing

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:** `ehrhartRing(ideal I);`

**Return:** The exponent vectors of the leading monomials of the elements of  $I$  are considered as points of a lattice polytope  $P$ .

The Ehrhart ring of a (lattice) polytope  $P$  is the monoid algebra defined by the monoid of lattice points in the cone over the polytope  $P$ ; see Bruns and Gubeladze, *Polytopes, Rings, and K-theory*, Springer 2009, pp. 228, 229.

The function returns a list of ideals:

(i) If the last ring variable is not used by the monomials, it is treated as the auxiliary

variable of the Ehrhart ring. The function returns two ideals, the first containing the monomials representing all the lattice points of the polytope, the second containing the algebra generators of the Ehrhart ring over the coefficient field.

(ii) If the last ring variable is used by the monomials, the list returned contains only one ideal, namely the monomials representing the lattice points of the polytope.

The function returns the a list containing the input ideal  $I$  if an option blocking the computation of the Hilbert basis has been activated. However, in this case some numerical invariants are computed, and some other data may be contained in files that you can read into Singular (see [Section D.4.24.15 \[showNuminvs\]](#), page 1222, [Section D.4.24.16 \[exportNuminvs\]](#), page 1223).

**Note:** A mathematical remark: the Ehrhart ring depends on the list of monomials given, and not only on the ideal they generate!

**Example:**

```
LIB "normaliz.lib";
ring R=37,(x,y,t),dp;
ideal J=x3,x2y,y3,xy2t7;
ehrhartRing(J);
⇒ [1]:
⇒ _[1]=y3
⇒ _[2]=xy2
⇒ _[3]=xy2t
⇒ _[4]=xy2t2
⇒ _[5]=xy2t3
⇒ _[6]=xy2t4
⇒ _[7]=xy2t5
⇒ _[8]=xy2t6
⇒ _[9]=xy2t7
⇒ _[10]=x2y
⇒ _[11]=x2yt
⇒ _[12]=x2yt2
⇒ _[13]=x2yt3
⇒ _[14]=x3
```

See also: [Section D.4.24.6 \[intclMonIdeal\]](#), page 1215; [Section D.4.24.1 \[intclToricRing\]](#), page 1211; [Section D.4.24.2 \[normalToricRing\]](#), page 1212.

## D.4.24.6 intclMonIdeal

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:** `intclMonIdeal(ideal I);`  
`intclMonIdeal(ideal I, intvec grading);`

**Return:** The exponent vectors of the leading monomials of the elements of  $I$  are considered as generators of a monomial ideal for which the normalization of its Rees algebra is computed. For a Definition of the Rees algebra (or Rees ring) see Bruns and Herzog, Cohen-Macaulay rings, Cambridge University Press 1998, p. 182.

The function returns a list of ideals:

(i) If the last ring variable is not used by the monomials, it is treated as the auxiliary variable of the Rees algebra. The function returns two ideals, the first containing the monomials generating the integral closure of the monomial ideal, the second containing

the algebra generators of the normalization of the Rees algebra.

(ii) If the last ring variable is used by the monomials, the list returned contains only one ideal, namely the monomials generating the integral closure of the ideal.

The function returns the a list containing the input ideal  $I$  if an option blocking the computation of Hilbert bases has been activated. However, in this case some numerical invariants are computed, and some other data may be contained in files that you can read into Singular (see [Section D.4.24.15 \[showNuminvs\]](#), page 1222, [Section D.4.24.16 \[exportNuminvs\]](#), page 1223).

**Example:**

```
LIB "normaliz.lib";
ring R=0,(x,y,z,t),dp;
ideal I=x^2,y^2,z^3;
list l=intclMonIdeal(I);
l[1]; // integral closure of I
↳ _[1]=z3
↳ _[2]=yz2
↳ _[3]=y2
↳ _[4]=xz2
↳ _[5]=xy
↳ _[6]=x2
l[2]; // monomials generating the integral closure of the Rees algebra
↳ _[1]=z
↳ _[2]=z3t
↳ _[3]=y
↳ _[4]=yz2t
↳ _[5]=y2t
↳ _[6]=x
↳ _[7]=xz2t
↳ _[8]=xyt
↳ _[9]=x2t
```

See also: [Section D.4.24.5 \[ehrhartRing\]](#), page 1214; [Section D.4.24.1 \[intclToricRing\]](#), page 1211; [Section D.4.24.2 \[normalToricRing\]](#), page 1212.

### D.4.24.7 definingBinomialIdeal

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:**      `definingBinomialIdeal(ideal I);`

**Return:**    The toric ring  $S$  is the subalgebra of the basering generated by the leading monomials of the elements of  $I$  (considered as a list of polynomials). The function computes the defining binomial ideal  $J$  of  $S$  with respect to the generators and returns it, together with  $J$ . DSee eample.

**Note:**      A mathematical remark: the toric ring depends on the list of monomials given, and not only on the ideal they generate! This function requires the previous setting of an `NmzFilename`. The function does not delete the written files.

**Example:**

```
LIB "normaliz.lib";
ring R = 37,(x,y,z,w),dp;
ideal I = x4,x3y,x2y2,xy3,y4;
setNmzFilename("binomials");
```

```

def S = definingBinomialIdeal(I);
↪ 1
↪ 1
setring S;
J;
↪ J[1]=-x(3)*x(4)+x(2)*x(5)
↪ J[2]=-x(2)*x(4)+x(1)*x(5)
↪ J[3]=x(2)*x(3)-x(1)*x(4)
↪ J[4]=-x(2)^2+x(1)*x(3)
↪ J[5]=-x(4)^2+x(3)*x(5)
↪ J[6]=x(3)^2-x(2)*x(4)

```

See also: [Section D.4.24.3 \[normalToricRingFromBinomials\]](#), page 1213.

### D.4.24.8 latticeIdeal

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:**      `latticeIdeal(ideal I);`

**Return:**      Returns the lattice ideal defined by the elements of `I` which have to be binomials. The lattice ideal is the restriction to the polynomial ring of the ideal of the Laurent polynomial ring generated by `I`

**Note:**        This function requires the previous setting of an `NmzFilename`. The function does not delete the written files.

**Example:**

```

LIB "normaliz.lib";
setNmzFilename("binomials");
ring S = 37, (u,v,w,x,y,z), dp;
ideal I = u2-v2, x2-y2, y2-vw, z2-xy;
latticeIdeal(I);
↪ 1
↪ 1
↪ _[1]=x2-y2
↪ _[2]=u2-v2
↪ _[3]=-xy+z2
↪ _[4]=vw-y2

```

See also: [Section D.4.24.34 \[intmat2binomials\]](#), page 1233.

### D.4.24.9 groebnerBasis

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:**      `groebnerBasis(ideal I);`

**Return:**      Returns a Gr"obner basis of the lattice ideal defined by the elements of `I` which have to be binomials. The lattice ideal is the restriction to the polynomial ring of the ideal of the Laurent polynomial ring generated by `I`. The default monomial order is `DegRevLex` where `Deg` is the total degree on the ambient polynomial ring. `Lex` and `DegLex` orders can be chosen via options.

**Note:**        This function requires the previous setting of an `NmzFilename`. The function does not delete the written files.

**Example:**

```

LIB "normaliz.lib";
setNmzFilename("binomials");
ring S = 37,(u,v,w,x,y,z),dp;
ideal I = u2-v2, x2-y2, y2-vw, z2-xy;
groebnerBasis(I);
  ↳ _[1]=xy-z2
  ↳ _[2]=x2-y2
  ↳ _[3]=vw-y2
  ↳ _[4]=u2-v2
  ↳ _[5]=y3-xz2
setNmzOption("lex",1);
  ↳ 1
groebnerBasis(I);
  ↳ _[1]=y4-z4
  ↳ _[2]=-y3+xz2
  ↳ _[3]=xy-z2
  ↳ _[4]=x2-y2
  ↳ _[5]=vw-y2
  ↳ _[6]=u2-v2
setNmzOption("lex",0);
  ↳ 1

```

See also: [Section D.4.24.8 \[latticeIdeal\]](#), page 1217.

**D.4.24.10 torusInvariants**

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:**     `torusInvariants(intmat A);`  
               `torusInvariants(intmat A, intvec grading);`

**Return:**   Returns an ideal representing the list of monomials generating the ring of invariants as an algebra over the coefficient field.  $R^T$ .  
 The function returns the ideal given by the input matrix `A` if one of the options `supp`, `triang`, `volume`, or `hseries` has been activated. However, in this case some numerical invariants are computed, and some other data may be contained in files that you can read into Singular (see [Section D.4.24.15 \[showNuminvs\]](#), page 1222, [Section D.4.24.16 \[exportNuminvs\]](#), page 1223).

**Background:**

Let  $T = (K^*)^r$  be the  $r$ -dimensional torus acting on the polynomial ring  $R = K[X_1, \dots, X_n]$  diagonally. Such an action can be described as follows: there are integers  $a_{i,j}$ ,  $i = 1, \dots, r$ ,  $j = 1, \dots, n$ , such that  $(\lambda_1, \dots, \lambda_r) \in T$  acts by the substitution

$$X_j \mapsto \lambda_1^{a_{1,j}} \cdots \lambda_r^{a_{r,j}} X_j, \quad j = 1, \dots, n.$$

In order to compute the ring of invariants  $R^T$  one must specify the matrix  $A = (a_{i,j})$ .

**Example:**

```

LIB "normaliz.lib";
ring R=0,(x,y,z,w),dp;
intmat E[2][4] = -1,-1,2,0, 1,1,-2,-1;
torusInvariants(E);
  ↳ _[1]=y2z

```



```

↳ _[2]=xyz
↳ _[3]=x2z

```

See also: [Section D.4.24.12 \[diagInvariants\]](#), page 1220; [Section D.4.24.11 \[finiteDiagInvariants\]](#), page 1219; [Section D.4.24.14 \[intersectionValRingIdeals\]](#), page 1221; [Section D.4.24.13 \[intersectionValRings\]](#), page 1221.

### D.4.24.11 finiteDiagInvariants

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:**     `finiteDiagInvariants(intmat U);`  
               `finiteDiagInvariants(intmat U, intvec grading);`

**Return:**    This function computes the ring of invariants of a finite abelian group  $G$  acting diagonally on the surrounding polynomial ring  $K[X_1, \dots, X_n]$ . The group is the direct product of cyclic groups generated by finitely many elements  $g_1, \dots, g_w$ . The element  $g_i$  acts on the indeterminate  $X_j$  by  $g_i(X_j) = \lambda_i^{u_{ij}} X_j$  where  $\lambda_i$  is a primitive root of unity of order equal to  $\text{ord}(g_i)$ . The ring of invariants is generated by all monomials satisfying the system  $u_{i1}a_1 + \dots + u_{in}a_n \equiv 0 \pmod{\text{ord}(g_i)}$ ,  $i = 1, \dots, w$ . The input to the function is the  $w \times (n+1)$  matrix  $U$  with rows  $u_{i1} \dots u_{in} \text{ord}(g_i)$ ,  $i = 1, \dots, w$ . The output is a monomial ideal listing the algebra generators of the subalgebra of invariants  $R^G = \{f \in R : g_i f = f \text{ for all } i = 1, \dots, w\}$ .

The function returns the ideal given by the input matrix  $C$  if one of the options `supp`, `triang`, `volume`, or `hseries` has been activated. However, in this case some numerical invariants are computed, and some other data may be contained in files that you can read into Singular (see [Section D.4.24.15 \[showNuminvs\]](#), page 1222, [Section D.4.24.16 \[exportNuminvs\]](#), page 1223).

**Note:**

**Example:**

```

LIB "normaliz.lib";
ring R = 0,(x,y,z,w),dp;
intmat C[2][5] = 1,1,1,1,5, 1,0,2,0,7;
finiteDiagInvariants(C);
↳ _[1]=w5
↳ _[2]=z7w3
↳ _[3]=z14w
↳ _[4]=z35
↳ _[5]=yw4
↳ _[6]=yz7w2
↳ _[7]=yz14
↳ _[8]=y2w3
↳ _[9]=y2z7w
↳ _[10]=y3w2
↳ _[11]=y3z7
↳ _[12]=y4w
↳ _[13]=y5
↳ _[14]=xz3w
↳ _[15]=xz24
↳ _[16]=xyz3
↳ _[17]=x2z13

```

```

↳ _[18]=x3z2
↳ _[19]=x5zw4
↳ _[20]=x5yzw3
↳ _[21]=x5y2zw2
↳ _[22]=x5y3zw
↳ _[23]=x5y4z
↳ _[24]=x7w3
↳ _[25]=x7yw2
↳ _[26]=x7y2w
↳ _[27]=x7y3
↳ _[28]=x12zw2
↳ _[29]=x12yzw
↳ _[30]=x12y2z
↳ _[31]=x14w
↳ _[32]=x14y
↳ _[33]=x19z
↳ _[34]=x35

```

See also: [Section D.4.24.12 \[diagInvariants\]](#), page 1220; [Section D.4.24.14 \[intersectionValRingIdeals\]](#), page 1221; [Section D.4.24.13 \[intersectionValRings\]](#), page 1221; [Section D.4.24.10 \[torusInvariants\]](#), page 1218.

### D.4.24.12 diagInvariants

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:**      `diagInvariants(intmat A, intmat U);`  
               `diagInvariants(intmat A, intmat U, intvec grading);`

**Return:**    This function computes the ring of invariants of a diagonalizable group  $D = T \times G$  where  $T$  is a torus and  $G$  is a finite abelian group, both acting diagonally on the polynomial ring  $K[X_1, \dots, X_n]$ . The group actions are specified by the input matrices  $A$  and  $U$ . The first matrix specifies the torus action, the second the action of the finite group. See `torusInvariants` and `finiteDiagInvariants` for more detail. The output is a monomial ideal listing the algebra generators of the subalgebra of invariants.

The function returns the ideal given by the input matrix  $A$  if one of the options `supp`, `triang`, `volume`, or `hseries` has been activated. However, in this case some numerical invariants are computed, and some other data may be contained in files that you can read into Singular (see [Section D.4.24.15 \[showNuminvs\]](#), page 1222, [Section D.4.24.16 \[exportNuminvs\]](#), page 1223).

**Example:**

```

LIB "normaliz.lib";
ring R=0,(x,y,z,w),dp;
intmat E[2][4] = -1,-1,2,0, 1,1,-2,-1;
intmat C[2][5] = 1,1,1,1,5, 1,0,2,0,7;
diagInvariants(E,C);
↳ _[1]=y70z35
↳ _[2]=xy19z10
↳ _[3]=x4y6z5
↳ _[4]=x15y5z10
↳ _[5]=x26y4z15
↳ _[6]=x37y3z20

```

```

↳ _[7]=x48y2z25
↳ _[8]=x59yz30
↳ _[9]=x70z35

```

See also: [Section D.4.24.11 \[finiteDiagInvariants\]](#), page 1219; [Section D.4.24.14 \[intersectionValRingIdeals\]](#), page 1221; [Section D.4.24.13 \[intersectionValRings\]](#), page 1221; [Section D.4.24.10 \[torusInvariants\]](#), page 1218.

### D.4.24.13 intersectionValRings

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:** `intersectionValRings(intmat V, intvec grading);`

**Return:** The function returns a monomial ideal, to be considered as the list of monomials generating  $S$  as an algebra over the coefficient field.

**Background:**

A discrete monomial valuation  $v$  on  $R = K[X_1, \dots, X_n]$  is determined by the values  $v(X_j)$  of the indeterminates. This function computes the subalgebra  $S = \{f \in R : v_i(f) \geq 0, i = 1, \dots, r\}$  for several such valuations  $v_i, i = 1, \dots, r$ . It needs the matrix  $V = (v_i(X_j))$  as its input.

The function returns the ideal given by the input matrix  $V$  if one of the options `supp`, `triang`, `volume`, or `hseries` has been activated. However, in this case some numerical invariants are computed, and some other data may be contained in files that you can read into Singular (see [Section D.4.24.15 \[showNuminvs\]](#), page 1222, [Section D.4.24.16 \[exportNuminvs\]](#), page 1223).

**Example:**

```

LIB "normaliz.lib";
ring R=0,(x,y,z,w),dp;
intmat V0[2][4]=0,1,2,3, -1,1,2,1;
intersectionValRings(V0);
↳ _[1]=w
↳ _[2]=z
↳ _[3]=y
↳ _[4]=xw
↳ _[5]=xz
↳ _[6]=xy
↳ _[7]=x2z

```

See also: [Section D.4.24.12 \[diagInvariants\]](#), page 1220; [Section D.4.24.11 \[finiteDiagInvariants\]](#), page 1219; [Section D.4.24.14 \[intersectionValRingIdeals\]](#), page 1221; [Section D.4.24.10 \[torusInvariants\]](#), page 1218.

### D.4.24.14 intersectionValRingIdeals

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:** `intersectionValRingIdeals(intmat V);`  
`intersectionValRingIdeals(intmat V, intvec grading);`

**Return:** The function returns two ideals, both to be considered as lists of monomials. The first is the system of monomial generators of  $S$ , the second the system of generators of  $M$ .

The function returns a list consisting of the ideal given by the blocking the computation of Hilbert bases has been activated. However, in this case some numerical invariants are computed, and some other data may be contained in files that you can read into Singular (see [Section D.4.24.15 \[showNuminvs\]](#), page 1222, [Section D.4.24.16 \[exportNuminvs\]](#), page 1223).

**Background:**

A discrete monomial valuation  $v$  on  $R = K[X_1, \dots, X_n]$  is determined by the values  $v(X_j)$  of the indeterminates. This function computes the subalgebra  $S = \{f \in R : v_i(f) \geq 0, i = 1, \dots, r\}$  for several such valuations  $v_i, i = 1, \dots, r$ . It needs the matrix  $V = (v_i(X_j))$  as its input.

This function simultaneously determines the  $S$ -submodule  $M = \{f \in R : v_i(f) \geq w_i, i = 1, \dots, r\}$  for integers  $w_1, \dots, w_r$ . (If  $w_i \geq 0$  for all  $i$ ,  $M$  is an ideal of  $S$ .) The numbers  $w_i$  form the  $(n + 1)$ th column of the input matrix.

**Note:** The function also gives an error message if the matrix  $V$  has the wrong number of columns.

**Example:**

```
LIB "normaliz.lib";
ring R=0,(x,y,z,w),dp;
intmat V[2][5]=0,1,2,3,4, -1,1,2,1,3;
intersectionValRingIdeals(V);
⇒ [1]:
⇒ _[1]=w
⇒ _[2]=z
⇒ _[3]=y
⇒ _[4]=xw
⇒ _[5]=xz
⇒ _[6]=xy
⇒ _[7]=x2z
⇒ [2]:
⇒ _[1]=w3
⇒ _[2]=zw
⇒ _[3]=z2
⇒ _[4]=yw2
⇒ _[5]=y2w
⇒ _[6]=y2z
⇒ _[7]=y4
⇒ _[8]=xz2
⇒ _[9]=xy2z
⇒ _[10]=xy4
```

See also: [Section D.4.24.12 \[diagInvariants\]](#), page 1220; [Section D.4.24.11 \[finiteDiagInvariants\]](#), page 1219; [Section D.4.24.13 \[intersectionValRings\]](#), page 1221; [Section D.4.24.10 \[torusInvariants\]](#), page 1218.

### D.4.24.15 showNuminvs

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:** `showNuminvs();`

**Purpose:** prints the numerical invariants

**Example:**

```

LIB "normaliz.lib";
ring R=0,(x,y,z,t),dp;
ideal I=x3,x2y,y3;
list l=intclMonIdeal(I);
showNuminvs();
⇒ hilbert_basis_elements : 7
⇒ number_extreme_rays : 5
⇒ dim_max_subspace : 0
⇒ embedding_dim : 4
⇒ rank : 4
⇒ external_index : 1
⇒ internal_index : 1
⇒ number_support_hyperplanes : 5
⇒ size_triangulation : 3
⇒ sum_dets : 4
⇒ integrally_closed : 0
⇒ inhomogeneous : 0
⇒ graded : 1
⇒ degree_1_elements : 7
⇒ grading : 1,1,1,-2
⇒ grading_denom : 1
⇒ multiplicity : 4
⇒ multiplicity_denom : 1
⇒ hilbert_series_num : 1,3
⇒ hilbert_series_denom : 1,1,1,1
⇒ primary : 0
⇒ class_group : 1

```

See also: [Section D.4.24.16 \[exportNuminvs\]](#), page 1223.

#### D.4.24.16 exportNuminvs

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:**      `exportNuminvs();`

**Create:**     Creates top-level variables which contain the numerical invariants. Depending on the options of `normaliz` different invariants are calculated. Use `showNuminvs` ([Section D.4.24.15 \[showNuminvs\]](#), page 1222) to see which invariants are available.

**Example:**

```

LIB "normaliz.lib";
ring R=0,(x,y,z,t),dp;
ideal I=x3,x2y,y3;
list l=intclMonIdeal(I);
exportNuminvs();
// for example, now the following variables are set:
nmz_hilbert_basis_elements;
⇒ 7
nmz_number_extreme_rays;
⇒ 5
nmz_rank;
⇒ 4
nmz_number_support_hyperplanes;
⇒ 5

```

```

nmz_multiplicity;
⇒ 4
nmz_primary;
⇒ 0

```

See also: [Section D.4.24.15 \[showNuminvs\]](#), page 1222.

#### D.4.24.17 allNmzOptions

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:** `proc allNmzOptions();`

**Purpose:** The function prints a list of the Normaliz options that are available in this library: the string naming the option, the default value, and the option sent to Normaliz.

**Example:**

```

LIB "normaliz.lib";
allNmzOptions();
⇒ supp 0 -s
⇒ triang 0 -T
⇒ volume 0 -v
⇒ hvect 0 -p -- name deprecated, use hvect_deg1
⇒ hvect_deg1 0 -p
⇒ only_hvect 0 -q
⇒ fvect 0 --FVector
⇒ height1 0 -1
⇒ Gorenst 0 -G
⇒ intclosed 0 --IsIntegrallyClosed
⇒ witness 0 --w
⇒ classgroup 0 -C
⇒ normal 0 -n -- name deprecated, use hilbbasvol
⇒ hilbbasvol 0 -n
⇒ normal_l 0 -N -- name deprecated, use hilbbas
⇒ hilbbas 0 -N
⇒ hilb 0 -h -- name deprecated, use hilbbas_hvect
⇒ hilbbas_hvect 0 -h
⇒ dual 0 -d
⇒ markov 0 --MarkovBasis --BinomialsPacked
⇒ groebner 0 --GroebnerBasis --BinomialsPacked
⇒ lex 0 --Lex
⇒ revlex 0 --RevLex
⇒ lex 0 --Lex
⇒ deglex 0 --DegLex
⇒ genoverori 0 -M --mod
⇒ type 0 --typ
⇒ control 0 -c -- name deprecated, use verbose
⇒ verbose 0 -c
⇒ allf 0 -a
⇒ bigint 0 -B
⇒ threads 8 -x=
⇒ errorcheck 0 -e -- allowed, but ignored by Normaliz
⇒

```

See also: [Section D.4.24.18 \[setNmzOption\]](#), page 1225; [Section D.4.24.20 \[showNmzOptions\]](#), page 1226.

### D.4.24.18 setNmzOption

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:** `setNmzOption(string s, int onoff);`

**Purpose:** If `onoff=1` the option `s` is activated, and if `onoff=0` it is deactivated. The predefined Normaliz options are accessible via the following names:

```
-s: supp
-t: triang
-v: volume
-p: hvect deprecated, replacement:
-p: hvect_deg1
-q: only_hvect
--FVector: fvect
-l: height1
-G: Gorenst
-w: witness
--IsIntegrallyClosed: intclosed
-C: classgroup
-n: normal deprecated, replacement:
-n: hilbbasvol
-N: normal_l deprecated, replacement:
-N: hilbbas
-h: hilb deprecated, replacement:
-h: hilbbas_hvect
--MarkovBasis: groebner
--GroebnerBasis: markov
--Lex: lex
--RevLex: revlex
--DegLex: deglex
-d: dual
-M: genoverori
-a: allf
--typ: type
-c: control deprecated, replacement:
-c: verbose
-e: errorcheck allowed, but ignored
-B: bigint Use GMP for arbitrary precision integers
-x=N: threads In this case the int parameter is used to set the number of threads N,
default 8, 0 means no explicit limiting.
```

Further Normaliz options can be added to the list by `addNmzOption`.

**Example:**

```
LIB "normaliz.lib";
setNmzOption("only_hvect",1);
↪ 1
showNmzOptions();
↪ -f -q -x=8
setNmzOption("only_hvect",0);
↪ 1
showNmzOptions();
```

⇒ -f -x=8

See also: [Section D.4.24.19 \[addNmzOption\]](#), page 1226; [Section D.4.24.17 \[allNmzOptions\]](#), page 1224; [Section D.4.24.21 \[resetNmzOptions\]](#), page 1226; [Section D.4.24.20 \[showNmzOptions\]](#), page 1226.

### D.4.24.19 addNmzOption

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:**     `addNmzOption(string short_cut, string nmz_option)`

**Note:**     This function allows the addition of Normaliz options in addition to the predefined ones. Adding an option does not activate it.  
               Note: The function prefixes a single letter option by - and multiletter options by --.

**Example:**

```
LIB "normaliz.lib";
addNmzOption("FL", "FaceLattice");
setNmzOption("FL", 1);
⇒ 1
showNmzOptions();
⇒ -f -x=8 --FaceLattice
resetNmzOptions();
```

See also: [Section D.4.24.17 \[allNmzOptions\]](#), page 1224; [Section D.4.24.21 \[resetNmzOptions\]](#), page 1226; [Section D.4.24.18 \[setNmzOption\]](#), page 1225; [Section D.4.24.20 \[showNmzOptions\]](#), page 1226.

### D.4.24.20 showNmzOptions

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:**     `showNmzOptions();`

**Return:**    Returns the string of activated options.

**Note:**     This string is used as parameter when calling Normaliz.

**Example:**

```
LIB "normaliz.lib";
setNmzOption("hilb",1);
⇒ 1
showNmzOptions();
⇒ -f -h -x=8
setNmzOption("hilb",0);
⇒ 1
showNmzOptions();
⇒ -f -x=8
```

See also: [Section D.4.24.18 \[setNmzOption\]](#), page 1225.

### D.4.24.21 resetNmzOptions

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:**     `resetNmzOptions();`

**Purpose:**    Resets the options to the default value.



**Example:**

```

LIB "normaliz.lib";
setNmzOption("only_hvect",1);
↪ 1
showNmzOptions();
↪ -f -q -x=8
resetNmzOptions();
showNmzOptions();
↪ -f -x=8

```

See also: [Section D.4.24.19 \[addNmzOption\]](#), page 1226; [Section D.4.24.17 \[allNmzOptions\]](#), page 1224; [Section D.4.24.18 \[setNmzOption\]](#), page 1225; [Section D.4.24.20 \[showNmzOptions\]](#), page 1226.

**D.4.24.22 normaliz**

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Return:** The function applies Normaliz to the matrix `input_mat` of type `nmz_type`. One can add further pairs "value, type" such as vectors or numerical parameters. See Normaliz manual for possible types and values. The function returns the `intmat` defined by the file `ith` suffix `gen`, provided it has been computed. Otherwise it returns the `input_mat`.

**Note:** You will find procedures for many applications of Normaliz in this library, so the explicit call of this procedure may not be necessary.

**Example:**

```

LIB "normaliz.lib";
ring R=0,(x,y,z),dp;
intmat M[3][2]=3,1,
3,2,
1,3;
normaliz(M,"cone");
↪ 1,1,
↪ 1,2,
↪ 1,3,
↪ 2,1,
↪ 3,1
intmat Inequalities[2][3] = 2,-1,0, // 2x-y >= 0
1, 1,0; // x+y >= 0
intmat Equation[1][3] = 0,1,-1; // y = z
intmat Congruence[1][4] = 1,0,0,3; // x = 0 (3)
normaliz(Inequalities,"inequalities",Equation,"equations",Congruence,"congruences");
↪ 3,-3,-3,
↪ 3,-2,-2,
↪ 3,-1,-1,
↪ 3,0,0,
↪ 3,1,1,
↪ 3,2,2,
↪ 3,3,3,
↪ 3,4,4,
↪ 3,5,5,
↪ 3,6,6

```

See also: [Section D.4.24.12 \[diagInvariants\]](#), page 1220; [Section D.4.24.5 \[ehrhartRing\]](#), page 1214; [Section D.4.24.11 \[finiteDiagInvariants\]](#), page 1219; [Section D.4.24.6 \[intclMonIdeal\]](#), page 1215;

Section D.4.24.1 [intcToricRing], page 1211; Section D.4.24.14 [intersectionValRingIdeals], page 1221; Section D.4.24.13 [intersectionValRings], page 1221; Section D.4.24.2 [normalToricRing], page 1212; Section D.4.24.10 [torusInvariants], page 1218.

### D.4.24.23 setNmzExecPath

Procedure from library `normaliz.lib` (see Section D.4.24 [normaliz.lib], page 1210).

**Usage:** `setNmzExecPath(string s);` `s` path to the Normaliz executable

**Create:** `Normaliz::nmz_exec_path` to save the given path `s`

**Note:** It is not necessary to use this function if the Normaliz executable is in the search path of the system.

**Example:**

```
LIB "normaliz.lib";
setNmzExecPath("../Normaliz/");
```

See also: Section D.4.24.18 [setNmzOption], page 1225.

### D.4.24.24 writeNmzData

Procedure from library `normaliz.lib` (see Section D.4.24 [normaliz.lib], page 1210).

**Usage:** `writeNmzData(intmat M, string nmz_type);`  
`writeNmzData(intmat M, string nmz_type, intmat M2, string nmz_type2, ...);`

**Create:** Creates an input file for Normaliz from the matrix `M`. The second parameter sets the type. How the matrix is interpreted depends on the type. See the Normaliz documentation for more information.

It is also possible to give more than one pair of matrix and type. In

**Note:** Needs an explicit filename set. The filename is created from the current filename. Note that all high level functions in `normaliz.lib` write and read their data automatically to and from the hard disk so that `writeNmzData` will hardly ever be used explicitly.

**Example:**

```
LIB "normaliz.lib";
setNmzFilename("VeryInteresting");
intmat sgr[3][3]=1,2,3,4,5,6,7,8,10;
writeNmzData(sgr,"cone_and_lattice");
int dummy=system("sh","cat VeryInteresting.in");
⇒ amb_space auto
⇒ cone_and_lattice
⇒ [
⇒ [1, 2, 3]
⇒ [4, 5, 6]
⇒ [7, 8, 10]
⇒ ]
intmat Inequalities[2][3] = 2,-1,0, // 2x-y >= 0
1, 1,0; // x+y >= 0
intmat Equation[1][3] = 0,1,-1; // y = z
intmat Congruence[1][4] = 1,0,0,3; // x = 0 (3)
writeNmzData(Inequalities,"inequalities",Equation,"equations",Congruence,"congruences");
dummy=system("sh","cat VeryInteresting.in");
```

```

⇒ amb_space auto
⇒ inequalities
⇒ [
⇒ [2, -1, 0]
⇒ [1, 1, 0]
⇒ ]
⇒ equations
⇒ [
⇒ [0, 1, -1]
⇒ ]
⇒ congruences
⇒ [
⇒ [1, 0, 0, 3]
⇒ ]

```

See also: [Section D.4.24.25 \[readNmzData\]](#), page 1229; [Section D.4.24.30 \[rmNmzFiles\]](#), page 1231; [Section D.4.24.27 \[setNmzDataPath\]](#), page 1230; [Section D.4.24.26 \[setNmzFilename\]](#), page 1230.

### D.4.24.25 readNmzData

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:**      `readNmzData(string suffix);`

**Return:**      Reads an output file of Normaliz containing an integer matrix and returns it as an `intmat`. For example, this function is useful if one wants to inspect the support hyperplanes. The filename is created from the current filename and the suffix given to the function. In addition to file suffixes, also `sup`, `equ` and `cgr` are allowed. They extract the support hyperplanes, equations and congruences, respectively, from the `cst` file.

**Note:**      Needs an explicit filename set by `setNmzFilename`.  
Note that all functions in `normaliz.lib` write and read their data automatically so that `readNmzData` will usually not be used explicitly.  
This function reads only the first matrix in a file! But see `su`, `equ`, `cgr` above.)  
It is the responsibility of the user to make sure that an output file read by this function has been created for the current input file of Normaliz. Files are not automatically removed before a new computation starts.  
Not every output file can be read by this function.

**Example:**

```

LIB "normaliz.lib";
setNmzFilename("VeryInteresting");
intmat sgr[3][3]=1,2,3,4,5,6,7,8,9;
intmat sgrnormal=normaliz(sgr,"cone");
readNmzData("sup");
⇒ -8,7,0,
⇒ 2,-1,0
readNmzData("equ");
⇒ 1,-2,1

```

See also: [Section D.4.24.30 \[rmNmzFiles\]](#), page 1231; [Section D.4.24.27 \[setNmzDataPath\]](#), page 1230; [Section D.4.24.26 \[setNmzFilename\]](#), page 1230; [Section D.4.24.24 \[writeNmzData\]](#), page 1228.

### D.4.24.26 setNmzFilename

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:** `setNmzFilename(string s);`

**Create:** `Normaliz::nmz_filename` to save the given filename `s`

**Note:** The function sets the filename for the exchange of data. Unless a path is set by `setNmzDataPath`, files will be created in the current directory. If a non-empty filename is set, the files created for and by Normaliz are kept. This is mandatory for the data access functions (see [Section D.4.24.24 \[writeNmzData\]](#), page 1228 and [Section D.4.24.25 \[readNmzData\]](#), page 1229). Resetting the filename by `setNmzFilename("")` forces the library to return to deletion of temporary files, but the files created while the filename had been set will not be erased.

**Example:**

```
LIB "normaliz.lib";
setNmzDataPath("examples/");
setNmzFilename("example1");
//now the files for the exchange with Normaliz are examples/example1.SUFFIX
```

See also: [Section D.4.24.25 \[readNmzData\]](#), page 1229; [Section D.4.24.30 \[rmNmzFiles\]](#), page 1231; [Section D.4.24.27 \[setNmzDataPath\]](#), page 1230; [Section D.4.24.24 \[writeNmzData\]](#), page 1228.

### D.4.24.27 setNmzDataPath

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:** `setNmzDataPath(string s);`

**Create:** `Normaliz::nmz_data_path` to save the given path `s`

**Note:** The function sets the path for the exchange of data. By default the files will be created in the current directory. It seems that Singular cannot use filenames starting with `~` or `$HOME` in its input/output functions. You must also avoid path names starting with `/` if you work under Cygwin, since Singular and Normaliz interpret them in different ways.

**Example:**

```
LIB "normaliz.lib";
setNmzDataPath("examples/");
setNmzFilename("example1");
//now the files for the exchange with Normalize are examples/example1.SUFFIX
```

See also: [Section D.4.24.25 \[readNmzData\]](#), page 1229; [Section D.4.24.30 \[rmNmzFiles\]](#), page 1231; [Section D.4.24.26 \[setNmzFilename\]](#), page 1230; [Section D.4.24.24 \[writeNmzData\]](#), page 1228.

### D.4.24.28 writeNmzPaths

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Create:** the file `nmz_sing_exec.path` where the path to the Normaliz executable is saved  
the file `nmz_sing_data.path` where the directory for the exchange of data is saved

**Note:** Both files are saved in the current directory. If one of the names has not been defined, the corresponding file is created, but contains nothing.

**Example:**

```
LIB "normaliz.lib";
setNmzExecPath("../Normaliz/");
writeNmzPaths();
int dummy=system("sh","cat nmz_sing_exec.path");
↳ ../Normaliz/
dummy=system("sh","cat nmz_sing_data.path");
↳
```

See also: [Section D.4.24.27 \[setNmzDataPath\]](#), page 1230; [Section D.4.24.23 \[setNmzExecPath\]](#), page 1228; [Section D.4.24.29 \[startNmz\]](#), page 1231.

**D.4.24.29 startNmz**

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:** `startNmz();`

**Purpose:** This function reads the files written by `writeNmzPaths()`, retrieves the path names, and types them on the standard output (as far as they have been set). Thus, once the path names have been stored, a Normaliz session can simply be opened by this function.

**Example:**

```
LIB "normaliz.lib";
writeNmzPaths();
startNmz();
↳ nmz_exec_path not set
↳ nmz_data_path not set
```

See also: [Section D.4.24.27 \[setNmzDataPath\]](#), page 1230; [Section D.4.24.23 \[setNmzExecPath\]](#), page 1228; [Section D.4.24.28 \[writeNmzPaths\]](#), page 1230.

**D.4.24.30 rmNmzFiles**

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:** `rmNmzFiles();`

**Purpose:** This function removes the files created for and by Normaliz, using the last filename specified. It needs an explicit filename set (see [Section D.4.24.26 \[setNmzFilename\]](#), page 1230).

**Example:**

```
LIB "normaliz.lib";
setNmzFilename("VeryInteresting");
rmNmzFiles();
```

See also: [Section D.4.24.25 \[readNmzData\]](#), page 1229; [Section D.4.24.27 \[setNmzDataPath\]](#), page 1230; [Section D.4.24.26 \[setNmzFilename\]](#), page 1230; [Section D.4.24.24 \[writeNmzData\]](#), page 1228.

**D.4.24.31 mons2intmat**

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:** `mons2intmat(ideal I);`

**Return:** Returns the intmat whose rows represent the leading exponents of the (non-zero) elements of  $I$ . The length of each row is `nvars(basering)`.

**Example:**

```
LIB "normaliz.lib";
ring R=0,(x,y,z),dp;
ideal I=x2,y2,x2yz3;
mons2intmat(I);
↦ 2,0,0,
↦ 0,2,0,
↦ 2,1,3
```

See also: [Section D.4.24.32 \[intmat2mons\]](#), page 1232.

### D.4.24.32 intmat2mons

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:** `intmat2mons(intmat M);`

**Return:** an ideal generated by the monomials which correspond to the exponent vectors given by the rows of  $M$

**Note:** The number of variables in the basering `nvars(basering)` has to be at least the number of columns `ncols(M)`, otherwise the function exits with an error. is thrown (see [Section 5.1.30 \[ERROR\]](#), page 177).

**Example:**

```
LIB "normaliz.lib";
ring R=0,(x,y,z),dp;
intmat expo_vecs[3][3] =
2,0,0,
0,2,0,
2,1,3;
intmat2mons(expo_vecs);
↦ _[1]=x2
↦ _[2]=y2
↦ _[3]=x2yz3
```

See also: [Section D.4.24.31 \[mons2intmat\]](#), page 1231.

### D.4.24.33 binomials2intmat

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:** `binomials2intmat(ideal I);`

**Return:** Returns the intmat whose rows represent the exponents of the (non-zero) elements of  $I$  which have to be binomials.  
The length of each row is `nvars(basering)`.

**Example:**

```
LIB "normaliz.lib";
ring S = 37,(u,v,w,x,y,z),dp;
ideal I = u2v-xyz, ux2-vyz, uvw-y2z;
binomials2intmat(I);
↦ 2,1,0,-1,-1,-1,
```

$\mapsto 1, -1, 0, 2, -1, -1,$   
 $\mapsto 1, 1, 1, 0, -2, -1$

See also: [Section D.4.24.34 \[intmat2binomials\]](#), page 1233; [Section D.4.24.32 \[intmat2mons\]](#), page 1232; [Section D.4.24.31 \[mons2intmat\]](#), page 1231.

### D.4.24.34 intmat2binomials

Procedure from library `normaliz.lib` (see [Section D.4.24 \[normaliz.lib\]](#), page 1210).

**Usage:** `intmat2binomials(intmat M);`

**Return:** an ideal generated by the binomials which correspond to the exponent vectors given by the rows of `M`

**Note:** The number of variables in the basering `nvars(basering)` has to be at least the number of columns `ncols(M)`, otherwise the function exits with an error is thrown (see [Section 5.1.30 \[ERROR\]](#), page 177). The vector with all entries zero represents 1.

**Example:**

```
LIB "normaliz.lib";
ring S = 37,(u,v,w,x,y,z),dp;
intmat M[2][6] = 1,0,1,-1,-2,0, -3,4,5,0,0,1;
intmat2binomials(M);
 $\mapsto$  _[1]=-xy2+uw
 $\mapsto$  _[2]=v4w5z-u3
```

See also: [Section D.4.24.33 \[binomials2intmat\]](#), page 1232.

### D.4.25 pointid\_lib

**Library:** `pointid.lib`

**Purpose:** Procedures for computing a factorized lex GB of the vanishing ideal of a set of points via the Axis-of-Evil Theorem (M.G. Marinari, T. Mora)

**Author:** Stefan Steidel, [steidel@mathematik.uni-kl.de](mailto:steidel@mathematik.uni-kl.de)

**Overview:** The algorithm of Cerlienco-Mureddu [Marinari M.G., Mora T., A remark on a remark by Macaulay or Enhancing Lazard Structural Theorem. Bull. of the Iranian Math. Soc., 29 (2003), 103-145] associates to each ordered set of points  $A := \{a_1, \dots, a_s\}$  in  $K^n$ ,  $a_i := (a_{i1}, \dots, a_{in})$

- a set of monomials  $N$  and

- a bijection  $\phi: A \rightarrow N$ .

Here  $I(A) := \{f \in K[x(1), \dots, x(n)] \mid f(a_i) = 0, \text{ for all } 1 \leq i \leq s\}$  denotes the vanishing ideal of  $A$  and  $N = \text{Mon}(x(1), \dots, x(n))$ .  $\{LM(f) \mid f \in I(A)\}$  is the set of monomials which do not lie in the leading ideal of  $I(A)$  (w.r.t. the lexicographical ordering with  $x(n) > \dots > x(1)$ ).  $N$  is also called the set of non-monomials of  $I(A)$ . NOTE:  $\#A = \#N$  and  $N$  is a monomial basis of  $K[x(1..n)]/I(A)$ . In particular, this allows to deduce the set of corner-monomials, i.e. the minimal basis  $M := \{m_1, \dots, m_r\}$ ,  $m_1 < \dots < m_r$ , of its associated monomial ideal  $M(I(A))$ , such that

$M(I(A)) = \{k * m_i \mid k \in \text{Mon}(x(1), \dots, x(n)), m_i \in M\}$ ,

and (by interpolation) the unique reduced lexicographical Groebner basis  $G := \{f_1, \dots, f_r\}$  such that  $LM(f_i) = m_i$  for each  $i$ , that is,  $I(A) = \langle G \rangle$ . Moreover, a variation of this algorithm allows to deduce a canonical linear factorization of each element of such a Groebner basis in the sense of the Axis-of-Evil Theorem by M.G. Marinari and

T. Mora. More precisely, a combinatorial algorithm and interpolation allow to deduce polynomials

$$y\_mdi = x(m) - g\_mdi(x(1), \dots, x(m-1)),$$

$i=1, \dots, r; m=1, \dots, n; d$  in a finite index-set  $F$ , satisfying

$$f_i = (\text{product of } y\_mdi) \text{ modulo } (f_1, \dots, f_{i-1})$$

where the product runs over all  $m=1, \dots, n$ ; and all  $d$  in  $F$ .

#### Procedures:

##### D.4.25.1 nonMonomials

Procedure from library `pointid.lib` (see [Section D.4.25 \[pointid.lib\]](#), page 1233).

- Usage:** `nonMonomials(id)`;  $id = \langle \text{list of vectors} \rangle$  or  $\langle \text{list of lists} \rangle$  or  $\langle \text{module} \rangle$  or  $\langle \text{matrix} \rangle$ .  
 Let  $A = \{a_1, \dots, a_s\}$  be a set of points in  $K^n$ ,  $a_i := (a_{i1}, \dots, a_{in})$ , then  $A$  can be given as
- a list of vectors (the  $a_i$  are vectors) or
  - a list of lists (the  $a_i$  are lists of numbers) or
  - a module s.t. the  $a_i$  are generators or
  - a matrix s.t. the  $a_i$  are columns
- Assume:** basering must have ordering `ip`, i.e., be of the form `0,x(1..n),ip`; (the first entry of a point belongs to the lex-smallest variable, etc.)
- Return:** ideal, the non-monomials of the vanishing ideal  $I(A)$  of  $A$
- Purpose:** compute the set of non-monomials  $\text{Mon}(x(1), \dots, x(n)) \setminus \{LM(f) \mid f \in I(A)\}$  of the vanishing ideal  $I(A)$  of the given set of points  $A$  in  $K^n$ , where  $K[x(1), \dots, x(n)]$  is equipped with the lexicographical ordering induced by  $x(1) < \dots < x(n)$  by using the algorithm of Cerlienco-Mureddu

#### Example:

```
LIB "pointid.lib";
ring R1 = 0,x(1..3),ip;
vector a1 = [4,0,0];
vector a2 = [2,1,4];
vector a3 = [2,4,0];
vector a4 = [3,0,1];
vector a5 = [2,1,3];
vector a6 = [1,3,4];
vector a7 = [2,4,3];
vector a8 = [2,4,2];
vector a9 = [1,0,2];
list A = a1,a2,a3,a4,a5,a6,a7,a8,a9;
nonMonomials(A);
⇒ _[1]=1
⇒ _[2]=x(1)
⇒ _[3]=x(2)
⇒ _[4]=x(1)^2
⇒ _[5]=x(3)
⇒ _[6]=x(1)^3
⇒ _[7]=x(2)*x(3)
⇒ _[8]=x(3)^2
⇒ _[9]=x(1)*x(2)
```



```

matrix MAT[9][3] = 4,0,0,2,1,4,2,4,0,3,0,1,2,1,3,1,3,4,2,4,3,2,4,2,1,0,2;
MAT = transpose(MAT);
print(MAT);
↳ 4,2,2,3,2,1,2,2,1,
↳ 0,1,4,0,1,3,4,4,0,
↳ 0,4,0,1,3,4,3,2,2
nonMonomials(MAT);
↳ _[1]=1
↳ _[2]=x(1)
↳ _[3]=x(2)
↳ _[4]=x(1)^2
↳ _[5]=x(3)
↳ _[6]=x(1)^3
↳ _[7]=x(2)*x(3)
↳ _[8]=x(3)^2
↳ _[9]=x(1)*x(2)
module MOD = gen(3),gen(2)-2*gen(3),2*gen(1)+2*gen(3),2*gen(2)-2*gen(3),gen(1)+3*gen(2);
print(MOD);
↳ 0,0, 2,0, 1,1,1,
↳ 0,1, 0,2, 0,1,1,
↳ 1,-2,2,-2,3,3,1
nonMonomials(MOD);
↳ _[1]=1
↳ _[2]=x(2)
↳ _[3]=x(1)
↳ _[4]=x(2)^2
↳ _[5]=x(1)^2
↳ _[6]=x(1)*x(2)
↳ _[7]=x(3)
ring R2 = 0,x(1..2),ip;
list l1 = 0,0;
list l2 = 0,1;
list l3 = 2,0;
list l4 = 0,2;
list l5 = 1,0;
list l6 = 1,1;
list L = l1,l2,l3,l4,l5,l6;
nonMonomials(L);
↳ _[1]=1
↳ _[2]=x(2)
↳ _[3]=x(1)
↳ _[4]=x(2)^2
↳ _[5]=x(1)^2
↳ _[6]=x(1)*x(2)

```

### D.4.25.2 cornerMonomials

Procedure from library `pointid.lib` (see [Section D.4.25 \[pointid.lib\]](#), page 1233).

**Usage:**      `cornerMonomials(N);` N ideal

**Assume:**    N is given by monomials satisfying the condition that if a monomial is in N then any of its factors is in N (N is then called an order ideal)

**Return:** ideal, the corner-monomials of the order ideal N  
The corner-monomials are the leading monomials of an ideal I s.t. N is a basis of basering/I.

**Note:** In our applications, I is the vanishing ideal of a finite set of points.

**Example:**

```
LIB "pointid.lib";
ring R = 0,x(1..3),ip;
poly n1 = 1;
poly n2 = x(1);
poly n3 = x(2);
poly n4 = x(1)^2;
poly n5 = x(3);
poly n6 = x(1)^3;
poly n7 = x(2)*x(3);
poly n8 = x(3)^2;
poly n9 = x(1)*x(2);
ideal N = n1,n2,n3,n4,n5,n6,n7,n8,n9;
cornerMonomials(N);
↦ _[1]=x(1)^4
↦ _[2]=x(1)^2*x(2)
↦ _[3]=x(2)^2
↦ _[4]=x(1)*x(3)
↦ _[5]=x(2)*x(3)^2
↦ _[6]=x(3)^3
```

### D.4.25.3 facGBIdeal

Procedure from library `pointid.lib` (see [Section D.4.25 \[pointid.lib\]](#), page 1233).

**Usage:** `facGBIdeal(id)`; `id = <list of vectors>` or `<list of lists>` or `<module>` or `<matrix>`.  
Let  $A = \{a_1, \dots, a_s\}$  be a set of points in  $K^n$ ,  $a_i = (a_{i1}, \dots, a_{in})$ , then A can be given as

- a list of vectors (the  $a_i$  are vectors) or
- a list of lists (the  $a_i$  are lists of numbers) or
- a module s.t. the  $a_i$  are generators or
- a matrix s.t. the  $a_i$  are columns

**Assume:** basering must have ordering `ip`, i.e., be of the form `0,x(1..n),ip`; (the first entry of a point belongs to the lex-smallest variable, etc.)

**Return:** a list where the first entry contains the Groebner basis G of  $I(A)$  and the second entry contains the linear factors of each element of G

**Note:** combinatorial algorithm due to the Axis-of-Evil Theorem of M.G. Marinari, T. Mora

**Example:**

```
LIB "pointid.lib";
ring R = 0,x(1..3),ip;
vector a1 = [4,0,0];
vector a2 = [2,1,4];
vector a3 = [2,4,0];
vector a4 = [3,0,1];
vector a5 = [2,1,3];
vector a6 = [1,3,4];
vector a7 = [2,4,3];
```

```

vector a8 = [2,4,2];
vector a9 = [1,0,2];
list A = a1,a2,a3,a4,a5,a6,a7,a8,a9;
facGBIdeal(A);
↳ [1]:
↳ _[1]=x(1)^4-10*x(1)^3+35*x(1)^2-50*x(1)+24
↳ _[2]=x(1)^2*x(2)-3*x(1)*x(2)+2*x(2)
↳ _[3]=x(2)^2-1/2*x(1)^2*x(2)-1/2*x(1)*x(2)-2*x(2)+2*x(1)^3-16*x(1)^2+38\
*x(1)-24
↳ _[4]=x(1)*x(3)-2*x(3)-2/3*x(1)*x(2)+4/3*x(2)+1/6*x(1)^3-1/2*x(1)^2-5/3\
*x(1)+4
↳ _[5]=x(2)*x(3)^2-4*x(3)^2-2/3*x(2)^2*x(3)-5/6*x(1)^3*x(2)*x(3)+41/6*x(\
1)^2*x(2)*x(3)-16*x(1)*x(2)*x(3)+23/3*x(2)*x(3)+10/3*x(1)^3*x(3)-82/3*x(1\
)^2*x(3)+64*x(1)*x(3)-20*x(3)+2*x(2)^2+5/2*x(1)^3*x(2)-41/2*x(1)^2*x(2)+4\
8*x(1)*x(2)-32*x(2)-10*x(1)^3+82*x(1)^2-192*x(1)+96
↳ _[6]=x(3)^3+4/3*x(2)*x(3)^2-5/6*x(1)^3*x(3)^2+35/6*x(1)^2*x(3)^2-9*x(1\
)*x(3)^2-9*x(3)^2-20/3*x(2)*x(3)+25/6*x(1)^3*x(3)-175/6*x(1)^2*x(3)+45*x(\
1)*x(3)+26*x(3)+8*x(2)-5*x(1)^3+35*x(1)^2-54*x(1)-24
↳ [2]:
↳ [1]:
↳ _[1]=x(1)-4
↳ _[2]=x(1)-2
↳ _[3]=x(1)-3
↳ _[4]=x(1)-1
↳ [2]:
↳ _[1]=x(1)-2
↳ _[2]=x(1)-1
↳ _[3]=x(2)
↳ [3]:
↳ _[1]=x(2)-4*x(1)+4
↳ _[2]=2*x(2)-x(1)^2+7*x(1)-12
↳ [4]:
↳ _[1]=x(1)-2
↳ _[2]=6*x(3)-4*x(2)+x(1)^2-x(1)-12
↳ [5]:
↳ _[1]=x(2)-4
↳ _[2]=x(3)-3
↳ _[3]=6*x(3)-4*x(2)-5*x(1)^3+41*x(1)^2-96*x(1)+48
↳ [6]:
↳ _[1]=x(3)-2
↳ _[2]=x(3)-3
↳ _[3]=6*x(3)+8*x(2)-5*x(1)^3+35*x(1)^2-54*x(1)-24
matrix MAT[9][3] = 4,0,0,2,1,4,2,4,0,3,0,1,2,1,3,1,3,4,2,4,3,2,4,2,1,0,2;
MAT = transpose(MAT);
print(MAT);
↳ 4,2,2,3,2,1,2,2,1,
↳ 0,1,4,0,1,3,4,4,0,
↳ 0,4,0,1,3,4,3,2,2
facGBIdeal(MAT);
↳ [1]:
↳ _[1]=x(1)^4-10*x(1)^3+35*x(1)^2-50*x(1)+24
↳ _[2]=x(1)^2*x(2)-3*x(1)*x(2)+2*x(2)
↳ _[3]=x(2)^2-1/2*x(1)^2*x(2)-1/2*x(1)*x(2)-2*x(2)+2*x(1)^3-16*x(1)^2+38\

```

```

*x(1)-24
⇒ _[4]=x(1)*x(3)-2*x(3)-2/3*x(1)*x(2)+4/3*x(2)+1/6*x(1)^3-1/2*x(1)^2-5/3\
*x(1)+4
⇒ _[5]=x(2)*x(3)^2-4*x(3)^2-2/3*x(2)^2*x(3)-5/6*x(1)^3*x(2)*x(3)+41/6*x(\
1)^2*x(2)*x(3)-16*x(1)*x(2)*x(3)+23/3*x(2)*x(3)+10/3*x(1)^3*x(3)-82/3*x(1\
)^2*x(3)+64*x(1)*x(3)-20*x(3)+2*x(2)^2+5/2*x(1)^3*x(2)-41/2*x(1)^2*x(2)+4\
8*x(1)*x(2)-32*x(2)-10*x(1)^3+82*x(1)^2-192*x(1)+96
⇒ _[6]=x(3)^3+4/3*x(2)*x(3)^2-5/6*x(1)^3*x(3)^2+35/6*x(1)^2*x(3)^2-9*x(1\
)*x(3)^2-9*x(3)^2-20/3*x(2)*x(3)+25/6*x(1)^3*x(3)-175/6*x(1)^2*x(3)+45*x(\
1)*x(3)+26*x(3)+8*x(2)-5*x(1)^3+35*x(1)^2-54*x(1)-24
⇒ [2]:
⇒ [1]:
⇒ _[1]=x(1)-4
⇒ _[2]=x(1)-2
⇒ _[3]=x(1)-3
⇒ _[4]=x(1)-1
⇒ [2]:
⇒ _[1]=x(1)-2
⇒ _[2]=x(1)-1
⇒ _[3]=x(2)
⇒ [3]:
⇒ _[1]=x(2)-4*x(1)+4
⇒ _[2]=2*x(2)-x(1)^2+7*x(1)-12
⇒ [4]:
⇒ _[1]=x(1)-2
⇒ _[2]=6*x(3)-4*x(2)+x(1)^2-x(1)-12
⇒ [5]:
⇒ _[1]=x(2)-4
⇒ _[2]=x(3)-3
⇒ _[3]=6*x(3)-4*x(2)-5*x(1)^3+41*x(1)^2-96*x(1)+48
⇒ [6]:
⇒ _[1]=x(3)-2
⇒ _[2]=x(3)-3
⇒ _[3]=6*x(3)+8*x(2)-5*x(1)^3+35*x(1)^2-54*x(1)-24
module MOD = gen(3),gen(2)-2*gen(3),2*gen(1)+2*gen(3),2*gen(2)-2*gen(3),gen(1)+3*gen
print(MOD);
⇒ 0,0, 2,0, 1,1,1,
⇒ 0,1, 0,2, 0,1,1,
⇒ 1,-2,2,-2,3,3,1
facGBIdeal(MOD);
⇒ [1]:
⇒ _[1]=x(1)^3-3*x(1)^2+2*x(1)
⇒ _[2]=x(1)^2*x(2)-x(1)*x(2)
⇒ _[3]=x(1)*x(2)^2-x(1)*x(2)
⇒ _[4]=x(2)^3-3*x(2)^2+2*x(2)
⇒ _[5]=x(1)*x(3)-x(3)-3/2*x(1)*x(2)^2+3/2*x(2)^2+9/2*x(1)*x(2)-9/2*x(2)-\
1/2*x(1)^2-1/2*x(1)+1
⇒ _[6]=x(2)*x(3)-x(3)+3/2*x(2)^2+3/2*x(1)^2*x(2)-7/2*x(1)*x(2)-5/2*x(2)-\
3/2*x(1)^2+7/2*x(1)+1
⇒ _[7]=x(3)^2+3*x(1)^2*x(3)-8*x(1)*x(3)+x(3)-3*x(1)^2+8*x(1)-2
⇒ [2]:
⇒ [1]:
⇒ _[1]=x(1)

```

```

↳      _[2]=x(1)-2
↳      _[3]=x(1)-1
↳      [2]:
↳      _[1]=x(1)
↳      _[2]=x(1)-1
↳      _[3]=x(2)
↳      [3]:
↳      _[1]=x(1)
↳      _[2]=x(2)-1
↳      _[3]=x(2)
↳      [4]:
↳      _[1]=x(2)-2
↳      _[2]=x(2)-1
↳      _[3]=x(2)
↳      [5]:
↳      _[1]=x(1)-1
↳      _[2]=2*x(3)-3*x(2)^2+9*x(2)-x(1)-2
↳      [6]:
↳      _[1]=x(2)-1
↳      _[2]=2*x(3)+3*x(2)+3*x(1)^2-7*x(1)-2
↳      [7]:
↳      _[1]=x(3)-1
↳      _[2]=x(3)+3*x(1)^2-8*x(1)+2
list l1 = 0,0,1;
list l2 = 0,1,-2;
list l3 = 2,0,2;
list l4 = 0,2,-2;
list l5 = 1,0,3;
list l6 = 1,1,3;
list L = l1,l2,l3,l4,l5,l6;
facGBIdeal(L);
↳ [1]:
↳      _[1]=x(1)^3-3*x(1)^2+2*x(1)
↳      _[2]=x(1)^2*x(2)-x(1)*x(2)
↳      _[3]=x(1)*x(2)^2-x(1)*x(2)
↳      _[4]=x(2)^3-3*x(2)^2+2*x(2)
↳      _[5]=x(3)-3/2*x(2)^2-3*x(1)*x(2)+9/2*x(2)+3/2*x(1)^2-7/2*x(1)-1
↳ [2]:
↳      [1]:
↳      _[1]=x(1)
↳      _[2]=x(1)-2
↳      _[3]=x(1)-1
↳      [2]:
↳      _[1]=x(1)
↳      _[2]=x(1)-1
↳      _[3]=x(2)
↳      [3]:
↳      _[1]=x(1)
↳      _[2]=x(2)-1
↳      _[3]=x(2)
↳      [4]:
↳      _[1]=x(2)-2
↳      _[2]=x(2)-1

```

```

⇒      _[3]=x(2)
⇒      [5]:
⇒      _[1]=2*x(3)-3*x(2)^2-6*x(1)*x(2)+9*x(2)+3*x(1)^2-7*x(1)-2

```

### D.4.26 primdec\_lib

**Library:** primdec.lib

**Purpose:** Primary Decomposition and Radical of Ideals

**Authors:** Gerhard Pfister, pfister@mathematik.uni-kl.de (GTZ)  
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**Overview:** Algorithms for primary decomposition based on the ideas of Gianni, Trager and Zacharias (implementation by Gerhard Pfister), respectively based on the ideas of Shimoyama and Yokoyama (implementation by Wolfram Decker and Hans Schoenemann). The procedures are implemented to be used in characteristic 0. They also work in positive characteristic  $>> 0$ . In small characteristic and for algebraic extensions, primdecGTZ may not terminate. Algorithms for the computation of the radical based on the ideas of Krick, Logar, Laplagne and Kemper (implementation by Gerhard Pfister and Santiago Laplagne). They work in any characteristic. Baserings must have a global ordering and no quotient ideal. Exceptions: primdecGTZ, absPrimdecGTZ, minAssGTZ, primdecSY, minAssChar, radical accept non-global ordering.

**Procedures:** See also: [Section D.4.27 \[primdecint\\_lib\]](#), page 1253.

#### D.4.26.1 Ann

Procedure from library `primdec.lib` (see [Section D.4.26 \[primdec\\_lib\]](#), page 1240).

**Usage:** Ann(M); M module

**Return:** ideal, the annihilator of `coker(M)`

**Note:** The output is the ideal of all elements  $a$  of the basering  $R$  such that  $a * R^m$  is contained in  $M$  ( $m$ =number of rows of  $M$ ).

**Example:**

```

LIB "primdec.lib";
ring r = 0,(x,y,z),lp;
module M = x2-y2,z3;
Ann(M);
⇒ _[1]=z3
⇒ _[2]=x2-y2
M = [1,x2],[y,x];
Ann(M);
⇒ _[1]=x2y-x
qring Q=std(xy-1);
module M=imap(r,M);
Ann(M);
⇒ _[1]=0

```

### D.4.26.2 primdecGTZ

Procedure from library `primdec.lib` (see [Section D.4.26 \[primdec.lib\]](#), page 1240).

**Usage:** `primdecGTZ(I); I ideal`

**Return:** a list `pr` of primary ideals and their associated primes for a proper ideal `I`, otherwise  
`pr = list( list( ideal(1), ideal(1) )`  
`pr[i][1]` the *i*-th primary component,  
`pr[i][2]` the *i*-th prime component.

**Note:**

- Algorithm of Gianni/Trager/Zacharias.
- Designed for characteristic 0, works also in char  $k > 0$ , if it terminates (may result in an infinite loop in small characteristic!) - For local orderings, the result is considered in the localization of the polynomial ring, not in the power series ring
- For local and mixed orderings, the decomposition in the corresponding global ring is returned if the string 'global' is specified as second argument

**Example:**

```
LIB "primdec.lib";
ring r = 0,(x,y,z),lp;
poly p = z2+1;
poly q = z3+2;
ideal i = p*q^2,y-z2;
list pr = primdecGTZ(i);
pr;
⇒ [1]:
⇒ [1]:
⇒ _[1]=z2+1
⇒ _[2]=y-z2
⇒ [2]:
⇒ _[1]=z2+1
⇒ _[2]=y-z2
⇒ [2]:
⇒ [1]:
⇒ _[1]=z6+4z3+4
⇒ _[2]=y-z2
⇒ [2]:
⇒ _[1]=z3+2
⇒ _[2]=y-z2
```

### D.4.26.3 primdecGTZE

Procedure from library `primdec.lib` (see [Section D.4.26 \[primdec.lib\]](#), page 1240).

**Usage:** `primdecGTZE(I); i ideal`

**Return:** a list `pr` of primary ideals and their associated primes for a proper ideal, and an empty list for the unit ideal.  
`pr[i][1]` the *i*-th primary component,  
`pr[i][2]` the *i*-th prime component.

**Note:**

- Algorithm of Gianni/Trager/Zacharias.
- Designed for characteristic 0, works also in char  $k > 0$ , if it terminates (may result in an infinite loop in small characteristic!) - For local orderings, the result is considered

in the localization of the polynomial ring, not in the power series ring  
 - For local and mixed orderings, the decomposition in the corresponding global ring is returned if the string 'global' is specified as second argument

**Example:**

```
LIB "primdec.lib";
ring r = 0, (x,y,z), lp;
poly p = z2+1;
poly q = z3+2;
ideal I = p*q^2,y-z2;
list pr = primdecGTZE(I);
pr;
⇒ [1]:
⇒ [1]:
⇒ [1]=z2+1
⇒ [2]=y-z2
⇒ [2]:
⇒ [1]=z2+1
⇒ [2]=y-z2
⇒ [2]:
⇒ [1]:
⇒ [1]=z6+4z3+4
⇒ [2]=y-z2
⇒ [2]:
⇒ [1]=z3+2
⇒ [2]=y-z2
ideal J = 1;
list premtyp = primdecGTZE(J);
premtyp;
⇒ empty list
```

**D.4.26.4 primdecSY**

Procedure from library `primdec.lib` (see [Section D.4.26 \[primdec.lib\]](#), page 1240).

**Usage:** `primdecSY(I, c);` I ideal, c int (optional)

**Return:** a list pr of primary ideals and their associated primes for proper ideal I, otherwise pr[1] is list( ideal(1),ideal(1) )'  
     pr[i][1] the i-th primary component,  
     pr[i][2] the i-th prime component.

**Note:** Algorithm of Shimoyama/Yokoyama.

if c=0, the given ordering of the variables is used,  
 if c=1, minAssChar tries to use an optimal ordering (default),  
 if c=2, minAssGTZ is used,  
 if c=3, minAssGTZ and facstd are used.

For local orderings, the result is considered in the localization of the polynomial ring, not in the power series ring.

For local and mixed orderings, the decomposition in the corresponding global ring is returned if the string 'global' is specified as third argument

**Example:**



```

LIB "primdec.lib";
ring r = 0,(x,y,z),lp;
poly p = z2+1;
poly q = z3+2;
ideal i = p*q^2,y-z2;
list pr = primdecSY(i);
pr;
⇒ [1]:
⇒ [1]:
⇒ [1]=z6+4z3+4
⇒ [2]=y-z2
⇒ [2]:
⇒ [1]=z3+2
⇒ [2]=y-z2
⇒ [2]:
⇒ [1]:
⇒ [1]=z2+1
⇒ [2]=y-z2
⇒ [2]:
⇒ [1]=z2+1
⇒ [2]=y+1

```

#### D.4.26.5 primdecSYE

Procedure from library `primdec.lib` (see [Section D.4.26 \[primdec.lib\]](#), page 1240).

**Usage:** `primdecSYE(I, c);` I ideal, c int (optional)

**Return:** a list pr of primary ideals and their associated primes:

pr[i][1] the i-th primary component,  
pr[i][2] the i-th prime component.

If I is the unit ideal returns an empty list.

**Note:** Algorithm of Shimoyama/Yokoyama.

if c=0, the given ordering of the variables is used,  
if c=1, `minAssChar` tries to use an optimal ordering (default),  
if c=2, `minAssGTZ` is used,  
if c=3, `minAssGTZ` and `facstd` are used.

For local orderings, the result is considered in the localization of the polynomial ring, not in the power series ring.

For local and mixed orderings, the decomposition in the corresponding global ring is returned if the string 'global' is specified as third argument

**Example:**

```

LIB "primdec.lib";
ring r = 0,(x,y,z),lp;
poly p = z2+1;
poly q = z3+2;
ideal I = p*q^2,y-z2;
list pr = primdecSYE(I);
pr;
⇒ [1]:
⇒ [1]:
⇒ [1]=z6+4z3+4

```

```

↳      _[2]=y-z2
↳      [2]:
↳      _[1]=z3+2
↳      _[2]=y-z2
↳ [2]:
↳      [1]:
↳      _[1]=z2+1
↳      _[2]=y-z2
↳      [2]:
↳      _[1]=z2+1
↳      _[2]=y+1
ideal J = x;
list prUnit = primdecSYE(J);
prUnit;
↳ [1]:
↳      [1]:
↳      _[1]=x
↳      [2]:
↳      _[1]=x

```

#### D.4.26.6 minAssGTZ

Procedure from library `primdec.lib` (see [Section D.4.26 \[primdec.lib\]](#), page 1240).

**Usage:**     `minAssGTZ(I[, l]);` I ideal, l list (optional)  
               Optional parameters in list l (can be entered in any order):  
               0, "facstd" -> uses facstd to first decompose the ideal (default)  
               1, "noFacstd" -> does not use facstd  
               "GTZ" -> the original algorithm by Gianni, Trager and Zacharias is used  
               "SL" -> GTZ algorithm with modifications by Laplagne is used (default)

**Return:**    a list, the minimal associated prime ideals of proper ideal I, otherwise ideal(1)

**Note:**       - Designed for characteristic 0, works also in char  $k > 0$  based on an algorithm of Yokoyama  
               - For local orderings, the result is considered in the localization of the polynomial ring, not in the power series ring  
               - For local and mixed orderings, the decomposition in the corresponding global ring is returned if the string 'global' is specified as second argument

**Example:**

```

LIB "primdec.lib";
ring r = 0,(x,y,z),dp;
poly p = z2+1;
poly q = z3+2;
ideal i = p*q^2,y-z2;
list pr = minAssGTZ(i);
pr;
↳ [1]:
↳      _[1]=z3+2
↳      _[2]=-z2+y
↳ [2]:
↳      _[1]=z2+1
↳      _[2]=-z2+y

```

### D.4.26.7 minAssGTZE

Procedure from library `primdec.lib` (see [Section D.4.26 \[primdec.lib\]](#), page 1240).

**Usage:**     `minAssGTZE(I[, l]);` I ideal, l list (optional)  
               Optional parameters in list l (can be entered in any order):  
               0, "facstd" -> uses facstd to first decompose the ideal (default)  
               1, "noFacstd" -> does not use facstd  
               "GTZ" -> the original algorithm by Gianni, Trager and Zacharias is used  
               "SL" -> GTZ algorithm with modifications by Laplagne is used (default)

**Return:**    a list, the minimal associated prime ideals of I.

**Note:**     - Designed for characteristic 0, works also in char  $k > 0$  based on an algorithm of Yokoyama  
               - For local orderings, the result is considered in the localization of the polynomial ring, not in the power series ring  
               - For local and mixed orderings, the decomposition in the corresponding global ring is returned if the string 'global' is specified as second argument

**Example:**

```
LIB "primdec.lib";
ring r = 0,(x,y,z),dp;
poly p = z2+1;
poly q = z3+2;
ideal I = p*q^2,y-z2;
list pr = minAssGTZE(I);
pr;
↳ [1]:
↳   _[1]=z3+2
↳   _[2]=-z2+y
↳ [2]:
↳   _[1]=z2+1
↳   _[2]=-z2+y
ideal J = 1;
list preempty = minAssGTZE(J);
preempty;
↳ empty list
```

### D.4.26.8 minAssChar

Procedure from library `primdec.lib` (see [Section D.4.26 \[primdec.lib\]](#), page 1240).

**Usage:**     `minAssChar(I[,c]);` i ideal, c int (optional).

**Return:**    list, the minimal associated prime ideals of I. If I is the unit ideal returns list( ideal(1) )

**Note:**     If  $c=0$ , the given ordering of the variables is used.  
               Otherwise, the system tries to find an optimal ordering, which in some cases may considerably speed up the algorithm.  
               For local orderings, the result is considered in the localization of the polynomial ring, not in the power series ring  
               For local and mixed orderings, the decomposition in the corresponding global ring is returned if the string 'global' is specified as third argument

**Example:**

```

LIB "primdec.lib";
ring r = 0, (x,y,z), dp;
poly p = z2+1;
poly q = z3+2;
ideal i = p*q^2, y-z2;
list pr = minAssChar(i);
pr;
⇒ [1]:
⇒   _[1]=y+1
⇒   _[2]=z2+1
⇒ [2]:
⇒   _[1]=z2-y
⇒   _[2]=yz+2
⇒   _[3]=y2+2z

```

**D.4.26.9 minAssCharE**

Procedure from library `primdec.lib` (see [Section D.4.26 \[primdec.lib\]](#), page 1240).

**Usage:** `minAssCharE(I[,c]);` *i* ideal, *c* int (optional).

**Return:** list, the minimal associated prime ideals of *I*. If *I* is the unit ideal returns an empty list.

**Note:** If *c*=0, the given ordering of the variables is used.  
 Otherwise, the system tries to find an optimal ordering, which in some cases may considerably speed up the algorithm.  
 For local orderings, the result is considered in the localization of the polynomial ring, not in the power series ring  
 For local and mixed orderings, the decomposition in the corresponding global ring is returned if the string 'global' is specified as third argument

**Example:**

```

LIB "primdec.lib";
ring r = 0, (x,y,z), dp;
poly p = z2+1;
poly q = z3+2;
ideal I = p*q^2, y-z2;
list pr = minAssCharE(I);
pr;
⇒ [1]:
⇒   _[1]=y+1
⇒   _[2]=z2+1
⇒ [2]:
⇒   _[1]=z2-y
⇒   _[2]=yz+2
⇒   _[3]=y2+2z
ideal J = 5;
list premtyp = minAssCharE(J);
premtyp;
⇒ empty list

```

### D.4.26.10 testPrimary

Procedure from library `primdec.lib` (see [Section D.4.26 \[primdec.lib\]](#), page 1240).

**Usage:** `testPrimary(pr,k)`; `pr` a list, `k` an ideal.

**Assume:** `pr` is the result of `primdecGTZ(k)` or `primdecSY(k)`.

**Return:** `int`, 1 if the intersection of the ideals in `pr` is `k`, 0 if not

**Example:**

```
LIB "primdec.lib";
ring r = 32003,(x,y,z),dp;
poly p = z2+1;
poly q = z4+2;
ideal i = p^2*q^3,(y-z3)^3,(x-yz+z4)^4;
list pr = primdecGTZ(i);
testPrimary(pr,i);
↪ 1
```

### D.4.26.11 testPrimaryE

Procedure from library `primdec.lib` (see [Section D.4.26 \[primdec.lib\]](#), page 1240).

**Usage:** `testPrimaryE(pr,k)`; `pr` a list, `k` an ideal.

**Assume:** `pr` is the result of a primary decomposition and may be empty ( for the unit ideal)

**Return:** `int`, 1 if the intersection of the ideals in `pr` is `k`, 0 if not

**Example:**

```
LIB "primdec.lib";
ring r = 32003,(x,y,z),dp;
poly p = z2+1;
poly q = z4+2;
ideal i = p^2*q^3,(y-z3)^3,(x-yz+z4)^4;
list pr = primdecGTZ(i);
testPrimaryE(pr,i);
↪ 1
```

### D.4.26.12 radical

Procedure from library `primdec.lib` (see [Section D.4.26 \[primdec.lib\]](#), page 1240).

**Usage:** `radical(I[, l])`; `I` ideal, `l` list (optional)

Optional parameters in list `l` (can be entered in any order):

0, "fullRad" -> full radical is computed (default)

1, "equiRad" -> equiRadical is computed

"KL" -> Krick/Logar algorithm is used

"SL" -> modifications by Laplagne are used (default)

"facstd" -> uses facstd to first decompose the ideal (default for non homogeneous ideals)

"noFacstd" -> does not use facstd (default for homogeneous ideals)

**Return:** ideal, the radical of `I` (or the equiradical if required in the input parameters)

**Note:** A combination of the algorithms of Krick/Logar (with modifications by Laplagne) and Kemper is used. Works also in positive characteristic (Kempers algorithm).

**Example:**

```

LIB "primdec.lib";
ring r = 0,(x,y,z),dp;
poly p = z2+1;
poly q = z3+2;
ideal i = p*q^2,y-z2;
ideal pr = radical(i);
pr;
↪ pr[1]=z2-y
↪ pr[2]=y2z+z3+2z2+2

```

**D.4.26.13 radicalEHV**

Procedure from library `primdec.lib` (see [Section D.4.26 \[primdec.lib\]](#), page 1240).

**Usage:** `radicalEHV(i); i ideal.`

**Return:** ideal, the radical of `i`.

**Note:** Uses the algorithm of Eisenbud/Huneke/Vasconcelos, which reduces the computation to the complete intersection case, by taking, in the general case, a generic linear combination of the input.  
Works only in characteristic 0 or  $p$  large.

**Example:**

```

LIB "primdec.lib";
ring r = 0,(x,y,z),dp;
poly p = z2+1;
poly q = z3+2;
ideal i = p*q^2,y-z2;
ideal pr= radicalEHV(i);
pr;
↪ pr[1]=z2-y
↪ pr[2]=y2z+yz+2y+2
↪ pr[3]=y3+y2+2yz+2z

```

**D.4.26.14 equiRadical**

Procedure from library `primdec.lib` (see [Section D.4.26 \[primdec.lib\]](#), page 1240).

**Usage:** `equiRadical(I); I ideal`

**Return:** ideal, intersection of associated primes of `I` of maximal dimension.

**Note:** A combination of the algorithms of Krick/Logar (with modifications by Laplagne) and Kemper is used. Works also in positive characteristic (Kempers algorithm).

**Example:**

```

LIB "primdec.lib";
ring r = 0,(x,y,z),dp;
poly p = z2+1;
poly q = z3+2;
ideal i = p*q^2,y-z2;
ideal pr= equiRadical(i);
pr;
↪ pr[1]=z2-y
↪ pr[2]=y2z+z3+2z2+2

```

### D.4.26.15 prepareAss

Procedure from library `primdec.lib` (see [Section D.4.26 \[primdec.lib\], page 1240](#)).

**Usage:**      `prepareAss(I)`;  $I$  ideal

**Return:**     list of radicals of the equidimensional components of  $I$

**Note:**       Uses algorithm of Eisenbud/Huneke/Vasconcelos.

**Example:**

```
LIB "primdec.lib";
ring r = 0,(x,y,z),dp;
poly p = z2+1;
poly q = z3+2;
ideal i = p*q^2,y-z2;
list pr = prepareAss(i);
pr;
⇒ [1]:
⇒   _[1]=z2-y
⇒   _[2]=y2z+z3+2z2+2
```

### D.4.26.16 equidim

Procedure from library `primdec.lib` (see [Section D.4.26 \[primdec.lib\], page 1240](#)).

**Usage:**      `equidim(I)` or `equidim(I,1)` ;  $I$  ideal

**Return:**     list of equidimensional ideals  $a[1], \dots, a[s]$  with:  
 -  $a[s]$  the equidimensional locus of  $I$ , i.e. the intersection of the primary ideals of dimension of  $I$ , except  $I$  is unit ideal. -  $a[1], \dots, a[s-1]$  the lower dimensional equidimensional loci. If  $I$  is the unit ideal, a list containing the unit ideal as  $a[1]$  is returned.

**Note:**       An embedded component  $q$  (primary ideal) of  $I$  can be replaced in the decomposition by a primary ideal  $q_1$  with the same radical as  $q$ .  
`equidim(I,1)` uses the algorithm of Eisenbud/Huneke/Vasconcelos.

**Example:**

```
LIB "primdec.lib";
ring r = 32003,(x,y,z),dp;
ideal i = intersect(ideal(z),ideal(x,y),ideal(x2,z2),ideal(x5,y5,z5));
equidim(i);
⇒ [1]:
⇒   _[1]=z4
⇒   _[2]=y5
⇒   _[3]=x5
⇒   _[4]=x3z3
⇒   _[5]=x4y4
⇒ [2]:
⇒   _[1]=yz
⇒   _[2]=xz
⇒   _[3]=x2
⇒ [3]:
⇒   _[1]=z
```

### D.4.26.17 equidimMax

Procedure from library `primdec.lib` (see [Section D.4.26 \[primdec.lib\], page 1240](#)).

**Usage:** `equidimMax(i); i ideal`

**Return:** ideal of equidimensional locus (of maximal dimension) of `i`.

**Example:**

```
LIB "primdec.lib";
ring r = 32003,(x,y,z),dp;
ideal i = intersect(ideal(z),ideal(x,y),ideal(x2,z2),ideal(x5,y5,z5));
equidimMax(i);
⇒ _[1]=z
```

### D.4.26.18 equidimMaxEHV

Procedure from library `primdec.lib` (see [Section D.4.26 \[primdec.lib\], page 1240](#)).

**Usage:** `equidimMaxEHV(I); I ideal`

**Return:** ideal, the equidimensional component (of maximal dimension) of `I`.

**Note:** Uses algorithm of Eisenbud, Huneke and Vasconcelos.

**Example:**

```
LIB "primdec.lib";
ring r = 0,(x,y,z),dp;
ideal i=intersect(ideal(z),ideal(x,y),ideal(x2,z2),ideal(x5,y5,z5));
equidimMaxEHV(i);
⇒ _[1]=z
```

### D.4.26.19 zerodec

Procedure from library `primdec.lib` (see [Section D.4.26 \[primdec.lib\], page 1240](#)).

**Usage:** `zerodec(I); I ideal`

**Assume:** `I` is zero-dimensional, the characteristic of the ground field is 0

**Return:** list of primary ideals, the zero-dimensional decomposition of `I`

**Note:** The algorithm (of Monico), works well only for a small total number of solutions (`vdim(std(I))` should be  $< 100$ ) and without parameters. In practice, it works also in large characteristic  $p > 0$  but may fail for small  $p$ .

If `printlevel > 0` (default = 0) additional information is displayed.

**Example:**

```
LIB "primdec.lib";
ring r = 0,(x,y),dp;
ideal i = x2-2,y2-2;
list pr = zerodec(i);
pr;
⇒ [1]:
⇒ _[1]=y2-2
⇒ _[2]=xy-2
⇒ _[3]=x2-2
⇒ [2]:
⇒ _[1]=y2-2
⇒ _[2]=xy+2
⇒ _[3]=x2-2
```



### D.4.26.20 absPrimdecGTZ

Procedure from library `primdec.lib` (see [Section D.4.26 \[primdec.lib\]](#), page 1240).

**Usage:** `absPrimdecGTZ(I);`  $I$  ideal

**Assume:** Ground field has characteristic 0.

**Return:** a ring containing two lists: `absolute_primes`, the absolute prime components of  $I$ , and `primary_decomp`, the output of `primdecGTZ(I)`. The list `absolute_primes` has to be interpreted as follows: each entry describes a class of conjugated absolute primes,

`absolute_primes[i][1]` the absolute prime component,  
`absolute_primes[i][2]` the number of conjugates.

The first entry of `absolute_primes[i][1]` is the minimal polynomial of a minimal finite field extension over which the absolute prime component is defined.

For local orderings, the result is considered in the localization of the polynomial ring, not in the power series ring. For local and mixed orderings, the decomposition in the corresponding global ring is returned if the string 'global' is specified as second argument

**Note:** Algorithm of Gianni/Trager/Zacharias combined with the `absFactorize` command.

**Example:**

```
LIB "primdec.lib";
ring r = 0,(x,y,z),lp;
poly p = z2+1;
poly q = z3+2;
ideal i = p*q^2,y-z2;
def S = absPrimdecGTZ(i);
⇒
⇒ // 'absPrimdecGTZ' created a ring, in which two lists absolute_primes (th\
e
⇒ // absolute prime components) and primary_decomp (the primary and prime
⇒ // components over the current basering) are stored.
⇒ // To access the list of absolute prime components, type (if the name S w\
as
⇒ // assigned to the return value):
⇒          setring S; absolute_primes;
setring S;
absolute_primes;
⇒ [1]:
⇒   [1]:
⇒       _[1]=a2+1
⇒       _[2]=z-a
⇒       _[3]=y+1
⇒   [2]:
⇒       2
⇒ [2]:
⇒   [1]:
⇒       _[1]=a3+2
⇒       _[2]=z-a
⇒       _[3]=y-a2
⇒   [2]:
⇒       3
```

See also: [Section D.4.1.1 \[absFactorize\]](#), page 1002; [Section D.4.26.2 \[primdecGTZ\]](#), page 1241.

### D.4.26.21 absPrimdecGTZE

Procedure from library `primdec.lib` (see [Section D.4.26 \[primdec.lib\]](#), page 1240).

**Usage:** `absPrimdecGTZE(I);` I ideal

**Assume:** Ground field has characteristic 0.

**Return:** a ring containing two lists: `absolute_primes`, the absolute prime components of I, and `primary_decomp`, the output of `primdecGTZ(I)`. Will fail for unit ideal. The list `absolute_primes` has to be interpreted as follows: each entry describes a class of conjugated absolute primes,

`absolute_primes[i][1]` the absolute prime component,  
`absolute_primes[i][2]` the number of conjugates.

The first entry of `absolute_primes[i][1]` is the minimal polynomial of a minimal finite field extension over which the absolute prime component is defined.

For local orderings, the result is considered in the localization of the polynomial ring, not in the power series ring. For local and mixed orderings, the decomposition in the corresponding global ring is returned if the string 'global' is specified as second argument

**Note:** Algorithm of Gianni/Trager/Zacharias combined with the `absFactorize` command.

**Example:**

```
LIB "primdec.lib";
ring r = 0,(x,y,z),lp;
poly p = z2+1;
poly q = z3+2;
ideal I = p*q^2,y-z2;
def S = absPrimdecGTZE(I);
⇒
⇒ // 'absPrimdecGTZ' created a ring, in which two lists absolute_primes (th\
e
⇒ // absolute prime components) and primary_decomp (the primary and prime
⇒ // components over the current basering) are stored.
⇒ // To access the list of absolute prime components, type (if the name S w\
as
⇒ // assigned to the return value):
⇒      setring S; absolute_primes;
setring S;
absolute_primes;
⇒ [1]:
⇒      [1]:
⇒          _[1]=a2+1
⇒          _[2]=z-a
⇒          _[3]=y+1
⇒      [2]:
⇒          2
⇒ [2]:
⇒      [1]:
⇒          _[1]=a3+2
⇒          _[2]=z-a
⇒          _[3]=y-a2
⇒      [2]:
⇒          3
```

See also: [Section D.4.1.1 \[absFactorize\]](#), page 1002; [Section D.4.26.2 \[primdecGTZ\]](#), page 1241.

### D.4.26.22 sep

Procedure from library `primdec.lib` (see [Section D.4.26 \[primdec.lib\]](#), page 1240).

**Example:**

```
LIB "primdec.lib";
ring R=(5,t,s),(x,y,z),dp;
poly f=(x^25-t*x^5+t)*(x^3+s);
sep(f,1);
↳ [1]:
↳ x8+(s5)*x5+(-t)*x4+(t)*x3+(-ts5)*x+(ts5)
↳ [2]:
↳ 1
```

### D.4.27 primdecint.lib

**Library:** `primdecint.lib`

**Purpose:** primary decomposition of an ideal in the polynomial ring over the integers

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**Overview:** A library for computing the primary decomposition of an ideal in the polynomial ring over the integers,  $\mathbb{Z}[x_1, \dots, x_n]$ .  
The first procedure 'primdecZ' can be used in parallel. The coefficients must always be  $\mathbb{Z}$ .

Reference: Pfister, Sadiq, Steidel, "An Algorithm for primary decomposition in polynomial rings over the integers", arXiv:1008.2074

**Procedures:** See also: [Section D.4.26 \[primdec.lib\]](#), page 1240.

#### D.4.27.1 primdecZ

Procedure from library `primdecint.lib` (see [Section D.4.27 \[primdecint.lib\]](#), page 1253).

**Usage:** `primdecZ(I, n)`; I ideal, n integer (number of processors)

**Note:** If `size(#)` > 0, then `#[1]` is the number of available processors for the computation.  
The coefficients must be  $\mathbb{Z}$ .

**Return:** a list `pr` of primary ideals and their associated primes:

`pr[i][1]` the i-th primary component,  
`pr[i][2]` the i-th prime component.

**Example:**

```
LIB "primdecint.lib";
ring R=integer,(a,b,c,d),dp;
ideal I1=9,a,b;
ideal I2=3,c;
ideal I3=11,2a,7b;
ideal I4=13a2,17b4;
ideal I5=9c5,6d5;
```

```

ideal I6=17,a15,b15,c15,d15;
ideal I=intersectZ(I1,I2);
I=intersectZ(I,I3);
I=intersectZ(I,I4);
I=intersectZ(I,I5);
I=intersectZ(I,I6);
primdecZ(I);
↳ [1]:
↳   [1]:
↳     _[1]=d5
↳     _[2]=c5
↳   [2]:
↳     _[1]=d
↳     _[2]=c
↳ [2]:
↳   [1]:
↳     _[1]=a2
↳     _[2]=b4
↳   [2]:
↳     _[1]=b
↳     _[2]=a
↳ [3]:
↳   [1]:
↳     _[1]=2
↳     _[2]=c5
↳   [2]:
↳     _[1]=2
↳     _[2]=c
↳ [4]:
↳   [1]:
↳     _[1]=3
↳   [2]:
↳     _[1]=3
↳ [5]:
↳   [1]:
↳     _[1]=13
↳     _[2]=b4
↳   [2]:
↳     _[1]=13
↳     _[2]=b
↳ [6]:
↳   [1]:
↳     _[1]=17
↳     _[2]=a2
↳   [2]:
↳     _[1]=17
↳     _[2]=a
↳ [7]:
↳   [1]:
↳     _[1]=17
↳     _[2]=d15
↳     _[3]=c15
↳     _[4]=b15

```

```

⇒      _[5]=a15
⇒      [2]:
⇒      _[1]=17
⇒      _[2]=d
⇒      _[3]=c
⇒      _[4]=b
⇒      _[5]=a
⇒ [8]:
⇒      [1]:
⇒      _[1]=9
⇒      _[2]=3d5
⇒      _[3]=d10
⇒      [2]:
⇒      _[1]=3
⇒      _[2]=d
ideal J=intersectZ(ideal(17,a),ideal(17,a2,b));
primdecZ(J);
⇒ [1]:
⇒      [1]:
⇒      _[1]=17
⇒      _[2]=a
⇒      [2]:
⇒      _[1]=17
⇒      _[2]=a
⇒ [2]:
⇒      [1]:
⇒      _[1]=17
⇒      _[2]=b
⇒      _[3]=a2
⇒      [2]:
⇒      _[1]=17
⇒      _[2]=b
⇒      _[3]=a
ideal K=intersectZ(ideal(9,a+3),ideal(9,b+3));
primdecZ(K);
⇒ [1]:
⇒      [1]:
⇒      _[1]=9
⇒      _[2]=b+3
⇒      [2]:
⇒      _[1]=3
⇒      _[2]=b
⇒ [2]:
⇒      [1]:
⇒      _[1]=9
⇒      _[2]=a+3
⇒      [2]:
⇒      _[1]=3
⇒      _[2]=a

```

#### D.4.27.2 primdecZM

Procedure from library `primdecint.lib` (see [Section D.4.27 \[primdecint.lib\]](#), page 1253).

**Usage:**     `primdecZM(N)`;  $N$  module  
               The coefficients must be  $\mathbb{Z}\mathbb{Z}$ .

**Return:**    a list `pr` of primary modules and their associated primes:  
               `pr[i][1]`   the  $i$ -th primary component,  
               `pr[i][2]`   the  $i$ -th prime component.

**Example:**

```
LIB "primdecint.lib";
ring R=integer,(x,y),(c,lp);
module N=[0,0,xy2-x2-xy],[0,y,x],[0,x,2xy-x],[x,0,-xy],[0,0,18x];
primdecZM(N);
⇒ [1]:
⇒   [1]:
⇒   _[1]=[0,0,x]
⇒   _[2]=[0,1]
⇒   _[3]=[x]
⇒   [2]:
⇒   _[1]=x
⇒ [2]:
⇒   [1]:
⇒   _[1]=[0,0,y3]
⇒   _[2]=[0,0,18x]
⇒   _[3]=[0,0,xy2]
⇒   _[4]=[0,0,x2+xy]
⇒   _[5]=[0,y,x]
⇒   _[6]=[0,x,2xy-x]
⇒   _[7]=[y3]
⇒   _[8]=[x,0,-xy]
⇒   [2]:
⇒   _[1]=y
⇒   _[2]=x
```

### D.4.27.3 minAssZ

Procedure from library `primdecint.lib` (see [Section D.4.27 \[primdecint.lib\]](#), page 1253).

**Usage:**     `minAssZ(I)`;  $I$  ideal  
               The coefficients must be  $\mathbb{Z}\mathbb{Z}$ .

**Return:**    a list `pr` of associated primes:

**Example:**

```
LIB "primdecint.lib";
ring R=integer,(a,b,c,d),dp;
ideal I1=9,a,b;
ideal I2=3,c;
ideal I3=11,2a,7b;
ideal I4=13a2,17b4;
ideal I5=9c5,6d5;
ideal I6=17,a15,b15,c15,d15;
ideal I=intersectZ(I1,I2);
I=intersectZ(I,I3);
I=intersectZ(I,I4);
I=intersectZ(I,I5);
```

```

I=intersectZ(I,I6);
minAssZ(I);
↳ [1]:
↳   _[1]=d
↳   _[2]=c
↳ [2]:
↳   _[1]=b
↳   _[2]=a
↳ [3]:
↳   _[1]=2
↳   _[2]=c
↳ [4]:
↳   _[1]=3
↳ [5]:
↳   _[1]=13
↳   _[2]=b
↳ [6]:
↳   _[1]=17
↳   _[2]=a
ideal J=intersectZ(ideal(17,a),ideal(17,a2,b));
minAssZ(J);
↳ [1]:
↳   _[1]=17
↳   _[2]=a
ideal K=intersectZ(ideal(9,a+3),ideal(9,b+3));
minAssZ(K);
↳ [1]:
↳   _[1]=3
↳   _[2]=b
↳ [2]:
↳   _[1]=3
↳   _[2]=a

```

#### D.4.27.4 radicalZ

Procedure from library `primdecint.lib` (see [Section D.4.27 \[primdecint.lib\]](#), page 1253).

**Usage:**      `radicalZ(I);` I ideal  
                  The coefficients must be ZZ.

**Return:**    the radcal of the input ideal

**Example:**

```

LIB "primdecint.lib";
ring R=integer,(a,b,c,d),dp;
ideal I1=9,a,b;
ideal I2=3,c;
ideal I3=11,2a,7b;
ideal I4=13a2,17b4;
ideal I5=9c5,6d5;
ideal I6=17,a15,b15,c15,d15;
ideal I=intersectZ(I1,I2);
I=intersectZ(I,I3);
I=intersectZ(I,I4);

```

```

I=intersectZ(I,I5);
I=intersectZ(I,I6);
radicalZ(I);
↪ _[1]=102bd
↪ _[2]=78ad
↪ _[3]=51bc
↪ _[4]=39ac
↪ _[5]=6abd
↪ _[6]=3abc
ideal J=intersectZ(ideal(17,a),ideal(17,a2,b));
radicalZ(J);
↪ _[1]=17
↪ _[2]=a

```

#### D.4.27.5 heightZ

Procedure from library `primdecint.lib` (see [Section D.4.27 \[primdecint.lib\]](#), page 1253).

**Usage:**      `heightZ(I);` I ideal  
                  The coefficients must be ZZ.

**Return:**     the height of the input ideal

**Example:**

```

LIB "primdecint.lib";
ring R=integer,(a,b,c,d),dp;
ideal I1=9,a,b;
ideal I2=3,c;
ideal I3=11,2a,7b;
ideal I4=13a2,17b4;
ideal I5=9c5,6d5;
ideal I6=17,a15,b15,c15,d15;
ideal I=intersectZ(I1,I2);
I=intersectZ(I,I3);
I=intersectZ(I,I4);
I=intersectZ(I,I5);
I=intersectZ(I,I6);
heightZ(I);
↪ 1

```

#### D.4.27.6 equidimZ

Procedure from library `primdecint.lib` (see [Section D.4.27 \[primdecint.lib\]](#), page 1253).

**Usage:**      `equidimZ(I);` I ideal  
                  The coefficients must be ZZ.

**Return:**     the part of minimal height

**Example:**

```

LIB "primdecint.lib";
ring R=integer,(a,b,c,d),dp;
ideal I1=9,a,b;
ideal I2=3,c;
ideal I3=11,2a,7b;

```



```

ideal I4=13a2,17b4;
ideal I5=9c5,6d5;
ideal I6=17,a15,b15,c15,d15;
ideal I=intersectZ(I1,I2);
I=intersectZ(I,I3);
I=intersectZ(I,I4);
I=intersectZ(I,I5);
I=intersectZ(I,I6);
equidimZ(I);
↪ _[1]=3

```

### D.4.27.7 intersectZ

Procedure from library `primdecint.lib` (see [Section D.4.27 \[primdecint.lib\]](#), page 1253).

**Return:** the intersection of the input ideals

**Note:** this is an alternative to `intersect(I,J)` over integers, is faster for some examples and should be kept for debug purposes.

**Example:**

```

LIB "primdecint.lib";
ring R=integer,(a,b,c,d),dp;
ideal I1=9,a,b;
ideal I2=3,c;
ideal I3=11,2a,7b;
ideal I4=13a2,17b4;
ideal I5=9c5,6d5;
ideal I6=17,a15,b15,c15,d15;
ideal I=intersectZ(I1,I2); I;
↪ I[1]=9
↪ I[2]=3b
↪ I[3]=3a
↪ I[4]=bc
↪ I[5]=ac
I=intersectZ(I,I3); I;
↪ I[1]=99
↪ I[2]=3b
↪ I[3]=3a
↪ I[4]=bc
↪ I[5]=ac
I=intersectZ(I,I4); I;
↪ I[1]=39a2
↪ I[2]=13a2c
↪ I[3]=51b4
↪ I[4]=17b4c
↪ I[5]=3a2b4
↪ I[6]=a2b4c
I=intersectZ(I,I5); I;
↪ I[1]=78a2d5
↪ I[2]=117a2c5
↪ I[3]=102b4d5
↪ I[4]=153b4c5
↪ I[5]=6a2b4d5

```

```

⇒ I[6]=9a2b4c5
⇒ I[7]=39a2c5d5
⇒ I[8]=51b4c5d5
⇒ I[9]=3a2b4c5d5
I=intersectZ(I,I6); I;
⇒ I[1]=1326a2d5
⇒ I[2]=1989a2c5
⇒ I[3]=102b4d5
⇒ I[4]=153b4c5
⇒ I[5]=663a2c5d5
⇒ I[6]=51b4c5d5
⇒ I[7]=78a2d15
⇒ I[8]=117a2c15
⇒ I[9]=78a15d5
⇒ I[10]=117a15c5
⇒ I[11]=6a2b4d15
⇒ I[12]=9a2b4c15
⇒ I[13]=39a2c5d15
⇒ I[14]=39a2c15d5
⇒ I[15]=6a2b15d5
⇒ I[16]=9a2b15c5
⇒ I[17]=6a15b4d5
⇒ I[18]=9a15b4c5
⇒ I[19]=39a15c5d5
⇒ I[20]=3a2b4c5d15
⇒ I[21]=3a2b4c15d5
⇒ I[22]=3a2b15c5d5
⇒ I[23]=3a15b4c5d5

```

### D.4.28 `primitiv_lib`

**Library:** `primitiv.lib`

**Purpose:** Computing a Primitive Element

**Author:** Martin Lamm, email: lamm@mathematik.uni-kl.de

**Procedures:**

#### D.4.28.1 `primitive`

Procedure from library `primitiv.lib` (see [Section D.4.28 \[primitiv\\_lib\]](#), page 1260).

**Usage:** `primitive(i); i ideal`

**Assume:** `i` is given by generators `m[1],...,m[n]` such that for `j=1,...,n`

- `m[j]` is a polynomial in `k[x(1),...,x(j)]`
- `m[j](a[1],...,a[j-1],x(j))` is the minimal polynomial for `a[j]` over `k(a[1],...,a[j-1])` (`k` the ground field of the current basering and `x(1),...,x(n)` the ring variables).

**Return:** ideal `j` in `k[x(n)]` with

- `j[1]` a minimal polynomial for a primitive element `b` of `k(a[1],...,a[n])` over `k`,
- `j[2],...,j[n+1]` polynomials in `k[x(n)]` such that `j[i+1](b)=a[i]` for `i=1,...,n`.

**Note:** the number of variables in the basering has to be exactly `n`, the number of given generators (i.e., minimal polynomials).

If the ground field  $k$  has only a few elements it may happen that no linear combination of  $a[1], \dots, a[n]$  is a primitive element. In this case `primitive(i)` returns the zero ideal, and one should use `primitive_extra(i)` instead.

**Example:**

```
LIB "primitiv.lib";
ring exring=0,(x,y),dp;
ideal i=x2+1,y2-x;           // compute Q(i,i^(1/2))=:L
ideal j=primitive(i);        // L=Q(a) with a=(-1)^(1/4)
j[1];                         // i=a^2
⇨ y4+1
j[2];                         // i=a^2
⇨ y2
j[3];                         // i^(1/2)=a
⇨ y
// the 2nd element was already primitive!
j=primitive(ideal(x2-2,y2-3)); // compute Q(sqrt(2),sqrt(3))
j[1];
⇨ y4-10y2+1
j[2];
⇨ 1/2y3-9/2y
j[3];
⇨ -1/2y3+11/2y
// no element was primitive -- the calculation of primitive elements
// is based on a random choice.
```

See also: [Section D.4.28.2 \[primitive\\_extra\], page 1261](#).

### D.4.28.2 primitive\_extra

Procedure from library `primitiv.lib` (see [Section D.4.28 \[primitiv.lib\], page 1260](#)).

**Usage:** `primitive_extra(i); i ideal`

**Assume:** The ground field of the basering is  $k=\mathbb{Q}$  or  $k=\mathbb{Z}/p\mathbb{Z}$  and the ideal  $i$  is given by 2 generators  $f, g$  with the following properties:

$f$  is the minimal polynomial of  $a$  in  $k[x]$ ,  
 $g$  is a polynomial in  $k[x, y]$  s.th.  $g(a, y)$  is the minpoly of  $b$  in  $k(a)[y]$ .

Here,  $x$  is the name of the first ring variable,  $y$  the name of the second.

**Return:** ideal  $j$  in  $k[y]$  such that  
 $j[1]$  is the minimal polynomial for a primitive element  $c$  of  $k(a, b)$  over  $k$ ,  
 $j[2]$  is a polynomial s.th.  $j[2](c)=a$ .

**Note:** While `primitive(i)` may fail for finite fields, `primitive_extra(i)` tries all elements of  $k(a, b)$  and, hence, always finds a primitive element.  
In order to do this (try all elements), field extensions like  $\mathbb{Z}/p\mathbb{Z}(a)$  are not allowed for the ground field  $k$ .  
`primitive_extra(i)` assumes that the second generator,  $g$ , is monic as polynomial in  $(k[x])[y]$ .

**Example:**

```
LIB "primitiv.lib";
ring exring=3,(x,y),dp;
ideal i=x2+1,y3+y2-1;
```

```

primitive_extra(i);
↳ _[1]=y6-y5+y4-y3-y-1
↳ _[2]=y5+y4+y2+y+1
ring extension=(3,y),x,dp;
minpoly=y6-y5+y4-y3-y-1;
number a=y5+y4+y2+y+1;
a^2;
↳ -1
factorize(x2+1);
↳ [1]:
↳   _[1]=1
↳   _[2]=x+(y5+y4+y2+y+1)
↳   _[3]=x+(-y5-y4-y2-y-1)
↳ [2]:
↳   1,1,1
factorize(x3+x2-1);
↳ [1]:
↳   _[1]=1
↳   _[2]=x+(y3+y+1)
↳   _[3]=x+(y5+y4+y2+1)
↳   _[4]=x+(-y5-y4-y3-y2-y-1)
↳ [2]:
↳   1,1,1,1

```

### D.4.28.3 splitring

Procedure from library `primitiv.lib` (see [Section D.4.28 \[primitiv.lib\]](#), page 1260).

**Usage:** `splitring(f[,L]);`  $f$  poly,  $L$  list of polys and/or ideals (optional)

**Assume:**  $f$  is univariate and irreducible over the active ring.  
The active ring must allow an algebraic extension (e.g., it cannot be a transcendent ring extension of  $\mathbb{Q}$  or  $\mathbb{Z}/p$ ).

**Return:** ring;  
if called with a nonempty second parameter  $L$ , then in the output ring there is defined a list `erg` ( $=L$  mapped to the new ring); if the minpoly of the active ring is non-zero, then the image of the primitive root of  $f$  in the output ring is appended as last entry of the list `erg`.

**Note:** If the old ring has no parameter, the name  $a$  is chosen for the parameter of  $R$  (if  $a$  is no ring variable; if it is,  $b$  is chosen, etc.; if  $a, b, c, o$  are ring variables, `splitring(f[,L])` produces an error message), otherwise the name of the parameter is kept and only the minimal polynomial is changed.

The names of the ring variables and the orderings are not affected.

#### Example:

```

LIB "primitiv.lib";
ring r=0,(x,y),dp;
def r1=splitring(x2-2);
setring r1; basering; // change to Q(sqrt(2))
↳ // coefficients: QQ[a]/(a2-2)
↳ // number of vars : 2
↳ //          block 1 : ordering dp

```

```

⇒ //          : names      x y
⇒ //          block    2 : ordering C
// change to Q(sqrt(2),sqrt(sqrt(2)))=Q(a) and return the transformed
// old parameter:
def r2=splitring(x2-a,a);
⇒ // new minimal polynomial: a4-2
setring r2; basering; erg;
⇒ // coefficients: QQ[a]/(a4-2)
⇒ // number of vars : 2
⇒ //          block    1 : ordering dp
⇒ //          : names      x y
⇒ //          block    2 : ordering C
⇒ [1]:
⇒      (a2)
⇒ [2]:
⇒      (a)
// the result is (a)^2 = (sqrt(sqrt(2)))^2
kill r1; kill r2;

```

## D.4.29 realrad\_lib

**Library:** realrad.lib

**Purpose:** Computation of real radicals

**Author :** Silke Spang

**Overview:** Algorithms about the computation of the real radical of an arbitrary ideal over the rational numbers and transcendental extensions thereof

**Procedures:**

### D.4.29.1 realpoly

Procedure from library `realrad.lib` (see [Section D.4.29 \[realrad\\_lib\]](#), page 1263).

**Usage:** realpoly(f); a univariate polynomial f;

**Return:** poly f, where f is the real part of the input f

**Example:**

```

LIB "realrad.lib";
ring r1 = 0,x,dp;
poly f=x5+16x2+x+1;
realpoly(f);
⇒ x5+16x2+x+1
realpoly(f*(x4+2));
⇒ x5+16x2+x+1
ring r2=0,(x,y),dp;
poly f=x6-3x4y2 + y6 + x2y2 -6y+5;
realpoly(f);
⇒ x6-3x4y2+y6+x2y2-6y+5
ring r3=0,(x,y,z),dp;
poly f=x4y4-2x5y3z2+x6y2z4+2x2y3z-4x3y2z3+2x4yz5+z2y2-2z4yx+z6x2;
realpoly(f);

```

```

↪ x3yz2-x2y2+xz3-yz
realpoly(f*(x2+y2+1));
↪ x3yz2-x2y2+xz3-yz

```

### D.4.29.2 realzero

Procedure from library `realrad.lib` (see [Section D.4.29 \[realrad.lib\]](#), page 1263).

**Usage:** `realzero(j)`; a zero-dimensional ideal `j`

**Return:** `j`: a zero dimensional ideal, which is the real radical of `i`, if `dim(i)=0`  
 0: otherwise  
 this acts via  
 primary decomposition (`i=1`)  
 listdecomp (`i=2`) or facstd (`i=3`)

**Example:**

```

LIB "realrad.lib";
//in non parametric fields
ring r=0,(x,y),dp;
ideal i=(y3+3y2+y+1)*(y2+4y+4)*(x2+1),(x2+y)*(x2-y2)*(x2+2xy+y2)*(y2+y+1);
realzero(i);
↪ _[1]=y4+5y3+7y2+3y+2
↪ _[2]=x4-x2y2+x2y-y3
ideal j=(y3+3y2+y+1)*(y2-2y+1),(x2+y)*(x2-y2);
realzero(j);
↪ _[1]=y4+2y3-2y2-1
↪ _[2]=x2y3+3x2y2+x2y-y3+x2-3y2-y-1
↪ _[3]=x4-x2y2+x2y-y3
//to get every path
ring r1=(0,t),(x,y),lp;
ideal m1=x2+1-t,y3+t2;
ideal m2=x2+t2+1,y2+t;
ideal m3=x2+1-t,y2-t;
ideal m4=x^2+1+t,y2-t;
ideal i=intersect(m1,m2,m3,m4);
realzero(i);
↪ _[1]=y5+(-t)*y3+(t2)*y2+(-t3)
↪ _[2]=x2+(-t+1)

```

### D.4.29.3 realrad

Procedure from library `realrad.lib` (see [Section D.4.29 \[realrad.lib\]](#), page 1263).

**Usage:** `realrad(id)`, `id` an ideal of arbitrary dimension

**Return:** the real radical of `id`

**Example:** `example realrad`; shows an example

**Example:**

```

LIB "realrad.lib";
ring r1=0,(x,y,z),lp;
//dimension 0
ideal i0=(x2+1)*(x3-2),(y3-2)*(y2+y+1),z3+2;
//dimension 1

```

```

ideal i1=(y3+3y2+y+1)*(y2+4y+4)*(x2+1),(x2+y)*(x2-y2)*(x2+2xy+y2)*(y2+y+1);
ideal i=intersect(i0,i1);
realrad(i);
⇒ _[1]=2x4y+5x4-2x2y3-3x2y2+5x2y-2y4-5y3
⇒ _[2]=-3x4+4x2y4+20x2y3+31x2y2+9x2y+8x2+4y5-4y4z2+4y4z+20y4-20y3z2+20y3z+3\
1y3-28y2z2+28y2z+12y2-12yz2+12yz+8y-8z2+8z
⇒ _[3]=y4z3+2y4+5y3z3+10y3+7y2z3+14y2+3yz3+6y+2z3+4
⇒ _[4]=59y5z2+38y5z-74y5+333y4z2+116y4z-488y4+603y3z2-104y3z-1108y3+443y2z2\
-404y2z-1048y2+232yz2-146yz-502y+76z2-148z-236
⇒ _[5]=59xy4z2+38xy4z-74xy4+295xy3z2+190xy3z-370xy3+413xy2z2+266xy2z-518xy2\
+177xyz2+114xyz-222xy+118xz2+76xz-148x+38y4z2-74y4z-118y4+190y3z2-370y3z-\
590y3+266y2z2-518y2z-826y2+114yz2-222yz-354y+76z2-148z-236

```

### D.4.30 reesclos.lib

**Library:** reesclos.lib

**Purpose:** procedures to compute the int. closure of an ideal

**Author:** Tobias Hirsch, email: hirsch@math.tu-cottbus.de  
 Janko Boehm, email: boehm@mathematik.uni-kl.de  
 Magdaleen Marais, email: magdaleen@aims.ac.za

**Overview:** A library to compute the integral closure of an ideal  $I$  in a polynomial ring  $R=k[x(1),\dots,x(n)]$  using the Rees Algebra  $R[It]$  of  $I$ . It computes the integral closure of  $R[It]$ , which is a graded subalgebra of  $R[t]$ . The degree- $k$ -component is the integral closure of the  $k$ -th power of  $I$ .

In contrast to the previous version, the library uses 'normal.lib' to compute the integral closure of  $R[It]$ . This improves the performance considerably.

**Procedures:**

#### D.4.30.1 ReesAlgebra

Procedure from library `reesclos.lib` (see [Section D.4.30 \[reesclos.lib\]](#), page 1265).

**Usage:** ReesAlgebra (I); I = ideal

**Return:** The Rees algebra  $R[It]$  as an affine ring, where  $I$  is an ideal in  $R$ . The procedure returns a list containing two rings:

[1]: a ring, say  $RR$ ; in the ring an ideal  $\ker$  such that  $R[It]=RR/\ker$

[2]: a ring, say  $Kxt$ ; the basering with additional variable  $t$  containing an ideal  $\text{mapI}$  that defines the map  $RR \rightarrow Kxt$

**Example:**

```

LIB "reesclos.lib";
ring R = 0,(x,y),dp;
ideal I = x2,xy4,y5;
list L = ReesAlgebra(I);
def Rees = L[1];          // defines the ring Rees, containing the ideal ker
setring Rees;            // passes to the ring Rees
Rees;
⇒ // coefficients: QQ
⇒ // number of vars : 5

```

```

⇒ //          block   1 : ordering dp
⇒ //          : names   x y U(1) U(2) U(3)
⇒ //          block   2 : ordering C
ker;          // R[It] is isomorphic to Rees/ker
⇒ ker[1]=y*U(2)-x*U(3)
⇒ ker[2]=y^3*U(1)*U(3)-U(2)^2
⇒ ker[3]=y^4*U(1)-x*U(2)
⇒ ker[4]=x*y^2*U(1)*U(3)^2-U(2)^3
⇒ ker[5]=x^2*y*U(1)*U(3)^3-U(2)^4
⇒ ker[6]=x^3*U(1)*U(3)^4-U(2)^5

```

### D.4.30.2 normalI

Procedure from library `reesclos.lib` (see [Section D.4.30 \[reesclos.lib\]](#), page 1265).

**Usage:**     `normalI (I [,p [,r [,l]]])`; I an ideal, p, r, and l optional integers

**Return:**    the integral closure of I, ...,  $I^p$ , where I is an ideal in the polynomial ring  $R=k[x(1),\dots,x(n)]$ . If p is not given, or  $p==0$ , compute the closure of all powers up to the maximum degree in t occurring in the closure of  $R[It]$  (so this is the last power whose closure is not just the sum/product of the smaller). If r is given and  $r==1$ , `normalI` starts with a check whether I is already a radical ideal. If  $l==1$  then `locNormal` instead of `normal` is used to compute normalization. The result is a list containing the closure of the desired powers of I as ideals of the basering.

**Display:**   The procedure displays more comments for higher `printlevel`.

**Example:**

```

LIB "reesclos.lib";
ring R=0,(x,y),dp;
ideal I = x2,xy4,y5;
list J = normalI(I);
I;
⇒ I[1]=x2
⇒ I[2]=xy4
⇒ I[3]=y5
J;                               // J[1] is the integral closure of I
⇒ [1]:
⇒   _[1]=x2
⇒   _[2]=xy4
⇒   _[3]=y5
⇒   _[4]=xy3

```

### D.4.31 rstandard.lib

**Library:**    `rstandard.lib`

**Purpose:**     Computes Janet bases and border bases for ideals

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**Overview:**  Computing Janet bases and border bases for any ordering using the idea of r-standard bases (defined by V. Gerdt)



**References:**

- [1] A. Kehrein, M. Kreuzer, L. Robbiano: An algebrists view on border bases, in: A. Dickenstein and I. Emiris (eds.), Solving Polynomial Equations: Foundations, Algorithms and Applications, Springer, Heidelberg 2005, 169-202.
- [2] V.P. Gerdt: Involute Algorithms for Computing Groebner Bases, In Computational Commutative and Non-Computational Algebra Geometry, S.Conjocaru, G. Pfister and V. Ufnarovski (Eds.), NATO Science Series, 105 Press 2005, 199-255.

**Procedures:****D.4.31.1 borderBasis**

Procedure from library `rstandard.lib` (see [Section D.4.31 \[rstandard.lib\]](#), page 1266).

**Usage:**      `borderBasis(I)`; `I` is an ideal.

**Return:**     ideal, a border basis for `I`.

**Purpose:**     Computes a border basis for the ideal given by the generators in `I`.

**Example:**

```
LIB "rstandard.lib";
ring R=32003,(x,y,z),ds;
poly f=x3y+x5+x3y2+2x2y3+x2yz2+xy5+x12+y16+z20;
ideal i= jacob(f);
i=i,f;
ideal j=borderBasis(i);  j;
⇨ j[1]=y4z20
⇨ j[2]=y5z19
⇨ j[3]=xy4z19
⇨ j[4]=y6z18
⇨ j[5]=xy5z18
⇨ j[6]=y3z20
⇨ j[7]=xy3z19
⇨ j[8]=xy4z18
⇨ j[9]=y6z17-12804y4z19
⇨ j[10]=xy5z17
⇨ j[11]=y7z16+16001y5z18
⇨ j[12]=xy6z16
⇨ j[13]=y8z15-6401y4z19
⇨ j[14]=y9z14
⇨ j[15]=xy8z14
⇨ j[16]=z22
⇨ j[17]=yz21-2461y4z19
⇨ j[18]=xz21
⇨ j[19]=y2z20
⇨ j[20]=xyz20
⇨ j[21]=xy2z19+y4z19
⇨ j[22]=xy3z18+8001y5z18
⇨ j[23]=xy4z17-12801y4z19
⇨ j[24]=xy5z16
⇨ j[25]=y7z15+16001y5z17+y3z19+6153y4z19
⇨ j[26]=xy6z15
⇨ j[27]=xy7z14
⇨ j[28]=y9z13
```

```

⇒ j[29]=xy8z13
⇒ j[30]=xz20
⇒ j[31]=xyz19+y3z19+6153y4z19
⇒ j[32]=xy2z18-y8z14-16001y6z16
⇒ j[33]=xy3z17+8001y5z17+16001y3z19+12925y4z19
⇒ j[34]=xy4z16-16001y8z14
⇒ j[35]=xy5z15
⇒ j[36]=xy6z14
⇒ j[37]=xy7z13
⇒ j[38]=y9z12
⇒ j[39]=xy8z12
⇒ j[40]=xz19+y2z19+13336z21-10668y3z19-2051y4z19
⇒ j[41]=xyz18-y7z14-16001y5z16
⇒ j[42]=x2z18+15999y8z14
⇒ j[43]=xy2z17-y8z13-16001y6z15
⇒ j[44]=xy3z16-16001y7z14
⇒ j[45]=xy4z15-16001y8z13
⇒ j[46]=xy5z14
⇒ j[47]=xy6z13
⇒ j[48]=xy7z12
⇒ j[49]=y9z11
⇒ j[50]=xy8z11
⇒ j[51]=xyz17-y7z13-16001y5z15
⇒ j[52]=x2z17+15999y8z13
⇒ j[53]=xy2z16-y8z12-16001y6z14
⇒ j[54]=xy3z15-16001y7z13
⇒ j[55]=xy4z14-16001y8z12
⇒ j[56]=xy5z13
⇒ j[57]=xy6z12
⇒ j[58]=xy7z11
⇒ j[59]=y9z10
⇒ j[60]=xy8z10
⇒ j[61]=xyz16-y7z12-16001y5z14
⇒ j[62]=x2z16+15999y8z12
⇒ j[63]=xy2z15-y8z11-16001y6z13
⇒ j[64]=xy3z14-16001y7z12
⇒ j[65]=xy4z13-16001y8z11
⇒ j[66]=xy5z12
⇒ j[67]=xy6z11
⇒ j[68]=xy7z10
⇒ j[69]=y9z9
⇒ j[70]=xy8z9
⇒ j[71]=xyz15-y7z11-16001y5z13
⇒ j[72]=x2z15+15999y8z11
⇒ j[73]=xy2z14-y8z10-16001y6z12
⇒ j[74]=xy3z13-16001y7z11
⇒ j[75]=xy4z12-16001y8z10
⇒ j[76]=xy5z11
⇒ j[77]=xy6z10
⇒ j[78]=xy7z9
⇒ j[79]=y9z8
⇒ j[80]=xy8z8
⇒ j[81]=xyz14-y7z10-16001y5z12

```

```

⇒ j[82]=x2z14+15999y8z10
⇒ j[83]=xy2z13-y8z9-16001y6z11
⇒ j[84]=xy3z12-16001y7z10
⇒ j[85]=xy4z11-16001y8z9
⇒ j[86]=xy5z10
⇒ j[87]=xy6z9
⇒ j[88]=xy7z8
⇒ j[89]=y9z7
⇒ j[90]=xy8z7
⇒ j[91]=xyz13-y7z9-16001y5z11
⇒ j[92]=x2z13+15999y8z9
⇒ j[93]=xy2z12-y8z8-16001y6z10
⇒ j[94]=xy3z11-16001y7z9
⇒ j[95]=xy4z10-16001y8z8
⇒ j[96]=xy5z9
⇒ j[97]=xy6z8
⇒ j[98]=xy7z7
⇒ j[99]=y9z6
⇒ j[100]=xy8z6
⇒ j[101]=xyz12-y7z8-16001y5z10
⇒ j[102]=x2z12+15999y8z8
⇒ j[103]=xy2z11-y8z7-16001y6z9
⇒ j[104]=xy3z10-16001y7z8
⇒ j[105]=xy4z9-16001y8z7
⇒ j[106]=xy5z8
⇒ j[107]=xy6z7
⇒ j[108]=xy7z6
⇒ j[109]=y9z5
⇒ j[110]=xy8z5
⇒ j[111]=xyz11-y7z7-16001y5z9
⇒ j[112]=x2z11+15999y8z7
⇒ j[113]=xy2z10-y8z6-16001y6z8
⇒ j[114]=xy3z9-16001y7z7
⇒ j[115]=xy4z8-16001y8z6
⇒ j[116]=xy5z7
⇒ j[117]=xy6z6
⇒ j[118]=xy7z5
⇒ j[119]=y9z4
⇒ j[120]=xy8z4
⇒ j[121]=xyz10-y7z6-16001y5z8
⇒ j[122]=x2z10+15999y8z6
⇒ j[123]=xy2z9-y8z5-16001y6z7
⇒ j[124]=xy3z8-16001y7z6
⇒ j[125]=xy4z7-16001y8z5
⇒ j[126]=xy5z6
⇒ j[127]=xy6z5
⇒ j[128]=xy7z4
⇒ j[129]=y9z3
⇒ j[130]=xy8z3
⇒ j[131]=xyz9-y7z5-16001y5z7
⇒ j[132]=x2z9+15999y8z5
⇒ j[133]=xy2z8-y8z4-16001y6z6
⇒ j[134]=xy3z7-16001y7z5

```

```

⇒ j[135]=xy4z6-16001y8z4
⇒ j[136]=xy5z5
⇒ j[137]=xy6z4
⇒ j[138]=xy7z3
⇒ j[139]=y9z2
⇒ j[140]=xy8z2
⇒ j[141]=xyz8-y7z4-16001y5z6
⇒ j[142]=x2z8+15999y8z4
⇒ j[143]=xy2z7-y8z3-16001y6z5
⇒ j[144]=xy3z6-16001y7z4
⇒ j[145]=xy4z5-16001y8z3
⇒ j[146]=xy5z4
⇒ j[147]=xy6z3
⇒ j[148]=xy7z2
⇒ j[149]=y9z+10y4z19
⇒ j[150]=xy8z
⇒ j[151]=xyz7-y7z3-16001y5z5
⇒ j[152]=x2z7+15999y8z3-10672z21-10671y3z19+11493y4z19
⇒ j[153]=xy2z6-y8z2-16001y6z4
⇒ j[154]=x2yz6
⇒ j[155]=x3z6
⇒ j[156]=xy3z5-16001y7z3
⇒ j[157]=xy4z4-16001y8z2
⇒ j[158]=xy5z3
⇒ j[159]=xy6z2
⇒ j[160]=xy7z-10y4z19
⇒ j[161]=y9-13yz20
⇒ j[162]=xy8
⇒ j[163]=xyz6-y7z2-16001y5z4
⇒ j[164]=xy2z5-y8z-16001y6z3-10y3z19-7333y4z19
⇒ j[165]=x2yz5
⇒ j[166]=x3z5+10672z21+10671y3z19-11493y4z19
⇒ j[167]=xy3z4-16001y7z2
⇒ j[168]=xy4z3-16001y8z+5y3z19-14791y4z19
⇒ j[169]=xy5z2
⇒ j[170]=xy6z-10y3z19+7388y4z19
⇒ j[171]=xy7+13yz20
⇒ j[172]=xyz5-y7z-16001y5z3-10y2z19+5333z21-10616y3z19+391y4z19
⇒ j[173]=xy2z4-y8-16001y6z2-3x2z6-15994y8z2+13z20+15952yz20
⇒ j[174]=x2yz4
⇒ j[175]=x3z4+x2z6+15999y8z2
⇒ j[176]=xy3z3-16001y7z+5y2z19+10666z21+10646y3z19-9017y4z19
⇒ j[177]=x2y2z3-7393y4z19
⇒ j[178]=xy4z2-16001y8-16000x2z6+7997y8z2+15995z20+8018yz20
⇒ j[179]=xy5z-10y2z19-5333z21+10671y3z19-11493y4z19
⇒ j[180]=xy6-3x2z6-15994y8z2+13z20+15yz20
⇒ j[181]=x2yz3+10z21
⇒ j[182]=x3z3+x2z5+15999y8z-25y3z19-9699y4z19
⇒ j[183]=xy3z2-16001xyz4+8001y5z2+8002x2z6+11998y8z2+15994z20+15994yz20
⇒ j[184]=x2y2z2+10yz20
⇒ j[185]=xy4z-16001xy2z3+8001y6z+15994yz19+15994y2z19+15989y4z19
⇒ j[186]=xy5-xyz4+y7+16001y5z2+15987x2z6+8037y8z2-15937z20-15931yz20
⇒ j[187]=x2y4+3x2z6+15994y8z2-13z20+15983yz20

```

```

⇒ j[188]=x2yz2+10z20
⇒ j[189]=x3z2+x2z4+15999y8+15994x2z6-7982y8z2-15969z20-8147yz20
⇒ j[190]=xy3z-16001xyz3+8001y5z+8002x2z5+11998y8z+15994z19+15994yz19+7957y3\
z19+14472y4z19
⇒ j[191]=x2y2z+10yz19
⇒ j[192]=x2y3+xyz4-y7-16001y5z2-15986x2z6+7962y8z2-68z20+15931yz20
⇒ j[193]=x2yz+10z19
⇒ j[194]=x3z+x2z3+15999xy2z3-8002y6z+15979yz19-15944y2z19+10696z21+10661y3z\
19+7047y4z19
⇒ j[195]=x2y2+10669xy4-10667xy2z2+10668y6-xyz4+y7+16001y5z2+15986x2z6-7962y\
8z2+68z20+5426yz20
⇒ j[196]=x2y+10669xy3-10667xyz2-10669xy4+10668y5+10667xy2z2-10668y6-10666x2\
z4+xyz4-y7-16001y5z2-5338y8-2x2z6+5y8z2-5326z20-13868yz20
⇒ j[197]=x3+x2z2-3xy4-4xy2z2-2y6+8xyz4-8y7+4y5z2+122x2z6-305y8z2-507z20-694\
yz20

```

See also: [Section D.4.31.2 \[modBorder\]](#), page 1271.

### D.4.31.2 modBorder

Procedure from library `rstandard.lib` (see [Section D.4.31 \[rstandard.lib\]](#), page 1266).

**Usage:** `modBorder(I,i)`;  $I$  is an ideal,  $i$  an integer.

**Return:** ideal, a border basis for  $I$  using modular methods.

**Purpose:** Computes a border basis for the ideal given by the generators in  $I$  using modular techniques.  
If second argument is 0 then the result is not verified.

**Example:**

```

LIB "rstandard.lib";
ring R=0,(x,y,z),ds;
poly f=x3y+x5+x3y2+2x2y3+x2yz2+xy5+x12+y16+z20;
ideal i= jacob(f);
i=i,f;
ideal j=modBorder(i,1); j;
⇒ j[1]=y4z20
⇒ j[2]=y5z19
⇒ j[3]=xy4z19
⇒ j[4]=y6z18
⇒ j[5]=xy5z18
⇒ j[6]=y3z20
⇒ j[7]=xy3z19
⇒ j[8]=xy4z18
⇒ j[9]=y6z17-14/5y4z19
⇒ j[10]=xy5z17
⇒ j[11]=y7z16-1/2y5z18
⇒ j[12]=xy6z16
⇒ j[13]=y8z15-2/5y4z19
⇒ j[14]=y9z14
⇒ j[15]=xy8z14
⇒ j[16]=z22
⇒ j[17]=yz21+10/13y4z19
⇒ j[18]=xz21

```

```

⇒ j[19]=y2z20
⇒ j[20]=xyz20
⇒ j[21]=xy2z19+y4z19
⇒ j[22]=xy3z18+1/4y5z18
⇒ j[23]=xy4z17+1/5y4z19
⇒ j[24]=xy5z16
⇒ j[25]=y7z15-1/2y5z17+y3z19-37/26y4z19
⇒ j[26]=xy6z15
⇒ j[27]=xy7z14
⇒ j[28]=y9z13
⇒ j[29]=xy8z13
⇒ j[30]=xz20
⇒ j[31]=xyz19+y3z19-37/26y4z19
⇒ j[32]=xy2z18-y8z14+1/2y6z16
⇒ j[33]=xy3z17+1/4y5z17-1/2y3z19+37/52y4z19
⇒ j[34]=xy4z16+1/2y8z14
⇒ j[35]=xy5z15
⇒ j[36]=xy6z14
⇒ j[37]=xy7z13
⇒ j[38]=y9z12
⇒ j[39]=xy8z12
⇒ j[40]=xz19+y2z19+17/12z21-1/3y3z19+37/78y4z19
⇒ j[41]=xyz18-y7z14+1/2y5z16
⇒ j[42]=x2z18-5/2y8z14
⇒ j[43]=xy2z17-y8z13+1/2y6z15
⇒ j[44]=xy3z16+1/2y7z14
⇒ j[45]=xy4z15+1/2y8z13
⇒ j[46]=xy5z14
⇒ j[47]=xy6z13
⇒ j[48]=xy7z12
⇒ j[49]=y9z11
⇒ j[50]=xy8z11
⇒ j[51]=xyz17-y7z13+1/2y5z15
⇒ j[52]=x2z17-5/2y8z13
⇒ j[53]=xy2z16-y8z12+1/2y6z14
⇒ j[54]=xy3z15+1/2y7z13
⇒ j[55]=xy4z14+1/2y8z12
⇒ j[56]=xy5z13
⇒ j[57]=xy6z12
⇒ j[58]=xy7z11
⇒ j[59]=y9z10
⇒ j[60]=xy8z10
⇒ j[61]=xyz16-y7z12+1/2y5z14
⇒ j[62]=x2z16-5/2y8z12
⇒ j[63]=xy2z15-y8z11+1/2y6z13
⇒ j[64]=xy3z14+1/2y7z12
⇒ j[65]=xy4z13+1/2y8z11
⇒ j[66]=xy5z12
⇒ j[67]=xy6z11
⇒ j[68]=xy7z10
⇒ j[69]=y9z9
⇒ j[70]=xy8z9
⇒ j[71]=xyz15-y7z11+1/2y5z13

```

```

⇒ j[72]=x2z15-5/2y8z11
⇒ j[73]=xy2z14-y8z10+1/2y6z12
⇒ j[74]=xy3z13+1/2y7z11
⇒ j[75]=xy4z12+1/2y8z10
⇒ j[76]=xy5z11
⇒ j[77]=xy6z10
⇒ j[78]=xy7z9
⇒ j[79]=y9z8
⇒ j[80]=xy8z8
⇒ j[81]=xyz14-y7z10+1/2y5z12
⇒ j[82]=x2z14-5/2y8z10
⇒ j[83]=xy2z13-y8z9+1/2y6z11
⇒ j[84]=xy3z12+1/2y7z10
⇒ j[85]=xy4z11+1/2y8z9
⇒ j[86]=xy5z10
⇒ j[87]=xy6z9
⇒ j[88]=xy7z8
⇒ j[89]=y9z7
⇒ j[90]=xy8z7
⇒ j[91]=xyz13-y7z9+1/2y5z11
⇒ j[92]=x2z13-5/2y8z9
⇒ j[93]=xy2z12-y8z8+1/2y6z10
⇒ j[94]=xy3z11+1/2y7z9
⇒ j[95]=xy4z10+1/2y8z8
⇒ j[96]=xy5z9
⇒ j[97]=xy6z8
⇒ j[98]=xy7z7
⇒ j[99]=y9z6
⇒ j[100]=xy8z6
⇒ j[101]=xyz12-y7z8+1/2y5z10
⇒ j[102]=x2z12-5/2y8z8
⇒ j[103]=xy2z11-y8z7+1/2y6z9
⇒ j[104]=xy3z10+1/2y7z8
⇒ j[105]=xy4z9+1/2y8z7
⇒ j[106]=xy5z8
⇒ j[107]=xy6z7
⇒ j[108]=xy7z6
⇒ j[109]=y9z5
⇒ j[110]=xy8z5
⇒ j[111]=xyz11-y7z7+1/2y5z9
⇒ j[112]=x2z11-5/2y8z7
⇒ j[113]=xy2z10-y8z6+1/2y6z8
⇒ j[114]=xy3z9+1/2y7z7
⇒ j[115]=xy4z8+1/2y8z6
⇒ j[116]=xy5z7
⇒ j[117]=xy6z6
⇒ j[118]=xy7z5
⇒ j[119]=y9z4
⇒ j[120]=xy8z4
⇒ j[121]=xyz10-y7z6+1/2y5z8
⇒ j[122]=x2z10-5/2y8z6
⇒ j[123]=xy2z9-y8z5+1/2y6z7
⇒ j[124]=xy3z8+1/2y7z6

```

```

⇒ j [125]=xy4z7+1/2y8z5
⇒ j [126]=xy5z6
⇒ j [127]=xy6z5
⇒ j [128]=xy7z4
⇒ j [129]=y9z3
⇒ j [130]=xy8z3
⇒ j [131]=xyz9-y7z5+1/2y5z7
⇒ j [132]=x2z9-5/2y8z5
⇒ j [133]=xy2z8-y8z4+1/2y6z6
⇒ j [134]=xy3z7+1/2y7z5
⇒ j [135]=xy4z6+1/2y8z4
⇒ j [136]=xy5z5
⇒ j [137]=xy6z4
⇒ j [138]=xy7z3
⇒ j [139]=y9z2
⇒ j [140]=xy8z2
⇒ j [141]=xyz8-y7z4+1/2y5z6
⇒ j [142]=x2z8-5/2y8z4
⇒ j [143]=xy2z7-y8z3+1/2y6z5
⇒ j [144]=xy3z6+1/2y7z4
⇒ j [145]=xy4z5+1/2y8z3
⇒ j [146]=xy5z4
⇒ j [147]=xy6z3
⇒ j [148]=xy7z2
⇒ j [149]=y9z+10y4z19
⇒ j [150]=xy8z
⇒ j [151]=xyz7-y7z3+1/2y5z5
⇒ j [152]=x2z7-5/2y8z3-13/3z21-10/3y3z19+185/39y4z19
⇒ j [153]=xy2z6-y8z2+1/2y6z4
⇒ j [154]=x2yz6
⇒ j [155]=x3z6
⇒ j [156]=xy3z5+1/2y7z3
⇒ j [157]=xy4z4+1/2y8z2
⇒ j [158]=xy5z3
⇒ j [159]=xy6z2
⇒ j [160]=xy7z-10y4z19
⇒ j [161]=y9-13yz20
⇒ j [162]=xy8
⇒ j [163]=xyz6-y7z2+1/2y5z4
⇒ j [164]=xy2z5-y8z+1/2y6z3-10y3z19+680/13y4z19
⇒ j [165]=x2yz5
⇒ j [166]=x3z5+13/3z21+10/3y3z19-185/39y4z19
⇒ j [167]=xy3z4+1/2y7z2
⇒ j [168]=xy4z3+1/2y8z+5y3z19-265/13y4z19
⇒ j [169]=xy5z2
⇒ j [170]=xy6z-10y3z19+35/13y4z19
⇒ j [171]=xy7+13yz20
⇒ j [172]=xyz5-y7z+1/2y5z3-10y2z19-5/6z21+155/3y3z19-1505/78y4z19
⇒ j [173]=xy2z4-y8+1/2y6z2-3x2z6+15/2y8z2+13z20-99/2yz20
⇒ j [174]=x2yz4
⇒ j [175]=x3z4+x2z6-5/2y8z2
⇒ j [176]=xy3z3+1/2y7z+5y2z19-5/3z21-65/3y3z19+370/39y4z19
⇒ j [177]=x2y2z3-100/13y4z19

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⇒ j[178]=xy4z2+1/2y8+3/2x2z6-15/4y8z2-13/2z20+69/4yz20
⇒ j[179]=xy5z-10y2z19+5/6z21+10/3y3z19-185/39y4z19
⇒ j[180]=xy6-3x2z6+15/2y8z2+13z20+15yz20
⇒ j[181]=x2yz3+10z21
⇒ j[182]=x3z3+x2z5-5/2y8z-25y3z19+1925/13y4z19
⇒ j[183]=xy3z2+1/2xyz4+1/4y5z2+5/4x2z6-25/8y8z2-15/2z20-15/2yz20
⇒ j[184]=x2y2z2+10yz20
⇒ j[185]=xy4z+1/2xy2z3+1/4y6z-15/2yz19-15/2y2z19-25/2y4z19
⇒ j[186]=xy5-xyz4+y7-1/2y5z2-29/2x2z6+145/4y8z2+129/2z20+141/2yz20
⇒ j[187]=x2y4+3x2z6-15/2y8z2-13z20-37/2yz20
⇒ j[188]=x2yz2+10z20
⇒ j[189]=x3z2+x2z4-5/2y8-15/2x2z6+75/4y8z2+65/2z20-585/4yz20
⇒ j[190]=xy3z+1/2xyz3+1/4y5z+5/4x2z5-25/8y8z-15/2z19-15/2yz19-175/4y3z19+16\
475/52y4z19
⇒ j[191]=x2y2z+10yz19
⇒ j[192]=x2y3+xyz4-y7+1/2y5z2+31/2x2z6-155/4y8z2-68z20-141/2yz20
⇒ j[193]=x2yz+10z19
⇒ j[194]=x3z+x2z3-5/2xy2z3-5/4y6z-45/2yz19+115/2y2z19+85/3z21-20/3y3z19+561\
5/78y4z19
⇒ j[195]=x2y2+4/3xy4+2/3xy2z2+1/3y6-xyz4+y7-1/2y5z2-31/2x2z6+155/4y8z2+68z2\
0+553/6yz20
⇒ j[196]=x2y+4/3xy3+2/3xyz2-4/3xy4+1/3y5-2/3xy2z2-1/3y6+5/3x2z4+xyz4-y7+1/2\
y5z2-25/6y8-2x2z6+5y8z2+47/6z20-6401/12yz20
⇒ j[197]=x3+x2z2-3xy4-4xy2z2-2y6+8xyz4-8y7+4y5z2+122x2z6-305y8z2-507z20-694\
yz20
ring S=0,(x,y,z),ds;
ideal i=
3x2+6xy+3y2+6xz+6yz+3z2+3x2yz+4xy2z+y3z+4xyz2+2y2z2+yz3+10x9,
3x2+6xy+3y2+6xz+6yz+3z2+x3z+4x2yz+3xy2z+2x2z2+4xyz2+xz3+10y9,
3x2+6xy+3y2+6xz+6yz+3z2+x3y+2x2y2+xy3+4x2yz+4xy2z+3xyz2+10z9;
ideal j=modBorder(i,0); j;
⇒ j[1]=z25
⇒ j[2]=yz24
⇒ j[3]=xz24
⇒ j[4]=yz23+1/2z24
⇒ j[5]=xz23+1/2z24
⇒ j[6]=y2z22+1/2z24
⇒ j[7]=xyz22-z24
⇒ j[8]=xz22+yz22+z23
⇒ j[9]=y2z21+yz22+z23
⇒ j[10]=xyz21-z23
⇒ j[11]=xz21+yz21+z22
⇒ j[12]=y2z20+yz21+z22
⇒ j[13]=xyz20-z22
⇒ j[14]=xz20+yz20+z21
⇒ j[15]=y2z19+yz20+z21
⇒ j[16]=xyz19-z21
⇒ j[17]=xz19+yz19+z20
⇒ j[18]=y2z18+yz19+z20
⇒ j[19]=xyz18-z20
⇒ j[20]=xz18+yz18+z19+5z24
⇒ j[21]=y2z17+yz18+z19-22805/228z24
⇒ j[22]=xyz17-z19+22235/228z24

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$\mapsto j[23] = xz17 + yz17 + z18 + 5z23$   
 $\mapsto j[24] = y2z16 + yz17 + z18 + 5yz22 - 22235/228z23 + 3371625/32z24$   
 $\mapsto j[25] = xyz16 - z18 + 22235/228z23 - 3371625/32z24$   
 $\mapsto j[26] = y3z15 - z18 + 13985/228yz22 + 40345/228z23 - 10114875/64z24$   
 $\mapsto j[27] = xy2z15 - yz17 - 16265/228yz22 - 9625/114z23 + 3371625/64z24$   
 $\mapsto j[28] = xz16 + yz16 + z17 + 5z22$   
 $\mapsto j[29] = xyz15 + y2z15 + yz16 + 5yz21$   
 $\mapsto j[30] = y3z14 + 3/2y2z15 + 3/2yz16 + 1/2z17 + 15695/228yz21 + 13985/456z22 - 70875/32yz \backslash$   
 $22 - 70875/64z23$   
 $\mapsto j[31] = xy2z14 - 1/2y2z15 - 3/2yz16 - 1/2z17 - 16835/228yz21 - 16265/456z22 + 70875/32y \backslash$   
 $z22 + 70875/64z23$   
 $\mapsto j[32] = y4z13 - 39/35y2z15 - 74/35yz16 - 39/35z17 - 114311/798yz21 - 126587/1596z22 + 7 \backslash$   
 $0875/16yz22 - 7729425/224z23 + 14531794875/896z24$   
 $\mapsto j[33] = xy3z13 - 27/70y2z15 + 43/70yz16 + 43/70z17 + 118757/1596yz21 + 171239/3192z22 \backslash$   
 $- 70875/32yz22 + 15954975/448z23 - 14531794875/896z24$   
 $\mapsto j[34] = xz15 + yz15 + z16 + 5z21 - 151575/8z24$   
 $\mapsto j[35] = xyz14 + y2z14 + yz15 + 5yz20 + 151575/16z24$   
 $\mapsto j[36] = xy2z13 + y3z13 + y2z14 - 5yz20 - 5z21 - 111675/16z24$   
 $\mapsto j[37] = y4z12 + 2y3z13 + 66/35y2z14 + 31/35yz15 - 4/35z16 - 512487/2156y2z15 - 512487/2 \backslash$   
 $156yz16 - 512487/2156z17 - 39/7yz20 - 2391/133z21 - 2562435/2156yz21 - 58384395/431 \backslash$   
 $2z22 - 4757545125/539z23 + 913078227343185/327712z24$   
 $\mapsto j[38] = xy3z12 - y3z13 - 66/35y2z14 - 31/35yz15 + 4/35z16 + 512487/2156y2z15 + 512487/2 \backslash$   
 $156yz16 + 512487/2156z17 + 39/7yz20 + 3056/133z21 + 2562435/2156yz21 + 58384395/431 \backslash$   
 $2z22 + 4757545125/539z23 - 913076348631735/327712z24$   
 $\mapsto j[39] = y5z11 - 30/17y3z13 - 281/119y2z14 - 162/119yz15 + 48/119z16 + 2562435/4312y2z \backslash$   
 $15 + 2562435/4312yz16 + 2562435/4312z17 + 8140/357yz20 + 354395/6783z21 + 823087575 \backslash$   
 $/146608yz21 + 10312820775/293216z22 - 39639375/2176yz22 + 51740725336875/234572 \backslash$   
 $8z23 - 77611672933260075/11142208z24$   
 $\mapsto j[40] = xy4z11 - 4/17y3z13 + 283/595y2z14 + 283/595yz15 - 172/595z16 - 1537461/4312y2 \backslash$   
 $z15 - 1537461/4312yz16 - 1537461/4312z17 - 4366/357yz20 - 232454/6783z21 - 64884199 \backslash$   
 $5/146608yz21 - 6342681915/293216z22 + 39639375/2176yz22 - 31035888952875/234572 \backslash$   
 $8z23 + 46566963638175885/11142208z24$   
 $\mapsto j[41] = y6z10 + 5/17y3z13 + 78/119y2z14 + 78/119yz15 - 76/119z16 - 3144177/4312y2z15 - \backslash$   
 $3144177/4312yz16 - 3144177/4312z17 - 4825/119yz20 - 183885/2261z21 - 1696930965/1 \backslash$   
 $46608yz21 - 13353125805/293216z22 + 118918125/2176yz22 - 65428421864625/2345728 \backslash$   
 $z23 + 199935960229764075/22284416z24$   
 $\mapsto j[42] = y7z9 + 232/187y3z13 + 348/187y2z14 + 161/187yz15 + 116/187z16 + 41553/88y2z15 \backslash$   
 $+ 41553/88yz16 + 41553/88z17 + 7785/187yz20 + 290710/3553z21 + 42390135/2992yz21 + 1 \backslash$   
 $97137395/5984z22 - 2972953125/23936yz22 + 977771887875/47872z23 - 6387511968556 \backslash$   
 $275/909568z24$   
 $\mapsto j[43] = xy6z9 - 287/187y3z13 - 3294/1309y2z14 - 1985/1309yz15 + 24/1309z16 + 138510/5 \backslash$   
 $39y2z15 + 138510/539yz16 + 138510/539z17 - 1420/1309yz20 + 112120/24871z21 - 190092 \backslash$   
 $825/73304yz21 + 1846696725/146608z22 + 832426875/11968yz22 + 8758799679375/1172 \backslash$   
 $864z23 - 86883916940081475/44568832z24$   
 $\mapsto j[44] = xz14 + yz14 + z15 + 5z20 - 151575/8z23 + 647139375/64z24$   
 $\mapsto j[45] = xyz13 + y2z13 + yz14 + 5yz19 + 23925/8yz22 + 43875/4z23 - 647139375/128z24$   
 $\mapsto j[46] = xy2z12 + y3z12 + y2z13 - 5yz19 - 5z20 - 23925/8yz22 - 8475z23 + 319696875/128z24$   
 $\mapsto j[47] = xy3z11 + y4z11 + y3z12 + 5z20 + 16875/8yz22 + 13575/2z23 - 155975625/128z24$   
 $\mapsto j[48] = y5z10 + 5/2y4z11 + 55/17y3z12 + 40/17y2z13 + 29/34yz14 + 2/17z15 - 438615/12716 \backslash$   
 $y3z13 - 1315845/25432y2z14 - 1315845/25432yz15 - 438615/25432z16 + 905/102yz19 + 74 \backslash$   
 $5/102z20 + 1707075/6358yz20 + 13407525/50864z21 + 2959284375/50864yz21 + 29592843 \backslash$   
 $75/101728z22 - 7018033348175/406912yz22 - 7019757749975/813824z23 + 82848936375 \backslash$   
 $/813824z24$

- $j[49] = xy4z10 - 3/2y4z11 - 55/17y3z12 - 40/17y2z13 - 29/34yz14 - 2/17z15 + 438615/1271 \backslash$   
 $6y3z13 + 1315845/25432y2z14 + 1315845/25432yz15 + 438615/25432z16 - 395/102yz19 - 7 \backslash$   
 $45/102z20 - 1707075/6358yz20 - 13407525/50864z21 - 2959284375/50864yz21 - 2959284 \backslash$   
 $375/101728z22 + 7017533609375/406912yz22 + 7015637765975/813824z23 + 3391275498 \backslash$   
 $75/813824z24$
- $j[50] = xy5z9 + y6z9 - 5/2y4z11 - 55/17y3z12 - 40/17y2z13 - 29/34yz14 - 2/17z15 + 438615/ \backslash$   
 $12716y3z13 + 1315845/25432y2z14 + 1315845/25432yz15 + 438615/25432z16 - 1415/102y \backslash$   
 $z19 - 1255/102z20 - 1707075/6358yz20 - 13407525/50864z21 - 2959284375/50864yz21 - 2 \backslash$   
 $959284375/101728z22 + 7018189754975/406912yz22 + 7022473887575/813824z23 - 1707 \backslash$   
 $20470125/813824z24$
- $j[51] = y7z8 + 7/2y6z9 - 35/4y4z11 - 2848/187y3z12 - 4617/374y2z13 - 3437/748yz14 - 115 \backslash$   
 $/187z15 + 58551255/279752y3z13 + 175653765/559504y2z14 + 175653765/559504yz15 + 5 \backslash$   
 $8551255/559504z16 - 38545/748yz19 - 34055/748z20 - 107445825/69938yz20 - 17378354 \backslash$   
 $25/1119008z21 - 443675323125/1119008yz21 - 443675323125/2238016z22 + 5625720982 \backslash$   
 $51725/8952064yz22 + 562790878490325/17904128z23 - 4547863128375/17904128z24$
- $j[52] = xy6z8 - 5/2y6z9 + 35/4y4z11 + 2848/187y3z12 + 4617/374y2z13 + 3437/748yz14 + 11 \backslash$   
 $5/187z15 - 58551255/279752y3z13 - 175653765/559504y2z14 - 175653765/559504yz15 - \backslash$   
 $58551255/559504z16 + 38545/748yz19 + 37795/748z20 + 107445825/69938yz20 + 1737835 \backslash$   
 $425/1119008z21 + 443675323125/1119008yz21 + 443675323125/2238016z22 - 562568321 \backslash$   
 $599725/8952064yz22 - 562820420301525/17904128z23 + 2614689385875/17904128z24$
- $j[53] = y8z7 - 878/141y6z9 + 9877/517y4z11 + 915172/26367y3z12 + 253943/8789y2z13 + 9 \backslash$   
 $4823/8789yz14 + 37210/26367z15 - 847642725/1643543y3z13 - 2534922297/3287086y2z \backslash$   
 $14 - 2534922297/3287086yz15 - 839636847/3287086z16 + 5435755695/3503192y2z15 + 54 \backslash$   
 $35755695/3503192yz16 + 5435755695/3503192z17 + 1063395/8789yz19 + 5880395/52734 \backslash$   
 $z20 + 6007578570/1643543yz20 + 25156075905/6574172z21 + 1064589124796775/101242 \backslash$   
 $2488yz21 + 1075884396851925/2024844976z22 - 2383632331137825/26296688yz22 + 103 \backslash$   
 $195950091159325/1012422488z23 - 274018430271493924425/16198759808z24$
- $j[54] = xy7z7 + 769/282y6z9 - 21413/2068y4z11 - 30212/1551y3z12 - 17111/1034y2z13 - 1 \backslash$   
 $2809/2068yz14 - 1235/1551z15 + 237013695/773432y3z13 + 707273613/1546864y2z14 + 7 \backslash$   
 $07273613/1546864yz15 + 233246223/1546864z16 - 5435755695/3503192y2z15 - 5435755 \backslash$   
 $695/3503192yz16 - 5435755695/3503192z17 - 133305/2068yz19 - 409355/6204z20 - 4097 \backslash$   
 $17845/193358yz20 - 639413595/281248z21 - 156040912046925/238217056yz21 - 158698 \backslash$   
 $623118725/476434112z22 + 688041879438525/24749824yz22 - 508310646770040175/38 \backslash$   
 $11472896z23 + 64476566767453351725/3811472896z24$
- $j[55] = xz13 + yz13 + z14 - 20007/154y2z15 - 20007/154yz16 - 20007/154z17 + 5z19 - 100035 \backslash$   
 $/154yz21 - 1933395/308z22 - 4406592375/1232z23 + 576566423518975/561792z24$
- $j[56] = xyz12 + y2z12 + yz13 + 20007/308y2z15 + 20007/308yz16 + 20007/308z17 + 5yz18 + 20 \backslash$   
 $42295/616yz21 + 5709015/1232z22 + 2581875/64yz22 + 17825173875/9856z23 - 57656547 \backslash$   
 $5494975/1123584z24$
- $j[57] = xy2z11 + y3z11 + y2z12 - 13851/308y2z15 - 13851/308yz16 - 13851/308z17 - 5yz18 - \backslash$   
 $5z19 - 1980735/616yz21 - 5038095/1232z22 - 2581875/64yz22 - 22282522875/9856z23 + 9 \backslash$   
 $15874327042225/1123584z24$
- $j[58] = xy3z10 + y4z10 + y3z11 + 1539/44y2z15 + 1539/44yz16 + 1539/44z17 + 5z19 + 201015/ \backslash$   
 $88yz21 + 594255/176z22 + 7745625/64yz22 + 3558400875/1408z23 - 77537732695475/802 \backslash$   
 $56z24$
- $j[59] = xy4z9 + y5z9 + y4z10 - 7695/308y2z15 - 7695/308yz16 - 7695/308z17 + 5yz18 - 83347 \backslash$   
 $5/616yz21 - 3235275/1232z22 - 12909375/64yz22 - 21828234375/9856z23 + 92636468662 \backslash$   
 $2125/1123584z24$
- $j[60] = xy5z8 + y6z8 + y5z9 + 4617/308y2z15 + 4617/308yz16 + 4617/308z17 - 5yz18 - 5z19 + 2 \backslash$   
 $82945/616yz21 + 2310765/1232z22 + 13820625/64yz22 + 15566792625/9856z23 - 6027924 \backslash$   
 $06737525/1123584z24$
- $j[61] = xy6z7 + y7z7 + y6z8 - 1539/308y2z15 - 1539/308yz16 - 1539/308z17 + 5z19 + 244485/ \backslash$   
 $616yz21 - 1432455/1232z22 - 10479375/64yz22 - 7060030875/9856z23 + 10437976377287 \backslash$

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5/561792z24
⇒ j [62]=y8z6+4y7z7+1096/141y6z8+438/47y5z9+3827/517y4z10+130/33y3z11+745/51\
7y2z12+185/517yz13+74/1551z14-626565/48598y6z9+36597195/1069156y4z11+2938\
45620/4543913y3z12+531935685/9087826y2z13+441719055/18175652yz14+30706050\
/4543913z15-824462422425/617972168y3z13+11079250419375/13595387696y2z14+1\
1079250419375/13595387696yz15+29217423712725/13595387696z16+6498875861433\
477/3622300528y2z15+6498875861433477/3622300528yz16+6498875861433477/3622\
300528z17-115/47yz18-715/282z19+568770225/1652332yz19+75725325/150212z20+\
83154721712625/3398846924yz20+2507243939897625/27190775392z21+23491579901\
268980805/2093689705184yz21+16367386229217586595/523422426296z22-13196224\
488747115125/19775109376yz22+1268637865899370688210625/8374758820736z23-1\
7691473521227817276849545175/904473952639488z24
⇒ j [63]=xy7z6-3y7z7-1096/141y6z8-438/47y5z9-3827/517y4z10-130/33y3z11-745/5\
17y2z12-185/517yz13-74/1551z14+626565/48598y6z9-36597195/1069156y4z11-293\
845620/4543913y3z12-531935685/9087826y2z13-441719055/18175652yz14-3070605\
0/4543913z15+824462422425/617972168y3z13-11079250419375/13595387696y2z14-\
11079250419375/13595387696yz15-29217423712725/13595387696z16-649889396117\
5401/3622300528y2z15-6498893961175401/3622300528yz16-6498893961175401/362\
2300528z17+350/47yz18+715/282z19-568770225/1652332yz19-75725325/150212z20\
-83154721712625/3398846924yz20-2507243939897625/27190775392z21-2349420352\
2192320265/2093689705184yz21-32734262338246025995/1046844852592z22+131967\
40688623697625/19775109376yz22-634320342549593324852625/4187379410368z23+\
336141189847485260140967097625/17185005100150272z24
⇒ j [64]=y9z5-52/5y7z7-6904/235y6z8-10137/235y5z9-20582/517y4z10-1327/55y3z1\
1-25293/2585y2z12-7006/2585yz13-1072/2585z14+5639085/97196y6z9-329374755/\
2138312y4z11-1284909129/4543913y3z12-4563044199/18175652y2z13-3526717563/\
36351304yz14-238958289/9087826z15+7263199915155/1235944336y3z13-104892996\
034485/27190775392y2z14-104892996034485/27190775392yz15-264683394167895/2\
7190775392z16-292449214667345301/36223005280y2z15-292449214667345301/3622\
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⇒ j [71]=xy2z10+y3z10+y2z11+286/423y6z9-320/141y4z11-31210/7191y3z12-9196/23\
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⇒ j [72]=xy3z9+y4z9+y3z10-286/423y6z9+320/141y4z11+31210/7191y3z12+9196/2397\
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⇒ j [76]=xy7z5+y8z5+y7z6-286/423y6z9+320/141y4z11+31210/7191y3z12+9196/2397y\
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⇒ j [78]=xy8z4-7/2y8z5-38/5y7z6-28/5y6z7+6/5y5z8+13/2y4z9+32/5y3z10+33/10y2z\
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- $\mapsto j[79] = xy9z3 + y10z3 - 9/2y8z5 - 38/5y7z6 - 28/5y6z7 + 6/5y5z8 + 13/2y4z9 + 32/5y3z10 + 33/10y2z11 + 11/10yz12 + 1/5z13 + 243/10y7z7 + 1701/20y6z8 + 1863/20y5z9 + 81/4y4z10 - 15/39/20y3z11 - 1863/20y2z12 - 891/20yz13 - 81/10z14 + 143/2115y6z9 - 32/141y4z11 + 1581/57940/79101y3z12 + 790860199/263670y2z13 + 263640013/87890yz14 + 158189543/1582/02z15 - 404057047923/13148344y3z13 - 8489953497915/184076816y2z14 - 84899534979/15/184076816yz15 - 2833154826993/184076816z16 - 232919344017/11146520y2z15 - 23/2919344017/11146520yz16 - 232919344017/11146520z17 + 7yz17 + 7/4z18 - 486yz18 - 170/1/4z19 + 2317144343/158202yz19 - 4662916/26367z20 - 603006984939855/92038408yz2/0 - 1163846020930875/368153632z21 + 132236704098135963/2577075424yz21 + 1303312/19207468751/5154150848z22 - 1159686946509974573885/71947738368yz22 - 68677251/539404951638665/7050878360064z23 + 303844033154752813237905/1566861857792z2/4$
- $\mapsto j[80] = y11z2 + 11/2y10z3 - 165/4y8z5 - 88y7z6 - 77y6z7 + 275/4y4z9 + 149/2y3z10 + 161/4y/2z11 + 55/4yz12 + 5/2z13 + 1701/4y7z7 + 11907/8y6z8 + 13041/8y5z9 + 2835/8y4z10 - 10773/8y3z11 - 13041/8y2z12 - 6237/8yz13 - 567/4z14 + 7865/846y6z9 - 4400/141y4z11 + 27636/44230/79101y3z12 + 5531165905/105468y2z13 + 1844819035/35156yz14 + 5535979285/3/16404z15 - 14155627908105/26296688y3z13 - 297065656926729/368153632y2z14 - 2970/65656926729/368153632yz15 - 98886866213259/368153632z16 + 467371398141/445860/8y2z15 + 467371398141/4458608yz16 + 467371398141/4458608z17 + 100yz17 - 275/8z18 - 8505yz18 - 59535/8z19 + 81043973125/316404yz19 - 86090350/26367z20 - 211109592183/47385/184076816yz20 - 40725259662179865/736307264z21 + 4706734673877922755/51/54150848yz21 + 4766400819649905885/10308301696z22 - 2049414034886720541775/75/73446144yz22 - 1836647027390449275817975/14101756720128z23 - 6941017444566613/228827975/3133723715584z24$
- $\mapsto j[81] = xy10z2 - 9/2y10z3 + 165/4y8z5 + 88y7z6 + 77y6z7 - 275/4y4z9 - 149/2y3z10 - 161/4y/2z11 - 55/4yz12 - 5/2z13 - 1701/4y7z7 - 11907/8y6z8 - 13041/8y5z9 - 2835/8y4z10 + 10773/8y3z11 + 13041/8y2z12 + 6237/8yz13 + 567/4z14 - 2431/282y6z9 + 1360/47y4z11 - 921329/180/26367y3z12 - 1843856843/35156y2z13 - 1844874123/35156yz14 - 1845344615/1054/68z15 + 14155483898745/26296688y3z13 + 297092613853737/368153632y2z14 + 2970926/13853737/368153632yz15 + 98915839271307/368153632z16 - 270887899131/4458608y2/z15 - 270887899131/4458608yz16 - 270887899131/4458608z17 - 95yz17 + 275/8z18 + 8505/yz18 + 59535/8z19 - 27016215455/105468yz19 + 28572080/8789z20 + 21110683794699105/184076816yz20 + 40726306493438745/736307264z21 - 4698753505995194955/5154150/848yz21 - 4748618833102825785/10308301696z22 + 39114939183464627699725/143895/476736yz22 + 1893501137798116652057275/14101756720128z23 + 536564751106962074/4877275/3133723715584z24$
- $\mapsto j[82] = xz11 + yz11 + z12 + 572/423y6z8 + 572/141y5z9 + 790/141y4z10 + 40/9y3z11 + 88/47y/2z12 + 46/141yz13 + 56/423z14 - 582770/6627y6z9 + 5608430/24299y4z11 + 540007510/12/39249y3z12 + 140696995/413083y2z13 + 45353685/413083yz14 - 8938535/1239249z15 - 1/27805831775/14044822y3z13 - 26678525255325/2162902588y2z14 - 26678525255325/2/162902588yz15 - 6996427161975/2162902588z16 + 965701597805963/1152550168y2z15/ + 965701597805963/1152550168yz16 + 965707360556803/1152550168z17 + 910/141yz18/ + 530/423z19 + 235982375/112659yz19 + 147407075/75106z20 + 41856813119250/540725/647yz20 + 466506627125625/4325805176z21 + 813428824867634820/41635874819yz21 + 360803350499543037395/18985958917464z22 - 2044981435549759625/449434304yz22/ + 379820494972020072979875/5329391976832z23 - 74930243087916747964230028075/8201934252344448z24$
- $\mapsto j[83] = xyz10 + y2z10 + yz11 - 44/3y7z7 - 22000/423y6z8 - 12694/141y5z9 - 4440/47y4z10 - 572/9y3z11 - 1266/47y2z12 - 368/47yz13 - 592/423z14 + 291385/6627y6z9 - 2804215/242/99y4z11 - 354995030/1239249y3z12 - 112844135/413083y2z13 - 65172480/413083yz14 - 38026370/1239249z15 - 4104216934875/154493042y3z13 - 87610866574725/216290258/8y2z14 - 87610866574725/2162902588yz15 - 30151829486475/2162902588z16 - 9657361/51858727/2305100336y2z15 - 965724626357047/2305100336yz16 - 965736151858727/2/$

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⇒ j [84]=xy2z9+y3z9+y2z10+44/3y7z7+22000/423y6z8+12694/141y5z9+4440/47y4z10+\
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 $z13+51289774263611/2617293888yz14+21727413902435/5888911248z15-1323759792\backslash$   
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714970145775/104219962z22+1652695466624865838625/1058054976yz22+834251599\
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4
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82975/1013851790336z24
↳ j [112]=xy4z5+y5z5+y4z6+xyz8+y2z8+yz9-50y10z3+270y8z5+292y7z6-238y6z7-894y\
5z8-1010y4z9-628y3z10-231y2z11-87yz12+11z13-5346y7z7-18711y6z8-20493y5z9-\
4455y4z10+16929y3z11+20493y2z12+9801yz13+1782z14-107426/1269y6z9+1238705/\
4653y4z11-104279949410/237303y3z12-52166850719/79101y2z13-17395705898/263\
67yz14-52200514085/237303z15+22263444302265/3287086y3z13+467423176131315/\
46019204y2z14+467423176131315/46019204yz15+155734955899605/46019204z16-93\
56490298143/6130586y2z15-9356490298143/6130586yz16-9356490298143/6130586z\
17+485yz17+1840z18+106920yz18+93555z19-764254156340/237303yz19+3168673535\
/79101z20+16589299116393375/11504801yz20+64014825565742775/92038408z21-16\
2451373868908926605/14173914832yz21-20514628345434412995/3543478708z22+12\
2304080714970234219725/35973869184yz22+30999660984074405039458075/1938991\
5490176z23+211553781983683424781193575/8617740217856z24
↳ j [113]=xy5z4+y6z4+y5z5-xyz8-y2z8-yz9+75y10z3-675/2y8z5-498y7z6-168y6z7+30\
6y5z8+795/2y4z9+192y3z10+63/2y2z11+51/2yz12-24z13+12393/2y7z7+86751/4y6z8\
+95013/4y5z9+20655/4y4z10-78489/4y3z11-95013/4y2z12-45441/4yz13-4131/2z14\
+902/47y6z9-20940/517y4z11+4481086570/8789y3z12+13444813493/17578y2z13+13\
445349673/17578yz14+4482215115/17578z15-854678163465/108664y3z13-21714741\
66899745/184076816y2z14-2171474166899745/184076816yz15-723649357990035/18\
4076816z16+106352070291801/49044688y2z15+106352070291801/49044688yz16+106\
352070291801/49044688z17-915yz17-8715/4z18-123930yz18-433755/4z19+5967905\
885/1598yz19-71510445/1598z20-153838794976230825/92038408yz20-29680965044\
8392825/368153632z21+379100754580207174965/28347829664yz21+19102901522990\
0486715/28347829664z22-8465085153731213643475/2180234496yz22-228639664680\

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12z24
⇒ j [114]=xy6z3+y7z3+y6z4+xyz8+y2z8+yz9-355/2y10z3+4095/4y8z5+1804y7z6+959y6\
z7-1038y5z8-8765/4y4z9-3587/2y3z10-3363/4y2z11-1161/4yz12-41/2z13-58563/4\
y7z7-409941/8y6z8-448983/8y5z9-97605/8y4z10+370899/8y3z11+448983/8y2z12+2\
14731/8yz13+19521/4z14-456709/2538y6z9+2648150/4653y4z11-16803562855/1395\
9y3z12-33620602921/18612y2z13-1019124317/564yz14-33637230565/55836z15+287\
15244673095/1546864y3z13+602655259692735/21656096y2z14+602655259692735/21\
656096yz15+200641834269405/21656096z16-769409234887941/98089376y2z15-7694\
09234887941/98089376yz16-769409234887941/98089376z17-505yz17+27365/8z18+2\
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181783475/10828048yz20+82486202679399975/43312192z21-10548438775411220419\
5/3335038784yz21-53480743988955730695/3335038784z22+775647419365109709660\
25/8464439808yz22+18278145748430711894029175/4562333056512z23+22714341902\
5045581602456175/2027703580672z24
⇒ j [115]=xy7z2+y8z2+y7z3-xyz8-y2z8-yz9+1575/4y10z3-21105/8y8z5-5310y7z6-850\
5/2y6z7+450y5z8+33255/8y4z9+16605/4y3z10+17505/8y2z11+6075/8yz12+405/4z13\
+295245/8y7z7+2066715/16y6z8+2263545/16y5z9+492075/16y4z10-1869885/16y3z1\
1-2263545/16y2z12-1082565/16yz13-98415/8z14+292655/564y6z9-1745765/1034y4\
z11+79997752370/26367y3z12+320135288395/70312y2z13+320254000415/70312yz14\
+320314393255/210936z15-2460867528553425/52593376y3z13-4695016674413115/6\
6937024y2z14-4695016674413115/66937024yz15-17193038018796315/736307264z16\
+3522639307208265/196178752y2z15+3522639307208265/196178752yz16+352263930\
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24170922643625/68941921742848z24
⇒ j [116]=xy8z+y9z+y8z2+xyz8+y2z8+yz9-5335/8y10z3+59625/16y8z5+7111y7z6+2066\
9/4y6z7-2859/2y5z8-99575/16y4z9-47057/8y3z10-50433/16y2z11-17211/16yz12-1\
121/8z13-1086453/16y7z7-7605171/32y6z8-8329473/32y5z9-1810755/32y4z10+688\
0869/32y3z11+8329473/32y2z12+3983661/32yz13+362151/16z14-7832869/10152y6z\
9+94544155/37224y4z11-1324972802525/237303y3z12-10603655889767/1265616y2z\
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17+242915/32z18+5432265/4yz18+38025855/32z19-155350412628845/3796848yz19+\
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198479515/226782637312z22+24251267359343224416321425/575581906944yz22+292\
591298284635838904393275/16328349886464z23+73556186438621967928969329225/\
137883843485696z24
⇒ j [117]=xy9+y10+y9z+y10z2+5/3y9z3-5/3y8z4-50/3y7z5-42y6z6-182/3y5z7-170/3y\
4z8-35y3z9-41/3y2z10-11/3yz11-2/3z12+4765/8y10z3-20835/16y8z5+7895y7z6+13\
0465/4y6z7+114225/2y5z8+948125/16y4z9+314315/8y3z10+269835/16y2z11+78825/\
16yz12+6155/8z13+10426687/144y7z7+30873054127/121824y6z8+11274323719/4060\
8y5z9+2458368725/40608y4z10-593601799/2592y3z11-3754171309/13536y2z12-538\
6884667/40608yz13-1469170099/60912z14+157775815/59643y6z9-32234694805/349\
9056y4z11+132459543842675/22306482y3z12+530472611040205/59483952y2z13+885\
03433475315/9913992yz14+531270309400015/178451856z15-41040285445422225/44\
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331934448384z17+36475/4yz17-677825/32z18-7346578175/5076yz18-154292180075\
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566425/4724314129350402048z24
→ j [118]=x2y8-y10-2y9z-y8z2-2xyz8-2y2z8-2yz9-2y10z2-10/3y9z3+10/3y8z4+100/3\
y7z5+84y6z6+364/3y5z7+340/3y4z8+70y3z9+82/3y2z10+22/3yz11+4/3z12+285/2y10\
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2+165809429/10152yz13+45230621/15228z14-100544483/26508y6z9+237709835/176\
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865611471994802315/994173848769024z22+7037175725963290260484475/563590617\
216yz22+1956021496085718380059557275/883710087794688z23+12534642939968108\
162393838568625/9760979606095872z24
→ j [119]=x2z7+2xyz7+y2z7+2xz8+2yz8+z9+30800/423y6z8+30800/141y5z9+486470/15\
51y4z10+26060/99y3z11+70670/517y2z12+64340/1551yz13+88810/4653z14-7405525\
/24299y6z9+353409675/534578y4z11+5600660600/4543913y3z12+5809184925/45439\
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15+9366558850788075/47583856936z16+214662048107506095/1584756481y2z15+214\
662048107506095/1584756481yz16+214662048107506095/1584756481z17+49000/141\
yz18+91175/423z19+8122630625/826166yz19+1475494125/75106z20+1556636160857\
9625/11895964234yz20+612200237651841375/95167713872z21+535252907767487420\
2025/7327913968144yz21+87107704943898257223025/43967483808864z22-15588448\
3661694095625/9887554688yz22+1379179562234115213518045625/117246623490304\
z23-266465097028654098023229668060875/180442553551577856z24
→ j [120]=xy2z6+y3z6+xyz7+2y2z7+yz8-20y10z2-100y9z3-315y8z4-660y7z5-840y6z6-\
630y5z7-210y4z8+60y3z9+90y2z10+35yz11+25z12-1852/3y7z7-2785226/1269y6z8-1\
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03/4653yz13-949063/13959z14-200660825/72897y6z9+2647176050/267289y4z11+24\
7943744230/13631739y3z12+79951936540/4543913y2z13+34949943690/4543913yz14\
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7583856936z16-11452257803653034639/50712207392y2z15-11452223572913045039/\
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5/2538z19+161736531775/2478498yz19+43666153975/413083z20-1080912810452488\
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55887045/417691096184208yz21+234245080384114055828470205/1002458630842099\
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5/468986493961216z23+666402566762256623805660342503825/270663830327366784\
z24

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$\mapsto j[121]=x^2yz^6-y^3z^6-2y^2z^7-yz^8+40y^10z^2+200y^9z^3+630y^8z^4+1320y^7z^5+1680y^6z^6+126\backslash$   
 $0y^5z^7+420y^4z^8-120y^3z^9-180y^2z^{10}-70yz^{11}-50z^{12}+3392/3y^7z^7+5062336/1269y^6z^8+2\backslash$   
 $917726/423y^5z^9+33733390/4653y^4z^{10}+1453558/297y^3z^{11}+3249508/1551y^2z^{12}+2849\backslash$   
 $128/4653yz^{13}+1667198/13959z^{14}+824859875/145794y^6z^9-21530818075/1069156y^4z\backslash$   
 $11-503166594530/13631739y^3z^{12}-324495046255/9087826y^2z^{13}-282966185235/1817\backslash$   
 $5652yz^{14}-60700373165/13631739z^{15}+273674327539725/56179288y^3z^{13}+1075367159\backslash$   
 $69527575/13595387696y^2z^{14}+107536715969527575/13595387696yz^{15}+413075287049\backslash$   
 $14125/13595387696z^{16}+1390709606701412201/3622300528y^2z^{15}+1390704716595699\backslash$   
 $401/3622300528yz^{16}+1390706346630937001/3622300528z^{17}+929435/423yz^{18}-20988\backslash$   
 $05/1269z^{19}-60756900425/450636yz^{19}-365571595775/1652332z^{20}+153631004775148\backslash$   
 $125/849711731yz^{20}+2397615030608356125/27190775392z^{21}-10677692774577796473\backslash$   
 $522745/119340313195488yz^{21}-8543001261389272550615555/179010469793232z^{22}-1\backslash$   
 $533912321925569677241875/19775109376yz^{22}+15562075862209915374841875/83747\backslash$   
 $58820736z^{23}-647400216245483540020440549874525/154665045901352448z^{24}$

$\mapsto j[122]=xy^3z^5+y^4z^5+y^3z^6-xyz^7-y^2z^7-yz^8+30y^10z^2+135y^9z^3+405y^8z^4+720y^7z^5+630y\backslash$   
 $6z^6-630y^4z^8-720y^3z^9-405y^2z^{10}-120yz^{11}-45z^{12}+2078y^7z^7+340566/47y^6z^8+583893\backslash$   
 $47y^5z^9+6709595/517y^4z^{10}+95958/11y^3z^{11}+1933377/517y^2z^{12}+569694/517yz^{13}+102\backslash$   
 $533/517z^{14}-94912875/97196y^6z^9-2911102875/2138312y^4z^{11}-13490258445/4543913\backslash$   
 $y^3z^{12}-178197534495/18175652y^2z^{13}-306906320115/36351304yz^{14}-53243896545/90\backslash$   
 $87826z^{15}+5204006726201775/1235944336y^3z^{13}+171211027298041575/27190775392y\backslash$   
 $2z^{14}+171211027298041575/27190775392yz^{15}+56722879321602525/27190775392z^{16}-\backslash$   
 $12628824906906423/1034943008y^2z^{15}-12630222079967223/1034943008yz^{16}-126295\backslash$   
 $23493436823/1034943008z^{17}+341835/94yz^{18}-290635/94z^{19}+13423502475/3304664y\backslash$   
 $z^{19}-314243831475/3304664z^{20}+320302952555061375/1699423462yz^{20}+42384348010\backslash$   
 $01854125/54381550784z^{21}-1026776842710772806212955/11365744113856yz^{21}-5814\backslash$   
 $88683445294414836705/11365744113856z^{22}-3080895255663027470128125/39550218\backslash$   
 $752yz^{22}-164562449972300743426396875/4785576468992z^{23}+21707136779621806841\backslash$   
 $0739043975/1636667152395264z^{24}$

$\mapsto j[123]=xy^4z^4+y^5z^4+y^4z^5+xyz^7+y^2z^7+yz^8-50y^10z^2-220y^9z^3-720y^8z^4-1380y^7z^5-147\backslash$   
 $0y^6z^6-630y^5z^7+420y^4z^8+780y^3z^9+495y^2z^{10}+155yz^{11}+55z^{12}-9736/3y^7z^7-14313938\backslash$   
 $1269y^6z^8-8159288/423y^5z^9-93567545/4653y^4z^{10}-4008014/297y^3z^{11}-8977919/1551\backslash$   
 $y^2z^{12}-7971314/4653yz^{13}-4227799/13959z^{14}+68457475/291588y^6z^9+13681240225/2\backslash$   
 $138312y^4z^{11}+177807853615/13631739y^3z^{12}+366210664405/18175652y^2z^{13}+4998402\backslash$   
 $44985/36351304yz^{14}+187299515465/27263478z^{15}-724354328812725/112358576y^3z^{1}\backslash$   
 $3-2076609628894013775/190335427744y^2z^{14}-2076609628894013775/190335427744y\backslash$   
 $z^{15}-849553395885257625/190335427744z^{16}-160256670164141259683/202848829568\backslash$   
 $y^2z^{15}-160256259395261384483/202848829568yz^{16}-160256441959207995683/202848\backslash$   
 $829568z^{17}-2308780/423yz^{18}+6216265/1269z^{19}+8685932675/901272yz^{19}+322019002\backslash$   
 $025/3304664z^{20}-27489769156849798125/95167713872yz^{20}-58343927494571985375\backslash$   
 $380670855488z^{21}+440042360916873913967311355/3341528769473664yz^{21}+13866977\backslash$   
 $9041579387287118415/2110439222825472z^{22}+4618755900045255937019375/3955021\backslash$   
 $8752yz^{22}-36008438343336428263980999375/1875945975844864z^{23}+18651077950560\backslash$   
 $763379965724423502925/2165310642618934272z^{24}$

$\mapsto j[124]=xy^5z^3+y^6z^3+y^5z^4-xyz^7-y^2z^7-yz^8+75y^10z^2+255y^9z^3+810y^8z^4+1440y^7z^5+126\backslash$   
 $0y^6z^6-1260y^4z^8-1440y^3z^9-810y^2z^{10}-255yz^{11}-75z^{12}+5806y^7z^7+2841226/141y^6z^8+1\backslash$   
 $616641/47y^5z^9+18500915/517y^4z^{10}+790633/33y^3z^{11}+5298074/517y^2z^{12}+1573338/5\backslash$   
 $17yz^{13}+822383/1551z^{14}-1051504275/194392y^6z^9+18635231025/4276624y^4z^{11}+2532\backslash$   
 $457335/534578y^3z^{12}-19479647115/2138312y^2z^{13}-57594525255/4276624yz^{14}-90214\backslash$   
 $64715/1069156z^{15}+1178008818946425/145405216y^3z^{13}+334460480269289625/22392\backslash$   
 $403264y^2z^{14}+334460480269289625/22392403264yz^{15}+153047122151540175/2239240\backslash$   
 $3264z^{16}+719571432155612807313/405697659136y^2z^{15}+719570336771933140113/405\backslash$   
 $697659136yz^{16}+719570884463772973713/405697659136z^{17}+444335/47yz^{18}-2577985\backslash$   
 $/282z^{19}+41330722575/388784yz^{19}+21221903925/388784z^{20}+4500217777107531375\backslash$

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82164476575/44612964z20+142035481156802119875/2471888672yz20+248372839402\
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3481001789504610481352075/439389334947024z22-2110368281564678373555572125\
/40578524439552yz22-33244263077213633196406501625/130189432576896z23+4689\
2356386637393626600336845176375/3037059083153829888z24
→ j [155]=x2z2+2xyz2+y2z2+2xz3+2yz3+z4-2/3xy2z3-2/3y3z3-2/3xyz4-4/3y2z4-2/3y\
z5-20/3y10z-80/3y9z2-90y8z3-160y7z4-140y6z5+140y4z7+160y3z8+90y2z9+80/3yz\
10+10z11-830/27y10z3+275y8z5+880/27y7z6-38500/27y6z7-29920/9y5z8-105875/2\
7y4z9-75970/27y3z10-11315/9y2z11-3485/9yz12-1810/27z13+413325y7z7+1359707\
25/94y6z8+235722975/94y5z9+2744424075/1034y4z10+40093425/22y3z11+83109997\

```

$$\frac{5}{1034y^2z^{12}+252250575/1034yz^{13}+21991875/517z^{14}+49791120211145/35427942y^6z^9-981707768208925/259804908y^4z^{11}-26915262096774350/3312512577y^3z^{12}-174152/27294983795/2208341718y^2z^{13}-6047140843471955/1472227812yz^{14}-3863048585443/400/3312512577z^{15}+99220216387124175/617972168y^3z^{13}-1845524026948536245/95/167713872y^2z^{14}-1845495476634374645/95167713872yz^{15}-17125423075408578395/9/5167713872z^{16}-16814191835989165659975/101424414784y^2z^{15}-16814191835989165/659975/101424414784yz^{16}-16814191835989165659975/101424414784z^{17}-43000/27y/z^{17}+65075/54z^{18}+45335250/47yz^{18}-45342375/94z^{19}-53204132191387525/12045500/28yz^{19}-21036974719117025/401516676z^{20}+335196368084571524425/428254712424y/z^{20}-232347108822197506862575/32547358144224z^{21}-32253993343737504882347625/29311655872576yz^{21}-354637525372144350327946125/117246623490304z^{22}+732545/6958222560929731798875/91301679988992yz^{22}-6014622677565739087337412588900/9875/4330621285237868544z^{23}+12078241497674805171302189500227875/668305753/8947328z^{24}}$$

$\mapsto j[156]=x^2yz+2xy^2z+y^3z+2xyz^2+2y^2z^2+yz^3+1/3xy^2z^3+1/3y^3z^3+1/3xyz^4+2/3y^2z^4+1/3yz^5+45xy^8+35y^9-180y^7z^2-420y^6z^3-630y^5z^4-630y^4z^5-420y^3z^6+90xyz^7-90y^2z^7+45x/z^8+90yz^8+40z^9+10/9y^{10}z+20/9y^9z^2-5y^8z^3-160/3y^7z^4-490/3y^6z^5-280y^5z^6-910/3y^4/z^7-640/3y^3z^8-95y^2z^9-220/9yz^{10}-65/9z^{11}+48600y^9z^3+218700y^8z^4+988200y^7z^5+243/8100y^6z^6+3669300y^5z^7+3588300y^4z^8+2316600y^3z^9+959850y^2z^{10}+271350yz^{11}+44550/z^{12}+1870/81y^{10}z^3-1925/9y^8z^5+880/81y^7z^6+100100/81y^6z^7+75680/27y^5z^8+265375/81y^4z^9+189530/81y^3z^{10}+28135/27y^2z^{11}+8665/27yz^{12}+4490/81z^{13}-6326265y^7z^7-18/580308065/846y^6z^8-10574680565/282y^5z^9-120965644625/3102y^4z^{10}-5171874545/1/98y^3z^{11}-11529185295/1034y^2z^{12}-10319323445/3102yz^{13}-2632816985/4653z^{14}+140/67277731418885/850270608y^6z^9-267328523142314525/6235317792y^4z^{11}-294173585/1974370725/39750150924y^3z^{12}-2490411503534248985/53000201232y^2z^{13}-14541270/4549717015/35333467488yz^{14}+832563980759702225/79500301848z^{15}-142645030789/52862325/4943777344y^3z^{13}-5119706412424471629935/2284025132928y^2z^{14}-511970/6755028241569135/2284025132928yz^{15}+490164644639193222205/761341710976z^{16}+1078547677678234556755035/811395318272y^2z^{15}+1078548318477687162067035/811/395318272yz^{16}+1078547036878781951443035/811395318272z^{17}+105500/81yz^{17}-162/175/162z^{18}-1499168825/141yz^{18}+8738470825/846z^{19}-9024426703351886825/28909/200672yz^{19}-2037318893266015825/9636400224z^{20}-2464013188446215782366225/10/278113098176yz^{20}-41789846500130973998889725/781136595461376z^{21}+4311656607/55395824844433078675/4455371692631552yz^{21}+3497756262768292493018323747225/534644460311578624z^{22}+176529455932853120329574124096625/2191240319735808y/z^{22}+16074902003602993181695479936506184125/103934910845708845056z^{23}-41834/546022107683399753037302991167375/2887080856825245696z^{24}$

$\mapsto j[157]=x^2y^2+2xy^3+y^4+2xy^2z+2y^3z+y^2z^2-2/3xy^2z^3-2/3y^3z^3-2/3xyz^4-4/3y^2z^4-2/3y/z^5+10y^9-10z^9-80/3y^9z^2-90y^8z^3-160y^7z^4-140y^6z^5+140y^4z^7+160y^3z^8+90y^2z^9+20yz^{10}+10z^{11}-665/27y^{10}z^3+1375/6y^8z^5-1760/27y^7z^6-40810/27y^6z^7-29920/9y^5z^8-20762/5/54y^4z^9-73735/27y^3z^{10}-21805/18y^2z^{11}-6695/18yz^{12}-1735/27z^{13}+736795/2y^7z^7+726842665/564y^6z^8+419688265/188y^5z^9+4881546825/2068y^4z^{10}+213665245/132y^3z^{11}+1473666285/2068y^2z^{12}+446844345/2068yz^{13}+116587885/3102z^{14}+467289960822/385/283423536y^6z^9-9127602561460025/2078439264y^4z^{11}-120995451431889425/132/50050308y^3z^{12}-154502068214333885/17666733744y^2z^{13}-51278297627949115/11777/822496yz^{14}-33453810794614975/26500100616z^{15}+906953634241417525/4943777344/y^3z^{13}-53912042597130809835/761341710976y^2z^{14}-161735480650938099905/228402/5132928yz^{15}-580748326140071838055/2284025132928z^{16}-1788471347732991366897/15/811395318272y^2z^{15}-178847134773299136689715/811395318272yz^{16}-1788471347/73299136689715/811395318272z^{17}-40150/27yz^{17}+125725/108z^{18}+39832975/47yz^{18}-248449325/564z^{19}-481438957885768325/9636400224yz^{19}-204441736415955325/32/12133408z^{20}-1119979083259190232725/3426037699392yz^{20}-26201042343147240768/46225/260378865153792z^{21}-331129424884733448349858425/234493246980608yz^{21}-$

```

3648918601091100687569986825/937972987922432z22+6697590805725544159757041\
7125/730413439911936yz22-643543449144748754616730497333700375/34644970281\
902948352z23+2312528707662090601738417382851770625/962360285608415232z24
↳ j [158]=x2z+2xyz+y2z+2xz2+2yz2+z3-2/3xy2z2-2/3y3z2-2/3xyz3-4/3y2z3-2/3yz4-\
20/3y10-80/3y9z-90y8z2-160y7z3-140y6z4+140y4z6+160y3z7-60xyz8+30y2z8-100/\
3yz9+10z10-2450/27y10z2-9250/27y9z3-2150/3y8z4-7300/9y7z5-2380/9y6z6+1820\
/3y5z7+9500/9y4z8+7550/9y3z9+1090/3y2z10+2690/27yz11+580/27z12+54375/4y10\
z3+851175/8y8z5+312090y7z6+715155/2y6z7+84195y5z8-1966425/8y4z9-1281615/4\
y3z10-1319535/8y2z11-433845/8yz12-46095/4z13+559449685/648y7z7+1656592613\
605/548208y6z8+604442226925/182736y5z9+130593991175/182736y4z10-319676320\
45/11664y3z11-201910849615/60912y2z12-289649124265/182736yz13-78992436625\
/274104z14-17472459515/357858y6z9+842146700825/5248584y4z11+2390269317429\
425/33459723y3z12+9544837831361185/89225928y2z13+397105055726965/3717747y\
z14+9525468506032975/267677784z15-2752812051085923525/2471888672y3z13-824\
4367425143148375/4943777344y2z14-8244367425143148375/4943777344yz15-27387\
43322971301325/4943777344z16+211821064966441572815/71128810368y2z15+21182\
1100926006814415/71128810368yz16+211821037700397598415/71128810368z17-176\
4975/2yz17-2997075/16z18-394881744125/22842yz18-8291014780025/548208z19+6\
9884274190681925/133838892yz19-438520928661025/89225928z20-57726703117712\
8798875/2471888672yz20-1112473953175012074375/9887554688z21+5993695670872\
60601425167775/292926223298016yz21+14885829279357080546349916825/14060458\
718304768z22-16623359014950125055050528375/40578524439552yz22+27477279038\
4510449804245496125/4166061842460672z23-915031956057938001524614109807421\
25/3037059083153829888z24
↳ j [159]=x2y+2xy2+y3+2xyz+2y2z+yz2-2/3xy3z-2/3y4z-2/3xy2z2-4/3y3z2-2/3y2z3+\
10y10+80/3y9z+90y8z2+160y7z3+140y6z4-140y4z6-160y3z7+60xyz8-30y2z8+100/3y\
z9-20/3z10+2410/27y10z2+9650/27y9z3+2350/3y8z4+8900/9y7z5+5180/9y6z6-700/\
3y5z7-6700/9y4z8-5950/9y3z9-890/3y2z10-2170/27yz11-500/27z12-54375/4y10z3\
-851175/8y8z5-312090y7z6-715155/2y6z7-84195y5z8+1966425/8y4z9+1281615/4y3\
z10+1319535/8y2z11+433845/8yz12+46095/4z13-559569365/648y7z7-165694140336\
5/548208y6z8-604639142765/182736y5z9-130797216775/182736y4z10+31959013805\
/11664y3z11+201891765455/60912y2z12+289631931305/182736yz13+78988107665/2\
74104z14+18058968265/357858y6z9-867303535525/5248584y4z11-239056339916312\
5/33459723y3z12-9545414450904185/89225928y2z13-794222524233355/7435494yz1\
4-9525284419337375/267677784z15+2753042527186728525/2471888672y3z13+57728\
361763254800025/34606441408y2z14+57728361763254800025/34606441408yz15+191\
85766382640600675/34606441408z16-1364048854658712353965/497901672576y2z15\
-1364049100843428238765/497901672576yz16-1364048696989849371565/497901672\
576z17+1764975/2yz17+2997075/16z18+394875663325/22842yz18+8291192536825/5\
48208z19-69888585468885175/133838892yz19+436943255084525/89225928z20+4040\
801094796307983125/17303220704yz20+7787846744821476359625/69212882816z21-\
1047672181803023795345498225/512620890771528yz21-103846347836692127525249\
141075/98423211028133376z22+16678095795938225381262763375/40578524439552y\
z22-1287321999810395201311669625375/29162432897224704z23+5852694774994777\
83471587033435652875/21259413582076809216z24
↳ j [160]=x2+2xy+y2+2xz+2yz+z2-2/3xy2z-2/3y3z-2/3xyz2-4/3y2z2-2/3yz3-1/3xy2z\
3-1/3y3z3-1/3xyz4-2/3y2z4-1/3yz5-15xy8-25/3y9+60y7z2+140y6z3+210y5z4+210y\
4z5+140y3z6-30xyz7+30y2z7-15xz8-30yz8-40/3z9-10/9y10z-20/9y9z2+5y8z3+160/\
3y7z4+490/3y6z5+280y5z6+910/3y4z7+640/3y3z8+95y2z9+220/9yz10+65/9z11-1620\
0y9z3-72900y8z4-329400y7z5-812700y6z6-1223100y5z7-1196100y4z8-772200y3z9-\
319950y2z10-90450yz11-14850z12-1870/81y10z3+1925/9y8z5-880/81y7z6-100100/\
81y6z7-75680/27y5z8-265375/81y4z9-189530/81y3z10-28135/27y2z11-8665/27yz1\
2-4490/81z13+2245305y7z7+6598212305/846y6z8+3759783605/282y5z9+4307153082\

```

```

5/3102y4z10+1845502865/198y3z11+4126709415/1034y2z12+3701280965/3102yz13+\
947136845/4653z14-4520669493739885/850270608y6z9+85503196850075525/623531\
7792y4z11+919902607150906925/39750150924y3z12+746481393750364385/53000201\
232y2z13+13136147016481615/35333467488yz14-297400526011588025/79500301848\
z15+4888765901631375325/4943777344y3z13+1818903882360545256935/2284025132\
928y2z14+1818904224964315196135/2284025132928yz15-146568629352993015205/7\
61341710976z16-355094819337109366083795/811395318272y2z15-355095032936926\
901187795/811395318272yz16-355094605737291830979795/811395318272z17-10550\
0/81yz17+162175/162z18+548903525/141yz18-3026775025/846z19+27820127662598\
75825/28909200672yz19+631251774268012825/9636400224z20+834549760512521153\
551225/10278113098176yz20+14530392406701081768114725/781136595461376z21-1\
43805121095377365823474118475/4455371692631552yz21-1168361866569917419774\
687429825/53464460311578624z22-58813484594869708628716919721625/219124031\
9735808yz22-5298255598048491212044632289975509125/103934910845708845056z2\
3+13773306630404801119508893030689754375/2887080856825245696z24

```

See also: [Section D.4.31.1 \[borderBasis\]](#), page 1267.

### D.4.31.3 rJanet

Procedure from library `rstandard.lib` (see [Section D.4.31 \[rstandard.lib\]](#), page 1266).

**Usage:** `rJanet(I)`; `I` is an ideal.

**Return:** ideal, a Janet basis for `I`.

**Purpose:** Computes a Janet basis for the ideal given by the generators in `I` for any ordering.

**Example:**

```

LIB "rstandard.lib";
ring R= 32003,(a,b,c,d,e,f,g),dp;
ideal i=
a+b+c+d+e+f+g,
ab+bc+cd+de+ef+fg+ga,
abc+bcd+cde+fde+efg+fga+gab,
abcd+bcde+cdef+defg+efga+fgab+gabc,
abcde+bcdef+cdefg+defga+efgab+fgabc+gabcd,
abcdef+bcdefg+cdefga+defgab+efgabc+fgabcd+gabcde,
abcdefg-1;
ideal j = rJanet(i);  j;
→ Length of Janet basis: 210
→ j[1]=a+b+c+d+e+f+g
→ j[2]=b2+bd-cd+be-de+bf-ef+2bg+cg+dg+eg+g2
→ j[3]=bc2-bcd+c2d+cef-def-c2g-ceg+deg+bfg+cfg+dfg+efg+f2g-bg2-2cg2-dg2-eg2\
+fg2-g3
→ j[4]=c2d2+cd3+bd2e+cd2e+d3e-2bde2-2cde2-de3+2bcd+cd2f+2cdef-2be2f-ce2f-3\
de2f-e3f+bef2-cef2+def2-e2f2-2bcdg+2cd2g+2bceg+cdeg+2d2eg+de2g-3bcfg+2bdf\
g-cdfg+2d2fg-befg-cefg-3defg-e2fg-2cf2g+df2g-4ef2g-f3g+bcg2-c2g2+cdg2-2be\
g2-3ceg2-2e2g2+5bfg2+5cfg2+7dfg2+4efg2+f2g2-2bg3-3cg3-2dg3-4eg3+5fg3-2g4
→ j[5]=bcd2-bcde+cd2e-cdef+d2ef-cd2g+cdeg-d2eg+bcfg-bdfg-d2fg+befg+cefg+def\
g+e2fg-df2g+2ef2g-bcg2+bdg2+d2g2-2bfg2-2cfg2-3dfg2-efg2-f2g2+bg3+cg3+2dg3\
+eg3-2fg3+g4
→ j[6]=c3d+cd3+bcde+2c2de+bd2e-cd2e+d3e-2bde2-2cde2-de3+2bcd+cd2f+cd2f+bce\
f+c2ef-bdef+2cdef-d2ef-2be2f-4de2f-e3f+bef2-e2f2-c3g-3bcdg+3c2dg+4cd2g+bc\
eg-2c2eg+bdeg+cdeg+3d2eg-ce2g+2de2g-3bcfg-c2fg+3bdfg+3d2fg-befg+cefg-4def\
g-e2fg+bfg2-cf2g+3df2g-3ef2g-5c2g2-bdg2+cdg2-d2g2-3beg2-8ceg2+deg2-3e2g2+

```



```

9bfg2+7cfg2+11dfg2+6efg2+6f2g2-5bg3-9cg3-6dg3-8eg3+8fg3-5g4
⇒ j [7]=bde2f-11823cde2f-7526d2e2f-2154be3f+7864ce3f+15572de3f+15266e4f+1331\
0c3f2+12479bcd2f+8143c2df2+4270bd2f2+10092cd2f2-5691d3f2-14681bcef2+14474\
c2ef2-11522bdef2-179cdef2+14676d2ef2-2804be2f2+5761ce2f2+5413de2f2-9491e3\
f2+8304bcf3+1258c2f3+8248bdf3-6374cdf3-6552d2f3+9252bef3+12153cef3-15012d\
ef3-13865e2f3+6826bf4+2029cf4-4303df4+1572ef4-3519f5-11980c4g+9748bd3g+86\
60cd3g+15466d4g+6444c3eg-4232bcdeg+4568c2deg-14626bd2eg-786cd2eg-3625d3eg\
+1987bce2g+4552c2e2g-11880bde2g-15172cde2g-15100d2e2g+1684be3g-2235ce3g-2\
457de3g+7861e4g-3828c3fg+11118bcd2fg+2913c2dfg+6653bd2fg-11706cd2fg-9675d3\
fg+13393bcefg-8499c2efg+4752bdefg-3700cdefg+13113d2efg-2959be2fg-5611ce2f\
g+15743de2fg+7880e3fg+8042bcf2g+15356c2f2g+4811bdf2g-8874cdf2g+12656d2f2g\
+941bef2g+7466cef2g+10054def2g-9344e2f2g-10679bf3g-14371cf3g-12656df3g-51\
04ef3g+3433f4g-5370c3g2-3464bcdg2-8570c2dg2+6077bd2g2+3404cd2g2-6569d3g2-\
9558bceg2+15307c2eg2-14132bdeg2-10957cdeg2-8341d2eg2+8321be2g2+1980ce2g2-\
8582de2g2+8677e3g2-11191bcfg2+10521c2fg2+9012bdfg2-13634cdfg2-3808d2fg2+3\
919befg2+6111cef2g+10508defg2+2202e2fg2+3217bf2g2+10056cf2g2+2240df2g2-14\
043ef2g2-9755f3g2+5900bcg3+15877c2g3+5228bdg3-3217cdg3-4407d2g3-6835beg3+\
9399ceg3+882deg3-8551e2g3+12586bfg3-11500cfg3+8923dfg3+7550efg3-6574f2g3-\
14049bg4+13640cg4+15435dg4+4111eg4-15602fg4-10030g5
⇒ j [8]=c2e2f-8933cde2f-4268d2e2f-9066be3f-1129ce3f-2379de3f-14284e4f-13185c\
3f2+7955bcd2f-7656c2df2-5995bd2f2+9705cd2f2-7658d3f2-10633bcef2+2156c2ef2\
+3772bdef2+511cdef2-14468d2ef2-2189be2f2+1185ce2f2-1262de2f2-8051e3f2+729\
9bcf3-11885c2f3-4273bdf3-7906cdf3+6861d2f3+11955bef3+4630cef3+2964def3+71\
31e2f3+5282bf4-2891cf4+14815df4-8785ef4+13656f5-10635c4g+8702bd3g+9201cd3\
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→ j [35]=d4e+8045cde2f+6931d2e2f+4847be3f+10826ce3f+13320de3f-4586e4f+13524c\
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5330bcf3-6383c2f3+1323bdf3-1616cdf3+6022d2f3+7581bef3+10611cef3-8171def3+\
5943e2f3+493bf4+10246cf4-13718df4-5602ef4-6434f5+1007c4g+732bd3g-898cd3g-\
12935d4g-11619c3eg-11984bcdeg-15074c2deg-14890bd2eg+5908cd2eg+8075d3eg+38\
04bce2g-10819c2e2g+3687bde2g+14669cde2g-3234d2e2g-9745be3g-7758ce3g-981de\
3g+389e4g-1918c3fg+4383bcd2fg+10047c2dfg-5103bd2fg+9341cd2fg-14370d3fg+598\
4bcefg-1890c2efg+7792bdefg+11684cdefg-14510d2efg+15933be2fg+5012ce2fg+741\
6de2fg-13231e3fg+2459bcf2g+6903c2f2g-5105bdf2g+5301cdf2g-14769d2f2g+15493\
bef2g-15328cef2g+3441def2g-9776e2f2g+10234bf3g+9476cf3g-11720df3g+2409ef3\
g+11290f4g-2554c3g2+11813bcdg2+3787c2dg2-9913bd2g2+11743cd2g2+10811d3g2-5\
858bceg2-9974c2eg2+12799bdeg2-12588cdeg2-13894d2eg2+5209be2g2-6912ce2g2+1\
1171de2g2-11593e3g2+11203bcfg2-10106c2fg2-4152bdfg2+1621cdfg2+6972d2fg2-1\
846befg2+955cefg2+1734defg2-5827e2fg2+452bf2g2-13047cf2g2+13055df2g2+1340\
5ef2g2-7606f3g2+5624bcg3-3155c2g3-1939bdg3+9838cdg3-14929d2g3+14178beg3+4\
471ceg3-5172deg3+13976e2g3-2269bfg3+10707cfg3-9861dfg3-6666efg3-5928f2g3+\
13494bg4+15880cg4+14976dg4-11010eg4+15663fg4-11945g5
→ j [36]=cd3e+4164cde2f-4930d2e2f+3210be3f+3237ce3f+10270de3f-1747e4f-12028c\
3f2-10171bcd2f-4117c2df2-3246bd2f2+10118cd2f2+13368d3f2+15093bcef2+2511c2\
ef2-1532bdef2-5068cdef2+4553d2ef2-9879be2f2-7174ce2f2+7979de2f2+5294e3f2-\
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+13572e2f3+15029bf4-2661cf4-1177df4+5739ef4+4410f5+8778c4g-815bd3g+13969c\
d3g+3249d4g-4983c3eg-1679bcdeg-13644c2deg-15272bd2eg+12666cd2eg+681d3eg+1\
1899bce2g+7945c2e2g+4618bde2g-13102cde2g-5212d2e2g+1050be3g+5569ce3g-1061\

```

```

6de3g-750e4g+2998c3fg-12846bcdg+8679c2dfg-10221bd2fg-5335cd2fg+15700d3fg\
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-5927de2fg-3407e3fg-8306bcf2g+7339c2f2g-4316bdf2g-7982cdf2g-14844d2f2g-11\
321bef2g+5074cef2g+9430def2g-13286e2f2g-215bf3g+2453cf3g-971df3g+10700ef3\
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0367bg4+11912cg4-13117dg4-14989eg4-2553fg4+6229g5
↳ j [37]=bd3e+15585cde2f+6620d2e2f-7236be3f-1779ce3f-3872de3f-11984e4f+6234c\
3f2-9765bcd2f-13477c2df2-7706bd2f2-10200cd2f2-1451d3f2+4482bcef2+13094c2e\
f2-10213bdef2+5773cdef2+13283d2ef2+2117be2f2+5240ce2f2+6017de2f2-7063e3f2\
-8750bcf3-2512c2f3+7988bdf3-4476cdf3+6604d2f3-6655bef3-6050cef3-1868def3-\
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050befg2-13100cef2g+11416defg2-12178e2fg2-9546bf2g2+6626cf2g2-7988df2g2-6\
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3+2883ceg3-15920deg3-9615e2g3-10170bfg3+13107cfg3+15511dfg3+6932efg3+4959\
f2g3-12598bg4+12426cg4-12774dg4+4956eg4+11380fg4-5820g5
↳ j [38]=c4e-2049cde2f-7326d2e2f-10573be3f+2613ce3f-11325de3f+14057e4f-7555c\
3f2+11545bcd2f-6424c2df2+4808bd2f2-4641cd2f2-14450d3f2-485bcef2-13326c2ef\
2-3488bdef2+10652cdef2+2864d2ef2-10458be2f2+7572ce2f2-15303de2f2+13349e3f\
2-10148bcf3+3156c2f3+6222bdf3+14603cdf3-10387d2f3-8676bef3-10860cef3+3787\
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-10571bce2g-13050c2e2g-3278bde2g-1891cde2g+10956d2e2g+12443be3g+4867ce3g+\
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3g+12925ef3g+6887f4g+2042c3g2+14586bcdg2-8333c2dg2-14248bd2g2-13584cd2g2+\
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3180d2fg2-12207befg2-9336cefg2+5625defg2-10581e2fg2-14232bf2g2+7978cf2g2+\
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2g3+14306beg3+8029ceg3+147deg3+12206e2g3+13160bfg3-8463cfg3-4398dfg3+6510\
efg3+14733f2g3+11498bg4-7351cg4+15056dg4-15654eg4+14335fg4+2427g5
↳ j [39]=d5-15168cde2f+3159d2e2f+1949be3f+14043ce3f-3040de3f+9184e4f+2047c3f\
2-11177bcd2f+3537c2df2-3258bd2f2+7956cd2f2+12260d3f2+10841bcef2+4185c2ef2\
-6750bdef2+1578cdef2+12262d2ef2-15506be2f2-13776ce2f2-11480de2f2+10151e3f\
2+7725bcf3-7251c2f3+1408bdf3-4974cdf3-4406d2f3-5569bef3-3175cef3+15239def\
3-1082e2f3+5405bf4+1808cf4+12728df4+13678ef4-5938f5+1694c4g-6857bd3g-1083\
4cd3g-5318d4g+13531c3eg-9105bcdeg+6836c2deg-1784bd2eg-279cd2eg-15693d3eg+\

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12272bce2g+5390c2e2g-2183bde2g+13801cde2g-11427d2e2g+12799be3g+9491ce3g+1\
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30d3fg-8349bcefg+17c2efg-15169bdefg-13211cdefg-9587d2efg-15565be2fg-1280c\
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g-6639bef2g-11466cef2g+13188def2g+7942e2f2g+12889bf3g-13145cf3g-4525df3g+\
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g2-12119de2g2-7266e3g2+14218bcfg2+636c2fg2+4028bdfg2-2160cdfg2-5281d2fg2+\
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g3-13774ceg3-8993deg3-2919e2g3-8918bfg3-2442cfg3+6950dfg3+10639efg3-387f2\
g3+8575bg4+9643cg4-11324dg4+9266eg4-13885fg4-5109g5
⇒ j [40]=cd4+5627cde2f-4828d2e2f-10130be3f-13957ce3f+6572de3f+13887e4f+3012c\
3f2+5585bcd2f-5375c2df2-15476bd2f2-727cd2f2-10898d3f2+3400bce2f-15405c2ef\
2+9947bdef2+14208cdef2+1621d2ef2-13560be2f2-8513ce2f2-7894de2f2-7978e3f2-\
2171bcf3+8552c2f3+13135bdf3+10454cdf3-3060d2f3-3452bef3-1638cef3-13088def\
3-13179e2f3-13316bf4-15123cf4+12463df4+9796ef4+15892f5-10666c4g+8810bd3g+\
2189cd3g+12857d4g-4929c3eg+9066bcdeg-970c2deg-14960bd2eg-6795cd2eg-7567d3\
eg-15819bce2g-9300c2e2g+4402bde2g+13891cde2g-2966d2e2g+1632be3g+3687ce3g-\
11696de3g+14667e4g-10988c3fg-14422bcd2fg+5287c2dfg+5403bd2fg-2489cd2fg-136\
01d3fg-14256bcefg-1462c2efg-10351bdefg-2286cdefg+1107d2efg-1879be2fg+5683\
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d2f2g+15409bef2g-15312cef2g-3318def2g+5270e2f2g-607bf3g+8476cf3g+3501df3g\
+5288ef3g+14664f4g+9738c3g2+2746bcdg2+2376c2dg2-615bd2g2-11568cd2g2-282d3\
g2-15934bceg2-525c2eg2+7185bdeg2+13482cdeg2-15291d2eg2+9193be2g2+15237ce2\
g2-15673de2g2-14105e3g2+807bcfg2+9511c2fg2+12547bdfg2-4284cdfg2+9853d2fg2\
+14570befg2+12576cefg2+13505defg2+3568e2fg2+2773bf2g2+5472cf2g2-1148df2g2\
+8476ef2g2-6791f3g2-86bcg3+12001c2g3-4348bdg3-9404cdg3-1830d2g3-15274beg3\
+11575ceg3+5812deg3+4833e2g3+7234bfg3-15669cfg3-7850dfg3-13148efg3-4347f2\
g3-10966bg4+8783cg4-4238dg4-8475eg4-12970fg4+12634g5
⇒ j [41]=bd4-5475cde2f+6113d2e2f+1236be3f+11914ce3f+14708de3f-390e4f-6751c3f\
2-12882bcd2f-8073c2df2+5960bd2f2-2583cd2f2-6400d3f2-2667bce2f+10942c2ef2-\
7132bdef2-5133cdef2-6185d2ef2-12971be2f2-7836ce2f2+5141de2f2+11815e3f2+86\
17bcf3+10105c2f3+7388bdf3+1692cdf3+9153d2f3-10297bef3+7518cef3+682def3-88\
37e2f3-13183bf4+400cf4+1059df4-7382ef4+1582f5+14048c4g-13952bd3g+7861cd3g\
-9678d4g+14339c3eg-1827bcdeg-1543c2deg+9656bd2eg+10919cd2eg-8391d3eg-1049\
1bce2g+2878c2e2g-737bde2g+14792cde2g+6129d2e2g+2302be3g-422ce3g-8013de3g-\
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9bcefg-195c2efg+7008bdefg+11938cdefg+7668d2efg-15014be2fg-5739ce2fg+732de\
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ef2g+10279cef2g-5184def2g-13461e2f2g+7428bf3g+3522cf3g-7828df3g-9630ef3g+\
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e2g2-4363e3g2-1319bcfg2-15923c2fg2+1327bdfg2-10617cdfg2-5406d2fg2+9986bef\
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g2+13303f3g2-8255bcg3+4989c2g3+7646bdg3+3092cdg3-15168d2g3-13798beg3-6151\
ceg3-8667deg3-13687e2g3+817bfg3-3879cfg3+302dfg3+15699efg3-5184f2g3-223bg\
4-8315cg4+2750dg4-11635eg4-11711fg4+7894g5
⇒ j [42]=c4d-6838cde2f-12575d2e2f-4314be3f+5389ce3f-4665de3f-10315e4f+10370c\
3f2+8148bcd2f+1022c2df2+12956bd2f2-7418cd2f2-9669d3f2-2293bce2f-4547c2ef2\
-12684bdef2-5889cdef2-5650d2ef2+14762be2f2+15112ce2f2-6069de2f2+12381e3f2\
+13475bcf3-7427c2f3-453bdf3+8550cdf3-7850d2f3+15106bef3+6862cef3-7071def3\
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3cd3g+8777d4g-3040c3eg-10513bcddeg+4423c2deg+10305bd2eg-13846cd2eg+108d3eg\
+10171bce2g+2827c2e2g-14304bde2g+10615cde2g-13508d2e2g+14030be3g+13666ce3\
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2f2g+15466bef2g+14845cef2g-12534def2g-4372e2f2g-3530bf3g-3037cf3g-197df3g\
+14779ef3g-5128f4g+8744c3g2+4326bcdg2+1648c2dg2-13667bd2g2-10907cd2g2-496\
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beg3-8041ceg3-13699deg3-12231e2g3+11009bfg3-12366cfg3-2676dfg3+12465efg3+\
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→ j [43]=c5-627cde2f+12983d2e2f+13461be3f-12722ce3f-1902de3f-13705e4f+10635c\
3f2-4278bcd2f-10929c2df2-7789bd2f2-2142cd2f2-15773d3f2-13716bcef2-13959c2\
ef2-7597bdef2-947cdef2-7258d2ef2+8759be2f2+5114ce2f2+8895de2f2+11401e3f2+\
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3+5874e2f3+10300bf4+7095cf4+2349df4+8876ef4-7970f5+5890c4g-12962bd3g-1260\
1cd3g-11920d4g-2991c3eg+3603bcdeg+1229c2deg-10323bd2eg+1979cd2eg-12915d3e\
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11ceg3-779deg3+15761e2g3-2102bfg3-9023cfg3-4924dfg3+8477efg3-15267f2g3-10\
021bg4-9011cg4-12981dg4-7663eg4+13207fg4-15883g5
→ j [44]=f6+14502cde2fg-13623d2e2fg-9624be3fg-5286ce3fg-14763de3fg+6291e4fg-\
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72d2fg3-4977befg3-1058cef3g3+15557defg3+3565e2fg3+308bf2g3-3915cf2g3-13681\
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→ j [45]=ef5+11572cde2fg-3802d2e2fg-2963be3fg+7487ce3fg+11634de3fg+1481e4fg-\
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→ j [46]=df5+8372cde2fg-7557d2e2fg-2916be3fg-14748ce3fg+13461de3fg+6868e4fg+\
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fg+15171c3f2g-5551bcd2fg-14159c2df2g-13104bd2f2g+13569cd2f2g+14064d3f2g-1\
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f2g4-5227bg5+15821cg5+13545dg5+9117eg5+6388fg5-1550g6
⇒ j [49]=e2f4+3118cde2fg-1019d2e2fg+6886be3fg+15789ce3fg+9943de3fg-7500e4fg-\
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fg4-189f2g4+9054bg5-6520cg5-12610dg5-5733eg5+6074fg5+12891g6
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↳ j [85]=de2f2g4-11965e3f2g4-6571bcf3g4-11437c2f3g4-10159bdf3g4-3273cdf3g4+4\
295d2f3g4+14252bef3g4+1254cef3g4+11711def3g4+1493e2f3g4+8761bf4g4+12621cf\
4g4+11735df4g4-3533ef4g4+5084f5g4-2530c4g5+4213bd3g5-3638cd3g5-12794d4g5+\
4582c3eg5+6054bcdeg5-140c2deg5+5072bd2eg5-3793cd2eg5-11433d3eg5-12336bce2\
g5-270c2e2g5-8546bde2g5+9133cde2g5-646d2e2g5+5007be3g5-11031ce3g5+9652de3\
g5+15086e4g5+4240c3fg5-1076bcdfg5+12192c2dfg5+9373bd2fg5-11824cd2fg5+2230\
d3fg5+6322bcefg5+1767c2efg5+15107bdefg5-13673cdefg5-6437d2efg5-3127be2fg5\
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f2g5-4734d2f2g5-6674bef2g5+1791cef2g5+14489def2g5-866e2f2g5+2921bf3g5-385\
1cf3g5+1466df3g5+4395ef3g5-6728f4g5-15158c3g6-770bcdg6-7232c2dg6+10496bd2\
g6-8432cd2g6+7372d3g6-3645bceg6-13437c2eg6-10710bdeg6+13101cdeg6-10189d2e\
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516cdfg6+7128d2fg6-14947befg6-4523cefg6+2813defg6+10490e2fg6+5393bf2g6+12\
829cf2g6+3488df2g6+11366ef2g6+1654f3g6-12197bcg7+8383c2g7-788bdg7-13956cd\
g7+14402d2g7+11974beg7-9580ceg7-14671deg7-15604e2g7-7877bfg7+12264cfg7-69\
48dfg7-11692efg7-3006f2g7+6111bg8-8848cg8+10314dg8+2521eg8+2938fg8+9550g9\
-9515bc-15857c2+14931bd+6522cd-879d2+4652be-9025ce-3420de-7315e2+14919bf+\
15569cf+8733df-13238ef+370f2+12708bg-1253cg-5342dg-9585eg-7004fg+775g2
↳ j [86]=ce2f2g4+15814e3f2g4-392bcf3g4-857c2f3g4+13445bdf3g4+1530cdf3g4+1378\
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4-15692df4g4+9903ef4g4+1807f5g4-14522c4g5+3420bd3g5+11613cd3g5+5533d4g5-1\
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8fg8-14325g9-6580bc-14693c2-8236bd-5213cd+8723d2-4314be+11583ce+14819de-9\
06e2+3047bf-15172cf-9597df+8928ef-14526f2-4495bg-502cg-5390dg+4583eg+1210\
fg-3476g2

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$\mapsto j[87] = \text{be2f2g4} - 3146\text{e3f2g4} + 7057\text{bcf3g4} + 15530\text{c2f3g4} + 1604\text{bdf3g4} + 14540\text{cdf3g4} + 12\backslash$   
 $454\text{d2f3g4} - 12607\text{bef3g4} + 10482\text{cef3g4} + 88\text{def3g4} + 15727\text{e2f3g4} + 3612\text{bf4g4} - 1873\text{cf4g}\backslash$   
 $4 + 2649\text{df4g4} - 10634\text{ef4g4} + 3118\text{f5g4} + 15765\text{c4g5} - 14637\text{bd3g5} + 612\text{cd3g5} + 9822\text{d4g5} + 11\backslash$   
 $147\text{c3eg5} - 6972\text{bcdeg5} + 9961\text{c2deg5} - 3577\text{bd2eg5} + 8810\text{cd2eg5} - 14415\text{d3eg5} - 14831\text{bce2}\backslash$   
 $\text{g5} + 10862\text{c2e2g5} + 7299\text{bde2g5} - 3060\text{cde2g5} - 10442\text{d2e2g5} + 4695\text{be3g5} + 531\text{ce3g5} + 3475\text{d}\backslash$   
 $\text{e3g5} + 7122\text{e4g5} - 11995\text{c3fg5} + 10479\text{bcdfg5} - 8694\text{c2dfg5} + 12496\text{bd2fg5} + 1695\text{cd2fg5} + 14\backslash$   
 $22\text{d3fg5} - 10473\text{bcefg5} + 4193\text{c2efg5} + 3863\text{bdefg5} + 9052\text{cdefg5} - 8861\text{d2efg5} + 14140\text{be2f}\backslash$   
 $\text{g5} - 10386\text{ce2fg5} + 11879\text{de2fg5} + 11159\text{e3fg5} + 7412\text{bcf2g5} + 681\text{c2f2g5} - 6321\text{bdf2g5} - 915\backslash$   
 $9\text{cdf2g5} + 15306\text{d2f2g5} - 5653\text{bef2g5} + 13351\text{cef2g5} + 135\text{def2g5} - 15405\text{e2f2g5} + 12679\text{bf3}\backslash$   
 $\text{g5} + 6310\text{cf3g5} + 2203\text{df3g5} + 9721\text{ef3g5} + 7493\text{f4g5} - 15953\text{c3g6} + 11280\text{bcdg6} - 5064\text{c2dg6} - \backslash$   
 $11894\text{bd2g6} + 13093\text{cd2g6} - 15731\text{d3g6} + 11341\text{bceg6} + 4507\text{c2eg6} - 15017\text{bdeg6} - 767\text{cdeg6} - \backslash$   
 $13014\text{d2eg6} - 2136\text{be2g6} + 7624\text{ce2g6} + 11387\text{de2g6} - 198\text{e3g6} - 8478\text{bcfg6} + 2667\text{c2fg6} + 597\backslash$   
 $8\text{bdfg6} + 7073\text{cdfg6} - 14689\text{d2fg6} + 7019\text{befg6} - 322\text{cefg6} - 14294\text{defg6} + 13337\text{e2fg6} - 1280\backslash$   
 $6\text{bf2g6} - 10649\text{cf2g6} + 4436\text{df2g6} - 14805\text{ef2g6} + 7653\text{f3g6} - 4127\text{bcg7} - 5910\text{c2g7} - 13206\text{bd}\backslash$   
 $\text{g7} + 8994\text{cdg7} - 6636\text{d2g7} - 6840\text{beg7} + 1450\text{ceg7} - 1802\text{deg7} + 10914\text{e2g7} - 12118\text{bfg7} - 9759\text{c}\backslash$   
 $\text{fg7} + 8624\text{dfg7} - 10949\text{efg7} + 10670\text{f2g7} - 3494\text{bg8} - 10039\text{cg8} + 8045\text{dg8} + 505\text{eg8} - 12273\text{fg8}\backslash$   
 $+ 8397\text{g9} - 3334\text{bc} + 4156\text{c2} + 7377\text{bd} - 2564\text{cd} - 2878\text{d2} - 14781\text{be} - 8192\text{ce} + 50\text{de} - 1925\text{e2} - 930\backslash$   
 $7\text{bf} + 15622\text{cf} - 8765\text{df} - 7079\text{ef} - 9597\text{f2} + 271\text{bg} - 7398\text{cg} - 2771\text{dg} + 4143\text{eg} - 12249\text{fg} - 2294\text{g}\backslash$   
 $2$

$\mapsto j[88] = \text{d2ef2g4} + 15618\text{e3f2g4} + 8689\text{bcf3g4} - 3945\text{c2f3g4} + 13923\text{bdf3g4} + 2575\text{cdf3g4} + 12\backslash$   
 $729\text{d2f3g4} - 1696\text{bef3g4} - 6365\text{cef3g4} - 429\text{def3g4} - 6259\text{e2f3g4} + 3038\text{bf4g4} - 2524\text{cf4g4} - \backslash$   
 $11099\text{df4g4} + 5915\text{ef4g4} - 9139\text{f5g4} + 5649\text{c4g5} + 8296\text{bd3g5} - 3517\text{cd3g5} - 14508\text{d4g5} - 9055\backslash$   
 $\text{c3eg5} + 11219\text{bcdeg5} - 15536\text{c2deg5} + 5640\text{bd2eg5} + 14226\text{cd2eg5} + 9414\text{d3eg5} + 2190\text{bce2g5}\backslash$   
 $+ 3497\text{c2e2g5} + 5650\text{bde2g5} + 12776\text{cde2g5} + 8965\text{d2e2g5} + 14648\text{be3g5} + 7311\text{ce3g5} - 9779\text{de}\backslash$   
 $\text{3g5} - 10458\text{e4g5} - 10287\text{c3fg5} - 15711\text{bcdfg5} + 12163\text{c2dfg5} + 9596\text{bd2fg5} + 10074\text{cd2fg5} + 1\backslash$   
 $241\text{d3fg5} + 15464\text{bcefg5} + 14635\text{c2efg5} + 7888\text{bdefg5} + 7379\text{cdefg5} - 13087\text{d2efg5} + 817\text{be2}\backslash$   
 $\text{fg5} - 7554\text{ce2fg5} - 15937\text{de2fg5} - 15833\text{e3fg5} + 10289\text{bcf2g5} + 6209\text{c2f2g5} - 14694\text{bdf2g5} + \backslash$   
 $3848\text{cdf2g5} + 8679\text{d2f2g5} + 3622\text{bef2g5} + 4081\text{cef2g5} - 9784\text{def2g5} + 13882\text{e2f2g5} + 14142\text{b}\backslash$   
 $\text{f3g5} - 9131\text{cf3g5} + 1634\text{df3g5} - 6962\text{ef3g5} - 1555\text{f4g5} + 6808\text{c3g6} - 12611\text{bcdg6} + 2119\text{c2dg6}\backslash$   
 $- 5411\text{bd2g6} + 7196\text{cd2g6} + 14382\text{d3g6} + 5135\text{bceg6} - 14560\text{c2eg6} - 6014\text{bdeg6} - 11201\text{cdeg6} + \backslash$   
 $13589\text{d2eg6} + 1452\text{be2g6} - 7509\text{ce2g6} - 8059\text{de2g6} - 12560\text{e3g6} - 4111\text{bcfg6} - 3420\text{c2fg6} + 97\backslash$   
 $53\text{bdfg6} - 5429\text{cdfg6} + 14292\text{d2fg6} - 4971\text{befg6} - 10081\text{cefg6} - 6047\text{defg6} - 12741\text{e2fg6} + 23\backslash$   
 $2\text{bf2g6} - 7256\text{cf2g6} + 9358\text{df2g6} + 15315\text{ef2g6} - 9708\text{f3g6} - 7326\text{bcg7} - 4488\text{c2g7} - 5107\text{bdg7}\backslash$   
 $+ 4040\text{cdg7} - 14485\text{d2g7} - 6516\text{beg7} - 10658\text{ceg7} + 13756\text{deg7} + 10306\text{e2g7} - 15219\text{bfg7} - 1173\backslash$   
 $2\text{cfg7} + 14210\text{dfg7} - 8599\text{efg7} + 223\text{f2g7} - 11453\text{bg8} + 2308\text{cg8} - 553\text{dg8} - 2504\text{eg8} + 15324\text{fg8}\backslash$   
 $- 5197\text{g9} + 5372\text{bc} - 13179\text{c2} - 508\text{bd} - 1204\text{cd} + 13796\text{d2} + 12589\text{be} - 4722\text{ce} - 10035\text{de} + 2932\text{e2}\backslash$   
 $+ 8935\text{bf} - 6801\text{cf} + 9838\text{df} + 11150\text{ef} + 7097\text{f2} - 3225\text{bg} + 15116\text{cg} + 7091\text{dg} - 5929\text{eg} - 894\text{fg} + 9\backslash$   
 $547\text{g2}$

$\mapsto j[89] = \text{cdef2g4} + 8633\text{e3f2g4} - 15395\text{bcf3g4} + 2088\text{c2f3g4} - 7495\text{bdf3g4} - 10035\text{cdf3g4} - 28\backslash$   
 $82\text{d2f3g4} + 2859\text{bef3g4} + 15084\text{cef3g4} + 7917\text{def3g4} - 11741\text{e2f3g4} + 4885\text{bf4g4} + 3981\text{cf4g}\backslash$   
 $4 + 6028\text{df4g4} + 7795\text{ef4g4} + 501\text{f5g4} + 8168\text{c4g5} + 5650\text{bd3g5} + 8320\text{cd3g5} - 5698\text{d4g5} - 8735\text{c}\backslash$   
 $3\text{eg5} + 12896\text{bcdeg5} - 7153\text{c2deg5} - 10240\text{bd2eg5} + 11810\text{cd2eg5} + 8889\text{d3eg5} + 14204\text{bce2g5}\backslash$   
 $- 5079\text{c2e2g5} - 732\text{bde2g5} + 15741\text{cde2g5} + 2062\text{d2e2g5} + 15464\text{be3g5} + 10636\text{ce3g5} - 4978\text{de}\backslash$   
 $\text{3g5} + 12266\text{e4g5} + 1517\text{c3fg5} + 7302\text{bcdfg5} - 6411\text{c2dfg5} + 4009\text{bd2fg5} + 197\text{cd2fg5} - 4595\text{d3}\backslash$   
 $\text{fg5} - 7519\text{bcefg5} - 12179\text{c2efg5} - 15484\text{bdefg5} - 9077\text{cdefg5} + 12470\text{d2efg5} - 10963\text{be2fg5}\backslash$   
 $+ 9360\text{ce2fg5} + 11070\text{de2fg5} + 6508\text{e3fg5} + 15971\text{bcf2g5} + 9874\text{c2f2g5} + 8667\text{bdf2g5} + 2579\text{c}\backslash$   
 $\text{df2g5} + 4377\text{d2f2g5} - 1625\text{bef2g5} + 7485\text{cef2g5} + 11369\text{def2g5} - 4262\text{e2f2g5} + 10756\text{bf3g5} + \backslash$   
 $4138\text{cf3g5} + 6829\text{df3g5} + 2517\text{ef3g5} - 523\text{f4g5} - 4475\text{c3g6} - 8030\text{bcdg6} + 4046\text{c2dg6} - 2880\text{bd}\backslash$   
 $\text{2g6} - 8787\text{cd2g6} + 11037\text{d3g6} - 3836\text{bceg6} - 5719\text{c2eg6} - 6036\text{bdeg6} - 10212\text{cdeg6} - 12459\text{d2e}\backslash$   
 $\text{g6} + 13255\text{be2g6} - 9325\text{ce2g6} + 3725\text{de2g6} - 8855\text{e3g6} + 6772\text{bcfg6} - 918\text{c2fg6} - 14458\text{bdfg6} - \backslash$   
 $9568\text{cdfg6} + 15673\text{d2fg6} - 7541\text{befg6} - 3852\text{cefg6} - 14078\text{defg6} + 5293\text{e2fg6} + 14869\text{bf2g6} + \backslash$   
 $13085\text{cf2g6} - 14667\text{df2g6} + 14807\text{ef2g6} + 9634\text{f3g6} + 11132\text{bcg7} + 13994\text{c2g7} + 144\text{bdg7} - 391\backslash$   
 $7\text{cdg7} + 6751\text{d2g7} + 11610\text{beg7} + 4392\text{ceg7} + 6830\text{deg7} - 7843\text{e2g7} + 6506\text{bfg7} + 11457\text{cfg7} + 45\backslash$

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2752bf-693cf+4105df+7790ef+13499f2-13847bg+4860cg-4987dg+2989eg+14496fg-1\
5180g2
↳ j [90]=bdef2g4-8790e3f2g4+258bcf3g4-10989c2f3g4+6662bdf3g4+2512cdf3g4-7965\
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-1809cf-15779df+11998ef+12849f2-279bg-8019cg+8163dg-8683eg+6612fg-6617g2
↳ j [91]=c2ef2g4-745e3f2g4-5383bcf3g4+12613c2f3g4+13434bdf3g4+891cdf3g4+4748\
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e2+10275ce2-8352de2+15288e3-8574bcf-7603c2f+15097bdf-9709cdf+1658d2f-1529\
9bef-6060cef-2546def-2095e2f+4228bf2-9350cf2+1742df2-4024ef2-2075f3+9731b\
cg-10387c2g+10128bdg-5880cdg+6332d2g+12940beg+13848ceg+10138deg-9685e2g+7\
653bfg+5241cfg+3731dfg-5779efg+15606f2g-190bg2-3077cg2+11171dg2-4050eg2+4\
146fg2-10058g3
↪ j [110]=cd3g6+8069d2e2g6+2172be3g6+9608ce3g6-10519de3g6+2434e4g6+5663c3fg6\

```



```

+14642bcdg6-15484c2dfg6-11593bd2fg6-7785cd2fg6+11274d3fg6+8773bcefg6-146\
99c2efg6-9100bdefg6+14555cdefg6-1101d2efg6+14884be2fg6+4186ce2fg6-14928de\
2fg6-14540e3fg6+14608bcf2g6+13891c2f2g6+4802bdf2g6+15898cdf2g6+14307d2f2g\
6+6510bef2g6+12015cef2g6-11787def2g6-4245e2f2g6+1339bf3g6+7509cf3g6-2837d\
f3g6-4624ef3g6-13264f4g6-4157c3g7-12090bcdg7-11640c2dg7+4020bd2g7-13397cd\
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7+15369ce2g7-11738de2g7+12859e3g7-11454bcfg7+8238c2fg7+9161bdfg7-7011cdfg\
7-8131d2fg7-10469befg7+4834cefg7+373defg7+14164e2fg7-2895bf2g7+12266cf2g7\
+6511df2g7-3720ef2g7-9140f3g7+8700bcg8+12518c2g8-4475bdg8-11629cdg8-8447d\
2g8-11505beg8-5899ceg8+7257deg8-9042e2g8+6859bfg8-983cfg8+11928dfg8+12935\
efg8-7074f2g8+1436bg9+8254cg9-9069dg9+5548eg9+13024fg9-14059g10+4938c3-14\
530bcd-4209c2d+11928bd2-5071cd2-3970d3-7581bce+1016c2e+9856bde+15964cde+5\
52d2e-9442be2+10873ce2+12498de2-13812e3-14397bcf+13508c2f+11348bdf+34cdf+\
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ef2-1547f3-9162bcg+7280c2g+13832bdg-14988cdg+12365d2g+2998beg+8240ceg-373\
deg-11829e2g-11427bfg-12619cfg-10429dfg-14844efg-9172f2g+14335bg2+5993cg2\
-3441dg2-11617eg2-13625fg2-15719g3
↳ j[111]=bd3g6-5519d2e2g6-1115be3g6+6336ce3g6-2069de3g6+2629e4g6+15344c3fg6\
+6470bcdg6-10902c2dfg6-1658bd2fg6+10724cd2fg6+14832d3fg6-13520bcefg6+962\
3c2efg6+7786bdefg6-15791cdefg6-10914d2efg6+1690be2fg6+2498ce2fg6-2410de2f\
g6+1627e3fg6-15385bcf2g6+7042c2f2g6-7786bdf2g6+13334cdf2g6-7629d2f2g6+149\
05bef2g6-15392cef2g6-6534def2g6-13924e2f2g6+13562bf3g6+14423cf3g6+10846df\
3g6+12131ef3g6-6187f4g6+3311c3g7-4717bcdg7+14916c2dg7+6338bd2g7-4934cd2g7\
-13329d3g7-695bceg7-8327c2eg7+15569bdeg7-11427cdeg7-13441d2eg7-15017be2g7\
-273ce2g7+11750de2g7-7341e3g7-11764bcfg7-7831c2fg7-10611bdfg7-7281cdfg7+3\
925d2fg7-12870befg7+9363cefg7-15556defg7-14641e2fg7+10741bf2g7-2222cf2g7+\
8884df2g7+1133ef2g7-13257f3g7-13206bcg8-11206c2g8+3434bdg8-13768cdg8+1038\
1d2g8-3204beg8-11494ceg8-3629deg8+10685e2g8-8862bfg8+8588cfg8-10306dfg8+4\
417efg8+12211f2g8-7933bg9+9250cg9+6382dg9+4064eg9+14713fg9+1352g10+12798c\
3+11816bcd-2372c2d+8470bd2+9430cd2+6332d3+14288bce+13945c2e-2081bde-11849\
cde-3576d2e-13041be2+14593ce2-12725de2+13481e3-12681bcf-8375c2f+7268bdf-3\
649cdf+13505d2f+321bef-6341cef-6760def-3487e2f+1550bf2-1889cf2+7039df2-21\
45ef2+12251f3-7535bcg-13639c2g+11649bdg+11287cdg-4180d2g-14051beg+3441ceg\
+14440deg+12247e2g+12265bfg+15877cfg+13083dfg+8455efg+15228f2g-2394bg2+15\
671cg2+13794dg2-14642eg2-1538fg2+9102g3
↳ j[112]=c4g6+13123d2e2g6-920be3g6+5453ce3g6-6616de3g6+11823e4g6+9895c3fg6-\
7219bcdg6-8607c2dfg6+1779bd2fg6-11714cd2fg6+2192d3fg6-15845bcefg6-11203c\
2efg6+67bdefg6+10724cdefg6+10061d2efg6-8196be2fg6-6271ce2fg6+9476de2fg6-8\
53e3fg6+10327bcf2g6-12783c2f2g6+5722bdf2g6-15750cdf2g6+3810d2f2g6+15606be\
f2g6-8534cef2g6-12229def2g6-1286e2f2g6-3818bf3g6+13581cf3g6-7471df3g6-192\
2ef3g6-6164f4g6+11513c3g7+217bcdg7-13041c2dg7-1948bd2g7+9318cd2g7-6822d3g\
7-14136bceg7-12921c2eg7-961bdeg7-15312cdeg7+7872d2eg7-3170be2g7-14619ce2g\
7-9886de2g7+2518e3g7+5913bcfg7-9689c2fg7+549bdfg7-13105cdfg7-9202d2fg7+15\
297befg7-2997cefg7+4978defg7-4630e2fg7-950bf2g7-2061cf2g7+1169df2g7-365ef\
2g7+6512f3g7-9204bcg8+12028c2g8+146bdg8+821cdg8-757d2g8-15555beg8+6783ceg\
8-14648deg8+7508e2g8+7391bfg8+12189cfg8-8110dfg8-585efg8-14067f2g8-1768bg\
9+3827cg9+921dg9-8135eg9+147fg9-1919g10-472c3+182bcd-13192c2d-7860bd2+291\
9cd2-8369d3+8316bce+177c2e-4107bde-5041cde+13043d2e-9423be2-10786ce2-8800\
de2-3070e3+3645bcf+12422c2f-12246bdf+7171cdf-1852d2f-4121bef+2111cef-6748\
def-3887e2f-9913bf2-5170cf2+14115df2+10542ef2+11748f3+5783bcg+500c2g-2068\
bdg-5732cdg-371d2g-8940beg+14944ceg+11085deg+5386e2g-12356bfg-10248cfg+89\
82dfg-6885efg+5640f2g+9778bg2+1695cg2+74dg2-15956eg2+7741fg2+8309g3
↳ j[113]=f5g5+1303d2e2g6+14970be3g6+5805ce3g6-2426de3g6-5905e4g6+1402c3fg6+\

```

```

5293bcdfig6+15561c2dfg6-10532bd2fg6+12164cd2fg6+670d3fg6-3835bcefg6-1580c2\
efg6-15855bdefg6-6072cdefg6+1050d2efg6+7565be2fg6+80ce2fg6+381de2fg6+2488\
e3fg6-13012bcf2g6+3012c2f2g6-13069bdf2g6-5163cdf2g6+596d2f2g6-8371bef2g6-\
4886cef2g6+2708def2g6+12414e2f2g6-13268bf3g6+15388cf3g6+13169df3g6+3485ef\
3g6-6229f4g6+8241c3g7+11996bcdg7-12865c2dg7-189bd2g7-756cd2g7-13873d3g7+1\
1748bceg7-5650c2eg7-4920bdeg7-925cdeg7+11657d2eg7+6616be2g7+1276ce2g7+198\
5de2g7+15637e3g7+10154bcfg7+2399c2fg7-11686bdfg7+11423cdfg7+6270d2fg7+517\
0befg7+5754defg7-337e2fg7-13782bf2g7+15470cf2g7-15887df2g7+10684ef2g7-113\
f3g7+7951bcg8+6900c2g8-223bdg8-2994cdg8-2954d2g8-4120beg8+13776ceg8-12256\
deg8-4002e2g8-13238bfg8+3508cfg8+2953dfg8-5953efg8-2955f2g8+7152bg9+11664\
cg9+9852dg9+6251eg9-554fg9-14474g10-11560c3-13244bcd+10072c2d-5851bd2-335\
9cd2+856d3+14458bce-15027c2e+2864bde+15277cde-949d2e-10013be2+7334ce2-484\
de2-8801e3+6227bcf-15793c2f-9029bdf+4336cdf-14907d2f+9592bef+9119cef-5762\
def-6165e2f-12bf2-5502cf2+15868df2-6395ef2-11156f3+5001bcg-15966c2g+8690b\
dg-8434cdg+6095d2g+3258beg-9010ceg+5385deg+1741e2g-11740bfg-2662cfg+1777d\
fg+15152efg+975f2g+5281bg2+5750cg2+3471dg2+1775eg2+6399fg2-14012g3
→ j[114]=ef4g5-9255d2e2g6-12750be3g6+5767ce3g6-5628de3g6+11998e4g6-3305c3fg\
6+3637bcdfig6-2399c2dfg6+3428bd2fg6+1975cd2fg6+1198d3fg6+12769bcefg6+5871c\
2efg6+8387bdefg6-13694cdefg6-8795d2efg6+66be2fg6+15882ce2fg6-2285de2fg6-1\
4465e3fg6-11903bcf2g6+1689c2f2g6-15890bdf2g6+4954cdf2g6-13745d2f2g6+6536b\
ef2g6-8412cef2g6+1924def2g6+13713e2f2g6-3074bf3g6+15608cf3g6+10766df3g6-7\
988ef3g6-4042f4g6+7928c3g7-4643bcdg7-15612c2dg7-15499bd2g7-6152cd2g7+8986\
d3g7+10972bceg7+3495c2eg7+4120bdeg7-11416cdeg7-14592d2eg7+6772be2g7+14708\
ce2g7+9453de2g7+593e3g7-9494bcfg7+13003c2fg7+8842bdfg7-4674cdfg7-7366d2fg\
7-14284befg7+3460cefg7-1715defg7+937e2fg7-8267bf2g7-3595cf2g7-5910df2g7-1\
5495ef2g7-15883f3g7+13518bcg8+7581c2g8-14135bdg8-5611cdg8+15940d2g8-7822b\
eg8-15302ceg8-62deg8+13910e2g8-12642bfg8+64cfg8+15803dfg8-2547efg8-15062f\
2g8+12374bg9-1029cg9-13737dg9+818eg9+15113fg9-14941g10+14493c3+7775bcd+87\
11c2d+6866bd2+9369cd2-4448d3-13220bce-5220c2e-9833bde+378cde+13051d2e-264\
2be2+5046ce2-15959de2+9102e3+6595bcf-13771c2f+2871bdf-7907cdf+13852d2f+13\
173bef+3273cef-2361def-5733e2f+11301bf2-3205cf2-517df2+4889ef2-10517f3+15\
108bcg+8221c2g-929bdg+907cdg-2660d2g+14beg+10415ceg+9981deg+13973e2g+1262\
1bfg+224cfg-6763dfg+3745efg-3216f2g-10718bg2+4956cg2+12172dg2-687eg2-4854\
fg2+14639g3
→ j[115]=df4g5-7211d2e2g6-7989be3g6+13706ce3g6+13184de3g6-8500e4g6+1391c3fg\
6-5897bcdfig6+7682c2dfg6-2260bd2fg6-9442cd2fg6-12429d3fg6-11989bcefg6-1337\
1c2efg6+6288bdefg6+7751cdefg6-12538d2efg6+11269be2fg6+7088ce2fg6+8646de2f\
g6-13717e3fg6+557bcf2g6-471c2f2g6+8522bdf2g6-15516cdf2g6-13720d2f2g6-7273\
bef2g6-12439cef2g6+5973def2g6-3875e2f2g6-4238bf3g6+6638cf3g6-11527df3g6-3\
562ef3g6-10893f4g6-14629c3g7-1580bcdg7-13891c2dg7+15348bd2g7-9205cd2g7-14\
277d3g7+12908bceg7-14431c2eg7+12749bdeg7+1549cdeg7+12267d2eg7+15540be2g7+\
6728ce2g7+5056de2g7-3160e3g7-11221bcfg7+11419c2fg7+4605bdfg7-3130cdfg7+59\
93d2fg7+7975befg7+13689cefg7-9517defg7+13166e2fg7+3364bf2g7-3480cf2g7-143\
13df2g7+9149ef2g7+14712f3g7-8532bcg8+13937c2g8-2582bdg8+14000cdg8+3655d2g\
8+2353beg8+11013ceg8-5856deg8-15262e2g8-13097bfg8+12542cfg8+12359dfg8-129\
49efg8-2523f2g8-12384bg9-15198cg9-12309dg9-7282eg9+1550fg9-2579g10-9768c3\
+13226bcd-13921c2d+11722bd2-10172cd2-7591d3-9297bce-7344c2e-10053bde+1495\
4cde-9570d2e+11246be2+5934ce2-13403de2+5321e3-10318bcf+14312c2f-676bdf+12\
445cdf-9277d2f+14315bef+6261cef+8916def-15708e2f-10170bf2-13372cf2+10402d\
f2-10941ef2+6825f3-12511bcg+7413c2g+15624bdg+14964cdg+5651d2g+8313beg+791\
4ceg-3580deg-15218e2g-11939bfg-3477cfg-8592dfg-4964efg+12942f2g+10085bg2+\
11678cg2+15540dg2+13883eg2+11351fg2-5456g3
→ j[116]=cf4g5+14890d2e2g6-8449be3g6-12484ce3g6-1303de3g6-10710e4g6-4213c3f\

```

```

g6-14589bcdfg6-14413c2dfg6+9725bd2fg6+9692cd2fg6+11668d3fg6+11761bcefg6-1\
1195c2efg6+8056bdefg6+6719cdefg6-11512d2efg6-762be2fg6+1117ce2fg6+13122de\
2fg6+e3fg6+5634bcf2g6-13427c2f2g6-2234bdf2g6-529cdf2g6-8318d2f2g6-13529be\
f2g6+7832cef2g6+9208def2g6+11418e2f2g6+4802bf3g6-9748cf3g6-14651df3g6+950\
1ef3g6+13257f4g6-13251c3g7-2752bcdg7-15658c2dg7+15507bd2g7-9691cd2g7+1126\
d3g7-12726bceg7+6734c2eg7-5889bdeg7-15139cdeg7-11957d2eg7+10742be2g7-1524\
4ce2g7+4341de2g7+5000e3g7+10553bcfg7+10135c2fg7-2020bdfg7-1740cdfg7-11857\
d2fg7+3993befg7+10411cefg7-11226defg7-4560e2fg7+3119bf2g7-15389cf2g7+4252\
df2g7+3698ef2g7+6383f3g7-13551bcg8-7637c2g8+12744bdg8+9878cdg8-14530d2g8-\
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12659f2g8-12480bg9-3461cg9-10341dg9-1491eg9-10461fg9-4247g10-5826c3+15477\
bcd-13111c2d-3830bd2+1084cd2-9870d3+10649bce+13526c2e-41bde-2479cde+997d2\
e-5978be2+5843ce2+14997de2+11490e3-14544bcf+2299c2f-11807bdf+2184cdf+2672\
d2f+8381bef+5574cef+5737def-12257e2f+13791bf2+7651cf2+10376df2-12291ef2+1\
0028f3-8385bcg+1120c2g-13248bdg-13354cdg-12971d2g-6022beg-2929ceg+11930de\
g-4237e2g+1329bfg+789cfg+6529dfg-14675efg-935f2g-7958bg2-6221cg2+8415dg2-\
3556eg2-13707fg2+9320g3
→ j [117]=bf4g5-872d2e2g6+2360be3g6-7674ce3g6+13232de3g6-3451e4g6-424c3fg6-1\
2359bcdfg6+6567c2dfg6+6213bd2fg6+2590cd2fg6+14611d3fg6-2626bcefg6-3822c2e\
fg6+6519bdefg6-10659cdefg6+10481d2efg6-6698be2fg6-2493ce2fg6-8015de2fg6+9\
884e3fg6+1206bcf2g6-11250c2f2g6-2318bdf2g6-5567cdf2g6-6892d2f2g6-6344bef2\
g6+79cef2g6-1285def2g6+4898e2f2g6+61bf3g6+11349cf3g6-1354df3g6+4606ef3g6-\
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39bceg7+12641c2eg7-4614bdeg7-5230cdeg7-14820d2eg7+13987be2g7+12415ce2g7+1\
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ceg8+10530deg8+6988e2g8-7718bfg8+15078cfg8-6352dfg8-3998efg8-10388f2g8+10\
766bg9+10679cg9-15843dg9+2286eg9+8148fg9+5788g10-15608c3-13968bcd+4113c2d\
+7071bd2+4527cd2+3633d3+5499bce+2613c2e+10701bde+14757cde+13232d2e-3375be\
2-14408ce2-14558de2+13626e3+429bcf+6648c2f-8640bdf-1755cdf+5208d2f+3117be\
f-2046cef-7024def+8262e2f+14397bf2+5424cf2-5942df2+5158ef2+673f3-14925bcg\
-195c2g+12457bdg-15207cdg-10494d2g+900beg-13799ceg+7881deg+14292e2g+1515b\
fg-11769cfg+1589dfg-8084efg-2151f2g-3446bg2-12723cg2+3215dg2+10286eg2-663\
fg2+6938g3
→ j [118]=e2f3g5-421d2e2g6+1230be3g6+4014ce3g6+8796de3g6+4958e4g6+741c3fg6+4\
017bcdfg6-10254c2dfg6-4430bd2fg6+5622cd2fg6+7207d3fg6-9588bcefg6-9224c2ef\
g6+1835bdefg6+6665cdefg6-4819d2efg6-5392be2fg6-7435ce2fg6+8640de2fg6-1561\
6e3fg6+15363bcf2g6-2501c2f2g6+605bdf2g6-10329cdf2g6-7911d2f2g6+7712bef2g6\
-6988cef2g6-15223def2g6+327e2f2g6-4042bf3g6-11394cf3g6-293df3g6-3750ef3g6\
+8753f4g6-11288c3g7+13645bcdg7-9447c2dg7-2854bd2g7+14014cd2g7-3297d3g7-15\
821bceg7+12823c2eg7+14558bdeg7-8797cdeg7-3412d2eg7+2028be2g7-15681ce2g7+1\
5292de2g7+3317e3g7-9351bcfg7-13930c2fg7-9672bdfg7+7835cdfg7-3977d2fg7-490\
3befg7+14479cefg7+12201defg7-5375e2fg7+13140bf2g7+1103cf2g7-9508df2g7-211\
9ef2g7-8166f3g7-1371bcg8+11272c2g8-1555bdg8+6296cdg8-10804d2g8-2685beg8-6\
899ceg8-1024deg8-11339e2g8+12491bfg8-13792cfg8-5791dfg8+3429efg8-12121f2g\
8-2142bg9+14836cg9-8155dg9+10053eg9+5078fg9+15733g10+9306c3-12017bcd-4668\
c2d-15759bd2-4377cd2+7801d3+3239bce+15244c2e-9022bde+4438cde+10225d2e-202\
be2+8968ce2+13144de2+148e3+13702bcf-9063c2f-12368bdf+11714cdf-9621d2f-124\
95bef+229cef-15059def-5561e2f+14591bf2-11089cf2+9829df2-6597ef2-10762f3-9\
988bcg+6614c2g-5893bdg+9978cdg-8681d2g+10258beg-8873ceg-13794deg+1859e2g-\
4650bfg+9954cfg+9083dfg-11999efg-1535f2g-4217bg2-10204cg2+9888dg2+93eg2-1\
2424fg2+9381g3

```

$\mapsto j[119] = \text{def}3g5 + 6034d2e2g6 - 5377be3g6 + 6547ce3g6 - 11350de3g6 + 11954e4g6 - 2402c3f \backslash$   
 $g6 + 1536bcdfg6 + 303c2dfg6 - 970bd2fg6 + 996cd2fg6 - 5378d3fg6 + 4965bcefg6 - 15565c2e \backslash$   
 $fg6 - 80bdefg6 - 776cdefg6 - 13776d2efg6 + 10088be2fg6 - 13747ce2fg6 - 6944de2fg6 - 122 \backslash$   
 $4e3fg6 - 9785bcf2g6 + 2642c2f2g6 + 2394bdf2g6 - 6187cdf2g6 - 12024d2f2g6 - 4917bef2g6 \backslash$   
 $-14583cef2g6 + 2530def2g6 - 1227e2f2g6 - 2176bf3g6 + 1033cf3g6 - 4535df3g6 + 12189ef3 \backslash$   
 $g6 - 7514f4g6 - 6920c3g7 + 3724bcdg7 - 1428c2dg7 + 5748bd2g7 - 6906cd2g7 + 3017d3g7 + 892 \backslash$   
 $9bceg7 + 9101c2eg7 - 15541bdeg7 - 4540cdeg7 - 3610d2eg7 - 10698be2g7 - 15664ce2g7 - 118 \backslash$   
 $66de2g7 + 13543e3g7 + 13221bcfg7 + 116c2fg7 - 3673bdfg7 + 10192cdfg7 + 2220d2fg7 + 1063 \backslash$   
 $6befg7 - 9989cefg7 + 14662defg7 - 5977e2fg7 + 10119bf2g7 - 1624cf2g7 - 10286df2g7 + 710 \backslash$   
 $0ef2g7 + 5098f3g7 + 3762bcg8 - 3990c2g8 - 14190bdg8 - 14646cdg8 - 11650d2g8 - 14412beg8 \backslash$   
 $-13548ceg8 - 14983deg8 + 5387e2g8 + 6152bfg8 + 13757cfg8 + 337dfg8 + 2398efg8 + 15005f2 \backslash$   
 $g8 + 8565bg9 - 3202cg9 + 7019dg9 - 11905eg9 - 1376fg9 - 11968g10 + 4009c3 - 15202bcd + 8842 \backslash$   
 $c2d - 4988bd2 - 8860cd2 + 15626d3 + 14875bce + 12202c2e - 8694bde + 14728cde - 1960d2e + 11 \backslash$   
 $421be2 + 815ce2 + 15315de2 + 3310e3 - 2314bcf + 3941c2f + 400bdf + 12821cdf - 13422d2f + 10 \backslash$   
 $193bef - 6617cef - 12985def + 8499e2f + 9938bf2 + 9270cf2 + 4081df2 - 11767ef2 - 11722f3 - \backslash$   
 $3324bcg + 14974c2g + 11884bdg + 3505cdg + 868d2g + 15247beg - 4892ceg - 298deg + 308e2g + 3 \backslash$   
 $522bfg + 13519cfg - 6202dfg - 10273efg - 11070f2g + 13028bg2 + 13386cg2 - 6280dg2 - 4832e \backslash$   
 $g2 - 6953fg2 - 7766g3$

$\mapsto j[120] = \text{cef}3g5 - 6537d2e2g6 - 7888be3g6 + 12464ce3g6 - 3103de3g6 - 2810e4g6 - 3175c3fg \backslash$   
 $6 + 9990bcdfg6 + 8320c2dfg6 - 1772bd2fg6 - 7966cd2fg6 + 7471d3fg6 - 966bcefg6 - 252c2ef \backslash$   
 $g6 + 13251bdefg6 + 12087cdefg6 + 6103d2efg6 + 3973be2fg6 + 15851ce2fg6 - 1039de2fg6 - 9 \backslash$   
 $496e3fg6 - 14055bcf2g6 - 4970c2f2g6 - 2815bdf2g6 - 10950cdf2g6 - 3690d2f2g6 - 13106be \backslash$   
 $f2g6 + 1652cef2g6 - 13537def2g6 + 13752e2f2g6 - 24bf3g6 + 8402cf3g6 - 15388df3g6 - 9560 \backslash$   
 $ef3g6 + 5396f4g6 + 13382c3g7 - 8986bcdg7 - 14963c2dg7 - 12793bd2g7 + 8320cd2g7 - 6255d3 \backslash$   
 $g7 - 14633bceg7 - 4370c2eg7 - 13630bdeg7 - 8775cdeg7 - 10293d2eg7 + 56be2g7 - 2262ce2g7 \backslash$   
 $-3718de2g7 - 13208e3g7 - 3176bcfg7 + 9c2fg7 - 11134bdfg7 - 1038cdfg7 + 13776d2fg7 + 101 \backslash$   
 $85befg7 + 4606cefg7 + 3268defg7 - 1614e2fg7 + 13193bf2g7 - 15398cf2g7 + 3540df2g7 + 120 \backslash$   
 $0ef2g7 + 14189f3g7 + 1008bcg8 + 10096c2g8 - 11449bdg8 - 7015cdg8 - 9247d2g8 + 10207beg8 \backslash$   
 $-13593ceg8 + 8233deg8 - 15481e2g8 - 14793bfg8 - 13438cfg8 - 1378dfg8 - 5973efg8 + 1867f \backslash$   
 $2g8 - 11588bg9 + 14261cg9 - 7997dg9 + 14831eg9 + 14252fg9 + 3047g10 - 5694c3 - 149bcd - 429 \backslash$   
 $0c2d - 11454bd2 + 15329cd2 + 12137d3 + 12476bce - 542c2e - 3641bde + 11574cde + 5882d2e - 9 \backslash$   
 $364be2 - 2728ce2 + 4118de2 + 9647e3 - 2548bcf - 9689c2f - 1858bdf + 5649cdf - 10598d2f + 14 \backslash$   
 $942bef + 1441cef - 11936def + 8126e2f - 9852bf2 + 4066cf2 - 7572df2 + 8482ef2 - 14506f3 + 3 \backslash$   
 $655bcg - 10684c2g - 409bdg + 12986cdg + 5122d2g + 8895beg - 4315ceg - 5045deg - 7620e2g - 8 \backslash$   
 $798bfg - 261cfg - 5773dfg + 6126efg + 832f2g - 6959bg2 - 526cg2 - 9254dg2 + 6310eg2 + 15771 \backslash$   
 $fg2 + 5548g3$

$\mapsto j[121] = \text{bef}3g5 - 11837d2e2g6 - 14441be3g6 + 5998ce3g6 + 10898de3g6 - 8786e4g6 - 4659c3 \backslash$   
 $fg6 - 11397bcdfg6 + 1497c2dfg6 - 2328bd2fg6 + 2064cd2fg6 - 6620d3fg6 - 3311bcefg6 - 228 \backslash$   
 $c2efg6 + 15716bdefg6 + 5688cdefg6 + 12620d2efg6 + 2653be2fg6 - 1426ce2fg6 + 15885de2f \backslash$   
 $g6 + 52e3fg6 - 8420bcf2g6 + 3798c2f2g6 + 8313bdf2g6 + 978cdf2g6 + 1522d2f2g6 + 4806bef2 \backslash$   
 $g6 - 631cef2g6 - 9132def2g6 + 7841e2f2g6 - 12922bf3g6 - 6831cf3g6 - 141df3g6 - 7173ef3g \backslash$   
 $6 - 15691f4g6 - 8081c3g7 + 15735bcdg7 - 11199c2dg7 - 12324bd2g7 - 2978cd2g7 + 12456d3g7 \backslash$   
 $-10968bceg7 + 1882c2eg7 + 3914bdeg7 + 11056cdeg7 - 11218d2eg7 + 2066be2g7 - 4706ce2g7 \backslash$   
 $-3276de2g7 + 8280e3g7 + 692bcfg7 + 11247c2fg7 - 4494bdfg7 - 6645cdfg7 - 1542d2fg7 - 100 \backslash$   
 $50befg7 - 13168cefg7 + 13891defg7 - 51e2fg7 + 12521bf2g7 + 1158cf2g7 + 13739df2g7 + 533 \backslash$   
 $5ef2g7 - 5016f3g7 + 5392bcg8 + 10234c2g8 + 4499bdg8 + 2166cdg8 + 1831d2g8 - 7069beg8 + 75 \backslash$   
 $80ceg8 + 11924deg8 - 5865e2g8 + 7263bfg8 + 356cfg8 - 9120dfg8 + 1788efg8 + 12657f2g8 - 10 \backslash$   
 $395bg9 - 9327cg9 + 13890dg9 + 13316eg9 + 4704fg9 - 891g10 - 12175c3 + 12885bcd - 6028c2d - \backslash$   
 $4652bd2 + 6405cd2 - 1919d3 + 15688bce - 754c2e - 3156bde - 11829cde - 15387d2e - 14430be2 \backslash$   
 $-9374ce2 - 14211de2 + 5230e3 + 375bcf - 8464c2f - 15310bdf + 15690cdf - 8527d2f - 4814bef \backslash$   
 $+355cef - 1497def + 14240e2f + 9481bf2 + 2813cf2 + 4006df2 - 14643ef2 - 1752f3 + 7352bcg - \backslash$   
 $13616c2g + 2734bdg + 9229cdg + 11169d2g + 11468beg + 14293ceg - 11964deg + 876e2g + 351bf \backslash$   
 $g + 6523cfg + 8921dfg - 1009efg - 8827f2g + 6576bg2 + 10300cg2 + 9728dg2 - 6142eg2 + 10878f \backslash$



g2-12628g3

→ j [122]=d2f3g5+13897d2e2g6+4299be3g6-14765ce3g6-1275de3g6+6604e4g6+7658c3f\g6+10769bcdfg6+794c2dfg6+12176bd2fg6+15940cd2fg6+7784d3fg6-10780bcefg6-97\00c2efg6+782bdefg6+2296cdefg6+14121d2efg6-6353be2fg6-7067ce2fg6+6221de2fg\6-1415e3fg6-7331bcf2g6-12085c2f2g6+5244bdf2g6-5055cdf2g6-13047d2f2g6-1341\0bef2g6-7577cef2g6+4207def2g6+7951e2f2g6+2846bf3g6+3470cf3g6-4021df3g6-18\62ef3g6+5900f4g6-3933c3g7-933bcdg7+9064c2dg7-6054bd2g7-12458cd2g7+10791d3\g7-7515bceg7+3705c2eg7-3632bdeg7-11357cdeg7-6579d2eg7+13407be2g7-4686ce2g\7+8033de2g7-9645e3g7+13032bcfg7-4808c2fg7-9706bdfg7+13639cdfg7+3633d2fg7+\7304befg7+13044cefg7+8947defg7+6404e2fg7-15143bf2g7-1745cf2g7+5013df2g7-3\238ef2g7+9165f3g7+9171bcg8-4423c2g8-2396bdg8+10113cdg8-4005d2g8+15103beg8\ -9251ceg8+9480deg8-7792e2g8-6330bfg8+6280cfg8+11780dfg8+6345efg8-5385f2g8\ +669bg9-6474cg9-12921dg9-6753eg9+4530fg9-11300g10-2781c3+15517bcd+13417c2\ d-13605bd2+7611cd2+9392d3-9116bce-11820c2e+3267bde+11027cde+12552de+1349\ be2-350ce2-13641de2+1563e3+4125bcf-2805c2f+7971bdf+12567cdf+14319d2f-1088\ 4bef+527cef-7256def-861e2f-14690bf2+9396cf2-5847df2-292ef2+2027f3-7370bcg\ +3c2g-2197bdg-11284cdg-15096d2g-2050beg-9052ceg-13912deg-10309e2g-11886bf\ g+14232cfg+4765dfg-10131efg-5683f2g+2347bg2+15479cg2+5667dg2+3444eg2-1412\ 7fg2-2926g3

→ j [123]=cdf3g5+9304d2e2g6+8650be3g6-15250ce3g6+13233de3g6+15726e4g6+7491c3\fg6+2216bcdfg6+13992c2dfg6+13166bd2fg6-14664cd2fg6+4569d3fg6-3201bcefg6-2\645c2efg6+4718bdefg6-8811cdefg6-1007d2efg6-13388be2fg6-375ce2fg6+9736de2f\g6-8550e3fg6-4413bcf2g6+7230c2f2g6-7319bdf2g6-6919cdf2g6-11672d2f2g6+5281\ bef2g6+2904cef2g6-160def2g6+7243e2f2g6-3936bf3g6-15807cf3g6+13787df3g6+34\ 0ef3g6-7376f4g6-3732c3g7-5483bcdg7-11248c2dg7-3783bd2g7-14836cd2g7+4106d3\g7+2772bceg7+11596c2eg7-13332bdeg7-13932cdeg7-6769d2eg7+744be2g7-12405ce2\g7+24de2g7+11045e3g7-71bcfg7-12842c2fg7+13975bdfg7-6316cdfg7-15309d2fg7+1\0839befg7-15841cefg7+14617defg7-10010e2fg7+42bf2g7-14430cf2g7+9719df2g7-2\654ef2g7+4269f3g7-12274bcg8-7362c2g8-12367bdg8+14634cdg8+2565d2g8+1784beg\8-13954ceg8+14848deg8-5278e2g8+4368bfg8+11764cfg8+1178dfg8+11193efg8+9420\ f2g8-8079bg9-8976cg9+4256dg9+15525eg9-14541fg9+10552g10+1808c3+8912bcd-11\655c2d-9985bd2-13996cd2-15439d3+4789bce-1517c2e+9801bde+3508cde+13190d2e+\3688be2+7433ce2-11741de2-1814e3+11143bcf+2525c2f+3933bdf+9186cdf-5209d2f+\14995bef-9861cef-9209def+11728e2f+3089bf2+12560cf2-14438df2-7592ef2+12725\ f3-8271bcg-1690c2g+2556bdg-13137cdg+9255d2g-4347beg+12324ceg+1247deg-1892\ e2g+5942bfg-15427cfg-774dfg-2000efg+3377f2g+4849bg2+6822cg2+4433dg2-8256e\g2-9507fg2+5831g3

→ j [124]=bdf3g5-14560d2e2g6-9156be3g6-1387ce3g6+11788de3g6-4923e4g6+7701c3f\g6+4565bcdfg6-11358c2dfg6+5148bd2fg6-11059cd2fg6-3916d3fg6-12171bcefg6+98\37c2efg6+13429bdefg6+15057cdefg6+3882d2efg6+5298be2fg6+2727ce2fg6+9746de2\fg6-5681e3fg6+6556bcf2g6+10051c2f2g6+2837bdf2g6+8973cdf2g6-1684d2f2g6-920\8bef2g6+12036cef2g6-656def2g6-2656e2f2g6+4576bf3g6+2086cf3g6+2387df3g6+71\36ef3g6+13723f4g6+4430c3g7+8732bcdg7+4766c2dg7+4930bd2g7+12397cd2g7-177d3\g7+7040bceg7-4384c2eg7+12642bdeg7-3014cdeg7+13295d2eg7-11685be2g7-12844ce\2g7+10263de2g7+6052e3g7-5671bcfg7+9726c2fg7-10244bdfg7-4051cdfg7+11659d2f\g7+11330befg7-4615cefg7+6072defg7+11125e2fg7+8325bf2g7+10393cf2g7-7581df2\g7-1756ef2g7-2598f3g7-14139bcg8+356c2g8-99bdg8+3655cdg8+13387d2g8-533beg8\ -15315ceg8-10959deg8-1865e2g8+4082bfg8-6384cfg8-7930dfg8-11789efg8+15544f\2g8-2327bg9+5252cg9-4249dg9-14340eg9-14017fg9-760g10-5051c3+2990bcd-14011\c2d+5411bd2+7841cd2-14301d3+5485bce+1077c2e-3257bde+5456cde-10134d2e+58be\2-7122ce2+9281de2+12486e3-13225bcf+1975c2f-13004bdf+15141cdf+3759d2f-5618\ bef-228cef-2530def-5051e2f+1423bf2+13591cf2-12820df2-11184ef2-10655f3+135\26bcg-8030c2g+14772bdg-14314cdg+11024d2g-9918beg-5041ceg-1729deg+4132e2g-\

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2192bfg-7427cfg+11771dfg-411efg-4152f2g+11294bg2-3369cg2+8902dg2-11394eg2\
-12730fg2-13746g3
⇒ j[125]=c2f3g5+14199d2e2g6+6910be3g6-11092ce3g6+472de3g6-15338e4g6-1809c3f\
g6-4673bcdfg6-9475c2dfg6+9272bd2fg6-7651cd2fg6-1445d3fg6-10773bcefg6+1591\
9c2efg6+4734bdefg6+4754cdefg6+6427d2efg6-3098be2fg6-9682ce2fg6-12074de2fg\
6+7585e3fg6-312bcf2g6+9373c2f2g6+3520bdf2g6-4635cdf2g6-6105d2f2g6-907bef2\
g6-4331cef2g6+7529def2g6+13651e2f2g6+15993bf3g6+12130cf3g6+7341df3g6-1281\
3ef3g6+7738f4g6+10071c3g7+15879bcdg7+3399c2dg7+9559bd2g7+11466cd2g7+513d3\
g7-11824bceg7-227c2eg7+15734bdeg7+1030cdeg7+5615d2eg7+10234be2g7+457ce2g7\
-14239de2g7-8867e3g7-6200bcfg7+142c2fg7+1768bdfg7-1872cdfg7+9507d2fg7-333\
0befg7-11557cefg7+3634defg7-9159e2fg7+2798bf2g7+4414cf2g7-2878df2g7-2777e\
f2g7-11284f3g7-789bcg8+3084c2g8+1663bdg8-9321cdg8+15945d2g8+6317beg8+8317\
ceg8+13581deg8+7651e2g8+10384bfg8+12623cfg8-3171dfg8-5543efg8+12820f2g8-6\
585bg9+4646cg9-714dg9+2089eg9-3034fg9-9705g10-11864c3+9178bcd-11489c2d-41\
8bd2-11151cd2-2477d3-12925bce+5026c2e-15911bde-8873cde-12431d2e+4592be2+3\
96ce2-9345de2-9802e3-11638bcf-7845c2f-3530bdf-14988cdf+3644d2f-15752bef+4\
966cef+9324def-5474e2f+6474bf2-13436cf2-9808df2-10837ef2-3428f3+7361bcg+1\
4974c2g+2556bdg-329cdg-7498d2g-5516beg-14153ceg-3586deg+3138e2g-12772bfg-\
2219cfg-1929dfg-7177efg+15015f2g+14030bg2-362cg2-7167dg2+8278eg2+12514fg2\
-938g3
⇒ j[126]=bcf3g5+6820d2e2g6+5187be3g6-4585ce3g6+518de3g6+7291e4g6-11308c3fg6\
+3958bcdfg6+6530c2dfg6+2942bd2fg6+7417cd2fg6-9059d3fg6+4867bcefg6-2454c2e\
fg6-10772bdefg6-4967cdefg6+4622d2efg6+11542be2fg6-7524ce2fg6-92de2fg6+308\
6e3fg6+7266bcf2g6-15084c2f2g6-1145bdf2g6-10961cdf2g6-4590d2f2g6-2316bef2g\
6-9717cef2g6+7441def2g6+15816e2f2g6-213bf3g6-5606cf3g6+10285df3g6-6009ef3\
g6+2896f4g6+6038c3g7+5132bcdg7+9284c2dg7+10660bd2g7-13382cd2g7+2252d3g7+3\
908bceg7-768c2eg7+9017bdeg7+6445cdeg7+12527d2eg7+14849be2g7+2842ce2g7+515\
9de2g7+9193e3g7-1402bcfg7-13099c2fg7-15035bdfg7+9450cdfg7+13318d2fg7-1151\
8befg7+1303cefg7+8675defg7-9235e2fg7-12694bf2g7-678cf2g7+10335df2g7+5560e\
f2g7+6354f3g7+1874bcg8-11102c2g8+12855bdg8-5545cdg8+8619d2g8+14898beg8-15\
780ceg8-4268deg8-3540e2g8-6006bfg8+2900cfg8+14955dfg8+12997efg8-3715f2g8+\
6921bg9+11999cg9+15527dg9-5978eg9-15796fg9+1892g10-7949c3+1145bcd+6281c2d\
-11884bd2+11009cd2+4359d3-7199bce-11215c2e-9769bde-880cde-15422d2e+12668b\
e2+376ce2+10453de2+2767e3+1864bcf-14238c2f+7359bdf-2754cdf-15221d2f+15712\
bef+6841cef+8526def-12722e2f+953bf2-1868cf2+8649df2-12708ef2-14967f3+1848\
bcg-15752c2g+12591bdg-214cdg+9948d2g-11072beg-6236ceg-5578deg+6670e2g-390\
8bfg-8621cfg-4165dfg+2632efg-1832f2g-1078bg2-7358cg2+8581dg2-13219eg2+133\
12fg2-14992g3
⇒ j[127]=e3f2g5-1796d2e2g6+11352be3g6-3580ce3g6+5996de3g6-5103e4g6-6965c3fg\
6+3905bcdfg6+8283c2dfg6+700bd2fg6+14770cd2fg6-13314d3fg6-4300bcefg6+6626c\
2efg6-14379bdefg6+15897cdefg6+11609d2efg6-806be2fg6-2340ce2fg6+6903de2fg6\
+10799e3fg6-8321bcf2g6+192c2f2g6+943bdf2g6-5783cdf2g6+5359d2f2g6+4120bef2\
g6-11803cef2g6-315def2g6+1851e2f2g6-1693bf3g6-14999cf3g6-13198df3g6-10221\
ef3g6-8581f4g6+1404c3g7+4120bcdg7-10616c2dg7-8041bd2g7+4059cd2g7-9557d3g7\
-76bceg7-10400c2eg7+438bdeg7+8911cdeg7-7936d2eg7+13361be2g7+2752ce2g7-816\
1de2g7+10214e3g7-13491bcfg7+14992c2fg7+1343bdfg7+1104cdfg7-14732d2fg7+485\
0befg7-14011cef7+5759def7+3112e2fg7-12690bf2g7-12085cf2g7-15618df2g7-45\
46ef2g7-5270f3g7+3079bcg8-927c2g8+7312bdg8-5887cdg8-2847d2g8+15817beg8+12\
609ceg8+13662deg8-13174e2g8+4275bfg8+5621cfg8+4872dfg8-507efg8-3787f2g8-1\
4790bg9-11080cg9+3041dg9-5002eg9-901fg9-1083g10-7235c3-13139bcd-10407c2d+\
13361bd2+13403cd2-10479d3+3168bce-10379c2e-13448bde+7747cde+14290d2e-1551\
5be2-3621ce2+9339de2+12827e3-13964bcf+11096c2f-13948bdf-8972cdf-2872d2f-6\
934bef+5952cef+6591def+8724e2f+7798bf2-9537cf2-15741df2+839ef2+4203f3+105\

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47bcg-14110c2g+2223bdg-12537cdg+11278d2g-11126beg+6835ceg+12991deg-9765e2\
g+491bfg+972cfg-4910dfg+6548efg+10195f2g-10548bg2-15098cg2+6228dg2-4582eg\
2-14797fg2-15298g3
↳ j [128]=de2g8-3906e3g8+11290bcfg8-5212c2fg8-12729bdfg8-3843cdfg8-9429d2fg8\
+5327befg8-12702cefg8+9832defg8-9658e2fg8+8454bf2g8+15955cf2g8-14224df2g8\
+11462ef2g8+7488f3g8+7264bcg9+10370c2g9+1398bdg9+540cdg9+2597d2g9+8268beg\
9-11891ceg9-5031deg9+7694e2g9-3427bfg9+9333cfg9-13787dfg9+12181efg9+3762f\
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4-9554c3f+11186bcd-7465c2df-6799bd2f+9079cd2f-11867d3f+357bcef+1667c2ef+\
12673bdef+14340cdef-5748d2ef-4074be2f+12479ce2f-15227de2f+7483e3f+1376bcf\
2-8512c2f2-15233bdf2-15616cdf2+13178d2f2-13611bef2+4870cef2+4936def2+1521\
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g+1550bd2g-2489cd2g+14811d3g-1179bceg-9413c2eg+11308bdeg-809cdeg+11453d2e\
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2g+4152ef2g-4028f3g-3298bcg2+15598c2g2+10868bdg2-12120cdg2-9520d2g2-10077\
beg2-6858ceg2-12994deg2-7784e2g2-8493bfg2+362cfg2+11303dfg2+8620efg2-9078\
f2g2+12591bg3-13914cg3-755dg3-14035eg3-9161fg3+10742g4
↳ j [129]=ce2g8+10163e3g8+4152bcfg8+3803c2fg8-3304bdfg8+959cdfg8+10360d2fg8-\
14462befg8+2604cefg8+11337defg8+13998e2fg8-6769bf2g8-15187cf2g8+14846df2g\
8+8718ef2g8-11103f3g8-912bcg9+12503c2g9-6212bdg9+2813cdg9-12550d2g9-15923\
beg9+5531ceg9+1531deg9+3706e2g9+13537bfg9-8457cfg9+13417dfg9-627efg9+607f\
2g9-10104bg10+5579cg10+199dg10+10816eg10-3586fg10-11626g11+5671c4-3004bd3\
+6491cd3-4383d4+14997c3e+3553bcde-14631c2de+1590bd2e-6516cd2e+15229d3e+12\
801bce2+6311c2e2-11423bde2-12906cde2+13009d2e2-9815be3-10784ce3+13368de3+\
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2c2ef-11121bdef+12101cdef+7274d2ef+4700be2f-7879ce2f-842de2f-13708e3f+991\
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42df2g+12081ef2g-8140f3g+1258bcg2+11009c2g2-11671bdg2+12573cdg2-3357d2g2-\
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↳ j [130]=be2g8+9955e3g8-1388bcfg8+6753c2fg8+5625bdfg8+14147cdfg8+8324d2fg8-\
3459befg8-12978cefg8+11256defg8-6738e2fg8-5535bf2g8-3823cf2g8+7120df2g8+1\
124ef2g8-5154f3g8+3306bcg9+8244c2g9+6177bdg9+2930cdg9+7087d2g9+10115beg9-\
7538ceg9-1559deg9-15681e2g9+2068bfg9-14455cfg9-6499dfg9+5582efg9-1665f2g9\
+2779bg10-4590cg10+9645dg10-7150eg10+9856fg10+13886g11-2438c4+14315bd3+12\
955cd3-8049d4+5316c3e+13335bcde-10037c2de+6127bd2e-11948cd2e+6662d3e+183b\
ce2+6793c2e2-11878bde2-3323cde2-8652d2e2-6801be3-15596ce3-10478de3+9056e4\
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736bdef+7120cdef-4229d2ef+14349be2f+8984ce2f-4732de2f-7909e3f+7350bcf2+14\
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g+9855ce2g-299de2g-244e3g+366bcfg-10016c2fg+4930bdfg-3564cdfg-7823d2fg+12\
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2g+14757f3g+9643bcg2+2707c2g2-4488bdg2+7795cdg2-4149d2g2+2788beg2-11551ce\
g2-15149deg2-11393e2g2-999bfg2-7461cfg2+15120dfg2+4237efg2-12090f2g2-4070\

```

```

bg3-7194cg3-13607dg3+9423eg3-9598fg3-13934g4
⇒ j [131]=d2eg8+8063e3g8+14280bcfg8+15811c2fg8+12244bdfg8-12867cdfg8-5837d2f\
g8-3747befg8+11294cefg8-909defg8+10840e2fg8+10820bf2g8-9748cf2g8-10839df2\
g8-13985ef2g8+3786f3g8+8585bcg9+8149c2g9-4041bdg9-10568cdg9-12894d2g9-139\
67beg9-12246ceg9+11305deg9-15842e2g9-1388bfg9+345cfg9+15245dfg9-12330efg9\
+6706f2g9-4739bg10-3729cg10+11724dg10-3821eg10-8514fg10+1919g11+1064c4+14\
735bd3+478cd3+9813d4-1825c3e+15947bcde+12134c2de-4336bd2e+10569cd2e+9373d\
3e-2093bce2-13602c2e2+2470bde2-10120cde2-6778d2e2+5147be3+254ce3+8772de3+\
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730c2ef-10877bdef+10959cdef-4534d2ef-3031be2f-12992ce2f-7061de2f+1324e3f-\
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-9262e2f2-11690bf3+252cf3-15128df3+15709ef3-11250f4-12682c3g-12152bcdg+38\
83c2dg-1122bd2g-12474cd2g+12655d3g+12473bceg+1488c2eg+12011bdeg-14114cdeg\
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g-1208cdfg+2581d2fg+11708befg-4996cefg+14346defg+12282e2fg-14760bf2g+7512\
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4d2g2+12645beg2-3395ceg2-3163deg2-14576e2g2+4947bfg2-11994cfg2+15375dfg2+\
14443efg2+2238f2g2-8519bg3+11872cg3-6171dg3+11741eg3+8621fg3-3254g4
⇒ j [132]=cdeg8+9878e3g8-15593bcfg8+5661c2fg8-7825bdfg8+771cdfg8-5581d2fg8-1\
5117befg8-11903cefg8-10552defg8+9930e2fg8+14001bf2g8-5017cf2g8+3732df2g8-\
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g9+10641ceg9-11498deg9+2562e2g9+6946bfg9+10318cfg9+1854dfg9+9761efg9+5614\
f2g9-6326bg10-4680cg10+5847dg10+14882eg10+2518fg10-4646g11+8690c4-141bd3-\
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bce2+8410c2e2-8208bde2-9905cde2+4298d2e2-11704be3+1589ce3-7172de3-9790e4-\
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123ceg2-9193deg2+5836e2g2+5788bfg2+14253cfg2-11274dfg2+15528efg2+5898f2g2\
+7975bg3-4904cg3-912dg3+4581eg3+5109fg3-9975g4
⇒ j [133]=bdeg8+93e3g8+3974bcfg8-15854c2fg8-13931bdfg8+3498cdfg8+4488d2fg8+8\
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1ef2g8-6191f3g8-201bcg9+8719c2g9-1552bdg9+14722cdg9-11804d2g9-8274beg9-11\
892ceg9+11030deg9-15263e2g9+13721bfg9-1426cfg9+1418dfg9+6885efg9-610f2g9-\
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0cd3+9587d4+6315c3e+12556bcde-12145c2de+2046bd2e-12817cd2e-7328d3e+6805bc\
e2+8037c2e2+13326bde2+8305cde2-7246d2e2-13433be3+4277ce3-10681de3+12336e4\
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+6700bg3-11940cg3-13622dg3-6003eg3-13345fg3+12846g4
⇒ j [134]=c2eg8-7391e3g8-3890bcfg8+12893c2fg8+9119bdfg8-12175cdfg8-13105d2fg\

```



```

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4f2g9+1218bg10+14446cg10+9058dg10-8876eg10-9710fg10+11864g11+2290c4+8637b\
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49ceg2+3297deg2-6533e2g2+11919bfg2+2343cfg2+8732dfg2+12447efg2+12288f2g2-\
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→ j [135]=bceg8-7059e3g8+2345bcfg8+4854c2fg8+7647bdfg8-11395cdfg8+7056d2fg8-\
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+15495ef2g8+5083f3g8+7617bcg9-11636c2g9+11700bdg9-6590cdg9+1522d2g9+2405b\
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459f2g9+2006bg10-4247cg10+12803dg10+10394eg10-10721fg10+12183g11-14102c4-\
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3e-15985bce2-1354c2e2-758bde2+15535cde2+7952d2e2-1798be3+14510ce3-14242de\
3-11448e4+10796c3f+6172bcd-9102c2df-1703bd2f+3690cd2f-6302d3f+11850bcef-\
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47bcf2-6764c2f2+15519bdf2-13125cdf2+305d2f2+3221bef2+15919cef2+8739def2-6\
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+13200be2g+3579ce2g+13323de2g+13395e3g-1846bcfg+3371c2fg+5513bdfg+5008cdf\
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f2g2-640bg3-4118cg3+7941dg3-5810eg3+13790fg3+8032g4
→ j [136]=d3g8+6672e3g8+5297bcfg8+11913c2fg8-13099bdfg8+1197cdfg8+13057d2fg8\
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d2g+15893cd2g-8482d3g-14374bceg-12542c2eg+13125bdeg+1396cdeg-13590d2eg-35\
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ceg2-1687deg2-735e2g2-6323bfg2+8857cfg2+9721dfg2+11830efg2-110f2g2+9651bg\
3+10090cg3-13262dg3+6429eg3+15865fg3+1893g4
→ j [137]=cd2g8-11401e3g8-2467bcfg8+528c2fg8-9708bdfg8+840cdfg8-15037d2fg8+1\
5924befg8+6718cefg8-11348defg8+10851e2fg8+11205bf2g8-6336cf2g8-3972df2g8+\
5943ef2g8-9248f3g8-12238bcg9-9094c2g9-15989bdg9+15725cdg9+14925d2g9+3250b\

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eg9-9096ceg9+6977deg9-10001e2g9-4842bfg9+10348cfg9+6991dfg9+13409efg9-108\
45f2g9+6727bg10+10051cg10+7031dg10+11353eg10-13954fg10+9938g11+2417c4+914\
7bd3-204cd3+10392d4-368c3e+12279bcde+13684c2de-10703bd2e+9247cd2e+950d3e+\
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⇒ j [151]=d2f2g7-12384e3g8-6496bcfg8-1407c2fg8+1963bdfg8-3399cdfg8-9936d2fg8\
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g2-13280ceg2+11722deg2+1593e2g2-11696bfg2-7617cfg2-14703dfg2-11725efg2+21\
96f2g2+15559bg3-5738cg3-11028dg3+15983eg3+15246fg3-12645g4
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+13628ef2g8-9408f3g8-11792bcg9+2335c2g9-7757bdg9+14314cdg9-7420d2g9+8688b\
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8+7268ef2g8+14085f3g8+12363bcg9-7873c2g9-6270bdg9-5828cdg9-2294d2g9+2163b\
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⇒ j[160]=d2efg7+9708e3g8-11505bcfg8-8680c2fg8+3689bdfg8-9569cdfg8+2226d2fg8\
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⇒ j[162]=bdefg7-13002e3g8+15232bcfg8+8452c2fg8-9443bdfg8-12031cdfg8-7219d2f\
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⇒ j[165]=d3fg7-4997e3g8-1201bcfg8+9152c2fg8+8391bdfg8-15976cdfg8-1491d2fg8+\
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⇒ j[168]=c2dfg7-2054e3g8-11396bcfg8+14982c2fg8+2063bdfg8-13440cdfg8-13212d2\
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36cg3-2170dg3+14497eg3+296fg3+6801g4
⇒ j [171]=e4g7+5785e3g8-3232bcfg8+11011c2fg8-5705bdfg8-6222cdfg8-10516d2fg8+\
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→ j [173]=ce3g7+3622e3g8+4327bcfg8-15430c2fg8+4235bdfg8-4520cdfg8+8272d2fg8-\
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→ j [174]=be3g7+5098e3g8+11267bcfg8+1479c2fg8-1470bdfg8+3559cdfg8-7552d2fg8-\
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3-8830bg4+3060cg4+14945dg4+9029eg4-273fg4+59g5
↳ j [177]=fg11+5602cde2f-11890d2e2f+15224be3f-12933ce3f-3700de3f+7888e4f-111\
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fg+15260de2fg-13510e3fg+10860bcf2g-15083c2f2g-6298bdf2g-12346cdf2g-12736d\
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3+2479bg4+7371cg4+9952dg4-14261eg4-10976fg4-11673g5
↳ j [178]=eg11-15587cde2f+13430d2e2f+9844be3f+3029ce3f-4882de3f-14025e4f-401\
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f2-8178bdef2+7394cdef2+4215d2ef2-1836be2f2-15220ce2f2+11875de2f2+11647e3f\
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3-9878bg4-3116cg4-11502dg4+9506eg4+12006fg4-4826g5
↳ j [179]=dg11-4180cde2f+2770d2e2f-5543be3f-14850ce3f+8296de3f+152e4f+12947c\
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+14124bdef2+13946cdef2+5058d2ef2+10864be2f2+7163ce2f2+2154de2f2-12394e3f2\
+7863bcf3-7457c2f3-5307bdf3-9461cdf3+10188d2f3+5828bef3-7037cef3-7608def3\
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3293dg4+13411eg4+3299fg4+8236g5
↳ j [180]=cg11+13711cde2f-103d2e2f-9261be3f-5141ce3f-12148de3f+3069e4f+3713c\
3f2+7067bcd2f-12214c2df2+6813bd2f2-7505cd2f2+4312d3f2+10000bcef2+10425c2e\
f2+5479bdef2-990cdef2+2868d2ef2+13713be2f2+4679ce2f2-2018de2f2-736e3f2+11\
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d3g-659d4g-14178c3eg+3104bcdeg-4758c2deg+6574bd2eg-9349cd2eg-1913d3eg-152\
37bce2g-15710c2e2g-406bde2g+3295cde2g-1387d2e2g-13497be3g+15361ce3g-3653d\
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```

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4-6248dg4-4011eg4-4269fg4+8494g5
↳ j [181]=bg11-5655cde2f+5532d2e2f-9000be3f-12426ce3f-9841de3f+13305e4f+1195\
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2ef2-1582bdef2-11438cdef2-235d2ef2-13340be2f2-55ce2f2+5326de2f2+3199e3f2+\
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def3+15946e2f3-364bf4+12474cf4+10151df4-4028ef4+14639f5+15480c4g-12620bd3\
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e3g-10257de3g+6431e4g-1259c3fg+1583bcd2fg-239c2dfg+12503bd2fg+533cd2fg+688\
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efg3-13744f2g3+139bg4-2460cg4+14263dg4-4597eg4-10906fg4-10473g5
↳ j [182]=f2g10-288cde2f-9668d2e2f-9632be3f-6993ce3f-2452de3f-14525e4f+4090c\
3f2+15029bcd2f-14961c2df2-6066bd2f2+1215cd2f2-3674d3f2-13667bcef2-4437c2e\
f2+13112bdef2+1161cdef2+10234d2ef2+11139be2f2-10492ce2f2+7818de2f2+8953e3\
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-5699bg4+1157cg4-9570dg4-5196eg4+5236fg4-6624g5
↳ j [183]=efg10-14268cde2f-12497d2e2f-7654be3f-1037ce3f-8496de3f+4013e4f+466\
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1e2f3+15630bf4+7526cf4+13741df4-3589ef4+8666f5-8325c4g-272bd3g-14440cd3g+\
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29e4g-9291c3fg-6898bcd2fg-8113c2dfg-6680bd2fg-4397cd2fg-13911d3fg+5845bcef\
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+6538e3fg+2406bcf2g+8797c2f2g-9823bdf2g+4035cdf2g-4228d2f2g-14525bef2g-20\
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5cg4+3206dg4-12174eg4-6767fg4-3485g5
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beg3-6688ceg3-12728deg3-12994e2g3-854bfg3+10212cfg3+15197dfg3-84efg3+1304\
2f2g3+15903bg4-14391cg4+12464dg4+7581eg4+11257fg4+2325g5
→ j [185]=cfg10-14195cde2f+706d2e2f+9078be3f-11099ce3f-12380de3f+13844e4f+10\
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→ j[187]=e2g10-5439cde2f+1817d2e2f-2680be3f-587ce3f+15569de3f-4506e4f+3438c\
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→ j[188]=deg10-4710cde2f-14675d2e2f-6578be3f+6001ce3f+2285de3f+1638e4f-2115\
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```

```

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g2-14407f3g2-6702bcg3+11675c2g3+4584bdg3-12688cdg3-9704d2g3+13923beg3-649\
2ceg3-7473deg3+14348e2g3+11313bfg3-7987cfg3+13638dfg3-10211efg3-7052f2g3+\
11689bg4-2392cg4-9899dg4-57eg4+2221fg4+5838g5
↳ j [207]=bdfg9+6180cde2f-9339d2e2f+8979be3f+6372ce3f+11737de3f-9558e4f+9770\

```



```

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df2g+6211ef2g2-563f3g2-2601bcg3+3090c2g3+9339bdg3+11184cdg3-14444d2g3+53\
98beg3+1504ceg3-7236deg3-10467e2g3-2032bfg3-9353cfg3-10391dfg3+3118efg3+1\
4061f2g3+3186bg4+696cg4-2408dg4-8032eg4+265fg4+5090g5
→ j [208]=c2fg9+15357cde2f+1738d2e2f-9507be3f-3626ce3f-15619de3f-686e4f-5065\
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→ j [209]=bcfg9-15466cde2f+6718d2e2f+2038be3f+10674ce3f+13682de3f-6829e4f+83\
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0efg3+13364f2g3+11902bg4-7739cg4+5263dg4+9970eg4+2637fg4+1403g5

```

See also: [Section D.4.31.4 \[modJanet\]](#), page 1376.

Procedure from library `rstandard.lib` (see [Section D.4.31 \[rstandard.lib\]](#), page 1266).

**Return:** ideal, a Janet basis for I using modular methods.

If second argument is 0 then the result is not verified.

[illegible]

```

↳ j[2]=tz
↳ j[3]=ty
ring S=0,(x,y,z),dp;
poly p1 =x2y*(47x5y7z3+28xy5z8+63+91x5y3z7);
poly p2 =xyz*(57y6+21x2yz9+51y2z2+15x2z4);
poly p3 =xy4z*(74y+32x6z7+53x5y2z+17x2y3z);
poly p4 =y3z*(21x2z6+32x10y6z5+23x5y5z7+27y2);
poly p5 =xz*(36y2z2+81x9y10+19x2y5z4+79x4z6);
ideal i =p1,p2,p3,p4,p5;
ideal j=modJanet(i,0); j;
↳ Length of Janet basis: 9
↳ Length of Janet basis: 9
↳ Length of Janet basis: 9
↳ Length of Janet basis: 9
↳ Length of Janet basis: 9
↳ Length of Janet basis: 9
↳ Length of Janet basis: 9
↳ Length of Janet basis: 9
↳ Length of Janet basis: 9
↳ Length of Janet basis: 9
↳ Length of Janet basis: 9
↳ Length of Janet basis: 9
↳ Length of Janet basis: 9
↳ Length of Janet basis: 9
↳ j[1]=x2y
↳ j[2]=x3y
↳ j[3]=x4y
↳ j[4]=y5z
↳ j[5]=x5y
↳ j[6]=xy3z3
↳ j[7]=xy5z
↳ j[8]=xy4z3
↳ j[9]=x5z7+36/79xy2z3

```

See also: [Section D.4.31.3 \[rJanet\]](#), page 1306.

## D.4.32 sagbi.lib

**Library:** sagbi.lib

**Purpose:** Compute SAGBI basis (subalgebra bases analogous to Groebner bases for ideals) of a subalgebra

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 Viktor Levandovskyy, levandov@math.rwth-aachen.de

**Overview:** SAGBI stands for 'subalgebra bases analogous to Groebner bases for ideals'. SAGBI bases provide important tools for working with finitely presented subalgebras of a polynomial ring. Note, that in contrast to Groebner bases, SAGBI bases may be infinite.

**References:**

Ana Bravo: Some Facts About Canonical Subalgebra Bases, MSRI Publications 51, p. 247-254

**Procedures:** See also: [Section D.4.2 \[algebra.lib\]](#), page 1004.

### D.4.32.1 sagbiSPoly

Procedure from library `sagbi.lib` (see [Section D.4.32 \[sagbi.lib\], page 1377](#)).

- Usage:** `sagbiSPoly(A[, returnRing, meth]);` A is an ideal, returnRing and meth are integers.
- Return:** ideal or ring
- Assume:** basering is not a qring
- Purpose:** Returns SAGBI S-polynomials of the leading terms of a given ideal A if returnRing=0. Otherwise returns a new ring containing the ideals algebraicRelations and spolynomial, where these objects are explained by their name. See the example on how to access these objects. The other optional argument meth determines which method is used for computing the algebraic relations.
- If meth=0 (default), the procedure std is used.
  - If meth=1, the procedure slimgb is used.
  - If meth=2, the procedure uses toric\_ideal.

**Example:**

```
LIB "sagbi.lib";
ring r= 0,(x,y),dp;
ideal A=x*y+x,x*y^2,y^2+y,x^2+x;
//----- Compute the SAGBI S-polynomials only
sagbiSPoly(A);
⇒ _[1]=x2y-xy2+x2-xy
⇒ _[2]=x2y3+1/2xy4+1/2x2y2+xy3+1/2xy2
//----- Extended ring is to be returned, which contains
// the ideal of algebraic relations and the ideal of the S-polynomials
def rNew=sagbiSPoly(A,1); setring rNew;
spolynomial;
⇒ spolynomial[1]=x^2*y-x*y^2+x^2-x*y
⇒ spolynomial[2]=x^2*y^3+1/2*x*y^4+1/2*x^2*y^2+x*y^3+1/2*x*y^2
algebraicRelations;
⇒ algebraicRelations[1]=@y(1)^2-@y(3)*@y(4)
⇒ algebraicRelations[2]=@y(3)^2*@y(4)-@y(2)^2
//----- Now we verify that the substitution of A[i] into @y(i)
// results in the spolynomial listed above
ideal A=fetch(r,A);
map phi=rNew,x,y,A;
ideal spolynomial2=simplify(phi(algebraicRelations),1);
spolynomial2;
⇒ spolynomial2[1]=x^2*y-x*y^2+x^2-x*y
⇒ spolynomial2[2]=x^2*y^3+1/2*x*y^4+1/2*x^2*y^2+x*y^3+1/2*x*y^2
```

### D.4.32.2 sagbiReduce

Procedure from library `sagbi.lib` (see [Section D.4.32 \[sagbi.lib\], page 1377](#)).

- Usage:** `sagbiReduce(I, A[, tr, mt]);` I, A ideals, tr, mt optional integers
- Return:** ideal of remainders of I after SAGBI reduction by A
- Assume:** basering is not a qring
- Purpose:**



The optional argument `tr=tailred` determines whether tail reduction will be performed.

- If (`tailred=0`), no tail reduction is done.
- If (`tailred<>0`), tail reduction is done.

The other optional argument `meth` determines which method is used for Groebner basis computations.

- If `mt=0` (default), the procedure `std` is used.
- If `mt=1`, the procedure `slimgb` is used.

**Example:**

```
LIB "sagbi.lib";
ring r=0,(x,y,z),dp;
ideal A=x2,2*x2y+y,x3y2;
poly p1=x^5+x2y+y;
poly p2=x^16+x^12*y^5+6*x^8*y^4+x^6+y^4+3;
ideal P=p1,p2;
//-----
//SAGBI reduction of polynomial p1 by algebra A.
//Default call, that is, no tail-reduction is done.
sagbiReduce(p1,A);
↪ x5+x2y+y
//-----
//SAGBI reduction of set of polynomials P by algebra A,
//now tail-reduction is done.
sagbiReduce(P,A,1);
↪ _[1]=x5+1/2y
↪ _[2]=x6y5-8y4
```

### D.4.32.3 sagbi

Procedure from library `sagbi.lib` (see [Section D.4.32 \[sagbi.lib\]](#), page 1377).

**Usage:** `sagbi(A[, tr, mt]);` A ideal, tr, mt optional integers

**Return:** ideal, a SAGBI basis for A

**Assume:** basering is not a qring

**Purpose:** Computes a SAGBI basis for the subalgebra given by the generators in A.

The optional argument `tr=tailred` determines whether tail reduction will be performed.

- If (`tailred=0`), no tail reduction is performed,
- If (`tailred<>0`), tail reduction is performed.

The other optional argument `meth` determines which method is used for Groebner basis computations.

- If `mt=0` (default), the procedure `std` is used.
- If `mt=1`, the procedure `slimgb` is used.

**Example:**

```
LIB "sagbi.lib";
ring r= 0,(x,y,z),dp;
ideal A=x2,y2,xy+y;
//Default call, no tail-reduction is done.
sagbi(A);
↪ _[1]=x2
↪ _[2]=y2
↪ _[3]=xy+y
```

```

↳ _[4]=xy2+1/2y2
//-----
//Call with tail-reduction and method specified.
sagbi(A,1,0);
↳ _[1]=x2
↳ _[2]=y2
↳ _[3]=xy+y
↳ _[4]=xy2

```

#### D.4.32.4 sagbiPart

Procedure from library `sagbi.lib` (see [Section D.4.32 \[sagbi.lib\]](#), page 1377).

**Usage:** `sagbiPart(A, k,[tr, mt]);` A is an ideal, k, tr and mt are integers

**Return:** ideal

**Assume:** basering is not a qring

**Purpose:** Performs k iterations of the SAGBI construction algorithm for the subalgebra given by the generators given by A.

The optional argument `tr=tailred` determines if tail reduction will be performed.

- If (`tailred=0`), no tail reduction is performed,

- If (`tailred<>0`), tail reduction is performed.

The other optional argument `meth` determines which method is used for Groebner basis computations.

- If `mt=0` (default), the procedure `std` is used.

- If `mt=1`, the procedure `slimgb` is used.

**Example:**

```

LIB "sagbi.lib";
ring r= 0,(x,y,z),dp;
//The following algebra does not have a finite SAGBI basis.
ideal A=x,xy-y2,xy2;
//-----
//Call with two iterations, no tail-reduction is done.
sagbiPart(A,2);
↳ //SAGBI construction algorithm stopped as it reached the limit of 2 itera\
  tions.
↳ //In general the returned generators are no SAGBI basis for the given alg\
  ebra.
↳ _[1]=x
↳ _[2]=xy-y2
↳ _[3]=xy2
↳ _[4]=2xy3-y4
↳ _[5]=3xy5-y6
↳ _[6]=xy4
//-----
//Call with three iterations, tail-reduction and method 0.
sagbiPart(A,3,1,0);
↳ //SAGBI construction algorithm stopped as it reached the limit of 3 itera\
  tions.
↳ //In general the returned generators are no SAGBI basis for the given alg\
  ebra.
↳ _[1]=x

```

```

⇒ _[2]=xy-y2
⇒ _[3]=xy2
⇒ _[4]=2xy3-y4
⇒ _[5]=3xy5-y6
⇒ _[6]=xy4
⇒ _[7]=5xy9-y10
⇒ _[8]=xy8
⇒ _[9]=4xy7-y8
⇒ _[10]=xy6

```

### D.4.32.5 algebraicDependence

Procedure from library `sagbi.lib` (see [Section D.4.32 \[sagbi.lib\]](#), page 1377).

**Usage:**      `algebraicDependence(I,it)`; I an ideal, it is an integer

**Return:**     ring

**Assume:**    basering is not a qring

**Purpose:**     Returns a ring containing the ideal `algDep`, which contains possibly some algebraic dependencies of the elements of I obtained through it iterations of the SAGBI construction algorithms. See the example on how to access these objects.

**Example:**

```

LIB "sagbi.lib";
ring r= 0,(x,y),dp;
//The following algebra does not have a finite SAGBI basis.
ideal I=x^2, xy-y2, xy2;
//-----
//Call with two iterations
def DI = algebraicDependence(I,2);
⇒ //AlgDep-1- initialisation and precomputation
⇒ //AlgDep-2- call of SAGBI construction algorithm
⇒ //SAGBI construction algorithm stopped as it reached the limit of 2 itera\
tions.
⇒ //In general the returned generators are no SAGBI basis for the given alg\
ebra.
⇒ //AlgDep-3- postprocessing of results
setring DI; algDep;
⇒ algDep[1]=0
// we see that no dependency has been seen so far
//-----
//Call with two iterations
setring r; kill DI;
def DI = algebraicDependence(I,3);
⇒ //AlgDep-1- initialisation and precomputation
⇒ //AlgDep-2- call of SAGBI construction algorithm
⇒ //SAGBI construction algorithm stopped as it reached the limit of 3 itera\
tions.
⇒ //In general the returned generators are no SAGBI basis for the given alg\
ebra.
⇒ //AlgDep-3- postprocessing of results
setring DI; algDep;

```

```

↳ algDep[1]=0
map F = DI,x,y,x^2, xy-y2, xy2;
F(algDep); // we see that it is a dependence indeed
↳ _[1]=0

```

### D.4.33 sing4ti2.lib

**Library:** sing4ti2.lib

**Purpose:** Communication Interface to 4ti2

**Authors:** Thomas Kahle , kahle@mis.mpg.de  
Anne Fruehbis-Krueger, anne@math.uni-hannover.de

**Note:** This library uses the external program 4ti2 for calculations and the standard unix tools sed and awk for conversion of the returned result

**Procedures:**

#### D.4.33.1 markov4ti2

Procedure from library `sing4ti2.lib` (see [Section D.4.33 \[sing4ti2.lib\]](#), page 1382).

**Usage:** markov4ti2(A[,i]);  
A=intmat  
i=int

**Assume:** - A is a matrix with integer entries which describes the lattice as  $\ker(A)$ , if second argument is not present,  
as left image  $\text{Im}(A) = \{zA, z \in \mathbb{Z}^k\}(!)$ , if second argument is a positive integer  
- number of variables of basering equals number of columns of A  
(for  $\ker(A)$ ) resp. of rows of A (for  $\text{Im}(A)$ )

**Create:** files sing4ti2.mat, sing4ti2.lat, sing4ti2.mar in the current directory (I/O files for communication with 4ti2)

**Note:** input rules for 4ti2 also apply to input to this procedure  
hence  $\ker(A) = \{x \mid Ax=0\}$  and  $\text{Im}(A) = \{xA\}$

**Return:** toric ideal specified by Markov basis thereof

**Example:**

```

LIB "sing4ti2.lib";
ring r=0,(x,y,z),dp;
matrix M[2][3]=0,1,2,2,1,0;
markov4ti2(M);
↳ _[1]=-y2+xz
matrix N[1][3]=1,2,1;
markov4ti2(N,1);
↳ _[1]=xy2z-1
↳ _[2]=xy2z-1

```

### D.4.33.2 hilbert4ti2

Procedure from library `sing4ti2.lib` (see [Section D.4.33 \[sing4ti2.lib\]](#), page 1382).

**Usage:**     `hilbert4ti2(A[,i]);`  
               `A=intmat`  
               `i=int`

**Assume:**   -  $A$  is a matrix with integer entries which describes the lattice  
               as  $\ker(A)$ , if second argument is not present,  
               as the left image  $\operatorname{Im}(A) = \{zA : z \in \mathbb{Z}^k\}$ , if second argument is a positive integer  
               - number of variables of basering equals number of columns of  $A$   
               (for  $\ker(A)$ ) resp. of rows of  $A$  (for  $\operatorname{Im}(A)$ )

**Create:**    temporary files `sing4ti2.mat`, `sing4ti2.lat`, `sing4ti2.mar`  
               in the current directory (I/O files for communication with 4ti2)

**Note:**      input rules for 4ti2 also apply to input to this procedure  
               hence  $\ker(A) = \{x \mid Ax = 0\}$  and  $\operatorname{Im}(A) = \{xA\}$

**Return:**    toric ideal specified by Hilbert basis thereof

**Example:**

```
LIB "sing4ti2.lib";
ring r=0, (x1,x2,x3,x4,x5,x6,x7,x8,x9), dp;
matrix M[7][9]=1,1,1,-1,-1,-1,0,0,0,1,1,1,0,0,0,-1,-1,-1,0,1,1,-1,0,0,-1,0,0,1,0,1,0;
hilbert4ti2(M);
⇨ _[1]=x1^2*x3*x5*x6^2*x7*x8^2-1
⇨ _[2]=x1*x3^2*x4^2*x5*x8^2*x9-1
⇨ _[3]=x2^2*x3*x4^2*x5*x7*x9^2-1
⇨ _[4]=x1*x2^2*x5*x6^2*x7^2*x9-1
⇨ _[5]=x1*x2*x3*x4*x5*x6*x7*x8*x9-1
```

### D.4.33.3 graver4ti2

Procedure from library `sing4ti2.lib` (see [Section D.4.33 \[sing4ti2.lib\]](#), page 1382).

**Usage:**     `graver4ti2(A[,i]);`  
               `A=intmat`  
               `i=int`

**Assume:**   -  $A$  is a matrix with integer entries which describes the lattice  
               as  $\ker(A)$ , if second argument is not present,  
               as the left image  $\operatorname{Im}(A) = \{zA : z \in \mathbb{Z}^k\}$ , if second argument is a positive integer  
               - number of variables of basering equals number of columns of  $A$   
               (for  $\ker(A)$ ) resp. of rows of  $A$  (for  $\operatorname{Im}(A)$ )

**Create:**    temporary files `sing4ti2.mat`, `sing4ti2.lat`, `sing4ti2.gra`  
               in the current directory (I/O files for communication with 4ti2)

**Note:**      input rules for 4ti2 also apply to input to this procedure  
               hence  $\ker(A) = \{x \mid Ax = 0\}$  and  $\operatorname{Im}(A) = \{xA\}$

**Return:**    toric ideal specified by Graver basis thereof

**Example:**

```

LIB "sing4ti2.lib";
ring r=0,(x,y,z,w),dp;
matrix M[2][4]=0,1,2,3,3,2,1,0;
graver4ti2(M);
↪ _[1]=-y2+xz
↪ _[2]=-y3+x2w
↪ _[3]=-yz+xw
↪ _[4]=-z2+yw
↪ _[5]=-z3+xw2

```

### D.4.34 symodstd.lib

**Library:** symodstd.lib

**Purpose:** Procedures for computing Groebner basis of ideals being invariant under certain variable permutations.

**Author:** Stefan Steidel, steidel@mathematik.uni-kl.de

**Overview:** A library for computing the Groebner basis of an ideal in the polynomial ring over the rational numbers, that is invariant under certain permutations of the variables, using the symmetry and modular methods. More precisely let  $I = \langle f_1, \dots, f_r \rangle$  be an ideal in  $\mathbb{Q}[x(1), \dots, x(n)]$  and  $\sigma$  a permutation of order  $k$  in  $\text{Sym}(n)$  such that  $\sigma(I) = I$ . We assume that  $\sigma(\{f_1, \dots, f_r\}) = \{f_1, \dots, f_r\}$ . This can always be obtained by adding  $\sigma(f_i)$  to  $\{f_1, \dots, f_r\}$ .

To compute a standard basis of  $I$  we apply a modification of the modular version of the standard basis algorithm (improving the calculations in positive characteristic). Therefore we only allow primes  $p$  such that  $p-1$  is divisible by  $k$ . This guarantees the existence of a  $k$ -th primitive root of unity in  $\mathbb{Z}/p\mathbb{Z}$ .

**Procedures:**

#### D.4.34.1 genSymId

Procedure from library `symodstd.lib` (see [Section D.4.34 \[symodstd.lib\]](#), page 1384).

**Usage:** `genSymId(I, sigma);`  $I$  ideal,  $\sigma$  intvec

**Assume:** `size(sigma) = nvars(basering) =: n`

**Return:** ideal  $J$  such that  $\sigma(J) = J$  and  $J$  includes  $I$

**Note:**  $\sigma$  is a permutation of the variables of the basering, i.e.  
 $\sigma: \text{var}(i) \rightarrow \text{var}(\sigma[i]), 1 \leq i \leq n$ .

**Example:**

```

LIB "symodstd.lib";
ring R = 0,(u,v,w,x,y),dp;
intvec pi = 2,3,4,5,1;
ideal I = u2v + x3y - w2;
genSymId(I, pi);
↪ _[1]=x3y+u2v-w2
↪ _[2]=uy3+v2w-x2
↪ _[3]=u3v+w2x-y2
↪ _[4]=v3w+x2y-u2
↪ _[5]=w3x+uy2-v2

```

### D.4.34.2 isSymmetric

Procedure from library `symodstd.lib` (see [Section D.4.34 \[symodstd.lib\], page 1384](#)).

**Usage:** `isSymmetric(I,sigma)`; I ideal, sigma intvec

**Assume:** `size(sigma) = nvars(basing) =: n`

**Return:** 1, if the set of generators of I is invariant under sigma;  
0, if the set of generators of I is not invariant under sigma

**Note:** sigma is a permutation of the variables of the basing, i.e.  
sigma:  $\text{var}(i) \rightarrow \text{var}(\text{sigma}[i])$ ,  $1 \leq i \leq n$ .

**Example:**

```
LIB "symodstd.lib";
ring R = 0,x(1..5),dp;
ideal I = cyclic(5);
intvec pi = 2,3,4,5,1;
isSymmetric(I,pi);
↪ 1
intvec tau = 2,5,1,4,3;
isSymmetric(I,tau);
↪ 0
```

### D.4.34.3 primRoot

Procedure from library `symodstd.lib` (see [Section D.4.34 \[symodstd.lib\], page 1384](#)).

**Usage:** `primRoot(p,k)`; p,k integers

**Assume:** p is a prime and k divides p-1.

**Return:** int: a k-th primitive root of unity in  $\mathbb{Z}/p\mathbb{Z}$

**Example:**

```
LIB "symodstd.lib";
primRoot(181,10);
↪ 56
ring R = 2147482801, x, lp;
number a = primRoot(2147482801,5);
a;
↪ -159774741
a^2;
↪ 140354890
a^3;
↪ 260989846
a^4;
↪ -241569996
a^5;
↪ 1
```

### D.4.34.4 eigenvalues

Procedure from library `symodstd.lib` (see [Section D.4.34 \[symodstd.lib\], page 1384](#)).

**Usage:** `eigenvalues(I,sigma)`; I ideal, sigma intvec

**Assume:**  $\text{size}(\text{sigma}) = \text{nvars}(\text{basing}) =: n$

**Return:** list of eigenvalues of generators of  $I$  under permutation  $\text{sigma}$

**Note:**  $\text{sigma}$  is a permutation of the variables of the basing, i.e.  $\text{sigma: var}(i) \longrightarrow \text{var}(\text{sigma}[i]), 1 \leq i \leq n$ .

**Example:**

```
LIB "symodstd.lib";
ring R = 11, x(1..5), dp;
poly p1 = x(1)+x(2)+x(3)+x(4)+x(5);
poly p2 = x(1)+4*x(2)+5*x(3)-2*x(4)+3*x(5);
poly p3 = x(1)+5*x(2)+3*x(3)+4*x(4)-2*x(5);
poly p4 = x(1)-2*x(2)+4*x(3)+3*x(4)+5*x(5);
poly p5 = x(1)+3*x(2)-2*x(3)+5*x(4)+4*x(5);
ideal I = p1,p2,p3,p4,p5;
intvec tau = 2,3,4,5,1;
eigenvalues(I,tau);
↪ [1]:
↪ 1
↪ [2]:
↪ 3
↪ [3]:
↪ -2
↪ [4]:
↪ 5
↪ [5]:
↪ 4
```

#### D.4.34.5 symmStd

Procedure from library `symodstd.lib` (see [Section D.4.34 \[symodstd.lib\]](#), page 1384).

**Usage:** `symmStd(I,sigma,#);`  $I$  ideal,  $\text{sigma}$  intvec

**Assume:**  $\text{size}(\text{sigma}) = \text{nvars}(\text{basing}) =: n$ ,  $\text{basing}$  has an  $\text{order}(\text{sigma})$ -th primitive root of unity  $a$  (if  $\text{char}(\text{basing}) > 0$ ) and  $\text{sigma}(I) = I$

**Return:** ideal, a standard basis of  $I$

**Note:** Assuming that the ideal  $I$  is invariant under the variable permutation  $\text{sigma}$  and the basing has an  $\text{order}(\text{sigma})$ -th primitive root of unity the procedure uses linear transformation of variables in order to improve standard basis computation.  
If  $\text{char}(\text{basing}) = 0$  all computations are done in the polynomial ring over the smallest field extension that has an  $\text{order}(\text{sigma})$ -th primitive root of unity.

**Example:**

```
LIB "symodstd.lib";
ring R = 0, x(1..4), dp;
ideal I = cyclic(4);
I;
↪ I[1]=x(1)+x(2)+x(3)+x(4)
↪ I[2]=x(1)*x(2)+x(2)*x(3)+x(1)*x(4)+x(3)*x(4)
↪ I[3]=x(1)*x(2)*x(3)+x(1)*x(2)*x(4)+x(1)*x(3)*x(4)+x(2)*x(3)*x(4)
↪ I[4]=x(1)*x(2)*x(3)*x(4)-1
intvec pi = 2,3,4,1;
```



```

ideal sI = symmStd(I,pi);
sI;
↪ sI[1]=x(1)+x(2)+x(3)+x(4)
↪ sI[2]=x(2)^2+2*x(2)*x(4)+x(4)^2
↪ sI[3]=x(2)*x(3)^2+x(3)^2*x(4)-x(2)*x(4)^2-x(4)^3
↪ sI[4]=x(2)*x(3)*x(4)^2+x(3)^2*x(4)^2-x(2)*x(4)^3+x(3)*x(4)^3-x(4)^4-1
↪ sI[5]=x(2)*x(4)^4+x(4)^5-x(2)-x(4)
↪ sI[6]=x(3)^3*x(4)^2+x(3)^2*x(4)^3-x(3)-x(4)
↪ sI[7]=x(3)^2*x(4)^4+x(2)*x(3)-x(2)*x(4)+x(3)*x(4)-2*x(4)^2
ring S = 31, (x,y,z), dp;
ideal J;
J[1] = xy-y2+xz;
J[2] = xy+yz-z2;
J[3] = -x2+xz+yz;
intvec tau = 3,1,2;
ideal sJ = symmStd(J,tau);
sJ;
↪ sJ[1]=y2-xz+yz-z2
↪ sJ[2]=xy+yz-z2
↪ sJ[3]=x2-xz-yz
↪ sJ[4]=yz2-z3
↪ sJ[5]=xz2
↪ sJ[6]=z4

```

#### D.4.34.6 syModStd

Procedure from library `symodstd.lib` (see [Section D.4.34 \[symodstd.lib\]](#), page 1384).

**Usage:** `syModStd(I,sigma);` I ideal, sigma intvec

**Assume:** `size(sigma) = nvars(basering)` and `sigma(I) = I`. If `size(#)` > 0, then # contains either 1, 2 or 4 integers such that

- `#[1]` is the number of available processors for the computation,
- `#[2]` is an optional parameter for the exactness of the computation, if `#[2] = 1`, the procedure computes a standard basis for sure,
- `#[3]` is the number of primes until the first lifting,
- `#[4]` is the constant number of primes between two liftings until the computation stops.

**Return:** ideal, a standard basis of I if no warning appears;

**Note:** The procedure computes a standard basis of the ideal I (over the rational numbers) by using modular methods and the fact that I is invariant under the variable permutation sigma.

By default the procedure computes a standard basis of I for sure, but if the optional parameter `#[2] = 0`, it computes a standard basis of I with high probability.

The procedure distinguishes between different variants for the standard basis computation in positive characteristic depending on the ordering of the basering, the parameter `#[2]` and if the ideal I is homogeneous.

- variant = 1, if I is homogeneous,
- variant = 2, if I is not homogeneous, 1-block-ordering,
- variant = 3, if I is not homogeneous, complicated ordering (lp or > 1 block),
- variant = 4, if I is not homogeneous, ordering lp, `dim(I) = 0`.

**Example:**

```

LIB "symodstd.lib";
ring R1 = 0, (x,y,z), dp;
ideal I;
I[1] = -2xyz4+xz5+xz;
I[2] = -2xyz4+yz5+yz;
intvec sigma = 2,1,3;
ideal sI = syModStd(I,sigma);
sI;
↪ sI[1]=x2yz-xy2z
↪ sI[2]=xz5-yz5+xz-yz
↪ sI[3]=xyz4-1/2yz5-1/2yz
↪ sI[4]=y2z5-1/2yz6-xyz+y2z-1/2yz2
ring R2 = 0, x(1..4), dp;
ideal I = cyclic(4);
I;
↪ I[1]=x(1)+x(2)+x(3)+x(4)
↪ I[2]=x(1)*x(2)+x(2)*x(3)+x(1)*x(4)+x(3)*x(4)
↪ I[3]=x(1)*x(2)*x(3)+x(1)*x(2)*x(4)+x(1)*x(3)*x(4)+x(2)*x(3)*x(4)
↪ I[4]=x(1)*x(2)*x(3)*x(4)-1
intvec pi = 2,3,4,1;
ideal sJ1 = syModStd(I,pi,1);
ideal sJ2 = syModStd(I,pi,1,0);
size(reduce(sJ1,sJ2));
↪ 0
size(reduce(sJ2,sJ1));
↪ 0

```

### D.4.35 toric\_lib

**Library:** toric.lib

**Purpose:** Standard Basis of Toric Ideals

**Author:** Christine Theis, email: ctheis@math.uni-sb.de

**Procedures:**

#### D.4.35.1 toric\_ideal

Procedure from library `toric.lib` (see [Section D.4.35 \[toric\\_lib\]](#), page 1388).

**Usage:**     `toric_ideal(A,alg);` A intmat, alg string  
               `toric_ideal(A,alg,prsv);` A intmat, alg string, prsv intvec

**Return:**    ideal: standard basis of the toric ideal of A

**Note:**     These procedures return the standard basis of the toric ideal of A with respect to the term ordering in the current basering. Not all term orderings are supported: The usual global term orderings may be used, but no block orderings combining them.

One may call the procedure with several different algorithms:

- the algorithm of Conti/Traverso using elimination (ect),
- the algorithm of Pottier (pt),
- an algorithm of Bigatti/La Scala/Robbiano (blr),
- the algorithm of Hosten/Sturmfels (hs),
- the algorithm of DiBiase/Urbanke (du).

The argument ‘alg’ should be the abbreviation for an algorithm as above: ect, pt, blr, hs or du.

If ‘alg’ is chosen to be ‘blr’ or ‘hs’, the algorithm needs a vector with positive coefficients in the row space of A.

If no row of A contains only positive entries, one has to use the second version of `toric_ideal` which takes such a vector as its third argument.

For the mathematical background, see

[Section C.6 \[Toric ideals and integer programming\], page 782.](#)

**Example:**

```
LIB "toric.lib";
ring r=0,(x,y,z),dp;
// call with two arguments
intmat A[2][3]=1,1,0,0,1,1;
A;
  ↳ 1,1,0,
  ↳ 0,1,1
ideal I=toric_ideal(A,"du");
I;
  ↳ I[1]=xz-y
I=toric_ideal(A,"blr");
  ↳ ? The chosen algorithm needs a positive vector in the row space of the\
    matrix.
  ↳ ? leaving toric.lib::toric_ideal_1 (0)
  ↳ ? leaving toric.lib::toric_ideal (704)
I;
  ↳ I[1]=xz-y
// call with three arguments
intvec prsv=1,2,1;
I=toric_ideal(A,"blr",prsv);
I;
  ↳ I[1]=xz-y
```

See also: [Section C.6.1 \[Toric ideals\], page 782](#); [Section D.4.13 \[intprog.lib\], page 1119](#); [Section D.4.35.2 \[toric\\_std\], page 1389](#).

### D.4.35.2 toric\_std

Procedure from library `toric.lib` (see [Section D.4.35 \[toric.lib\], page 1388](#)).

**Usage:** `toric_std(I);` I ideal

**Return:** ideal: standard basis of I

**Note:** This procedure computes the standard basis of I using a specialized Buchberger algorithm. The generating system by which I is given has to consist of binomials of the form  $x^u - x^v$ . There is no real check if I is toric. If I is generated by binomials of the above form, but not toric, `toric_std` computes an ideal ‘between’ I and its saturation with respect to all variables.

For the mathematical background, see

[Section C.6 \[Toric ideals and integer programming\], page 782.](#)

**Example:**

```

LIB "toric.lib";
ring r=0,(x,y,z),wp(3,2,1);
// call with toric ideal (of the matrix A=(1,1,1))
ideal I=x-y,x-z;
ideal J=toric_std(I);
J;
⇒ J[1]=y-z
⇒ J[2]=x-z
// call with the same ideal, but badly chosen generators:
// 1) not only binomials
I=x-y,2x-y-z;
J=toric_std(I);
⇒      ? Generator 2 of the input ideal is no difference of monomials.
⇒      ? leaving toric.lib::toric_std (0)
// 2) binomials whose monomials are not relatively prime
I=x-y,xy-yz,y-z;
J=toric_std(I);
⇒ Warning: The monomials of generator 2 of the input ideal are not relative\
    ly prime.
J;
⇒ J[1]=y-z
⇒ J[2]=x-z
// call with a non-toric ideal that seems to be toric
I=x-yz,xy-z;
J=toric_std(I);
J;
⇒ J[1]=y2-1
⇒ J[2]=x-yz
// comparison with real standard basis and saturation
ideal H=std(I);
H;
⇒ H[1]=x-yz
⇒ H[2]=y2z-z
LIB "elim.lib";
sat_with_exp(H,xyz);
⇒ [1]:
⇒      _[1]=x-yz
⇒      _[2]=y2-1
⇒ [2]:
⇒      1

```

See also: [Section C.6.1 \[Toric ideals\]](#), page 782; [Section D.4.13 \[intprog.lib\]](#), page 1119; [Section D.4.35.1 \[toric\\_ideal\]](#), page 1388; [Section D.4.35 \[toric.lib\]](#), page 1388.

## D.5 Algebraic geometry

### D.5.1 brillnoether.lib

**Library:** brillnoether.lib

**Purpose:** Riemann-Roch spaces of divisors on curves

**Authors:** I. Stenger: stenger@mathematik.uni-kl.de  
 Janko Boehm boehm@mathematik.uni-kl.de

**Procedures:** See also: [Section D.10.1 \[brnoeth\\_lib\]](#), page 1922.

### D.5.1.1 RiemannRochBN

Procedure from library `brillnoether.lib` (see [Section D.5.1 \[brillnoether\\_lib\]](#), page 1390).

**Usage:** `RiemannRochBN(C,I,J)`; ideal C, ideal I, ideal J

**Assume:** C is a homogeneous ideal defining a projective curve. If C is a non-planar curve, then C is assumed to be nonsingular. This assumption is not checked.  
The ideals I and J represent a divisor D on C.

**Return:** A vector space basis of the Riemann-Roch space of D, stored in a list `RRBasis`. The list `RRBasis` contains a list `IH` and a form `F`. The vector space basis of  $L(D)$  consists of all rational functions  $G/F$ , where  $G$  is an element of `IH`.

**Example:**

```
LIB "brillnoether.lib";
ring R = 0,(x,y,z),dp;
poly f = y^2+x^2-1;
f = homog(f,z);
ideal C = f;
ideal P1 = x,y-z;
ideal P2 = x^2+y^2,z;
ideal I = intersect(P1^3,P2^2);
ideal P3 = x+z,y;
ideal J = P3^2;
RiemannRochBN(C,I,J);
⇒ [1]:
⇒ _[1]=65y2z2-81xz3-81z4
⇒ _[2]=65xyz2+63xz3+65yz3+63z4
⇒ _[3]=65y3z-81xyz2-81yz3
⇒ _[4]=65xy2z-81x2z2-81xz3
⇒ _[5]=65y4-81xy2z-81y2z2
⇒ _[6]=65xy3-81x2yz-81xyz2
⇒ [2]:
⇒ 8xyz2-y2z2-8xz3+2yz3-z4
```

### D.5.2 chern\_lib

**Library:** `chern.lib`

**Purpose:** Symbolic Computations with Chern classes, Computation of Chern classes

**Author:** Oleksandr Iena, o.g.yena@gmail.com

**Overview:** A toolbox for symbolic computations with Chern classes. The Aluffi's algorithms for computation of characteristic classes of algebraic varieties (Segre, Fulton, Chern-Schwartz-MacPherson classes) are implemented as well.

**References:**

- [1] Aluffi, Paolo Computing characteristic classes of projective schemes. *Journal of Symbolic Computation*, 35 (2003), 3-19. [2] Iena, Oleksandr, On symbolic computations with Chern classes: remarks on the library `chern.lib` for Singular,

<http://hdl.handle.net/10993/22395>, 2015.

[3] Lascoux, Alain, Classes de Chern d'un produit tensoriel. C. R. Acad. Sci., Paris, Ser. A 286, 385-387 (1978). [4] Manivel, Laurent Chern classes of tensor products, arXiv 1012.0014, 2010.

#### Procedures:

##### D.5.2.1 `symm`

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\]](#), page 1391).

**Usage:** `symm(l [n]);` `l` a list of polynomials, `n` integer

**Return:** list of polynomials

**Purpose:** computes the list of elementary symmetric functions in the entries of `l`

**Note:** makes sense only for a list of polynomials

#### Example:

```
LIB "chern.lib";
// elementary symmetric functions in x, y, z:
ring r = 0, (x, y, z), dp;
list l=(x, y, z);
print(symm(l));
⇒ [1]:
⇒ x+y+z
⇒ [2]:
⇒ xy+xz+yz
⇒ [3]:
⇒ xyz
//now let us compute only the first two symmetric polynomials in a(1), ... , a(10)
ring q= 0,(a(1..10)), dp;
list l=a(1..10);
print(symm(l, 2));
⇒ [1]:
⇒ a(1)+a(2)+a(3)+a(4)+a(5)+a(6)+a(7)+a(8)+a(9)+a(10)
⇒ [2]:
⇒ a(1)*a(2)+a(1)*a(3)+a(2)*a(3)+a(1)*a(4)+a(2)*a(4)+a(3)*a(4)+a(1)*a(5)+\
a(2)*a(5)+a(3)*a(5)+a(4)*a(5)+a(1)*a(6)+a(2)*a(6)+a(3)*a(6)+a(4)*a(6)+a(5)\
)*a(6)+a(1)*a(7)+a(2)*a(7)+a(3)*a(7)+a(4)*a(7)+a(5)*a(7)+a(6)*a(7)+a(1)*a(\
8)+a(2)*a(8)+a(3)*a(8)+a(4)*a(8)+a(5)*a(8)+a(6)*a(8)+a(7)*a(8)+a(1)*a(9)\
+a(2)*a(9)+a(3)*a(9)+a(4)*a(9)+a(5)*a(9)+a(6)*a(9)+a(7)*a(9)+a(8)*a(9)+a(\
1)*a(10)+a(2)*a(10)+a(3)*a(10)+a(4)*a(10)+a(5)*a(10)+a(6)*a(10)+a(7)*a(10\
)+a(8)*a(10)+a(9)*a(10)
```

##### D.5.2.2 `symNsym`

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\]](#), page 1391).

**Usage:** `symNsym(f, c);` `f` polynomial; `c` list of polynomials

**Return:** list with 2 poly entries

**Purpose:** computes a symmetric and a non-symmetric part of `f` in terms of the elementary symmetric functions from `c` as well a non-symmetric remainder

**Note:** constants are considered symmetric

**Example:**

```
LIB "chern.lib";
ring r=0, (x,y,z, c(1..3)), dp;
list l=c(1..3);
// The symmetric part of f = 3x2 + 3y2 + 3z2 + 7xyz + y
// in terms of the elementary symmetric functions c(1), c(2), c(3)
// and the remainder
poly f = 3x2 + 3y2 + 3z2 + 7xyz + y;
print( symNsym(f, l) );
⇒ [1]:
⇒ 3*c(1)^2-6*c(2)+7*c(3)
⇒ [2]:
⇒ y
// Take a symmetric polynomial in variables x and z
f=x2+xz+z2;
// Express it in terms of the elementary the symmetric functions
print( symNsym(f, l)[1] );
⇒ c(1)^2-c(2)
```

### D.5.2.3 CompleteHomog

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** CompleteHomog(N, c); N integer, c list of polynomials

**Return:** list of polynomials

**Purpose:** computes the list of the complete homogeneous symmetric polynomials in terms of the elementary symmetric polynomials (entries of c)

**Note:**

**Example:**

```
LIB "chern.lib";
ring r = 0, (x(1..3)), dp;
list l=x(1..3);
//Complete homogeneous symmetric polynomials up to degree 3 in variables x(1), x(2),
print( CompleteHomog(3, l) );
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ x(1)
⇒ [3]:
⇒ x(1)^2-x(2)
⇒ [4]:
⇒ x(1)^3-2*x(1)*x(2)+x(3)
```

### D.5.2.4 segre

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** segre(c[, N]); c list of polynomials, N integer

**Return:** list of polynomials

**Purpose:** computes the list of the Segre classes up to degree N in terms of the Chern classes from c

**Note:**

**Example:**

```
LIB "chern.lib";
ring r = 0, (c(1..3)), dp;
list l=c(1..3);
//Segre classes up to degree 5 in Chern classes c(1), c(2), c(3)
print( segre(l, 5) );
⇒ [1]:
⇒ -c(1)
⇒ [2]:
⇒ c(1)^2-c(2)
⇒ [3]:
⇒ -c(1)^3+2*c(1)*c(2)-c(3)
⇒ [4]:
⇒ c(1)^4-3*c(1)^2*c(2)+c(2)^2+2*c(1)*c(3)
⇒ [5]:
⇒ -c(1)^5+4*c(1)^3*c(2)-3*c(1)*c(2)^2-3*c(1)^2*c(3)+2*c(2)*c(3)
```

### D.5.2.5 chern

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\]](#), page 1391).

**Usage:** chern(s); s list of polynomials

**Return:** list of polynomials

**Purpose:** computes the list of the Chern classes up to degree N in terms of the Segre classes from s

**Note:**

**Example:**

```
LIB "chern.lib";
ring r = 0, (s(1..3)), dp;
list l=s(1..3);
// Chern classes in Segre classes s(1), s(2), s(3)
print( chern(l) );
⇒ [1]:
⇒ -s(1)
⇒ [2]:
⇒ s(1)^2-s(2)
⇒ [3]:
⇒ -s(1)^3+2*s(1)*s(2)-s(3)
// This procedure is inverse to segre(...). Indeed:
print( segre(chern(l), 3) );
⇒ [1]:
⇒ s(1)
⇒ [2]:
⇒ s(2)
⇒ [3]:
⇒ s(3)
```



### D.5.2.6 chNum

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `chNum(N, c);` N integer, c list

**Return:** list

**Purpose:** computes the Chern numbers of a vector bundle with Chern classes c on a complex manifold (variety) of dimension N,  
the zeroes corresponding to the higher zero Chern classes are ignored

**Note:** computes basically the partitions of N  
in summands not greater than the length of c

**Example:**

```
LIB "chern.lib";
ring r = 0, (c(1..2)), dp;
list l=c(1..2);
// Let c(1) be a variable of degree 1, let c(2) be a variable of degree 2.
// The monomials in c(1) and c(2) of weighted degree 5 are:
print( chNum( 5, l ) );
⇒ [1]:
⇒ c(1)^5
⇒ [2]:
⇒ c(1)^3*c(2)
⇒ [3]:
⇒ c(1)*c(2)^2
// Compare the result to the output of chNumbers(...):
print( chNumbers(5, l) );
⇒ [1]:
⇒ c(1)^5
⇒ [2]:
⇒ c(1)^3*c(2)
⇒ [3]:
⇒ c(1)*c(2)^2
⇒ [4]:
⇒ 0
⇒ [5]:
⇒ 0
⇒ [6]:
⇒ 0
⇒ [7]:
⇒ 0
```

### D.5.2.7 chNumbers

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `chNumbers(r, c);` r integer, c list

**Return:** list

**Purpose:** computes the Chern numbers of a vector bundle with Chern classes c on a complex manifold (variety) of dimension r

**Note:** computes basically the partitions of r

**Example:**

```

LIB "chern.lib";
ring r = 0, (c(1..3)), dp;
list l=c(1..3);
// The Chern numbers of a vector bundle with Chern classes c(1), c(2), c(3)
// on a 3-fold:
print( chNumbers( 3, l ) );
⇒ [1]:
⇒ c(1)^3
⇒ [2]:
⇒ c(1)*c(2)
⇒ [3]:
⇒ c(3)
// If the highest Chern class is zero, the Chern numbers are:
l=c(1..2);
print( chNumbers( 3, l ) );
⇒ [1]:
⇒ c(1)^3
⇒ [2]:
⇒ c(1)*c(2)
⇒ [3]:
⇒ 0
// Compare this to the output of chNum(...):
print( chNum( 3, l ) );
⇒ [1]:
⇒ c(1)^3
⇒ [2]:
⇒ c(1)*c(2)

```

**D.5.2.8 sum\_of\_powers**

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `sum_of_powers(k, l)`;  $k$  non-negative integer,  $l$  list of polynomials

**Return:** polynomial

**Purpose:** computes the sum of  $k$ -th powers of the entries of  $l$

**Note:** returns 0 if  $k$  is negative

**Example:**

```

LIB "chern.lib";
ring r = 0, (x, y, z), dp;
list l=x, y, z;
//sum of 7-th powers of x, y, z
print( sum_of_powers(7, l) );
⇒ x7+y7+z7

```

**D.5.2.9 powSumSym**

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `powSumSym(l [,N])`;  $l$  a list of polynomials,  $N$  integer

**Return:** list of polynomials

**Purpose:** computes the expressions for the sums of powers [up to degree N] in terms of the elementary symmetric polynomials (entries of l),

**Note:** returns the terms of the Chern character multiplied by the corresponding factorials

**Example:**

```
LIB "chern.lib";
// the expressions of the first 3 sums of powers of 3 variables a(1), a(2), a(3)
// in terms of the elementary symmetric polynomials c(1), c(2), c(3):
ring r = 0, (c(1..3)), dp;
list l=(c(1..3));
print(powSumSym(l));
⇒ [1]:
⇒ c(1)
⇒ [2]:
⇒ c(1)^2-2*c(2)
⇒ [3]:
⇒ c(1)^3-3*c(1)*c(2)+3*c(3)
// The first 5 sums in the same situation
print(powSumSym(l, 5));
⇒ [1]:
⇒ c(1)
⇒ [2]:
⇒ c(1)^2-2*c(2)
⇒ [3]:
⇒ c(1)^3-3*c(1)*c(2)+3*c(3)
⇒ [4]:
⇒ c(1)^4-4*c(1)^2*c(2)+2*c(2)^2+4*c(1)*c(3)
⇒ [5]:
⇒ c(1)^5-5*c(1)^3*c(2)+5*c(1)*c(2)^2+5*c(1)^2*c(3)-5*c(2)*c(3)
```

### D.5.2.10 chAll

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\]](#), page 1391).

**Usage:** `chAll(l [,N]);` l a list of polynomials, N integer

**Return:** list of polynomials

**Purpose:** computes the list of terms of positive degree [up to degree N] of the Chern character, where the entries of l are considered as the Chern classes

**Note:** makes sense only for a list of polynomials

**Example:**

```
LIB "chern.lib";
// Chern character (terms of degree 1, 2, 3)
// corresponding to the Chern classes c(1), c(2), c(3):
ring r = 0, (c(1..3)), dp;
list l=(c(1..3));
print(chAll(l));
⇒ [1]:
⇒ c(1)
⇒ [2]:
⇒ 1/2*c(1)^2-c(2)
```

```

⇒ [3]:
⇒ 1/6*c(1)^3-1/2*c(1)*c(2)+1/2*c(3)
// terms up to degree 5 in the same situation
print(chAll(1, 5));
⇒ [1]:
⇒ c(1)
⇒ [2]:
⇒ 1/2*c(1)^2-c(2)
⇒ [3]:
⇒ 1/6*c(1)^3-1/2*c(1)*c(2)+1/2*c(3)
⇒ [4]:
⇒ 1/24*c(1)^4-1/6*c(1)^2*c(2)+1/12*c(2)^2+1/6*c(1)*c(3)
⇒ [5]:
⇒ 1/120*c(1)^5-1/24*c(1)^3*c(2)+1/24*c(1)*c(2)^2+1/24*c(1)^2*c(3)-1/24*c(2)*c(3)

```

### D.5.2.11 chAllInv

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `chAllInv(l);` l a list of polynomials

**Return:** list of polynomials

**Purpose:** procedure inverse to `chAll()`, computes the list of Chern classes from the list of terms of positive degree of the Chern character

**Note:** makes sense only for a list of polynomials

**Example:**

```

LIB "chern.lib";
// first 3 Chern classes in terms of the first 3 terms
// of the Chern character Chern ch(1), ch(2), ch(3):
ring r = 0, (ch(1..3)), dp;
list l=(ch(1..3));
print(chAllInv(l));
⇒ [1]:
⇒ ch(1)
⇒ [2]:
⇒ 1/2*ch(1)^2-ch(2)
⇒ [3]:
⇒ 1/6*ch(1)^3-ch(1)*ch(2)+2*ch(3)
// let's see that chAllInv() is inverse to chAll()
print( chAll( chAllInv(l) ) );
⇒ [1]:
⇒ ch(1)
⇒ [2]:
⇒ ch(2)
⇒ [3]:
⇒ ch(3)

```

### D.5.2.12 chHE

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `chHE(c);` c list of polynomials

**Return:** polynomial

**Purpose:** computes the highest relevant term of the Chern character

**Note:** uses the elimination and is extremely inefficient,  
is included just for comparison with `chAll(c)`

**Example:**

```
LIB "chern.lib";
ring r = 0, (c(1..3)), dp;
list l=c(1..3);
//the third degree term of the Chern character
print( chHE(l) );
⇒ 1/6*c(1)^3-1/2*c(1)*c(2)+1/2*c(3)
```

### D.5.2.13 ChernRootsSum

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** ChernRootsSum(a, b); a, b lists of polynomials

**Return:** list of polynomials

**Purpose:** computes the Chern roots of the direct (Whitney) sum of a vector bundle with Chern roots a and a vector bundle with Chern roots b

**Note:**

**Example:**

```
LIB "chern.lib";
ring r = 0, (a(1..3), b(1..2)), dp;
// assume a(1), a(2), a(3) are the Chern roots of a vector bundle E
// assume b(1), b(2) are the Chern roots of a vector bundle F
list l=a(1..3);
list L=b(1..2);
// the Chern roots of their direct sum is
print( ChernRootsSum(l, L) );
⇒ [1]:
⇒ a(1)
⇒ [2]:
⇒ a(2)
⇒ [3]:
⇒ a(3)
⇒ [4]:
⇒ b(1)
⇒ [5]:
⇒ b(2)
```

### D.5.2.14 chSum

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** chSum(c, C); c, C lists of polynomials

**Return:** list of polynomials

**Purpose:** computes the Chern classes of a direct sum of two vector bundles

**Note:**

**Example:**

```
LIB "chern.lib";
ring r = 0, (c(1..3), C(1..2)), dp;
// Let E be a vector bundle with Chern classes c(1), c(2), c(3).
// Let F be a vector bundle with Chern classes C(1), C(2).
list l=c(1..3);
list L=C(1..2);
// Then the Chern classes of their direct sum are
print( chSum(l, L) );
⇒ [1]:
⇒ c(1)+C(1)
⇒ [2]:
⇒ c(1)*C(1)+c(2)+C(2)
⇒ [3]:
⇒ c(2)*C(1)+c(1)*C(2)+c(3)
⇒ [4]:
⇒ c(3)*C(1)+c(2)*C(2)
⇒ [5]:
⇒ c(3)*C(2)
```

### D.5.2.15 ChernRootsDual

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\]](#), page 1391).

**Usage:** ChernRootsDual(l); l a list of polynomials

**Return:** list of polynomials

**Purpose:** computes the Chern roots of the dual vector bundle  
of a vector bundle with Chern roots from l

**Note:**

**Example:**

```
LIB "chern.lib";
ring r = 0, (a(1..3)), dp;
// assume a(1), a(2), a(3) are the Chern roots of a vector bundle
list l=a(1..3);
// the Chern roots of the dual vector bundle
print( ChernRootsDual(l) );
⇒ [1]:
⇒ -a(1)
⇒ [2]:
⇒ -a(2)
⇒ [3]:
⇒ -a(3)
```

### D.5.2.16 chDual

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\]](#), page 1391).

**Usage:** chDual(c); c list of polynomials

**Return:** list of polynomials

**Purpose:** computes the list of Chern classes of the dual vector bundle

**Note:**

**Example:**

```
LIB "chern.lib";
// Chern classes of a vector bundle that is dual to a vector bundle
// with Chern classes c(1), c(2), c(3)
ring r=0, (c(1..3)), dp;
list l=c(1..3);
print(chDual(l));
⇒ [1]:
⇒ -c(1)
⇒ [2]:
⇒ c(2)
⇒ [3]:
⇒ -c(3)
```

### D.5.2.17 ChernRootsProd

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** ChernRootsProd(a, b); a, b lists of polynomials

**Return:** list of polynomials

**Purpose:** computes the Chern roots of the tensor product of a vector bundle with Chern roots a and a vector bundles with Chern roots b

**Note:**

**Example:**

```
LIB "chern.lib";
ring r=0, (a(1..2), b(1..3)), dp;
list l=a(1..2);
list L=b(1..3);
// Chern roots of the tensor product of a vector bundle with Chern roots a(1), a(2)
// and a vector bundle with Chern roots b(1), b(2), b(3)
print(ChernRootsProd(l, L));
⇒ [1]:
⇒ a(1)+b(1)
⇒ [2]:
⇒ a(1)+b(2)
⇒ [3]:
⇒ a(1)+b(3)
⇒ [4]:
⇒ a(2)+b(1)
⇒ [5]:
⇒ a(2)+b(2)
⇒ [6]:
⇒ a(2)+b(3)
```

### D.5.2.18 chProd

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** chProd(r, c, R, C [, N]); r, R polynomials (integers); c, C lists of polynomials, N integer

**Return:** list of polynomials

**Purpose:** computes the list of Chern classes of the product of two vector bundles in terms of their ranks and Chern classes [up to degree N]

**Note:**

**Example:**

```
LIB "chern.lib";
ring H = 0, ( r, R, c(1..3), C(1..2) ), dp;
list l=c(1..3);
list L=C(1..2);
// the Chern classes of the tensor product of a vector bundle E of rank 3
// with Chern classes c(1), c(2), c(3)
// and a vector bundle F of rank 2 with Chern classes C(1) and C(2):
print( chProd(3, l, 2, L) );
⇒ [1]:
⇒ 2*c(1)+3*C(1)
⇒ [2]:
⇒ c(1)^2+5*c(1)*C(1)+3*C(1)^2+2*c(2)+3*C(2)
⇒ [3]:
⇒ 2*c(1)^2*C(1)+4*c(1)*C(1)^2+C(1)^3+2*c(1)*c(2)+4*c(2)*C(1)+4*c(1)*C(2)\
+6*C(1)*C(2)+2*c(3)
⇒ [4]:
⇒ c(1)^2*C(1)^2+c(1)*C(1)^3+3*c(1)*c(2)*C(1)+3*c(2)*C(1)^2+2*c(1)^2*C(2)\
+6*c(1)*C(1)*C(2)+3*C(1)^2*C(2)+c(2)^2+2*c(1)*c(3)+3*c(3)*C(1)+3*C(2)^2
⇒ [5]:
⇒ c(1)*c(2)*C(1)^2+c(2)*C(1)^3+2*c(1)^2*C(1)*C(2)+2*c(1)*C(1)^2*C(2)+c(2)\
^2*C(1)+2*c(1)*c(3)*C(1)+3*c(3)*C(1)^2+2*c(1)*c(2)*C(2)+2*c(1)*C(2)^2+3*\
C(1)*C(2)^2+2*c(2)*c(3)-6*c(3)*C(2)
⇒ [6]:
⇒ c(1)*c(3)*C(1)^2+c(3)*C(1)^3+c(1)*c(2)*C(1)*C(2)+c(2)*C(1)^2*C(2)+c(1)\
^2*C(2)^2+c(1)*C(1)*C(2)^2+c(2)*c(3)*C(1)+c(2)^2*C(2)-2*c(1)*c(3)*C(2)-3*\
c(3)*C(1)*C(2)-2*c(2)*C(2)^2+C(2)^3+c(3)^2
// the first two Chern classes of the tensor product
// of a vector bundle E of rank r with Chern classes c(1) and c(2)
// and a vector bundle G of rank R with Chern classes C(1) and C(2)
// this gives the Chern classes of a tensor product on a complex surface
l=c(1..2);
L=C(1..2);
print( chProd(r, l, R, L, 2) );
⇒ [1]:
⇒ R*c(1)+r*C(1)
⇒ [2]:
⇒ 1/2*R^2*c(1)^2+r*R*c(1)*C(1)+1/2*r^2*C(1)^2-1/2*R*c(1)^2-1/2*r*C(1)^2+\
R*c(2)-c(1)*C(1)+r*C(2)
```

### D.5.2.19 chProdE

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\]](#), page 1391).

**Usage:** `chProdE(c, C)`; `c, C` lists of polynomials

**Return:** list of polynomials



**Purpose:** computes the list of Chern classes of the product of two vector bundles in terms of their Chern classes

**Note:** makes sense only for (lists of) polynomials;  
uses elimination, hence very inefficient;  
included only for comparison with `chProd(...)`

**Example:**

```
LIB "chern.lib";
ring H = 0, ( c(1..3), C(1..2) ), dp;
list l=c(1..3);
list L=C(1..2);
// the Chern classes of the tensor product of a vector bundle E of rank 3
// with Chern classes c(1), c(2), c(3)
// and a vector bundle F of rank 2 with Chern classes C(1) and C(2):
print( chProdE(l, L) );
⇒ [1]:
⇒ 2*c(1)+3*C(1)
⇒ [2]:
⇒ c(1)^2+5*c(1)*C(1)+3*C(1)^2+2*c(2)+3*C(2)
⇒ [3]:
⇒ 2*c(1)^2*C(1)+4*c(1)*C(1)^2+C(1)^3+2*c(1)*c(2)+4*c(2)*C(1)+4*c(1)*C(2)\
+6*C(1)*C(2)+2*c(3)
⇒ [4]:
⇒ c(1)^2*C(1)^2+c(1)*C(1)^3+3*c(1)*c(2)*C(1)+3*c(2)*C(1)^2+2*c(1)^2*C(2)\
+6*c(1)*C(1)*C(2)+3*C(1)^2*C(2)+c(2)^2+2*c(1)*c(3)+3*c(3)*C(1)+3*C(2)^2
⇒ [5]:
⇒ c(1)*c(2)*C(1)^2+c(2)*C(1)^3+2*c(1)^2*C(1)*C(2)+2*c(1)*C(1)^2*C(2)+c(2)\
)^2*C(1)+2*c(1)*c(3)*C(1)+3*c(3)*C(1)^2+2*c(1)*c(2)*C(2)+2*c(1)*C(2)^2+3*\
C(1)*C(2)^2+2*c(2)*c(3)-6*c(3)*C(2)
⇒ [6]:
⇒ c(1)*c(3)*C(1)^2+c(3)*C(1)^3+c(1)*c(2)*C(1)*C(2)+c(2)*C(1)^2*C(2)+c(1)\
^2*C(2)^2+c(1)*C(1)*C(2)^2+c(2)*c(3)*C(1)+c(2)^2*C(2)-2*c(1)*c(3)*C(2)-3*\
c(3)*C(1)*C(2)-2*c(2)*C(2)^2+C(2)^3+c(3)^2
```

### D.5.2.20 chProdL

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\]](#), page 1391).

**Usage:** `chProdL(r, c, R, C)`; `r, R` integers; `c, C` lists of polynomials

**Return:** list

**Purpose:** computes the list of Chern classes of the product of two vector bundles in terms of their Chern classes

**Note:** Implementation of the formula of Lascoux, the Schur polynomials are computed using the second Jacobi-Trudi formula (in terms of the Chern classes)

**Example:**

```
LIB "chern.lib";
// The Chern classes of the tensor product of a vector bundle of rank 3
// with Chern classes c(1), c(2), c(3) and a vector bundle of rank 1 with
// Chern class C(1)
ring r = 0, ( c(1..3), C(1) ), dp;
list c=c(1..3);
```

```

list C=C(1);
print( chProdL(3,c,1,C) );
↳ [1]:
↳ c(1)+3*C(1)
↳ [2]:
↳ 2*c(1)*C(1)+3*C(1)^2+c(2)
↳ [3]:
↳ c(1)*C(1)^2+C(1)^3+c(2)*C(1)+c(3)

```

### D.5.2.21 chProdLP

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `chProdLP(r, c, R, C)`;  $r, R$  integers;  $c, C$  lists of polynomials

**Return:** polynomial

**Purpose:** computes the total Chern class of the product of two vector bundles in terms of their ranks and Chern classes

**Note:** Implementation of the formula of Lascoux, the Schur polynomials are computed using the second Jacobi-Trudi formula (in terms of the Chern classes)

**Example:**

```

LIB "chern.lib";
// The total Chern class of the tensor product of a vector bundle of rank 3
// with Chern classes c(1), c(2), c(3) and a vector bundle of rank 1 with
// Chern class C(1)
ring r = 0, ( c(1..3), C(1)), ws(1,2,3, 1);
list c=c(1..3);
list C=C(1);
print( chProdLP(3,c,1,C) );
↳ 1+c(1)+3*C(1)+c(2)+2*c(1)*C(1)+3*C(1)^2+c(3)+c(2)*C(1)+c(1)*C(1)^2+C(1)^3

```

### D.5.2.22 chProdM

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `chProdM(r, c, R, C)`;  $r, R$  integers;  $c, C$  lists of polynomials

**Return:** list

**Purpose:** computes the list of Chern classes of the product of two vector bundles in terms of their Chern classes

**Note:** Implementation of the formula of Manivel

**Example:**

```

LIB "chern.lib";
// The Chern classes of the tensor product of a vector bundle of rank 3
// with Chern classes c(1), c(2), c(3) and a vector bundle of rank 1 with
// Chern class C(1)
ring r = 0, ( c(1..3), C(1)), dp;
list c=c(1..3);
list C=C(1);
print( chProdM(3,c,1,C) );
↳ [1]:

```

```

⇒      c(1)+3*C(1)
⇒ [2]:
⇒      2*c(1)*C(1)+3*C(1)^2+c(2)
⇒ [3]:
⇒      c(1)*C(1)^2+C(1)^3+c(2)*C(1)+c(3)

```

### D.5.2.23 chProdMP

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `chProdMP(r, c, R, C);`  $r, R$  integers;  $c, C$  lists of polynomials

**Return:** polynomial

**Purpose:** computes the total Chern class of the product of two vector bundles in terms of their ranks and Chern classes

**Note:** Implementation of the formula of Lascoux, the Schur polynomials are computed using the second Jacobi-Trudi formula (in terms of the Chern classes)

**Example:**

```

LIB "chern.lib";
// The total Chern class of the tensor product of a vector bundle of rank 3
// with Chern classes c(1), c(2), c(3) and a vector bundle of rank 1 with
// Chern class C(1)
ring r = 0, ( c(1..3), C(1)), ws(1,2,3, 1);
list c=c(1..3);
list C=C(1);
print( chProdMP(3,c,1,C) );
⇒ 1+c(1)+3*C(1)+c(2)+2*c(1)*C(1)+3*C(1)^2+c(3)+c(2)*C(1)+c(1)*C(1)^2+C(1)^3

```

### D.5.2.24 ChernRootsHom

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `ChernRootsHom(a, b);`  $a, b$  lists of polynomials

**Return:** list of polynomials

**Purpose:** for a vector bundle  $E$  with Chern roots  $a$  and a vector bundle  $F$  with Chern roots  $b$ , computes the Chern roots of  $\text{Hom}(E, F)$

**Note:**

**Example:**

```

LIB "chern.lib";
ring r=0, (a(1..2), b(1..3)), dp;
list l=a(1..2);
list L=b(1..3);
// Let E be a vector bundle with Chern roots a(1), a(2),
// let F be a vector bundle with Chern roots b(1), b(2), b(3).
// Then the Chern roots of Hom(E, F) are
print(ChernRootsHom(l, L));
⇒ [1]:
⇒      -a(1)+b(1)
⇒ [2]:
⇒      -a(1)+b(2)

```

```

⇒ [3]:
⇒      -a(1)+b(3)
⇒ [4]:
⇒      -a(2)+b(1)
⇒ [5]:
⇒      -a(2)+b(2)
⇒ [6]:
⇒      -a(2)+b(3)

```

### D.5.2.25 chHom

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `chHom(r, c, R, C [, N]);` `r, R` polynomials (integers); `c, C` lists of polynomials, `N` integer

**Return:** list of polynomials

**Purpose:** computes [up to degree `N`] the list of Chern classes of the vector bundle  $\text{Hom}(E, F)$  in terms of the ranks and the Chern classes of  $E$  and  $F$

**Note:**

**Example:**

```

LIB "chern.lib";
ring H = 0, ( r, R, c(1..3), C(1..2) ), dp;
list l=c(1..3);
list L=C(1..2);
// the Chern classes of Hom(E, F) for a vector bundle E of rank 3
// with Chern classes c(1), c(2), c(3)
// and a vector bundle F of rank 2 with Chern classes C(1) and C(2):
print( chHom(3, l, 2, L) );
⇒ [1]:
⇒      -2*c(1)+3*C(1)
⇒ [2]:
⇒      c(1)^2-5*c(1)*C(1)+3*C(1)^2+2*c(2)+3*C(2)
⇒ [3]:
⇒      2*c(1)^2*C(1)-4*c(1)*C(1)^2+C(1)^3-2*c(1)*c(2)+4*c(2)*C(1)-4*c(1)*C(2)\
+6*C(1)*C(2)-2*c(3)
⇒ [4]:
⇒      c(1)^2*C(1)^2-c(1)*C(1)^3-3*c(1)*c(2)*C(1)+3*c(2)*C(1)^2+2*c(1)^2*C(2)\
-6*c(1)*C(1)*C(2)+3*C(1)^2*C(2)+c(2)^2+2*c(1)*c(3)-3*c(3)*C(1)+3*C(2)^2
⇒ [5]:
⇒      -c(1)*c(2)*C(1)^2+c(2)*C(1)^3+2*c(1)^2*C(1)*C(2)-2*c(1)*C(1)^2*C(2)+c(\
2)^2*C(1)+2*c(1)*c(3)*C(1)-3*c(3)*C(1)^2-2*c(1)*c(2)*C(2)-2*c(1)*C(2)^2+3*\
*C(1)*C(2)^2-2*c(2)*c(3)+6*c(3)*C(2)
⇒ [6]:
⇒      c(1)*c(3)*C(1)^2-c(3)*C(1)^3-c(1)*c(2)*C(1)*C(2)+c(2)*C(1)^2*C(2)+c(1)\
^2*C(2)^2-c(1)*C(1)*C(2)^2-c(2)*c(3)*C(1)+c(2)^2*C(2)-2*c(1)*c(3)*C(2)+3*\
c(3)*C(1)*C(2)-2*c(2)*C(2)^2+C(2)^3+c(3)^2
// the first two Chern classes of Hom(E, F) for a vector bundle E of rank r
// with Chern classes c(1) and c(2)
// and a vector bundle G of rank R with Chern classes C(1) and C(2)
// this gives the Chern classes of a tensor product on a complex surface
l=c(1..2);
L=C(1..2);

```

```

print( chHom(r, l, R, L, 2 ) );
⇒ [1]:
⇒      -R*c(1)+r*C(1)
⇒ [2]:
⇒      1/2*R^2*c(1)^2-r*R*c(1)*C(1)+1/2*r^2*C(1)^2-1/2*R*c(1)^2-1/2*r*C(1)^2+\
      R*c(2)+c(1)*C(1)+r*C(2)

```

### D.5.2.26 ChernRootsSymm

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** ChernRootsSymm(m, l); m integer, l a list of polynomials

**Return:** list of polynomials

**Purpose:** computes the Chern roots of m-th symmetric power of a vector bundle with Chern roots from l

**Note:**

**Example:**

```

LIB "chern.lib";
ring r=0, (a(1..3)), dp;
list l=a(1..3);
// the Chern roots of the second symmetric power of a vector bundle
// with Chern roots a(1), a(2), a(3)
print( ChernRootsSymm(2, l) );
⇒ [1]:
⇒      2*a(1)
⇒ [2]:
⇒      a(1)+a(2)
⇒ [3]:
⇒      a(1)+a(3)
⇒ [4]:
⇒      2*a(2)
⇒ [5]:
⇒      a(2)+a(3)
⇒ [6]:
⇒      2*a(3)

```

### D.5.2.27 ChernRootsWedge

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** ChernRootsWedge(m, l); m integer, l a list of polynomials

**Return:** list of polynomials

**Purpose:** computes the Chern roots of m-th exterior power of a vector bundle with Chern roots from l

**Note:** makes sense only for list of polynomials

**Example:**

```

LIB "chern.lib";
ring r=0, (a(1..3)), dp;
list l=a(1..3);

```

```
// the Chern roots of the second exterior power of a vector bundle
// with Chern roots a(1), a(2), a(3)
print( ChernRootsWedge(2, 1) );
⇒ [1]:
⇒ a(2)+a(3)
⇒ [2]:
⇒ a(1)+a(3)
⇒ [3]:
⇒ a(1)+a(2)
```

### D.5.2.28 chSymm

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `chSymm(k, r, c[, pos]);` `k, r` integers, `c` list of polynomials, `pos` list of integers

**Return:** list with entries: `int N`, list of polynomials `l`

**Purpose:** computes the rank and the Chern classes of the symmetric power of a vector bundle

**Note:** for the second symmetric power `chSymm2L(...)` could be faster

**Example:**

```
LIB "chern.lib";
ring r=0, (c(1..5)), dp;
list l=c(1..5);
// the rank and the Chern classes of the second symmetric power of a vector bundle of rank 3
print( chSymm(2, 3, 1) );
⇒ [1]:
⇒ 6
⇒ [2]:
⇒ [1]:
⇒ 4*c(1)
⇒ [2]:
⇒ 5*c(1)^2+5*c(2)
⇒ [3]:
⇒ 2*c(1)^3+11*c(1)*c(2)+7*c(3)
⇒ [4]:
⇒ 6*c(1)^2*c(2)+4*c(2)^2+14*c(1)*c(3)
⇒ [5]:
⇒ 4*c(1)*c(2)^2+8*c(1)^2*c(3)+4*c(2)*c(3)
⇒ [6]:
⇒ 8*c(1)*c(2)*c(3)-8*c(3)^2
// the rank and the first 3 Chern classes
// of the second symmetric power of a vector bundle of rank 5
print( chSymm(2, 5, 1, 1, 2, 3) );
⇒ [1]:
⇒ 15
⇒ [2]:
⇒ [1]:
⇒ 6*c(1)
⇒ [2]:
⇒ 14*c(1)^2+7*c(2)
⇒ [3]:
⇒ 16*c(1)^3+31*c(1)*c(2)+9*c(3)
```

### D.5.2.29 chSymm2L

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `chSymm2L(r, c)`;  $r$  integer,  $c$  list of polynomials

**Return:** list of polynomials

**Purpose:** computes the Chern classes of the second symmetric power of a vector bundle

**Note:** Implementation of the formula of Lascoux, the Schur polynomials are computed using the second Jacobi-Trudi formula (in terms of the Chern classes)

**Example:**

```
LIB "chern.lib";
ring r=0, (c(1..2)), dp;
list l=c(1..2);
// the Chern classes of the second symmetric power of a vector bundle of rank 2
print( chSymm2L(2, l));
⇒ [1]:
⇒ 3
⇒ [2]:
⇒ [1]:
⇒ 3*c(1)
⇒ [2]:
⇒ 2*c(1)^2+4*c(2)
⇒ [3]:
⇒ 4*c(1)*c(2)
```

### D.5.2.30 chSymm2LP

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `chSymm2LP(r, c)`;  $r$  integer,  $c$  list of polynomials

**Return:** poly

**Purpose:** computes the total Chern class of the second symmetric power of a vector bundle

**Note:** Implementation of the formula of Lascoux, the Schur polynomials are computed using the second Jacobi-Trudi formula (in terms of the Chern classes)

**Example:**

```
LIB "chern.lib";
ring r=0, (c(1..2)), ws(1, 2);
list l=c(1..2);
// the total Chern class of the second symmetric power of a vector bundle of rank 2
print( chSymm2LP(2, l));
⇒ 1+3*c(1)+2*c(1)^2+4*c(2)+4*c(1)*c(2)
```

### D.5.2.31 chWedge

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `chWedge(k, r, c [,pos])`;  $k, r$  integers,  $c$  list of polynomials,  $pos$  list of integers

**Return:** list with entries: int  $N$ , list of polynomials  $l$

**Purpose:** computes the rank and the Chern classes of the exterior power of a vector bundle

**Note:** for the second exterior power `chWedge2L(...)` could be faster

**Example:**

```
LIB "chern.lib";
ring r=0, (c(1..5)), dp;
list l=c(1..5);
// the rank and the Chern classes of the second exterior power of a vector bundle of
print( chWedge(2, 3, 1) );
⇒ [1]:
⇒ 3
⇒ [2]:
⇒ [1]:
⇒ 2*c(1)
⇒ [2]:
⇒ c(1)^2+c(2)
⇒ [3]:
⇒ c(1)*c(2)-c(3)
// the rank and the first 3 Chern classes
// of the fourth exterior power of a vector bundle of rank 5
print( chWedge(4, 5, 1, 1, 2, 3) );
⇒ [1]:
⇒ 5
⇒ [2]:
⇒ [1]:
⇒ 4*c(1)
⇒ [2]:
⇒ 6*c(1)^2+c(2)
⇒ [3]:
⇒ 4*c(1)^3+3*c(1)*c(2)-c(3)
```

### D.5.2.32 chWedge2L

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\]](#), page 1391).

**Usage:** `chWedge2L(r, c)`; `r` integer, `c` list of polynomials

**Return:** list of polynomials

**Purpose:** computes the Chern classes of the second exterior power of a vector bundle

**Note:** Implementation of the formula of Lascoux, the Schur polynomials are computed using the second Jacobi-Trudi formula (in terms of the Chern classes)

**Example:**

```
LIB "chern.lib";
ring r=0, (c(1..3)), dp;
list l=c(1..3);
// the Chern classes of the second exterior power of a vector bundle of rank 3
print(chWedge2L(3, 1));
⇒ [1]:
⇒ 3
⇒ [2]:
⇒ [1]:
⇒ 2*c(1)
⇒ [2]:
⇒ c(1)^2+c(2)
```



```

⇒      [3]:
⇒      c(1)*c(2)-c(3)

```

### D.5.2.33 chWedge2LP

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `chWedge2LP(r, c );` `r` integer, `c` list of polynomials

**Return:** `poly`

**Purpose:** computes the total Chern class of the second exterior power of a vector bundle

**Note:** Implementation of the formula of Lascoux, the Schur polynomials are computed using the second Jacobi-Trudi formula (in terms of the Chern classes)

**Example:**

```

LIB "chern.lib";
ring r=0, (c(1..3)), ws(1,2,3);
list l=c(1..3);
// the total Chern class of the second exterior power of a vector bundle of rank 3
print(chWedge2LP(3, l));
⇒ 1+2*c(1)+c(1)^2+c(2)+c(1)*c(2)-c(3)

```

### D.5.2.34 todd

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `todd(l [, n ] );` `l` a list of polynomials, `n` integer

**Return:** list of polynomials

**Purpose:** computes [the first `n`] terms of the Todd class

**Note:** returns an empty list if `l` is empty

**Example:**

```

LIB "chern.lib";
// the terms of the Todd class up to degree 5
// in terms of the Chern classes c(1), c(2), c(3), c(4), c(5)
ring r=0, (c(1..5)), dp;
list l=c(1..5);
print( todd( l ) );
⇒ [1]:
⇒ 1/2*c(1)
⇒ [2]:
⇒ 1/12*c(1)^2+1/12*c(2)
⇒ [3]:
⇒ 1/24*c(1)*c(2)
⇒ [4]:
⇒ -1/720*c(1)^4+1/180*c(1)^2*c(2)+1/240*c(2)^2+1/720*c(1)*c(3)-1/720*c(4\
)
⇒ [5]:
⇒ -1/1440*c(1)^3*c(2)+1/480*c(1)*c(2)^2+1/1440*c(1)^2*c(3)-1/1440*c(1)*c\
(4)
// in the same situation compute only first two terms
print( todd(1, 2) );

```

```

⇒ [1]:
⇒ 1/2*c(1)
⇒ [2]:
⇒ 1/12*c(1)^2+1/12*c(2)
// compute the first 5 terms corresponding to the Chern classes c(1), c(2)
l=c(1..2);
print( todd(1, 5) );
⇒ [1]:
⇒ 1/2*c(1)
⇒ [2]:
⇒ 1/12*c(1)^2+1/12*c(2)
⇒ [3]:
⇒ 1/24*c(1)*c(2)
⇒ [4]:
⇒ -1/720*c(1)^4+1/180*c(1)^2*c(2)+1/240*c(2)^2
⇒ [5]:
⇒ -1/1440*c(1)^3*c(2)+1/480*c(1)*c(2)^2

```

### D.5.2.35 toddE

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `toddE(l)`; `l` a list of polynomials

**Return:** polynomial

**Purpose:** computes the highest relevant term of the Todd class

**Note:** returns an empty list if `l` is empty,  
very inefficient because the elimination is used, included for comparison with `todd(c)`

**Example:**

```

LIB "chern.lib";
// first 3 terms of the Todd class in terms of the Chern classes c(1), c(2), c(3)
ring r=0, (c(1..3)), dp;
list l;
//first term
l=c(1);
print( toddE( l ) );
⇒ 1/2*c(1)
// second term
l=c(1..2);
print( toddE( l ) );
⇒ 1/12*c(1)^2+1/12*c(2)
// third term
l=c(1..3);
print( toddE( l ) );
⇒ 1/24*c(1)*c(2)

```

### D.5.2.36 Bern

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `Bern(n)`; `n` non-negative integer

**Return:** list of numbers

**Purpose:** computes the list of (second) Bernoulli numbers from  $B(0)$  to  $B(n)$

**Note:** needs a base ring to be defined, returns an empty list if  $n$  is negative, uses the Akiyama-Tanigawa algorithm

**Example:**

```
LIB "chern.lib";
// first 10 Bernoulli numbers: B(0), ..., B(9)
ring r=0,(t), dp;
print( Bern(9) );
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 1/2
⇒ [3]:
⇒ 1/6
⇒ [4]:
⇒ 0
⇒ [5]:
⇒ -1/30
⇒ [6]:
⇒ 0
⇒ [7]:
⇒ 1/42
⇒ [8]:
⇒ 0
⇒ [9]:
⇒ -1/30
⇒ [10]:
⇒ 0
```

### D.5.2.37 tdCf

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\]](#), page 1391).

**Usage:** `tdCf(n)`;  $n$  integer

**Return:** list of rational numbers

**Purpose:** computes up to degree  $n$  the coefficients of the Todd class of a line bundle

**Note:**

**Example:**

```
LIB "chern.lib";
// first 5 coefficients
ring r=0,(t), dp;
print( tdCf(4) );
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 1/2
⇒ [3]:
⇒ 1/12
⇒ [4]:
⇒ 0
```

```

⇒ [5]:
⇒ -1/720

```

### D.5.2.38 tdTerms

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `tdTerms(n, f)`;  $n$  integer,  $f$  polynomial

**Return:** list of polynomials

**Purpose:** computes the terms of the Todd class of the line bundle with the Chern root  $f$

**Note:**

**Example:**

```

LIB "chern.lib";
ring r=0, (t), ls;;
// the terms of the Todd class of a line bundle with Chern root t up to degree 4
print( tdTerms(4, t) );
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 1/2t
⇒ [3]:
⇒ 1/12t2
⇒ [4]:
⇒ 0
⇒ [5]:
⇒ -1/720t4

```

### D.5.2.39 tdFactor

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `tdFactor(n, a)`;  $n$  integer,  $a$  polynomial

**Return:** polynomial

**Purpose:** computes up to degree  $n$  the Todd class of the line bundle corresponding to the Chern root  $t$

**Note:** returns 0 if  $n$  is negative

**Example:**

```

LIB "chern.lib";
// the Todd class up to degree 4
ring r=0,(t), ls;
print( tdFactor(4, t) );
⇒ 1+1/2t+1/12t2-1/720t4

```

### D.5.2.40 cProj

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `cProj(n)`;  $n$  integer

**Return:** list of integers

**Purpose:** computes the terms of positive degree of the total Chern class of the tangent bundle on the complex projective space

**Note:**

**Example:**

```
LIB "chern.lib";
ring r=0, (t), dp;
// the coefficients of the total Chern class of the complex projective line
print( cProj(1) );
⇒ [1]:
⇒ 2
// the coefficients of the total Chern class of the complex projective plane
print( cProj(2) );
⇒ [1]:
⇒ 3
⇒ [2]:
⇒ 3
// the coefficients of the total Chern class of the complex projective space
// of dimension three
print( cProj(3) );
⇒ [1]:
⇒ 4
⇒ [2]:
⇒ 6
⇒ [3]:
⇒ 4
```

#### D.5.2.41 chProj

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\]](#), page 1391).

**Usage:** `chProj(n)`;  $n$  integer

**Return:** list of (rational) numbers

**Purpose:** computes the terms of the Chern character of the tangent bundle on the complex projective space

**Note:**

**Example:**

```
LIB "chern.lib";
ring r=0, (t), dp;
// the coefficients of the Chern character of the complex projective line
print( chProj(1) );
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 2
// the coefficients of the Chern character of the complex projective plane
print( chProj(2) );
⇒ [1]:
⇒ 2
⇒ [2]:
⇒ 3
```

```

⇒ [3]:
⇒ 3/2
// the coefficients of the Chern character of the complex 3-dimensional projective space
print( chProj(3) );
⇒ [1]:
⇒ 3
⇒ [2]:
⇒ 4
⇒ [3]:
⇒ 2
⇒ [4]:
⇒ 2/3

```

### D.5.2.42 tdProj

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\]](#), page 1391).

**Usage:** `tdProj(n)`;  $n$  integer

**Return:** list of (rational) numbers

**Purpose:** computes the terms of the Todd class  
of the (tangent bundle of the) complex projective space

**Note:**

**Example:**

```

LIB "chern.lib";
ring r=0, (t), dp;
// the coefficients of the Todd class of the complex projective line
print( tdProj(1) );
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 1
// the coefficients of the Todd class of the complex projective line
print( tdProj(2) );
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 3/2
⇒ [3]:
⇒ 1
// the coefficients of the Todd class of the complex projective line
print( tdProj(3) );
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 2
⇒ [3]:
⇒ 11/6
⇒ [4]:
⇒ 1

```

### D.5.2.43 eulerChProj

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `eulerChProj(n, r, c);`  $n$  integer,  $r$  polynomial (or integer),  $c$  list of polynomials

**Return:** polynomial

**Purpose:** computes the Euler characteristic of a vector bundle on  $P_n$  in terms of its rank and Chern classes

**Note:**

**Example:**

```
LIB "chern.lib";
ring h=0, (r, c(1..3)), ws(0,1,2,3);
list l=c(1..3);
// the Euler characteristic of a vector bundle on the projective line
print( eulerChProj(1, r, l) );
⇒ r+c(1)
// the Euler characteristic of a vector bundle on the projective plane
print( eulerChProj(2, r, l) );
⇒ r+3/2*c(1)+1/2*c(1)^2-c(2)
// the Euler characteristic of a vector bundle on P_3
print( eulerChProj(3, r, l) );
⇒ r+11/6*c(1)+c(1)^2-2*c(2)+1/6*c(1)^3-1/2*c(1)*c(2)+1/2*c(3)
// assume now that we have a bundle framed at a subplane of P_3
// this implies c(1)=c(2)=0
l= 0, 0, c(3);
// the Euler characteristic is
print( eulerChProj(3, r, l) );
⇒ r+1/2*c(3)
// which implies that c(3) must be even in this case
```

### D.5.2.44 chNumbersProj

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `chNumbersProj(n);`  $n$  integer

**Return:** list of integers

**Purpose:** computes the Chern numbers of the projective space  $P_n$

**Note:**

**Example:**

```
LIB "chern.lib";
ring h=0, (t), dp;
// The Chern numbers of the projective plane P_2:
print( chNumbersProj(2) );
⇒ [1]:
⇒ 9
⇒ [2]:
⇒ 3
// The Chern numbers of P_3:
print( chNumbersProj(3) );
⇒ [1]:
```

```

↦      64
↦ [2] :
↦      24
↦ [3] :
↦      4

```

#### D.5.2.45 classpoly

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `classpoly(l, t)`; `l` list of polynomials, `t` polynomial

**Return:** polynomial

**Purpose:** computes the polynomial in `t` with coefficients being the entries of `l`

**Note:**

**Example:**

```

LIB "chern.lib";
ring r=0, (c(1..5), t), ds;
list l=c(1..5);
// get the polynomial c(1)*t + c(2)*t^2 + ... + c(5)*t^5
print( classpoly(l, t) );
↦ c(1)*t+c(2)*t^2+c(3)*t^3+c(4)*t^4+c(5)*t^5

```

#### D.5.2.46 chernPoly

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `chernPoly(c, t)`; `c` list of polynomials, `t` polynomial

**Return:** polynomial

**Purpose:** computes the Chern polynomial in `t`

**Note:** does the same as `toddPoly(...)`

**Example:**

```

LIB "chern.lib";
ring r=0, (c(1..5), t), ds;
list l=c(1..5);
// get the Chern polynomial 1 + c(1)*t + c(2)*t^2 + ... + c(5)*t^5
print( chernPoly(l, t) );
↦ 1+c(1)*t+c(2)*t^2+c(3)*t^3+c(4)*t^4+c(5)*t^5

```

#### D.5.2.47 chernCharPoly

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `chernCharPoly(r, ch, t)`; `r` polynomial, `ch` list of polynomials, `t` polynomial

**Return:** polynomial

**Purpose:** computes the polynomial in `t` corresponding to the Chern character

**Note:**

**Example:**



```

LIB "chern.lib";
ring h=0, (r, ch(1..5), t), ds;
list l=ch(1..5);
// get the polynomial r + ch(1)*t + ch(2)*t^2 + ... + ch(5)*t^5
print( chernCharPoly(r, l, t) );
↪ r+ch(1)*t+ch(2)*t^2+ch(3)*t^3+ch(4)*t^4+ch(5)*t^5

```

### D.5.2.48 toddPoly

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `toddPoly(td, t)`; `td` list of polynomials, `t` polynomial

**Return:** polynomial

**Purpose:** computes the polynomial in `t` corresponding to the Todd class

**Note:** does the same as `chernPoly(...)`

**Example:**

```

LIB "chern.lib";
ring r=0, (td(1..5), c(1..5), t), ds;
list l=td(1..5);
// get the polynomial 1 + td(1)*t + td(2)*t^2 + ... + td(5)*t^5
print( toddPoly(l, t) );
↪ 1+td(1)*t+td(2)*t^2+td(3)*t^3+td(4)*t^4+td(5)*t^5

```

### D.5.2.49 rHRR

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `rHRR( N, ch, td)`; `N` integer, `ch`, `td` lists of polynomials

**Return:** polynomial

**Purpose:** computes the the main ingredient of the right-hand side of the Hirzebruch-Riemann-Roch formula

**Note:** in order to get the right-hand side of the HRR formula  
one needs to be able to compute the degree of the output of this procedure

**Example:**

```

LIB "chern.lib";
ring r=0, (td(0..3), ch(0..3)), dp;
// Let ch(0), ch(1), ch(2), ch(3) be the terms of the Chern character
// of a vector bundle E on a 3-fold X.
list c = ch(0..3);
// Let td(0), td(1), td(2), td(3) be the terms of the Todd class of X.
list t = td(0..3);
// Then the highest term of the product ch(E).td(X) is:
print( rHRR(3, c, t) );
↪ td(3)*ch(0)+td(2)*ch(1)+td(1)*ch(2)+td(0)*ch(3)

```

### D.5.2.50 SchurS

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** `SchurS(I, S)`; `I` list of integers representing a partition, `S` list of polynomials

**Return:** poly

**Purpose:** computes the Schur polynomial in the Segre classes  $S$  (of the dual vector bundle), i.e., in the complete homogeneous symmetric polynomials, with respect to the partition  $I$

**Note:** if  $S$  are the Segre classes of the tautological bundle on a grassmanian, this gives the cohomology class of a Schubert cycle

**Example:**

```
LIB "chern.lib";
// The Schur polynomial corresponding to the partition 1,2,4
// and the Segre classes 1, s(1), s(2),..., s(6)
ring r=0,(s(1..6)), dp;
list I=1,2,4;
list S=s(1..6);
print( SchurS(I, S) );
↪ s(1)*s(2)*s(4)-s(1)^2*s(5)-s(3)*s(4)+s(1)*s(6)
// compare this with the Schur polynomial computed using Chern classes
list C=chDual(chern(S));
print( SchurCh(I, C) );
↪ s(1)*s(2)*s(4)-s(1)^2*s(5)-s(3)*s(4)+s(1)*s(6)
```

#### D.5.2.51 SchurCh

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** SchurCh(I, C); I list of integers representing a partition, C list of polynomials

**Return:** poly

**Purpose:** computes the Schur polynomial in the Chern classes  $C$ , i.e., in the elementary symmetric polynomials, with respect to the partition  $I$

**Note:** if  $C$  are the Chern classes of the tautological bundle on a grassmanian, this gives the cohomology class of a Schubert cycle

**Example:**

```
LIB "chern.lib";
// The Schur polynomial corresponding to the partition 1,2,4
// and the Chern classes c(1), c(2), c(3)
ring r=0,(c(1..3)), dp;
list I=1,2,4;
list C=c(1..3);
print( SchurCh(I, C) );
↪ c(1)^2*c(2)*c(3)-c(2)^2*c(3)-c(1)*c(3)^2
// Compare this with the Schur polynomial computed using Segre classes
list S=segre( chDual( list(c(1..3)) ), 6 );
print(SchurS(I,S));
↪ c(1)^2*c(2)*c(3)-c(2)^2*c(3)-c(1)*c(3)^2
```

#### D.5.2.52 part

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** part( m, n ); m positive integer, n non-negative integer

**Return:** list of lists

**Purpose:** computes all partitions of integers not exceeding  $n$  into  $m$  non-negative summands

**Note:** if  $n$  is negative or  $m$  is non-positive, the list with one empty entry is returned

**Example:**

```
LIB "chern.lib";
// partitions into 3 summands of numbers not exceeding 1
print( part(3, 1) );
⇒ [1]:
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ 0
⇒ [2]:
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ 1
⇒ [3]:
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ 1
⇒ [4]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ 1
```

### D.5.2.53 dualPart

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\]](#), page 1391).

**Usage:** `dualPart( I [,N] );`  $I$  list of integers,  $N$  integer

**Return:** list of integers

**Purpose:** computes the partition dual (conjugate) to  $I$

**Note:** the result is extended by zeroes to length  $N$  if an optional integer parameter  $N$  is given and the length of the computed dual partition is smaller than  $N$

**Example:**

```
LIB "chern.lib";
// dual partition to (1, 3, 4):
list I = 1, 3, 4;
print( dualPart(I) );
⇒ [1]:
```

```

⇒      1
⇒ [2]:
⇒      2
⇒ [3]:
⇒      2
⇒ [4]:
⇒      3

```

#### D.5.2.54 PartC

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:**     `PartC( I, m);` I list of integers, m integer

**Return:**    list of integers

**Purpose:**     computes the complement of a partition with respect to m

**Note:**       returns the zero partition if the maximal element of the partition is smaller than m

**Example:**

```

LIB "chern.lib";
// Complement of the partition (1, 3, 4) with respect to 5
list I = 1, 3, 4;
print( PartC(I, 5) );
⇒ [1]:
⇒      1
⇒ [2]:
⇒      2
⇒ [3]:
⇒      4

```

#### D.5.2.55 partOver

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:**     `partOver( n, J);` n integer, J list of integers (partition)

**Return:**    list of lists

**Purpose:**     computes the partitions over a given one with summands not exceeding n

**Note:**

**Example:**

```

LIB "chern.lib";
// Partitions over the partition (3, 3, 4) with summands not exceeding 4
list I = 3, 3, 4;
print( partOver(4, I) );
⇒ [1]:
⇒      [1]:
⇒          3
⇒      [2]:
⇒          3
⇒      [3]:
⇒          4
⇒ [2]:

```

```

⇒      [1]:
⇒      3
⇒      [2]:
⇒      4
⇒      [3]:
⇒      4
⇒ [3]:
⇒      [1]:
⇒      4
⇒      [2]:
⇒      4
⇒      [3]:
⇒      4

```

### D.5.2.56 partUnder

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:**      `partUnder(J)`; J list of integers (partition)

**Return:**     list of lists

**Purpose:**     computes the partitions under a given one

**Note:**

**Example:**

```

LIB "chern.lib";
// Partitions under the partition (0, 1, 1)
list I = 0, 1, 1;
print( partUnder(I) );
⇒ [1]:
⇒      [1]:
⇒      0
⇒      [2]:
⇒      0
⇒      [3]:
⇒      0
⇒ [2]:
⇒      [1]:
⇒      0
⇒      [2]:
⇒      0
⇒      [3]:
⇒      1
⇒ [3]:
⇒      [1]:
⇒      0
⇒      [2]:
⇒      1
⇒      [3]:
⇒      1

```

### D.5.2.57 SegreA

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** SegreA(I); I an ideal

**Return:** list of integers

**Purpose:** computes the Segre classes of the subscheme defined by I

**Note:**

**Example:**

```
LIB "chern.lib";
// Consider a 3-dimensional projective space
ring r = 0, (x, y, z, w), dp;
// Consider 3 non-coplanar lines through one point and compute the Segre class
ideal I=xy, xz, yz;
I;
⇒ I[1]=xy
⇒ I[2]=xz
⇒ I[3]=yz
SegreA(I);
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ 3
⇒ [4]:
⇒ -10
// Now consider 3 coplanar lines through one point and its Segre class
ideal J=w, x*y*(x+y);
J;
⇒ J[1]=w
⇒ J[2]=x2y+xy2
SegreA(J);
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ 3
⇒ [4]:
⇒ -12
```

#### D.5.2.58 FultonA

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\]](#), page 1391).

**Usage:** FultonA(I); I an ideal

**Return:** list of integers

**Purpose:** computes the Fulton classes of the subscheme defined by I

**Note:**

**Example:**

```
LIB "chern.lib";
// Consider a 3-dimensional projective space
```

```

ring r = 0, (x, y, z, w), dp;
// Consider 3 non-coplanar lines trough one point and compute the Fulton class
ideal I=xy, xz, yz;
I;
↪ I[1]=xy
↪ I[2]=xz
↪ I[3]=yz
FultonA(I);
↪ [1]:
↪ 0
↪ [2]:
↪ 0
↪ [3]:
↪ 3
↪ [4]:
↪ 2
// Now consider 3 coplanar lines trough one point and its Fulton class
ideal J=w, x*y*(x+y);
J;
↪ J[1]=w
↪ J[2]=x2y+xy2
FultonA(J);
↪ [1]:
↪ 0
↪ [2]:
↪ 0
↪ [3]:
↪ 3
↪ [4]:
↪ 0

```

### D.5.2.59 CSMA

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\]](#), page 1391).

**Usage:** CSMA(I); I an ideal

**Return:** list of integers

**Purpose:** computes the Chern-Schwartz-MacPherson classes of the variety defined by I

**Note:**

**Example:**

```

LIB "chern.lib";
// consider the projective plane with homogeneous coordinates x, y, z
ring r = 0, (x, y, z), dp;
// the Chern-Schwartz-MacPherson class of a smooth cubic:
ideal I=x3+y3+z3;
I;
↪ I[1]=x3+y3+z3
CSMA(I);
↪ [1]:
↪ 0
↪ [2]:

```

```

⇒      3
⇒ [3]:
⇒      0
// the Chern-Schwartz-MacPherson class of singular cubic
// that is a union of 3 non-collinear lines:
ideal J=x*y*z;
J;
⇒ J[1]=xyz
CSMA(J);
⇒ [1]:
⇒      0
⇒ [2]:
⇒      3
⇒ [3]:
⇒      3
// the Chern-Schwartz-MacPherson class of singular cubic
// that is a union of 3 lines passing through one point
ideal K=x*y*(x+y);
K;
⇒ K[1]=x2y+xy2
CSMA(K);
⇒ [1]:
⇒      0
⇒ [2]:
⇒      3
⇒ [3]:
⇒      4

```

#### D.5.2.60 EulerAff

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** EulerAff(I); I an ideal

**Return:** integer

**Purpose:** computes the Euler characteristic of the affine variety defined by I

**Note:**

**Example:**

```

LIB "chern.lib";
ring r = 0, (x, y), dp;
// compute the Euler characteristic of the affine elliptic curve y^2=x^3+x+1;
ideal I=y2-x3-x-1;
EulerAff(I);
⇒ -1

```

#### D.5.2.61 EulerProj

Procedure from library `chern.lib` (see [Section D.5.2 \[chern.lib\], page 1391](#)).

**Usage:** EulerProj(I); I an ideal

**Return:** integer



**Purpose:** computes the highest degree term of the Chern-Schwartz-MacPherson class of the variety defined by  $I$ , which equals the Euler characteristic

**Note:** uses CSMA(...)

**Example:**

```
LIB "chern.lib";
// consider the projective plane with homogeneous coordinates x, y, z
ring r = 0, (x, y, z), dp;
// Euler characteristic of a smooth cubic:
ideal I=x3+y3+z3;
I;
⇨ I[1]=x3+y3+z3
EulerProj(I);
⇨ 0
// Euler characteristic of 3 non-collinear lines:
ideal J=x*y*z;
J;
⇨ J[1]=xyz
EulerProj(J);
⇨ 3
// Euler characteristic of 3 lines passing through one point
ideal K=x*y*(x+y);
K;
⇨ K[1]=x2y+xy2
EulerProj(K);
⇨ 4
```

### D.5.3 deRham\_lib

**Library:** deRham.lib

**Purpose:** Computation of deRham cohomology

**Authors:** Cornelia Rottner, rottner@mathematik.uni-kl.de

**Overview:** A library for computing the de Rham cohomology of complements of complex affine varieties.

**References:**

- [OT] Oaku, T.; Takayama, N.: Algorithms of D-modules - restriction, tensor product, localization, and local cohomology groups}, J. Pure Appl. Algebra 156, 267-308 (2001)
- [R] Rottner, C.: Computing de Rham Cohomology, diploma thesis (2012)
- [W1] Walther, U.: Algorithmic computation of local cohomology modules and the local cohomological dimension of algebraic varieties}, J. Pure Appl. Algebra 139, 303-321 (1999)
- [W2] Walther, U.: Algorithmic computation of de Rham Cohomology of Complements of Complex Affine Varieties}, J. Symbolic Computation 29, 796-839 (2000)
- [W3] Walther, U.: Computing the cup product structure for complements of complex affine varieties, J. Pure Appl. Algebra 164, 247-273 (2001)

**Procedures:**

#### D.5.3.1 deRhamCohomology

Procedure from library `deRham.lib` (see [Section D.5.3 \[deRham\\_lib\]](#), page 1427).

**Usage:** `deRhamCohomology(L[,choices]);` L a list consisting of polynomials, choices optional list consisting of one up to three strings  
 The optional strings may be one of the strings  
 -'noCE': compute quasi-isomorphic complexes without using Cartan-Eilenberg resolutions  
 -'Vdres': compute quasi-isomorphic complexes using Cartan-Eilenberg resolutions; the CE resolutions are computed via V\_d-homogenization and without using Schreyer's method  
 -'Sres': compute quasi-isomorphic complexes using Cartan-Eilenberg resolutions in the homogenized Weyl algebra via Schreyer's method  
 one of the strings  
 -'iterativeloc': compute localizations by factorizing the polynomials and successive localization of the factors  
 -'no iterativeloc': compute localizations by directly localizing the product and one of the strings  
 -'onlybounds': computes bounds for the minimal and maximal integer roots of the global b-function  
 -'exactroots' computes the minimal and maximal integer root of the global b-function  
 The default is 'noCE', 'iterativeloc' and 'onlybounds'.

**Assume:** -The basering must be a polynomial ring over the field of rational numbers

**Return:** list, where the ith entry is the (i-1)st de Rham cohomology group of the complement of the complex affine variety given by the polynomials in L

**Example:**

```
LIB "deRham.lib";
ring r = 0,(x,y,z),dp;
list L=(xy,xz);
deRhamCohomology(L);
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ 0
⇒ [4]:
⇒ 1
⇒ [5]:
⇒ 1
```

### D.5.3.2 MVComplex

Procedure from library `deRham.lib` (see [Section D.5.3 \[deRham.lib\]](#), page 1427).

**Usage:** `MVComplex(L);` L a list of polynomials

**Assume:** -Basing is a polynomial ring with n vvariables and rational coefficients -L is a list of non-constant polynomials

**Return:** ring W: the nth Weyl algebra  
 W contains a list MV, which represents the Mayer-Vietoris complex ( $C^i, d^i$ ) of the polynomials contained in L as follows:

the  $C^i$  are given by  $D_n \cdot \text{ncols}(C[2^*i-1]) / \text{im}(C[2^*i-1])$  and the differentials  $d^i$  are given by  $C[2^*i]$

**Example:**

```
LIB "deRham.lib";
ring r = 0,(x,y,z),dp;
list L=xy,xz;
def C=MVComplex(L);
setring C;
MV;
⇨ [1]:
⇨   _[1,1]=D(3)
⇨   _[1,2]=0
⇨   _[2,1]=x(1)*D(1)+1
⇨   _[2,2]=0
⇨   _[3,1]=-x(2)*D(2)-1
⇨   _[3,2]=0
⇨   _[4,1]=0
⇨   _[4,2]=D(2)
⇨   _[5,1]=0
⇨   _[5,2]=x(1)*D(1)+1
⇨   _[6,1]=0
⇨   _[6,2]=-x(3)*D(3)-1
⇨ [2]:
⇨   _[1,1]=-x(1)*x(3)
⇨   _[2,1]=x(1)*x(2)
⇨ [3]:
⇨   _[1,1]=x(2)*D(2)+1
⇨   _[2,1]=x(1)*D(1)+2
⇨   _[3,1]=-x(3)*D(3)-1
⇨ [4]:
⇨   _[1,1]=0
```

### D.5.4 divisors.lib

**Library:** divisors.lib

**Purpose:** Divisors and P-Divisors

**Authors:** Janko Boehm boehm@mathematik.uni-kl.de  
 Lars Kastner kastner@math.fu-berlin.de  
 Benjamin Lorenz blorenz@math.uni-frankfurt.de  
 Hans Schoenemann hannes@mathematik.uni-kl.de  
 Yue Ren ren@mathematik.uni-kl.de

**Overview:** We implement a class divisor on an algebraic variety and methods for computing with them. Divisors are represented by tuples of ideals defining the positive and the negative part. In particular, we implement the group structure on divisors, computing global sections and testing linear equivalence.

In addition to this we provide a class formaldivisor which implements integer formal sums of divisors (not necessarily prime). A formal divisor can be evaluated to a divisor, and a divisor can be decomposed into a formal sum.

Finally we provide a class pdivisor which implements polyhedral formal sums of divisors (P-divisors) where the coefficients are assumed to be polyhedra with fixed tail cone.

There is a function to evaluate a P-divisor on a vector in the dual of the tail cone. The result will be a formal divisor.

**References:**

For the class divisor we closely follow Macaulay2's tutorial on divisors.

**Procedures:**

### D.5.4.1 makeDivisor

Procedure from library `divisors.lib` (see [Section D.5.4 \[divisors.lib\], page 1429](#)).

**Usage:**      `makeDivisor(I ,J);` I = ideal, J = ideal.

**Assume:**    I and J are ideals in a qring Q of a smooth irreducible variety X such that any ideal in Q satisfies the S2 condition.

**Return:**     a divisor on X

**Theory:**     The procedure will eliminate all components which are not of codimension 1. The S2 condition requires that every proper nonzero principal ideal has pure codimension 1.

**Example:**

```
LIB "divisors.lib";
ring r=31991,(x,y,z),dp;
ideal I = y^2*z - x*(x-z)*(x+3*z);
qring Q = std(I);
divisor P = makeDivisor(ideal(x,z),ideal(1));
```

### D.5.4.2 divisorplus

Procedure from library `divisors.lib` (see [Section D.5.4 \[divisors.lib\], page 1429](#)).

**Usage:**      `divisorplus(A ,B);` A + B; A = divisor, B = divisor.

**Assume:**    A and B are divisors on X.

**Return:**     a divisor on X

**Theory:**     The procedure will compute the product of the numerator and denominator ideals, respectively.

**Example:**

```
LIB "divisors.lib";
ring r=31991,(x,y,z),dp;
ideal I = y^2*z - x*(x-z)*(x+3*z);
qring Q = std(I);
divisor A = makeDivisor(ideal(x,z),ideal(1));
divisor B = makeDivisor(ideal(x,y),ideal(1));
A+B;
 $\mapsto (x,yz) - (1)$ 
 $\mapsto$ 
```

### D.5.4.3 multdivisor

Procedure from library `divisors.lib` (see [Section D.5.4 \[divisors.lib\]](#), page 1429).

**Usage:** `multdivisor(n,A);`  $A^n$ ;  $n = \text{integer}$ ,  $A = \text{divisor}$ .

**Assume:**  $n$  is an integer and  $A$  is a divisor on  $X$ .

**Return:** a divisor on  $X$

**Theory:** The procedure will compute the  $n$ -th power of the numerator and denominator ideals, respectively.

**Example:**

```
LIB "divisors.lib";
ring r=31991,(x,y,z),dp;
ideal I = y^2*z - x*(x-z)*(x+3*z);
qring Q = std(I);
divisor A = makeDivisor(ideal(x,z),ideal(1));
A;
 $\mapsto (z,x) - (1)$ 
 $\mapsto$ 
divisor D = multdivisor(4,A);
D;
 $\mapsto (z^2,xz) - (1)$ 
 $\mapsto$ 
 $A^4$ ;
 $\mapsto (z^2,xz) - (1)$ 
 $\mapsto$ 
```

### D.5.4.4 negativedivisor

Procedure from library `divisors.lib` (see [Section D.5.4 \[divisors.lib\]](#), page 1429).

**Usage:** `negativedivisor(A);`  $A^{(-1)}$ ;  $A = \text{divisor}$ .

**Assume:**  $A$  is a divisor on  $X$ .

**Return:** a divisor on  $X$

**Theory:** The procedure will interchange the numerator and denominator ideals.

**Example:**

```
LIB "divisors.lib";
ring r=31991,(x,y,z),dp;
ideal I = y^2*z - x*(x-z)*(x+3*z);
qring Q = std(I);
divisor A = makeDivisor(ideal(x,z),ideal(1));
A;
 $\mapsto (z,x) - (1)$ 
 $\mapsto$ 
divisor D = negativedivisor(A);
D;
 $\mapsto (1) - (z,x)$ 
 $\mapsto$ 
```

### D.5.4.5 normalForm

Procedure from library `divisors.lib` (see [Section D.5.4 \[divisors.lib\]](#), page 1429).

**Usage:**      `normalForm(A)`;  $A = \text{divisor}$ .

**Assume:**     $A$  is a divisor on  $X$ .

**Return:**     different representative of the same divisor on  $X$

**Theory:**     The procedure will cancel common components of numerator and denominator.

**Example:**

```
LIB "divisors.lib";
ring r=31991,(x,y,z),dp;
ideal I = y^2*z - x*(x-z)*(x+3*z);
qring Q = std(I);
divisor A = makeDivisor(ideal(x,z),ideal(1));
divisor B = makeDivisor(ideal(x,y),ideal(1));
divisor D = (A+B)+multdivisor(-1,B);
D;
 $\mapsto (x,yz) - (y,x)$ 
 $\mapsto$ 
normalForm(D);
 $\mapsto (z,x) - (1)$ 
 $\mapsto$ 
```

### D.5.4.6 isEqualDivisor

Procedure from library `divisors.lib` (see [Section D.5.4 \[divisors.lib\]](#), page 1429).

**Usage:**      `isEqualDivisor(A,B)`;  $A = \text{divisor}$ ,  $B = \text{divisor}$ .

**Assume:**     $A$  and  $B$  are divisors on  $X$ .

**Return:**     int 0 or 1, checks equality of  $A$  and  $B$ .

**Theory:**     The procedure will compute the normal forms of  $A$  and  $B$  and compare.

**Example:**

```
LIB "divisors.lib";
ring r=31991,(x,y,z),dp;
ideal I = y^2*z - x*(x-z)*(x+3*z);
qring Q = std(I);
divisor A = makeDivisor(ideal(x,z),ideal(1));
divisor B = makeDivisor(ideal(x,y),ideal(1));
divisor D = (A+B)+multdivisor(-1,B);
isEqualDivisor(A,D);
 $\mapsto 1$ 
```

### D.5.4.7 globalSections

Procedure from library `divisors.lib` (see [Section D.5.4 \[divisors.lib\]](#), page 1429).

**Usage:**      `globalSections(A)`;  $A = \text{divisor}$ .

**Assume:**  $A$  is a divisor on  $X$ .

**Return:** a list with a basis of the space of global sections of  $D$ .

**Theory:** We assume that the qring of  $X$  satisfies the S2-condition and that  $X$  is smooth. We compute  $\text{sat}((f^*J) : I) / f$  where  $D = (I) - (J)$ .

**Example:**

```
LIB "divisors.lib";
ring r=31991,(x,y,z),dp;
ideal I = y^2*z - x*(x-z)*(x+3*z);
qring Q = std(I);
divisor P = makeDivisor(ideal(x,z),ideal(1));
divisor D = multdivisor(4,P);
globalSections(D);
⇒ [1]:
⇒  _[1]=x2
⇒  _[2]=z2
⇒  _[3]=yz
⇒  _[4]=xz
⇒ [2]:
⇒  z2
```

#### D.5.4.8 degreeDivisor

Procedure from library `divisors.lib` (see [Section D.5.4 \[divisors.lib\], page 1429](#)).

**Usage:** `degreeDivisor(A)`;  $A = \text{divisor}$ .

**Assume:**  $A$  is a divisor on  $X$ .

**Return:** The degree of  $A$ .

**Theory:** We compute difference of the degrees of the numerator and denominator ideals.

**Example:**

```
LIB "divisors.lib";
ring r=31991,(x,y,z),dp;
ideal I = y^2*z - x*(x-z)*(x+3*z);
qring Q = std(I);
divisor P = makeDivisor(ideal(x,z),ideal(1));
degreeDivisor(P);
⇒ 1
```

#### D.5.4.9 linearlyEquivalent

Procedure from library `divisors.lib` (see [Section D.5.4 \[divisors.lib\], page 1429](#)).

**Usage:** `linearlyEquivalent(A,B)`;  $A = \text{divisor}$ ,  $B = \text{divisor}$ .

**Assume:**  $A$  and  $B$  are divisors on  $X$ .

**Return:** list if  $A$  and  $B$  are linearly equivalent and `int 0` otherwise.

**Theory:** Checks whether  $A - B$  is principle. If yes, returns a list  $L = (f, g)$  where  $A - B = (f/g)$ .

**Example:**

```

LIB "divisors.lib";
ring r=31991,(x,y,z),dp;
ideal I = y^2*z - x*(x-z)*(x+3*z);
qring Q = std(I);
divisor A = makeDivisor(ideal(x,z),ideal(1));
divisor B = makeDivisor(ideal(x,y),ideal(1));
linearlyEquivalent(A,B);
 $\mapsto$  0
linearlyEquivalent(multdivisor(2,A),multdivisor(2,B));
 $\mapsto$  [1]:
 $\mapsto$  x
 $\mapsto$  [2]:
 $\mapsto$  z

```

#### D.5.4.10 effective

Procedure from library `divisors.lib` (see [Section D.5.4 \[divisors.lib\]](#), page 1429).

**Usage:** `effective(A)`;  $A$  = divisor.

**Assume:**  $A$  is a divisor on  $X$  which is linearly equivalent to an effective divisor.

**Return:** divisor on  $X$ .

**Theory:** We compute an effective divisor linearly equivalent to  $A$ .

**Example:**

```

LIB "divisors.lib";
ring r=31991,(x,y,z),dp;
ideal I = y^2*z - x*(x-z)*(x+3*z);
qring Q = std(I);
divisor A = makeDivisor(ideal(x,z),ideal(1));
divisor B = makeDivisor(ideal(x,y),ideal(1));
divisor D = divisorplus(multdivisor(2,B),negativedivisor(A));
effective(D);
 $\mapsto$  (z,x) - (1)
 $\mapsto$ 

```

#### D.5.4.11 makeFormalDivisor

Procedure from library `divisors.lib` (see [Section D.5.4 \[divisors.lib\]](#), page 1429).

**Usage:** `makeFormalDivisor(L)`;  $L$  = list.

**Assume:**  $L$  is a list of tuples of an integer and a divisor.

**Return:** a formal divisor on  $X$

**Theory:** Represents an integer formal sum of divisors.

**Example:**

```

LIB "divisors.lib";
ring r=31991,(x,y,z),dp;
ideal I = y^2*z - x*(x-z)*(x+3*z);
qring Q = std(I);

```



```

divisor A = makeDivisor(ideal(x,z),ideal(1));
divisor B = makeDivisor(ideal(x,y),ideal(1));
makeFormalDivisor(list(list(-5,A),list(2,B)));
 $\mapsto -5*(z,x) - (1)$ 
 $\mapsto +2*(y,x) - (1)$ 
 $\mapsto$ 

```

#### D.5.4.12 evaluateFormalDivisor

Procedure from library `divisors.lib` (see [Section D.5.4 \[divisors.lib\]](#), page 1429).

**Usage:** `evaluateFormalDivisor(D)`; D = formal divisor.

**Assume:** D is a formal divisor on X.

**Return:** a divisor on X

**Theory:** Will evaluate the formal sum.

**Example:**

```

LIB "divisors.lib";
ring r=31991,(x,y,z),dp;
ideal I = y^2*z - x*(x-z)*(x+3*z);
qring Q = std(I);
divisor A = makeDivisor(ideal(x,z),ideal(1));
divisor B = makeDivisor(ideal(x,y),ideal(1));
formaldivisor fE= makeFormalDivisor(list(list(-5,A),list(2,B)));
evaluateFormalDivisor(fE);
 $\mapsto (x,y^2) - (z^2,x^2z)$ 
 $\mapsto$ 

```

#### D.5.4.13 formaldivisorplus

Procedure from library `divisors.lib` (see [Section D.5.4 \[divisors.lib\]](#), page 1429).

**Usage:** `formaldivisorplus(A ,B)`; A + B; A = formaldivisor, B = formaldivisor.

**Assume:** A and B are formal divisors on X.

**Return:** a formal divisor on X

**Theory:** The procedure will add the formal sums.

**Example:**

```

LIB "divisors.lib";
ring r=31991,(x,y,z),dp;
ideal I = y^2*z - x*(x-z)*(x+3*z);
qring Q = std(I);
divisor A = makeDivisor(ideal(x,z),ideal(1));
divisor B = makeDivisor(ideal(x,y),ideal(1));
divisor C = makeDivisor(ideal(x-z,y),ideal(1));
formaldivisor fE= makeFormalDivisor(list(list(-5,A),list(2,B)));
formaldivisor fE2= makeFormalDivisor(list(list(-5,A),list(2,C)));
formaldivisorplus(fE,fE2);
 $\mapsto [1]:$ 

```

```

⇒      [1]:
⇒      -10
⇒      [2]:
⇒      (z,x) - (1)
⇒
⇒ [2]:
⇒      [1]:
⇒      2
⇒      [2]:
⇒      (y,x) - (1)
⇒
⇒ [3]:
⇒      [1]:
⇒      2
⇒      [2]:
⇒      (y,x-z) - (1)
⇒

```

#### D.5.4.14 negativeformaldivisor

Procedure from library `divisors.lib` (see [Section D.5.4 \[divisors.lib\]](#), page 1429).

**Usage:** `negativeformaldivisor(A)`;  $A$  = formaldivisor.

**Assume:**  $A$  is a formaldivisor on  $X$ .

**Return:** a formal divisor on  $X$

**Theory:** The procedure will change the signs of the coefficients.

**Example:**

```

LIB "divisors.lib";
ring r=31991,(x,y,z),dp;
ideal I = y^2*z - x*(x-z)*(x+3*z);
qring Q = std(I);
divisor A = makeDivisor(ideal(x,z),ideal(1));
divisor B = makeDivisor(ideal(x,y),ideal(1));
formaldivisor fE= makeFormalDivisor(list(list(-5,A),list(2,B)));
negativeformaldivisor(fE);
⇒ 5*( (z,x) - (1) )
⇒ -2*( (y,x) - (1) )
⇒

```

#### D.5.4.15 multformaldivisor

Procedure from library `divisors.lib` (see [Section D.5.4 \[divisors.lib\]](#), page 1429).

**Usage:** `multformaldivisor(n ,A)`;  $A^n$ ;  $n$  = integer,  $A$  = formaldivisor.

**Assume:**  $n$  is an integer and  $A$  is a formal divisor on  $X$ .

**Return:** a formal divisor on  $X$

**Theory:** The procedure will multiply the formal sum with  $n$ .

**Example:**

```

LIB "divisors.lib";
ring r=31991,(x,y,z),dp;
ideal I = y^2*z - x*(x-z)*(x+3*z);
qring Q = std(I);
divisor A = makeDivisor(ideal(x,z),ideal(1));
divisor B = makeDivisor(ideal(x,y),ideal(1));
formaldivisor fE= makeFormalDivisor(list(list(-5,A),list(2,B)));
fE*2;
↪ -10*( (z,x) - (1) )
↪ +4*( (y,x) - (1) )
↪

```

#### D.5.4.16 degreeFormalDivisor

Procedure from library `divisors.lib` (see [Section D.5.4 \[divisors.lib\]](#), page 1429).

**Usage:**      `degreeFormalDivisor(A)`;  $A$  = formaldivisor.

**Assume:**     $A$  is a formaldivisor on  $X$ .

**Return:**     The degree of  $A$ .

**Theory:**     We compute degrees of the summands and return the weighted sum.

**Example:**

```

LIB "divisors.lib";
ring r=31991,(x,y,z),dp;
ideal I = y^2*z - x*(x-z)*(x+3*z);
qring Q = std(I);
divisor A = makeDivisor(ideal(x,z),ideal(1));
divisor B = makeDivisor(ideal(x,y),ideal(1));
formaldivisor fE= makeFormalDivisor(list(list(-5,A),list(2,B)));
degreeFormalDivisor(fE);
↪ -3

```

#### D.5.4.17 makePDivisor

Procedure from library `divisors.lib` (see [Section D.5.4 \[divisors.lib\]](#), page 1429).

**Usage:**      `makePDivisor(L)`;  $L$  = list.

**Assume:**     $L$  is a list of tuples of a integral polyhedron and a divisor such that all polyhedra have the same tail cone.

**Return:**     a pdivisor on  $X$

**Theory:**     Represents an polyhedral formal sum of divisors.

**Example:**

```

LIB "divisors.lib";
ring r=31991,(x,y,z),dp;
ideal I = y^2*z - x*(x-z)*(x+3*z);
qring Q = std(I);
divisor A = makeDivisor(ideal(x,z),ideal(1));
divisor B = makeDivisor(ideal(x,y),ideal(1));

```

```

intmat M[4][4]= 1,4,0,0,
1,0,3,0,
0,0,0,2,
1,1,1,1;
polytope PP = polytopeViaPoints(M);
makePDivisor(list(list(PP,A),list(PP,B)));
↳ tail=<cone>
↳ summands=<list>

```

### D.5.5 goettsche\_lib

**Library:** goettsche.lib

**Purpose:** Drezet's formula for the Betti numbers of the moduli space of Kronecker modules; Goettsche's formula for the Betti numbers of the Hilbert scheme of points on a surface; Nakajima's and Yoshioka's formula for the Betti numbers of the punctual Quot-schemes on a plane or, equivalently, of the moduli spaces of the framed torsion-free planar sheaves; Macdonald's formula for the symmetric product

**Author:** Oleksandr Iena, o.g.yena@gmail.com

**References:**

- [1] Drezet, Jean-Marc Cohomologie des variétés de modules de hauteur nulle. Mathematische Annalen: 281, 43-85, (1988).
- [2] Goettsche, Lothar, The Betti numbers of the Hilbert scheme of points on a smooth projective surface. Mathematische Annalen: 286, 193-208, (1990).
- [3] Macdonald, I. G., The Poincare polynomial of a symmetric product, Mathematical proceedings of the Cambridge Philosophical Society: 58, 563-568, (1962).
- [4] Nakajima, Hiraku; Lectures on instanton counting, CRM Proceedings and Lecture Notes, Yoshioka, Kota Volume 88, 31-101, (2004).

**Procedures:**

#### D.5.5.1 GoettscheF

Procedure from library goettsche.lib (see [Section D.5.5 \[goettsche\\_lib\]](#), page 1438).

**Usage:** GoettscheF(z, t, n, b); z, t polynomials, n integer, b list of non-negative integers

**Return:** polynomial in z and t

**Purpose:** computes the Goettsche's formula up to degree n in t

**Note:** zero is returned if n<0 or b is not a list of non-negative integers or if there are not enough Betti numbers

**Example:**

```

LIB "goettsche.lib";
ring r=0, (t, z), ls;
// consider the projective plane with Betti numbers 1,0,1,0,1
list b=1,0,1,0,1;
// get the Goettsche's formula up to degree 3
print( GoettscheF(z, t, 3, b) );
↳ 1+t+tz2+tz4+t2+2t2z2+3t2z4+2t2z6+t2z8+t3+2t3z2+5t3z4+6t3z6+5t3z8+2t3z10+t\
3z12

```

### D.5.5.2 PPolyH

Procedure from library `goettsche.lib` (see [Section D.5.5 \[goettsche\\_lib\], page 1438](#)).

**Usage:** `PPolyH(z, n, b)`; `z` polynomial, `n` integer, `b` list of non-negative integers

**Return:** polynomial in `z`

**Purpose:** computes the Poincare polynomial of the Hilbert scheme of `n` points on a surface with Betti numbers `b`

**Note:** zero is returned if `n < 0` or `b` is not a list of non-negative integers or if there are not enough Betti numbers

**Example:**

```
LIB "goettsche.lib";
ring r=0, (z), ls;
// consider the projective plane P_2 with Betti numbers 1,0,1,0,1
list b=1,0,1,0,1;
// get the Poincare polynomial of the Hilbert scheme of 3 points on P_2
print( PPolyH(z, 3, b) );
↪ 1+2z2+5z4+6z6+5z8+2z10+z12
```

### D.5.5.3 BettiNumsH

Procedure from library `goettsche.lib` (see [Section D.5.5 \[goettsche\\_lib\], page 1438](#)).

**Usage:** `BettiNumsH(n, b)`; `n` integer, `b` list of non-negative integers

**Return:** list of non-negative integers

**Purpose:** computes the Betti numbers of the Hilbert scheme of `n` points on a surface with Betti numbers `b`

**Note:** an empty list is returned if `n < 0` or `b` is not a list of non-negative integers or if there are not enough Betti numbers

**Example:**

```
LIB "goettsche.lib";
ring r=0, (z), ls;
// consider the projective plane P_2 with Betti numbers 1,0,1,0,1
list b=1,0,1,0,1;
// get the Betti numbers of the Hilbert scheme of 3 points on P_2
print( BettiNumsH(3, b) );
↪ [1]:
↪ 1
↪ [2]:
↪ 0
↪ [3]:
↪ 2
↪ [4]:
↪ 0
↪ [5]:
↪ 5
↪ [6]:
↪ 0
↪ [7]:
```

```

⇒      6
⇒ [8]:
⇒      0
⇒ [9]:
⇒      5
⇒ [10]:
⇒      0
⇒ [11]:
⇒      2
⇒ [12]:
⇒      0
⇒ [13]:
⇒      1

```

#### D.5.5.4 NakYoshF

Procedure from library `goettsche.lib` (see [Section D.5.5 \[goettsche.lib\]](#), page 1438).

**Usage:** NakYoshF(z, t, r, n); z, t polynomials, r, n integers

**Return:** polynomial in z and t

**Purpose:** computes the formula of Nakajima and Yoshioka  
up to degree n in t

**Note:** zero is returned if  $n < 0$  or  $r \leq 0$

**Example:**

```

LIB "goettsche.lib";
ring r=0, (t, z), ls;
// get the Nakajima-Yoshioka formula for r=1 up to degree 3, i.e.,
// the generating function for the Poincare polynomials of the
// punctual Hilbert schemes of n planar points
print( NakYoshF(z, t, 1, 3) );
⇒ 1+t+t2+t2z2+t3+t3z2+t3z4

```

#### D.5.5.5 PPolyQp

Procedure from library `goettsche.lib` (see [Section D.5.5 \[goettsche.lib\]](#), page 1438).

**Usage:** PPolyQp(z, r, n); z polynomial, r, n integers

**Return:** polynomial in z

**Purpose:** computes the Poincare polynomial of the punctual Quot-scheme of rank r on n planar points

**Note:** zero is returned if  $n < 0$  or  $r \leq 0$

**Example:**

```

LIB "goettsche.lib";
ring r=0, (z), ls;
// get the Poincare polynomial of the punctual Hilbert scheme (r=1)
// of 3 planar points
print( PPolyQp(z, 1, 3) );
⇒ 1+z2+z4

```

### D.5.5.6 BettiNumsQp

Procedure from library `goettsche.lib` (see [Section D.5.5 \[goettsche.lib\], page 1438](#)).

**Usage:** `BettiNumsQp(r, n);`  $n, r$  integers

**Return:** list of non-negative integers

**Purpose:** computes the Betti numbers of the punctual Quot-scheme of rank  $r$  on  $n$  points on a plane

**Note:** an empty list is returned if  $n < 0$  or  $r \leq 0$

**Example:**

```
LIB "goettsche.lib";
ring r=0, (z), ls;
// get the Betti numbers of the punctual Hilbert scheme (r=1)
// of 3 points on a plane
print( BettiNumsQp(1, 3) );
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ 1
⇒ [4]:
⇒ 0
⇒ [5]:
⇒ 1
```

### D.5.5.7 MacdonaldF

Procedure from library `goettsche.lib` (see [Section D.5.5 \[goettsche.lib\], page 1438](#)).

**Usage:** `MacdonaldF(z, t, n, b);`  $z, t$  polynomials,  $n$  integer,  $b$  list of non-negative integers

**Return:** polynomial in  $z$  and  $t$  with integer coefficients

**Purpose:** computes the Macdonald's formula up to degree  $n$  in  $t$

**Note:** zero is returned if  $n < 0$  or  $b$  is not a list of non-negative integers

**Example:**

```
LIB "goettsche.lib";
ring r=0, (t, z), ls;
// consider the projective plane with Betti numbers 1,0,1,0,1
list b=1,0,1,0,1;
// get the Macdonald's formula up to degree 3
print( MacdonaldF(z, t, 3, b) );
⇒ 1+t+tz2+tz4+t2+t2z2+2t2z4+t2z6+t2z8+t3+t3z2+2t3z4+2t3z6+2t3z8+t3z10+t3z12
```

### D.5.5.8 PPolyS

Procedure from library `goettsche.lib` (see [Section D.5.5 \[goettsche.lib\], page 1438](#)).

**Usage:** `PPolyS(z, n, b);`  $z$  polynomial,  $n$  integer,  $b$  list of non-negative integers

**Return:** polynomial in  $z$  with integer coefficients

**Purpose:** computes the Poincare polynomial of the  $n$ -th symmetric power of a variety with Betti numbers  $b$

**Note:** zero is returned if  $n < 0$  or  $b$  is not a list of non-negative integers

**Example:**

```
LIB "goettsche.lib";
ring r=0, (z), ls;
// consider the projective plane P_2 with Betti numbers 1,0,1,0,1
list b=1,0,1,0,1;
// get the Poincare polynomial of the third symmetric power of P_2
print( PPolyS(z, 3, b) );
↪ 1+z2+2z4+2z6+2z8+z10+z12
```

### D.5.5.9 BettiNumsS

Procedure from library `goettsche.lib` (see [Section D.5.5 \[goettsche.lib\]](#), page 1438).

**Usage:** BettiNumsS( $n$ ,  $b$ );  $n$  integer,  $b$  list of non-negative integers

**Return:** list of non-negative integers

**Purpose:** computes the Betti numbers of the  $n$ -th symmetric power of a variety with Betti numbers  $b$

**Note:** an empty list is returned if  $n < 0$  or  $b$  is not a list of non-negative integers

**Example:**

```
LIB "goettsche.lib";
ring r=0, (z), ls;
// consider a complex torus T (elliptic curve) with Betti numbers 1,2,1
list b=1,2,1;
// get the Betti numbers of the second symmetric power of T
print( BettiNumsS(2, b) );
↪ [1]:
↪ 1
↪ [2]:
↪ 2
↪ [3]:
↪ 2
↪ [4]:
↪ 2
↪ [5]:
↪ 1
// consider a projective plane P_2 with Betti numbers 1,0,1,0,1
b=1,0,1,0,1;
// get the Betti numbers of the third symmetric power of P_2
print( BettiNumsS(3, b) );
↪ [1]:
↪ 1
↪ [2]:
↪ 0
↪ [3]:
↪ 1
↪ [4]:
↪ 0
```



```

⇒ [5]:
⇒ 2
⇒ [6]:
⇒ 0
⇒ [7]:
⇒ 2
⇒ [8]:
⇒ 0
⇒ [9]:
⇒ 2
⇒ [10]:
⇒ 0
⇒ [11]:
⇒ 1
⇒ [12]:
⇒ 0
⇒ [13]:
⇒ 1

```

#### D.5.5.10 PPolyN

Procedure from library `goettsche.lib` (see [Section D.5.5 \[goettsche.lib\], page 1438](#)).

**Usage:** PPolyN( $t, q, m, n$ );  $t$  polynomial,  $q, m, n$  integers

**Return:** polynomial in  $t$

**Purpose:** computes the Poincare polynomial of the moduli space of Kronecker modules  $N(q; m, n)$

**Note:** if  $m$  and  $n$  are not coprime, the result does not necessary make sense

**Example:**

```

LIB "goettsche.lib";
ring r=0, (t), ls;
// get the Poincare polynomial of N(3; 2, 3)
print( PPolyN(t, 3, 2, 3) );
⇒ 1+t2+3t4+3t6+3t8+t10+t12

```

#### D.5.5.11 BettiNumsN

Procedure from library `goettsche.lib` (see [Section D.5.5 \[goettsche.lib\], page 1438](#)).

**Usage:** BettiNumsN( $q, m, n$ );  $q, m, n$  integers

**Return:** list of integers

**Purpose:** computes the Betti numbers of the moduli space of Kronecker modules  $N(q; m, n)$

**Note:** if  $m$  and  $n$  are not coprime, the result does not necessary make sense

**Example:**

```

LIB "goettsche.lib";
ring r=0, (t), dp;
// get the Betti numbers of N(3; 2, 3)
print( BettiNumsN(3, 2, 3) );

```

```

⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ 1
⇒ [4]:
⇒ 0
⇒ [5]:
⇒ 3
⇒ [6]:
⇒ 0
⇒ [7]:
⇒ 3
⇒ [8]:
⇒ 0
⇒ [9]:
⇒ 3
⇒ [10]:
⇒ 0
⇒ [11]:
⇒ 1
⇒ [12]:
⇒ 0
⇒ [13]:
⇒ 1

```

### D.5.6 graal\_lib

**Library:** graal.lib

**Purpose:** localization at prime ideals and their associated graded rings

**Author:** Magdaleen Marais, magdaleen@aims.ac.za  
Yue Ren, ren@mathematik.uni-kl.de

**Overview:** This library is on a computational treatment of localizations at prime ideals and their associated graded rings based on a work of Mora. Not only does it construct a ring isomorphic to the localization of an affine coordinate ring at a prime ideal, the algorithms in this library aim to exploit the topology in the localization by computing first and foremost in the associated graded ring and lifting the result to the localization afterwards.  
Features include a check for regularity and the resolution of ideals.

**References:**

Mora, Teo: La queste del Saint Gr<sub>a</sub>(A.L): A computational approach to local algebra  
Marais, Magdaleen and Ren, Yue: Mora's holy graal: Algorithms for computing in localizations at prime ideals

**Procedures:**

#### D.5.6.1 graalMixed

Procedure from library `graal.lib` (see [Section D.5.6 \[graal.lib\]](#), page 1444).

**Usage:** `graalMixed(L,t)`; L ideal, t int (optional)

**Return:** graalBearer with all the necessary structures for our machinery if  $t$  specified and  $t > 0$ , puts an upper time limit on finding a necessary transformation to map an intermediate ideal into general position.

**Note:** assumes that the current basering is a domain and that  $L$  is a prime ideal.

**Example:**

```
LIB "graal.lib";
// see [Mora] Example 6.5
ring Q = 0,(x,y,z),dp;
ideal H = y2-xz;
qring A = std(H);
ideal L = x3-yz,x2y-z2;
graalBearer Gr = graalMixed(L); Gr;
↳ affine coordinate ring:
↳   (QQ),(x,y,z),(dp(3),C)
↳   mod <y2-xz>
↳
↳ ideal defining the subvariety:
↳   <x3-yz,x2y-z2>
↳
↳ A1:
↳   (0,z),(Y(1),Y(2),x,y),(ds(2),c,dp(2))
↳   mod <x^3+(-z)*y-Y(1),x^2*y+(-z^2)-Y(2),y^2+(-z)*x,(z)*Y(1)-Y(2)*y>
↳ graal:
↳   (0,z),(Y(1),Y(2),y),(c,dp(2),dp(1))
↳   mod <(z)*Y(1)-Y(2)*y,y^5+(-z^4)>
↳   where
↳   Y(1) represents generator x3-yz
↳   Y(2) represents generator x2y-z2
↳   and x,y in A1 are mapped to 1/(z)*y^2,y in Graal
↳
```

### D.5.6.2 dimensionOfLocalization

Procedure from library `graal.lib` (see [Section D.5.6 \[graal.lib\]](#), page 1444).

**Usage:** `dimensionOfLocalization(L)`;  $L$  ideal or `graalBearer`

**Return:** `int`, the dimension of the localization  $A_L$  of  $A$  at  $L$ .

**Example:**

```
LIB "graal.lib";
ring Q = 0,(X(1),X(2)),dp;
ideal H = X(2)^2-(X(1)-1)*X(1)*(X(1)+1);
ideal J = std(X(1),X(2));
↳ // ** _ is no standard basis
qring A = std(H);
ideal L = fetch(Q,J);
graalBearer Gr = graalMixed(L);
// def fA = Gr.fA; setring fA;
dimensionOfLocalization(Gr); // = 1
↳ // ** redefining scriptIn ( ideal scriptIn = Gr.scriptIn;) graal.li\
  b::dimensionOfLocalization:582
↳ 1
```

### D.5.6.3 systemOfParametersOfLocalization

Procedure from library `graal.lib` (see [Section D.5.6 \[graal.lib\], page 1444](#)).

**Usage:** `systemOfParametersOfLocalization(def L);` L ideal or `graalBearer`

**Return:** ideal, a system of parameter of the localization  $A_L$  of  $A$  at  $L$ .

**Example:**

```
LIB "graal.lib";
ring Q = 0, (X(1), X(2)), dp;
ideal H = X(2)^2 - (X(1)-1)*X(1)*(X(1)+1);
ideal J = X(1), X(2);
qring A = std(H);
ideal L = fetch(Q, J);
graalBearer Gr = graalMixed(L);
systemOfParametersOfLocalization(Gr); // = 1
⇒ // ** redefining J (    ideal J = Gr.J; ideal ret;) graal.lib::systemOfPa\
    rametersOfLocalization:647
⇒ _[1]=X(2)
```

### D.5.6.4 isLocalizationRegular

Procedure from library `graal.lib` (see [Section D.5.6 \[graal.lib\], page 1444](#)).

**Usage:** `isLocalizationRegular(def L);` L ideal or `graalBearer`

**Return:** int, 1 if the localization  $A_L$  of  $A$  at  $L$  is regular,  
0 otherwise.

**Example:**

```
LIB "graal.lib";
ring Q = 0, (X(1), X(2)), dp;
ideal H = X(2)^2 - (X(1)-1)*X(1)*(X(1)+1);
ideal J = X(1), X(2);
qring A = std(H);
ideal L = fetch(Q, J);
graalBearer Gr = graalMixed(L);
isLocalizationRegular(Gr); // = 1
⇒ 1
```

### D.5.6.5 warkedPreimageStd

Procedure from library `graal.lib` (see [Section D.5.6 \[graal.lib\], page 1444](#)).

**Usage:** `warkedPreimageStd(wM);` M warkedModule

**Return:** given wM consisting of:

- wM.Gr a `graalBearer` containing all relevant global structures - wM.modQ0y generating set G of a module M over Q0y - wM.stdmodQ0y empty
- wM.qQ0y empty
- wM.modKy corresponding generating set H of M.in over Ky - wM.stdmodKy empty
- wM.qKy empty
- wM.w weights on M

returns the same warkedModule, except following differences: - wM.stdmodQ0y contains a subset G such that for any standard basis L of the kernel  $G + L$  is a standard

basis of  $\text{modQ0y} + \text{kernel}$

-  $\text{wM.qQ0y}$  contains a transformation matrix such that  $\text{stdmodAy} = \text{QAy} * \text{modQ0y}$

-  $\text{wM.stdmodKy}$  contains a standardbasis of  $\text{modKy}$

-  $\text{wM.qKy}$  contains a transformation matrix such that  $\text{stdmodKy} = \text{QKy} * \text{modKy}$

**Note:** the standard basis of  $\text{modAy}$  is computed by lifting a corresponding Groebner basis of  $\text{modKy}$

**Example:**

```
LIB "graal.lib";
ring Q = 0,(x,y,z),dp;
ideal H = y2-xz;
qring A = std(H);
ideal L = x3-yz,x2y-z2;
graalBearer Gr = graalMixed(L);
def Q0y = Gr.Q0y; setring Q0y;
module M = (Y(1)*y+y^2-1)*gen(1)+(Y(2)*z+z^2-1)*gen(2), Y(1)*y*gen(1)+Y(2)*z*gen(2);
/* This is M: */
print(matrix(M));
  ↳ y^2-1+Y(1)*y,    Y(1)*y,
  ↳ (z^2-1)+(z)*Y(2), (z)*Y(2)
intvec w = 1,1,1;
warkedModule wM;
wM.Gr = Gr;
wM.modQ0y = M;
wM.w = w;
def Ky = Gr.Ky; setring Ky;
module Min = (y^2-1)*gen(1)+(z^2-1)*gen(2),Y(1)*y*gen(1)+Y(2)*z*gen(2);
/* This is M_in: */
print(matrix(Min));
  ↳ y^2-1,    Y(1)*y,
  ↳ (z^2-1), (z)*Y(2)
wM.modKy = Min;
/* warkedPreimageStd yields the same standard basis as std: */
warkedModule wN = warkedPreimageStd(wM); wN;
  ↳ // ** redefining scriptIin ( ideal scriptIin = Gr.scriptIin;) graal.lib:\
      :warkedPreimageStd:892
  ↳ // ** redefining scriptI ( ideal scriptI = Gr.scriptI;) graal.lib::warke\
      dPreimageStd:921
  ↳ // ** redefining a ( poly a = L[1][1,1];) graal.lib::normalizeInY:810
  ↳ module over Q^0[Y] = (0,z),(Y(1),Y(2),x,y),(ds(2),c,dp(2)) / <0>:
  ↳ y^2-1+Y(1)*y,    Y(1)*y,
  ↳ (z^2-1)+(z)*Y(2), (z)*Y(2)
  ↳ standard basis:
  ↳ _[1,1],_[1,2],0,          0,          0,          0,
  ↳ _[2,1],_[2,2],x^3+(-z)*y-Y(1),x^2*y+(-z^2)-Y(2),y^2+(-z)*x,(z)*Y(1)-Y(2)*\
      y
  ↳ module over K[Y] = (0,z),(Y(1),Y(2),y),(c,dp(2),dp(1)) / <y^5+(-z^4)>:
  ↳ y^2-1,    Y(1)*y,
  ↳ (z^2-1), (z)*Y(2)
  ↳ weights on the unit vectors: 1,1,1
  ↳
setring Q0y;
module stdM = std(M);
```

```

print(matrix(stdM));
⇒ y^2-1+Y(1)*y,      Y(1)-Y(1)^2*y,Y(1)^2*y^2,
⇒ (z^2-1)+(z)*Y(2),_ [2,2],      _ [2,3]

```

### D.5.6.6 resolutionInLocalization

Procedure from library `graal.lib` (see [Section D.5.6 \[graal.lib\], page 1444](#)).

**Usage:**      `resolutionInLocalization(I,L)`; I ideal, L ideal or `graalBearer`

**Return:**    the resolution of  $I \cdot A_L$ , where  
 $A_L$  is the localization of the current basering (possibly a quotient ring) at a prime ideal L.

**Example:**

```

LIB "graal.lib";
ring Q = 0,(x,y,z,w),dp;
ideal circle = (x-1)^2+y^2-3,z;
ideal twistedCubic = xz-y^2,yw-z^2,xw-yz,z;
ideal I = std(intersect(circle,twistedCubic));
// the resolution is more complicated due to the twisted cubic
res(I,0);
⇒ 1      4      5      2
⇒ Q <--  Q <--  Q <--  Q
⇒
⇒ 0      1      2      3
⇒ resolution not minimized yet
⇒
// however if we localize outside of the twisted cubic,
// it should become very easy again.
ideal L = std(I+ideal(x-1));
graalBearer Gr = graalMixed(L); Gr;
⇒ affine coordinate ring:
⇒ (QQ),(x,y,z,w),(dp(4),C)
⇒
⇒ ideal defining the subvariety:
⇒ <z,x-1,y^2w-3w,y^4-3y^2>
⇒
⇒ A1:
⇒ (0,w),(Y(1),Y(2),Y(3),Y(4),x,y,z),(ds(4),c,dp(3))
⇒ mod <(w)*y^2+(-3*w)-Y(3),x-1-Y(2),z-Y(1),(3*w)*Y(3)+(-w^2)*Y(4)+Y(3)\
^2>
⇒ graal:
⇒ (0,w),(Y(1),Y(2),Y(3),Y(4),z),(c,dp(4),dp(1))
⇒ mod <3*Y(3)+(-w)*Y(4),z^2+10*z-2>
⇒ where
⇒ Y(1) represents generator z
⇒ Y(2) represents generator x-1
⇒ Y(3) represents generator y^2w-3w
⇒ Y(4) represents generator y^4-3y^2
⇒ and x,y,z in A1 are mapped to 1,1/3*z+5/3,0 in Graal
⇒
markedResolution mr = resolutionInLocalization(I,Gr);
⇒ // ** full resolution in a qring may be infinite, setting max length to 5

```

```

mr;
⇒ resolution over A1:
⇒      1      2      1
⇒ A1 <--  A1 <--  A1
⇒
⇒      0      1      2
⇒ resolution not minimized yet
⇒
⇒ k=1
⇒ Y(1), (w^2)*Y(4)+(3*w^2)*Y(2)^2*x+(3*w)*Y(2)*Y(3)-Y(3)^2
⇒
⇒ k=2
⇒ _[1,1],
⇒ -Y(1)
⇒
⇒ resolution over Graal:
⇒      1      2      1
⇒ Graal <--  Graal <--  Graal
⇒
⇒      0      1      2
⇒
⇒ k=1
⇒ Y(1), (w^2)*Y(4)
⇒
⇒ k=2
⇒ (w^2)*Y(4),
⇒ -Y(1)
⇒
⇒

```

### D.5.7 hess\_lib

**Library:** hess.lib

**Purpose:** Riemann-Roch space of divisors on function fields and curves

**Authors:** I. Stenger: stenger@mathematik.uni-kl.de

**Overview:** Let  $f$  be an absolutely irreducible polynomial in two variables  $x, y$ . Assume that  $f$  is monic as a polynomial in  $y$ . Let  $F = \text{Quot}(k[x, y]/f)$  be the function field defined by  $f$ . Define  $O_F = \text{IntCl}(k[x], F)$  and  $O_-(F, \text{inf}) = \text{IntCl}(k[1/x], F)$ . We represent a divisor  $D$  on  $F$  by two fractional ideals  $I$  and  $J$  of  $O_F$  and  $O_-(F, \text{inf})$ , respectively. The Riemann-Roch space  $L(D)$  is then the intersection of  $I^{(-1)}$  and  $J^{(-1)}$ .

**Procedures:**

#### D.5.7.1 RiemannRochHess

Procedure from library `hess.lib` (see [Section D.5.7 \[hess.lib\]](#), page 1449).

**Note:** All fractional ideals must be represented by a list of size two. The first element is an ideal of  $k[x, y]$  and the second element the common denominator, i.e., a polynomial of  $k[x]$ .

**Assume:** The base ring  $R$  must be a ring in two variables, say  $x, y$ , or three variables, say  $x, y, z$ . If  $\text{nvars}(R) = 2$ :

-  $f$  is an absolutely irreducible polynomial, monic as a polynomial in  $y$ .  
 - List `divisorD` describes a divisor  $D$  of  $F = \text{Quot}(k[x,y]/f)$ . If ( $s = \text{"ideals"}$ )  
 $D$  is given in ideal representation, i.e., `divisorD` is a list of size 2.  
`divisorD[1]` is the finite ideal of  $D$ , i.e., the  
 fractional ideal of  $D$  of  $\text{IntCl}(k[x], F)$ .  
`divisorD[2]` is the infinite ideal of  $D$ , i.e.,  
 the fractional ideal of  $D$  of  $\text{IntCl}(k[1/x], F)$ .  
 If ( $s = \text{"free"}$ )  
 $D$  is given in free representation, i.e., `divisorD` is a list of size 2, containing the finite  
 and infinite places of  $D$  with exponents.  
`divisorD[i]`,  $i = 1, 2$ , is a list. Each element of the list is again a list. The first entry is  
 a fractional ideal, and the second an integer, the exponent of the place. If  $\text{nvars}(R) = 3$ :  
 -  $f$  is an absolutely irreducible homogeneous polynomial describing the projective plane  
 curve corresponding to the function field  $F$ . We assume that the dehomogenization of  
 $f$  w.r.t.  $z$  is monic as a polynomial in  $y$ .  
 List `divisorD` describes a divisor  $D$  of  $F$ .  
 If ( $s = \text{"ideals"}$ )  
 $D$  is given in ideal representation, i.e., `divisorD` is a list of size 2. `divisorD[1]` is an ideal  
 of the base ring representing the positive divisor of  $D$  and  
`divisorD[2]` is an ideal of the base ring representing the negative divisor. (i.e.  $D = (I) - (J)$ ).  
 If ( $s = \text{"free"}$ )  
 $D$  is given in free representation, i.e., `divisorD` is a list of places of  $D$ . `D[i][1]` is an prime  
 ideal and `D[i][2]` an integer, the exponent of the place.

**Return:** A vector space basis of the Riemann-Roch space of  $D$ ,  
 stored in a list `RRBasis`. The list `RRBasis` contains a list, say `rbasis`, and a polynomial,  
 say `den`. The basis of  $L(D)$  consists of all rational functions  $g/\text{den}$ , where  $g$  is an element  
 of `rbasis`.

**Example:**

```
LIB "hess.lib";
ring R = 0, (x,y), dp;
poly f = y^2*(y-1)^3-x^5;
list A1 = list(ideal(x,y-1),1),2;
list A2 = list(ideal(y^2-y+1,x),1),3;
list A3 = list(ideal(1,y-x),x),-2;
list D = A1,A2;
list E = list(A3);
RiemannRochHess(f,list(D,E),"free");
⇒ // ** redefining Infin (    list Infin = maxorderInfinite(f_aff);) hess.lib::RiemannRochHess:171
⇒ [1]:
⇒      [1]:
⇒      3x5+x3y2-4x2y3-x3y+4x2y2+3y4-6y3+3y2
⇒      [2]:
⇒      5x3y2-5x2y3-2x3y+5x2y2
⇒      [3]:
⇒      5x4y-5x3y2+3x3y
⇒      [4]:
⇒      5x5-5x4y+3x3y
```



```

⇒      [5]:
⇒      x5+2x4y-3xy4+6xy3-3xy2
⇒ [2]:
⇒      x5
ring S = 0,(x,y,z),dp;
poly f = y^2*(y-1)^3-x^5;
f = homog(f,z);
ideal P1 = x,y-z;
ideal P2 = y^2-yz+z^2,x;
ideal P3 = x-y,z;
list B1 = P1,2;
list B2 = P2,3;
list B3 = P3,-2;
list Ddivisor = B1,B2,B3;
RiemannRochHess(f,Ddivisor,"free");
⇒ [1]:
⇒      [1]:
⇒      3x5+x3y2-4x2y3-x3y+4x2y2+3y4-6y3+3y2
⇒      [2]:
⇒      5x3y2-5x2y3-2x3y+5x2y2
⇒      [3]:
⇒      5x4y-5x3y2+3x3y
⇒      [4]:
⇒      5x5-5x4y+3x3y
⇒      [5]:
⇒      x5+2x4y-3xy4+6xy3-3xy2
⇒ [2]:
⇒      x5
ideal I = intersect(P1^2,P2^3);
ideal J = P3^2;
RiemannRochHess(f,list(I,J),"ideals");
⇒ [1]:
⇒      [1]:
⇒      2x5-x3y2-x2y3+x2y2+2y4-4y3+2y2
⇒      [2]:
⇒      5x3y2-5x2y3-2x3y+5x2y2
⇒      [3]:
⇒      5x4y-5x3y2+3x3y
⇒      [4]:
⇒      5x5-5x4y+3x3y
⇒      [5]:
⇒      x5+2x4y-3xy4+6xy3-3xy2
⇒ [2]:
⇒      x5

```

### D.5.8 numerAlg\_lib

Todos/Issues:

- does not follow the naming convention
- syntax errors in examples
- no test suite

**Library:** NumerAlg.lib

**Purpose:** Numerical Algebraic Algorithm

**Overview:** The library contains procedures to test the inclusion, the equality of two ideals defined by polynomial systems, compute the degree of a pure  $i$ -dimensional component of an algebraic variety defined by a polynomial system, compute the local dimension of an algebraic variety defined by a polynomial system at a point computed as an approximate value. The use of the library requires to install Bertini (<http://www.nd.edu/~sommese/bertini>).

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**Procedures:**

### D.5.8.1 Incl

Procedure from library `numerAlg.lib` (see [Section D.5.8 \[numerAlg.lib\]](#), page 1451).

**Usage:** `Incl(ideal I, ideal J)`;  $I, J$  ideals

**Return:**  $t=1$  if the algebraic variety defined by  $I$  contains the algebraic variety defined by  $J$ , otherwise  $t=0$

**Example:**

```
LIB "numerAlg.lib";
ring r=0,(x,y,z),dp;
poly f1=(x2+y2+z2-6)*(x-y)*(x-1);
poly f2=(x2+y2+z2-6)*(x-z)*(y-2);
poly f3=(x2+y2+z2-6)*(x-y)*(x-z)*(z-3);
ideal I=f1,f2,f3;
poly g1=(x2+y2+z2-6)*(x-1);
poly g2=(x2+y2+z2-6)*(y-2);
poly g3=(x2+y2+z2-6)*(z-3);
ideal J=g1,g2,g3;
def W=Incl(I,J);
==>
Inclusion:
0
def W=Incl(J,I);
==>
Inclusion:
1
```

### D.5.8.2 Equal

Procedure from library `numerAlg.lib` (see [Section D.5.8 \[numerAlg.lib\]](#), page 1451).

**Usage:** `Equal(ideal I, ideal J)`;  $I, J$  ideals

**Return:**  $t=1$  if the algebraic variety defined by  $I$  equals to the algebraic variety defined by  $J$ , otherwise  $t=0$

**Example:**

```
LIB "numerAlg.lib";
ring r=0,(x,y,z),dp;
poly f1=(x2+y2+z2-6)*(x-y)*(x-1);
poly f2=(x2+y2+z2-6)*(x-z)*(y-2);
```

```

poly f3=(x2+y2+z2-6)*(x-y)*(x-z)*(z-3);
ideal I=f1,f2,f3;
poly g1=(x2+y2+z2-6)*(x-1);
poly g2=(x2+y2+z2-6)*(y-2);
poly g3=(x2+y2+z2-6)*(z-3);
ideal J=g1,g2,g3;
def W=Equal(I,J);
==>
Equality:
0
def W=Equal(J,J);
==>
Equality:
1

```

### D.5.8.3 DegreePure

Procedure from library `numerAlg.lib` (see [Section D.5.8 \[numerAlg.lib\], page 1451](#)).

**Usage:** DegreePure(ideal I,int i); I ideal, i positive integer

**Return:** the degree of the pure i-dimensional component of the algebraic variety defined by I

**Example:**

```

LIB "numerAlg.lib";
ring r=0,(x,y,z),dp;
poly f1=(x2+y2+z2-6)*(x-y)*(x-1);
poly f2=(x2+y2+z2-6)*(x-z)*(y-2);
poly f3=(x2+y2+z2-6)*(x-y)*(x-z)*(z-3);
ideal I=f1,f2,f3;
def W=DegreePure(I,1);
==>
The Degree of Component
3
def W=DegreePure(I,2);
==>
The Degree of Component
2

```

### D.5.8.4 NumLocalDim

Procedure from library `numerAlg.lib` (see [Section D.5.8 \[numerAlg.lib\], page 1451](#)).

**Usage:** NumLocalDim(ideal J, list w, int e); J ideal,  
w list of an approximate value of a point v in the algebraic variety defined by J, e  
integer

**Return:** the local dimension of the algebraic variety defined by J at v

**Example:**

```

LIB "numerAlg.lib";
int e=14;
ring r=(complex,e,I),(x,y,z),dp;
poly f1=(x2+y2+z2-6)*(x-y)*(x-1);
poly f2=(x2+y2+z2-6)*(x-z)*(y-2);

```

```

poly f3=(x2+y2+z2-6)*(x-y)*(x-z)*(z-3);
ideal J=f1,f2,f3;
list p0=0.999999999999999+I*0.000000000000001,2,3+I*0.000000000000001;
list p2=1,0.999999999999998,2;
list p1=5+I,4.999999999999998+I,5+I;
def D=NumLocalDim(J,p0,e);
==>
The Local Dimension:
0
def D=NumLocalDim(J,p1,e);
==>
The Local Dimension:
1
def D=NumLocalDim(J,p2,e);
==>
The Local Dimension:
2

```

### D.5.9 numerDecom.lib

Todos/Issues:

- does not follow the naming convention
- syntax errors in examples
- no test suite

**Library:** NumDecom.lib

**Purpose:** Numerical Decomposition of Ideals

**Overview:** The library contains procedures to compute numerical irreducible decomposition, and numerical primary decomposition of an algebraic variety defined by a polynomial system. The use of the library requires to install Bertini (<http://www.nd.edu/~sommese/bertini>).

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**Procedures:**

#### D.5.9.1 re2squ

Procedure from library `numerDecom.lib` (see [Section D.5.9 \[numerDecom.lib\]](#), page 1454).

**Usage:** `re2squ(ideal I);` I ideal

**Return:** ideal J defined by the polynomial system of the same number of polynomials and unknowns

**Example:**

```

LIB "numerDecom.lib";
ring r=0,(x,y,z),dp;
ideal I= x3+y4,z4+yx,xz+3x,x2y+z;
def D=re2squ(I);
setring D;
J;

```

### D.5.9.2 UseBertini

Procedure from library `numerDecom.lib` (see [Section D.5.9 \[numerDecom.lib\], page 1454](#)).

**Usage:**     `UseBertini(ideal H,string sv);`  
               H ideal, sv string of the variable of ring

**Return:**    text file called input used in Bertini to compute the solution of the homotopy function H that existed in file input.text

**Note:**       Need to define a start solution of H

**Example:**

```
LIB "numerDecom.lib";
ring r=0,(x,y,z),dp;
ideal I= x3+y4,z4+yx,xz+3x,x2y+z;
string sv=varstr(basing);
def A=UseBertini(I,sv);
```

### D.5.9.3 Singular2bertini

Procedure from library `numerDecom.lib` (see [Section D.5.9 \[numerDecom.lib\], page 1454](#)).

**Usage:**     `Singular2bertini(list L);`L a list

**Return:**    text file called start

**Note:**       adopting the list L to be as a start solution of the homotopy function in Bertini

**Example:**

```
LIB "numerDecom.lib";
ring r=(complex,16,I),(x,y,z),dp;
list L=list(1,2,3),list(4,5,6+I*2);
def D=Singular2bertini(L);
```

### D.5.9.4 bertini2Singular

Procedure from library `numerDecom.lib` (see [Section D.5.9 \[numerDecom.lib\], page 1454](#)).

**Usage:**     `bertini2Singular(string snp, int q);`  
               snp string, q=nvars(basing) integer

**Return:**    re list of the solutions of the homotopy function computed by Bertini

**Example:**

```
LIB "numerDecom.lib";
ring r = 0,(a,b,c),ds;
int q=nvars(basing);
def T=bertini2Singular("nonsingular_solutions",q);
re;
```

### D.5.9.5 ReJunkUseHomo

Procedure from library `numerDecom.lib` (see [Section D.5.9 \[numerDecom.lib\], page 1454](#)).

**Usage:**     `ReJunkUseHomo(ideal I, ideal L, list W, list w);`  
               I ideal, L list of generic linear polynomials  $\{l_1, \dots, l_i\}$ , W list of a subset of the solution set of the generic slicing  $V(L)$  with  $V(J)$ , w list of a point in  $V(J)$

**Return:**  $t=1$  if  $w$  on an  $i$ -dimensional component of  $V(I)$ ,  
otherwise  $t=0$ . Where  $i=\text{size}(L)$

**Example:**

```
LIB "numerDecom.lib";
ring r=(complex,16,I),(x,y,z),dp;
poly f1=(x2+y2+z2-6)*(x-y)*(x-1);
poly f2=(x2+y2+z2-6)*(x-z)*(y-2);
poly f3=(x2+y2+z2-6)*(x-y)*(x-z)*(z-3);
ideal J=f1,f2,f3;
poly l1=15x+16y+6z+17;
poly l2=2x+14y+4z+18;
ideal L=l1,l2;
list W1=list(0.5372775295412116,-0.7105339291010922,-2.2817700129167831+I*0),list(0.
list w=list(2,2,-131666666/100000000);
def D=ReJunkUseHomo(J,L,W1,w);
setring D;
t;
```

### D.5.9.6 JuReTopDim

Procedure from library `numerDecom.lib` (see [Section D.5.9 \[numerDecom.lib\], page 1454](#)).

**Return:**  $t=1$  if  $w$  on a  $d$ -dimensional component of  $V(I)$ , otherwise  $t=0$ .

**Example:**

```
LIB "numerDecom.lib";
ring r=(complex,16,I),(x,y,z),dp;
poly f1=(x2+y2+z2-6)*(x-y)*(x-1);
poly f2=(x2+y2+z2-6)*(x-z)*(y-2);
poly f3=(x2+y2+z2-6)*(x-y)*(x-z)*(z-3);
ideal J=f1,f2,f3;
list w=list(0.5372775295412116,-0.7105339291010922,-2.2817700129167831);
def D=JuReTopDim(J,w,2,2);
setring D;
t;
```

### D.5.9.7 JuReZeroDim

Procedure from library `numerDecom.lib` (see [Section D.5.9 \[numerDecom.lib\], page 1454](#)).

**Return:**  $t=1$  if  $w$  on a positive-dimensional component of  $V(I)$ , i.e  $w$  is not isolated point in  $V(J)$

**Example:**

```
LIB "numerDecom.lib";
ring r=(complex,16,I),(x,y,z),dp;
poly f1=(x2+y2+z2-6)*(x-y)*(x-1);
poly f2=(x2+y2+z2-6)*(x-z)*(y-2);
poly f3=(x2+y2+z2-6)*(x-y)*(x-z)*(z-3);
ideal J=f1,f2,f3;
list w1=list(0.5372775295412116,-0.7105339291010922,-2.2817700129167831);
def D1=JuReZeroDim(J,w1,2);
setring D1;
t;
```

### D.5.9.8 WitSupSet

Procedure from library `numerDecom.lib` (see [Section D.5.9 \[numerDecom.lib\], page 1454](#)).

**Usage:**     `WitSupSet(ideal I); I ideal`

**Return:**    list of Witness point Super Sets  $W(i)$  for  $i=1,\dots,\dim(V(I))$ ,  $L$  list of generic linear polynomials and  $N(0)$  list of a polynomial system of the same number of polynomials and unknowns. // if  $W(i) = x$ , then  $V(I)$  has no component of dimension  $i$

**Example:**

```
LIB "numerDecom.lib";
ring r=0,(x,y,z),dp;
poly f1=(x2+y2+z2-6)*(x-y)*(x-1);
poly f2=(x2+y2+z2-6)*(x-z)*(y-2);
poly f3=(x2+y2+z2-6)*(x-y)*(x-z)*(z-3);
ideal I=f1,f2,f3;
def W=WitSupSet(I);
setring W;
W(2);
// witness point super set of a pure 2-dimensional component of V(I)
W(1);
// witness point super set of a pure 1-dimensional component of V(I)
W(0);
// witness point super set of a pure 0-dimensional component of V(I)
L;
// list of generic linear polynomials
```

### D.5.9.9 WitSet

Procedure from library `numerDecom.lib` (see [Section D.5.9 \[numerDecom.lib\], page 1454](#)).

**Usage:**     `WitSet(ideal I); I ideal`

**Return:**    lists  $W(0..d)$  of witness point sets of  $i$ -dimensional components of  $V(J)$  for  $i=0,\dots,d$  respectively, where  $d$  the dimension of  $V(J)$ ,  $L$  list of generic linear polynomials

**Note:**      if  $W(i)=x$ , then  $V(J)$  has no component of dimension  $i$

**Example:**

```
LIB "numerDecom.lib";
ring r=0,(x,y,z),dp;
poly f1=(x3+z)*(x2-y);
poly f2=(x3+y)*(x2-z);
poly f3=(x3+y)*(x3+z)*(z2-y);
ideal I=f1,f2,f3;
def W=WitSet(I);
setring W;
W(1);
// witness point set of a pure 1-dimensional component of V(I)
W(0);
// witness point set of a pure 0-dimensional component of V(I)
L;
// list of generic linear polynomials
```

### D.5.9.10 NumIrrDecom

Procedure from library `numerDecom.lib` (see [Section D.5.9 \[numerDecom.lib\]](#), page 1454).

**Return:**  $w(1), \dots, w(t)$  lists of irreducible witness point sets of irreducible components of  $V(J)$

**Example:**

```
LIB "numerDecom.lib";
ring r=0,(x,y,z),dp;
poly f1=(x2+y2+z2-6)*(x-y)*(x-1);
poly f2=(x2+y2+z2-6)*(x-z)*(y-2);
poly f3=(x2+y2+z2-6)*(x-y)*(x-z)*(z-3);
ideal I=f1,f2,f3;
list W=NumIrrDecom(I);
==>
Dimension
0
Number of Components
1
Dimension
1
Number of Components
3
Dimension
2
Number of Components
1
def A(0)=W[1];
// corresponding 0-dimensional components
setring A(0);
w(1);
// corresponding 0-dimensional irreducible component
==> 0-Witness point set (one point)
def A(1)=W[2];
// corresponding 1-dimensional components
setring A(1);
w(1);
// corresponding 1-dimensional irreducible component
==> 1-Witness point set (one point)
w(2);
// corresponding 1-dimensional irreducible component
==> 1-Witness point set (one point)
w(3);
// corresponding 1-dimensional irreducible component
==> 1-Witness point set (one point)
def A(2)=W[3];
// corresponding 2-dimensional components
setring A(2);
w(1);
// corresponding 2-dimensional irreducible component
==> 1-Witness point set (two points)
```



### D.5.9.11 defl

Procedure from library `numerDecom.lib` (see [Section D.5.9 \[numerDecom.lib\]](#), page 1454).

**Usage:** `defl(ideal I, int d);` I ideal, int d order of the deflation

**Return:** deflation ideal DI of I

**Example:**

```
LIB "numerDecom.lib";
ring r=0,(x,y,z),dp;
poly f1=z^2;
poly f2=z*(x^2+y);
ideal I=f1,f2;
def D=defl(I,1);
setring D;
DI;
```

### D.5.9.12 NumPrimDecom

Procedure from library `numerDecom.lib` (see [Section D.5.9 \[numerDecom.lib\]](#), page 1454).

**Usage:** `NumPrimDecom(ideal I,int d);` I ideal, d order of the deflation

**Return:** lists of the numerical primary decomposition

**Example:**

```
LIB "numerDecom.lib";
ring r=0,(x,y),dp;
poly f1=yx;
poly f2=x2;
ideal I=f1,f2;
def W=NumPrimDecom(I,1);
setring W;
w(1);
==> 1-Witness point set (one point)
w(2);
==> 1-Witness point set (one point)
```

### D.5.10 orbitparam.lib

**Library:** `orbitparam.lib`

**Purpose:** Parametrizing orbits of unipotent actions

**Authors:** J. Boehm, boehm at mathematik.uni-kl.de  
S. Papadakis, papadakis at math.ist.utl.pt

**Overview:** This library implements the theorem of Chevalley-Rosenlicht as stated in Theorem 3.1.4 of [Corwin, Greenleaf]. Given a set of strictly upper triangular  $n \times n$  matrices  $L_1, \dots, L_c$  which generate a Lie algebra as a vector space, and a vector  $v$  of size  $n$ , the function `parametrizeOrbit` constructs a parametrization of the orbit of  $v$  under the action of  $\exp(\langle L_1, \dots, L_c \rangle)$ .

To compute  $\exp$  of the Lie algebra elements corresponding to the parameters we require that the characteristic of the base field is zero or larger than  $n$ .

By determining the parameters from bottom to top  
this allows you to find an element in the orbit with (at least) as many zeros as the  
dimension of the orbit.

Note: Theorem 3.1.4 of [Corwin, Greenleaf] uses strictly lower triangular matrices.

#### References:

Laurence Corwin, Frederick P. Greenleaf: Representations of Nilpotent Lie Groups and their Applications: Volume 1, Part 1, Basic Theory and Examples, Cambridge University Press (2004).

#### Procedures:

### D.5.10.1 tangentGens

Procedure from library `orbitparam.lib` (see [Section D.5.10 \[orbitparam.lib\]](#), [page 1459](#)).

**Usage:** `tangentGens(L,v)`; L list, v matrix.

**Assume:** L is a list of strictly upper triangular  $n \times n$  matrices of same size. The vector space  $\langle L \rangle$  generated by the elements of L should be closed under the Lie bracket.  
v is matrix of constants of size  $n \times 1$ .

**Return:** list, with four entries  
- first entry is the dimension of the orbit of under the action of  $\exp(\langle L \rangle)$   
- second entry is a list generators of the tangent space of the orbit of v at v under the action of  $\exp(\langle L \rangle)$ . If the characteristic p of the ground field is positive, then n has to be smaller than p. The generators are elements of  $\langle L \rangle$ .  
- third entry is the list of matrices with the coefficient to obtain the generators as a linear combination of the elements of L  
- fourth entry is list of integers with entries in v which can be made zero by the action of  $\exp(\langle L \rangle)$

**Theory:** We apply the theorem of Chevalley-Rosenlicht.

#### Example:

```
LIB "orbitparam.lib";
ring R = 0,(x),dp;
matrix L1[3][3] = 0,1,0, 0,0,0, 0,0,0;
matrix L2[3][3] = 0,0,1, 0,0,0, 0,0,0;
matrix L3[3][3] = 0,1,1, 0,0,1, 0,0,0;
list L = L1,L2,L3;
matrix v[3][1] = 1,2,3;
tangentGens(L,v);
⇒ [1]:
⇒ 2
⇒ [2]:
⇒ [1]:
⇒ _[1,1]=0
⇒ _[1,2]=1/3
⇒ _[1,3]=1/3
⇒ _[2,1]=0
⇒ _[2,2]=0
⇒ _[2,3]=1/3
⇒ _[3,1]=0
⇒ _[3,2]=0
```

```

⇒      _[3,3]=0
⇒      [2]:
⇒      _[1,1]=0
⇒      _[1,2]=1/2
⇒      _[1,3]=0
⇒      _[2,1]=0
⇒      _[2,2]=0
⇒      _[2,3]=0
⇒      _[3,1]=0
⇒      _[3,2]=0
⇒      _[3,3]=0
⇒ [3]:
⇒      [1]:
⇒      _[1,1]=0
⇒      _[2,1]=0
⇒      _[3,1]=1/3
⇒      [2]:
⇒      _[1,1]=1/2
⇒      _[2,1]=0
⇒      _[3,1]=0
⇒ [4]:
⇒      [1]:
⇒      2
⇒      [2]:
⇒      1

```

### D.5.10.2 matrixExp

Procedure from library `orbitparam.lib` (see [Section D.5.10 \[orbitparam.lib\]](#), page 1459).

**Usage:**      `matrixExp(A)`;  $A$  matrix.

**Assume:**     $A$  is a nilpotent  $n \times n$  matrix.

If the characteristic  $p$  of the ground field is positive, then  $n$  has to be smaller than  $p$ .

**Return:**    `matrix, exp(A)`

**Theory:**    We compute the power series, which terminates since  $A$  is nilpotent.

**Example:**

```

LIB "orbitparam.lib";
ring R = 0,(x),dp;
matrix A[4][4] = 0,0,1,0, 0,0,1,0, 0,0,0,0;
matrixExp(A);
⇒ _[1,1]=1
⇒ _[1,2]=0
⇒ _[1,3]=1
⇒ _[1,4]=0
⇒ _[2,1]=0
⇒ _[2,2]=1
⇒ _[2,3]=1
⇒ _[2,4]=0
⇒ _[3,1]=0
⇒ _[3,2]=0
⇒ _[3,3]=1

```

```

↦ _[3,4]=0
↦ _[4,1]=0
↦ _[4,2]=0
↦ _[4,3]=0
↦ _[4,4]=1

```

### D.5.10.3 matrixLog

Procedure from library `orbitparam.lib` (see [Section D.5.10 \[orbitparam.lib\], page 1459](#)).

**Usage:** `matrixLog(A)`; A matrix.

**Assume:** A-E is a nilpotent  $n \times n$  matrix.  
If the characteristic  $p$  of the ground field is positive, then  $n$  has to be smaller than  $p$ .

**Return:** matrix,  $\log(A)$

**Theory:** We compute the power series, which terminates since A-E is nilpotent.

**Example:**

```

LIB "orbitparam.lib";
ring R = 0,(s,t),dp;
matrix A[3][3] = 1,s,st/2, 0,1,t, 0,0,1;
matrixLog(A);
↦ _[1,1]=0
↦ _[1,2]=s
↦ _[1,3]=0
↦ _[2,1]=0
↦ _[2,2]=0
↦ _[2,3]=t
↦ _[3,1]=0
↦ _[3,2]=0
↦ _[3,3]=0

```

### D.5.10.4 parametrizeOrbit

Procedure from library `orbitparam.lib` (see [Section D.5.10 \[orbitparam.lib\], page 1459](#)).

**Usage:** `parametrizeOrbit(L,v)`; L list, v matrix.

**Assume:** L is a list of strictly upper triangular  $n \times n$  matrices of same size. The vector space  $\langle L \rangle$  generated by the elements of L should be closed under the Lie bracket.  
v is matrix of constants of size  $n \times 1$ .  
The basering has at least  $\text{size}(L)$  variables. However we will only use  $\text{tangentGens}(L,v)[1]$  many of them.

**Return:** list, with four entries  
- int, dimension of the orbit  
- matrix A over the basering giving a parametrization of the orbit of v under the action of  $\exp(\langle L \rangle)$ .  
- list of integers, with the (row)-indices of entries which can be deleted by the action  
- the variables of the parametrization to solve for

**Theory:** We apply the theorem of Chevalley-Rosenlicht. First we determine tangent space generators, then apply `matrixExp` to the generators, and finally take the product to obtain the parametrization.

**Example:**

```

LIB "orbitparam.lib";
ring R = 0,(t(1..3)),dp;
matrix L1[3][3] = 0,1,0, 0,0,0, 0,0,0;
matrix L2[3][3] = 0,0,1, 0,0,0, 0,0,0;
matrix L3[3][3] = 0,1,1, 0,0,1, 0,0,0;
list L = L1,L2,L3;
matrix v[3][1] = 1,2,3;
parametrizeOrbit(L,v);
↳ [1]:
↳ 2
↳ [2]:
↳ _[1,1]=1/6*t(1)^2+5/3*t(1)+t(2)+1
↳ _[2,1]=t(1)+2
↳ _[3,1]=3
↳ [3]:
↳ [1]:
↳ 2
↳ [2]:
↳ 1
↳ [4]:
↳ [1]:
↳ t(1)
↳ [2]:
↳ t(2)
ring R1 = 0,(t(1..2)),dp;
matrix L1[4][4] = 0,1,0,0, 0,0,0,0, 0,0,0,1, 0,0,0,0;
matrix L2[4][4] = 0,0,1,0, 0,0,0,1, 0,0,0,0, 0,0,0,0;
list L = L1,L2;
matrix v[4][1] = 1,2,3,4;
parametrizeOrbit(L,v);
↳ [1]:
↳ 2
↳ [2]:
↳ _[1,1]=1/4*t(1)*t(2)+1/2*t(1)+3/4*t(2)+1
↳ _[2,1]=t(2)+2
↳ _[3,1]=t(1)+3
↳ _[4,1]=4
↳ [3]:
↳ [1]:
↳ 3
↳ [2]:
↳ 2
↳ [4]:
↳ [1]:
↳ t(1)
↳ [2]:
↳ t(2)

```

**D.5.10.5 maxZeros**

Procedure from library `orbitparam.lib` (see [Section D.5.10 \[orbitparam.lib\]](#), page 1459).

**Usage:** `maxZeros(L,v)`; L list, v matrix.

- Assume:**  $L$  is a list of strictly upper triangular  $n \times n$  matrices of same size. The vector space  $\langle L \rangle$  generated by the elements of  $L$  should be closed under the Lie bracket.  
 $v$  is matrix of constants of size  $n \times 1$ .  
The basering has at least  $\text{size}(L)$  variables. However we will only use  $\text{tangentGens}(L, v)[1]$  many of them.
- Return:** matrix of constants over the basering giving an element in the orbit of  $v$  under the action of  $\exp(\langle L \rangle)$  with (at least) as many zeros as the dimension of the orbit.
- Theory:** We apply `parametrizeOrbit` to obtain a parametrization of the orbit according to the theorem of Chevalley-Rosenlicht. By determining the parameters from bottom to top we find an element in the orbit with (at least) as many zeros as the dimension of the orbit.

**Example:**

```
LIB "orbitparam.lib";
ring R = 0,(x),dp;
matrix L1[3][3] = 0,1,0, 0,0,0, 0,0,0;
matrix L2[3][3] = 0,0,1, 0,0,0, 0,0,0;
matrix L3[3][3] = 0,1,1, 0,0,1, 0,0,0;
list L = L1,L2,L3;
matrix v[3][1] = 1,2,3;
maxZeros(L,v);
  ↪ _[1,1]=0
  ↪ _[2,1]=0
  ↪ _[3,1]=3
ring R1 = 0,(x),dp;
matrix L1[4][4] = 0,1,0,0, 0,0,0,0, 0,0,0,1, 0,0,0,0;
matrix L2[4][4] = 0,0,1,0, 0,0,0,1, 0,0,0,0, 0,0,0,0;
list L = L1,L2;
matrix v[4][1] = 1,2,3,4;
maxZeros(L,v);
  ↪ _[1,1]=-1/2
  ↪ _[2,1]=0
  ↪ _[3,1]=0
  ↪ _[4,1]=4
```

**D.5.11 paraplanecurves.lib****Library:** paraplanecurves.lib**Purpose:** Rational parametrization of rational plane curves

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**Overview:** Suppose  $C = \{f(x,y,z)=0\}$  is a rational plane curve, where  $f$  is homogeneous of degree  $n$  with coefficients in  $\mathbb{Q}$  and absolutely irreducible (these conditions are checked automatically.)  
After a first step, realized by a projective automorphism in the procedure `adjointIdeal`,  $C$  satisfies:  
-  $C$  does not have singularities at infinity  $z=0$ .

-  $C$  does not contain the point  $(0:1:0)$  (that is, the dehomogenization of  $f$  with respect to  $z$  is monic as a polynomial in  $y$ ).

Considering  $C$  in the chart  $z \neq 0$ , the algorithm regards  $x$  as transcendental and  $y$  as algebraic and computes an integral basis in  $C(x)[y]$  of the integral closure of  $C[x]$  in  $C(x,y)$  using the normalization algorithm from [Section D.4.23 \[normal\\_lib\], page 1184](#); see [Section D.4.12 \[integralbasis\\_lib\], page 1115](#). In a future edition of the library, also van Hoeij's algorithm for computing the integral basis will be available.

From the integral basis, the adjoint ideal is obtained by linear algebra. Alternatively, the algorithm starts with a local analysis of the singular locus of  $C$ . Then, for each primary component of the singular locus which does not correspond to ordinary multiple points or cusps, the integral basis algorithm is applied separately. The ordinary multiple points and cusps, in turn, are addressed by a straightforward direct algorithm. The adjoint ideal is obtained by intersecting all ideals obtained locally. The local variant of the algorithm is used by default.

The linear system corresponding to the adjoint ideal maps the curve birationally to a rational normal curve in  $P^{n-2}$ .

Iterating the anticanonical map, the algorithm projects the rational normal curve to  $PP1$  for  $n$  odd resp. to a conic  $C2$  in  $PP2$  for  $n$  even.

In case  $n$  is even, the algorithm tests whether there is a rational point on  $C2$  and if so gives a parametrization of  $C2$  which is defined over  $Q$ . Otherwise, the parametrization given is defined over a quadratic field extension of  $Q$ .

By inverting the birational map of  $C$  to  $PP1$  resp. to  $C2$ , a parametrization of  $C$  is obtained (defined over  $Q$  or the quadratic field extension).

#### References:

Janko Boehm: Parametrisierung rationaler Kurven, Diploma Thesis, <http://www.math.uni-sb.de/ag/schreyer/jb/diplom%20janko%20boehm.pdf>

Janko Boehm, Wolfram Decker, Santiago Laplagne, Gerhard Pfister: Local to global algorithms for the Gorenstein adjoint ideal of a curve, *Algorithmic and Experimental Methods in Algebra, Geometry, and Number Theory*, Springer 2018

Theo de Jong: An algorithm for computing the integral closure, *Journal of Symbolic Computation* 26 (3) (1998), p. 273-277

Gert-Martin Greuel, Santiago Laplagne, Frank Seelisch: Normalization of Rings, *Journal of Symbolic Computation* 9 (2010), p. 887-901

Mark van Hoeij: An Algorithm for Computing an Integral Basis in an Algebraic Function Field, *Journal of Symbolic Computation* 18 (1994), p. 353-363, <http://www.math.fsu.edu/~hoeij/papers/comments/jsc1994.html>

#### Procedures:

##### D.5.11.1 adjointIdeal

Procedure from library `paraplanecurves.lib` (see [Section D.5.11 \[paraplanecurves\\_lib\], page 1464](#)).

**Usage:** `adjointIdeal(f [, choices]);`  $f$  polynomial in three variables,  $choices$  optional list consisting of one integer or of one string or of one integer followed by one string.

Optional integer can be:

1: compute integral basis via normalization.

2: make local analysis of singularities first and apply normalization separately.

3: normalization via ideal quotient.

4: normalization via local ideal quotient.

The default is 2.

Optional string may contain substrings:

- rattestyes -> causes error message if curve is not rational.

- firstchecksdone -> prevents that check of assumptions will be done more than once.

**Assume:** The basering must be a polynomial ring in three variables, say  $x, y, z$ , with coefficients in  $\mathbb{Q}$ .

The polynomial  $f$  must be homogeneous and absolutely irreducible.

All these conditions will be checked automatically.

**Return:** ideal, the adjoint ideal of the curve defined by  $f$ .

**Theory:** Considering  $C$  in the chart  $z \neq 0$ , the algorithm regards  $x$  as transcendental and  $y$  as algebraic and computes an integral basis in  $C(x)[y]$  of the integral closure of  $C[x]$  in  $C(x, y)$  using the normalization algorithm from [Section D.4.23 \[normal.lib\], page 1184](#): see [Section D.4.12 \[integralbasis.lib\], page 1115](#). In a future edition of the library, also van Hoeij's algorithm for computing the integral basis will be available.

From the integral basis, the adjoint ideal is obtained by linear algebra. Alternatively, the algorithm starts with a local analysis of the singular locus of  $C$ . Then, for each primary component of the singular locus which does not correspond to ordinary multiple points or cusps, the integral basis algorithm is applied separately. The ordinary multiple points and cusps, in turn, are addressed by a straightforward direct algorithm. The adjoint ideal is obtained by intersecting all ideals obtained locally. The local variant of the algorithm is used by default.

**Example:**

```
LIB "paraplanecurves.lib";
ring R = 0, (x, y, z), dp;
poly f = y^8 - x^3 * (z + x)^5;
adjointIdeal(f);
↪ _[1]=y6
↪ _[2]=xy5+y5z
↪ _[3]=x2y4+xy4z
↪ _[4]=x3y3+2x2y3z+xy3z2
↪ _[5]=x4y2+3x3y2z+3x2y2z2+xy2z3
↪ _[6]=x5y+3x4yz+3x3yz2+x2yz3
↪ _[7]=x6+4x5z+6x4z2+4x3z3+x2z4
```

### D.5.11.2 invertBirMap

Procedure from library `paraplanecurves.lib` (see [Section D.5.11 \[paraplanecurves.lib\], page 1464](#)).

**Usage:** `invertBirMap(phi, I)`;  $\phi$  ideal,  $I$  ideal

**Assume:** The ideal  $\phi$  in the basering  $R$  represents a birational map of the variety given by the ideal  $I$  in  $R$  to its image in projective space  $P = \mathbb{P}^{(\text{size}(\phi)-1)}$ .

**Note:** The procedure might fail or give a wrong output if  $\phi$  does not define a birational map.

**Return:** ring, the coordinate ring of  $P$ , with an ideal named  $J$  and an ideal named  $\psi$ . The ideal  $J$  defines the image of  $\phi$ .



The ideal  $\psi$  gives the inverse of  $\phi$ .

Note that the entries of  $\psi$  should be considered as representatives of classes in the quotient ring  $R/J$ .

**Theory:** We compute the ideal  $I(G)$  in  $R^*S$  of the graph  $G$  of  $\phi$ .  
 The ideal  $J$  is given by the intersection of  $I(G)$  with  $S$ .  
 The map  $\psi$  is given by a relation mod  $J$  of those relations in  $I(G)$  which are linear in the variables of  $R$ .

**Example:**

```
LIB "paraplanecurves.lib";
ring R = 0,(x,y,z),dp;
poly f = y^8-x^3*(z+x)^5;
ideal adj = adjointIdeal(f);
def Rn = invertBirMap(adj,ideal(f));
⇒ // 'invertBirMap' created a ring together with two ideals J and psi.
⇒ // Supposing you typed, say, def RPn = invertBirMap(phi,I);
⇒ // you may access the ideals by typing
⇒ //      setring RPn; J; psi;
setring(Rn);
J;
⇒ J[1]=y(5)*y(6)-y(4)*y(7)
⇒ J[2]=y(4)*y(6)-y(3)*y(7)
⇒ J[3]=y(2)*y(6)-y(1)*y(7)
⇒ J[4]=y(4)*y(5)-y(2)*y(7)
⇒ J[5]=y(3)*y(5)-y(1)*y(7)
⇒ J[6]=y(1)*y(5)-y(7)^2
⇒ J[7]=y(4)^2-y(1)*y(7)
⇒ J[8]=y(3)*y(4)-y(1)*y(6)
⇒ J[9]=y(2)*y(4)-y(7)^2
⇒ J[10]=y(1)*y(4)-y(6)*y(7)
⇒ J[11]=y(2)*y(3)-y(6)*y(7)
⇒ J[12]=y(1)*y(3)-y(6)^2
⇒ J[13]=y(2)^2-y(5)*y(7)
⇒ J[14]=y(1)*y(2)-y(4)*y(7)
⇒ J[15]=y(1)^2-y(3)*y(7)
⇒ J[16]=y(1)*y(6)^2-y(3)^2*y(7)
⇒ J[17]=y(6)^4-y(3)^3*y(7)
psi;
⇒ psi[1]=-y(6)^2
⇒ psi[2]=-y(4)*y(7)
⇒ psi[3]=y(6)^2-y(5)*y(7)
```

### D.5.11.3 paraPlaneCurve

Procedure from library `paraplanecurves.lib` (see [Section D.5.11 \[paraplanecurves.lib\]](#), [page 1464](#)).

**Usage:** `paraPlaneCurve(f [, s]);`  $f$  poly,  $s$  optional string  
 optional string  $s$  can be:  
 'normal': compute integral basis via normalization.  
 'local': make local analysis of singularities first and apply normalization separately.

The default is 2.

- Assume:** The basering must be a polynomial ring in three variables, say  $x, y, z$ , with coefficients in  $\mathbb{Q}$ .  
 The polynomial  $f$  must be homogeneous and absolutely irreducible.  
 The curve  $C = \{f = 0\}$  must be rational, i.e., have geometric genus 0 (see [Section D.4.23.7 \[genus\]](#), [page 1201](#)).  
 These conditions will be checked automatically.
- Return:** ring with an ideal `PARA` which contains a rational parametrization of the rational plane curve given by  $f$ ; the ground field of the returned polynomial ring is either  $\mathbb{Q}$  or some algebraic extension  $\mathbb{Q}(a)$ ; `PARA` consists of three generators that parametrize the three coordinates of the rational curve
- Theory:** After a first step, realized by a projective automorphism in the procedure `adjointIdeal`,  $C$  satisfies:  
 -  $C$  does not have singularities at infinity  $z=0$ .  
 -  $C$  does not contain the point  $(0:1:0)$  (that is, the dehomogenization of  $f$  with respect to  $z$  is monic as a polynomial in  $y$ ).  
 Considering  $C$  in the chart  $z \neq 0$ , the algorithm regards  $x$  as transcendental and  $y$  as algebraic and computes an integral basis in  $\mathbb{C}(x)[y]$  of the integral closure of  $\mathbb{C}[x]$  in  $\mathbb{C}(x, y)$  using the normalization algorithm from [Section D.4.23 \[normal\\_lib\]](#), [page 1184](#): see [Section D.4.12 \[integralbasis\\_lib\]](#), [page 1115](#). In a future edition of the library, also van Hoeij's algorithm for computing the integral basis will be available.  
 From the integral basis, the adjoint ideal is obtained by linear algebra. Alternatively, the algorithm starts with a local analysis of the singular locus of  $C$ . Then, for each primary component of the singular locus which does not correspond to ordinary multiple points or cusps, the integral basis algorithm is applied separately. The ordinary multiple points and cusps, in turn, are addressed by a straightforward direct algorithm. The adjoint ideal is obtained by intersecting all ideals obtained locally. The local variant of the algorithm is used by default.  
 The linear system corresponding to the adjoint ideal maps the curve birationally to a rational normal curve in  $\mathbb{P}^{(n-2)}$ .  
 Iterating the anticanonical map, the algorithm projects the rational normal curve to  $\mathbb{P}^1$  for  $n$  odd resp. to a conic  $C_2$  in  $\mathbb{P}^2$  for  $n$  even.  
 In case  $n$  is even, the algorithm tests whether there is a rational point on  $C_2$  and if so gives a parametrization of  $C_2$  which is defined over  $\mathbb{Q}$ . Otherwise the parametrization is defined over a quadratic field extension of  $\mathbb{Q}$ .  
 By inverting the birational map of  $C$  to  $\mathbb{P}^1$  resp. to  $C_2$ , a parametrization of  $C$  is obtained (defined over  $\mathbb{Q}$  or the quadratic field extension).

**Example:**

```
LIB "paraplanecurves.lib";
ring R = 0,(x,y,z),dp;
poly f1 = 1/2*x^5+x^2*y*z^2+x^3*y*z+1/2*x*y^2*z^2-2*x*y^3*z+y^5;
def Rp1 = paraPlaneCurve(f1);
⇒ // 'paraPlaneCurve' created a ring together with an ideal PARA.
⇒ // Supposing you typed, say, def Rp1 = paraPlaneCurve(f);
⇒ // you may access the ideal by typing
⇒ //      setring Rp1; PARA;
setring Rp1;
PARA;
```

```

⇒ PARA[1]=-4s4t+2s3t2
⇒ PARA[2]=2s2t3-st4
⇒ PARA[3]=4s5-t5
setring R;
poly f2 = x6+3x4y2+3x2y4+y6-4x4z2-34x3yz2-7x2y2z2+12xy3z2+6y4z2;
f2 = f2+36x2z4+36xyz4+9y2z4;
def Rp2 = paraPlaneCurve(f2);
⇒ // 'paraPlaneCurve' created a ring together with an ideal PARA.
⇒ // Supposing you typed, say, def RP1 = paraPlaneCurve(f);
⇒ // you may access the ideal by typing
⇒ //      setring RP1; PARA;
setring Rp2;
PARA;
⇒ PARA[1]=(-a)*s6+(-3821910a+568836)*s5t+(-17261814423635a-17036342693900)*\
s4t2+(-40791433831085325700a-41443502254869224120)*s3t3+(7095798487219992\
1706914985a-63670892810117870548425400)*s2t4+(119395301366776391295025433\
032250a-70343024051936434031711433716044)*st5+(14536738745802049687386582\
3959936915531a+80558165687949544164912330361180107460)*t6
⇒ PARA[2]=s6+(568836a+3821910)*s5t+(20661581653300a+24799008638835)*s4t2+(5\
7466832045400548680a+45699794797119440900)*s3t3+(-82140322496947054109676\
600a+33367675319234688316737815)*s2t4+(-153764457242487188699049348352844\
a+37012366361272671947391736404550)*st5+(-1589516820449857995417712398245\
02169340a+125264680768653363798746666533010931669)*t6
⇒ PARA[3]=(90983469924655936000a+37676837794369664000)*s3t3+(18458128762166\
9915274432000a+46121405018308889263872000)*s2t4+(121948097419855203536634\
716352000a+11879229621327449840239705728000)*st5+(26268443692822012344779\
010112242496000a-1331506747881842263336367321772544000)*t6

```

#### D.5.11.4 rncAntiCanonicalMap

Procedure from library `paraplanecurves.lib` (see [Section D.5.11 \[paraplanecurves.lib\]](#), [page 1464](#)).

**Usage:** `rncAntiCanonicalMap(I);` I ideal

**Assume:** I is a homogeneous ideal in the basering  
defining a rational normal curve C in  $\mathbb{P}P^n$ .

**Note:** The procedure will fail or give a wrong output if I is not the ideal of a rational normal curve.

**Return:** ideal defining the anticanonical map  $C \rightarrow \mathbb{P}P^{(n-2)}$ .  
Note that the entries of the ideal should be considered as representatives of elements in  $R/I$ , where R is the basering.

**Theory:** The anti-canonical map of a rational normal curve  
maps C isomorphically to a rational normal curve in  $\mathbb{P}P^{(n-2)}$ .

**Example:**

```

LIB "paraplanecurves.lib";
ring R = 0,(x,y,z),dp;
poly f = y^8-x^3*(z+x)^5;
ideal adj = adjointIdeal(f);
def Rn = mapToRatNormCurve(f,adj);
⇒ //'mapToRatNorm' created a ring together with an ideal RNC.

```

```

⇒ // Supposing you typed, say, def RPn = mapToRatNorm(f,AI);
⇒ // you may access the ideal by typing
⇒ //      setring RPn; RNC;
setring(Rn);
RNC;
⇒ RNC[1]=y(5)*y(6)-y(4)*y(7)
⇒ RNC[2]=y(4)*y(6)-y(3)*y(7)
⇒ RNC[3]=y(2)*y(6)-y(1)*y(7)
⇒ RNC[4]=y(4)*y(5)-y(2)*y(7)
⇒ RNC[5]=y(3)*y(5)-y(1)*y(7)
⇒ RNC[6]=y(1)*y(5)-y(7)^2
⇒ RNC[7]=y(4)^2-y(1)*y(7)
⇒ RNC[8]=y(3)*y(4)-y(1)*y(6)
⇒ RNC[9]=y(2)*y(4)-y(1)*y(5)
⇒ RNC[10]=y(1)*y(4)-y(6)*y(7)
⇒ RNC[11]=y(2)*y(3)-y(6)*y(7)
⇒ RNC[12]=y(1)*y(3)-y(6)^2
⇒ RNC[13]=y(2)^2-y(5)*y(7)
⇒ RNC[14]=y(1)*y(2)-y(4)*y(7)
⇒ RNC[15]=y(1)^2-y(3)*y(7)
⇒ RNC[16]=y(1)*y(6)^2-y(3)^2*y(7)
⇒ RNC[17]=y(6)^4-y(3)^3*y(7)
rncAntiCanonicalMap(RNC);
⇒ _[1]=y(1)
⇒ _[2]=-y(2)
⇒ _[3]=-y(5)
⇒ _[4]=-y(4)
⇒ _[5]=-y(7)

```

### D.5.11.5 rationalPointConic

Procedure from library `paraplanecurves.lib` (see [Section D.5.11 \[paraplanecurves.lib\]](#), [page 1464](#)).

**Usage:** `rationalPointConic(p); p poly`

**Assume:** assumes that  $p$  is an irreducible quadratic polynomial in the first three ring variables; ground field is expected to be  $\mathbb{Q}$ .

**Return:** The method finds a point on the given conic. There are two possibilities:

- 1) There is a rational point on the curve.
- 2) There is no rational point on the curve.

In the second case, the method creates a modification of the current basering which is a polynomial ring over some quadratic field extension  $\mathbb{Q}(a)$  of  $\mathbb{Q}$ . Apart from the replacement of  $\mathbb{Q}$  by  $\mathbb{Q}(a)$ , the new polynomial ring,  $R$  say, is the same as the original basering. (In the first case,  $R$  is identical with the basering.) In both cases, the method will then define a  $(1 \times 3)$  matrix named 'point' which lives in  $R$  and which contains the coordinates of the desired point on  $q$ .

Finally, the method returns the ring  $R$  (which will in the 1st case be the original base ring).

**Example:**

```

LIB "paraplanecurves.lib";
ring R = 0, (x,y,z), dp;

```

```

system("random", 4711);
poly p = x^2 + 2*y^2 + 5*z^2 - 4*x*y + 3*x*z + 17*y*z;
def S = rationalPointConic(p); // quadratic field extension,
// minpoly = a^2 - 2
testPointConic(p, S);
⇒ conic: x2-4xy+2y2+3xz+17yz+5z2
⇒ point: (-1/4a), (-1/4a+1/4), 0
⇒ minpoly: (a2-2)
⇒ 1
setring R;
p = x^2 - 1857669520 * y^2 + 86709575222179747132487270400 * z^2;
S = rationalPointConic(p); // same as current basering,
// no extension needed
testPointConic(p, S);
⇒ conic: x2-1857669520y2+86709575222179747132487270400z2
⇒ point: 819398304609269135405871919690911280, 4998995645707055256041604614\
16220, 73117135886813712113057
⇒ 1

```

### D.5.11.6 mapToRatNormCurve

Procedure from library `paraplanecurves.lib` (see [Section D.5.11 \[paraplanecurves.lib\]](#), [page 1464](#)).

**Usage:** `mapToRatNormCurve(f, AI);`  $f$  polynomial,  $AI$  ideal

**Assume:** The polynomial  $f$  is homogeneous in three variables and absolutely irreducible.  
The plane curve  $C$  defined by  $f$  is rational.  
The ideal  $AI$  is the adjoint ideal of  $C$ .

**Return:** ring with an ideal  $RNC$ .

**Example:**

```

LIB "paraplanecurves.lib";
ring R = 0,(x,y,z),dp;
poly f = y^8-x^3*(z+x)^5;
ideal adj = adjointIdeal(f);
def Rn = mapToRatNormCurve(f,adj);
⇒ //'mapToRatNorm' created a ring together with an ideal RNC.
⇒ // Supposing you typed, say, def RPn = mapToRatNorm(f,AI);
⇒ // you may access the ideal by typing
⇒ //      setring RPn; RNC;
setring(Rn);
RNC;
⇒ RNC[1]=y(5)*y(6)-y(4)*y(7)
⇒ RNC[2]=y(4)*y(6)-y(3)*y(7)
⇒ RNC[3]=y(2)*y(6)-y(1)*y(7)
⇒ RNC[4]=y(4)*y(5)-y(2)*y(7)
⇒ RNC[5]=y(3)*y(5)-y(1)*y(7)
⇒ RNC[6]=y(1)*y(5)-y(7)^2
⇒ RNC[7]=y(4)^2-y(1)*y(7)
⇒ RNC[8]=y(3)*y(4)-y(1)*y(6)
⇒ RNC[9]=y(2)*y(4)-y(1)*y(5)
⇒ RNC[10]=y(1)*y(4)-y(6)*y(7)
⇒ RNC[11]=y(2)*y(3)-y(6)*y(7)

```

```

⇒ RNC[12]=y(1)*y(3)-y(6)^2
⇒ RNC[13]=y(2)^2-y(5)*y(7)
⇒ RNC[14]=y(1)*y(2)-y(4)*y(7)
⇒ RNC[15]=y(1)^2-y(3)*y(7)
⇒ RNC[16]=y(1)*y(6)^2-y(3)^2*y(7)
⇒ RNC[17]=y(6)^4-y(3)^3*y(7)

```

### D.5.11.7 rncItProjOdd

Procedure from library `paraplanecurves.lib` (see [Section D.5.11 \[paraplanecurves.lib\]](#), [page 1464](#)).

**Usage:** `rncItProjOdd(I);` I ideal

**Assume:** I is a homogeneous ideal in the basering with  $n+1$  variables defining a rational normal curve C in  $\mathbb{P}^n$  with  $n$  odd.

**Note:** The procedure will fail or give a wrong output if I is not the ideal of a rational normal curve. It will test whether  $n$  is odd.

**Return:** ideal PHI defining an isomorphic projection of C to  $\mathbb{P}^1$ .  
Note that the entries of PHI should be considered as representatives of elements in  $R/I$ , where  $R$  is the basering.

**Theory:** We iterate the procedure [Section D.5.11.4 \[rncAntiCanonicalMap\]](#), [page 1469](#) to obtain PHI.

**Example:**

```

LIB "paraplanecurves.lib";
ring R = 0, (x,y,z), dp;
poly f = -x7-10x5y2-10x4y3-3x3y4+8x2y5+7xy6+11y7+3x6+10x5y +30x4y2
+26x3y3-13x2y4-29xy5-33y6-3x5-20x4y-33x3y2-8x2y3+37xy4+33y5
+x4+10x3y+13x2y2-15xy3-11y4;
f = homog(f,z);
ideal adj = adjointIdeal(f);
def Rn = mapToRatNormCurve(f,adj);
⇒ // 'mapToRatNorm' created a ring together with an ideal RNC.
⇒ // Supposing you typed, say, def RPn = mapToRatNorm(f,AI);
⇒ // you may access the ideal by typing
⇒ //      setring RPn; RNC;
setring(Rn);
RNC;
⇒ RNC[1]=y(4)*y(5)-y(3)*y(6)
⇒ RNC[2]=y(2)*y(5)-y(1)*y(6)
⇒ RNC[3]=y(4)^2-y(2)*y(6)
⇒ RNC[4]=y(3)*y(4)-y(1)*y(6)
⇒ RNC[5]=11*y(1)*y(4)+7*y(2)*y(4)+8*y(1)*y(6)-3*y(2)*y(6)-10*y(3)*y(6)-10*y\
(4)*y(6)-y(6)^2
⇒ RNC[6]=y(3)^2-y(1)*y(5)
⇒ RNC[7]=11*y(2)*y(3)+7*y(2)*y(4)+8*y(1)*y(6)-3*y(2)*y(6)-10*y(3)*y(6)-10*y\
(4)*y(6)-y(6)^2
⇒ RNC[8]=11*y(1)*y(3)+7*y(1)*y(4)+8*y(1)*y(5)-10*y(3)*y(5)-3*y(1)*y(6)-10*y\
(3)*y(6)-y(5)*y(6)
⇒ RNC[9]=121*y(1)*y(2)+77*y(2)^2-89*y(2)*y(4)-174*y(1)*y(6)-86*y(2)*y(6)+80\
*y(3)*y(6)+69*y(4)*y(6)+8*y(6)^2

```

```

    ↪ RNC[10]=1331*y(1)^2-539*y(2)^2+1246*y(2)*y(4)-1914*y(1)*y(5)+880*y(3)*y(5\
      )+984*y(1)*y(6)+335*y(2)*y(6)-691*y(3)*y(6)-1373*y(4)*y(6)+88*y(5)*y(6)-1\
      45*y(6)^2
    rncItProjOdd(RNC);
    ↪ _[1]=121*y(3)+77*y(4)
    ↪ _[2]=-11*y(5)-7*y(6)

```

See also: [Section D.5.11.8 \[rncItProjEven\]](#), page 1473.

### D.5.11.8 rncItProjEven

Procedure from library `paraplanecurves.lib` (see [Section D.5.11 \[paraplanecurves.lib\]](#), page 1464).

**Usage:** `rncItProjEven(I); I ideal`

**Assume:** `I` is a homogeneous ideal in the basering with  $n+1$  variables defining a rational normal curve  $C$  in  $\mathbb{P}^n$  with  $n$  even.

**Note:** The procedure will fail or give a wrong output if `I` is not the ideal of a rational normal curve. It will test whether  $n$  is odd.

**Return:** ring with an ideal `CONIC` defining a conic  $C_2$  in  $\mathbb{P}^2$ .  
In addition, an ideal `PHI` in the basering defining an isomorphic projection of  $C$  to  $C_2$  will be exported.  
Note that the entries of `PHI` should be considered as representatives of elements in  $R/I$ , where  $R$  is the basering.

**Theory:** We iterate the procedure [Section D.5.11.4 \[rncAntiCanonicalMap\]](#), page 1469 to obtain `PHI`.

**Example:**

```

LIB "paraplanecurves.lib";
ring R = 0,(x,y,z),dp;
poly f = y^8-x^3*(z+x)^5;
ideal adj = adjointIdeal(f);
def Rn = mapToRatNormCurve(f,adj);
↪ //'mapToRatNorm' created a ring together with an ideal RNC.
↪ // Supposing you typed, say, def RPn = mapToRatNorm(f,AI);
↪ // you may access the ideal by typing
↪ //      setring RPn; RNC;
setring(Rn);
RNC;
↪ RNC[1]=y(5)*y(6)-y(4)*y(7)
↪ RNC[2]=y(4)*y(6)-y(3)*y(7)
↪ RNC[3]=y(2)*y(6)-y(1)*y(7)
↪ RNC[4]=y(4)*y(5)-y(2)*y(7)
↪ RNC[5]=y(3)*y(5)-y(1)*y(7)
↪ RNC[6]=y(1)*y(5)-y(7)^2
↪ RNC[7]=y(4)^2-y(1)*y(7)
↪ RNC[8]=y(3)*y(4)-y(1)*y(6)
↪ RNC[9]=y(2)*y(4)-y(1)*y(5)
↪ RNC[10]=y(1)*y(4)-y(6)*y(7)
↪ RNC[11]=y(2)*y(3)-y(6)*y(7)
↪ RNC[12]=y(1)*y(3)-y(6)^2
↪ RNC[13]=y(2)^2-y(5)*y(7)

```

```

    ↪ RNC[14]=y(1)*y(2)-y(4)*y(7)
    ↪ RNC[15]=y(1)^2-y(3)*y(7)
    ↪ RNC[16]=y(1)*y(6)^2-y(3)^2*y(7)
    ↪ RNC[17]=y(6)^4-y(3)^3*y(7)
    def Rc = rncItProjEven(RNC);
    PHI;
    ↪ PHI[1]=-y(7)
    ↪ PHI[2]=-y(2)
    ↪ PHI[3]=-y(5)
    setring Rc;
    CONIC;
    ↪ y(2)^2-y(1)*y(3)

```

See also: [Section D.5.11.7 \[rncItProjOdd\]](#), page 1472.

### D.5.11.9 paraConic

Procedure from library `paraplanecurves.lib` (see [Section D.5.11 \[paraplanecurves.lib\]](#), page 1464).

**Usage:** `paraConic(q); q poly`

**Assume:** The basering must be a polynomial ring in three variables with coefficients in  $\mathbb{Q}$ . The polynomial  $q$  must be homogeneous of degree 2 and absolutely irreducible.

**Note:** The procedure might fail or give a wrong output if the assumptions do not hold.

**Return:** ring with an ideal `PARACONIC`. The ring should be considered as the homogeneous coordinate ring of  $\mathbb{P}^1$ , the ideal defines a rational parametrization  $\mathbb{P}^1 \rightarrow C2 = \{q=0\}$ .

**Theory:** We compute a point on  $C2$  via [Section D.5.11.5 \[rationalPointConic\]](#), page 1470. The pencil of lines through this point projects  $C2$  birationally to  $\mathbb{P}^1$ . Inverting the projection gives the result.

**Example:**

```

LIB "paraplanecurves.lib";
ring R = 0,(x,y,z),dp;
poly f = y^8-x^3*(z+x)^5;
ideal adj = adjointIdeal(f);
def Rn = invertBirMap(adj,ideal(f));
↪ // 'invertBirMap' created a ring together with two ideals J and psi.
↪ // Supposing you typed, say, def RPn = invertBirMap(phi,I);
↪ // you may access the ideals by typing
↪ //      setring RPn; J; psi;
setring(Rn);
J;
↪ J[1]=y(5)*y(6)-y(4)*y(7)
↪ J[2]=y(4)*y(6)-y(3)*y(7)
↪ J[3]=y(2)*y(6)-y(1)*y(7)
↪ J[4]=y(4)*y(5)-y(2)*y(7)
↪ J[5]=y(3)*y(5)-y(1)*y(7)
↪ J[6]=y(1)*y(5)-y(7)^2
↪ J[7]=y(4)^2-y(1)*y(7)
↪ J[8]=y(3)*y(4)-y(1)*y(6)

```



```

⇒ J[9]=y(2)*y(4)-y(7)^2
⇒ J[10]=y(1)*y(4)-y(6)*y(7)
⇒ J[11]=y(2)*y(3)-y(6)*y(7)
⇒ J[12]=y(1)*y(3)-y(6)^2
⇒ J[13]=y(2)^2-y(5)*y(7)
⇒ J[14]=y(1)*y(2)-y(4)*y(7)
⇒ J[15]=y(1)^2-y(3)*y(7)
⇒ J[16]=y(1)*y(6)^2-y(3)^2*y(7)
⇒ J[17]=y(6)^4-y(3)^3*y(7)
def Rc = rncItProjEven(J);
PHI;
⇒ PHI[1]=-y(7)
⇒ PHI[2]=-y(2)
⇒ PHI[3]=-y(5)
setring Rc;
CONIC;
⇒ y(2)^2-y(1)*y(3)
def RPc = paraConic(CONIC);
⇒ // 'paraConic' created a ring together with an ideal RNC.
⇒ // Supposing you typed, say, def RP1 = paraConic(q);
⇒ // you may access the ideal by typing
⇒ //      setring RP1; PARACONIC;
setring RPc;
PARACONIC;
⇒ PARACONIC[1]=s2
⇒ PARACONIC[2]=st
⇒ PARACONIC[3]=t2

```

See also: [Section D.5.11.5 \[rationalPointConic\]](#), page 1470.

### D.5.11.10 testParametrization

Procedure from library `paraplanecurves.lib` (see [Section D.5.11 \[paraplanecurves.lib\]](#), page 1464).

**Usage:** `testParametrization(f, rTT)`;  $f$  poly,  $rTT$  ring

**Assume:** The assumptions on the basering and the polynomial  $f$  are as required by [Section D.5.11.3 \[paraPlaneCurve\]](#), page 1467. The ring  $rTT$  has two variables and contains an ideal  $PARA$  (such as the ring obtained by applying [Section D.5.11.3 \[paraPlaneCurve\]](#), page 1467 to  $f$ ).

**Return:** int which is 1 if  $PARA$  defines a parametrization of the curve  $\{f=0\}$  and 0, otherwise.

**Theory:** We compute the polynomial defining the image of  $PARA$  and compare it with  $f$ .

**Example:**

```

LIB "paraplanecurves.lib";
ring R = 0,(x,y,z),dp;
poly f = y^8-x^3*(z+x)^5;
def RP1 = paraPlaneCurve(f);
⇒ // 'paraPlaneCurve' created a ring together with an ideal PARA.
⇒ // Supposing you typed, say, def RP1 = paraPlaneCurve(f);
⇒ // you may access the ideal by typing
⇒ //      setring RP1; PARA;

```

```
testParametrization(f, RP1);
↪ 1
```

#### D.5.11.11 testPointConic

Procedure from library `paraplanecurves.lib` (see [Section D.5.11 \[paraplanecurves.lib\]](#), [page 1464](#)).

**Usage:** `testPointConic(p, r);` p poly, r ring

**Assume:** assumes that p is a homogeneous quadratic polynomial in the first three ring variables of the current basering;  
Assumes that there is a (1x3) matrix named 'point' in r with entries from the ground field of r.

**Return:** returns 1 iff the point named 'point', residing in r, lies on the conic given by p; 0 otherwise

**Note:** This method temporarily changes the basering to r. Afterwards, the basering will be the same as before.

**Example:**

```
LIB "paraplanecurves.lib";
ring R = 0, (x,y,z), dp;
system("random", 4711);
poly p = x^2 + 2*y^2 + 5*z^2 - 4*x*y + 3*x*z + 17*y*z;
def S = rationalPointConic(p);
if (testPointConic(p, S) == 1)
↪ conic: x2-4xy+2y2+3xz+17yz+5z2
↪ point: (-1/4a), (-1/4a+1/4), 0
↪ minpoly: (a2-2)
{ "point lies on conic"; }
↪ point lies on conic
else
{ "point does not lie on conic"; }
```

#### D.5.12 resbinomial.lib

Todos/Issues:

formatting is inappropriate

avoid export

bad names(or should be static): identifyvars, elimrep, convertdata, lcmofall, genoutput, salida, iniD, reslist, sumlist, dividelist, createlist

**Library:** `resbinomial.lib`

**Purpose:** Combinatorial algorithm of resolution of singularities of binomial ideals in arbitrary characteristic. Binomial resolution algorithm of Blanco

**Authors:** R. Blanco, [mariarocio.blanco@uclm.es](mailto:mariarocio.blanco@uclm.es),  
G. Pfister, [pfister@mathematik.uni-kl.de](mailto:pfister@mathematik.uni-kl.de)

**Procedures:**

### D.5.12.1 BINresol

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:**     `BINresol(J); J ideal`

**Return:**    E-resolution of singularities of a binomial ideal `J` in terms of the affine charts, see example

**Example:**

```
LIB "resbinomial.lib";
ring r = 0,(x(1..2)),dp;
ideal J=x(1)^2-x(2)^3;
list B=BINresol(J);
B[1]; // list of final charts
⇒ [1]:
⇒ // coefficients: QQ
⇒ // number of vars : 2
⇒ //      block 1 : ordering dp
⇒ //      : names y(1) y(2)
⇒ //      block 2 : ordering C
⇒ [2]:
⇒ // coefficients: QQ
⇒ // number of vars : 2
⇒ //      block 1 : ordering dp
⇒ //      : names y(1) y(2)
⇒ //      block 2 : ordering C
⇒ [3]:
⇒ // coefficients: QQ
⇒ // number of vars : 2
⇒ //      block 1 : ordering dp
⇒ //      : names y(1) x(2)
⇒ //      block 2 : ordering C
⇒ [4]:
⇒ // coefficients: QQ
⇒ // number of vars : 2
⇒ //      block 1 : ordering dp
⇒ //      : names x(1) y(2)
⇒ //      block 2 : ordering C
B[2]; // list of all charts
⇒ [1]:
⇒ // coefficients: QQ
⇒ // number of vars : 2
⇒ //      block 1 : ordering dp
⇒ //      : names x(1) x(2)
⇒ //      block 2 : ordering C
⇒ [2]:
⇒ // coefficients: QQ
⇒ // number of vars : 2
⇒ //      block 1 : ordering dp
⇒ //      : names y(1) y(2)
⇒ //      block 2 : ordering C
⇒ [3]:
⇒ // coefficients: QQ
```

```

⇒ // number of vars : 2
⇒ //      block 1 : ordering dp
⇒ //      : names  x(1) x(2)
⇒ //      block 2 : ordering C
⇒ [4]:
⇒ // coefficients: QQ
⇒ // number of vars : 2
⇒ //      block 1 : ordering dp
⇒ //      : names  y(1) y(2)
⇒ //      block 2 : ordering C
⇒ [5]:
⇒ // coefficients: QQ
⇒ // number of vars : 2
⇒ //      block 1 : ordering dp
⇒ //      : names  x(1) x(2)
⇒ //      block 2 : ordering C
⇒ [6]:
⇒ // coefficients: QQ
⇒ // number of vars : 2
⇒ //      block 1 : ordering dp
⇒ //      : names  y(1) x(2)
⇒ //      block 2 : ordering C
⇒ [7]:
⇒ // coefficients: QQ
⇒ // number of vars : 2
⇒ //      block 1 : ordering dp
⇒ //      : names  x(1) y(2)
⇒ //      block 2 : ordering C
ring r = 2,(x(1..3)),dp;
⇒ // ** redefining r (ring r = 2,(x(1..3)),dp;) ./examples/BINresol.sing:7
ideal J=x(1)^2-x(2)^2*x(3)^2;
list B=BINresol(J);
⇒ // ** redefining B (list B=BINresol(J);) ./examples/BINresol.sing:9
B[2]; // list of all charts
⇒ [1]:
⇒ // coefficients: ZZ/2
⇒ // number of vars : 3
⇒ //      block 1: ordering dp
⇒ //      : names  x(1) x(2) x(3)
⇒ //      block 2 : ordering C
⇒ [2]:
⇒ // coefficients: ZZ/2
⇒ // number of vars : 3
⇒ //      block 1 : ordering dp
⇒ //      : names  y(1) y(2) y(3)
⇒ //      block 2 : ordering C
⇒ [3]:
⇒ // coefficients: ZZ/2
⇒ // number of vars : 3
⇒ //      block 1 : ordering dp
⇒ //      : names  x(1) x(2) x(3)
⇒ //      block 2 : ordering C
⇒ [4]:

```

```

⇒ // coefficients: ZZ/2
⇒ // number of vars : 3
⇒ //      block 1 : ordering dp
⇒ //      : names  x(1) x(2) x(3)
⇒ //      block 2 : ordering C
⇒ [5]:
⇒ // coefficients: ZZ/2
⇒ // number of vars : 3
⇒ //      block 1 : ordering dp
⇒ //      : names  x(1) y(2) y(3)
⇒ //      block 2 : ordering C
⇒ [6]:
⇒ // coefficients: ZZ/2
⇒ // number of vars : 3
⇒ //      block 1 : ordering dp
⇒ //      : names  x(1) x(2) x(3)
⇒ //      block 2 : ordering C
⇒ [7]:
⇒ // coefficients: ZZ/2
⇒ // number of vars : 3
⇒ //      block 1 : ordering dp
⇒ //      : names  x(1) y(2) y(3)
⇒ //      block 2 : ordering C
⇒ [8]:
⇒ // coefficients: ZZ/2
⇒ // number of vars : 3
⇒ //      block 1 : ordering dp
⇒ //      : names  x(1) x(2) x(3)
⇒ //      block 2 : ordering C
⇒ [9]:
⇒ // coefficients: ZZ/2
⇒ // number of vars : 3
⇒ //      block 1 : ordering dp
⇒ //      : names  x(1) y(2) x(3)
⇒ //      block 2 : ordering C
⇒ [10]:
⇒ // coefficients: ZZ/2
⇒ // number of vars : 3
⇒ //      block 1 : ordering dp
⇒ //      : names  y(1) x(2) x(3)
⇒ //      block 2 : ordering C
⇒ [11]:
⇒ // coefficients: ZZ/2
⇒ // number of vars : 3
⇒ //      block 1 : ordering dp
⇒ //      : names  x(1) x(2) y(3)
⇒ //      block 2 : ordering C
⇒ [12]:
⇒ // coefficients: ZZ/2
⇒ // number of vars : 3
⇒ //      block 1 : ordering dp
⇒ //      : names  y(1) x(2) x(3)
⇒ //      block 2 : ordering C

```

### D.5.12.2 Eresol

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:**      `Eresol(J)`;  $J$  ideal

**Return:**     The E-resolution of singularities of  $J$  in terms of the affine charts, see example

**Example:**

```
LIB "resbinomial.lib";
ring r = 0,(x(1..2)),dp;
ideal J=x(1)^2-x(2)^3;
list L=Eresol(J);
"Please press return after each break point to see the next element of the output list"
⇒ Please press return after each break point to see the next element of the\
   output list
L[1][1]; // information of the first chart, L[1] list of charts
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ 0
⇒ [4]:
⇒ [1]:
⇒ [1]:
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 3
⇒ [2]:
⇒ [1]:
⇒ 2
⇒ [2]:
⇒ 0
⇒ [5]:
⇒ [1]:
⇒ [1]:
⇒ -1
⇒ [2]:
⇒ 1
⇒ [6]:
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 0
⇒ [7]:
⇒ 0
⇒ [8]:
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 0
⇒ [9]:
⇒ _[1]=--gen(2)
```

```

⇒ [10]:
⇒     empty list
⇒ [11]:
⇒     empty list
~;
⇒
⇒ -- break point in ./examples/Eresol.sing --

```

### D.5.12.3 determinecenter

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:** `determinecenter(Coef,expJ,c,n,Y,a,listmb,flag,control3,Hhist);` Coef, expJ, listmb, flag lists, c number, n, Y, control3 integers, a, Hhist intvec

**Compute:** next center of blowing up and related information, see example

**Return:** several lists defining the center and related information

**Example:**

```

LIB "resbinomial.lib";
ring r = 0,(x(1..4)),dp;
list flag=identifyvar();
ideal J=x(1)^2-x(2)^2*x(3)^5, x(1)*x(3)^3+x(4)^6;
list Lmb=1,list(0,0,0,0),list(0,0,0,0),list(0,0,0,0),iniD(4),iniD(4),list(0,0,0,0),-1;
list L=data(J,2,4);
list LL=determinecenter(L[1],L[2],2,4,0,0,Lmb,flag,0,-1); // Compute the first center
LL[1]; // index of variables in the center
⇒ [1]:
⇒     1
⇒ [2]:
⇒     4
⇒ [3]:
⇒     3
⇒ [4]:
⇒     2
LL[2]; // exponents of ideals J_4,J_3,J_2,J_1
⇒ [1]:
⇒     [1]:
⇒         [1]:
⇒             0
⇒         [2]:
⇒             2
⇒         [3]:
⇒             5
⇒         [4]:
⇒             0
⇒     [2]:
⇒         [1]:
⇒             2
⇒         [2]:
⇒             0
⇒         [3]:
⇒             0

```

```

    ↪          [4]:
    ↪          0
    ↪      [2]:
    ↪      [1]:
    ↪          [1]:
    ↪          0
    ↪          [2]:
    ↪          0
    ↪          [3]:
    ↪          0
    ↪          [4]:
    ↪          6
    ↪      [2]:
    ↪      [1]:
    ↪          1
    ↪          [2]:
    ↪          0
    ↪          [3]:
    ↪          3
    ↪          [4]:
    ↪          0
    ↪ [2]:
    ↪      [1]:
    ↪          [1]:
    ↪          [1]:
    ↪      0
    ↪          [2]:
    ↪      2
    ↪          [3]:
    ↪      5
    ↪          [4]:
    ↪      0
    ↪      [2]:
    ↪      [1]:
    ↪          [1]:
    ↪      0
    ↪          [2]:
    ↪      0
    ↪          [3]:
    ↪      0
    ↪          [4]:
    ↪      6
    ↪      [3]:
    ↪      [1]:
    ↪          [1]:
    ↪      0
    ↪          [2]:
    ↪      0
    ↪          [3]:
    ↪      6
    ↪          [4]:
    ↪      0
    ↪ [3]:

```



```

    ↪      [1]:
    ↪      [1]:
    ↪      [1]:
    ↪      0
    ↪      [2]:
    ↪      2
    ↪      [3]:
    ↪      5
    ↪      [4]:
    ↪      0
    ↪      [2]:
    ↪      [1]:
    ↪      [1]:
    ↪      0
    ↪      [2]:
    ↪      0
    ↪      [3]:
    ↪      6
    ↪      [4]:
    ↪      0
    ↪      [4]:
    ↪      [1]:
    ↪      [1]:
    ↪      [1]:
    ↪      0
    ↪      [2]:
    ↪      12
    ↪      [3]:
    ↪      0
    ↪      [4]:
    ↪      0
LL[3]; // list of orders of J_4,J_3,J_2,J_1
    ↪      [1]:
    ↪      2
    ↪      [2]:
    ↪      6
    ↪      [3]:
    ↪      6
    ↪      [4]:
    ↪      12
LL[4]; // list of critical values
    ↪      [1]:
    ↪      2
    ↪      [2]:
    ↪      2
    ↪      [3]:
    ↪      6
    ↪      [4]:
    ↪      6
LL[5]; // components of the resolution function t
    ↪      [1]:
    ↪      1
    ↪      [2]:

```

```

    ↪ 3
    ↪ [3]:
    ↪ 1
    ↪ [4]:
    ↪ 2
    LL[6]; // list of D_4,D_3,D_2,D_1
    ↪ [1]:
    ↪   [1]:
    ↪   0
    ↪   [2]:
    ↪   0
    ↪   [3]:
    ↪   0
    ↪   [4]:
    ↪   0
    ↪ [2]:
    ↪   [1]:
    ↪   0
    ↪   [2]:
    ↪   0
    ↪   [3]:
    ↪   0
    ↪   [4]:
    ↪   0
    ↪ [3]:
    ↪   [1]:
    ↪   0
    ↪   [2]:
    ↪   0
    ↪   [3]:
    ↪   0
    ↪   [4]:
    ↪   0
    ↪ [4]:
    ↪   [1]:
    ↪   0
    ↪   [2]:
    ↪   0
    ↪   [3]:
    ↪   0
    ↪   [4]:
    ↪   0
    LL[7]; // list of H_4,H_3,H_2,H_1 (exceptional divisors)
    ↪ [1]:
    ↪   [1]:
    ↪   0
    ↪   [2]:
    ↪   0
    ↪   [3]:
    ↪   0
    ↪   [4]:
    ↪   0
    ↪ [2]:

```

```

⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ 0
⇒ [4]:
⇒ 0
⇒ [3]:
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ 0
⇒ [4]:
⇒ 0
⇒ [4]:
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ 0
⇒ [4]:
⇒ 0
LL[8]; // list of all exceptional divisors accumulated
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ 0
⇒ [4]:
⇒ 0
LL[9]; // auxiliary invariant
⇒ [1]:
⇒ 0
LL[10]; // intvec pointing out the last step where the function t has dropped
⇒ -1,-1,-1,-1
ring r= 0,(x(1..4)),dp;
⇒ // ** redefining r (ring r= 0,(x(1..4)),dp;) ./examples/determinecenter.s\
    ing:18
list flag=identifyvar();
⇒ // ** redefining flag (list flag=identifyvar();) ./examples/determinecent\
    er.sing:19
ideal J=x(1)^3-x(2)^2*x(3)^5, x(1)*x(3)^3+x(4)^5;
list Lmb=2,list(0,0,0,0),list(0,0,0,0),list(0,0,0,0),iniD(4),iniD(4),list(0,0,0,0),-1;
⇒ // ** redefining Lmb (=2,list(0,0,0,0),list(0,0,0,0),list(0,0,0,0),iniD(4\
    ),iniD(4),list(0,0,0,0),-1;) ./examples/determinecenter.sing:21
list L2=data(J,2,4);
list L3=determinecenter(L2[1],L2[2],2,4,0,0,Lmb,flag,0,-1); // Example with rational
L3[1]; // index of variables in the center

```

```

↳ [1]:
↳ 1
↳ [2]:
↳ 3
↳ [3]:
↳ 4
L3[2]; // exponents of ideals J_4,J_3,J_2,J_1
↳ [1]:
↳ [1]:
↳ [1]:
↳ 0
↳ [2]:
↳ 2
↳ [3]:
↳ 5
↳ [4]:
↳ 0
↳ [2]:
↳ [1]:
↳ 3
↳ [2]:
↳ 0
↳ [3]:
↳ 0
↳ [4]:
↳ 0
↳ [2]:
↳ [1]:
↳ [1]:
↳ 0
↳ [2]:
↳ 0
↳ [3]:
↳ 0
↳ [4]:
↳ 5
↳ [2]:
↳ [1]:
↳ 1
↳ [2]:
↳ 0
↳ [3]:
↳ 3
↳ [4]:
↳ 0
↳ [2]:
↳ [1]:
↳ [1]:
↳ 0
↳ [2]:
↳ 2

```

```

    ↪          [3]:
    ↪ 5
    ↪          [4]:
    ↪ 0
    ↪    [2]:
    ↪      [1]:
    ↪        [1]:
    ↪ 0
    ↪          [2]:
    ↪ 0
    ↪          [3]:
    ↪ 0
    ↪          [4]:
    ↪ 5
    ↪    [3]:
    ↪      [1]:
    ↪        [1]:
    ↪ 0
    ↪          [2]:
    ↪ 0
    ↪          [3]:
    ↪ 9/2
    ↪          [4]:
    ↪ 0
    ↪    [3]:
    ↪      [1]:
    ↪        [1]:
    ↪          [1]:
    ↪ 0
    ↪          [2]:
    ↪ 0
    ↪          [3]:
    ↪ 0
    ↪          [4]:
    ↪ 5
    ↪    [4]:
    ↪      [1]:
    ↪        [1]:
    ↪          [1]:
    ↪ 0
    ↪          [2]:
    ↪ 0
    ↪          [3]:
    ↪ 0
    ↪          [4]:
    ↪ 0
    ↪ 0
    ↪ L3[3]; // list of orders of J_4,J_3,J_2,J_1
    ↪ [1]:
    ↪ 3
    ↪ [2]:
    ↪ 9/2
    ↪ [3]:
    ↪ 5

```

```

L3[4]; // list of critical values
⇒ [1]:
⇒ 2
⇒ [2]:
⇒ 3
⇒ [3]:
⇒ 9/2
L3[5]; // components of the resolution function
⇒ [1]:
⇒ 3/2
⇒ [2]:
⇒ 3/2
⇒ [3]:
⇒ 10/9

```

#### D.5.12.4 Blowupcenter

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:**      `Blowupcenter(center,id,nchart,infochart,c,n,cstep);`  
                  center, infochart lists, id, nchart, n, cstep integers, c number

**Compute:**    The blowing up at the chart IDCHART along the given center

**Return:**     new affine charts and related information, see example

**Example:**

```

LIB "resbinomial.lib";
ring r = 0,(x(1),y(2),x(3),y(4),x(5..7)),dp;
list flag=identifyvar();
ideal J=x(1)^3-x(3)^2*y(4)^2,x(1)*x(7)*y(2)-x(6)^3*x(5)*y(4)^3,x(5)^3-x(5)^3*y(2)^2;
list Lmb=2,list(0,0,0,0,0,0,0),list(0,0,0,0,0,0,0),list(0,0,0,0,0,0,0),iniD(7),iniD(7);
list L=data(J,3,7);
list L2=determinecenter(L[1],L[2],2,7,0,0,Lmb,flag,0,-1); // Computing the center
module auxpath=[0,-1];
list infochart=0,0,0,L[2],L[1],flag,0,list(0,0,0,0,0,0,0),auxpath,list(),list();
list L3=Blowupcenter(L2[1],1,1,infochart,2,7,0);
L3[1]; // current chart (parent,Y,center,expJ,Coef,flag,Hhist,blwhist,path,hipercoef)
⇒ [1]:
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ 0
⇒ [4]:
⇒ [1]:
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ 2
⇒ [4]:

```

```

↳      2
↳      [5]:
↳      0
↳      [6]:
↳      0
↳      [7]:
↳      0
↳      [2]:
↳      [1]:
↳      3
↳      [2]:
↳      0
↳      [3]:
↳      0
↳      [4]:
↳      0
↳      [5]:
↳      0
↳      [6]:
↳      0
↳      [7]:
↳      0
↳      [2]:
↳      [1]:
↳      [1]:
↳      0
↳      [2]:
↳      0
↳      [3]:
↳      0
↳      [4]:
↳      3
↳      [5]:
↳      1
↳      [6]:
↳      3
↳      [7]:
↳      0
↳      [2]:
↳      [1]:
↳      1
↳      [2]:
↳      1
↳      [3]:
↳      0
↳      [4]:
↳      0
↳      [5]:
↳      0
↳      [6]:
↳      0
↳      [7]:
↳      1

```

```

↳      [3]:
↳      [1]:
↳      [1]:
↳      0
↳      [2]:
↳      2
↳      [3]:
↳      0
↳      [4]:
↳      0
↳      [5]:
↳      3
↳      [6]:
↳      0
↳      [7]:
↳      0
↳      [2]:
↳      [1]:
↳      0
↳      [2]:
↳      0
↳      [3]:
↳      0
↳      [4]:
↳      0
↳      [5]:
↳      3
↳      [6]:
↳      0
↳      [7]:
↳      0
↳      [5]:
↳      [1]:
↳      [1]:
↳ -1
↳      [2]:
↳ 1
↳      [2]:
↳      [1]:
↳ -1
↳      [2]:
↳ 1
↳      [3]:
↳      [1]:
↳ -1
↳      [2]:
↳ 1
↳      [6]:
↳      [1]:
↳      0
↳      [2]:
↳      1
↳      [3]:

```



```

↳      0
↳      [4]:
↳      1
↳      [5]:
↳      0
↳      [6]:
↳      0
↳      [7]:
↳      0
↳      [7]:
↳      0
↳      [8]:
↳      [1]:
↳      0
↳      [2]:
↳      0
↳      [3]:
↳      0
↳      [4]:
↳      0
↳      [5]:
↳      0
↳      [6]:
↳      0
↳      [7]:
↳      0
↳      [9]:
↳      _[1]=--gen(2)
↳      [10]:
↳      empty list
↳      [11]:
↳      empty list
↳      [12]:
↳      2
↳      [13]:
↳      3
↳      [14]:
↳      4
↳      [15]:
↳      5
↳      [16]:
↳      6
L3[2][1]; // information of its first son, write L3[2][2],...,L3[2][5] to see the oth
↳ [1]:
↳ 1
↳ [2]:
↳ 3
↳ [3]:
↳ 3,1,7,5,6
↳ [4]:
↳ [1]:
↳ [1]:
↳ [1]:

```

```

↳      0
↳      [2]:
↳      0
↳      [3]:
↳ 0
↳      [4]:
↳      2
↳      [5]:
↳      0
↳      [6]:
↳      0
↳      [7]:
↳      0
↳      [2]:
↳      [1]:
↳      3
↳      [2]:
↳      0
↳      [3]:
↳ 1
↳      [4]:
↳      0
↳      [5]:
↳      0
↳      [6]:
↳      0
↳      [7]:
↳      0
↳      [2]:
↳      [1]:
↳      [1]:
↳      0
↳      [2]:
↳      0
↳      [3]:
↳ 2
↳      [4]:
↳      3
↳      [5]:
↳      1
↳      [6]:
↳      3
↳      [7]:
↳      0
↳      [2]:
↳      [1]:
↳      1
↳      [2]:
↳      1
↳      [3]:
↳ 0
↳      [4]:
↳      0

```

```

↳      [5]:
↳      0
↳      [6]:
↳      0
↳      [7]:
↳      1
↳      [3]:
↳      [1]:
↳      [1]:
↳      0
↳      [2]:
↳      2
↳      [3]:
↳ 1
↳      [4]:
↳      0
↳      [5]:
↳      3
↳      [6]:
↳      0
↳      [7]:
↳      0
↳      [2]:
↳      [1]:
↳      0
↳      [2]:
↳      0
↳      [3]:
↳ 1
↳      [4]:
↳      0
↳      [5]:
↳      3
↳      [6]:
↳      0
↳      [7]:
↳      0
↳ [5]:
↳ [1]:
↳ [1]:
↳ -1
↳ [2]:
↳ 1
↳ [2]:
↳ [1]:
↳ -1
↳ [2]:
↳ 1
↳ [3]:
↳ [1]:
↳ -1
↳ [2]:
↳ 1

```

```

    ↪ [6]:
    ↪   [1]:
    ↪     0
    ↪   [2]:
    ↪     1
    ↪   [3]:
    ↪     0
    ↪   [4]:
    ↪     1
    ↪   [5]:
    ↪     0
    ↪   [6]:
    ↪     0
    ↪   [7]:
    ↪     0
    ↪ [7]:
    ↪   0,3
    ↪ [8]:
    ↪   [1]:
    ↪     0,3
    ↪   [2]:
    ↪     0,0
    ↪   [3]:
    ↪     0,0
    ↪   [4]:
    ↪     0,0
    ↪   [5]:
    ↪     0,3
    ↪   [6]:
    ↪     0,3
    ↪   [7]:
    ↪     0,3
    ↪ [9]:
    ↪   _[1]=-gen(2)
    ↪   _[2]=gen(2)+gen(1)
    ↪ [10]:
    ↪   empty list
    ↪ [11]:
    ↪   empty list
    L3[3];    // current number of charts
    ↪ 6
    L3[4];    // step/level associated to each son
    ↪ [1]:
    ↪   1
    ↪ [2]:
    ↪   1
    ↪ [3]:
    ↪   1
    ↪ [4]:
    ↪   1
    ↪ [5]:
    ↪   1
    L3[5];    // number of variables in the center

```

↪ 5

### D.5.12.5 Nonhyp

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Compute:** The "ideal" generated by the non hyperbolic generators of J

**Return:** lists with the following information  
 newcoef,newJ: coefficients and exponents of the non hyperbolic generators total-  
 hyp,totalgen: coefficients and exponents of the hyperbolic generators flaglist: new list  
 saying status of variables

**Note:** the basering `r` is supposed to be a polynomial ring  $K[x,y]$ , in fact, we work in a localization of  $K[x,y]$ , of type  $K[x,y]_y$  with  $y$  invertible variables.

**Example:**

```
LIB "resbinomial.lib";
ring r = 0,(x(1),y(2),x(3),y(4),x(5..7)),dp;
list flag=identifyvar(); // List giving flag=1 to invertible variables: y(2),y(4)
ideal J=x(1)^3-x(3)^2*y(4)^2,x(1)*x(7)*y(2)-x(6)^3*x(5)*y(4)^3,1-x(5)^2*y(2)^2;
list L=data(J,3,7);
list L2=maxEord(L[1],L[2],3,7,flag);
L2[1]; // Maximum E-order
↪ 0
list New=Nonhyp(L[1],L[2],3,7,flag,L2[2]);
New[1]; // Coefficients of the non hyperbolic part
↪ [1]:
↪ [1]:
↪ -1
↪ [2]:
↪ 1
↪ [2]:
↪ [1]:
↪ -1
↪ [2]:
↪ 1
New[2]; // Exponents of the non hyperbolic part
↪ [1]:
↪ [1]:
↪ [1]:
↪ 0
↪ [2]:
↪ 0
↪ [3]:
↪ 2
↪ [4]:
↪ 2
↪ [5]:
↪ 0
↪ [6]:
↪ 0
↪ [7]:
↪ 0
↪ [2]:
```

```

⇨      [1]:
⇨      3
⇨      [2]:
⇨      0
⇨      [3]:
⇨      0
⇨      [4]:
⇨      0
⇨      [5]:
⇨      0
⇨      [6]:
⇨      0
⇨      [7]:
⇨      0
⇨ [2]:
⇨   [1]:
⇨   [1]:
⇨   0
⇨   [2]:
⇨   0
⇨   [3]:
⇨   0
⇨   [4]:
⇨   3
⇨   [5]:
⇨   1
⇨   [6]:
⇨   3
⇨   [7]:
⇨   0
⇨   [2]:
⇨   [1]:
⇨   1
⇨   [2]:
⇨   1
⇨   [3]:
⇨   0
⇨   [4]:
⇨   0
⇨   [5]:
⇨   0
⇨   [6]:
⇨   0
⇨   [7]:
⇨   1
New[3];    // Coefficients of the hyperbolic part
⇨ [1]:
⇨   [1]:
⇨ -1
⇨   [2]:
⇨ 1
New[4];    // New hyperbolic equations
⇨ [1]:

```

```

↳      [1]:
↳      [1]:
↳      0
↳      [2]:
↳      2
↳      [3]:
↳      0
↳      [4]:
↳      0
↳      [5]:
↳      2
↳      [6]:
↳      0
↳      [7]:
↳      0
↳      [2]:
↳      [1]:
↳      0
↳      [2]:
↳      0
↳      [3]:
↳      0
↳      [4]:
↳      0
↳      [5]:
↳      0
↳      [6]:
↳      0
↳      [7]:
↳      0
New[5];      // New list giving flag=1 to invertible variables: y(2),y(4),y(5)
↳ [1]:
↳ 0
↳ [2]:
↳ 1
↳ [3]:
↳ 0
↳ [4]:
↳ 1
↳ [5]:
↳ 1
↳ [6]:
↳ 0
↳ [7]:
↳ 0
ring r = 0,(x(1..4)),dp;
↳ // ** redefining r (ring r = 0,(x(1..4)),dp;) ./examples/Nonhyp.sing:14
list flag=identifyvar();
↳ // ** redefining flag (list flag=identifyvar();) ./examples/Nonhyp.sing:1\
5
ideal J=1-x(1)^5*x(2)^2*x(3)^5, x(1)^2*x(3)^3+x(1)^4*x(4)^6;
list L=data(J,2,4);
list L2=maxEord(L[1],L[2],2,4,flag);

```

```

L2[1];      // Maximum E-order
⇨ 0
list New=Nonhyp(L[1],L[2],2,4,flag,L2[2]);
New;
⇨ [1]:
⇨   empty list
⇨ [2]:
⇨   empty list
⇨ [3]:
⇨   [1]:
⇨       [1]:
⇨   -1
⇨       [2]:
⇨   1
⇨       [2]:
⇨       [1]:
⇨   1
⇨       [2]:
⇨   1
⇨ [4]:
⇨   [1]:
⇨       [1]:
⇨           [1]:
⇨           5
⇨           [2]:
⇨           2
⇨           [3]:
⇨           5
⇨           [4]:
⇨           0
⇨       [2]:
⇨           [1]:
⇨           0
⇨           [2]:
⇨           0
⇨           [3]:
⇨           0
⇨           [4]:
⇨           0
⇨   [2]:
⇨       [1]:
⇨           [1]:
⇨           4
⇨           [2]:
⇨           0
⇨           [3]:
⇨           0
⇨           [4]:
⇨           6
⇨       [2]:
⇨           [1]:
⇨           2
⇨       [2]:

```



```

↳          0
↳        [3]:
↳          3
↳        [4]:
↳          0
↳ [5]:
↳   [1]:
↳       1
↳   [2]:
↳       1
↳   [3]:
↳       1
↳   [4]:
↳       1

```

### D.5.12.6 identifyvar

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:**      `identifyvar();`

**Compute:**   Assign 0 to variables `x` and 1 to variables `y`, only necessary at the beginning

**Return:**    list, say `l`, of size the dimension of the basering  
               `l[i]` is: 0 if the *i*-th variable is `x(i)`,  
               1 if the *i*-th variable is `y(i)`

**Example:**

```

LIB "resbinomial.lib";
ring r = 0,(x(1),y(2),x(3),y(4),x(5..7),y(8)),dp;
identifyvar();
↳ [1]:
↳    0
↳ [2]:
↳    1
↳ [3]:
↳    0
↳ [4]:
↳    1
↳ [5]:
↳    0
↳ [6]:
↳    0
↳ [7]:
↳    0
↳ [8]:
↳    1

```

### D.5.12.7 Edatalist

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:**      `Edatalist(Coef,Exp,k,n,flaglist);`  
               `Coef,Exp,flaglist` lists, `k,n`, integers  
               `Exp` is a list of lists of exponents, `k=size(Exp)`

**Compute:** computes a list with the E-order of each term

**Return:** a list with the E-order of each term

**Example:**

```
LIB "resbinomial.lib";
ring r = 0,(x(1),y(2),x(3),y(4),x(5..7),y(8)),dp;
list flag=identifyvar();
ideal J=x(1)^3*x(3)-y(2)*y(4)^2,x(5)*y(2)-x(7)*y(4)^2,x(6)^2*(1-y(4)*y(8)^5);
list L=data(J,3,8);
list EL=EdataList(L[1],L[2],3,8,flag);
EL; // E-order of each term
↳ [1]:
↳ [1]:
↳ 4
↳ [2]:
↳ 0
↳ [2]:
↳ [1]:
↳ 1
↳ [2]:
↳ 1
↳ [3]:
↳ [1]:
↳ 2
↳ [2]:
↳ 2
ring r = 2,(x(1),y(2),x(3),y(4),x(5..7),y(8)),dp;
↳ // ** redefining r (ring r = 2,(x(1),y(2),x(3),y(4),x(5..7),y(8)),dp;) ./\
examples/EdataList.sing:8
list flag=identifyvar();
↳ // ** redefining flag (list flag=identifyvar();) ./examples/EdataList.sin\
g:9
ideal J=x(1)^3*x(3)-y(2)*y(4)^2,x(5)*y(2)-x(7)*y(4)^2,x(6)^2*(1-y(4)*y(8)^5);
list L=data(J,3,8);
list EL=EdataList(L[1],L[2],3,8,flag);
EL; // E-order of each term IN CHAR 2, COMPUTATIONS NEED TO BE DONE IN CHAR 0
↳ [1]:
↳ [1]:
↳ 0
↳ [2]:
↳ 0
↳ [2]:
↳ [1]:
↳ 1
↳ [2]:
↳ 1
↳ [3]:
↳ [1]:
↳ 0
↳ [2]:
↳ 0
ring r = 0,(x(1..3)),dp;
↳ // ** redefining r (ring r = 0,(x(1..3)),dp;) ./examples/EdataList.sing:1\
```

```

4
list flag=identifyvar();
⇒ // ** redefining flag (list flag=identifyvar();) ./examples/Edatalist.sin\
g:15
ideal J=x(1)^4*x(2)^2, x(1)^2-x(3)^3;
list L=data(J,2,3);
list EL=Edatalist(L[1],L[2],2,3,flag);
EL; // E-order of each term
⇒ [1]:
⇒ [1]:
⇒ 6
⇒ [2]:
⇒ [1]:
⇒ 3
⇒ [2]:
⇒ 2

```

### D.5.12.8 EOrdlist

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:** EOrdlist(Coef,Exp,k,n,flaglist);  
 Coef,Exp,flaglist lists, k,n, integers  
 Exp is a list of lists of exponents, k=size(Exp)

**Compute:** computes de E-order of an ideal given by a list (Coef,Exp) and extra information

**Return:** maximal E-order, and its position=number of generator and term

**Example:**

```

LIB "resbinomial.lib";
ring r = 0,(x(1),y(2),x(3),y(4),x(5..7),y(8)),dp;
list flag=identifyvar();
ideal J=x(1)^3*x(3)-y(2)*y(4)^2,x(5)*y(2)-x(7)*y(4)^2,x(6)^2*(1-y(4)*y(8)^5),x(7)^4*y(8)^5;
list L=data(J,4,8);
list Eo=EOrdlist(L[1],L[2],4,8,flag);
Eo[1]; // E-order
⇒ 0
Eo[2]; // generator giving the E-order
⇒ 1
Eo[3]; // term giving the E-order
⇒ 2

```

### D.5.12.9 maxEord

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:** maxEord(Coef,Exp,k,n,flaglist);  
 Coef,Exp,flaglist lists, k,n, integers  
 Exp is a list of lists of exponents, k=size(Exp)

**Return:** computes de maximal E-order of an ideal given by Coef,Exp

**Example:**

```

LIB "resbinomial.lib";
ring r = 0,(x(1),y(2),x(3),y(4),x(5..7),y(8)),dp;

```

```

list flag=identifyvar();
ideal J=x(1)^3*x(3)-y(2)*y(4)^2*x(3),x(5)*y(2)-x(7)*y(4)^2,x(6)^2*(1-y(4)*y(8)^5),x(
list L=data(J,4,8);
list M=maxEord(L[1],L[2],4,8,flag);
M[1]; // E-order
↳ 1

```

### D.5.12.10 ECoef

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:** ECoef(Coef,expP,sP,V,auxc,n,flaglist);  
 Coef, expP, flaglist lists, sP, V, n integers, auxc number

**Compute:** The ideal E-Coeff\_V(P), where V is a permissible hypersurface which belongs to the center

**Return:** list of exponents, list of coefficients and classification of the ideal E-Coeff\_V(P)

**Example:**

```

LIB "resbinomial.lib";
ring r = 0,(x(1),y(2),x(3),y(4),x(5..7)),dp;
list flag=identifyvar();
ideal P=x(1)^2*x(3)^5-x(5)^7*y(4),x(6)^3*y(2)^5-x(7)^5,x(5)^3*x(6)-y(4)^3*x(1)^5;
list L=data(P,3,7);
list L2=ECoef(L[1],L[2],3,1,3,7,flag);
L2[1]; // exponents of the E-Coefficient ideal respect to x(1)
↳ [1]:
↳ [1]:
↳ [1]:
↳ 0
↳ [2]:
↳ 0
↳ [3]:
↳ 0
↳ [4]:
↳ 0
↳ [5]:
↳ 7
↳ [6]:
↳ 0
↳ [7]:
↳ 0
↳ [2]:
↳ [1]:
↳ [1]:
↳ 0
↳ [2]:
↳ 0
↳ [3]:
↳ 15
↳ [4]:
↳ 0
↳ [5]:
↳ 0

```

```

    ↪      [6]:
    ↪ 0
    ↪      [7]:
    ↪ 0
    ↪ [3]:
    ↪   [1]:
    ↪   [1]:
    ↪ 0
    ↪   [2]:
    ↪ 5
    ↪   [3]:
    ↪ 0
    ↪   [4]:
    ↪ 0
    ↪   [5]:
    ↪ 0
    ↪   [6]:
    ↪ 3
    ↪   [7]:
    ↪ 0
    ↪   [2]:
    ↪   [1]:
    ↪ 0
    ↪   [2]:
    ↪ 0
    ↪   [3]:
    ↪ 0
    ↪   [4]:
    ↪ 0
    ↪   [5]:
    ↪ 0
    ↪   [6]:
    ↪ 0
    ↪   [7]:
    ↪ 5
    ↪ [4]:
    ↪   [1]:
    ↪   [1]:
    ↪ 0
    ↪   [2]:
    ↪   0
    ↪   [3]:
    ↪ 0
    ↪   [4]:
    ↪   0
    ↪   [5]:
    ↪ 3
    ↪   [6]:
    ↪ 1
    ↪   [7]:
    ↪ 0
    L2[2]; // its coefficients
    ↪ [1]:

```

```

⇒ -1
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ -1
⇒ [4]:
⇒ 1
L2[3]; // classify the type of ideal obtained
⇒ 0
ring r = 0,(x(1),y(2),x(3),y(4)),dp;
⇒ // ** redefining r (ring r = 0,(x(1),y(2),x(3),y(4)),dp;) ./examples/ECoe\
    f.sing:10
list flag=identifyvar();
⇒ // ** redefining flag (list flag=identifyvar();) ./examples/ECoeff.sing:11
ideal J=x(1)^3*(1-2*y(2)*y(4)^2); // Bold regular case
list L=data(J,1,4);
list L2=ECoeff(L[1],L[2],1,1,3,4,flag);
L2;
⇒ [1]:
⇒ empty list
⇒ [2]:
⇒ empty list
⇒ [3]:
⇒ 1
ring r = 0,(x(1),y(2),x(3),y(4),x(5..7)),dp;
⇒ // ** redefining r (ring r = 0,(x(1),y(2),x(3),y(4),x(5..7)),dp;) ./examp\
    les/ECoeff.sing:16
list flag=identifyvar();
⇒ // ** redefining flag (list flag=identifyvar();) ./examples/ECoeff.sing:17
ideal J=x(1)^3-x(3)^2*y(4)^2,x(1)*x(7)*y(2)-x(6)^3*x(5)*y(4)^3,x(5)^3-x(5)^3*y(2)^2;
list L=data(J,3,7);
list L2=ECoeff(L[1],L[2],3,1,2,7,flag);
⇒ // ** redefining L2 (list L2=ECoeff(L[1],L[2],3,1,2,7,flag);) ./examples/E\
    Coef.sing:20
L2;
⇒ [1]:
⇒ [1]:
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ 2
⇒ [4]:
⇒ 0
⇒ [5]:
⇒ 0
⇒ [6]:
⇒ 0

```

```

    ↪          [7]:
    ↪ 0
    ↪    [2]:
    ↪      [1]:
    ↪        [1]:
    ↪ 0
    ↪          [2]:
    ↪          0
    ↪          [3]:
    ↪ 0
    ↪          [4]:
    ↪          0
    ↪          [5]:
    ↪ 1
    ↪          [6]:
    ↪ 3
    ↪          [7]:
    ↪ 0
    ↪    [3]:
    ↪      [1]:
    ↪        [1]:
    ↪ 0
    ↪          [2]:
    ↪          0
    ↪          [3]:
    ↪ 0
    ↪          [4]:
    ↪          0
    ↪          [5]:
    ↪ 0
    ↪          [6]:
    ↪ 0
    ↪          [7]:
    ↪ 2
    ↪    [4]:
    ↪      [1]:
    ↪        [1]:
    ↪ 0
    ↪          [2]:
    ↪ 2
    ↪          [3]:
    ↪ 0
    ↪          [4]:
    ↪ 0
    ↪          [5]:
    ↪ 3
    ↪          [6]:
    ↪ 0
    ↪          [7]:
    ↪ 0
    ↪    [2]:
    ↪      [1]:
    ↪ 0

```

```

⇒          [2]:
⇒ 0
⇒          [3]:
⇒ 0
⇒          [4]:
⇒ 0
⇒          [5]:
⇒ 3
⇒          [6]:
⇒ 0
⇒          [7]:
⇒ 0
⇒ [2]:
⇒ [1]:
⇒ -1
⇒ [2]:
⇒ -1
⇒ [3]:
⇒ 1
⇒ [4]:
⇒ [1]:
⇒ -1
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ 0
ring r = 3,(x(1),y(2),x(3),y(4),x(5..7)),dp;
⇒ // ** redefining r (ring r = 3,(x(1),y(2),x(3),y(4),x(5..7)),dp;) ./examp\
    les/ECoeff.sing:22
list flag=identifyvar();
⇒ // ** redefining flag (list flag=identifyvar();) ./examples/ECoeff.sing:23
ideal J=x(1)^3-x(3)^2*y(4)^2,x(1)*x(7)*y(2)-x(6)^3*x(5)*y(4)^3,x(5)^3-x(5)^3*y(2)^2;
list L=data(J,3,7);
list L2=ECoeff(L[1],L[2],3,1,2,7,flag);
⇒ E-order zero!
L2; // THE COMPUTATIONS ARE NOT CORRECT IN CHARACTERISTIC p>0
⇒ [1]:
⇒ [1]:
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ -1
⇒ [4]:
⇒ 0
⇒ [5]:
⇒ 0
⇒ [6]:
⇒ 0
⇒ [7]:
⇒ 0

```



```

    ↪      [2]:
    ↪      [1]:
    ↪      [1]:
    ↪      0
    ↪      [2]:
    ↪      0
    ↪      [3]:
    ↪      0
    ↪      [4]:
    ↪      0
    ↪      [5]:
    ↪      0
    ↪      [6]:
    ↪      0
    ↪      [7]:
    ↪      0
    ↪      [3]:
    ↪      [1]:
    ↪      [1]:
    ↪      0
    ↪      [2]:
    ↪      0
    ↪      [3]:
    ↪      0
    ↪      [4]:
    ↪      0
    ↪      [5]:
    ↪      1
    ↪      [6]:
    ↪      0
    ↪      [7]:
    ↪      0
    ↪      [4]:
    ↪      [1]:
    ↪      [1]:
    ↪      0
    ↪      [2]:
    ↪      0
    ↪      [3]:
    ↪      0
    ↪      [4]:
    ↪      0
    ↪      [5]:
    ↪      0
    ↪      [6]:
    ↪      0
    ↪      [7]:
    ↪     -1
    ↪      [5]:
    ↪      [1]:
    ↪      [1]:
    ↪      0
    ↪      [2]:

```

```

↳ -1
↳      [3]:
↳ 0
↳      [4]:
↳ 0
↳      [5]:
↳ 0
↳      [6]:
↳ 0
↳      [7]:
↳ 0
↳      [2]:
↳      [1]:
↳ 0
↳      [2]:
↳ 0
↳      [3]:
↳ 0
↳      [4]:
↳ 0
↳      [5]:
↳ 0
↳      [6]:
↳ 0
↳      [7]:
↳ 0
↳ [2]:
↳      [1]:
↳ -1
↳      [2]:
↳ 1
↳      [3]:
↳ -1
↳      [4]:
↳ 1
↳      [5]:
↳      [1]:
↳ -1
↳      [2]:
↳ 1
↳ [3]:
↳      2
// because numbers are treated as 0 in assignments

```

#### D.5.12.11 elimrep

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:**      `elimrep(L)`; L is a list

**Compute:**   Eliminate repeated terms from a list

**Return:**     the same list without repeated terms

**Example:**

```

LIB "resbinomial.lib";
ring r = 0,(x(1..3)),dp;
list L=4,5,2,5,7,8,6,3,2;
elimrep(L);
↳ [1]:
↳ 4
↳ [2]:
↳ 5
↳ [3]:
↳ 2
↳ [4]:
↳ 7
↳ [5]:
↳ 8
↳ [6]:
↳ 6
↳ [7]:
↳ 3

```

### D.5.12.12 Emaxcont

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:**     Emaxcont(Coef,Exp,k,n,flag);  
               Coef,Exp,flag lists, k,n, integers  
               Exp is a list of lists of exponents, k=size(Exp)

**Compute:**   Identify ALL the variables of E-maximal contact

**Return:**     a list with the indexes of the variables of E-maximal contact

**Example:**

```

LIB "resbinomial.lib";
ring r = 0,(x(1),y(2),x(3),y(4),x(5..7),y(8)),dp;
list flag=identifyvar();
ideal J=x(1)^3*x(3)-y(2)*y(4)^2,x(5)*y(2)-x(7)*y(4)^2,x(6)^2*(1-y(4)*y(8)^5),x(7)^4*y(8)^5;
list L=data(J,4,8);
list hyp=Emaxcont(L[1],L[2],4,8,flag);
hyp[1]; // max E-order=0
↳ 0
hyp[2]; // There are no hypersurfaces of E-maximal contact
↳ empty list
ring r = 0,(x(1),y(2),x(3),y(4),x(5..7),y(8)),dp;
↳ // ** redefining r (ring r = 0,(x(1),y(2),x(3),y(4),x(5..7),y(8)),dp;) ./\
  examples/Emaxcont.sing:9
list flag=identifyvar();
↳ // ** redefining flag (list flag=identifyvar();) ./examples/Emaxcont.sing\
  :10
ideal J=x(1)^3*x(3)-y(2)*y(4)^2*x(3),x(5)*y(2)-x(7)*y(4)^2,x(6)^2*(1-y(4)*y(8)^5),x(7)^4*y(8)^5;
list L=data(J,4,8);
list hyp=Emaxcont(L[1],L[2],4,8,flag);
hyp[1]; // the E-order is 1
↳ 1
hyp[2]; // {x(3)=0},{x(5)=0},{x(7)=0} are hypersurfaces of E-maximal contact
↳ [1]:

```

```

⇒      3
⇒ [2]:
⇒      7
⇒ [3]:
⇒      5

```

### D.5.12.13 cleanunit

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Compute:** We clean (or forget) the units in a monomial, given by "y" variables

**Return:** The list defining the monomial ideal already cleaned

**Example:**

```

LIB "resbinomial.lib";
ring r = 0,(x(1),y(2),x(3),y(4)),dp;
list flag=identifyvar();
ideal J=x(1)^3*y(2)*x(3)^5*y(4)^8;
list L=data(J,1,4);
L[2][1][1]; // list of exponents of the monomial J
⇒ [1]:
⇒      3
⇒ [2]:
⇒      1
⇒ [3]:
⇒      5
⇒ [4]:
⇒      8
list M=cleanunit(L[2][1][1],4,flag);
M; // new list without units
⇒ [1]:
⇒      3
⇒ [2]:
⇒      0
⇒ [3]:
⇒      5
⇒ [4]:
⇒      0

```

### D.5.12.14 resfunction

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:** `resfunction(invariant,auxinv,nchart,n);`  
invariant, auxinv lists, nchart, n integers

**Compute:** Patch the resolution function

**Return:** The complete resolution function

**Example:**

```

LIB "resbinomial.lib";
ring r = 0,(x(1..2)),dp;
ideal J=x(1)^2-x(2)^3;
list L=Eresol(J);

```

```

L[3]; // incomplete resolution function
↳ [1]:
↳ [1]:
↳ 1
↳ [2]:
↳ 3/2
↳ [2]:
↳ @
↳ [3]:
↳ [1]:
↳ 1
↳ [2]:
↳ 2
↳ [4]:
↳ @
↳ [5]:
↳ [1]:
↳ 1
↳ [2]:
↳ 0
↳ [6]:
↳ #
↳ [7]:
↳ #
resfunction(L[3],L[7],7,2); // complete resolution function
↳ [1]:
↳ [1]:
↳ 1
↳ [2]:
↳ 3/2
↳ [2]:
↳ @
↳ [3]:
↳ [1]:
↳ 1
↳ [2]:
↳ 2
↳ [4]:
↳ @
↳ [5]:
↳ [1]:
↳ 1
↳ [2]:
↳ [1]:
↳ -1
↳ [2]:
↳ 1
↳ [3]:
↳ 1
↳ [6]:
↳ #
↳ [7]:
↳ #

```

### D.5.12.15 calculateI

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:**      `calculateI(Coef,expJ,c,n,Y,a,b,D);`  
                  Coef, expJ, D lists, c, b numbers, n,Y integers, a intvec

**Return:**     ideal I, non monomial part of J

**Example:**

```
LIB "resbinomial.lib";
ring r = 0,(x(1..3)),dp;
list flag=identifyvar();
ideal J=x(1)^4*x(2)^2, x(3)^3;
list Lmb=1,list(0,0,0),list(0,0,0),list(3),iniD(3),iniD(3),list(0,0,0),-1;
list L=data(J,2,3);
list LL=determinecenter(L[1],L[2],3,3,0,0,Lmb,flag,0,-1); // Calculate the center
module auxpath=[0,-1];
list infochart=0,0,0,L[2],L[1],flag,0,list(0,0,0),auxpath,list(),list();
list L3=Blowupcenter(LL[1],1,1,infochart,3,3,0); // blowing-up and looking to the x(
calculateI(L3[2][1][5],L3[2][1][4],3,3,3,L3[2][1][3],3,iniD(3)); // (I_3)
⇨ [1]:
⇨   [1]:
⇨       4
⇨   [2]:
⇨       2
⇨   [3]:
⇨   3
⇨ [2]:
⇨   [1]:
⇨       0
⇨   [2]:
⇨       0
⇨   [3]:
⇨   0
// looking to the x(1) chart
calculateI(L3[2][2][5],L3[2][2][4],3,3,1,L3[2][2][3],3,iniD(3)); // (I_3)
⇨ [1]:
⇨   [1]:
⇨       3
⇨   [2]:
⇨       2
⇨   [3]:
⇨       0
⇨ [2]:
⇨   [1]:
⇨       0
⇨   [2]:
⇨       0
⇨   [3]:
⇨       3
```

### D.5.12.16 Maxord

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:**      `Maxord(L,n)`; L list, n integer

**Compute:**   Find the maximal entry of a list, input is a list defining a monomial

**Return:**      maximum entry of a list and its position

**Example:**

```
LIB "resbinomial.lib";
ring r = 0,(x(1..3)),dp;
ideal J=x(1)^2*x(2)*x(3)^5;
list L=data(J,1,3);
L[2]; // list of exponents
⇒ [1]:
⇒ [1]:
⇒ 2
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ 5
Maxord(L[2][1][1],3);
⇒ 5 3
```

### D.5.12.17 Gamma

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:**      `Gamma(L,c,n)`; L list, c number, n integer

**Compute:**   The Gamma function, resolution function corresponding to the monomial case

**Return:**      lists of maximum exponents in L, value of Gamma function, center of blow up

**Example:**

```
LIB "resbinomial.lib";
ring r = 0,(x(1..5)),dp;
ideal J=x(1)^2*x(2)*x(3)^5*x(4)^2*x(5)^3;
list L=data(J,1,5);
list G=Gamma(L[2][1][1],9,5); // critical value c=9
G[1]; // maximum exponents in the ideal
⇒ [1]:
⇒ 5
⇒ [2]:
⇒ 3
⇒ [3]:
⇒ 2
G[2]; // maximal value of Gamma function
⇒ [1]:
⇒ -3
⇒ [2]:
⇒ 10/9
⇒ [3]:
⇒ 3,5,4
```

```

G[3]; // center given by Gamma
↳ [1]:
↳ 3
↳ [2]:
↳ 5
↳ [3]:
↳ 4

```

### D.5.12.18 convertdata

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:**      `convertdata(C,L,n,flaglist);`  
                  C, L, flaglist lists, n integer

**Compute:**    Compute the ideal corresponding to the given lists C,L

**Return:**      an ideal whose coefficients are given by C, exponents given by L

**Example:**

```

LIB "resbinomial.lib";
ring r = 0,(x(1..4),y(5)),dp;
list M=identifyvar();
ideal J=x(1)^2*y(5)^2-x(2)^2*x(3)^2,6*x(4)^2;
list L=data(J,2,5);
L[1]; // Coefficients
↳ [1]:
↳ [1]:
↳ -1
↳ [2]:
↳ 1
↳ [2]:
↳ [1]:
↳ 6
L[2]; // Exponents
↳ [1]:
↳ [1]:
↳ [1]:
↳ 0
↳ [2]:
↳ 2
↳ [3]:
↳ 2
↳ [4]:
↳ 0
↳ [5]:
↳ 0
↳ [2]:
↳ [1]:
↳ 2
↳ [2]:
↳ 0
↳ [3]:
↳ 0
↳ [4]:

```



```

⇒      0
⇒      [5]:
⇒      2
⇒ [2]:
⇒      [1]:
⇒      [1]:
⇒      0
⇒      [2]:
⇒      0
⇒      [3]:
⇒      0
⇒      [4]:
⇒      2
⇒      [5]:
⇒      0
ideal J2=convertdata(L[1],L[2],5,M);
J2;
⇒ J2[1]=-x(2)^2*x(3)^2+x(1)^2*y(5)^2
⇒ J2[2]=6*x(4)^2

```

### D.5.12.19 lcmofall

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:** `lcmofall(nchart,mobile);`  
`nchart` integer, `mobile` list of lists

**Compute:** Compute the lcm of the denominators of the E-orders of all the charts

**Return:** an integer given the lcm

**Note:** CALL BEFORE `salida`

**Example:**

```

LIB "resbinomial.lib";
ring r = 0,(x(1..2)),dp;
ideal J=x(1)^3-x(1)*x(2)^3;
list L=Eresol(J);
L[4]; // 8 charts, rational exponents
⇒ 8
L[8][2][2]; // E-orders at the first chart
⇒ [1]:
⇒ 3
⇒ [2]:
⇒ 9/2
lcmofall(8,L[8]);
⇒ 2

```

### D.5.12.20 computemcm

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:** `computemcm(Eolist);` `Eolist` list

**Return:** an integer, the least common multiple of the denominators of the E-orders

**Note:** Make the same as `lcmofall` but for one chart. NECESSARY BECAUSE THE E-ORDERS ARE OF TYPE NUMBER!!

**Example:**

```
LIB "resbinomial.lib";
ring r = 0,(x(1..2)),dp;
ideal J=x(1)^3-x(1)*x(2)^3;
list L=Eresol(J); // 8 charts, rational exponents
L[8][2][2]; // maximal E-order at the first chart
⇒ [1]:
⇒ 3
⇒ [2]:
⇒ 9/2
computemcm(L[8][2][2]);
⇒ 2
```

### D.5.12.21 constructH

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:** `constructH(Hhist,n,flag);`  
`Hhist` intvec, `n` integer, `flag` list

**Return:** the list of exceptional divisors accumulated at this chart

**Example:**

```
LIB "resbinomial.lib";
ring r = 0,(x(1..3)),dp;
list flag=identifyvar();
ideal J=x(1)^4*x(2)^2, x(1)^2+x(3)^3;
list L=Eresol(J); // 7 charts
// history of the exceptional divisors at the 7-th chart
L[1][7][7]; // blow ups at x(3)-th, x(1)-th and x(1)-th charts
⇒ 0,3,1,1
constructH(L[1][7][7],3,flag);
⇒ [1]:
⇒ _[1]=x(3)
⇒ [2]:
⇒ _[1]=1
⇒ [3]:
⇒ _[1]=x(1)
```

### D.5.12.22 constructblwup

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:** `constructblwup(blwhist,n,chy,flag);`  
`blwhist`, `flag` lists, `n` integer, `chy` ideal

**Return:** the ideal defining the map  $K[W] \rightarrow K[W_i]$ ,  
which gives the composition map of all the blowing up leading to this chart

**Note:** NECESSARY START WITH COLUMNS

**Example:**

```

LIB "resbinomial.lib";
ring r = 0,(x(1..3)),dp;
list flag=identifyvar();
ideal chy=maxideal(1);
ideal J=x(1)^4*x(2)^2, x(1)^2+x(3)^3;
list L=Eresol(J); // 7 charts
// history of the blow ups at the 7-th chart, center {x(1)=x(3)=0} every time
L[1][7][8]; // blow ups at x(3)-th, x(1)-th and x(1)-th charts
⇒ [1]:
⇒ 0,3,0,0
⇒ [2]:
⇒ 0,0,0,0
⇒ [3]:
⇒ 0,0,1,1
constructblwup(L[1][7][8],3,chy,flag);
⇒ _[1]=x(1)^3*x(3)
⇒ _[2]=x(2)
⇒ _[3]=x(1)^2*x(3)

```

### D.5.12.23 constructlastblwup

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:**      `constructlastblwup(blwhist,n,chy,flag);`  
              `blwhist, flag` lists, `n` integer, `chy` ideal

**Return:**    the ideal defining the last blow up

**Note:**      NECESSARY START WITH COLUMNS

**Example:**

```

LIB "resbinomial.lib";
ring r = 0,(x(1..3)),dp;
list flag=identifyvar();
ideal chy=maxideal(1);
ideal J=x(1)^4*x(2)^2, x(1)^2+x(3)^3;
list L=Eresol(J); // 7 charts
// history of the blow ups at the 7-th chart, center {x(1)=x(3)=0} every time
L[1][7][8]; // blow ups at x(3)-th, x(1)-th and x(1)-th charts
⇒ [1]:
⇒ 0,3,0,0
⇒ [2]:
⇒ 0,0,0,0
⇒ [3]:
⇒ 0,0,1,1
constructlastblwup(L[1][7][8],3,chy,flag);
⇒ _[1]=x(1)
⇒ _[2]=x(2)
⇒ _[3]=x(1)*x(3)

```

### D.5.12.24 genoutput

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:**      `genoutput(chart,mobile,nchart,nsons,n,q,p);`  
              `chart, mobile, nsons` lists, `nchart, n,q, p` integers

**Return:** two lists, the first one gives the rings corresponding to the final charts, the second one is the list of all rings corresponding to the affine charts of the resolution process

**Example:**

```
LIB "resbinomial.lib";
ring r = 0,(x(1..2)),dp;
ideal J=x(1)^3-x(1)*x(2)^3;
list L=Eresol(J);          // 8 charts, rational exponents
list B=genoutput(L[1],L[8],L[4],L[6],2,2,0); // generates the output
presentTree(B);
↳
↳ /////////////////////////////////////////////////// Final Chart 1 ///////////////////////////////////
↳ ===== History of this chart =====
↳
↳ Blow Up 1 :
↳   Center determined in L[2][1],
↳   Passing to chart 1 in resulting blow up.
↳
↳ ===== Data of this chart =====
↳
↳ ==== Ambient Space:
↳   _[1]=0
↳
↳ ==== Ideal of Variety:
↳   _[1]=-y(1)*y(2)^3+1
↳
↳ ==== Exceptional Divisors:
↳   [1]:
↳     _[1]=y(1)
↳
↳ ==== Images of variables of original ring:
↳   _[1]=y(1)
↳   _[2]=y(1)*y(2)
↳
↳ pause>
↳ /////////////////////////////////////////////////// Final Chart 2 ///////////////////////////////////
↳ ===== History of this chart =====
↳
↳ Blow Up 1 :
↳   Center determined in L[2][1],
↳   Passing to chart 2 in resulting blow up.
↳
↳ Blow Up 2 :
↳   Center determined in L[2][3],
↳   Passing to chart 1 in resulting blow up.
↳
↳ Blow Up 3 :
↳   Center determined in L[2][4],
↳   Passing to chart 1 in resulting blow up.
↳
↳ ===== Data of this chart =====
↳
↳ ==== Ambient Space:
```

```

⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=y(1)-1
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=1
⇒ [2]:
⇒   _[1]=y(1)
⇒ [3]:
⇒   _[1]=x(2)
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=y(1)^2*x(2)^3
⇒ _[2]=y(1)*x(2)^2
⇒
⇒ pause>
⇒ ////////////////////////////////// Final Chart 3 //////////////////////////////////
⇒ ===== History of this chart =====
⇒
⇒ Blow Up 1 :
⇒   Center determined in L[2][1],
⇒   Passing to chart 2 in resulting blow up.
⇒
⇒ Blow Up 2 :
⇒   Center determined in L[2][3],
⇒   Passing to chart 1 in resulting blow up.
⇒
⇒ Blow Up 3 :
⇒   Center determined in L[2][4],
⇒   Passing to chart 2 in resulting blow up.
⇒
⇒ ===== Data of this chart =====
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=-y(2)+1
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=y(2)
⇒ [2]:
⇒   _[1]=1
⇒ [3]:
⇒   _[1]=x(1)
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(1)^3*y(2)
⇒ _[2]=x(1)^2*y(2)
⇒

```

```

⇒ pause>
⇒ ////////////////////////////////// Final Chart 4 //////////////////////////////////
⇒ ===== History of this chart =====
⇒
⇒ Blow Up 1 :
⇒     Center determined in L[2][1],
⇒     Passing to chart 2 in resulting blow up.
⇒
⇒ Blow Up 2 :
⇒     Center determined in L[2][3],
⇒     Passing to chart 2 in resulting blow up.
⇒
⇒ Blow Up 3 :
⇒     Center determined in L[2][5],
⇒     Passing to chart 1 in resulting blow up.
⇒
⇒ ===== Data of this chart =====
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=y(1)^2*y(2)-1
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒     _[1]=1
⇒ [2]:
⇒     _[1]=y(2)
⇒ [3]:
⇒     _[1]=y(1)
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=y(1)*y(2)^2
⇒ _[2]=y(2)
⇒
⇒ pause>////////////////////////////////////\
/
⇒ For identification of exceptional divisors please use the tools
⇒ provided by reszeta.lib, e.g. collectDiv.
⇒ For viewing an illustration of the tree of charts please use the
⇒ procedure ResTree from resgraph.lib.
⇒ //////////////////////////////////
list iden0=collectDiv(B);
ResTree(B,iden0[1]);          // generates the resolution tree
⇒ sh: dot: Kommando nicht gefunden.
⇒ Currently showing graphics in separate window
⇒ Press <Return> to continue
⇒ pause>./examples/genoutput.sing 9> // Use presentTree(B); to see the fi\
    nal charts
⇒ sh: display: Kommando nicht gefunden.
// To see the tree type in another shell
// dot -Tjpg ResTree.dot -o ResTree.jpg

```

```
// /usr/bin/X11/xv ResTree.jpg
```

### D.5.12.25 salida

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:** `salida(idchart,chart,mobile,numson,previous,n,q,p);`  
`idchart, numson, n, q, p` integers, `chart, mobile, lists, previous` intvec

**Compute:** CONVERT THE OUTPUT OF A CHART IN A RING, WHERE DEFINE A BASIC OBJECT (BO)

**Return:** the ring corresponding to the chart

**Example:**

```
LIB "resbinomial.lib";
ring r = 0,(x(1..2)),dp;
ideal J=x(1)^2-x(2)^3;
list L=Eresol(J);
list B=salida(5,L[1][5],L[8][6],2,L[1][3][3],2,1,0); // chart 5
def RR=B[1];
setring RR;
B0;
⇒ [1]:
⇒   _[1]=0
⇒ [2]:
⇒   _[1]=x(1)-x(2)
⇒ [3]:
⇒   1,0
⇒ [4]:
⇒   [1]:
⇒     _[1]=x(2)
⇒   [2]:
⇒     _[1]=x(1)
⇒ [5]:
⇒     _[1]=x(1)^2*x(2)
⇒     _[2]=x(1)*x(2)
⇒ [6]:
⇒     0,0
⇒ [7]:
⇒     2,-1
⇒ [8]:
⇒     _[1,1]=0
⇒     _[1,2]=1
⇒     _[2,1]=0
⇒     _[2,2]=0
⇒ [9]:
⇒     0,0
"press return to see next example"; ~;
⇒ press return to see next example
⇒
⇒ -- break point in ./examples/salida.sing --
```

### D.5.12.26 iniD

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:**      `iniD(n)`;  $n$  integer

**Return:**     list of lists of zeros of size  $n$

**Example:**

```
LIB "resbinomial.lib";
iniD(3);
⇒ [1]:
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ 0
⇒ [2]:
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ 0
⇒ [3]:
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ 0
```

### D.5.12.27 `sumlist`

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:**      `sumlist(L1,L2)`;  $L1, L2$  lists, ( $\text{size}(L1) == \text{size}(L2)$ )

**Return:**     a list, sum of  $L1$  and  $L2$

**Example:**

```
LIB "resbinomial.lib";
list L1=1,2,3;
list L2=5,9,7;
sumlist(L1,L2);
⇒ [1]:
⇒ 6
⇒ [2]:
⇒ 11
⇒ [3]:
⇒ 10
```

### D.5.12.28 `reslist`

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:**      `reslist(L1,L2)`;  $L1, L2$  lists, ( $\text{size}(L1) == \text{size}(L2)$ )

**Return:**     a list, subtraction of  $L1$  and  $L2$



**Example:**

```

LIB "resbinomial.lib";
list L1=1,2,3;
list L2=5,9,7;
reslist(L1,L2);
⇒ [1]:
⇒ -4
⇒ [2]:
⇒ -7
⇒ [3]:
⇒ -4

```

**D.5.12.29 multiplylist**

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:**      `multiplylist(L,a)`; L list, a number

**Return:**    list of elements of type number, multiplication of L times a

**Example:**

```

LIB "resbinomial.lib";
ring r = 0,(x(1..3)),dp;
list L=1,2,3;
multiplylist(L,1/5);
⇒ [1]:
⇒ 1/5
⇒ [2]:
⇒ 2/5
⇒ [3]:
⇒ 3/5

```

**D.5.12.30 dividelist**

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:**      `dividelist(L1,L2)`; L1,L2 lists

**Return:**    list of elements of type number, division of L1 by L2

**Example:**

```

LIB "resbinomial.lib";
ring r = 0,(x(1..3)),dp;
list L1=1,2,3;
list L2=5,9,7;
dividelist(L1,L2);
⇒ [1]:
⇒ 1/5
⇒ [2]:
⇒ 2/9
⇒ [3]:
⇒ 3/7

```

### D.5.12.31 createlist

Procedure from library `resbinomial.lib` (see [Section D.5.12 \[resbinomial.lib\]](#), page 1476).

**Usage:** `createlist(L1,L2);` L1,L2 lists, (`size(L1)==size(L2)`)

**Return:** list of lists of two elements, the first one of L1 and the second of L2

**Example:**

```
LIB "resbinomial.lib";
list L1=1,2,3;
list L2=5,9,7;
createlist(L1,L2);
⇒ [1]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 5
⇒ [2]:
⇒ [1]:
⇒ 2
⇒ [2]:
⇒ 9
⇒ [3]:
⇒ [1]:
⇒ 3
⇒ [2]:
⇒ 7
```

### D.5.13 resgraph.lib

**Library:** `resgraph.lib`

**Purpose:** Visualization of Resolution Data

**Author:** A. Fruehbis-Krueger, [anne@mathematik.uni-kl.de](mailto:anne@mathematik.uni-kl.de),

**Note:** This library uses the external programs `surf`, `graphviz` and `imagemagick`.  
Input data is assumed to originate from `resolve.lib` and `reszeta.lib`

**Procedures:**

#### D.5.13.1 InterDiv

Procedure from library `resgraph.lib` (see [Section D.5.13 \[resgraph.lib\]](#), page 1524).

**Usage:** `InterDiv(M[,name]);`  
M = matrix  
name = string

**Assume:** - M is first list entry of output of 'intersectionDiv'  
from library `reszeta.lib`  
- write permission in the current directory or in the  
directory in which the file with name 'name' resides

**Create:** file 'name.jpg' containing dual graph of resolution  
if filename is given

- Note:** only available on UNIX-type systems and programs  
'display' (imagemagick package) and 'dot' (Graphviz package) need to be in the standard search PATH
- Return:** nothing, only generating graphics output in separate window

### D.5.13.2 ResTree

Procedure from library `resgraph.lib` (see [Section D.5.13 \[resgraph.lib\]](#), page 1524).

- Usage:** `ResTree(L,M[,name][,mark]);`  
`L` = list  
`M` = matrix  
`name` = string  
`mark` = intvec
- Assume:**
- `L` is the output of 'resolve' from `resolve.lib`
  - `M` is first entry of output of 'collectDiv(`L`);' from `reszeta.lib`
  - write permission in the current directory or in the directory in which the file with name '`name`' resides
  - `mark` intvec of size `size(L[2])`
  - `mark[i]=0` (default) border of box black
  - `mark[i]=1` border of box red
- Create:** file '`name.jpg`' containing the tree of charts of `L`  
if filename is given
- Note:** only available on UNIX-type systems and programs  
'display' (imagemagick package) and 'dot' (Graphviz package) need to be in the standard search PATH
- Return:** nothing, only generating graphics output in separate window

### D.5.13.3 finalCharts

Procedure from library `resgraph.lib` (see [Section D.5.13 \[resgraph.lib\]](#), page 1524).

- Usage:** `finalCharts(L1,L2,iv[,name]);`  
`L1` = list  
`L2` = list  
`iv` = intvec  
`name` = string
- Assume:**
- `L1` is the output of 'resolve' from `resolve.lib`
  - `L2` is the output of 'intersectionDiv(`L1`)' from `reszeta.lib`
  - `iv` is the first entry of the output of 'abstractR(`L1`)'
  - write permission in the current directory or in the directory in which the file with name '`name`' resides
- Create:**
- new windows in which surf-images of the final charts are presented
  - several '.ras' files in the directory in which '`name`' resides
- Note:** only available on UNIX-type systems  
external programs 'surf' and 'display' (imagemagick package) need to be in the standard search PATH
- Return:** nothing, only generating graphics output in separate window

### D.5.14 resjung\_lib

**Library:** resjung.lib

**Purpose:** Resolution of surface singularities (Desingularization) Algorithm of Jung

**Author:** Philipp Renner, philipp-renner@web.de

**Procedures:**

#### D.5.14.1 jungresolve

Procedure from library `resjung.lib` (see [Section D.5.14 \[resjung\\_lib\]](#), page 1526).

**Usage:**     jungresolve(ideal J[,is\_noeth]);  
               J = ideal  
               i = int

**Assume:**   J = two dimensional ideal

**Return:**    a list l of rings  
               l[i] is a ring containing two Ideals: QIdeal and BMap. BMap defines a birational morphism from  $V(\text{QIdeal}) \rightarrow V(J)$ , such that  $V(\text{QIdeal})$  is smooth. For this the algorithm computes first with `jungnormal` a representation of  $V(J)$  with Hirzebruch-Jung singularities and then it uses Villamayor's algorithm to resolve these singularities. If `is_noeth=1` the algorithm assumes J is in noether position with respect to the last two variables. As a default or if `is_noeth = 0` the algorithm computes a coordinate change such that J is in noether position. NOTE: since the noether position algorithm is randomized the performance can vary significantly.

**Example:**

```
LIB "resjung.lib";
//Computing a resolution of singularities of the variety z2-x3-y3
ring r = 0,(x,y,z),dp;
ideal I = z2-x3-y3;
//The ideal is in noether position
list l = jungresolve(I,1);
def R1 = l[1];
def R2 = l[2];
setring R1;
QIdeal;
⇨ QIdeal[1]=T(1)*x(2)^2*y(1)-T(2)*x
⇨ QIdeal[2]=T(2)*x(2)^2*y(1)-x
⇨ QIdeal[3]=x(2)^2*y(1)^2-x(2)^2*y(1)-T(1)*x
⇨ QIdeal[4]=T(1)^2-T(2)*y(1)+T(2)
⇨ QIdeal[5]=T(1)*T(2)-y(1)+1
⇨ QIdeal[6]=T(2)^2-T(1)
⇨ QIdeal[7]=x(2)^6*y(1)^4-x(2)^6*y(1)^3-x^3
BMap;
⇨ BMap[1]=x
⇨ BMap[2]=x(2)^2*y(1)
⇨ BMap[3]=x(2)^3*y(1)^2
setring R2;
QIdeal;
⇨ QIdeal[1]=T(2)*x(1)^2+x
⇨ QIdeal[2]=T(1)*x(1)^2*y(0)-T(2)*x
```

```

⇒ QIdeal[3]=x(1)^2*y(0)^2-x(1)^2*y(0)-T(1)*x
⇒ QIdeal[4]=T(1)^2-T(2)*y(0)+T(2)
⇒ QIdeal[5]=T(1)*T(2)+y(0)^2-y(0)
⇒ QIdeal[6]=T(2)^2+T(1)*y(0)
⇒ QIdeal[7]=x(1)^6*y(0)^3-x(1)^6*y(0)^2+x^3
BMap;
⇒ BMap[1]=x
⇒ BMap[2]=x(1)^2*y(0)
⇒ BMap[3]=x(1)^3*y(0)

```

### D.5.14.2 jungnormal

Procedure from library `resjung.lib` (see [Section D.5.14 \[resjung-lib\]](#), page 1526).

**Usage:**     jungnormal(ideal J[,is\_noeth]);  
               J = ideal  
               i = int

**Assume:**   J = two dimensional ideal

**Return:**    a list l of rings  
               l[i] is a ring containing two Ideals: QIdeal and BMap. BMap defines a birational morphism from  $V(\text{QIdeal}) \rightarrow V(J)$ , such that  $V(\text{QIdeal})$  has only singularities of Hizebuch-Jung type. If `is_noeth=1` the algorithm assumes J is in noether position with respect to the last two variables. As a default or if `is_noeth = 0` the algorithm computes a coordinate change such that J is in noether position. NOTE: since the noether position algorithm is randomized the performance can vary significantly.

**Example:**

```

LIB "resjung.lib";
//Computing a resolution of singularities of the variety z2-x3-y3
ring r = 0,(x,y,z),dp;
ideal I = z2-x3-y3;
//The ideal is in noether position
list l = jungnormal(I,1);
def R1 = l[1];
def R2 = l[2];
setring R1;
QIdeal;
⇒ QIdeal[1]=T(1)*x(2)^2*y(1)-T(2)*x
⇒ QIdeal[2]=T(2)*x(2)^2*y(1)-x
⇒ QIdeal[3]=-T(1)*x+x(2)^2*y(1)^2-x(2)^2*y(1)
⇒ QIdeal[4]=T(1)^2-T(2)*y(1)+T(2)
⇒ QIdeal[5]=T(1)*T(2)-y(1)+1
⇒ QIdeal[6]=T(2)^2-T(1)
⇒ QIdeal[7]=x(2)^6*y(1)^4-x(2)^6*y(1)^3-x^3
BMap;
⇒ BMap[1]=x
⇒ BMap[2]=x(2)^2*y(1)
⇒ BMap[3]=x(2)^3*y(1)^2
setring R2;
QIdeal;
⇒ QIdeal[1]=T(2)*x(1)^2+x
⇒ QIdeal[2]=T(1)*x(1)^2*y(0)-T(2)*x

```

```

⇒ QIdeal[3]=-T(1)*x+x(1)^2*y(0)^2-x(1)^2*y(0)
⇒ QIdeal[4]=T(1)^2-T(2)*y(0)+T(2)
⇒ QIdeal[5]=T(1)*T(2)+y(0)^2-y(0)
⇒ QIdeal[6]=T(2)^2+T(1)*y(0)
⇒ QIdeal[7]=x(1)^6*y(0)^3-x(1)^6*y(0)^2+x^3
BMap;
⇒ BMap[1]=x
⇒ BMap[2]=x(1)^2*y(0)
⇒ BMap[3]=x(1)^3*y(0)

```

### D.5.14.3 jungfib

Procedure from library `resjung.lib` (see [Section D.5.14 \[resjung\\_lib\]](#), page 1526).

**Usage:**     `jungfib(J[,is_noeth]);`  
               `J = ideal`  
               `j = int`

**Assume:**   `J = two dimensional ideal`

**Return:**    a list `l` of rings  
               `l[i]` is a ring containing two Ideals: `QIdeal` and `BMap`. `BMap` defines a birational morphism from  $V(QIdeal) \rightarrow V(J)$ , such that  $V(QIdeal)$  has only quasi-ordinary singularities.  
               If `is_noeth=1` the algorithm assumes `J` is in noether position with respect to the last two variables. As a default or if `is_noeth = 0` the algorithm computes a coordinate change such that `J` is in noether position. NOTE: since the noether position algorithm is randomized the performance can vary significantly.

**Example:**

```

LIB "resjung.lib";
//Computing a resolution of singularities of the variety z2-x3-y3
ring r = 0,(x,y,z),dp;
ideal I = z2-x3-y3;
//The ideal is in noether position
list l = jungfib(I,1);
def R1 = l[1];
def R2 = l[2];
setring R1;
QIdeal;
⇒ QIdeal[1]=x(2)^6*y(1)^4-x(2)^6*y(1)^3-x^3
BMap;
⇒ BMap[1]=x
⇒ BMap[2]=x(2)^2*y(1)
⇒ BMap[3]=x(2)^3*y(1)^2
setring R2;
QIdeal;
⇒ QIdeal[1]=x(1)^6*y(0)^3-x(1)^6*y(0)^2+x^3
BMap;
⇒ BMap[1]=x
⇒ BMap[2]=x(1)^2*y(0)
⇒ BMap[3]=x(1)^3*y(0)

```

### D.5.15 resolve\_lib

**Library:** resolve.lib

**Purpose:** Resolution of singularities (Desingularization) Algorithm of Villamayor

**Authors:** A. Fruehbs-Krueger, anne@mathematik.uni-kl.de,  
G. Pfister, pfister@mathematik.uni-kl.de

**References:**

- [1] J.Kollar: Lectures on Resolution of Singularities, Princeton University Press (2007)  
(contains large overview over various known methods for curves and surfaces as well as a detailed description of the approach in the general case)
- [2] A.Bravo, S.Encinas, O.Villamayor: A Simplified Proof of Desingularisation and Applications, Rev. Math. Iberoamericana 21 (2005), 349-458  
(description of the algorithmic proof of desingularization in characteristic zero which underlies this implementation)
- [3] A.Fruehbs-Krueger: Computational Aspects of Singularities, in J.-P. Brasselet, J.Damon et al.: Singularities in Geometry and Topology, World Scientific Publishing, 253–327 (2007)  
(chapter 4 contains a detailed discussion on algorithmic desingularization and efficiency aspects thereof)

**Procedures:**

### D.5.15.1 blowUp

Procedure from library `resolve.lib` (see [Section D.5.15 \[resolve.lib\]](#), page 1528).

**Usage:** blowUp(J,C[,W][,E]);  
W,J,C = ideals,  
E = list

**Assume:** J = ideal containing W ( W = 0 if not specified)  
C = ideal containing J  
E = list of smooth hypersurfaces (e.g. exceptional divisors)

**Note:** W the ideal of the ambient space, C the ideal of the center of the blowup and J the ideal of the variety  
Important difference to blowUp2:  
- the ambient space  $V(W)$  is blown up and  $V(J)$  transformed in it  
-  $V(C)$  is assumed to be non-singular

**Compute:** the blowing up of W in C, the exceptional locus, the strict transform of J and the blowup map

**Return:** list, say l, of size at most size(C),  
l[i] is the affine ring corresponding to the i-th chart each l[i] contains the ideals  
- aS, ideal of the blownup ambient space  
- sT, ideal of the strict transform  
- eD, ideal of the exceptional divisor  
- bM, ideal corresponding to the blowup map  
l[i] also contains a list BO, which can best be viewed with showBO(BO) detailed information on the data type BO can be viewed via the command showDataTypes();

**Example:**

```

LIB "resolve.lib";
ring R=0,(x,y),dp;
ideal J=x^2-y^3;
ideal C=x,y;
list blow=blowUp(J,C);
def Q=blow[1];
setring Q;
aS;
⇒ aS[1]=0
sT;
⇒ sT[1]=y(1)^2-x(2)
eD;
⇒ eD[1]=x(2)
bM;
⇒ bM[1]=x(2)*y(1)
⇒ bM[2]=x(2)

```

### D.5.15.2 blowUp2

Procedure from library `resolve.lib` (see [Section D.5.15 \[resolve.lib\]](#), page 1528).

**Usage:**      `blowUp2(J,C);`  
                  `J,C = ideals,`

**Assume:**    `C = ideal containing J`

**Note:**       `C` the ideal of the center of the blowup and `J` the ideal of the variety  
 Important differences to `blowUp`:  
 - `V(J)` itself is blown up, not the ambient space  
 - `C` is not assumed to be non-singular

**Compute:**   the blowing up of `J` in `C`, the exceptional locus and the blow-up map

**Return:**     list, say `l`, of size at most `size(C)`,  
                  `l[i]` is the affine ring corresponding to the `i`-th chart each `l[i]` contains the ideals  
                  - `Jnew`, ideal of the blowup `J`  
                  - `eD`, ideal of the new exceptional divisor  
                  - `bM`, ideal corresponding to the blowup map

**Example:**

```

LIB "resolve.lib";
ring r=0,(x,y,z),dp;
ideal I=z^2-x^3*y^2;
ideal C=z,xy;
list li=blowUp2(I,C);
size(li);                      // number of charts
⇒ 2
def S1=li[1];
setring S1;                     // chart 1
basing;
⇒ // coefficients: QQ
⇒ // number of vars : 3
⇒ //      block   1 : ordering dp
⇒ //                : names   x(1) x(3) y(2)
⇒ //      block   2 : ordering C

```



```

Jnew;
↪ Jnew[1]=x(1)*y(2)^2-1
eD;
↪ eD[1]=x(3)
↪ eD[2]=x(1)*y(2)^2-1
bM;
↪ bM[1]=x(1)
↪ bM[2]=x(3)*y(2)^3
↪ bM[3]=x(3)
def S2=li[2];
setring S2;                // chart 2
basing;
↪ // coefficients: QQ
↪ // number of vars : 2
↪ //          block  1 : ordering dp
↪ //          : names  x(2) y(1)
↪ //          block  2 : ordering C
Jnew;
↪ Jnew[1]=0
eD;
↪ eD[1]=x(2)*y(1)^2
bM;
↪ bM[1]=y(1)^2
↪ bM[2]=x(2)
↪ bM[3]=x(2)*y(1)^3

```

### D.5.15.3 Center

Procedure from library `resolve.lib` (see [Section D.5.15 \[resolve.lib\]](#), page 1528).

**Usage:**     Center(J,W)[,E])  
               J,W = ideals  
               E = list

**Assume:**   J = ideal containing W ( W = 0 if not specified)  
               E = list of smooth hypersurfaces (e.g. exceptional divisors)

**Compute:**   the center of the blow-up of J for the resolution algorithm of [Bravo,Encinas,Villamayor]

**Return:**    ideal, describing the center

**Example:**

```

LIB "resolve.lib";
ring R=0,(x,y),dp;
ideal J=x2-y3;
Center(J);
↪ _[1]=y
↪ _[2]=x

```

### D.5.15.4 resolve

Procedure from library `resolve.lib` (see [Section D.5.15 \[resolve.lib\]](#), page 1528).

**Usage:**     resolve (J); or resolve (J,i[,k]);  
               J ideal  
               i,k int

**Compute:** a resolution of  $J$ ,  
 if  $i > 0$  debugging is turned on according to the following switches:  
 j1: value 0 or 1; turn off or on correctness checks in all steps  
 j2: value 0 or 2; turn off or on debugCenter  
 j3: value 0 or 4; turn off or on debugBlowUp  
 j4: value 0 or 8; turn off or on debugCoeff  
 j5: value 0 or 16; turn off or on debugging of Intersection with  $E^\sim$   
 j6: value 0 or 32; turn off or on stop after pass through the loop  
 $i = j1 + j2 + j3 + j4 + j5 + j6$

**Return:** a list  $l$  of 2 lists of rings  
 $l[1][i]$  is a ring containing a basic object  $BO$ , the result of the resolution.  
 $l[2]$  contains all rings which occurred during the resolution process

**Note:** result may be viewed in a human readable form using `presentTree()`

**Example:**

```
LIB "resolve.lib";
ring R=0,(x,y,z),dp;
ideal J=x3+y5+yz2+xy4;
list L=resolve(J,0);
def Q=L[1][7];
setring Q;
showBO(BO);
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(1)^4*x(3)^2*y(1)+x(1)^2+y(1)+1
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒ _[1]=1
⇒ [2]:
⇒ _[1]=y(1)
⇒ [3]:
⇒ _[1]=1
⇒ [4]:
⇒ _[1]=x(1)
⇒ [5]:
⇒ _[1]=x(3)
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(1)^6*x(3)^5*y(1)^2
⇒ _[2]=x(1)^4*x(3)^3*y(1)
⇒ _[3]=x(1)^7*x(3)^6*y(1)^2
⇒
```

**D.5.15.5 showBO**

Procedure from library `resolve.lib` (see [Section D.5.15 \[resolve.lib\]](#), page 1528).

**Usage:** `showBO(BO);`  
 $BO$ =basic object, a list: ideal  $W$ ,

ideal J,  
 intvec b (already truncated for Coeff),  
 list Ex (already truncated for Coeff),  
 ideal ab,  
 intvec v,  
 intvec w (already truncated for Coeff),  
 matrix M

**Return:** nothing, only pretty printing

### D.5.15.6 presentTree

Procedure from library `resolve.lib` (see [Section D.5.15 \[resolve.lib\]](#), page 1528).

**Usage:** presentTree(L);  
 L=list, output of resolve

**Return:** nothing, only pretty printing of the output data of resolve()

### D.5.15.7 showDataTypes

Procedure from library `resolve.lib` (see [Section D.5.15 \[resolve.lib\]](#), page 1528).

**Usage:** showDataTypes();

**Return:** nothing, only pretty printing of extended version of help text

### D.5.15.8 blowUpBO

Procedure from library `resolve.lib` (see [Section D.5.15 \[resolve.lib\]](#), page 1528).

**Usage:** blowUpBO (BO,C,e);  
 BO = basic object, a list: ideal W,  
 ideal J,  
 intvec b,  
 list Ex,  
 ideal ab,  
 intvec v,  
 intvec w,  
 matrix M  
 C = ideal  
 e = integer (0 usual blowing up, 1 deleting extra charts, 2 deleting  
 no charts )

**Assume:** R = basering, a polynomial ring, W an ideal of R,  
 J = ideal containing W,  
 C = ideal containing J

**Compute:** the blowing up of BO[1] in C, the exceptional locus, the strict transform of BO[2]

**Note:** blowUpBO may be applied to basic objects in the sense of [Bravo, Encinas, Villamayor] in the following referred to as BO and to presentations in the sense of [Bierstone, Milman] in the following referred to as BM.

**Return:** a list  $l$  of length at most  $\text{size}(C)$ ,  
 $l[i]$  is a ring containing an object  $BO$  resp.  $BM$ :  
 $BO[1]=BM[1]$  an ideal, say  $W_i$ , defining the ambient space of the  $i$ -th chart of the blowing up  
 $BO[2]=BM[2]$  an ideal defining the strict transform  
 $BO[3]$  intvec, the first integer  $b$  such that in the original object  $(\Delta^b(BO[2]))==1$  the subsequent integers have the same property for Coeff-Objects of  $BO[2]$  used when determining the center  
 $BM[3]$  intvec,  $BM[3][i]$  is the assigned multiplicity of  $BM[2][i]$   
 $BO[4]=BM[4]$  the list of exceptional divisors  
 $BO[5]=BM[5]$  an ideal defining the map  $K[W] \longrightarrow K[W_i]$   
 $BO[6]=BM[6]$  an intvec  $BO[6][j]=1$  indicates that  $\langle BO[4][j], BO[2] \rangle = 1$ , i.e. the strict transform does not meet the  $j$ -th exceptional divisor  
 $BO[7]$  intvec, the index of the first blown-up object in the resolution process leading to this object for which the value of  $b$  was  $BO[3]$  the subsequent ones are the indices for the Coeff-Objects of  $BO[2]$  used when determining the center  
 $BM[7]$  intvec,  $BM[7][i]$  is the index at which the  $(2i-1)$ st entry of the invariant first reached its current maximal value  
 $BO[8]=BM[8]$  a matrix indicating that  $BO[4][i]$  meets  $BO[4][j]$  by  $BO[8][i,j]=1$  for  $i < j$   
 $BO[9]$  empty  
 $BM[9]$  the invariant

**Example:**

```
LIB "resolve.lib";
ring R=0,(x,y),dp;
ideal W;
ideal J=x^2-y^3;
intvec b=1;
list E;
ideal abb=maxideal(1);
intvec v;
intvec w=-1;
matrix M;
intvec ma;
list B0=W,J,b,E,abb,v,w,M,ma;
ideal C=CenterB0(B0)[1];
list blow=blowUpB0(B0,C,0);
def Q=blow[1];
setring Q;
B0;
⇒ [1]:
⇒   _[1]=0
⇒ [2]:
⇒   _[1]=y(1)^2-x(2)
⇒ [3]:
⇒   1
⇒ [4]:
⇒   [1]:
⇒   _[1]=x(2)
⇒ [5]:
⇒   _[1]=x(2)*y(1)
⇒   _[2]=x(2)
```

```

⇒ [6]:
⇒ 0
⇒ [7]:
⇒ -1
⇒ [8]:
⇒ _[1,1]=0
⇒ [9]:
⇒ 0

```

### D.5.15.9 createBO

Procedure from library `resolve.lib` (see [Section D.5.15 \[resolve.lib\]](#), page 1528).

**Usage:**     `createBO(J,W[,E]);`  
               `J,W = ideals`  
               `E = list`

**Assume:**   `J = ideal containing W ( W = 0 if not specified)`  
               `E = list of smooth hypersurfaces (e.g. exceptional divisors)`

**Return:**    list BO representing a basic object :  
               `BO[1]` ideal W, if W has been specified; `ideal(0)` otherwise `BO[2]` ideal J  
               `BO[3]` intvec  
               `BO[4]` the list E of exceptional divisors if specified; empty list otherwise  
               `BO[5]` an ideal defining the identity map  
               `BO[6]` an intvec  
               `BO[7]` intvec  
               `BO[8]` a matrix  
               entries 3,5,6,7,8 are initialized appropriately for use of `CenterBO` and `blowUpBO`

**Example:**

```

LIB "resolve.lib";
ring R=0,(x,y,z),dp;
ideal J=x2-y3;
createBO(J,ideal(z));
⇒ [1]:
⇒ _[1]=z
⇒ [2]:
⇒ _[1]=-y3+x2
⇒ [3]:
⇒ 0
⇒ [4]:
⇒ empty list
⇒ [5]:
⇒ _[1]=x
⇒ _[2]=y
⇒ _[3]=z
⇒ [6]:
⇒ 0
⇒ [7]:
⇒ -1
⇒ [8]:
⇒ _[1,1]=0

```

### D.5.15.10 CenterBO

Procedure from library `resolve.lib` (see [Section D.5.15 \[resolve.lib\], page 1528](#)).

**Usage:**      CenterBO(BO);  
              BO = basic object, a list: ideal W,  
              ideal J,  
              intvec b,  
              list Ex,  
              ideal ab,  
              intvec v,  
              intvec w,  
              matrix M

**Assume:**    R = basering, a polynomial ring, W an ideal of R,  
              J = ideal containing W

**Compute:**   the center of the next blow-up of BO in the resolution algorithm of  
              [Bravo,Encinas,Villamayor]

**Return:**     list l,  
              l[1]: ideal describing the center  
              l[2]: intvec w obtained in the process of determining l[1]  
              l[3]: intvec b obtained in the process of determining l[1]  
              l[4]: intvec inv obtained in the process of determining l[1]

**Example:**

```
LIB "resolve.lib";
ring R=0,(x,y),dp;
ideal W;
ideal J=x2-y3;
intvec b=1;
list E;
ideal abb=maxideal(1);
intvec v;
intvec w=-1;
matrix M;
list BO=W,J,b,E,abb,v,w,M,v;
CenterBO(BO);
⇒ [1]:
⇒   _[1]=y
⇒   _[2]=x
⇒ [2]:
⇒   -1
⇒ [3]:
⇒   2
⇒ [4]:
⇒   0
```

### D.5.15.11 Delta

Procedure from library `resolve.lib` (see [Section D.5.15 \[resolve.lib\], page 1528](#)).

**Usage:**      Delta (BO);  
              BO = basic object, a list: ideal W,

```

ideal J,
intvec b,
list Ex,
ideal ab,
intvec v,
intvec w,
matrix M

```

**Assume:**  $R = \text{basering}$ , a polynomial ring,  $W$  an ideal of  $R$ ,  
 $J = \text{ideal}$  containing  $W$

**Compute:** Delta-operator applied to  $J$  in the notation of  
[Bravo,Encinas,Villamayor]

**Return:** ideal

**Example:**

```

LIB "resolve.lib";
ring R=0,(x,y,z),dp;
ideal W=z^2-x;
ideal J=x*y^2+x^3;
intvec b=1;
list E;
ideal abb=maxideal(1);
intvec v;
intvec w=-1;
matrix M;
list BO=W,J,b,E,abb,v,w,M;
Delta(BO);
↪ _[1]=z2-x
↪ _[2]=xy
↪ _[3]=3x2z+y2z
↪ _[4]=x3

```

#### D.5.15.12 DeltaList

Procedure from library `resolve.lib` (see [Section D.5.15 \[resolve.lib\]](#), page 1528).

**Usage:** `DeltaList (BO);`  
 $BO = \text{basic object}$ , a list: ideal  $W$ ,  
ideal  $J$ ,  
intvec  $b$ ,  
list  $Ex$ ,  
ideal  $ab$ ,  
intvec  $v$ ,  
intvec  $w$ ,  
matrix  $M$

**Assume:**  $R = \text{basering}$ , a polynomial ring,  $W$  an ideal of  $R$ ,  
 $J = \text{ideal}$  containing  $W$

**Compute:** Delta-operator iteratively applied to  $J$  in the notation of [Bravo,Encinas,Villamayor]

**Return:** list  $l$  of length  $((\max w\text{-ord}) * b)$ ,  
 $l[i+1] = \text{Delta}^i(J)$

**Example:**

```

LIB "resolve.lib";
ring R=0,(x,y,z),dp;
ideal W=z^2-x;
ideal J=x*y^2+x^3;
intvec b=1;
list E;
ideal abb=maxideal(1);
intvec v;
intvec w=-1;
matrix M;
list B0=W,J,b,E,abb,v,w,M;
DeltaList(B0);
⇒ [1]:
⇒   _[1]=x3+xy2
⇒ [2]:
⇒   _[1]=z2-x
⇒   _[2]=xy
⇒   _[3]=3x2z+y2z
⇒   _[4]=x3
⇒ [3]:
⇒   _[1]=x
⇒   _[2]=z2
⇒   _[3]=yz
⇒   _[4]=y2
⇒ [4]:
⇒   _[1]=z
⇒   _[2]=y
⇒   _[3]=x

```

### D.5.16 reszeta.lib

**Library:** reszeta.lib

**Purpose:** topological Zeta-function and some other applications of desingularization

**Authors:** A. Fruehbis-Krueger, anne@mathematik.uni-kl.de,  
G. Pfister, pfister@mathematik.uni-kl.de

**References:**

- [1] Fruehbis-Krueger, A., Pfister, G.: Some Applications of Resolution of Singularities from a Practical Point of View, in Computational Commutative and Non-commutative Algebraic Geometry, NATO Science Series III, Computer and Systems Sciences 196, 104-117 (2005) [2] Fruehbis-Krueger: An Application of Resolution of Singularities: Computing the topological Zeta-function of isolated surface singularities in  $(C^3, 0)$ , in D. Cheniot, N. Dutertre et al. (Editors): Singularity Theory, World Scientific Publishing (2007)

**Procedures:**

#### D.5.16.1 intersectionDiv

Procedure from library `reszeta.lib` (see [Section D.5.16 \[reszeta.lib\]](#), page 1538).

**Usage:** intersectionDiv(L);  
L = list of rings



- Assume:** L is output of resolution of singularities  
(only case of isolated surface singularities)
- Compute:** intersection matrix and genera of the exceptional divisors (considered as curves on the strict transform)
- Return:** list l, where  
l[1]: intersection matrix of exceptional divisors  
l[2]: intvec, genera of exceptional divisors  
l[3]: divisorList, encoding the identification of the divisors

**Example:**

```
LIB "reszeta.lib";
ring r = 0,(x(1..3)),dp(3);
ideal J=x(3)^5+x(2)^4+x(1)^3+x(1)*x(2)*x(3);
list re=resolve(J);
list di=intersectionDiv(re);
di;
⇒ [1]:
⇒   -3,1,1,
⇒   1,-4,1,
⇒   1,1,-2
⇒ [2]:
⇒   0,0,0
⇒ [3]:
⇒   [1]:
⇒       [1]:
⇒       3,1,1
⇒       [2]:
⇒       5,1,1
⇒   [2]:
⇒       [1]:
⇒       3,1,2
⇒       [2]:
⇒       4,1,1
⇒       [3]:
⇒       5,1,2
⇒   [3]:
⇒       [1]:
⇒       4,2,1
⇒       [2]:
⇒       5,2,1
⇒ [4]:
⇒   1,1,1
```

**D.5.16.2 spectralNeg**

Procedure from library `reszeta.lib` (see [Section D.5.16 \[reszeta.lib\]](#), page 1538).

- Usage:** spectralNeg(L);  
L = list of rings
- Assume:** L is output of resolution of singularities
- Return:** list of numbers, each a spectral number in  $(-1,0]$

```
LIB "reszeta.lib";
ring R=0,(x,y,z),dp;
ideal I=x^3+y^4+z^5;
list L=resolve(I,"K");
spectralNeg(L);
--> [1]:
--> -13/60
--> [2]:
--> -1/60
LIB "gmssing.lib";
ring r=0,(x,y,z),ds;
poly f=x^3+y^4+z^5;
spectrum(f);
--> [1]:
--> _[1]=-13/60
--> _[2]=-1/60
--> _[3]=1/30
--> _[4]=7/60
--> _[5]=11/60
--> _[6]=7/30
--> _[7]=17/60
--> _[8]=19/60
--> _[9]=11/30
--> _[10]=23/60
--> _[11]=13/30
--> _[12]=29/60
--> _[13]=31/60
--> _[14]=17/30
--> _[15]=37/60
--> _[16]=19/30
--> _[17]=41/60
--> _[18]=43/60
--> _[19]=23/30
--> _[20]=49/60
--> _[21]=53/60
--> _[22]=29/30
--> _[23]=61/60
--> _[24]=73/60
--> [2]:
--> 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1
```

**Example:**

```

LIB "reszeta.lib";
ring R=0,(x,y,z),dp;
ideal I=x2+y2+z3;
list re=resolve(I);
discrepancy(re);
↦ 0,1,1

```

#### D.5.16.4 zetaDL

Procedure from library `reszeta.lib` (see [Section D.5.16 \[reszeta.lib\]](#), page 1538).

**Assume:** L is the output of resolution of singularities

**Compute:** local Denef-Loeser zeta function, if string `s1` is present and has the value 'local'; global Denef-Loeser zeta function otherwise  
if string `s1` or `s2` has the value "A", additionally the characteristic polynomial of the monodromy is computed

**Return:** list `l`  
if `a` is not present:  
`l[1]`: string specifying the top. zeta function  
`l[2]`: string specifying characteristic polynomial of monodromy, if "A" was specified  
if `a` is present:  
`l[1]`: string specifying the top. zeta function  
`l[2]`: list `ast`,  
`ast[1]`=`chi(Ei^*)`  
`ast[2]`=`chi(Eij^*)`  
`ast[3]`=`chi(Eijk^*)`  
`l[3]`: intvec `nu` of multiplicities as needed in computation of zeta function  
`l[4]`: intvec `N` of multiplicities as needed in computation of zeta function  
`l[5]`: string specifying characteristic polynomial of monodromy, if "A" was specified

**Example:**

```

LIB "reszeta.lib";
ring R=0,(x,y,z),dp;
ideal I=x2+y2+z3;
list re=resolve(I,"K");
zetaDL(re,1);
↦ [1]:
↦ (s+4)/(3s2+7s+4)
I=(xz+y2)*(xz+y2+x2)+z5;
list L=resolve(I,"K");
zetaDL(L,1);
↦ [1]:
↦ (20s2+130s+87)/(160s3+396s2+323s+87)
//===== expected zeta function =====
// (20s^2+130s+87)/((1+s)*(3+4s)*(29+40s))
//=====

```

#### D.5.16.5 collectDiv

Procedure from library `reszeta.lib` (see [Section D.5.16 \[reszeta.lib\]](#), page 1538).

**Usage:** `collectDiv(L);`  
L = list of rings

**Assume:** L is output of resolution of singularities

**Compute:** list representing the identification of the exceptional divisors in the various charts

**Return:** list l, where  
 l[1]: intmat, entry k in position i,j implies BO[4][j] of chart i is divisor k (if k!=0)  
 if k==0, no divisor corresponding to i,j  
 l[2]: list ll, where each entry of ll is a list of intvecs entry i,j in list ll[k] implies BO[4][j]  
 of chart i  
 is divisor k  
 l[3]: list L

**Example:**

```
LIB "reszeta.lib";
ring R=0,(x,y,z),dp;
ideal I=xyz+x4+y4+z4;
//we really need to blow up curves even if the generic point of
//the curve the total transform is n.c.
//this occurs here in r[2][5]
list re=resolve(I);
list di=collectDiv(re);
di[1];
  ↪ 0,0,0,
  ↪ 1,0,0,
  ↪ 1,0,0,
  ↪ 1,0,0,
  ↪ 1,2,0,
  ↪ 1,2,0,
  ↪ 1,3,0,
  ↪ 1,3,0,
  ↪ 1,4,0,
  ↪ 1,4,0,
  ↪ 0,2,5,
  ↪ 1,0,5,
  ↪ 0,2,5,
  ↪ 1,0,5,
  ↪ 0,3,6,
  ↪ 1,0,6,
  ↪ 0,3,6,
  ↪ 1,0,6,
  ↪ 0,4,7,
  ↪ 1,0,7,
  ↪ 0,4,7,
  ↪ 1,0,7
di[2];
  ↪ [1]:
  ↪     [1]:
  ↪         2,1
  ↪     [2]:
  ↪         3,1
  ↪     [3]:
  ↪         4,1
  ↪     [4]:
  ↪         5,1
```

```

    ↪      [5]:
    ↪      6,1
    ↪      [6]:
    ↪      7,1
    ↪      [7]:
    ↪      8,1
    ↪      [8]:
    ↪      9,1
    ↪      [9]:
    ↪      10,1
    ↪      [10]:
    ↪      12,1
    ↪      [11]:
    ↪      14,1
    ↪      [12]:
    ↪      16,1
    ↪      [13]:
    ↪      18,1
    ↪      [14]:
    ↪      20,1
    ↪      [15]:
    ↪      22,1
    ↪ [2]:
    ↪      [1]:
    ↪      5,2
    ↪      [2]:
    ↪      6,2
    ↪      [3]:
    ↪      11,2
    ↪      [4]:
    ↪      13,2
    ↪ [3]:
    ↪      [1]:
    ↪      7,2
    ↪      [2]:
    ↪      8,2
    ↪      [3]:
    ↪      15,2
    ↪      [4]:
    ↪      17,2
    ↪ [4]:
    ↪      [1]:
    ↪      9,2
    ↪      [2]:
    ↪      10,2
    ↪      [3]:
    ↪      19,2
    ↪      [4]:
    ↪      21,2
    ↪ [5]:
    ↪      [1]:
    ↪      11,3
    ↪      [2]:

```

```

↳      12,3
↳      [3]:
↳      13,3
↳      [4]:
↳      14,3
↳ [6]:
↳      [1]:
↳      15,3
↳      [2]:
↳      16,3
↳      [3]:
↳      17,3
↳      [4]:
↳      18,3
↳ [7]:
↳      [1]:
↳      19,3
↳      [2]:
↳      20,3
↳      [3]:
↳      21,3
↳      [4]:
↳      22,3
↳ [8]:
↳      [1]:
↳      11,0
↳      [2]:
↳      12,0
↳      [3]:
↳      13,0
↳      [4]:
↳      14,0
↳      [5]:
↳      15,0
↳      [6]:
↳      16,0
↳      [7]:
↳      17,0
↳      [8]:
↳      18,0
↳      [9]:
↳      19,0
↳      [10]:
↳      20,0
↳      [11]:
↳      21,0
↳      [12]:
↳      22,0

```

### D.5.16.6 prepEmbDiv

Procedure from library `reszeta.lib` (see [Section D.5.16 \[reszeta.lib\]](#), page 1538).

**Usage:**     `prepEmbDiv(L[,a]);`  
               `L` = list of rings  
               `a` = integer

**Assume:**   `L` is output of resolution of singularities

**Compute:**   if `a` is not present: exceptional divisors including components of the strict transform  
               otherwise: only exceptional divisors

**Return:**    list of  $\mathbb{Q}$ -irreducible exceptional divisors (embedded case)

**Example:**

```
LIB "reszeta.lib";
ring R=0,(x,y,z),dp;
ideal I=x2+y2+z11;
list L=resolve(I);
prepEmbDiv(L);
⇨ [1]:
⇨   [1]:
⇨     2,1
⇨   [2]:
⇨     3,1
⇨   [3]:
⇨     4,1
⇨ [2]:
⇨   [1]:
⇨     4,2
⇨   [2]:
⇨     5,2
⇨   [3]:
⇨     6,2
⇨ [3]:
⇨   [1]:
⇨     6,3
⇨   [2]:
⇨     7,3
⇨   [3]:
⇨     8,3
⇨ [4]:
⇨   [1]:
⇨     8,4
⇨   [2]:
⇨     9,4
⇨   [3]:
⇨    10,4
⇨ [5]:
⇨   [1]:
⇨    10,5
⇨   [2]:
⇨    11,5
⇨   [3]:
⇨    12,5
⇨   [4]:
⇨    13,5
⇨   [5]:
```

```

⇒      15,5
⇒      [6]:
⇒      17,5
⇒ [6]:
⇒      [1]:
⇒      12,6
⇒      [2]:
⇒      13,6
⇒      [3]:
⇒      14,6
⇒      [4]:
⇒      16,6
⇒ [7]:
⇒      [1]:
⇒      14,7
⇒      [2]:
⇒      15,7
⇒      [3]:
⇒      16,7
⇒      [4]:
⇒      17,7
⇒ [8]:
⇒      [1]:
⇒      3,2
⇒      [2]:
⇒      4,3
⇒      [3]:
⇒      6,4
⇒      [4]:
⇒      8,5
⇒      [5]:
⇒      10,6
⇒      [6]:
⇒      14,8
⇒      [7]:
⇒      15,8
⇒      [8]:
⇒      16,8
⇒      [9]:
⇒      17,8

```

### D.5.16.7 abstractR

Procedure from library `reszeta.lib` (see [Section D.5.16 \[reszeta.lib\]](#), page 1538).

**Usage:**     `abstractR(L);`  
               `L = list of rings`

**Assume:**   `L` is output of resolution of singularities

**Note:**      currently only implemented for isolated surface singularities

**Return:**    list `l`  
               `l[1]`: intvec, where  
               `l[1][i]=1` if the corresponding ring is a final chart



of non-embedded resolution  
 $l[1][i]=0$  otherwise  
 $l[2]$ : intvec, where  
 $l[2][i]=1$  if the corresponding ring does not occur  
in the non-embedded resolution  
 $l[2][i]=0$  otherwise  
 $l[3]$ : list L

**Example:**

```
LIB "reszeta.lib";
ring r = 0,(x,y,z),dp;
ideal I=x2+y2+z11;
list L=resolve(I);
list absR=abstractR(L);
absR[1];
⇒ 0,0,1,1,0,1,0,1,0,1,1,0,0,0,0,0,0
absR[2];
⇒ 0,0,0,0,0,0,0,0,0,0,0,1,1,1,1,1,1
```

**D.5.16.8 computeV**

Procedure from library `reszeta.lib` (see [Section D.5.16 \[reszeta.lib\]](#), page 1538).

**Usage:**      `computeV(L,DL);`  
 $L$  = list of rings  
 $DL$  = divisor list

**Assume:**     $L$  has structure of output of `resolve`  
 $DL$  has structure of output of `prepEmbDiv`

**Return:**    intvec,  
 $i$ -th entry is multiplicity of  $i$ -th divisor in  
pullback of volume form

**Example:**

```
LIB "reszeta.lib";
ring R=0,(x,y,z),dp;
ideal I=(x-y)*(x-z)*(y-z)-z4;
list re=resolve(I,1);
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 0
⇒ Total number of charts currently in chart-tree: 1
⇒ Index of the current chart in chart-tree: 1
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=-z4+x2y-xy2-x2z+y2z+xz2-yz2
⇒
⇒ ==== Exceptional Divisors:
⇒ empty list
⇒
```

```

↳ ==== Images of variables of original ring:
↳ _[1]=x
↳ _[2]=y
↳ _[3]=z
↳
↳ ----- Upcoming Center -----
↳ _[1]=z
↳ _[2]=y
↳ _[3]=x
↳ -----
↳ ++++++ Overview of Current Chart ++++++
↳ Current number of final charts: 0
↳ Total number of charts currently in chart-tree: 4
↳ Index of the current chart in chart-tree: 2
↳ ++++++
↳
↳ ==== Ambient Space:
↳ _[1]=0
↳
↳ ==== Ideal of Variety:
↳ _[1]=y(1)^2*y(2)-y(1)*y(2)^2-y(1)^2+y(2)^2+x(3)+y(1)-y(2)
↳
↳ ==== Exceptional Divisors:
↳ [1]:
↳   _[1]=x(3)
↳
↳ ==== Images of variables of original ring:
↳ _[1]=x(3)*y(2)
↳ _[2]=x(3)*y(1)
↳ _[3]=x(3)
↳
↳ ----- Upcoming Center -----
↳ _[1]=y(2)-1
↳ _[2]=y(1)-1
↳ _[3]=x(3)
↳ -----
↳ ++++++ Overview of Current Chart ++++++
↳ Current number of final charts: 0
↳ Total number of charts currently in chart-tree: 6
↳ Index of the current chart in chart-tree: 3
↳ ++++++
↳
↳ ==== Ambient Space:
↳ _[1]=0
↳
↳ ==== Ideal of Variety:
↳ _[1]=x(2)*y(0)^4-y(0)^2*y(2)+y(0)*y(2)^2+y(0)^2-y(2)^2-y(0)+y(2)
↳
↳ ==== Exceptional Divisors:
↳ [1]:
↳   _[1]=x(2)
↳
↳ ==== Images of variables of original ring:

```

```

↳ _[1]=x(2)*y(2)
↳ _[2]=x(2)
↳ _[3]=x(2)*y(0)
↳
↳ ----- Upcoming Center -----
↳ _[1]=y(2)-1
↳ _[2]=y(0)-1
↳ _[3]=x(2)
↳ -----
↳ ++++++ Overview of Current Chart ++++++
↳ Current number of final charts: 0
↳ Total number of charts currently in chart-tree: 8
↳ Index of the current chart in chart-tree: 4
↳ ++++++
↳
↳ ==== Ambient Space:
↳ _[1]=0
↳
↳ ==== Ideal of Variety:
↳ _[1]=x(1)*y(0)^4+y(0)^2*y(1)-y(0)*y(1)^2-y(0)^2+y(1)^2+y(0)-y(1)
↳
↳ ==== Exceptional Divisors:
↳ [1]:
↳   _[1]=x(1)
↳
↳ ==== Images of variables of original ring:
↳ _[1]=x(1)
↳ _[2]=x(1)*y(1)
↳ _[3]=x(1)*y(0)
↳
↳ ----- Upcoming Center -----
↳ _[1]=y(1)-1
↳ _[2]=y(0)-1
↳ _[3]=x(1)
↳ -----
↳ ++++++ Overview of Current Chart ++++++
↳ Current number of final charts: 0
↳ Total number of charts currently in chart-tree: 10
↳ Index of the current chart in chart-tree: 5
↳ ++++++
↳
↳ ==== Ambient Space:
↳ _[1]=0
↳
↳ ==== Ideal of Variety:
↳ _[1]=x(3)^2*y(1)^2-x(3)^2*y(1)-2*x(3)*y(1)^2+2*x(3)*y(1)+y(1)^2-y(1)+y(2)
↳
↳ ==== Exceptional Divisors:
↳ [1]:
↳   _[1]=y(2)
↳ [2]:
↳   _[1]=x(3)-1
↳

```

```

⇒ ==== Images of variables of original ring:
⇒ _[1]=x(3)^2*y(2)-x(3)*y(2)
⇒ _[2]=x(3)^2*y(1)*y(2)-2*x(3)*y(1)*y(2)+x(3)*y(2)+y(1)*y(2)-y(2)
⇒ _[3]=x(3)*y(2)-y(2)
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=y(2)
⇒ _[2]=x(3)-1
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 0
⇒ Total number of charts currently in chart-tree: 12
⇒ Index of the current chart in chart-tree: 6
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(2)^2*y(0)^2-x(2)^2*y(0)-2*x(2)*y(0)^2+2*x(2)*y(0)+y(0)^2-y(0)-y(2)
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=y(2)
⇒ [2]:
⇒   _[1]=x(2)-1
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(2)^2*y(0)*y(2)-2*x(2)*y(0)*y(2)+x(2)*y(2)+y(0)*y(2)-y(2)
⇒ _[2]=x(2)^2*y(2)-x(2)*y(2)
⇒ _[3]=x(2)*y(2)-y(2)
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=y(2)
⇒ _[2]=x(2)-1
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 0
⇒ Total number of charts currently in chart-tree: 14
⇒ Index of the current chart in chart-tree: 7
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(3)^4*y(1)^4*y(2)-4*x(3)^3*y(1)^4*y(2)+4*x(3)^3*y(1)^3*y(2)+6*x(3)^3*y(1)^2*y(2)-12*x(3)^2*y(1)^3*y(2)-4*x(3)*y(1)^4*y(2)+6*x(3)^2*y(1)^2*y(2)+12*x(3)*y(1)^3*y(2)+y(1)^4*y(2)-x(3)^2*y(1)^2-12*x(3)*y(1)^2*y(2)-4*y(1)^3*y(2)+x(3)^2*y(1)+2*x(3)*y(1)^2+4*x(3)*y(1)*y(2)+6*y(1)^2*y(2)-2*x(3)*y(1)-y(1)^2-4*y(1)*y(2)+y(1)+y(2)
⇒
⇒ ==== Exceptional Divisors:

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⇒ [1]:
⇒   _[1]=y(2)
⇒ [2]:
⇒   _[1]=x(3)-1
⇒
⇒ ==== Images of variables of original ring:
⇒   _[1]=x(3)^2*y(2)-x(3)*y(2)
⇒   _[2]=x(3)*y(2)-y(2)
⇒   _[3]=x(3)^2*y(1)*y(2)-2*x(3)*y(1)*y(2)+x(3)*y(2)+y(1)*y(2)-y(2)
⇒
⇒ ----- Upcoming Center -----
⇒   _[1]=y(2)
⇒   _[2]=x(3)-1
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 0
⇒ Total number of charts currently in chart-tree: 16
⇒ Index of the current chart in chart-tree: 8
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒   _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒   _[1]=x(2)^4*y(2)+x(2)^2*y(0)^2-x(2)^2*y(0)-2*x(2)*y(0)^2+2*x(2)*y(0)+y(0)\
    ^2-y(0)
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=y(2)
⇒ [2]:
⇒   _[1]=x(2)-1
⇒
⇒ ==== Images of variables of original ring:
⇒   _[1]=x(2)^2*y(0)*y(2)-2*x(2)*y(0)*y(2)+x(2)*y(2)+y(0)*y(2)-y(2)
⇒   _[2]=x(2)*y(2)-y(2)
⇒   _[3]=x(2)^2*y(2)-x(2)*y(2)
⇒
⇒ ----- Upcoming Center -----
⇒   _[1]=y(2)
⇒   _[2]=x(2)-1
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 0
⇒ Total number of charts currently in chart-tree: 18
⇒ Index of the current chart in chart-tree: 9
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒   _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒   _[1]=x(3)^4*y(1)^4*y(2)-4*x(3)^3*y(1)^4*y(2)+4*x(3)^3*y(1)^3*y(2)+6*x(3)^\

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2*y(1)^4*y(2)-12*x(3)^2*y(1)^3*y(2)-4*x(3)*y(1)^4*y(2)+6*x(3)^2*y(1)^2*y(\
2)+12*x(3)*y(1)^3*y(2)+y(1)^4*y(2)+x(3)^2*y(1)^2-12*x(3)*y(1)^2*y(2)-4*y(\
1)^3*y(2)-x(3)^2*y(1)-2*x(3)*y(1)^2+4*x(3)*y(1)*y(2)+6*y(1)^2*y(2)+2*x(3)\
*y(1)+y(1)^2-4*y(1)*y(2)-y(1)+y(2)
↳
↳ ===== Exceptional Divisors:
↳ [1]:
↳   _[1]=y(2)
↳ [2]:
↳   _[1]=x(3)-1
↳
↳ ===== Images of variables of original ring:
↳ _[1]=x(3)*y(2)-y(2)
↳ _[2]=x(3)^2*y(2)-x(3)*y(2)
↳ _[3]=x(3)^2*y(1)*y(2)-2*x(3)*y(1)*y(2)+x(3)*y(2)+y(1)*y(2)-y(2)
↳
↳ ----- Upcoming Center -----
↳ _[1]=y(2)
↳ _[2]=x(3)-1
↳ -----
↳ ++++++ Overview of Current Chart ++++++
↳ Current number of final charts: 0
↳ Total number of charts currently in chart-tree: 20
↳ Index of the current chart in chart-tree: 10
↳ ++++++
↳
↳ ===== Ambient Space:
↳ _[1]=0
↳
↳ ===== Ideal of Variety:
↳ _[1]=x(2)^4*y(2)-x(2)^2*y(0)^2+x(2)^2*y(0)+2*x(2)*y(0)^2-2*x(2)*y(0)-y(0)\
^2+y(0)
↳
↳ ===== Exceptional Divisors:
↳ [1]:
↳   _[1]=y(2)
↳ [2]:
↳   _[1]=x(2)-1
↳
↳ ===== Images of variables of original ring:
↳ _[1]=x(2)*y(2)-y(2)
↳ _[2]=x(2)^2*y(0)*y(2)-2*x(2)*y(0)*y(2)+x(2)*y(2)+y(0)*y(2)-y(2)
↳ _[3]=x(2)^2*y(2)-x(2)*y(2)
↳
↳ ----- Upcoming Center -----
↳ _[1]=y(2)
↳ _[2]=x(2)-1
↳ -----
↳ ++++++ Overview of Current Chart ++++++
↳ Current number of final charts: 0
↳ Total number of charts currently in chart-tree: 22
↳ Index of the current chart in chart-tree: 11
↳ ++++++

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↳
↳ ==== Ambient Space:
↳ _[1]=0
↳
↳ ==== Ideal of Variety:
↳ _[1]=x(2)^2*x(3)*y(1)^2-x(2)*x(3)*y(1)^2+1
↳
↳ ==== Exceptional Divisors:
↳ [1]:
↳   _[1]=1
↳ [2]:
↳   _[1]=y(1)
↳ [3]:
↳   _[1]=x(3)
↳
↳ ==== Images of variables of original ring:
↳ _[1]=x(3)^3*y(1)^2+x(3)^2*y(1)
↳ _[2]=x(2)*x(3)^3*y(1)^2+x(3)^2*y(1)
↳ _[3]=x(3)^2*y(1)
↳
↳ ----- Upcoming Center -----
↳ _[1]=x(2)^2*x(3)*y(1)^2-x(2)*x(3)*y(1)^2+1
↳ -----
↳ ++++++ Overview of Current Chart ++++++
↳ Current number of final charts: 0
↳ Total number of charts currently in chart-tree: 22
↳ Index of the current chart in chart-tree: 12
↳ ++++++
↳
↳ ==== Ambient Space:
↳ _[1]=0
↳
↳ ==== Ideal of Variety:
↳ _[1]=x(1)*x(2)^2-x(1)*x(2)-x(2)^2+x(2)+y(0)
↳
↳ ==== Exceptional Divisors:
↳ [1]:
↳   _[1]=y(0)
↳ [2]:
↳   _[1]=1
↳ [3]:
↳   _[1]=x(1)-1
↳
↳ ==== Images of variables of original ring:
↳ _[1]=x(1)^3*y(0)-2*x(1)^2*y(0)+x(1)*y(0)
↳ _[2]=x(1)^3*x(2)*y(0)-3*x(1)^2*x(2)*y(0)+x(1)^2*y(0)+3*x(1)*x(2)*y(0)-2*x\
    (1)*y(0)-x(2)*y(0)+y(0)
↳ _[3]=x(1)^2*y(0)-2*x(1)*y(0)+y(0)
↳
↳ ----- Upcoming Center -----
↳ _[1]=y(0)
↳ _[2]=x(1)-1
↳ -----

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⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 0
⇒ Total number of charts currently in chart-tree: 24
⇒ Index of the current chart in chart-tree: 13
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(2)^2*x(3)*y(1)^2-x(2)*x(3)*y(1)^2-1
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=1
⇒ [2]:
⇒   _[1]=y(1)
⇒ [3]:
⇒   _[1]=x(3)
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(2)*x(3)^3*y(1)^2+x(3)^2*y(1)
⇒ _[2]=x(3)^3*y(1)^2+x(3)^2*y(1)
⇒ _[3]=x(3)^2*y(1)
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=x(2)^2*x(3)*y(1)^2-x(2)*x(3)*y(1)^2-1
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 0
⇒ Total number of charts currently in chart-tree: 24
⇒ Index of the current chart in chart-tree: 14
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(1)*x(2)^2-x(1)*x(2)-x(2)^2+x(2)-y(0)
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=y(0)
⇒ [2]:
⇒   _[1]=1
⇒ [3]:
⇒   _[1]=x(1)-1
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(1)^3*x(2)*y(0)-3*x(1)^2*x(2)*y(0)+x(1)^2*y(0)+3*x(1)*x(2)*y(0)-2*x\
    (1)*y(0)-x(2)*y(0)+y(0)
⇒ _[2]=x(1)^3*y(0)-2*x(1)^2*y(0)+x(1)*y(0)
⇒ _[3]=x(1)^2*y(0)-2*x(1)*y(0)+y(0)

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↳
↳ ----- Upcoming Center -----
↳ _[1]=y(0)
↳ _[2]=x(1)-1
↳ -----
↳ ++++++ Overview of Current Chart ++++++
↳ Current number of final charts: 0
↳ Total number of charts currently in chart-tree: 26
↳ Index of the current chart in chart-tree: 15
↳ ++++++
↳
↳ ==== Ambient Space:
↳ _[1]=0
↳
↳ ==== Ideal of Variety:
↳ _[1]=x(2)^4*x(3)^4*y(1)^4+4*x(2)^3*x(3)^3*y(1)^3+6*x(2)^2*x(3)^2*y(1)^2-x\
(2)^2*x(3)*y(1)^2+x(2)*x(3)*y(1)^2+4*x(2)*x(3)*y(1)+1
↳
↳ ==== Exceptional Divisors:
↳ [1]:
↳   _[1]=1
↳ [2]:
↳   _[1]=y(1)
↳ [3]:
↳   _[1]=x(3)
↳
↳ ==== Images of variables of original ring:
↳ _[1]=x(3)^3*y(1)^2+x(3)^2*y(1)
↳ _[2]=x(3)^2*y(1)
↳ _[3]=x(2)*x(3)^3*y(1)^2+x(3)^2*y(1)
↳
↳ ----- Upcoming Center -----
↳ _[1]=x(2)^4*x(3)^4*y(1)^4+4*x(2)^3*x(3)^3*y(1)^3+6*x(2)^2*x(3)^2*y(1)^2-x\
(2)^2*x(3)*y(1)^2+x(2)*x(3)*y(1)^2+4*x(2)*x(3)*y(1)+1
↳ -----
↳ ++++++ Overview of Current Chart ++++++
↳ Current number of final charts: 0
↳ Total number of charts currently in chart-tree: 26
↳ Index of the current chart in chart-tree: 16
↳ ++++++
↳
↳ ==== Ambient Space:
↳ _[1]=0
↳
↳ ==== Ideal of Variety:
↳ _[1]=x(1)^4*x(2)^4*y(0)-4*x(1)^3*x(2)^4*y(0)+4*x(1)^3*x(2)^3*y(0)+6*x(1)^\
2*x(2)^4*y(0)-12*x(1)^2*x(2)^3*y(0)-4*x(1)*x(2)^4*y(0)+6*x(1)^2*x(2)^2*y(\
0)+12*x(1)*x(2)^3*y(0)+x(2)^4*y(0)-12*x(1)*x(2)^2*y(0)-4*x(2)^3*y(0)-x(1)\
*x(2)^2+4*x(1)*x(2)*y(0)+6*x(2)^2*y(0)+x(1)*x(2)+x(2)^2-4*x(2)*y(0)-x(2)+\
y(0)
↳
↳ ==== Exceptional Divisors:
↳ [1]:

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⇒      _[1]=y(0)
⇒ [2]:
⇒      _[1]=1
⇒ [3]:
⇒      _[1]=x(1)-1
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(1)^3*y(0)-2*x(1)^2*y(0)+x(1)*y(0)
⇒ _[2]=x(1)^2*y(0)-2*x(1)*y(0)+y(0)
⇒ _[3]=x(1)^3*x(2)*y(0)-3*x(1)^2*x(2)*y(0)+x(1)^2*y(0)+3*x(1)*x(2)*y(0)-2*x\
      (1)*y(0)-x(2)*y(0)+y(0)
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=y(0)
⇒ _[2]=x(1)-1
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 0
⇒ Total number of charts currently in chart-tree: 28
⇒ Index of the current chart in chart-tree: 17
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(3)^4*y(1)^4+4*x(3)^3*y(1)^3+x(2)^2*x(3)*y(1)^2-x(2)*x(3)*y(1)^2+6*\
      x(3)^2*y(1)^2+4*x(3)*y(1)+1
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒      _[1]=1
⇒ [2]:
⇒      _[1]=y(1)
⇒ [3]:
⇒      _[1]=x(3)
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(2)*x(3)^3*y(1)^2+x(3)^2*y(1)
⇒ _[2]=x(3)^2*y(1)
⇒ _[3]=x(3)^3*y(1)^2+x(3)^2*y(1)
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=x(3)^4*y(1)^4+4*x(3)^3*y(1)^3+x(2)^2*x(3)*y(1)^2-x(2)*x(3)*y(1)^2+6*\
      x(3)^2*y(1)^2+4*x(3)*y(1)+1
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 0
⇒ Total number of charts currently in chart-tree: 28
⇒ Index of the current chart in chart-tree: 18
⇒ ++++++
⇒
⇒ ==== Ambient Space:

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⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(1)^4*y(0)+x(1)*x(2)^2-x(1)*x(2)-x(2)^2+x(2)
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=y(0)
⇒ [2]:
⇒   _[1]=1
⇒ [3]:
⇒   _[1]=x(1)-1
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(1)^3*x(2)*y(0)-3*x(1)^2*x(2)*y(0)+x(1)^2*y(0)+3*x(1)*x(2)*y(0)-2*x\
  (1)*y(0)-x(2)*y(0)+y(0)
⇒ _[2]=x(1)^2*y(0)-2*x(1)*y(0)+y(0)
⇒ _[3]=x(1)^3*y(0)-2*x(1)^2*y(0)+x(1)*y(0)
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=y(0)
⇒ _[2]=x(1)-1
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 0
⇒ Total number of charts currently in chart-tree: 30
⇒ Index of the current chart in chart-tree: 19
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(2)^4*x(3)^4*y(1)^4+4*x(2)^3*x(3)^3*y(1)^3+6*x(2)^2*x(3)^2*y(1)^2+x\
  (2)^2*x(3)*y(1)^2-x(2)*x(3)*y(1)^2+4*x(2)*x(3)*y(1)+1
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=1
⇒ [2]:
⇒   _[1]=y(1)
⇒ [3]:
⇒   _[1]=x(3)
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(3)^2*y(1)
⇒ _[2]=x(3)^3*y(1)^2+x(3)^2*y(1)
⇒ _[3]=x(2)*x(3)^3*y(1)^2+x(3)^2*y(1)
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=x(2)^4*x(3)^4*y(1)^4+4*x(2)^3*x(3)^3*y(1)^3+6*x(2)^2*x(3)^2*y(1)^2+x\
  (2)^2*x(3)*y(1)^2-x(2)*x(3)*y(1)^2+4*x(2)*x(3)*y(1)+1
⇒ -----

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⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 0
⇒ Total number of charts currently in chart-tree: 30
⇒ Index of the current chart in chart-tree: 20
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(1)^4*x(2)^4*y(0)-4*x(1)^3*x(2)^4*y(0)+4*x(1)^3*x(2)^3*y(0)+6*x(1)^\
2*x(2)^4*y(0)-12*x(1)^2*x(2)^3*y(0)-4*x(1)*x(2)^4*y(0)+6*x(1)^2*x(2)^2*y(\
0)+12*x(1)*x(2)^3*y(0)+x(2)^4*y(0)-12*x(1)*x(2)^2*y(0)-4*x(2)^3*y(0)+x(1)\
*x(2)^2+4*x(1)*x(2)*y(0)+6*x(2)^2*y(0)-x(1)*x(2)-x(2)^2-4*x(2)*y(0)+x(2)+\
y(0)
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒ _[1]=y(0)
⇒ [2]:
⇒ _[1]=1
⇒ [3]:
⇒ _[1]=x(1)-1
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(1)^2*y(0)-2*x(1)*y(0)+y(0)
⇒ _[2]=x(1)^3*y(0)-2*x(1)^2*y(0)+x(1)*y(0)
⇒ _[3]=x(1)^3*x(2)*y(0)-3*x(1)^2*x(2)*y(0)+x(1)^2*y(0)+3*x(1)*x(2)*y(0)-2*x\
(1)*y(0)-x(2)*y(0)+y(0)
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=y(0)
⇒ _[2]=x(1)-1
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 0
⇒ Total number of charts currently in chart-tree: 32
⇒ Index of the current chart in chart-tree: 21
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(3)^4*y(1)^4+4*x(3)^3*y(1)^3-x(2)^2*x(3)*y(1)^2+x(2)*x(3)*y(1)^2+6*\
x(3)^2*y(1)^2+4*x(3)*y(1)+1
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒ _[1]=1
⇒ [2]:
⇒ _[1]=y(1)
⇒ [3]:

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⇒  _[1]=x(3)
⇒
⇒  ==== Images of variables of original ring:
⇒  _[1]=x(3)^2*y(1)
⇒  _[2]=x(2)*x(3)^3*y(1)^2+x(3)^2*y(1)
⇒  _[3]=x(3)^3*y(1)^2+x(3)^2*y(1)
⇒
⇒  ----- Upcoming Center -----
⇒  _[1]=x(3)^4*y(1)^4+4*x(3)^3*y(1)^3-x(2)^2*x(3)*y(1)^2+x(2)*x(3)*y(1)^2+6*\
    x(3)^2*y(1)^2+4*x(3)*y(1)+1
⇒  -----
⇒  ++++++ Overview of Current Chart ++++++
⇒  Current number of final charts: 0
⇒  Total number of charts currently in chart-tree: 32
⇒  Index of the current chart in chart-tree: 22
⇒  ++++++
⇒
⇒  ==== Ambient Space:
⇒  _[1]=0
⇒
⇒  ==== Ideal of Variety:
⇒  _[1]=x(1)^4*y(0)-x(1)*x(2)^2+x(1)*x(2)+x(2)^2-x(2)
⇒
⇒  ==== Exceptional Divisors:
⇒  [1]:
⇒  _[1]=y(0)
⇒  [2]:
⇒  _[1]=1
⇒  [3]:
⇒  _[1]=x(1)-1
⇒
⇒  ==== Images of variables of original ring:
⇒  _[1]=x(1)^2*y(0)-2*x(1)*y(0)+y(0)
⇒  _[2]=x(1)^3*x(2)*y(0)-3*x(1)^2*x(2)*y(0)+x(1)^2*y(0)+3*x(1)*x(2)*y(0)-2*x\
    (1)*y(0)-x(2)*y(0)+y(0)
⇒  _[3]=x(1)^3*y(0)-2*x(1)^2*y(0)+x(1)*y(0)
⇒
⇒  ----- Upcoming Center -----
⇒  _[1]=y(0)
⇒  _[2]=x(1)-1
⇒  -----
⇒  ++++++ Overview of Current Chart ++++++
⇒  Current number of final charts: 0
⇒  Total number of charts currently in chart-tree: 34
⇒  Index of the current chart in chart-tree: 23
⇒  ++++++
⇒
⇒  ==== Ambient Space:
⇒  _[1]=0
⇒
⇒  ==== Ideal of Variety:
⇒  _[1]=x(2)^2*y(1)-x(2)*y(1)+1
⇒

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⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=1
⇒ [2]:
⇒   _[1]=1
⇒ [3]:
⇒   _[1]=y(1)
⇒ [4]:
⇒   _[1]=x(3)
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(3)^4*y(1)^3+x(3)^3*y(1)^2
⇒ _[2]=x(2)*x(3)^4*y(1)^3+x(3)^3*y(1)^2
⇒ _[3]=x(3)^3*y(1)^2
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=x(2)^2*y(1)-x(2)*y(1)+1
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 1
⇒ Total number of charts currently in chart-tree: 34
⇒ Index of the current chart in chart-tree: 24
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(2)^2-x(2)+y(0)
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=y(0)
⇒ [2]:
⇒   _[1]=1
⇒ [3]:
⇒   _[1]=1
⇒ [4]:
⇒   _[1]=x(1)-1
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(1)^4*y(0)-3*x(1)^3*y(0)+3*x(1)^2*y(0)-x(1)*y(0)
⇒ _[2]=x(1)^4*x(2)*y(0)-4*x(1)^3*x(2)*y(0)+x(1)^3*y(0)+6*x(1)^2*x(2)*y(0)-3\
    *x(1)^2*y(0)-4*x(1)*x(2)*y(0)+3*x(1)*y(0)+x(2)*y(0)-y(0)
⇒ _[3]=x(1)^3*y(0)-3*x(1)^2*y(0)+3*x(1)*y(0)-y(0)
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=x(2)^2-x(2)+y(0)
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 2
⇒ Total number of charts currently in chart-tree: 34
⇒ Index of the current chart in chart-tree: 25

```

```

↳ ++++++
↳
↳ ==== Ambient Space:
↳ _[1]=0
↳
↳ ==== Ideal of Variety:
↳ _[1]=x(2)^2*y(1)-x(2)*y(1)-1
↳
↳ ==== Exceptional Divisors:
↳ [1]:
↳   _[1]=1
↳ [2]:
↳   _[1]=1
↳ [3]:
↳   _[1]=y(1)
↳ [4]:
↳   _[1]=x(3)
↳
↳ ==== Images of variables of original ring:
↳ _[1]=x(2)*x(3)^4*y(1)^3+x(3)^3*y(1)^2
↳ _[2]=x(3)^4*y(1)^3+x(3)^3*y(1)^2
↳ _[3]=x(3)^3*y(1)^2
↳
↳ ----- Upcoming Center -----
↳ _[1]=x(2)^2*y(1)-x(2)*y(1)-1
↳ -----
↳ ++++++ Overview of Current Chart ++++++
↳ Current number of final charts: 3
↳ Total number of charts currently in chart-tree: 34
↳ Index of the current chart in chart-tree: 26
↳ ++++++
↳
↳ ==== Ambient Space:
↳ _[1]=0
↳
↳ ==== Ideal of Variety:
↳ _[1]=x(2)^2-x(2)-y(0)
↳
↳ ==== Exceptional Divisors:
↳ [1]:
↳   _[1]=y(0)
↳ [2]:
↳   _[1]=1
↳ [3]:
↳   _[1]=1
↳ [4]:
↳   _[1]=x(1)-1
↳
↳ ==== Images of variables of original ring:
↳ _[1]=x(1)^4*x(2)*y(0)-4*x(1)^3*x(2)*y(0)+x(1)^3*y(0)+6*x(1)^2*x(2)*y(0)-3\
    *x(1)^2*y(0)-4*x(1)*x(2)*y(0)+3*x(1)*y(0)+x(2)*y(0)-y(0)
↳ _[2]=x(1)^4*y(0)-3*x(1)^3*y(0)+3*x(1)^2*y(0)-x(1)*y(0)
↳ _[3]=x(1)^3*y(0)-3*x(1)^2*y(0)+3*x(1)*y(0)-y(0)

```

```

↳
↳ ----- Upcoming Center -----
↳ _[1]=x(2)^2-x(2)-y(0)
↳ -----
↳ ++++++ Overview of Current Chart ++++++
↳ Current number of final charts: 4
↳ Total number of charts currently in chart-tree: 34
↳ Index of the current chart in chart-tree: 27
↳ ++++++
↳
↳ ==== Ambient Space:
↳ _[1]=0
↳
↳ ==== Ideal of Variety:
↳ _[1]=x(2)^4*x(3)^4*y(1)^4+4*x(2)^3*x(3)^3*y(1)^3+6*x(2)^2*x(3)^2*y(1)^2-x\
(2)^2*y(1)+4*x(2)*x(3)*y(1)+x(2)*y(1)+1
↳
↳ ==== Exceptional Divisors:
↳ [1]:
↳   _[1]=1
↳ [2]:
↳   _[1]=1
↳ [3]:
↳   _[1]=y(1)
↳ [4]:
↳   _[1]=x(3)
↳
↳ ==== Images of variables of original ring:
↳ _[1]=x(3)^4*y(1)^3+x(3)^3*y(1)^2
↳ _[2]=x(3)^3*y(1)^2
↳ _[3]=x(2)*x(3)^4*y(1)^3+x(3)^3*y(1)^2
↳
↳ ----- Upcoming Center -----
↳ _[1]=x(2)^4*x(3)^4*y(1)^4+4*x(2)^3*x(3)^3*y(1)^3+6*x(2)^2*x(3)^2*y(1)^2-x\
(2)^2*y(1)+4*x(2)*x(3)*y(1)+x(2)*y(1)+1
↳ -----
↳ ++++++ Overview of Current Chart ++++++
↳ Current number of final charts: 5
↳ Total number of charts currently in chart-tree: 34
↳ Index of the current chart in chart-tree: 28
↳ ++++++
↳
↳ ==== Ambient Space:
↳ _[1]=0
↳
↳ ==== Ideal of Variety:
↳ _[1]=x(1)^4*x(2)^4*y(0)-4*x(1)^3*x(2)^4*y(0)+4*x(1)^3*x(2)^3*y(0)+6*x(1)^2\
2*x(2)^4*y(0)-12*x(1)^2*x(2)^3*y(0)-4*x(1)*x(2)^4*y(0)+6*x(1)^2*x(2)^2*y(\
0)+12*x(1)*x(2)^3*y(0)+x(2)^4*y(0)-12*x(1)*x(2)^2*y(0)-4*x(2)^3*y(0)+4*x(\
1)*x(2)*y(0)+6*x(2)^2*y(0)-x(2)^2-4*x(2)*y(0)+x(2)+y(0)
↳
↳ ==== Exceptional Divisors:
↳ [1]:

```



```

⇒  _[1]=y(0)
⇒  [2]:
⇒  _[1]=1
⇒  [3]:
⇒  _[1]=1
⇒  [4]:
⇒  _[1]=x(1)-1
⇒
⇒  ==== Images of variables of original ring:
⇒  _[1]=x(1)^4*y(0)-3*x(1)^3*y(0)+3*x(1)^2*y(0)-x(1)*y(0)
⇒  _[2]=x(1)^3*y(0)-3*x(1)^2*y(0)+3*x(1)*y(0)-y(0)
⇒  _[3]=x(1)^4*x(2)*y(0)-4*x(1)^3*x(2)*y(0)+x(1)^3*y(0)+6*x(1)^2*x(2)*y(0)-3\
    *x(1)^2*y(0)-4*x(1)*x(2)*y(0)+3*x(1)*y(0)+x(2)*y(0)-y(0)
⇒
⇒  ----- Upcoming Center -----
⇒  _[1]=x(1)^4*x(2)^4*y(0)-4*x(1)^3*x(2)^4*y(0)+4*x(1)^3*x(2)^3*y(0)+6*x(1)^\
    2*x(2)^4*y(0)-12*x(1)^2*x(2)^3*y(0)-4*x(1)*x(2)^4*y(0)+6*x(1)^2*x(2)^2*y(\
    0)+12*x(1)*x(2)^3*y(0)+x(2)^4*y(0)-12*x(1)*x(2)^2*y(0)-4*x(2)^3*y(0)+4*x(\
    1)*x(2)*y(0)+6*x(2)^2*y(0)-x(2)^2-4*x(2)*y(0)+x(2)+y(0)
⇒
⇒  ++++++ Overview of Current Chart ++++++
⇒  Current number of final charts: 6
⇒  Total number of charts currently in chart-tree: 34
⇒  Index of the current chart in chart-tree: 29
⇒  ++++++
⇒
⇒  ==== Ambient Space:
⇒  _[1]=0
⇒
⇒  ==== Ideal of Variety:
⇒  _[1]=x(3)^4*y(1)^4+4*x(3)^3*y(1)^3+6*x(3)^2*y(1)^2+x(2)^2*y(1)-x(2)*y(1)+\
    4*x(3)*y(1)+1
⇒
⇒  ==== Exceptional Divisors:
⇒  [1]:
⇒  _[1]=1
⇒  [2]:
⇒  _[1]=1
⇒  [3]:
⇒  _[1]=y(1)
⇒  [4]:
⇒  _[1]=x(3)
⇒
⇒  ==== Images of variables of original ring:
⇒  _[1]=x(2)*x(3)^4*y(1)^3+x(3)^3*y(1)^2
⇒  _[2]=x(3)^3*y(1)^2
⇒  _[3]=x(3)^4*y(1)^3+x(3)^3*y(1)^2
⇒
⇒  ----- Upcoming Center -----
⇒  _[1]=x(3)^4*y(1)^4+4*x(3)^3*y(1)^3+6*x(3)^2*y(1)^2+x(2)^2*y(1)-x(2)*y(1)+\
    4*x(3)*y(1)+1
⇒
⇒  ++++++ Overview of Current Chart ++++++

```

```

⇒ Current number of final charts: 7
⇒ Total number of charts currently in chart-tree: 34
⇒ Index of the current chart in chart-tree: 30
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(1)^4*y(0)+x(2)^2-x(2)
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=y(0)
⇒ [2]:
⇒   _[1]=1
⇒ [3]:
⇒   _[1]=1
⇒ [4]:
⇒   _[1]=x(1)-1
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(1)^4*x(2)*y(0)-4*x(1)^3*x(2)*y(0)+x(1)^3*y(0)+6*x(1)^2*x(2)*y(0)-3\
  *x(1)^2*y(0)-4*x(1)*x(2)*y(0)+3*x(1)*y(0)+x(2)*y(0)-y(0)
⇒ _[2]=x(1)^3*y(0)-3*x(1)^2*y(0)+3*x(1)*y(0)-y(0)
⇒ _[3]=x(1)^4*y(0)-3*x(1)^3*y(0)+3*x(1)^2*y(0)-x(1)*y(0)
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=x(1)^4*y(0)+x(2)^2-x(2)
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 8
⇒ Total number of charts currently in chart-tree: 34
⇒ Index of the current chart in chart-tree: 31
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(2)^4*x(3)^4*y(1)^4+4*x(2)^3*x(3)^3*y(1)^3+6*x(2)^2*x(3)^2*y(1)^2+x\
  (2)^2*y(1)+4*x(2)*x(3)*y(1)-x(2)*y(1)+1
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=1
⇒ [2]:
⇒   _[1]=1
⇒ [3]:
⇒   _[1]=y(1)
⇒ [4]:
⇒   _[1]=x(3)
⇒

```

```

⇒ ==== Images of variables of original ring:
⇒ _[1]=x(3)^3*y(1)^2
⇒ _[2]=x(3)^4*y(1)^3+x(3)^3*y(1)^2
⇒ _[3]=x(2)*x(3)^4*y(1)^3+x(3)^3*y(1)^2
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=x(2)^4*x(3)^4*y(1)^4+4*x(2)^3*x(3)^3*y(1)^3+6*x(2)^2*x(3)^2*y(1)^2+x\
  (2)^2*y(1)+4*x(2)*x(3)*y(1)-x(2)*y(1)+1
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 9
⇒ Total number of charts currently in chart-tree: 34
⇒ Index of the current chart in chart-tree: 32
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(1)^4*x(2)^4*y(0)-4*x(1)^3*x(2)^4*y(0)+4*x(1)^3*x(2)^3*y(0)+6*x(1)^\
  2*x(2)^4*y(0)-12*x(1)^2*x(2)^3*y(0)-4*x(1)*x(2)^4*y(0)+6*x(1)^2*x(2)^2*y(\
  0)+12*x(1)*x(2)^3*y(0)+x(2)^4*y(0)-12*x(1)*x(2)^2*y(0)-4*x(2)^3*y(0)+4*x(\
  1)*x(2)*y(0)+6*x(2)^2*y(0)+x(2)^2-4*x(2)*y(0)-x(2)+y(0)
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒ _[1]=y(0)
⇒ [2]:
⇒ _[1]=1
⇒ [3]:
⇒ _[1]=1
⇒ [4]:
⇒ _[1]=x(1)-1
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(1)^3*y(0)-3*x(1)^2*y(0)+3*x(1)*y(0)-y(0)
⇒ _[2]=x(1)^4*y(0)-3*x(1)^3*y(0)+3*x(1)^2*y(0)-x(1)*y(0)
⇒ _[3]=x(1)^4*x(2)*y(0)-4*x(1)^3*x(2)*y(0)+x(1)^3*y(0)+6*x(1)^2*x(2)*y(0)-3\
  *x(1)^2*y(0)-4*x(1)*x(2)*y(0)+3*x(1)*y(0)+x(2)*y(0)-y(0)
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=x(1)^4*x(2)^4*y(0)-4*x(1)^3*x(2)^4*y(0)+4*x(1)^3*x(2)^3*y(0)+6*x(1)^\
  2*x(2)^4*y(0)-12*x(1)^2*x(2)^3*y(0)-4*x(1)*x(2)^4*y(0)+6*x(1)^2*x(2)^2*y(\
  0)+12*x(1)*x(2)^3*y(0)+x(2)^4*y(0)-12*x(1)*x(2)^2*y(0)-4*x(2)^3*y(0)+4*x(\
  1)*x(2)*y(0)+6*x(2)^2*y(0)+x(2)^2-4*x(2)*y(0)-x(2)+y(0)
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 10
⇒ Total number of charts currently in chart-tree: 34
⇒ Index of the current chart in chart-tree: 33
⇒ ++++++
⇒
⇒ ==== Ambient Space:

```

```

⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(3)^4*y(1)^4+4*x(3)^3*y(1)^3+6*x(3)^2*y(1)^2-x(2)^2*y(1)+x(2)*y(1)+\
  4*x(3)*y(1)+1
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=1
⇒ [2]:
⇒   _[1]=1
⇒ [3]:
⇒   _[1]=y(1)
⇒ [4]:
⇒   _[1]=x(3)
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(3)^3*y(1)^2
⇒ _[2]=x(2)*x(3)^4*y(1)^3+x(3)^3*y(1)^2
⇒ _[3]=x(3)^4*y(1)^3+x(3)^3*y(1)^2
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=x(3)^4*y(1)^4+4*x(3)^3*y(1)^3+6*x(3)^2*y(1)^2-x(2)^2*y(1)+x(2)*y(1)+\
  4*x(3)*y(1)+1
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 11
⇒ Total number of charts currently in chart-tree: 34
⇒ Index of the current chart in chart-tree: 34
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(1)^4*y(0)-x(2)^2+x(2)
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=y(0)
⇒ [2]:
⇒   _[1]=1
⇒ [3]:
⇒   _[1]=1
⇒ [4]:
⇒   _[1]=x(1)-1
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(1)^3*y(0)-3*x(1)^2*y(0)+3*x(1)*y(0)-y(0)
⇒ _[2]=x(1)^4*x(2)*y(0)-4*x(1)^3*x(2)*y(0)+x(1)^3*y(0)+6*x(1)^2*x(2)*y(0)-3\
  *x(1)^2*y(0)-4*x(1)*x(2)*y(0)+3*x(1)*y(0)+x(2)*y(0)-y(0)
⇒ _[3]=x(1)^4*y(0)-3*x(1)^3*y(0)+3*x(1)^2*y(0)-x(1)*y(0)
⇒

```

```

↳ ----- Upcoming Center -----
↳ _[1]=x(1)^4*y(0)-x(2)^2+x(2)
↳ -----
↳ ===== result will be tested =====
↳
↳ the number of charts obtained: 12
↳ ===== result is o.k. =====
list iden=prepEmbDiv(re);
intvec v=computeV(re, iden);
v;
↳ 3,5,8,11,1

```

### D.5.16.9 computeN

Procedure from library `reszeta.lib` (see [Section D.5.16 \[reszeta.lib\]](#), page 1538).

**Usage:**      `computeN(L,DL);`  
               `L` = list of rings  
               `DL` = divisor list

**Assume:**    `L` has structure of output of `resolve`  
               `DL` has structure of output of `prepEmbDiv`

**Return:**    `intvec`, `i`-th entry is multiplicity of `i`-th divisor  
               in total transform under resolution

**Example:**

```

LIB "reszeta.lib";
ring R=0,(x,y,z),dp;
ideal I=(x-y)*(x-z)*(y-z)-z4;
list re=resolve(I,1);
↳ ++++++ Overview of Current Chart ++++++
↳ Current number of final charts: 0
↳ Total number of charts currently in chart-tree: 1
↳ Index of the current chart in chart-tree: 1
↳ ++++++
↳
↳ ===== Ambient Space:
↳ _[1]=0
↳
↳ ===== Ideal of Variety:
↳ _[1]=-z4+x2y-xy2-x2z+y2z+xz2-yz2
↳
↳ ===== Exceptional Divisors:
↳ empty list
↳
↳ ===== Images of variables of original ring:
↳ _[1]=x
↳ _[2]=y
↳ _[3]=z
↳
↳ ----- Upcoming Center -----
↳ _[1]=z
↳ _[2]=y
↳ _[3]=x

```

```

↳ -----
↳ ++++++ Overview of Current Chart ++++++
↳ Current number of final charts: 0
↳ Total number of charts currently in chart-tree: 4
↳ Index of the current chart in chart-tree: 2
↳ ++++++
↳
↳ ==== Ambient Space:
↳ _[1]=0
↳
↳ ==== Ideal of Variety:
↳ _[1]=y(1)^2*y(2)-y(1)*y(2)^2-y(1)^2+y(2)^2+x(3)+y(1)-y(2)
↳
↳ ==== Exceptional Divisors:
↳ [1]:
↳   _[1]=x(3)
↳
↳ ==== Images of variables of original ring:
↳ _[1]=x(3)*y(2)
↳ _[2]=x(3)*y(1)
↳ _[3]=x(3)
↳
↳ ----- Upcoming Center -----
↳ _[1]=y(2)-1
↳ _[2]=y(1)-1
↳ _[3]=x(3)
↳ -----
↳ ++++++ Overview of Current Chart ++++++
↳ Current number of final charts: 0
↳ Total number of charts currently in chart-tree: 6
↳ Index of the current chart in chart-tree: 3
↳ ++++++
↳
↳ ==== Ambient Space:
↳ _[1]=0
↳
↳ ==== Ideal of Variety:
↳ _[1]=x(2)*y(0)^4-y(0)^2*y(2)+y(0)*y(2)^2+y(0)^2-y(2)^2-y(0)+y(2)
↳
↳ ==== Exceptional Divisors:
↳ [1]:
↳   _[1]=x(2)
↳
↳ ==== Images of variables of original ring:
↳ _[1]=x(2)*y(2)
↳ _[2]=x(2)
↳ _[3]=x(2)*y(0)
↳
↳ ----- Upcoming Center -----
↳ _[1]=y(2)-1
↳ _[2]=y(0)-1
↳ _[3]=x(2)
↳ -----

```

```

⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 0
⇒ Total number of charts currently in chart-tree: 8
⇒ Index of the current chart in chart-tree: 4
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(1)*y(0)^4+y(0)^2*y(1)-y(0)*y(1)^2-y(0)^2+y(1)^2+y(0)-y(1)
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=x(1)
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(1)
⇒ _[2]=x(1)*y(1)
⇒ _[3]=x(1)*y(0)
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=y(1)-1
⇒ _[2]=y(0)-1
⇒ _[3]=x(1)
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 0
⇒ Total number of charts currently in chart-tree: 10
⇒ Index of the current chart in chart-tree: 5
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(3)^2*y(1)^2-x(3)^2*y(1)-2*x(3)*y(1)^2+2*x(3)*y(1)+y(1)^2-y(1)+y(2)
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=y(2)
⇒ [2]:
⇒   _[1]=x(3)-1
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(3)^2*y(2)-x(3)*y(2)
⇒ _[2]=x(3)^2*y(1)*y(2)-2*x(3)*y(1)*y(2)+x(3)*y(2)+y(1)*y(2)-y(2)
⇒ _[3]=x(3)*y(2)-y(2)
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=y(2)
⇒ _[2]=x(3)-1
⇒ -----

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⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 0
⇒ Total number of charts currently in chart-tree: 12
⇒ Index of the current chart in chart-tree: 6
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(2)^2*y(0)^2-x(2)^2*y(0)-2*x(2)*y(0)^2+2*x(2)*y(0)+y(0)^2-y(0)-y(2)
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=y(2)
⇒ [2]:
⇒   _[1]=x(2)-1
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(2)^2*y(0)*y(2)-2*x(2)*y(0)*y(2)+x(2)*y(2)+y(0)*y(2)-y(2)
⇒ _[2]=x(2)^2*y(2)-x(2)*y(2)
⇒ _[3]=x(2)*y(2)-y(2)
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=y(2)
⇒ _[2]=x(2)-1
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 0
⇒ Total number of charts currently in chart-tree: 14
⇒ Index of the current chart in chart-tree: 7
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(3)^4*y(1)^4*y(2)-4*x(3)^3*y(1)^4*y(2)+4*x(3)^3*y(1)^3*y(2)+6*x(3)^2*y(1)^4*y(2)-12*x(3)^2*y(1)^3*y(2)-4*x(3)*y(1)^4*y(2)+6*x(3)^2*y(1)^2*y(\
2)+12*x(3)*y(1)^3*y(2)+y(1)^4*y(2)-x(3)^2*y(1)^2-12*x(3)*y(1)^2*y(2)-4*y(\
1)^3*y(2)+x(3)^2*y(1)+2*x(3)*y(1)^2+4*x(3)*y(1)*y(2)+6*y(1)^2*y(2)-2*x(3)\
*y(1)-y(1)^2-4*y(1)*y(2)+y(1)+y(2)
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=y(2)
⇒ [2]:
⇒   _[1]=x(3)-1
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(3)^2*y(2)-x(3)*y(2)
⇒ _[2]=x(3)*y(2)-y(2)
⇒ _[3]=x(3)^2*y(1)*y(2)-2*x(3)*y(1)*y(2)+x(3)*y(2)+y(1)*y(2)-y(2)

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↳
↳ ----- Upcoming Center -----
↳ _[1]=y(2)
↳ _[2]=x(3)-1
↳ -----
↳ ++++++ Overview of Current Chart ++++++
↳ Current number of final charts: 0
↳ Total number of charts currently in chart-tree: 16
↳ Index of the current chart in chart-tree: 8
↳ ++++++
↳
↳ ==== Ambient Space:
↳ _[1]=0
↳
↳ ==== Ideal of Variety:
↳ _[1]=x(2)^4*y(2)+x(2)^2*y(0)^2-x(2)^2*y(0)-2*x(2)*y(0)^2+2*x(2)*y(0)+y(0)\
^2-y(0)
↳
↳ ==== Exceptional Divisors:
↳ [1]:
↳   _[1]=y(2)
↳ [2]:
↳   _[1]=x(2)-1
↳
↳ ==== Images of variables of original ring:
↳ _[1]=x(2)^2*y(0)*y(2)-2*x(2)*y(0)*y(2)+x(2)*y(2)+y(0)*y(2)-y(2)
↳ _[2]=x(2)*y(2)-y(2)
↳ _[3]=x(2)^2*y(2)-x(2)*y(2)
↳
↳ ----- Upcoming Center -----
↳ _[1]=y(2)
↳ _[2]=x(2)-1
↳ -----
↳ ++++++ Overview of Current Chart ++++++
↳ Current number of final charts: 0
↳ Total number of charts currently in chart-tree: 18
↳ Index of the current chart in chart-tree: 9
↳ ++++++
↳
↳ ==== Ambient Space:
↳ _[1]=0
↳
↳ ==== Ideal of Variety:
↳ _[1]=x(3)^4*y(1)^4*y(2)-4*x(3)^3*y(1)^4*y(2)+4*x(3)^3*y(1)^3*y(2)+6*x(3)^\
2*y(1)^4*y(2)-12*x(3)^2*y(1)^3*y(2)-4*x(3)*y(1)^4*y(2)+6*x(3)^2*y(1)^2*y(\
2)+12*x(3)*y(1)^3*y(2)+y(1)^4*y(2)+x(3)^2*y(1)^2-12*x(3)*y(1)^2*y(2)-4*y(\
1)^3*y(2)-x(3)^2*y(1)-2*x(3)*y(1)^2+4*x(3)*y(1)*y(2)+6*y(1)^2*y(2)+2*x(3)\
*y(1)+y(1)^2-4*y(1)*y(2)-y(1)+y(2)
↳
↳ ==== Exceptional Divisors:
↳ [1]:
↳   _[1]=y(2)
↳ [2]:

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⇒      _[1]=x(3)-1
⇒
⇒ ===== Images of variables of original ring:
⇒      _[1]=x(3)*y(2)-y(2)
⇒      _[2]=x(3)^2*y(2)-x(3)*y(2)
⇒      _[3]=x(3)^2*y(1)*y(2)-2*x(3)*y(1)*y(2)+x(3)*y(2)+y(1)*y(2)-y(2)
⇒
⇒ ----- Upcoming Center -----
⇒      _[1]=y(2)
⇒      _[2]=x(3)-1
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 0
⇒ Total number of charts currently in chart-tree: 20
⇒ Index of the current chart in chart-tree: 10
⇒ ++++++
⇒
⇒ ===== Ambient Space:
⇒      _[1]=0
⇒
⇒ ===== Ideal of Variety:
⇒      _[1]=x(2)^4*y(2)-x(2)^2*y(0)^2+x(2)^2*y(0)+2*x(2)*y(0)^2-2*x(2)*y(0)-y(0)\
      ^2+y(0)
⇒
⇒ ===== Exceptional Divisors:
⇒      [1]:
⇒      _[1]=y(2)
⇒      [2]:
⇒      _[1]=x(2)-1
⇒
⇒ ===== Images of variables of original ring:
⇒      _[1]=x(2)*y(2)-y(2)
⇒      _[2]=x(2)^2*y(0)*y(2)-2*x(2)*y(0)*y(2)+x(2)*y(2)+y(0)*y(2)-y(2)
⇒      _[3]=x(2)^2*y(2)-x(2)*y(2)
⇒
⇒ ----- Upcoming Center -----
⇒      _[1]=y(2)
⇒      _[2]=x(2)-1
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 0
⇒ Total number of charts currently in chart-tree: 22
⇒ Index of the current chart in chart-tree: 11
⇒ ++++++
⇒
⇒ ===== Ambient Space:
⇒      _[1]=0
⇒
⇒ ===== Ideal of Variety:
⇒      _[1]=x(2)^2*x(3)*y(1)^2-x(2)*x(3)*y(1)^2+1
⇒
⇒ ===== Exceptional Divisors:
⇒      [1]:

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⇒  _[1]=1
⇒  [2]:
⇒  _[1]=y(1)
⇒  [3]:
⇒  _[1]=x(3)
⇒
⇒  ==== Images of variables of original ring:
⇒  _[1]=x(3)^3*y(1)^2+x(3)^2*y(1)
⇒  _[2]=x(2)*x(3)^3*y(1)^2+x(3)^2*y(1)
⇒  _[3]=x(3)^2*y(1)
⇒
⇒  ----- Upcoming Center -----
⇒  _[1]=x(2)^2*x(3)*y(1)^2-x(2)*x(3)*y(1)^2+1
⇒  -----
⇒  ++++++ Overview of Current Chart ++++++
⇒  Current number of final charts: 0
⇒  Total number of charts currently in chart-tree: 22
⇒  Index of the current chart in chart-tree: 12
⇒  ++++++
⇒
⇒  ==== Ambient Space:
⇒  _[1]=0
⇒
⇒  ==== Ideal of Variety:
⇒  _[1]=x(1)*x(2)^2-x(1)*x(2)-x(2)^2+x(2)+y(0)
⇒
⇒  ==== Exceptional Divisors:
⇒  [1]:
⇒  _[1]=y(0)
⇒  [2]:
⇒  _[1]=1
⇒  [3]:
⇒  _[1]=x(1)-1
⇒
⇒  ==== Images of variables of original ring:
⇒  _[1]=x(1)^3*y(0)-2*x(1)^2*y(0)+x(1)*y(0)
⇒  _[2]=x(1)^3*x(2)*y(0)-3*x(1)^2*x(2)*y(0)+x(1)^2*y(0)+3*x(1)*x(2)*y(0)-2*x\
    (1)*y(0)-x(2)*y(0)+y(0)
⇒  _[3]=x(1)^2*y(0)-2*x(1)*y(0)+y(0)
⇒
⇒  ----- Upcoming Center -----
⇒  _[1]=y(0)
⇒  _[2]=x(1)-1
⇒  -----
⇒  ++++++ Overview of Current Chart ++++++
⇒  Current number of final charts: 0
⇒  Total number of charts currently in chart-tree: 24
⇒  Index of the current chart in chart-tree: 13
⇒  ++++++
⇒
⇒  ==== Ambient Space:
⇒  _[1]=0
⇒

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⇒ ==== Ideal of Variety:
⇒ _[1]=x(2)^2*x(3)*y(1)^2-x(2)*x(3)*y(1)^2-1
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=1
⇒ [2]:
⇒   _[1]=y(1)
⇒ [3]:
⇒   _[1]=x(3)
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(2)*x(3)^3*y(1)^2+x(3)^2*y(1)
⇒ _[2]=x(3)^3*y(1)^2+x(3)^2*y(1)
⇒ _[3]=x(3)^2*y(1)
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=x(2)^2*x(3)*y(1)^2-x(2)*x(3)*y(1)^2-1
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 0
⇒ Total number of charts currently in chart-tree: 24
⇒ Index of the current chart in chart-tree: 14
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(1)*x(2)^2-x(1)*x(2)-x(2)^2+x(2)-y(0)
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=y(0)
⇒ [2]:
⇒   _[1]=1
⇒ [3]:
⇒   _[1]=x(1)-1
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(1)^3*x(2)*y(0)-3*x(1)^2*x(2)*y(0)+x(1)^2*y(0)+3*x(1)*x(2)*y(0)-2*x\
    (1)*y(0)-x(2)*y(0)+y(0)
⇒ _[2]=x(1)^3*y(0)-2*x(1)^2*y(0)+x(1)*y(0)
⇒ _[3]=x(1)^2*y(0)-2*x(1)*y(0)+y(0)
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=y(0)
⇒ _[2]=x(1)-1
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 0
⇒ Total number of charts currently in chart-tree: 26
⇒ Index of the current chart in chart-tree: 15

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⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(2)^4*x(3)^4*y(1)^4+4*x(2)^3*x(3)^3*y(1)^3+6*x(2)^2*x(3)^2*y(1)^2-x\
(2)^2*x(3)*y(1)^2+x(2)*x(3)*y(1)^2+4*x(2)*x(3)*y(1)+1
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒ _[1]=1
⇒ [2]:
⇒ _[1]=y(1)
⇒ [3]:
⇒ _[1]=x(3)
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(3)^3*y(1)^2+x(3)^2*y(1)
⇒ _[2]=x(3)^2*y(1)
⇒ _[3]=x(2)*x(3)^3*y(1)^2+x(3)^2*y(1)
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=x(2)^4*x(3)^4*y(1)^4+4*x(2)^3*x(3)^3*y(1)^3+6*x(2)^2*x(3)^2*y(1)^2-x\
(2)^2*x(3)*y(1)^2+x(2)*x(3)*y(1)^2+4*x(2)*x(3)*y(1)+1
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 0
⇒ Total number of charts currently in chart-tree: 26
⇒ Index of the current chart in chart-tree: 16
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(1)^4*x(2)^4*y(0)-4*x(1)^3*x(2)^4*y(0)+4*x(1)^3*x(2)^3*y(0)+6*x(1)^\
2*x(2)^4*y(0)-12*x(1)^2*x(2)^3*y(0)-4*x(1)*x(2)^4*y(0)+6*x(1)^2*x(2)^2*y(\
0)+12*x(1)*x(2)^3*y(0)+x(2)^4*y(0)-12*x(1)*x(2)^2*y(0)-4*x(2)^3*y(0)-x(1)\
*x(2)^2+4*x(1)*x(2)*y(0)+6*x(2)^2*y(0)+x(1)*x(2)+x(2)^2-4*x(2)*y(0)-x(2)+\
y(0)
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒ _[1]=y(0)
⇒ [2]:
⇒ _[1]=1
⇒ [3]:
⇒ _[1]=x(1)-1
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(1)^3*y(0)-2*x(1)^2*y(0)+x(1)*y(0)
⇒ _[2]=x(1)^2*y(0)-2*x(1)*y(0)+y(0)

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⇒  $\_ [3]=x(1)^3*x(2)*y(0)-3*x(1)^2*x(2)*y(0)+x(1)^2*y(0)+3*x(1)*x(2)*y(0)-2*x(1)*y(0)-x(2)*y(0)+y(0)$ 
⇒
⇒ ----- Upcoming Center -----
⇒  $\_ [1]=y(0)$ 
⇒  $\_ [2]=x(1)-1$ 
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 0
⇒ Total number of charts currently in chart-tree: 28
⇒ Index of the current chart in chart-tree: 17
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒  $\_ [1]=0$ 
⇒
⇒ ==== Ideal of Variety:
⇒  $\_ [1]=x(3)^4*y(1)^4+4*x(3)^3*y(1)^3+x(2)^2*x(3)*y(1)^2-x(2)*x(3)*y(1)^2+6*x(3)^2*y(1)^2+4*x(3)*y(1)+1$ 
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒  $\_ [1]=1$ 
⇒ [2]:
⇒  $\_ [1]=y(1)$ 
⇒ [3]:
⇒  $\_ [1]=x(3)$ 
⇒
⇒ ==== Images of variables of original ring:
⇒  $\_ [1]=x(2)*x(3)^3*y(1)^2+x(3)^2*y(1)$ 
⇒  $\_ [2]=x(3)^2*y(1)$ 
⇒  $\_ [3]=x(3)^3*y(1)^2+x(3)^2*y(1)$ 
⇒
⇒ ----- Upcoming Center -----
⇒  $\_ [1]=x(3)^4*y(1)^4+4*x(3)^3*y(1)^3+x(2)^2*x(3)*y(1)^2-x(2)*x(3)*y(1)^2+6*x(3)^2*y(1)^2+4*x(3)*y(1)+1$ 
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 0
⇒ Total number of charts currently in chart-tree: 28
⇒ Index of the current chart in chart-tree: 18
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒  $\_ [1]=0$ 
⇒
⇒ ==== Ideal of Variety:
⇒  $\_ [1]=x(1)^4*y(0)+x(1)*x(2)^2-x(1)*x(2)-x(2)^2+x(2)$ 
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒  $\_ [1]=y(0)$ 
⇒ [2]:

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⇒ _[1]=1
⇒ [3]:
⇒ _[1]=x(1)-1
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(1)^3*x(2)*y(0)-3*x(1)^2*x(2)*y(0)+x(1)^2*y(0)+3*x(1)*x(2)*y(0)-2*x\
  (1)*y(0)-x(2)*y(0)+y(0)
⇒ _[2]=x(1)^2*y(0)-2*x(1)*y(0)+y(0)
⇒ _[3]=x(1)^3*y(0)-2*x(1)^2*y(0)+x(1)*y(0)
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=y(0)
⇒ _[2]=x(1)-1
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 0
⇒ Total number of charts currently in chart-tree: 30
⇒ Index of the current chart in chart-tree: 19
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(2)^4*x(3)^4*y(1)^4+4*x(2)^3*x(3)^3*y(1)^3+6*x(2)^2*x(3)^2*y(1)^2+x\
  (2)^2*x(3)*y(1)^2-x(2)*x(3)*y(1)^2+4*x(2)*x(3)*y(1)+1
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒ _[1]=1
⇒ [2]:
⇒ _[1]=y(1)
⇒ [3]:
⇒ _[1]=x(3)
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(3)^2*y(1)
⇒ _[2]=x(3)^3*y(1)^2+x(3)^2*y(1)
⇒ _[3]=x(2)*x(3)^3*y(1)^2+x(3)^2*y(1)
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=x(2)^4*x(3)^4*y(1)^4+4*x(2)^3*x(3)^3*y(1)^3+6*x(2)^2*x(3)^2*y(1)^2+x\
  (2)^2*x(3)*y(1)^2-x(2)*x(3)*y(1)^2+4*x(2)*x(3)*y(1)+1
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 0
⇒ Total number of charts currently in chart-tree: 30
⇒ Index of the current chart in chart-tree: 20
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒

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⇒ ==== Ideal of Variety:
⇒ _[1]=x(1)^4*x(2)^4*y(0)-4*x(1)^3*x(2)^4*y(0)+4*x(1)^3*x(2)^3*y(0)+6*x(1)^2*
  2*x(2)^4*y(0)-12*x(1)^2*x(2)^3*y(0)-4*x(1)*x(2)^4*y(0)+6*x(1)^2*x(2)^2*y(
  0)+12*x(1)*x(2)^3*y(0)+x(2)^4*y(0)-12*x(1)*x(2)^2*y(0)-4*x(2)^3*y(0)+x(1)\
  *x(2)^2+4*x(1)*x(2)*y(0)+6*x(2)^2*y(0)-x(1)*x(2)-x(2)^2-4*x(2)*y(0)+x(2)+\
  y(0)
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=y(0)
⇒ [2]:
⇒   _[1]=1
⇒ [3]:
⇒   _[1]=x(1)-1
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(1)^2*y(0)-2*x(1)*y(0)+y(0)
⇒ _[2]=x(1)^3*y(0)-2*x(1)^2*y(0)+x(1)*y(0)
⇒ _[3]=x(1)^3*x(2)*y(0)-3*x(1)^2*x(2)*y(0)+x(1)^2*y(0)+3*x(1)*x(2)*y(0)-2*x\
  (1)*y(0)-x(2)*y(0)+y(0)
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=y(0)
⇒ _[2]=x(1)-1
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 0
⇒ Total number of charts currently in chart-tree: 32
⇒ Index of the current chart in chart-tree: 21
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(3)^4*y(1)^4+4*x(3)^3*y(1)^3-x(2)^2*x(3)*y(1)^2+x(2)*x(3)*y(1)^2+6*\
  x(3)^2*y(1)^2+4*x(3)*y(1)+1
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=1
⇒ [2]:
⇒   _[1]=y(1)
⇒ [3]:
⇒   _[1]=x(3)
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(3)^2*y(1)
⇒ _[2]=x(2)*x(3)^3*y(1)^2+x(3)^2*y(1)
⇒ _[3]=x(3)^3*y(1)^2+x(3)^2*y(1)
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=x(3)^4*y(1)^4+4*x(3)^3*y(1)^3-x(2)^2*x(3)*y(1)^2+x(2)*x(3)*y(1)^2+6*\

```



```

      x(3)^2*y(1)^2+4*x(3)*y(1)+1
    => -----
    => ++++++ Overview of Current Chart ++++++
    => Current number of final charts: 0
    => Total number of charts currently in chart-tree: 32
    => Index of the current chart in chart-tree: 22
    => ++++++
    =>
    => ==== Ambient Space:
    => _[1]=0
    =>
    => ==== Ideal of Variety:
    => _[1]=x(1)^4*y(0)-x(1)*x(2)^2+x(1)*x(2)+x(2)^2-x(2)
    =>
    => ==== Exceptional Divisors:
    => [1]:
    =>   _[1]=y(0)
    => [2]:
    =>   _[1]=1
    => [3]:
    =>   _[1]=x(1)-1
    =>
    => ==== Images of variables of original ring:
    => _[1]=x(1)^2*y(0)-2*x(1)*y(0)+y(0)
    => _[2]=x(1)^3*x(2)*y(0)-3*x(1)^2*x(2)*y(0)+x(1)^2*y(0)+3*x(1)*x(2)*y(0)-2*x\
      (1)*y(0)-x(2)*y(0)+y(0)
    => _[3]=x(1)^3*y(0)-2*x(1)^2*y(0)+x(1)*y(0)
    =>
    => ----- Upcoming Center -----
    => _[1]=y(0)
    => _[2]=x(1)-1
    => -----
    => ++++++ Overview of Current Chart ++++++
    => Current number of final charts: 0
    => Total number of charts currently in chart-tree: 34
    => Index of the current chart in chart-tree: 23
    => ++++++
    =>
    => ==== Ambient Space:
    => _[1]=0
    =>
    => ==== Ideal of Variety:
    => _[1]=x(2)^2*y(1)-x(2)*y(1)+1
    =>
    => ==== Exceptional Divisors:
    => [1]:
    =>   _[1]=1
    => [2]:
    =>   _[1]=1
    => [3]:
    =>   _[1]=y(1)
    => [4]:
    =>   _[1]=x(3)

```

```

↳
↳ ==== Images of variables of original ring:
↳ _[1]=x(3)^4*y(1)^3+x(3)^3*y(1)^2
↳ _[2]=x(2)*x(3)^4*y(1)^3+x(3)^3*y(1)^2
↳ _[3]=x(3)^3*y(1)^2
↳
↳ ----- Upcoming Center -----
↳ _[1]=x(2)^2*y(1)-x(2)*y(1)+1
↳ -----
↳ ++++++ Overview of Current Chart ++++++
↳ Current number of final charts: 1
↳ Total number of charts currently in chart-tree: 34
↳ Index of the current chart in chart-tree: 24
↳ ++++++
↳
↳ ==== Ambient Space:
↳ _[1]=0
↳
↳ ==== Ideal of Variety:
↳ _[1]=x(2)^2-x(2)+y(0)
↳
↳ ==== Exceptional Divisors:
↳ [1]:
↳   _[1]=y(0)
↳ [2]:
↳   _[1]=1
↳ [3]:
↳   _[1]=1
↳ [4]:
↳   _[1]=x(1)-1
↳
↳ ==== Images of variables of original ring:
↳ _[1]=x(1)^4*y(0)-3*x(1)^3*y(0)+3*x(1)^2*y(0)-x(1)*y(0)
↳ _[2]=x(1)^4*x(2)*y(0)-4*x(1)^3*x(2)*y(0)+x(1)^3*y(0)+6*x(1)^2*x(2)*y(0)-3\
    *x(1)^2*y(0)-4*x(1)*x(2)*y(0)+3*x(1)*y(0)+x(2)*y(0)-y(0)
↳ _[3]=x(1)^3*y(0)-3*x(1)^2*y(0)+3*x(1)*y(0)-y(0)
↳
↳ ----- Upcoming Center -----
↳ _[1]=x(2)^2-x(2)+y(0)
↳ -----
↳ ++++++ Overview of Current Chart ++++++
↳ Current number of final charts: 2
↳ Total number of charts currently in chart-tree: 34
↳ Index of the current chart in chart-tree: 25
↳ ++++++
↳
↳ ==== Ambient Space:
↳ _[1]=0
↳
↳ ==== Ideal of Variety:
↳ _[1]=x(2)^2*y(1)-x(2)*y(1)-1
↳
↳ ==== Exceptional Divisors:

```

```

⇒ [1]:
⇒   _[1]=1
⇒ [2]:
⇒   _[1]=1
⇒ [3]:
⇒   _[1]=y(1)
⇒ [4]:
⇒   _[1]=x(3)
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(2)*x(3)^4*y(1)^3+x(3)^3*y(1)^2
⇒ _[2]=x(3)^4*y(1)^3+x(3)^3*y(1)^2
⇒ _[3]=x(3)^3*y(1)^2
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=x(2)^2*y(1)-x(2)*y(1)-1
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 3
⇒ Total number of charts currently in chart-tree: 34
⇒ Index of the current chart in chart-tree: 26
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(2)^2-x(2)-y(0)
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=y(0)
⇒ [2]:
⇒   _[1]=1
⇒ [3]:
⇒   _[1]=1
⇒ [4]:
⇒   _[1]=x(1)-1
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(1)^4*x(2)*y(0)-4*x(1)^3*x(2)*y(0)+x(1)^3*y(0)+6*x(1)^2*x(2)*y(0)-3\
    *x(1)^2*y(0)-4*x(1)*x(2)*y(0)+3*x(1)*y(0)+x(2)*y(0)-y(0)
⇒ _[2]=x(1)^4*y(0)-3*x(1)^3*y(0)+3*x(1)^2*y(0)-x(1)*y(0)
⇒ _[3]=x(1)^3*y(0)-3*x(1)^2*y(0)+3*x(1)*y(0)-y(0)
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=x(2)^2-x(2)-y(0)
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 4
⇒ Total number of charts currently in chart-tree: 34
⇒ Index of the current chart in chart-tree: 27
⇒ ++++++

```

```

↳
↳ ==== Ambient Space:
↳ _[1]=0
↳
↳ ==== Ideal of Variety:
↳ _[1]=x(2)^4*x(3)^4*y(1)^4+4*x(2)^3*x(3)^3*y(1)^3+6*x(2)^2*x(3)^2*y(1)^2-x\
(2)^2*y(1)+4*x(2)*x(3)*y(1)+x(2)*y(1)+1
↳
↳ ==== Exceptional Divisors:
↳ [1]:
↳   _[1]=1
↳ [2]:
↳   _[1]=1
↳ [3]:
↳   _[1]=y(1)
↳ [4]:
↳   _[1]=x(3)
↳
↳ ==== Images of variables of original ring:
↳ _[1]=x(3)^4*y(1)^3+x(3)^3*y(1)^2
↳ _[2]=x(3)^3*y(1)^2
↳ _[3]=x(2)*x(3)^4*y(1)^3+x(3)^3*y(1)^2
↳
↳ ----- Upcoming Center -----
↳ _[1]=x(2)^4*x(3)^4*y(1)^4+4*x(2)^3*x(3)^3*y(1)^3+6*x(2)^2*x(3)^2*y(1)^2-x\
(2)^2*y(1)+4*x(2)*x(3)*y(1)+x(2)*y(1)+1
↳ -----
↳ ++++++ Overview of Current Chart ++++++
↳ Current number of final charts: 5
↳ Total number of charts currently in chart-tree: 34
↳ Index of the current chart in chart-tree: 28
↳ ++++++
↳
↳ ==== Ambient Space:
↳ _[1]=0
↳
↳ ==== Ideal of Variety:
↳ _[1]=x(1)^4*x(2)^4*y(0)-4*x(1)^3*x(2)^4*y(0)+4*x(1)^3*x(2)^3*y(0)+6*x(1)^\
2*x(2)^4*y(0)-12*x(1)^2*x(2)^3*y(0)-4*x(1)*x(2)^4*y(0)+6*x(1)^2*x(2)^2*y(\
0)+12*x(1)*x(2)^3*y(0)+x(2)^4*y(0)-12*x(1)*x(2)^2*y(0)-4*x(2)^3*y(0)+4*x(\
1)*x(2)*y(0)+6*x(2)^2*y(0)-x(2)^2-4*x(2)*y(0)+x(2)+y(0)
↳
↳ ==== Exceptional Divisors:
↳ [1]:
↳   _[1]=y(0)
↳ [2]:
↳   _[1]=1
↳ [3]:
↳   _[1]=1
↳ [4]:
↳   _[1]=x(1)-1
↳
↳ ==== Images of variables of original ring:

```

```

⇒ _[1]=x(1)^4*y(0)-3*x(1)^3*y(0)+3*x(1)^2*y(0)-x(1)*y(0)
⇒ _[2]=x(1)^3*y(0)-3*x(1)^2*y(0)+3*x(1)*y(0)-y(0)
⇒ _[3]=x(1)^4*x(2)*y(0)-4*x(1)^3*x(2)*y(0)+x(1)^3*y(0)+6*x(1)^2*x(2)*y(0)-3\
    *x(1)^2*y(0)-4*x(1)*x(2)*y(0)+3*x(1)*y(0)+x(2)*y(0)-y(0)
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=x(1)^4*x(2)^4*y(0)-4*x(1)^3*x(2)^4*y(0)+4*x(1)^3*x(2)^3*y(0)+6*x(1)^\
    2*x(2)^4*y(0)-12*x(1)^2*x(2)^3*y(0)-4*x(1)*x(2)^4*y(0)+6*x(1)^2*x(2)^2*y(\
    0)+12*x(1)*x(2)^3*y(0)+x(2)^4*y(0)-12*x(1)*x(2)^2*y(0)-4*x(2)^3*y(0)+4*x(\
    1)*x(2)*y(0)+6*x(2)^2*y(0)-x(2)^2-4*x(2)*y(0)+x(2)+y(0)
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 6
⇒ Total number of charts currently in chart-tree: 34
⇒ Index of the current chart in chart-tree: 29
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(3)^4*y(1)^4+4*x(3)^3*y(1)^3+6*x(3)^2*y(1)^2+x(2)^2*y(1)-x(2)*y(1)+\
    4*x(3)*y(1)+1
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒ _[1]=1
⇒ [2]:
⇒ _[1]=1
⇒ [3]:
⇒ _[1]=y(1)
⇒ [4]:
⇒ _[1]=x(3)
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(2)*x(3)^4*y(1)^3+x(3)^3*y(1)^2
⇒ _[2]=x(3)^3*y(1)^2
⇒ _[3]=x(3)^4*y(1)^3+x(3)^3*y(1)^2
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=x(3)^4*y(1)^4+4*x(3)^3*y(1)^3+6*x(3)^2*y(1)^2+x(2)^2*y(1)-x(2)*y(1)+\
    4*x(3)*y(1)+1
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 7
⇒ Total number of charts currently in chart-tree: 34
⇒ Index of the current chart in chart-tree: 30
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:

```

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⇒ _[1]=x(1)^4*y(0)+x(2)^2-x(2)
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=y(0)
⇒ [2]:
⇒   _[1]=1
⇒ [3]:
⇒   _[1]=1
⇒ [4]:
⇒   _[1]=x(1)-1
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(1)^4*x(2)*y(0)-4*x(1)^3*x(2)*y(0)+x(1)^3*y(0)+6*x(1)^2*x(2)*y(0)-3\
  *x(1)^2*y(0)-4*x(1)*x(2)*y(0)+3*x(1)*y(0)+x(2)*y(0)-y(0)
⇒ _[2]=x(1)^3*y(0)-3*x(1)^2*y(0)+3*x(1)*y(0)-y(0)
⇒ _[3]=x(1)^4*y(0)-3*x(1)^3*y(0)+3*x(1)^2*y(0)-x(1)*y(0)
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=x(1)^4*y(0)+x(2)^2-x(2)
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 8
⇒ Total number of charts currently in chart-tree: 34
⇒ Index of the current chart in chart-tree: 31
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(2)^4*x(3)^4*y(1)^4+4*x(2)^3*x(3)^3*y(1)^3+6*x(2)^2*x(3)^2*y(1)^2+x\
  (2)^2*y(1)+4*x(2)*x(3)*y(1)-x(2)*y(1)+1
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=1
⇒ [2]:
⇒   _[1]=1
⇒ [3]:
⇒   _[1]=y(1)
⇒ [4]:
⇒   _[1]=x(3)
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(3)^3*y(1)^2
⇒ _[2]=x(3)^4*y(1)^3+x(3)^3*y(1)^2
⇒ _[3]=x(2)*x(3)^4*y(1)^3+x(3)^3*y(1)^2
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=x(2)^4*x(3)^4*y(1)^4+4*x(2)^3*x(3)^3*y(1)^3+6*x(2)^2*x(3)^2*y(1)^2+x\
  (2)^2*y(1)+4*x(2)*x(3)*y(1)-x(2)*y(1)+1
⇒ -----

```

```

⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 9
⇒ Total number of charts currently in chart-tree: 34
⇒ Index of the current chart in chart-tree: 32
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(1)^4*x(2)^4*y(0)-4*x(1)^3*x(2)^4*y(0)+4*x(1)^3*x(2)^3*y(0)+6*x(1)^2*x(2)^4*y(0)-12*x(1)^2*x(2)^3*y(0)-4*x(1)*x(2)^4*y(0)+6*x(1)^2*x(2)^2*y(0)+12*x(1)*x(2)^3*y(0)+x(2)^4*y(0)-12*x(1)*x(2)^2*y(0)-4*x(2)^3*y(0)+4*x(1)*x(2)*y(0)+6*x(2)^2*y(0)+x(2)^2-4*x(2)*y(0)-x(2)+y(0)
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=y(0)
⇒ [2]:
⇒   _[1]=1
⇒ [3]:
⇒   _[1]=1
⇒ [4]:
⇒   _[1]=x(1)-1
⇒
⇒ ==== Images of variables of original ring:
⇒ _[1]=x(1)^3*y(0)-3*x(1)^2*y(0)+3*x(1)*y(0)-y(0)
⇒ _[2]=x(1)^4*y(0)-3*x(1)^3*y(0)+3*x(1)^2*y(0)-x(1)*y(0)
⇒ _[3]=x(1)^4*x(2)*y(0)-4*x(1)^3*x(2)*y(0)+x(1)^3*y(0)+6*x(1)^2*x(2)*y(0)-3*x(1)^2*y(0)-4*x(1)*x(2)*y(0)+3*x(1)*y(0)+x(2)*y(0)-y(0)
⇒
⇒ ----- Upcoming Center -----
⇒ _[1]=x(1)^4*x(2)^4*y(0)-4*x(1)^3*x(2)^4*y(0)+4*x(1)^3*x(2)^3*y(0)+6*x(1)^2*x(2)^4*y(0)-12*x(1)^2*x(2)^3*y(0)-4*x(1)*x(2)^4*y(0)+6*x(1)^2*x(2)^2*y(0)+12*x(1)*x(2)^3*y(0)+x(2)^4*y(0)-12*x(1)*x(2)^2*y(0)-4*x(2)^3*y(0)+4*x(1)*x(2)*y(0)+6*x(2)^2*y(0)+x(2)^2-4*x(2)*y(0)-x(2)+y(0)
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 10
⇒ Total number of charts currently in chart-tree: 34
⇒ Index of the current chart in chart-tree: 33
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒ _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒ _[1]=x(3)^4*y(1)^4+4*x(3)^3*y(1)^3+6*x(3)^2*y(1)^2-x(2)^2*y(1)+x(2)*y(1)+4*x(3)*y(1)+1
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=1

```

```

⇒ [2]:
⇒   _[1]=1
⇒ [3]:
⇒   _[1]=y(1)
⇒ [4]:
⇒   _[1]=x(3)
⇒
⇒ ==== Images of variables of original ring:
⇒   _[1]=x(3)^3*y(1)^2
⇒   _[2]=x(2)*x(3)^4*y(1)^3+x(3)^3*y(1)^2
⇒   _[3]=x(3)^4*y(1)^3+x(3)^3*y(1)^2
⇒
⇒ ----- Upcoming Center -----
⇒   _[1]=x(3)^4*y(1)^4+4*x(3)^3*y(1)^3+6*x(3)^2*y(1)^2-x(2)^2*y(1)+x(2)*y(1)+\
    4*x(3)*y(1)+1
⇒ -----
⇒ ++++++ Overview of Current Chart ++++++
⇒ Current number of final charts: 11
⇒ Total number of charts currently in chart-tree: 34
⇒ Index of the current chart in chart-tree: 34
⇒ ++++++
⇒
⇒ ==== Ambient Space:
⇒   _[1]=0
⇒
⇒ ==== Ideal of Variety:
⇒   _[1]=x(1)^4*y(0)-x(2)^2+x(2)
⇒
⇒ ==== Exceptional Divisors:
⇒ [1]:
⇒   _[1]=y(0)
⇒ [2]:
⇒   _[1]=1
⇒ [3]:
⇒   _[1]=1
⇒ [4]:
⇒   _[1]=x(1)-1
⇒
⇒ ==== Images of variables of original ring:
⇒   _[1]=x(1)^3*y(0)-3*x(1)^2*y(0)+3*x(1)*y(0)-y(0)
⇒   _[2]=x(1)^4*x(2)*y(0)-4*x(1)^3*x(2)*y(0)+x(1)^3*y(0)+6*x(1)^2*x(2)*y(0)-3\
    *x(1)^2*y(0)-4*x(1)*x(2)*y(0)+3*x(1)*y(0)+x(2)*y(0)-y(0)
⇒   _[3]=x(1)^4*y(0)-3*x(1)^3*y(0)+3*x(1)^2*y(0)-x(1)*y(0)
⇒
⇒ ----- Upcoming Center -----
⇒   _[1]=x(1)^4*y(0)-x(2)^2+x(2)
⇒ -----
⇒ ===== result will be tested =====
⇒
⇒ the number of charts obtained: 12
⇒ ===== result is o.k. =====
list iden=prepEmbDiv(re);
intvec v=computeN(re,iden);

```



```
v;
↳ 3,4,8,12,1
```

### D.5.17 schubert\_lib

**Library:** schubert.lib

**Purpose:** Procedures for Intersection Theory

**Author:** Hiep Dang, email: hiep@mathematik.uni-kl.de

**Overview:** We implement new classes (variety, sheaf, stack, graph) and methods for computing with them. An abstract variety is represented by a nonnegative integer which is its dimension and a graded ring which is its Chow ring. An abstract sheaf is represented by a variety and a polynomial which is its Chern character. In particular, we implement the concrete varieties such as projective spaces, Grassmannians, and projective bundles.

An important task of this library is related to the computation of Gromov-Witten invariants. In particular, we implement new tools for the computation in equivariant intersection theory. These tools are based on the localization of moduli spaces of stable maps and Bott's formula. They are useful for the computation of Gromov-Witten invariants. In order to do this, we have to deal with moduli spaces of stable maps, which were introduced by Kontsevich, and the graphs corresponding to the fixed point components of a torus action on the moduli spaces of stable maps.

As an insightful example, the numbers of rational curves on general complete intersection Calabi-Yau threefolds in projective spaces are computed up to degree 6. The results are all in agreement with predictions made from mirror symmetry computations.

**References:**

Hiep Dang, Intersection theory with applications to the computation of Gromov-Witten invariants, Ph.D thesis, TU Kaiserslautern, 2013.

Sheldon Katz and Stein A. Stromme, Schubert-A Maple package for intersection theory and enumerative geometry, 1992.

Daniel R. Grayson, Michael E. Stillman, Stein A. Stromme, David Eisenbud and Charley Crissman, Schubert2-A Macaulay2 package for computation in intersection theory, 2010.

Maxim Kontsevich, Enumeration of rational curves via torus actions, 1995.

**Procedures:**

#### D.5.17.1 makeVariety

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\]](#), page 1587).

**Usage:** makeVariety(d,i); d int, i ideal

**Assume:** d is a nonnegative integer, i is an ideal

**Return:** variety

**Theory:** create an abstract variety which has dimension d, and its Chow ring should be a quotient ring

**Example:**

```

LIB "schubert.lib";
ring r = 0,(h,e),wp(1,1);
ideal rels = he,h2+e2;
variety V = makeVariety(2,rels);
V;
⇒ A variety of dimension 2
⇒
V.dimension;
⇒ 2
V.relations;
⇒ _[1]=he
⇒ _[2]=h2+e2

```

### D.5.17.2 printVariety

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\], page 1587](#)).

**Usage:** `printVariety(V)`; V variety

**Assume:** V is an abstract variety

**Theory:** This is the print function used by Singular to print an abstract variety.

**Example:**

```

LIB "schubert.lib";
ring r = 0,(h,e),wp(1,1);
ideal rels = he,h2+e2;
variety V = makeVariety(2,rels);
V;
⇒ A variety of dimension 2
⇒

```

### D.5.17.3 productVariety

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\], page 1587](#)).

**Usage:** `productVariety(U,V)`; U variety, V variety

**Input:** two abstract varieties

**Output:** a product variety as an abstract variety

**Return:** variety

**Example:**

```

LIB "schubert.lib";
variety P = projectiveSpace(3);
variety G = Grassmannian(2,4);
variety W = productVariety(P,G);
W;
⇒ A variety of dimension 7
⇒
W.dimension == P.dimension + G.dimension;
⇒ 1
def r = W.baseRing;
setring r;
W.relations;

```

```

⇒ _[1]=h^4
⇒ _[2]=q(1)^3-2*q(1)*q(2)
⇒ _[3]=q(1)^4-3*q(1)^2*q(2)+q(2)^2

```

See also: [Section D.5.17.5 \[Grassmannian\]](#), page 1589; [Section D.5.17.7 \[projectiveBundle\]](#), page 1591; [Section D.5.17.6 \[projectiveSpace\]](#), page 1590.

### D.5.17.4 ChowRing

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\]](#), page 1587).

**Usage:** `ChowRing(V)`;  $V$  variety

**Assume:**  $V$  is an abstract variety

**Return:** `qring`

**Example:**

```

LIB "schubert.lib";
ring r = 0,(h,e),wp(1,1);
ideal rels = he,h2+e2;
int d = 2;
variety V = makeVariety(2,rels);
ChowRing(V);
⇒ // coefficients: QQ
⇒ // number of vars : 2
⇒ //      block   1 : ordering dp
⇒ //                : names     h e
⇒ //      block   2 : ordering C
⇒ // quotient ring from ideal ...

```

### D.5.17.5 Grassmannian

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\]](#), page 1587).

**Usage:** `Grassmannian(k,n)`;  $k$  int,  $n$  int

**Return:** variety

**Theory:** create a Grassmannian  $G(k,n)$  as an abstract variety. This abstract variety has dimension  $k(n-k)$  and its Chow ring is the quotient ring of a polynomial ring in  $n-k$  variables  $q(1), \dots, q(n-k)$ , which are the Chern classes of tautological quotient bundle on  $G(k,n)$ , modulo some ideal generated by  $n-k$  polynomials which come from the Giambelli formula. The monomial ordering of this Chow ring is 'wp' with vector  $(1..k, 1..n-k)$ . Moreover, we export the Chern characters of tautological subbundle and quotient bundle on  $G(k,n)$  (say 'subBundle' and 'quotientBundle').

**Example:**

```

LIB "schubert.lib";
variety G24 = Grassmannian(2,4);
G24;
⇒ A variety of dimension 4
⇒
def r = G24.baseRing;
setring r;
subBundle;

```

```

⇒ 1/6*q(1)*q(2)-1/2*q(1)^2+q(2)-q(1)+2
quotientBundle;
⇒ -1/6*q(1)*q(2)+1/2*q(1)^2-q(2)+q(1)+2
G24.dimension;
⇒ 4
G24.relations;
⇒ _[1]=q(1)^3-2*q(1)*q(2)
⇒ _[2]=q(1)^4-3*q(1)^2*q(2)+q(2)^2
ChowRing(G24);
⇒ // coefficients: QQ
⇒ // number of vars : 2
⇒ //      block   1 : ordering wp
⇒ //                : names    q(1) q(2)
⇒ //                : weights   1    2
⇒ //      block   2 : ordering C
⇒ // quotient ring from ideal ...

```

See also: [Section D.5.17.7 \[projectiveBundle\]](#), page 1591; [Section D.5.17.6 \[projectiveSpace\]](#), page 1590.

### D.5.17.6 projectiveSpace

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\]](#), page 1587).

**Usage:**     `projectiveSpace(n);`  $n$  int

**Return:**    variety

**Theory:**    create a projective space of dimension  $n$  as an abstract variety. Its Chow ring is a quotient ring in one variable  $h$  modulo the ideal generated by  $h^{(n+1)}$ .

**Example:**

```

LIB "schubert.lib";
variety P = projectiveSpace(3);
P;
⇒ A variety of dimension 3
⇒
P.dimension;
⇒ 3
def r = P.baseRing;
setring r;
P.relations;
⇒ _[1]=h4
ChowRing(P);
⇒ // coefficients: QQ
⇒ // number of vars : 1
⇒ //      block   1 : ordering wp
⇒ //                : names    h
⇒ //                : weights   1
⇒ //      block   2 : ordering C
⇒ // quotient ring from ideal ...

```

See also: [Section D.5.17.5 \[Grassmannian\]](#), page 1589; [Section D.5.17.7 \[projectiveBundle\]](#), page 1591.

### D.5.17.7 projectiveBundle

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\]](#), page 1587).

**Usage:** `projectiveBundle(S)`; S sheaf

**Input:** a sheaf on an abstract variety

**Return:** variety

**Theory:** create a projective bundle as an abstract variety. This is related to the enumeration of conics.

**Example:**

```
LIB "schubert.lib";
variety G = Grassmannian(3,5);
def r = G.baseRing;
setring r;
sheaf S = makeSheaf(G,subBundle);
sheaf B = dualSheaf(S)^2;
variety PB = projectiveBundle(B);
PB;
⇒ A variety of dimension 11
⇒
def R = PB.baseRing;
setring R;
QuotientBundle;
⇒ 1/1995840*z^5*q(2)^3-1/120960*z^5*q(1)*q(2)^2+1/20160*z^5*q(1)^2*q(2)+1/4\
0320*z^5*q(2)^2-1/10080*z^5*q(1)^3-1/2688*z^5*q(1)*q(2)+1/840*z^5*q(1)^2+\
1/1008*z^5*q(2)-1/180*z^5*q(1)+1/120*z^5-1/36288*z^4*q(2)^3+5/12096*z^4*q\
(1)*q(2)^2-1/448*z^4*q(1)^2*q(2)+5/8064*z^4*q(2)^2+1/252*z^4*q(1)^3+1/100\
8*z^4*q(1)*q(2)-1/72*z^4*q(1)^2+1/144*z^4*q(2)+1/24*z^4+43/72576*z^3*q(2)\
^3-47/8064*z^3*q(1)*q(2)^2+11/504*z^3*q(1)^2*q(2)-1/84*z^3*q(2)^2-1/36*z\
3*q(1)^3+5/144*z^3*q(1)*q(2)+1/6*z^3-1/192*z^2*q(2)^3+1/36*z^2*q(1)*q(2)\
^2-1/24*z^2*q(1)^2*q(2)+1/36*z^2*q(2)^2+1/2*z^2+1/63*z*q(2)^3-1/36*z*q(1)*\
q(2)^2+z+1
ChowRing(PB);
⇒ // coefficients: QQ
⇒ // number of vars : 3
⇒ //      block   1: ordering wp
⇒ //              : names     z
⇒ //              : weights   1
⇒ //      block   2: ordering wp
⇒ //              : names     q(1) q(2)
⇒ //              : weights   1     2
⇒ //      block   3: ordering C
⇒ // quotient ring from ideal ...
```

See also: [Section D.5.17.5 \[Grassmannian\]](#), page 1589; [Section D.5.17.6 \[projectiveSpace\]](#), page 1590.

### D.5.17.8 integral

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\]](#), page 1587).

**Usage:** `integral(V,f)`; V variety, f poly

**Input:** a abstract variety and a polynomial

**Return:** int

**Purpose:** computing intersection numbers.

**Example:**

```
LIB "schubert.lib";
variety G = Grassmannian(2,4);
def r = G.baseRing;
setring r;
integral(G,q(1)^4);
↪ 2
```

### D.5.17.9 makeSheaf

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\]](#), page 1587).

**Usage:** `makeSheaf(V,ch);` V variety, ch poly

**Return:** sheaf

**Theory:** create a sheaf on an abstract variety, and its Chern character is the polynomial ch.

**Example:**

```
LIB "schubert.lib";
variety X;
X.dimension = 4;
ring r = 0,(c(1..2),d(1..3)),wp(1..2,1..3);
setring r;
X.baseRing = r;
poly c = 1 + c(1) + c(2);
poly ch = 2 + logg(c,4);
sheaf S = makeSheaf(X,ch);
S;
↪ A sheaf of rank 2
↪
```

See also: [Section D.5.17.10 \[printSheaf\]](#), page 1592; [Section D.5.17.11 \[rankSheaf\]](#), page 1593.

### D.5.17.10 printSheaf

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\]](#), page 1587).

**Usage:** `printSheaf(S);` S sheaf

**Return:** string

**Input:** a sheaf

**Theory:** This is the print function used by Singular to print a sheaf.

**Example:**

```
LIB "schubert.lib";
variety X;
X.dimension = 4;
ring r = 0,(c(1..2),d(1..3)),wp(1..2,1..3);
setring r;
X.baseRing = r;
```

```

poly c = 1 + c(1) + c(2);
poly ch = 2 + logg(c,4);
sheaf S = makeSheaf(X,ch);
S;
↳ A sheaf of rank 2
↳

```

See also: [Section D.5.17.9 \[makeSheaf\]](#), page 1592; [Section D.5.17.11 \[rankSheaf\]](#), page 1593.

### D.5.17.11 rankSheaf

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\]](#), page 1587).

**Usage:** `rankSheaf(S);` S sheaf

**Return:** int

**Input:** S is a sheaf

**Output:** a positive integer which is the rank of a sheaf.

**Example:**

```

LIB "schubert.lib";
variety G = Grassmannian(2,4);
def R = G.baseRing;
setring R;
sheaf S = makeSheaf(G,subBundle);
rankSheaf(S);
↳ 2

```

See also: [Section D.5.17.9 \[makeSheaf\]](#), page 1592; [Section D.5.17.10 \[printSheaf\]](#), page 1592.

### D.5.17.12 totalChernClass

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\]](#), page 1587).

**Usage:** `totalChernClass(S);` S sheaf

**Return:** poly

**Input:** S is a sheaf

**Output:** a polynomial which is the total Chern class of a sheaf

**Example:**

```

LIB "schubert.lib";
variety X;
X.dimension = 4;
ring r = 0, (c(1..2), d(1..3)), wp(1..2, 1..3);
setring r;
X.baseRing = r;
poly c = 1 + c(1) + c(2);
poly ch = 2 + logg(c,4);
sheaf E = makeSheaf(X,ch);
sheaf S = E^3;
totalChernClass(S);
↳ 18*c(1)^2*c(2)+9*c(2)^2+6*c(1)^3+30*c(1)*c(2)+11*c(1)^2+10*c(2)+6*c(1)+1

```

See also: [Section D.5.17.13 \[ChernClass\]](#), page 1594; [Section D.5.17.14 \[topChernClass\]](#), page 1594; [Section D.5.17.15 \[totalSegreClass\]](#), page 1595.

### D.5.17.13 ChernClass

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\]](#), page 1587).

**Usage:** ChernClass(S,i); S sheaf, i int

**Input:** S is a sheaf, i is a nonnegative integer

**Return:** poly

**Theory:** This is the i-th Chern class of a sheaf

**Example:**

```
LIB "schubert.lib";
variety X;
X.dimension = 4;
ring r = 0, (c(1..2), d(1..3)), wp(1..2, 1..3);
setring r;
X.baseRing = r;
poly c = 1 + c(1) + c(2);
poly ch = 2 + logg(c, 4);
sheaf E = makeSheaf(X, ch);
sheaf S = E^3;
ChernClass(S, 1);
 $\mapsto 6*c(1)$ 
ChernClass(S, 2);
 $\mapsto 11*c(1)^2 + 10*c(2)$ 
ChernClass(S, 3);
 $\mapsto 6*c(1)^3 + 30*c(1)*c(2)$ 
ChernClass(S, 4);
 $\mapsto 18*c(1)^2*c(2) + 9*c(2)^2$ 
```

See also: [Section D.5.17.14 \[topChernClass\]](#), page 1594; [Section D.5.17.12 \[totalChernClass\]](#), page 1593.

### D.5.17.14 topChernClass

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\]](#), page 1587).

**Usage:** topChernClass(S); S sheaf

**Return:** poly

**Input:** S is a sheaf

**Theory:** This is the top Chern class of a sheaf

**Example:**

```
LIB "schubert.lib";
variety G = Grassmannian(2, 4);
def R = G.baseRing;
setring R;
sheaf S = makeSheaf(G, quotientBundle);
sheaf B = S^3;
topChernClass(B);
 $\mapsto 27*q(2)^2$ 
```

See also: [Section D.5.17.13 \[ChernClass\]](#), page 1594; [Section D.5.17.12 \[totalChernClass\]](#), page 1593.



**D.5.17.15 totalSegreClass**

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\]](#), page 1587).

**Usage:** `totalSegreClass(S)`; S sheaf

**Return:** poly

**Input:** S is a sheaf

**Theory:** This is the total Segre class of a sheaf.  
SEE ALSO: `totalChernClass`

**Example:**

```
LIB "schubert.lib";
variety G = Grassmannian(2,4);
def R = G.baseRing;
setring R;
sheaf S = makeSheaf(G,subBundle);
totalSegreClass(S);
↪ q(2)+q(1)+1
```

**D.5.17.16 dualSheaf**

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\]](#), page 1587).

**Usage:** `dualSheaf(S)`; S sheaf

**Return:** sheaf

**Theory:** This is the dual of a sheaf

**Example:**

```
LIB "schubert.lib";
variety G = Grassmannian(2,4);
def R = G.baseRing;
setring R;
sheaf S = makeSheaf(G,subBundle);
sheaf D = dualSheaf(S);
D;
↪ A sheaf of rank 2
↪
```

See also: [Section D.5.17.20 \[addSheaf\]](#), page 1597; [Section D.5.17.19 \[quotSheaf\]](#), page 1596; [Section D.5.17.18 \[symmetricPowerSheaf\]](#), page 1596; [Section D.5.17.17 \[tensorSheaf\]](#), page 1595.

**D.5.17.17 tensorSheaf**

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\]](#), page 1587).

**Usage:** `tensorSheaf(A,B)`; A sheaf, B sheaf

**Return:** sheaf

**Theory:** This is the tensor product of two sheaves

**Example:**

```

LIB "schubert.lib";
variety G = Grassmannian(3,4);
def R = G.baseRing;
setring R;
sheaf S = makeSheaf(G,subBundle);
sheaf Q = makeSheaf(G,quotientBundle);
sheaf T = S*Q;
T;
↳ A sheaf of rank 3
↳

```

See also: [Section D.5.17.20 \[addSheaf\]](#), page 1597; [Section D.5.17.16 \[dualSheaf\]](#), page 1595; [Section D.5.17.19 \[quotSheaf\]](#), page 1596; [Section D.5.17.18 \[symmetricPowerSheaf\]](#), page 1596.

### D.5.17.18 symmetricPowerSheaf

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\]](#), page 1587).

**Usage:** `symmetricPowerSheaf(S,n)`; S sheaf, n int

**Return:** sheaf

**Theory:** This is the n-th symmetric power of a sheaf

**Example:**

```

LIB "schubert.lib";
variety G = Grassmannian(2,4);
def R = G.baseRing;
setring R;
sheaf S = makeSheaf(G,quotientBundle);
sheaf B = symmetricPowerSheaf(S,3);
B;
↳ A sheaf of rank 4
↳
sheaf A = S^3;
A;
↳ A sheaf of rank 4
↳
A.ChernCharacter == B.ChernCharacter;
↳ 1

```

See also: [Section D.5.17.20 \[addSheaf\]](#), page 1597; [Section D.5.17.16 \[dualSheaf\]](#), page 1595; [Section D.5.17.19 \[quotSheaf\]](#), page 1596; [Section D.5.17.17 \[tensorSheaf\]](#), page 1595.

### D.5.17.19 quotSheaf

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\]](#), page 1587).

**Usage:** `quotSheaf(A,B)`; A sheaf, B sheaf

**Return:** sheaf

**Theory:** This is the quotient of two sheaves

**Example:**

```

LIB "schubert.lib";
variety G = Grassmannian(3,5);
def r = G.baseRing;

```

```

setring r;
sheaf S = makeSheaf(G,subBundle);
sheaf B = dualSheaf(S)^2;
sheaf B3 = dualSheaf(S)^3;
sheaf B5 = dualSheaf(S)^5;
variety PB = projectiveBundle(B);
def R = PB.baseRing;
setring R;
sheaf Q = makeSheaf(PB,QuotientBundle);
sheaf V = dualSheaf(Q)*B3;
sheaf A = B5 - V;
A;
⇒ A sheaf of rank 11
⇒

```

See also: [Section D.5.17.20 \[addSheaf\]](#), page 1597; [Section D.5.17.16 \[dualSheaf\]](#), page 1595; [Section D.5.17.18 \[symmetricPowerSheaf\]](#), page 1596; [Section D.5.17.17 \[tensorSheaf\]](#), page 1595.

### D.5.17.20 addSheaf

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\]](#), page 1587).

**Usage:** `addSheaf(A,B)`; A sheaf, B sheaf

**Return:** sheaf

**Theory:** This is the direct sum of two sheaves.

**Example:**

```

LIB "schubert.lib";
variety G = Grassmannian(3,5);
def r = G.baseRing;
setring r;
sheaf S = makeSheaf(G,subBundle);
sheaf Q = makeSheaf(G,quotientBundle);
sheaf D = S + Q;
D;
⇒ A sheaf of rank 5
⇒
D.ChernCharacter == rankSheaf(D);
⇒ 1
totalChernClass(D) == 1;
⇒ 1

```

See also: [Section D.5.17.16 \[dualSheaf\]](#), page 1595; [Section D.5.17.19 \[quotSheaf\]](#), page 1596; [Section D.5.17.18 \[symmetricPowerSheaf\]](#), page 1596; [Section D.5.17.17 \[tensorSheaf\]](#), page 1595.

### D.5.17.21 makeGraphVE

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\]](#), page 1587).

**Assume:** `v` is a list of vertices, `e` is a list of edges.

**Return:** graph with vertices `v` and edges `e`.

**Theory:** Creates a graph from a list of vertices and edges.

**Example:**

```
LIB "schubert.lib";
ring r = 0,x,dp;
graph G = makeGraphVE(list(list(0,1,list(0,1,2)),list(1,1,list(1,0,2))),
list(list(0,1,2)));
G;
↳ A graph with 2 vertices and 1 edges
↳
```

### D.5.17.22 printGraphG

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\], page 1587](#)).

**Usage:** `printGraphG(G);` G graph

**Assume:** G is a graph.

**Theory:** This is the print function used by Singular to print a graph.

**Example:**

```
LIB "schubert.lib";
ring r = 0,x,dp;
graph G = makeGraphVE(list(list(0,1,list(0,1,2)),list(1,1,list(1,0,2))),
list(list(0,1,2)));
G;
↳ A graph with 2 vertices and 1 edges
↳
```

### D.5.17.23 moduliSpace

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\], page 1587](#)).

**Usage:** `moduliSpace(V,d);` V variety, d int

**Assume:** V is a projective space and d is a positive integer.

**Theory:** This is the function used by Singular to create a moduli space of stable maps from a genus zero curve to a projective space.

**Example:**

```
LIB "schubert.lib";
ring r = 0,(x),dp;
variety P = projectiveSpace(4);
stack M = moduliSpace(P,2);
M;
↳ A moduli space of dimension 11
↳
```

### D.5.17.24 printStack

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\], page 1587](#)).

**Usage:** `printStack(M);` M stack

**Assume:** M is a moduli space of stable maps.

**Theory:** This is the print function used by Singular to print a stack.

**Example:**

```

LIB "schubert.lib";
ring r = 0,(x),dp;
variety P = projectiveSpace(4);
stack M = moduliSpace(P,2);
M;
↳ A moduli space of dimension 11
↳

```

### D.5.17.25 dimStack

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\], page 1587](#)).

**Usage:** `dimStack(M)`; M stack

**Return:** `int`

**Input:** M is a moduli space of stable maps.

**Output:** the dimension of moduli space of stable maps.

**Example:**

```

LIB "schubert.lib";
ring r = 0,(x),dp;
variety P = projectiveSpace(4);
stack M = moduliSpace(P,2);
dimStack(M);
↳ 11

```

### D.5.17.26 fixedPoints

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\], page 1587](#)).

**Usage:** `fixedPoints(M)`; M stack

**Return:** `list`

**Input:** M is a moduli space of stable maps.

**Output:** a list of graphs corresponding the fixed point components of a torus action on a moduli space of stable maps.

**Example:**

```

LIB "schubert.lib";
ring r = 0,x,dp;
variety P = projectiveSpace(4);
stack M = moduliSpace(P,2);
def F = fixedPoints(M);
size(F);
↳ 100
typeof(F[1]) == "list";
↳ 1
typeof(F[1][1]) == "graph";
↳ 1
typeof(F[1][2]) == "int";
↳ 1

```

### D.5.17.27 contributionBundle

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\]](#), page 1587).

**Usage:** `contributionBundle(M,G,#)`; M stack, G graph, # list

**Return:** number

**Input:** M is a moduli space of stable maps, G is a graph, # is a list.

**Output:** a number corresponding to the contribution bundle on a moduli space of stable maps at a fixed point component (graph)

**Example:**

```
LIB "schubert.lib";
ring r = 0,x,dp;
variety P = projectiveSpace(4);
stack M = moduliSpace(P,2);
def F = fixedPoints(M);
graph G = F[1][1];
number f = contributionBundle(M,G);
number g = contributionBundle(M,G,5);
f == g;
↦ 1
```

See also: [Section D.5.17.28 \[normalBundle\]](#), page 1600.

### D.5.17.28 normalBundle

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\]](#), page 1587).

**Return:** number

**Input:** M is a moduli space of stable maps, G is a graph

**Output:** a number corresponding to the normal bundle on a moduli space of stable maps at a graph  
hypersurfaces

**Example:**

```
LIB "schubert.lib";
ring r = 0,x,dp;
variety P = projectiveSpace(4);
stack M = moduliSpace(P,2);
def F = fixedPoints(M);
graph G = F[1][1];
number f = normalBundle(M,G);
f <> 0;
↦ 1
```

See also: [Section D.5.17.27 \[contributionBundle\]](#), page 1600.

### D.5.17.29 multipleCover

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\]](#), page 1587).

**Usage:** `multipleCover(d)`; d int

**Return:** number

**Theory:** This is the contribution of degree  $d$  multiple covers of a smooth rational curve as a Gromov-Witten invariant.

**Example:**

```
LIB "schubert.lib";
ring r = 0,x,dp;
multipleCover(1);
↦ 1
multipleCover(2);
↦ 1/8
multipleCover(3);
↦ 1/27
multipleCover(4);
↦ 1/64
multipleCover(5);
↦ 1/125
multipleCover(6);
↦ 1/216
```

See also: [Section D.5.17.30 \[linesHypersurface\]](#), page 1601; [Section D.5.17.31 \[rationalCurve\]](#), page 1602.

### D.5.17.30 linesHypersurface

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\]](#), page 1587).

**Usage:** `linesHypersurface(n); n int`

**Return:** number

**Theory:** This is the number of lines on a general hypersurface of degree  $d = 2n-3$  in an  $n$ -dimensional projective space.

**Example:**

```
LIB "schubert.lib";
ring r = 0,x,dp;
linesHypersurface(2);
↦ 1
linesHypersurface(3);
↦ 27
linesHypersurface(4);
↦ 2875
linesHypersurface(5);
↦ 698005
linesHypersurface(6);
↦ 305093061
linesHypersurface(7);
↦ 210480374951
linesHypersurface(8);
↦ 210776836330775
linesHypersurface(9);
↦ 289139638632755625
linesHypersurface(10);
↦ 520764738758073845321
```

See also: [Section D.5.17.30 \[linesHypersurface\]](#), page 1601; [Section D.5.17.29 \[multipleCover\]](#), page 1600.

### D.5.17.31 rationalCurve

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\]](#), page 1587).

**Usage:** `rationalCurve(d,#);` d int, # list

**Return:** number

**Theory:** This is the Gromov-Witten invariant corresponding the number of rational curves on a general Calabi-Yau threefold.

**Example:**

```
LIB "schubert.lib";
ring r = 0,x,dp;
rationalCurve(1);
⇨ 2875
/*
rationalCurve(2);
rationalCurve(3);
rationalCurve(4);
rationalCurve(1,list(4,2));
rationalCurve(1,list(3,3));
rationalCurve(1,list(3,2,2));
rationalCurve(1,list(2,2,2,2));
rationalCurve(2,list(4,2));
rationalCurve(2,list(3,3));
rationalCurve(2,list(3,2,2));
rationalCurve(2,list(2,2,2,2));
rationalCurve(3,list(4,2));
rationalCurve(3,list(3,3));
rationalCurve(3,list(3,2,2));
rationalCurve(3,list(2,2,2,2));
rationalCurve(4,list(4,2));
rationalCurve(4,list(3,3));
rationalCurve(4,list(3,2,2));
rationalCurve(4,list(2,2,2,2));
*/
```

See also: [Section D.5.17.30 \[linesHypersurface\]](#), page 1601; [Section D.5.17.29 \[multipleCover\]](#), page 1600.

### D.5.17.32 sumofquotients

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\]](#), page 1587).

**Usage:** `sumofquotient(M,F,#);` M stack, F list, # list

**Return:** number

**Theory:** This is useful for the parallel computation of rationalCurve.

**Example:**

```
LIB "schubert.lib";
ring r = 0,x,dp;
variety P = projectiveSpace(4);
stack M = moduliSpace(P,2);
list F = fixedPoints(M);
```



```

sumofquotients(M,F);
↦ 4876875/8
sumofquotients(M,F,list(5));
↦ 4876875/8

```

### D.5.17.33 homog\_part

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\], page 1587](#)).

**Usage:** `homog_part(f,n)`;  $f$  poly,  $n$  int

**Return:** poly

**Purpose:** computing the homogeneous component of a polynomial.

**Example:**

```

LIB "schubert.lib";
ring r = 0,(x,y,z),wp(1,2,3);
poly f = 1+x+x2+x3+x4+y+y2+y3+z+z2+xy+xz+yz+xyz;
homog_part(f,0);
↦ 1
homog_part(f,1);
↦ x
homog_part(f,2);
↦ x2+y
homog_part(f,3);
↦ x3+xy+z
homog_part(f,4);
↦ x4+y2+xz
homog_part(f,5);
↦ yz
homog_part(f,6);
↦ y3+xyz+z2

```

### D.5.17.34 homog\_parts

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\], page 1587](#)).

**Usage:** `homog_parts(f,i,j)`;  $f$  poly,  $i$  int,  $j$  int

**Return:** poly

**Theory:** computing a polynomial which is the sum of the homogeneous components of a polynomial.

**Example:**

```

LIB "schubert.lib";
ring r = 0,(x,y,z),wp(1,2,3);
poly f = 1+x+x2+x3+x4+y+y2+y3+z+z2+xy+xz+yz+xyz;
homog_parts(f,2,4);
↦ x4+y2+xz+x3+xy+z+x2+y

```

### D.5.17.35 logg

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\], page 1587](#)).

**Usage:** `logg(f,n)`;  $f$  poly,  $n$  int

**Return:** poly

**Theory:** computing Chern characters from total Chern classes.

**Example:**

```
LIB "schubert.lib";
ring r = 0,(x,y),wp(1,2);
poly f = 1+x+y;
logg(f,4);
 $\mapsto 1/24x^4-1/6x^2y+1/12y^2+1/6x^3-1/2xy+1/2x^2-y+x$ 
```

### D.5.17.36 exp

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\], page 1587](#)).

**Usage:** exp(f,n); f poly, n int

**Return:** poly

**Purpose:** computing total Chern classes from Chern characters.

**Example:**

```
LIB "schubert.lib";
ring r = 0,x,dp;
poly f = 3+x;
exp(f,3);
 $\mapsto 1/6x^3+1/2x^2+x+1$ 
```

### D.5.17.37 SchubertClass

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\], page 1587](#)).

**Usage:** SchubertClass(p); p list

**Input:** a list of integers which is a partition

**Return:** poly

**Purpose:** compute the Schubert classes on a Grassmannian.

**Example:**

```
LIB "schubert.lib";
variety G = Grassmannian(2,4);
def r = G.baseRing;
setring r;
list p = 1,1;
SchubertClass(p);
 $\mapsto q(1)^2-q(2)$ 
```

### D.5.17.38 dualPartition

Procedure from library `schubert.lib` (see [Section D.5.17 \[schubert.lib\], page 1587](#)).

**Usage:** dualPartition(k,n,p); k int, n int, p list

**Input:** two integers and a partition

**Return:** list

**Purpose:** compute the dual of a partition.

**Example:**

```
LIB "schubert.lib";
ring r = 0,(x),dp;
dualPartition(2,4,list(2,1));
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 0
```

See also: [Section D.5.17.37 \[SchubertClass\]](#), page 1604.

**D.5.18 sheafcoh\_lib**

**Library:** sheafcoh.lib

**Purpose:** Procedures for Computing Sheaf Cohomology

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**Procedures:****D.5.18.1 truncate**

Procedure from library `sheafcoh.lib` (see [Section D.5.18 \[sheafcoh\\_lib\]](#), page 1605).

**Usage:** `truncate(M,d)`;  $M$  module,  $d$  int

**Assume:**  $M$  is graded, and it comes assigned with an admissible degree vector as an attribute

**Return:** module

**Note:** Output is a presentation matrix for the truncation of  $\text{coker}(M)$  at degree  $d$ .

**Example:**

```
LIB "sheafcoh.lib";
ring R=0,(x,y,z),dp;
module M=maxideal(3);
homog(M);
⇒ 1
// compute presentation matrix for truncated module (R/<x,y,z>^3)_(≥2)
module M2=truncate(M,2);
print(M2);
⇒ z,0,0,0,0,0,y, 0, 0,0, 0,0,x, 0, 0, 0, 0, 0,
⇒ 0,z,0,0,0,0,-z,y, 0,0, 0,0,0, x, 0, 0, 0, 0,
⇒ 0,0,z,0,0,0,0, -z,y,0, 0,0,0, 0, x, 0, 0, 0,
⇒ 0,0,0,z,0,0,0, 0, 0,y, 0,0,-z,-y,0, x, 0, 0,
⇒ 0,0,0,0,z,0,0, 0, 0,-z,y,0,0, 0, -y,0, x, 0,
⇒ 0,0,0,0,0,z,0, 0, 0,0, 0,y,0, 0, 0, -z,-y,x
dimGradedPart(M2,1);
⇒ 0
dimGradedPart(M2,2);
⇒ 6
// this should coincide with:
dimGradedPart(M,2);
```

```

⇒ 6
// shift grading by 1:
intvec v=1;
attrib(M,"isHomog",v);
M2=truncate(M,2);
print(M2);
⇒ 0, -y,-z,z2,0, 0, yz,0, xz,y2,0, xy,x2,
⇒ -z,x, 0, 0, z2,0, 0, yz,0, 0, y2,0, 0,
⇒ y, 0, x, 0, 0, z2,0, 0, 0, 0, 0, 0, 0
dimGradedPart(M2,3);
⇒ 6

```

### D.5.18.2 truncateFast

Procedure from library `sheafcoh.lib` (see [Section D.5.18 \[sheafcoh.lib\]](#), page 1605).

**Usage:** `truncateFast(M,d)`; M module, d int

**Assume:** M is graded, and it comes assigned with an admissible degree vector as an attribute 'isHomog'

**Return:** module

**Note:** Output is a presentation matrix for the truncation of `coker(M)` at d.  
Fast + experimental version. M should be a SB!

**Display:** If `printlevel` ≥ 1, step-by step timings will be printed. If `printlevel` ≥ 2 we add progress debug messages if `printlevel` ≥ 3, even all intermediate results...

**Example:**

```

LIB "sheafcoh.lib";
ring R=0,(x,y,z,u,v),dp;
module M=maxideal(3);
homog(M);
⇒ 1
// compute presentation matrix for truncated module (R/<x,y,z,u>^3)_(≥2)
int t=timer;
module M2t=truncate(M,2);
t = timer - t;
"// Simple truncate: ", t;
⇒ // Simple truncate: 0
t=timer;
module M2=truncateFast(std(M),2);
t = timer - t;
"// Fast truncate: ", t;
⇒ // Fast truncate: 0
print(M2);
⇒ v,0,0,0,0,0,0,0,0,0,0,0,0,0,u, 0, 0, 0, 0, 0,0,0,0,0,0,0,0,0,z, 0, 0,\
  0, 0,
⇒ 0, 0, 0, 0, 0,0,0,0,0,0,0,y, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,0,0,x, 0,\
  0, 0,
⇒ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
⇒ 0,v,0,0,0,0,0,0,0,0,0,0,0,0,-v,u, 0, 0, 0, 0,0,0,0,0,0,0,0,0,0,z, 0,\
  0, 0,
⇒ 0, 0, 0, 0, 0,0,0,0,0,0,0,0, y, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,0,0,0, x,\
  0, 0,

```

```

⇒ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
⇒ 0,0,v,0,0,0,0,0,0,0,0,0,0,0,0, 0, u, 0, 0, 0,0,0,0,0,0,0,0,0,-v,-u,z,\
  0, 0,
⇒ 0, 0, 0, 0, 0,0,0,0,0,0,0, 0, y, 0, 0, 0, 0, 0, 0, 0, 0, 0,0,0,0, 0,\
  x, 0,
⇒ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
⇒ 0,0,0,v,0,0,0,0,0,0,0,0,0,0,0, 0, 0, u, 0, 0,0,0,0,0,0,0,0,0,0, 0, 0,\
  z, 0,
⇒ 0, 0, 0, 0, 0,0,0,0,0,0,-v,-u,-z,y, 0, 0, 0, 0, 0, 0, 0, 0, 0,0,0,0, 0,\
  0, x,
⇒ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
⇒ 0,0,0,0,v,0,0,0,0,0,0,0,0,0,0,0, 0, 0, 0, u, 0,0,0,0,0,0,0,0,0,0, 0, 0,\
  0, z,
⇒ 0, 0, 0, 0, 0,0,0,0,0,0,0, 0, 0, 0, y, 0, 0, 0, 0, 0, 0, 0, 0,0,0,-v,-u\
  ,-z,-y,
⇒ x, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
⇒ 0,0,0,0,0,v,0,0,0,0,0,0,0,0,0,0, -v,0, 0, 0, u,0,0,0,0,0,0,0,0,0, 0, 0,\
  0, 0,
⇒ z, 0, 0, 0, 0,0,0,0,0,0,0, 0, 0, 0, 0, y, 0, 0, 0, 0, 0, 0, 0, 0,0,0,0, 0,\
  0, 0,
⇒ 0, x, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
⇒ 0,0,0,0,0,0,v,0,0,0,0,0,0,0,0,0, 0, -v,0, 0, 0,u,0,0,0,0,0,0,0,0,0, 0, 0,\
  0, 0,
⇒ -u,z, 0, 0, 0,0,0,0,0,0,0, 0, 0, 0, 0, 0, y, 0, 0, 0, 0, 0, 0, 0,0,0,0, 0,\
  0, 0,
⇒ 0, 0, x, 0, 0, 0, 0, 0, 0, 0, 0, 0,
⇒ 0,0,0,0,0,0,0,v,0,0,0,0,0,0,0,0, 0, 0, -v,0, 0,0,u,0,0,0,0,0,0,0,0, 0, 0,\
  0, 0,
⇒ 0, 0, z, 0, 0,0,0,0,0,0,0, 0, 0, 0, 0, -u,-z,y, 0, 0, 0, 0, 0,0,0,0, 0,\
  0, 0,
⇒ 0, 0, 0, x, 0, 0, 0, 0, 0, 0, 0, 0,
⇒ 0,0,0,0,0,0,0,0,v,0,0,0,0,0,0,0, 0, 0, 0, -v,0,0,0,u,0,0,0,0,0,0,0, 0, 0,\
  0, 0,
⇒ 0, 0, 0, z, 0,0,0,0,0,0,0, 0, 0, 0, 0, 0, 0, 0, y, 0, 0, 0, 0,0,0,0, 0,\
  0, 0,
⇒ 0, -u,-z,-y,x, 0, 0, 0, 0, 0, 0, 0,
⇒ 0,0,0,0,0,0,0,0,0,v,0,0,0,0,0,0, 0, 0, 0, 0, 0,0,0,0,u,0,0,0,0,0, 0, -v\
  ,0, 0,
⇒ 0, -u,0, 0, z,0,0,0,0,0,0, 0, 0, 0, 0, 0, 0, 0, 0, y, 0, 0, 0,0,0,0, 0,\
  0, 0,
⇒ 0, 0, 0, 0, 0, x, 0, 0, 0, 0, 0, 0,
⇒ 0,0,0,0,0,0,0,0,0,0,v,0,0,0,0,0, 0, 0, 0, 0, 0,0,0,0,0,u,0,0,0,0, 0, 0,\
  -v,0,
⇒ 0, 0, -u,0, 0,z,0,0,0,0,0, 0, 0, 0, 0, 0, 0, 0, 0, -z,y, 0, 0,0,0,0, 0,\
  0, 0,
⇒ 0, 0, 0, 0, 0, 0, x, 0, 0, 0, 0, 0, 0,
⇒ 0,0,0,0,0,0,0,0,0,0,0,v,0,0,0,0, 0, 0, 0, 0, 0,0,0,0,0,0,u,0,0,0, 0, 0,\
  0, -v,
⇒ 0, 0, 0, -u,0,0,z,0,0,0,0, 0, 0, 0, 0, 0, 0, 0, 0, 0, y, 0,0,0,0, 0,\
  0, 0,
⇒ 0, 0, 0, 0, 0, -z,-y,x, 0, 0, 0, 0,
⇒ 0,0,0,0,0,0,0,0,0,0,0,v,0,0,0, 0, 0, 0, 0, 0,0,0,0,0,0,u,0,0,0, 0, 0,\
  0, 0,

```

```

⇒ 0, 0, 0, 0, 0,0,0,z,0,0,0, 0, 0, -v,0, 0, 0, -u,0, 0, -z,0, y,0,0,0, 0,\
  0, 0,
⇒ 0, 0, 0, 0, 0, 0, 0, 0, 0, x, 0, 0,
⇒ 0,0,0,0,0,0,0,0,0,0,0,0,0,0,v,0,0, 0, 0, 0, 0, 0,0,0,0,0,0,0,u,0,0, 0, 0,\
  0, 0,
⇒ 0, 0, 0, 0, 0,0,0,0,z,0,0, 0, 0, 0, -v,0, 0, 0, -u,0, 0, -z,0,y,0,0, 0,\
  0, 0,
⇒ 0, 0, 0, 0, 0, 0, 0, 0, 0, -y,x, 0,
⇒ 0,0,0,0,0,0,0,0,0,0,0,0,0,0,v,0, 0, 0, 0, 0, 0,0,0,0,0,0,0,0,u,0, 0, 0,\
  0, 0,
⇒ 0, 0, 0, 0, 0,0,0,0,0,z,0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,0,y,0, 0,\
  0, 0,
⇒ -v,0, 0, 0, -u,0, 0, -z,0, -y,x
"// Check: M2t == M2?: ", size(NF(M2, std(M2t))) + size(NF(M2t, std(M2)));
⇒ // Check: M2t == M2?: 0
dimGradedPart(M2,1);
⇒ 0
dimGradedPart(M2,2);
⇒ 15
// this should coincide with:
dimGradedPart(M,2);
⇒ 15
// shift grading by 1:
intvec v=1;
attrib(M,"isHomog",v);
t=timer;
M2t=truncate(M,2);
t = timer - t;
"// Simple truncate: ", t;
⇒ // Simple truncate: 0
t=timer;
M2=truncateFast(std(M),2);
t = timer - t;
"// Fast truncate: ", t;
⇒ // Fast truncate: 0
print(M2);
⇒ u, z, 0, y, 0, 0, x, 0, 0, 0, v2,0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\
  , 0,
⇒ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
⇒ -v,0, z, 0, y, 0, 0, x, 0, 0, 0, v2,0, 0, 0, uv,0, 0, 0, 0, 0, 0, 0, 0\
  , u2,
⇒ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
⇒ 0, -v,-u,0, 0, y, 0, 0, x, 0, 0, 0, v2,0, 0, 0, uv,0, 0, zv,0, 0, 0, 0, 0\
  , 0,
⇒ u2,0, 0, zu,0, 0, 0, 0, 0, z2,0, 0, 0, 0, 0, 0, 0, 0, 0,
⇒ 0, 0, 0, -v,-u,-z,0, 0, 0, x, 0, 0, 0, v2,0, 0, 0, uv,0, 0, zv,0, yv,0, 0\
  , 0,
⇒ 0, u2,0, 0, zu,0, yu,0, 0, 0, z2,0, yz,0, 0, y2,0, 0, 0,
⇒ 0, 0, 0, 0, 0, 0, -v,-u,-z,-y,0, 0, 0, 0, v2,0, 0, 0, uv,0, 0, zv,0, yv,x\
  v,0,
⇒ 0, 0, u2,0, 0, zu,0, yu,xu,0, 0, z2,0, yz,xz,0, y2,xy,x2
"// Check: M2t == M2?: ", size(NF(M2, std(M2t))) + size(NF(M2t, std(M2))); //?
⇒ // Check: M2t == M2?: 20

```

```
dimGradedPart(M2,3);
↳ 15
```

### D.5.18.3 CM\_regularity

Procedure from library `sheafcoh.lib` (see [Section D.5.18 \[sheafcoh.lib\]](#), page 1605).

**Usage:** `CM_regularity(M)`;  $M$  module

**Assume:**  $M$  is graded, and it comes assigned with an admissible degree vector as an attribute

**Return:** integer, the Castelnuovo-Mumford regularity of `coker(M)`

**Note:** procedure calls `mres`

**Example:**

```
LIB "sheafcoh.lib";
ring R=0,(x,y,z,u),dp;
resolution T1=mres(maxideal(1),0);
module M=T1[3];
intvec v=2,2,2,2,2,2;
attrib(M,"isHomog",v);
CM_regularity(M);
↳ 2
```

### D.5.18.4 sheafCohBGG

Procedure from library `sheafcoh.lib` (see [Section D.5.18 \[sheafcoh.lib\]](#), page 1605).

**Usage:** `sheafCohBGG(M,l,h)`;  $M$  module,  $l, h$  int

**Assume:**  $M$  is graded, and it comes assigned with an admissible degree vector as an attribute,  $h \geq 1$ , and the basering has  $n+1$  variables.

**Return:** intmat, cohomology of twists of the coherent sheaf  $F$  on  $P^n$  associated to `coker(M)`. The range of twists is determined by  $l, h$ .

**Display:** The intmat is displayed in a diagram of the following form: with `displayCohom(A,l,h,nvars(r)-1)`;

	$l$	$l+1$	.....	$h$
$n:$	$h^n(F(l))$	$h^n(F(l+1))$	.....	$h^n(F(h))$
	.....			
$1:$	$h^1(F(l))$	$h^1(F(l+1))$	.....	$h^1(F(h))$
$0:$	$h^0(F(l))$	$h^0(F(l+1))$	.....	$h^0(F(h))$
$\chi:$	$\chi(F(l))$	$\chi(F(l+1))$	.....	$\chi(F(h))$

A '−' in the diagram refers to a zero entry; a '\*' refers to a negative entry (= dimension not yet determined). refers to a not computed dimension.

**Note:** This procedure is based on the Bernstein-Gel'fand-Gel'fand correspondence and on Tate resolution ( see [Eisenbud, Floystad, Schreyer: Sheaf cohomology and free resolutions over exterior algebras, Trans AMS 355 (2003)] ).

`sheafCohBGG(M,l,h)` does not compute all values in the above table. To determine all values of  $h^i(F(d))$ ,  $d=1..h$ , use `sheafCohBGG(M,l-n,h+n)`.

**Example:**

```

LIB "sheafcoh.lib";
// cohomology of structure sheaf on P^4:
//-----
ring r=0,x(1..5),dp;
module M=0;
intmat A=sheafCohBGG(M,-9,4);
A;
⇒ 70,35,15,5,1,0,0,0,0,0,-1,-1,-1,-1,
⇒ -1,0,0,0,0,0,0,0,0,0,-1,-1,-1,
⇒ -1,-1,0,0,0,0,0,0,0,0,-1,-1,
⇒ -1,-1,-1,0,0,0,0,0,0,0,-1,
⇒ -1,-1,-1,-1,0,0,0,0,0,1,5,15,35,70
displayCohom(A,-9,4,nvars(r)-1);
⇒      -9  -8  -7  -6  -5  -4  -3  -2  -1   0   1   2   3   4
⇒ -----
⇒  4:  70  35  15   5   1   -   -   -   -   -   *   *   *   *
⇒  3:   *   -   -   -   -   -   -   -   -   -   -   *   *   *
⇒  2:   *   *   -   -   -   -   -   -   -   -   -   -   *   *
⇒  1:   *   *   *   -   -   -   -   -   -   -   -   -   -   *
⇒  0:   *   *   *   *   -   -   -   -   -   1   5  15  35  70
⇒ -----
⇒ chi:   *   *   *   *   1   0   0   0   0   1   *   *   *   *
// cohomology of cotangential bundle on P^3:
//-----
ring R=0,(x,y,z,u),dp;
resolution T1=mres(maxideal(1),0);
module M=T1[3];
intvec v=2,2,2,2,2,2;
attrib(M,"isHomog",v);
intmat B=sheafCohBGG(M,-8,4);
B;
⇒ 189,120,70,36,15,4,0,0,0,0,-1,-1,-1,
⇒ -1,0,0,0,0,0,0,0,0,0,-1,-1,
⇒ -1,-1,0,0,0,0,0,0,1,0,0,0,-1,
⇒ -1,-1,-1,0,0,0,0,0,0,0,6,20,45
displayCohom(B,-8,4,nvars(R)-1);
⇒      -8  -7  -6  -5  -4  -3  -2  -1   0   1   2   3   4
⇒ -----
⇒  3:  189  120   70   36   15   4   -   -   -   -   *   *   *
⇒  2:   *   -   -   -   -   -   -   -   -   -   -   -   *
⇒  1:   *   *   -   -   -   -   -   -   1   -   -   -   *
⇒  0:   *   *   *   -   -   -   -   -   -   -   6  20  45
⇒ -----
⇒ chi:   *   *   *  -36  -15  -4   0   0  -1   0   *   *   *

```

See also: [Section D.5.18.10 \[dimH\]](#), page 1619; [Section D.5.18.12 \[displayCohom\]](#), page 1620; [Section D.5.18.8 \[sheafCoh\]](#), page 1616.

**D.5.18.5 sheafCohBGGregul**

Procedure from library `sheafcoh.lib` (see [Section D.5.18 \[sheafcoh.lib\]](#), page 1605).

**Usage:** `sheafCohBGGregul(M,l,h);` M module, l,h int, reg int



- Assume:**  $M$  is graded, and it comes assigned with an admissible degree vector as an attribute,  $h \geq 1$ , and the basering has  $n+1$  variables.
- Return:** intmat, cohomology of twists of the coherent sheaf  $F$  on  $P^n$  associated to  $\text{coker}(M)$ . The range of twists is determined by  $l, h$ .
- Note:** This procedure is based on the Bernstein-Gel'fand-Gel'fand correspondence and on Tate resolution ( see [Eisenbud, Floystad, Schreyer: Sheaf cohomology and free resolutions over exterior algebras, Trans AMS 355 (2003)] ).  
`sheafCohBGG(M,l,h)` does not compute all values in the above table. To determine all values of  $h^i(F(d))$ ,  $d=1..h$ , use `sheafCohBGG(M,l-n,h+n)`.

**Example:**

```
LIB "sheafcoh.lib";
// cohomology of structure sheaf on P^4:
//-----
ring r=0,x(1..5),dp;
module M=0;
def A=sheafCohBGGregul(M,-9,4,CM_regularity(M));
A;
⇨ 70,35,15,5,1,0,0,0,0,0,-1,-1,-1,-1,
⇨ -1,0,0,0,0,0,0,0,0,0,-1,-1,-1,
⇨ -1,-1,0,0,0,0,0,0,0,0,-1,-1,
⇨ -1,-1,-1,0,0,0,0,0,0,0,-1,
⇨ -1,-1,-1,-1,0,0,0,0,0,1,5,15,35,70
// cohomology of cotangential bundle on P^3:
//-----
ring R=0,(x,y,z,u),dp;
resolution T1=mres(maxideal(1),0);
module M=T1[3];
intvec v=2,2,2,2,2,2;
attrib(M,"isHomog",v);
def B=sheafCohBGGregul(M,-8,4,CM_regularity(M));
B;
⇨ 189,120,70,36,15,4,0,0,0,0,-1,-1,-1,
⇨ -1,0,0,0,0,0,0,0,0,0,-1,-1,
⇨ -1,-1,0,0,0,0,0,0,1,0,0,0,-1,
⇨ -1,-1,-1,0,0,0,0,0,0,0,6,20,45
```

See also: [Section D.5.18.10 \[dimH\]](#), page 1619; [Section D.5.18.12 \[displayCohom\]](#), page 1620; [Section D.5.18.8 \[sheafCoh\]](#), page 1616.

**D.5.18.6 sheafCohBGGregul\_w**

Procedure from library `sheafcoh.lib` (see [Section D.5.18 \[sheafcoh.lib\]](#), page 1605).

- Usage:** `sheafCohBGGregul_w(M,l,h,w)`;  $M$  module,  $l, h$  int,  $reg$  int,  $w$  intvec
- Assume:**  $M$  is graded, and it comes assigned with an admissible degree vector  $w$ ,  $h \geq 1$ , and the basering has  $n+1$  variables.
- Return:** intmat, cohomology of twists of the coherent sheaf  $F$  on  $P^n$  associated to  $\text{coker}(M)$ . The range of twists is determined by  $l, h$ .
- Note:** This procedure is based on the Bernstein-Gel'fand-Gel'fand correspondence and on Tate resolution ( see [Eisenbud, Floystad, Schreyer: Sheaf cohomology and free resolutions

over exterior algebras, Trans AMS 355 (2003)] ).

`sheafCohBGG(M,l,h)` does not compute all values in the above table. To determine all values of  $h^i(F(d))$ ,  $d=1..h$ , use `sheafCohBGG(M,l-n,h+n)`.

**Example:**

```
LIB "sheafcoh.lib";
// cohomology of cotangential bundle on P^3:
//-----
ring R=0,(x,y,z,u),dp;
resolution T1=mres(maxideal(1),0);
module M=T1[3];
intvec v=2,2,2,2,2,2;
def B=sheafCohBGGregul_w(M,-8,4,CM_regularity(M),v);
B;
⇒ 189,120,70,36,15,4,0,0,0,0,-1,-1,-1,
⇒ -1,0,0,0,0,0,0,0,0,0,-1,-1,
⇒ -1,-1,0,0,0,0,0,0,1,0,0,0,-1,
⇒ -1,-1,-1,0,0,0,0,0,0,0,6,20,45
```

See also: [Section D.5.18.10 \[dimH\]](#), page 1619; [Section D.5.18.12 \[displayCohom\]](#), page 1620; [Section D.5.18.8 \[sheafCoh\]](#), page 1616.

### D.5.18.7 sheafCohBGG2

Procedure from library `sheafcoh.lib` (see [Section D.5.18 \[sheafcoh.lib\]](#), page 1605).

**Usage:** `sheafCohBGG2(M,l,h)`; M module, l,h int

**Assume:** M is graded, and it comes assigned with an admissible degree vector as an attribute,  $h \geq 1$ , and the basering has  $n+1$  variables.

**Return:** intmat, cohomology of twists of the coherent sheaf F on  $P^n$  associated to `coker(M)`. The range of twists is determined by l, h.

**Display:** The intmat is displayed in a diagram of the following form: with `displayCohom(A,l,h,nvars(r)-1)`;

	l	l+1	.....	h
n:	$h^n(F(l))$	$h^n(F(l+1))$	.....	$h^n(F(h))$
1:	$h^1(F(l))$	$h^1(F(l+1))$	.....	$h^1(F(h))$
0:	$h^0(F(l))$	$h^0(F(l+1))$	.....	$h^0(F(h))$
chi:	$\chi(F(l))$	$\chi(F(l+1))$	.....	$\chi(F(h))$

A '−' in the diagram refers to a zero entry; a '\*' refers to a negative entry (= dimension not yet determined). refers to a not computed dimension.

If `printlevel`  $\geq 1$ , step-by step timings will be printed. If `printlevel`  $\geq 2$  we add progress debug messages if `printlevel`  $\geq 3$ , even all intermediate results...

**Note:** This procedure is based on the Bernstein-Gel'fand-Gel'fand correspondence and on Tate resolution ( see [Eisenbud, Floystad, Schreyer: Sheaf cohomology and free resolutions over exterior algebras, Trans AMS 355 (2003)] ).  
`sheafCohBGG(M,l,h)` does not compute all values in the above table. To determine all

values of  $h^i(F(d))$ ,  $d=1..h$ , use `sheafCohBGG(M,l-n,h+n)`. Experimental version.  
Should require less memory.

**Example:**

```
LIB "sheafcoh.lib";
int pl = printlevel;
int l,h, t;
//-----
// cohomology of structure sheaf on P^4:
//-----
ring r=32001,x(1..5),dp;
⇒ // ** 32001 is invalid as characteristic of the ground field. 32003 is us\
   ed.
module M= getStructureSheaf(); // 00_P^4
l = -12; h = 12; // range of twists: 1..h
printlevel = 0;
////////////////////////////////////
t = timer;
def A = sheafCoh(M, l, h); // global Ext method:
displayCohom(A,l,h,nvars(basering)-1);
⇒      -12  -11  -10  -9  -8  -7  -6  -5  -4  -3  -2  \
      -1   0   1   2   3   4   5   6   7   8   9  10  \
      11  12
⇒ -----
⇒  4:   330  210  126  70  35  15  5  1  -  -  -  \
      -   -   -   -   -   -   -   -   -   -   -   -
⇒  3:   -   -   -   -   -   -   -   -   -   -   -  \
      -   -   -   -   -   -   -   -   -   -   -   -
⇒  2:   -   -   -   -   -   -   -   -   -   -   -  \
      -   -   -   -   -   -   -   -   -   -   -   -
⇒  1:   -   -   -   -   -   -   -   -   -   -   -  \
      -   -   -   -   -   -   -   -   -   -   -   -
⇒  0:   -   -   -   -   -   -   -   -   -   -   -  \
      -   1   5  15  35  70  126  210  330  495  715  1001  13\
      65  1820
⇒ -----
⇒ chi:   330  210  126  70  35  15  5  1  0  0  0  \
      0   1   5  15  35  70  126  210  330  495  715  1001  13\
      65  1820
"Time: ", timer - t;
⇒ Time: 4
////////////////////////////////////
t = timer;
A = sheafCohBGG(M, l, h); // BGG method (without optimization):
displayCohom(A,l,h,nvars(basering)-1);
⇒      -12  -11  -10  -9  -8  -7  -6  -5  -4  -3  -2  \
      -1   0   1   2   3   4   5   6   7   8   9  10  \
      11  12
⇒ -----
⇒  4:   330  210  126  70  35  15  5  1  -  -  -  \
      -   -   -   -   -   -   -   -   -   *   *   *   *
⇒  3:   *   -   -   -   -   -   -   -   -   -   -   -  \
      -   -   -   -   -   -   -   -   -   -   *   *   *
```

```

⇒ 2:  *  *  -  -  -  -  -  -  -  -  -  -  \
    -  -  -  -  -  -  -  -  -  -  -  -  *  *
⇒ 1:  *  *  *  -  -  -  -  -  -  -  -  -  -  \
    -  -  -  -  -  -  -  -  -  -  -  -  -  *
⇒ 0:  *  *  *  *  -  -  -  -  -  -  -  -  -  -  \
    -  1  5  15  35  70  126  210  330  495  715  1001  13\
    65  1820
⇒ -----
⇒ chi:  *  *  *  *  35  15  5  1  0  0  0  \
    0  1  5  15  35  70  126  210  330  495  *  *  \
    *  *
"Time: ", timer - t;
⇒ Time: 12
////////////////////////////////////
t = timer;
A = sheafCohBGG2(M, l, h); // BGG method (with optimization)
⇒ Cohomology table:
⇒      -12  -11  -10  -9  -8  -7  -6  -5  -4  -3  -2  \
    -1  0  1  2  3  4  5  6  7  8  9  10  \
    11  12
⇒ -----
⇒ 4:  330  210  126  70  35  15  5  1  -  -  -  \
    -  -  -  -  -  -  -  -  -  -  -  -
⇒ 3:  *  -  -  -  -  -  -  -  -  -  -  -  \
    -  -  -  -  -  -  -  -  -  -  -  -
⇒ 2:  *  *  -  -  -  -  -  -  -  -  -  -  \
    -  -  -  -  -  -  -  -  -  -  -  -
⇒ 1:  *  *  *  -  -  -  -  -  -  -  -  -  \
    -  -  -  -  -  -  -  -  -  -  -  -
⇒ 0:  *  *  *  *  -  -  -  -  -  -  -  -  -  \
    -  1  5  15  35  70  126  210  330  495  715  1001  13\
    65  1820
⇒ -----
⇒ chi:  *  *  *  *  35  15  5  1  0  0  0  \
    0  1  5  15  35  70  126  210  330  495  715  1001  13\
    65  1820
displayCohom(A,l,h,nvars(basing)-1);
⇒      -12  -11  -10  -9  -8  -7  -6  -5  -4  -3  -2  \
    -1  0  1  2  3  4  5  6  7  8  9  10  \
    11  12
⇒ -----
⇒ 4:  330  210  126  70  35  15  5  1  -  -  -  \
    -  -  -  -  -  -  -  -  -  -  -  -
⇒ 3:  *  -  -  -  -  -  -  -  -  -  -  -  \
    -  -  -  -  -  -  -  -  -  -  -  -
⇒ 2:  *  *  -  -  -  -  -  -  -  -  -  -  \
    -  -  -  -  -  -  -  -  -  -  -  -
⇒ 1:  *  *  *  -  -  -  -  -  -  -  -  -  \
    -  -  -  -  -  -  -  -  -  -  -  -
⇒ 0:  *  *  *  *  -  -  -  -  -  -  -  -  -  \
    -  1  5  15  35  70  126  210  330  495  715  1001  13\
    65  1820
⇒ -----

```

```

⇒ chi:      *      *      *      *      35      15      5      1      0      0      0      \
          0      1      5      15     35     70     126     210     330     495     715     1001     13\
          65     1820
"Time: ", timer - t;
⇒ Time: 4
////////////////////////////////////
printlevel = pl;
kill A, r;
//-----
// cohomology of cotangential bundle on P^3:
//-----
ring R=32001,(x,y,z,u),dp;
⇒ // ** 32001 is invalid as characteristic of the ground field. 32003 is us\
   ed.
module M = getCotangentialBundle();
l = -12; h = 11; // range of twists: l..h
////////////////////////////////////
printlevel = 0;
t = timer;
def B = sheafCoh(M, l, h); // global Ext method:
displayCohom(B,l,h,nvars(basing)-1);
⇒      -12  -11  -10  -9  -8  -7  -6  -5  -4  -3  -2  -1  0  \
      1    2    3    4    5    6    7    8    9   10   11
⇒ -----
⇒ 3:  715  540  396  280  189  120  70  36  15  4  -  -  -  \
      -  -  -  -  -  -  -  -  -  -  -  -  -  \
⇒ 2:  -  -  -  -  -  -  -  -  -  -  -  -  -  \
      -  -  -  -  -  -  -  -  -  -  -  -  -  \
⇒ 1:  -  -  -  -  -  -  -  -  -  -  -  -  1  \
      -  -  -  -  -  -  -  -  -  -  -  -  -  \
⇒ 0:  -  -  -  -  -  -  -  -  -  -  -  -  -  \
      -   6   20   45   84  140  216  315  440  594  780
⇒ -----
⇒ chi: -715 -540 -396 -280 -189 -120 -70 -36 -15 -4  0  0 -1  \
      0   6   20   45   84  140  216  315  440  594  780
"Time: ", timer - t;
⇒ Time: 1
////////////////////////////////////
t = timer;
B = sheafCohBGG(M, l, h); // BGG method (without optimization):
displayCohom(B,l,h,nvars(basing)-1);
⇒      -12  -11  -10  -9  -8  -7  -6  -5  -4  -3  -2  -1  0  \
      1    2    3    4    5    6    7    8    9   10   11
⇒ -----
⇒ 3:  715  540  396  280  189  120  70  36  15  4  -  -  -  \
      -  -  -  -  -  -  -  -  -  -  *  *  *  \
⇒ 2:  *  -  -  -  -  -  -  -  -  -  -  -  -  -  \
      -  -  -  -  -  -  -  -  -  -  *  *  -  \
⇒ 1:  *  *  -  -  -  -  -  -  -  -  -  -  -  1  \
      -  -  -  -  -  -  -  -  -  -  -  *  -  \
⇒ 0:  *  *  *  -  -  -  -  -  -  -  -  -  -  -  \
      -   6   20   45   84  140  216  315  440  594  780
⇒ -----

```

```

⇒ chi:      *      *      * -280 -189 -120 -70 -36 -15 -4  0  0 -1  \
      0  6  20  45  84  140  216  315  *  *  *
"Time: ", timer - t;
⇒ Time:  2
////////////////////////////////////
t = timer;
B = sheafCohBGG2(M, l, h); // BGG method (with optimization)
⇒ Cohomology table:
⇒      -12 -11 -10 -9 -8 -7 -6 -5 -4 -3 -2 -1  0  \
      1  2  3  4  5  6  7  8  9  10 11
⇒ -----
⇒ 3:  715  540  396  280  189  120  70  36  15  4  -  -  -  \
      -  -  -  -  -  -  -  -  -  -  -  -  -  \
⇒ 2:      *  -  -  -  -  -  -  -  -  -  -  -  -  \
      -  -  -  -  -  -  -  -  -  -  -  -  -  \
⇒ 1:      *  *  -  -  -  -  -  -  -  -  -  -  1  \
      -  -  -  -  -  -  -  -  -  -  -  -  -  \
⇒ 0:      *  *  *  -  -  -  -  -  -  -  -  -  -  \
      -  6  20  45  84  140  216  315  440  594  780
⇒ -----
⇒ chi:      *      *      * -280 -189 -120 -70 -36 -15 -4  0  0 -1  \
      0  6  20  45  84  140  216  315  440  594  780
displayCohom(B,l,h,nvars(basering)-1);
⇒      -12 -11 -10 -9 -8 -7 -6 -5 -4 -3 -2 -1  0  \
      1  2  3  4  5  6  7  8  9  10 11
⇒ -----
⇒ 3:  715  540  396  280  189  120  70  36  15  4  -  -  -  \
      -  -  -  -  -  -  -  -  -  -  -  -  -  \
⇒ 2:      *  -  -  -  -  -  -  -  -  -  -  -  -  \
      -  -  -  -  -  -  -  -  -  -  -  -  -  \
⇒ 1:      *  *  -  -  -  -  -  -  -  -  -  -  1  \
      -  -  -  -  -  -  -  -  -  -  -  -  -  \
⇒ 0:      *  *  *  -  -  -  -  -  -  -  -  -  -  \
      -  6  20  45  84  140  216  315  440  594  780
⇒ -----
⇒ chi:      *      *      * -280 -189 -120 -70 -36 -15 -4  0  0 -1  \
      0  6  20  45  84  140  216  315  440  594  780
"Time: ", timer - t;
⇒ Time:  0
////////////////////////////////////
printlevel = pl;

```

See also: [Section D.5.18.12 \[displayCohom\]](#), page 1620; [Section D.5.18.4 \[sheafCohBGG\]](#), page 1609.

### D.5.18.8 sheafCoh

Procedure from library `sheafcoh.lib` (see [Section D.5.18 \[sheafcoh.lib\]](#), page 1605).

**Usage:** `sheafCoh(M,l,h)`; M module, l,h int

**Assume:** M is graded, and it comes assigned with an admissible degree vector as an attribute,  $h \geq 1$ . The basering S has  $n+1$  variables.

**Return:** intmat, cohomology of twists of the coherent sheaf F on  $P^n$  associated to `coker(M)`. The range of twists is determined by l, h.

**Display:** The intmat can be displayed in a diagram of the following form: with `displayCohom(A,l,h,nvars(r)-1);`

	l	l+1	.....	h
n:	$h^n(F(l))$	$h^n(F(l+1))$	.....	$h^n(F(h))$
1:	$h^1(F(l))$	$h^1(F(l+1))$	.....	$h^1(F(h))$
0:	$h^0(F(l))$	$h^0(F(l+1))$	.....	$h^0(F(h))$
chi:	$\chi(F(l))$	$\chi(F(l+1))$	.....	$\chi(F(h))$

A '-' in the diagram refers to a zero entry.

**Note:** The procedure is based on local duality as described in [Eisenbud: Computing cohomology. In Vasconcelos: Computational methods in commutative algebra and algebraic geometry. Springer (1998)].

By default, the procedure uses `mres` to compute the Ext modules. If called with the additional parameter "`sres`", the `sres` command is used instead.

**Example:**

```
LIB "sheafcoh.lib";
//
// cohomology of structure sheaf on P^4:
//-----
ring r=0,x(1..5),dp;
module M=0;
intmat A=sheafCoh(0,-7,2);
A;
⇒ 15,5,1,0,0,0,0,0,0,0,
⇒ 0,0,0,0,0,0,0,0,0,0,
⇒ 0,0,0,0,0,0,0,0,0,0,
⇒ 0,0,0,0,0,0,0,0,0,0,
⇒ 0,0,0,0,0,0,0,0,1,5,15
displayCohom(A,-7,2,nvars(r)-1);
⇒      -7  -6  -5  -4  -3  -2  -1   0   1   2
⇒ -----
⇒ 4:  15   5   1   -   -   -   -   -   -   -
⇒ 3:   -   -   -   -   -   -   -   -   -   -
⇒ 2:   -   -   -   -   -   -   -   -   -   -
⇒ 1:   -   -   -   -   -   -   -   -   -   -
⇒ 0:   -   -   -   -   -   -   -   1   5  15
⇒ -----
⇒ chi:  15   5   1   0   0   0   0   1   5  15
//
// cohomology of cotangential bundle on P^3:
//-----
ring R=0,(x,y,z,u),dp;
resolution T1=mres(maxideal(1),0);
module M=T1[3];
intvec v=2,2,2,2,2,2;
attrib(M,"isHomog",v);
intmat B=sheafCoh(M,-6,2);
```

```

displayCohom(B,-6,2,nvars(R)-1);
↳      -6  -5  -4  -3  -2  -1   0   1   2
↳ -----
↳  3:  70  36  15   4   -   -   -   -   -
↳  2:   -   -   -   -   -   -   -   -   -
↳  1:   -   -   -   -   -   -   1   -   -
↳  0:   -   -   -   -   -   -   -   -   6
↳ -----
↳ chi: -70 -36 -15  -4   0   0  -1   0   6

```

See also: [Section D.5.18.10 \[dimH\], page 1619](#); [Section D.5.18.4 \[sheafCohBGG\], page 1609](#); [Section D.5.18.9 \[sheafCoh\\_w\], page 1618](#).

### D.5.18.9 sheafCoh\_w

Procedure from library `sheafcoh.lib` (see [Section D.5.18 \[sheafcoh.lib\], page 1605](#)).

**Usage:** `sheafCoh(M,l,h)`; `M` module, `l,h` int, `w` intvec

**Assume:** `M` is graded by `w`, `h` ≥ 1. The basering `S` has `n+1` variables.

**Return:** intmat, cohomology of twists of the coherent sheaf `F` on  $P^n$  associated to `coker(M)`. The range of twists is determined by `l, h`.

**Display:** The intmat can be displayed in a diagram of the following form: with `displayCohom(A,l,h,nvars(r)-1)`;

	l	l+1	.....	h
n:	$h^n(F(l))$	$h^n(F(l+1))$	.....	$h^n(F(h))$
	.....			
1:	$h^1(F(l))$	$h^1(F(l+1))$	.....	$h^1(F(h))$
0:	$h^0(F(l))$	$h^0(F(l+1))$	.....	$h^0(F(h))$
chi:	$chi(F(l))$	$chi(F(l+1))$	.....	$chi(F(h))$

A '-' in the diagram refers to a zero entry.

**Note:** The procedure is based on local duality as described in [Eisenbud: Computing cohomology. In Vasconcelos: Computational methods in commutative algebra and algebraic geometry. Springer (1998)].

By default, the procedure uses `mres` to compute the Ext modules. If called with the additional parameter "`sres`", the `sres` command is used instead.

**Example:**

```

LIB "sheafcoh.lib";
//
// cohomology of structure sheaf on P^4:
//-----
ring r=0,x(1..5),dp;
module M=0;
intmat A=sheafCoh_w(0,-7,2,intvec(0));
A;
↳ 15,5,1,0,0,0,0,0,0,0,
↳ 0,0,0,0,0,0,0,0,0,0,
↳ 0,0,0,0,0,0,0,0,0,0,

```



```

⇒ 0,0,0,0,0,0,0,0,0,0,
⇒ 0,0,0,0,0,0,0,0,1,5,15
displayCohom(A,-7,2,nvars(r)-1);
⇒      -7  -6  -5  -4  -3  -2  -1   0   1   2
⇒ -----
⇒ 4:  15   5   1   -   -   -   -   -   -   -
⇒ 3:   -   -   -   -   -   -   -   -   -   -
⇒ 2:   -   -   -   -   -   -   -   -   -   -
⇒ 1:   -   -   -   -   -   -   -   -   -   -
⇒ 0:   -   -   -   -   -   -   -   1   5  15
⇒ -----
⇒ chi:  15   5   1   0   0   0   0   1   5  15
//
// cohomology of cotangential bundle on P^3:
//-----
ring R=0,(x,y,z,u),dp;
resolution T1=mres(maxideal(1),0);
module M=T1[3];
intvec v=2,2,2,2,2,2;
intmat B=sheafCoh_w(M,-6,2,v);
displayCohom(B,-6,2,nvars(R)-1);
⇒      -6  -5  -4  -3  -2  -1   0   1   2
⇒ -----
⇒ 3:  70  36  15   4   -   -   -   -   -
⇒ 2:   -   -   -   -   -   -   -   -   -
⇒ 1:   -   -   -   -   -   -   1   -   -
⇒ 0:   -   -   -   -   -   -   -   -   6
⇒ -----
⇒ chi: -70 -36 -15  -4   0   0  -1   0   6

```

See also: [Section D.5.18.10 \[dimH\]](#), page 1619; [Section D.5.18.8 \[sheafCoh\]](#), page 1616; [Section D.5.18.4 \[sheafCohBGG\]](#), page 1609.

### D.5.18.10 dimH

Procedure from library `sheafcoh.lib` (see [Section D.5.18 \[sheafcoh.lib\]](#), page 1605).

**Usage:** `dimH(i,M,d)`;  $M$  module,  $i,d$  int

**Assume:**  $M$  is graded, and it comes assigned with an admissible degree vector as an attribute,  $h \geq 1$ , and the basering  $S$  has  $n+1$  variables.

**Return:** int, vector space dimension of  $H^i(F(d))$  for  $F$  the coherent sheaf on  $P^n$  associated to  $\text{coker}(M)$ .

**Note:** The procedure is based on local duality as described in [Eisenbud: Computing cohomology. In Vasconcelos: Computational methods in commutative algebra and algebraic geometry. Springer (1998)].

**Example:**

```

LIB "sheafcoh.lib";
ring R=0,(x,y,z,u),dp;
resolution T1=mres(maxideal(1),0);
module M=T1[3];
intvec v=2,2,2,2,2,2;
attrib(M,"isHomog",v);

```

```

dimH(0,M,2);
↳ 6
dimH(1,M,0);
↳ 1
dimH(2,M,1);
↳ 0
dimH(3,M,-5);
↳ 36

```

See also: [Section D.5.18.8 \[sheafCoh\]](#), page 1616; [Section D.5.18.4 \[sheafCohBGG\]](#), page 1609.

### D.5.18.11 dimGradedPart

Procedure from library `sheafcoh.lib` (see [Section D.5.18 \[sheafcoh.lib\]](#), page 1605).

**Usage:** `dimGradedPart(M,d)`;  $M$  module,  $d$  int

**Assume:**  $M$  is graded, and it comes assigned with an admissible degree vector as an attribute

**Return:** int

**Note:** Output is the vector space dimension of the graded part of degree  $d$  of  $\text{coker}(M)$ .

**Example:**

```

LIB "sheafcoh.lib";
ring R=0,(x,y,z),dp;
module M=maxideal(3);
// assign compatible weight vector (here: 0)
homog(M);
↳ 1
// compute dimension of graded pieces of  $R/\langle x,y,z \rangle^3$  :
dimGradedPart(M,0);
↳ 1
dimGradedPart(M,1);
↳ 3
dimGradedPart(M,2);
↳ 6
dimGradedPart(M,3);
↳ 0
// shift grading:
attrib(M,"isHomog",intvec(2));
dimGradedPart(M,2);
↳ 1

```

### D.5.18.12 displayCohom

Procedure from library `sheafcoh.lib` (see [Section D.5.18 \[sheafcoh.lib\]](#), page 1605).

**Usage:** `displayCohom(data,l,h,n)`;  $\text{data}$  intmat,  $l,h,n$  int

**Assume:**  $h \geq 1$ ,  $\text{data}$  is the return value of `sheafCoh(M,l,h)` or of `sheafCohBGG(M,l,h)`, and the basering has  $n+1$  variables.

**Return:** none

**Note:** The intmat is displayed in a diagram of the following form:

	l	l+1	.....	h
n:	$h^n(F(l))$	$h^n(F(l+1))$	.....	$h^n(F(h))$
1:	$h^1(F(l))$	$h^1(F(l+1))$	.....	$h^1(F(h))$
0:	$h^0(F(l))$	$h^0(F(l+1))$	.....	$h^0(F(h))$
chi:	$chi(F(l))$	$chi(F(l+1))$	.....	$chi(F(h))$

where  $F$  refers to the associated sheaf of  $M$  on  $P^n$ .

A '–' in the diagram refers to a zero entry, a '\*' refers to a negative entry (= dimension not yet determined).

### D.5.19 JMBTest.lib

**Library:** JMBTest.lib

**Purpose:** A library for Singular which performs JM basis test.

**Author:** Michela Ceria, email: [michela.ceria@unito.it](mailto:michela.ceria@unito.it)

**Overview:** The library performs the J-marked basis test, as described in [CR], [BCLR]. Such a test is performed via the criterion explained in [BCLR], concerning Eliahou-Kervaire polynomials (EK from now on). We point out that all the polynomials are homogeneous and they must be arranged by degree.

The fundamental steps are the following:

- construct the  $V_m$  polynomials, via the algorithm VConstructor explained in [CR];
- construct the Eliahou-Kervaire polynomials defined in [BCLR];
- reduce the Eliahou-Kervaire polynomials using the  $V_m$ 's;
- if it exist an Eliahou-Kervaire polynomial such that its reduction mod  $V_m$  is different from zero, the given one is not a J-Marked basis.

The algorithm terminates only if the ordering is ip. Anyway, the number of reduction steps is bounded.

**References:**

[CR] Francesca Cioffi, Margherita Roggero, Flat Families by Strongly Stable Ideals and a Generalization of Groebner Bases, J. Symbolic Comput. 46, 1070-1084, (2011).

[BCLR] Cristina Bertone, Francesca Cioffi, Paolo Lella, Margherita Roggero, Upgraded methods for the effective computation of marked schemes on a strongly stable ideal, Journal of Symbolic Computation (2012), <http://dx.doi.org/10.1016/j.jsc.2012.07.006>

**Procedures:** See also: [Section D.5.20 \[JMSConst.lib\]](#), page 1623.

#### D.5.19.1 Minimus

Procedure from library JMBTest.lib (see [Section D.5.19 \[JMBTest.lib\]](#), page 1621).

**Usage:** Minimus(L); G list, c int

**Return:** list: V

**Notes:** it returns the minimal variable generating the ideal L.  
The input must be an ideal generated by variables.

**Example:**

```
LIB "JMBTest.lib";
ring r=0, (x,y,z), ip;
ideal I=y,x,z;
Minimus(I);
↦ x
```

**D.5.19.2 Maximus**

Procedure from library `JMBTest.lib` (see [Section D.5.19 \[JMBTest.lib\]](#), page 1621).

**Usage:**      `Maximus(L); G list, c int`

**Return:**    `list: V`

**Notes:**     it returns the maximal variable generating the ideal L.  
               The input must be an ideal generated by variables.

**Example:**

```
LIB "JMBTest.lib";
ring r=0, (x,y,z), ip;
ideal I=y,x,z;
Maximus(I);
↦ z
```

**D.5.19.3 StartOrderingV**

Procedure from library `JMBTest.lib` (see [Section D.5.19 \[JMBTest.lib\]](#), page 1621).

**Usage:**      `StartOrdina(V,G); V list, G list`

**Return:**    `list: R`

**Note:**       Input Vm,G. This procedure uses `OrderingV` to get  
               the ordered polynomials as in [BCLR].

**Example:**

```
LIB "JMBTest.lib";
ring r=0, (x,y,z), ip;
jmp r1;
r1.h=z^3;
r1.t=poly(0);
jmp r2;
r2.h=z^2*y;
r2.t=poly(0);
jmp r3;
r3.h=z*y^2;
r3.t=-x^2*y;
jmp r4;
r4.h=y^5;
r4.t=poly(0);
list G2F=list(list(r1,r2,r3),list(r4));
StartOrderingV(VConst(G2F,4,basing)[1],G2F);
↦ [1]:
↦    [1]:
↦       1
```

```

⇒      [2]:
⇒      1
⇒      [3]:
⇒      3
⇒ [2]:
⇒      [1]:
⇒      1
⇒      [2]:
⇒      1
⇒      [3]:
⇒      2
⇒ [3]:
⇒      [1]:
⇒      1
⇒      [2]:
⇒      1
⇒      [3]:
⇒      1

```

#### D.5.19.4 TestJMark

Procedure from library `JMBTest.lib` (see [Section D.5.19 \[JMBTest.lib\]](#), page 1621).

**Usage:**      `TestJMark(G);` G list

**Return:**    `int: i`

**Note:**      This procedure performs J-marked basis test.  
               The input is a list of J-marked polynomials (`jmp`) arranged by degree, so `G1` is a list of list.  
               The output is a boolean evaluation:  
               `True=1/False=0`

**Example:**

```

LIB "JMBTest.lib";
ring r=0, (x,y,z), ip;
jmp r1;
r1.h=z^3;
r1.t=poly(0);
jmp r2;
r2.h=z^2*y;
r2.t=poly(0);
jmp r3;
r3.h=z*y^2 ;
r3.t=-x^2*y;
jmp r4;
r4.h=y^5;
r4.t=poly(0);
list G2F=list(list(r1,r2,r3),list(r4));
TestJMark(G2F,r);
⇒ NOT A BASIS
⇒ 0

```

#### D.5.20 JMSTConst.lib

**Library:** JMSConst.lib

**Purpose:** A library for Singular which constructs J-Marked Schemes.

**Author:** Michela Ceria, email: [michela.ceria@unito.it](mailto:michela.ceria@unito.it)

**Overview:** The library performs the J-marked computation, as described in [BCLR]. As in JMBTest.lib we construct the V polynomials and we reduce the EK polynomials w.r.t. them, putting the coefficients as results.

The algorithm terminates only if the ordering is ip. Anyway, the number of reduction steps is bounded.

**References:**

[CR] Francesca Cioffi, Margherita Roggero, Flat Families by Strongly Stable Ideals and a Generalization of Groebner Bases, J. Symbolic Comput. 46, 1070-1084, (2011).

[BCLR] Cristina Bertone, Francesca Cioffi, Paolo Lella, Margherita Roggero, Upgraded methods for the effective computation of marked schemes on a strongly stable ideal, Journal of Symbolic Computation (2012), <http://dx.doi.org/10.1016/j.jsc.2012.07.006>

**Procedures:** See also: [Section D.5.19 \[JMBTest.lib\], page 1621](#); [Section D.5.20 \[JMSConst.lib\], page 1623](#).

### D.5.20.1 BorelCheck

Procedure from library JMSConst.lib (see [Section D.5.20 \[JMSConst.lib\], page 1623](#)).

**Usage:** BorelCheck(Borid,r); Borid ideal, r ring

**Return:** int: d

**Note:** Input must be a monomial ideal.

The procedure checks whether the Borel moves produce elements belonging to Borid.

**Example:**

```
LIB "JMSConst.lib";
ring r=0, (x,y,z),ip;
ideal Borid=y^2*z,y*z^2,z^3,y^5;
BorelCheck(Borid,r);
↦ 1
```

### D.5.20.2 JMarkedScheme

Procedure from library JMSConst.lib (see [Section D.5.20 \[JMSConst.lib\], page 1623](#)).

**Usage:** JMarkedScheme(Borid, r); Borid ideal, r ring

**Return:** list: Jms

**Note:** This procedure performs automatically the whole construction of the J-marked scheme.

**Example:**

```
LIB "JMSConst.lib";
ring r=0, (x,y,z),ip;
ideal Borid=y^2*z,y*z^2,z^3,y^5;
JMarkedScheme(Borid,r);
↦ [1]:
```

```

⇒      (-c(1)*c(15)+c(1)*c(9)+c(8)^2+c(1)*c(4)*c(8)-c(2)*c(8)-c(1)*c(5)+c(1)*\
      c(2)*c(4)+c(3))
⇒ [2]:
⇒      (-c(1)*c(16)-c(10)+c(8)*c(9)+c(1)*c(4)*c(9)+c(6)-c(2)*c(5)-c(3)*c(4)+c\
      (2)^2*c(4))
⇒ [3]:
⇒      (-c(1)*c(17)+c(8)*c(10)+c(1)*c(4)*c(10)-c(2)*c(10)+c(3)*c(9)+c(7)-c(3)\
      *c(5)+c(2)*c(3)*c(4))
⇒ [4]:
⇒      (-c(11)-c(4)^2)
⇒ [5]:
⇒      (-c(1)*c(18)-c(12)+c(8)*c(11)+c(1)*c(4)*c(11)-c(2)*c(11)+c(4)*c(9)-2*c\
      (4)*c(5)+c(2)*c(4)^2)
⇒ [6]:
⇒      (-c(1)*c(19)-c(13)+c(8)*c(12)+c(1)*c(4)*c(12)-c(2)*c(12)+c(5)*c(9)-c(4)\
      )*c(6)-c(5)^2+c(2)*c(4)*c(5))
⇒ [7]:
⇒      (-c(1)*c(20)-c(14)+c(8)*c(13)+c(1)*c(4)*c(13)-c(2)*c(13)+c(6)*c(9)-c(4)\
      )*c(7)-c(5)*c(6)+c(2)*c(4)*c(6))
⇒ [8]:
⇒      (-c(1)*c(21)+c(8)*c(14)+c(1)*c(4)*c(14)-c(2)*c(14)+c(7)*c(9)-c(5)*c(7)\
      +c(2)*c(4)*c(7))
⇒ [9]:
⇒      (c(1)*c(16)-c(1)*c(12)+c(1)*c(8)*c(11)+c(1)*c(2)*c(11)+c(10)-c(8)*c(9)\
      )
⇒ [10]:
⇒      (-c(17)-c(8)*c(16)+c(2)*c(16)+c(9)*c(15)+c(13)-c(2)*c(12)+c(1)*c(9)*c(\
      11)-c(3)*c(11)+c(2)^2*c(11)-c(9)^2)
⇒ [11]:
⇒      (-c(8)*c(17)+c(3)*c(16)+c(10)*c(15)+c(14)-c(3)*c(12)+c(1)*c(10)*c(11)+\
      c(2)*c(3)*c(11)-c(9)*c(10))
⇒ [12]:
⇒      (-c(18)-c(4)*c(11))
⇒ [13]:
⇒      (-c(19)-c(8)*c(18)+c(4)*c(16)+c(11)*c(15)-c(4)*c(12)+c(1)*c(11)^2-c(9)\
      *c(11)-c(5)*c(11)+c(2)*c(4)*c(11))
⇒ [14]:
⇒      (-c(20)-c(8)*c(19)+c(5)*c(16)+c(12)*c(15)+c(1)*c(11)*c(12)-c(9)*c(12)-\
      c(5)*c(12)-c(6)*c(11)+c(2)*c(5)*c(11))
⇒ [15]:
⇒      (-c(21)-c(8)*c(20)+c(6)*c(16)+c(13)*c(15)+c(1)*c(11)*c(13)-c(9)*c(13)-\
      c(6)*c(12)-c(7)*c(11)+c(2)*c(6)*c(11))
⇒ [16]:
⇒      (-c(8)*c(21)+c(7)*c(16)+c(14)*c(15)+c(1)*c(11)*c(14)-c(9)*c(14)-c(7)*c\
      (12)+c(2)*c(7)*c(11))
⇒ [17]:
⇒      (-c(1)*c(27)+c(1)*c(8)*c(26)+c(1)*c(2)*c(26)-c(1)^2*c(9)*c(25)-c(1)*c(\
      8)^2*c(25)-c(1)*c(2)*c(8)*c(25)+c(1)*c(3)*c(25)-c(1)*c(2)^2*c(25)+c(24)-c\
      (8)*c(23)-c(1)*c(4)*c(23)-c(15)*c(22)-c(1)*c(11)*c(22)-c(4)*c(8)*c(22)+c(\
      5)*c(22)-c(2)*c(4)*c(22)-c(1)^2*c(10)+2*c(1)^2*c(8)*c(9)+2*c(1)^2*c(2)*c(\
      9)+c(1)*c(8)^3+c(1)*c(2)*c(8)^2-c(1)*c(3)*c(8)+c(1)*c(2)^2*c(8)-2*c(1)*c(\
      2)*c(3)+c(1)*c(2)^3)
⇒ [18]:

```

$\mapsto (c(28)-c(2)*c(27)+c(1)*c(9)*c(26)-c(3)*c(26)+c(2)^2*c(26)+c(1)*c(10)*c(25)-c(1)*c(8)*c(9)*c(25)-2*c(1)*c(2)*c(9)*c(25)+2*c(2)*c(3)*c(25)-c(2)^3*c(25)+c(4)*c(24)-c(1)*c(11)*c(23)-c(9)*c(23)+c(5)*c(23)-2*c(2)*c(4)*c(23)-c(16)*c(22)-c(4)*c(9)*c(22)-c(1)*c(8)*c(10)-2*c(1)*c(2)*c(10)+c(1)^2*c(9)^2+c(1)*c(8)^2*c(9)+2*c(1)*c(2)*c(8)*c(9)-2*c(1)*c(3)*c(9)+3*c(1)*c(2)^2*c(9)+c(3)^2-3*c(2)^2*c(3)+c(2)^4)$   
 $\mapsto [19]:$   
 $\mapsto (c(29)-c(3)*c(27)+c(1)*c(10)*c(26)+c(2)*c(3)*c(26)-c(1)*c(8)*c(10)*c(25)-c(1)*c(2)*c(10)*c(25)-c(1)*c(3)*c(9)*c(25)+c(3)^2*c(25)-c(2)^2*c(3)*c(25)-c(1)*c(11)*c(24)+c(5)*c(24)-c(2)*c(4)*c(24)-c(10)*c(23)-c(3)*c(4)*c(23)-c(17)*c(22)-c(4)*c(10)*c(22)+c(1)^2*c(9)*c(10)+c(1)*c(8)^2*c(10)+c(1)*c(2)*c(8)*c(10)-2*c(1)*c(3)*c(10)+c(1)*c(2)^2*c(10)+c(1)*c(3)*c(8)*c(9)+2*c(1)*c(2)*c(3)*c(9)-2*c(2)*c(3)^2+c(2)^3*c(3))$   
 $\mapsto [20]:$   
 $\mapsto (c(1)*c(12)-c(1)*c(8)*c(11)-c(1)*c(2)*c(11)-c(1)*c(4)*c(9)-c(6)+c(2)*c(5)+c(3)*c(4)-c(2)^2*c(4))$   
 $\mapsto [21]:$   
 $\mapsto (c(1)*c(12)*c(25)-c(1)*c(8)*c(11)*c(25)-c(1)*c(2)*c(11)*c(25)-c(1)*c(4)*c(9)*c(25)-c(6)*c(25)+c(2)*c(5)*c(25)+c(3)*c(4)*c(25)-c(2)^2*c(4)*c(25)-c(11)*c(23)-c(4)^2*c(23)-c(18)*c(22)-c(4)*c(11)*c(22)+c(1)*c(13)-c(1)*c(8)*c(12)-c(1)*c(2)*c(12)+c(1)^2*c(9)*c(11)+c(1)*c(8)^2*c(11)+c(1)*c(2)*c(8)*c(11)-c(1)*c(3)*c(11)+c(1)*c(2)^2*c(11)-c(1)*c(4)*c(10)+c(1)*c(4)*c(8)*c(9)-c(1)*c(5)*c(9)+2*c(1)*c(2)*c(4)*c(9)-c(7)+c(2)*c(6)+c(3)*c(5)-c(2)^2*c(5)-2*c(2)*c(3)*c(4)+c(2)^3*c(4))$   
 $\mapsto [22]:$   
 $\mapsto (c(4)*c(28)-c(1)*c(11)*c(27)-c(2)*c(4)*c(27)+c(1)*c(12)*c(26)-c(6)*c(26)+c(2)*c(5)*c(26)+c(1)*c(13)*c(25)-c(1)*c(8)*c(12)*c(25)-c(1)*c(2)*c(12)*c(25)-c(1)*c(5)*c(9)*c(25)-c(7)*c(25)+c(2)*c(6)*c(25)+c(3)*c(5)*c(25)-c(2)^2*c(5)*c(25)-c(12)*c(23)-c(4)*c(5)*c(23)-c(19)*c(22)-c(4)*c(12)*c(22)+c(1)*c(14)-c(1)*c(8)*c(13)-c(1)*c(2)*c(13)+c(1)^2*c(9)*c(12)+c(1)*c(8)^2*c(12)+c(1)*c(2)*c(8)*c(12)-c(1)*c(3)*c(12)+c(1)*c(2)^2*c(12)-c(1)*c(5)*c(10)+c(1)*c(5)*c(8)*c(9)-c(1)*c(6)*c(9)+2*c(1)*c(2)*c(5)*c(9)+c(2)*c(7)+c(3)*c(6)-c(2)^2*c(6)-2*c(2)*c(3)*c(5)+c(2)^3*c(5))$   
 $\mapsto [23]:$   
 $\mapsto (c(4)*c(29)-c(1)*c(11)*c(28)+c(5)*c(28)-c(2)*c(4)*c(28)-c(6)*c(27)+c(1)*c(13)*c(26)-c(7)*c(26)+c(2)*c(6)*c(26)+c(1)*c(14)*c(25)-c(1)*c(8)*c(13)*c(25)-c(1)*c(2)*c(13)*c(25)-c(1)*c(6)*c(9)*c(25)+c(2)*c(7)*c(25)+c(3)*c(6)*c(25)-c(2)^2*c(6)*c(25)-c(13)*c(23)-c(4)*c(6)*c(23)-c(20)*c(22)-c(4)*c(13)*c(22)-c(1)*c(8)*c(14)-c(1)*c(2)*c(14)+c(1)^2*c(9)*c(13)+c(1)*c(8)^2*c(13)+c(1)*c(2)*c(8)*c(13)-c(1)*c(3)*c(13)+c(1)*c(2)^2*c(13)-c(1)*c(6)*c(10)+c(1)*c(6)*c(8)*c(9)-c(1)*c(7)*c(9)+2*c(1)*c(2)*c(6)*c(9)+c(3)*c(7)-c(2)^2*c(7)-2*c(2)*c(3)*c(6)+c(2)^3*c(6))$   
 $\mapsto [24]:$   
 $\mapsto (-c(1)*c(11)*c(29)+c(5)*c(29)-c(2)*c(4)*c(29)-c(7)*c(27)+c(1)*c(14)*c(26)+c(2)*c(7)*c(26)-c(1)*c(8)*c(14)*c(25)-c(1)*c(2)*c(14)*c(25)-c(1)*c(7)*c(9)*c(25)+c(3)*c(7)*c(25)-c(2)^2*c(7)*c(25)-c(14)*c(23)-c(4)*c(7)*c(23)-c(21)*c(22)-c(4)*c(14)*c(22)+c(1)^2*c(9)*c(14)+c(1)*c(8)^2*c(14)+c(1)*c(2)*c(8)*c(14)-c(1)*c(3)*c(14)+c(1)*c(2)^2*c(14)-c(1)*c(7)*c(10)+c(1)*c(7)*c(8)*c(9)+2*c(1)*c(2)*c(7)*c(9)-2*c(2)*c(3)*c(7)+c(2)^3*c(7))$

## D.6 Singularities



### D.6.1 alexpoly\_lib

**Library:** alexpoly.lib

**Purpose:** Resolution Graph and Alexander Polynomial

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**Overview:** A library for computing the resolution graph of a plane curve singularity  $f$ , the total multiplicities of the total transforms of the branches of  $f$  along the exceptional divisors of a minimal good resolution of  $f$ , the Alexander polynomial of  $f$ , and the zeta function of its monodromy operator.

**Procedures:**

#### D.6.1.1 resolutiongraph

Procedure from library alexpoly.lib (see [Section D.6.1 \[alexpoly\\_lib\]](#), page 1627).

**Usage:** resolutiongraph(INPUT); INPUT poly or list

**Assume:** INPUT is either a REDUCED bivariate polynomial defining a plane curve singularity, or the output of `hnexpansion(f, "ess")`, or the list `hne` in the ring created by `hnexpansion(f, "ess")`, or the output of `develop(f)` resp. of `extdevelop(f, n)`, or a list containing the contact matrix and a list of integer vectors with the characteristic exponents of the branches of a plane curve singularity, or an integer vector containing the characteristic exponents of an irreducible plane curve singularity.

**Return:** intmat, the incidence matrix of the resolution graph of the plane curve defined by INPUT, where the entries on the diagonal are the weights of the vertices of the graph and a negative entry corresponds to the strict transform of a branch of the curve.

**Note:** In case the Hamburger-Noether expansion of the curve  $f$  is needed for other purposes as well it is better to calculate this first with the aid of `hnexpansion` and use it as input instead of the polynomial itself.  
If you are not sure whether the INPUT polynomial is reduced or not, use `squarefree(INPUT)` as input instead.

**Example:**

```
LIB "alexpoly.lib";
ring r=0,(x,y),ls;
poly f1=(y2-x3)^2-4x5y-x7;
poly f2=y2-x3;
poly f3=y3-x2;
resolutiongraph(f1*f2*f3);
⇒ 1,0,1,0,0,0,0,0,1,0,
⇒ 0,2,1,0,0,0,0,0,0,0,
⇒ 1,1,3,0,1,0,0,0,0,0,
⇒ 0,0,0,4,1,0,1,0,0,0,
⇒ 0,0,1,1,5,1,0,0,0,0,
⇒ 0,0,0,0,1,-1,0,0,0,0,
⇒ 0,0,0,1,0,0,-2,0,0,0,
⇒ 0,0,0,0,0,0,0,2,1,0,
⇒ 1,0,0,0,0,0,0,1,3,1,
⇒ 0,0,0,0,0,0,0,0,1,-3
```

See also: [Section D.6.1.3 \[alexanderpolynomial\]](#), page 1629; [Section D.6.15.2 \[develop\]](#), page 1722; [Section D.6.15.1 \[hnexpansion\]](#), page 1720; [Section D.6.1.2 \[totalmultiplicities\]](#), page 1628.

### D.6.1.2 totalmultiplicities

Procedure from library `alexpoly.lib` (see [Section D.6.1 \[alexpoly.lib\]](#), page 1627).

**Usage:** `totalmultiplicities(INPUT)`; INPUT poly or list

**Assume:** INPUT is either a REDUCED bivariate polynomial defining a plane curve singularity, or the output of `hnexpansion(f[,"ess"])`, or the list `hne` in the ring created by `hnexpansion(f[,"ess"])`, or the output of `develop(f)` resp. of `extdevelop(f,n)`, or a list containing the contact matrix and a list of integer vectors with the characteristic exponents of the branches of a plane curve singularity, or an integer vector containing the characteristic exponents of an irreducible plane curve singularity.

**Return:** list L of three integer matrices. L[1] is the incidence matrix of the resolution graph of the plane curve defined by INPUT, where the entries on the diagonal are the weights of the vertices of the graph and a negative entry corresponds to the strict transform of a branch of the curve. L[2] is an integer matrix, which has for each vertex in the graph a row and for each branch of the curve a column. The entry in position [i,j] contains the total multiplicity of the j-th branch (i.e. the branch with weight -j in L[1]) along the exceptional divisor corresponding to the i-th row in L[1]. In particular, the i-th row contains the total multiplicities of the branches of the plane curve (defined by INPUT) along the exceptional divisor which corresponds to the i-th row in the incidence matrix L[1]. L[3] is an integer matrix which contains the (strict) multiplicities of the branches of the curve along the exceptional divisors in the same way as L[2] contains the total multiplicities.

**Note:** The total multiplicity of a branch along an exceptional divisor is the multiplicity with which this exceptional divisor occurs in the total transform of this branch under the resolution corresponding to the resolution graph.

In case the Hamburger-Noether expansion of the curve f is needed for other purposes as well it is better to calculate this first with the aid of `hnexpansion` and use it as input instead of the polynomial itself.

If you are not sure whether the INPUT polynomial is reduced or not, use `squarefree(INPUT)` as input instead.

**Example:**

```
LIB "alexpoly.lib";
ring r=0,(x,y),ls;
poly f1=(y2-x3)^2-4x5y-x7;
poly f2=y2-x3;
poly f3=y3-x2;
totalmultiplicities(f1*f2*f3);
⇒ [1]:
⇒ 1,0,1,0,0,0,0,0,1,0,
⇒ 0,2,1,0,0,0,0,0,0,0,
⇒ 1,1,3,0,1,0,0,0,0,0,
⇒ 0,0,0,4,1,0,1,0,0,0,
⇒ 0,0,1,1,5,1,0,0,0,0,
⇒ 0,0,0,0,1,-1,0,0,0,0,
⇒ 0,0,0,1,0,0,-2,0,0,0,
⇒ 0,0,0,0,0,0,0,2,1,0,
⇒ 1,0,0,0,0,0,0,1,3,1,
⇒ 0,0,0,0,0,0,0,0,1,-3
⇒ [2]:
```

```

↳ 4,2,2,
↳ 6,3,2,
↳ 12,6,4,
↳ 13,7,4,
↳ 26,13,8,
↳ 0,0,0,
↳ 0,0,0,
↳ 4,2,3,
↳ 8,4,6,
↳ 0,0,0
↳ [3]:
↳ 4,2,2,
↳ 2,1,0,
↳ 2,1,0,
↳ 1,1,0,
↳ 1,0,0,
↳ 0,0,0,
↳ 0,0,0,
↳ 0,0,1,
↳ 0,0,1,
↳ 0,0,0

```

See also: [Section D.6.1.3 \[alexanderpolynomial\]](#), page 1629; [Section D.6.15.2 \[develop\]](#), page 1722; [Section D.6.15.1 \[hnexpansion\]](#), page 1720; [Section D.6.1.1 \[resolutiongraph\]](#), page 1627.

### D.6.1.3 alexanderpolynomial

Procedure from library `alexpoly.lib` (see [Section D.6.1 \[alexpoly.lib\]](#), page 1627).

**Usage:** `alexanderpolynomial(INPUT);` INPUT poly or list

**Assume:** INPUT is either a REDUCED bivariate polynomial defining a plane curve singularity, or the output of `hnexpansion(f[,"ess"])`, or the list `hne` in the ring created by `hnexpansion(f[,"ess"])`, or the output of `develop(f)` resp. of `extdevelop(f,n)`, or a list containing the contact matrix and a list of integer vectors with the characteristic exponents of the branches of a plane curve singularity, or an integer vector containing the characteristic exponents of an irreducible plane curve singularity.

**Create:** a ring with variables `t, t(1), ..., t(r)` (where `r` is the number of branches of the plane curve singularity `f` defined by INPUT) and ordering `ls` over the ground field of the basering.

Moreover, the ring contains the Alexander polynomial of `f` in variables `t(1), ..., t(r)` (`alexpoly`), the zeta function of the monodromy operator of `f` in the variable `t` (`zeta_monodromy`), and a list containing the factors of the Alexander polynomial with multiplicities (`alexfactors`).

**Return:** a list, say `ALEX`, where `ALEX[1]` is the created ring

**Note:** to use the ring type: `def ALEXring=ALEX[i]; setring ALEXring;`

Alternatively you may use the procedure `sethnering` and type: `sethnering(ALEX,"ALEXring");`

To access the Alexander polynomial resp. the zeta function resp. the factors of the Alexander polynomial type: `alexpoly` resp. `zeta_monodromy` resp. `alexfactors`.

In case the Hamburger-Noether expansion of the curve `f` is needed for other purposes as well it is better to calculate this first with the aid of `hnexpansion` and use it as

input instead of the polynomial itself.

If you are not sure whether the INPUT polynomial is reduced or not, use `squarefree(INPUT)` as input instead.

**Example:**

```
LIB "alexpoly.lib";
ring r=0,(x,y),ls;
poly f1=(y2-x3)^2-4x5y-x7;
poly f2=y2-x3;
poly f3=y3-x2;
list ALEX=alexanderpolynomial(f1*f2*f3);
def ALEXring=ALEX[1];
setring ALEXring;
alexfactors;
⇒ [1]:
⇒ [1]:
⇒ -t(1)^6*t(2)^3*t(3)^2+1
⇒ [2]:
⇒ -1
⇒ [2]:
⇒ [1]:
⇒ -t(1)^12*t(2)^6*t(3)^4+1
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ [1]:
⇒ -t(1)^26*t(2)^13*t(3)^8+1
⇒ [2]:
⇒ 1
⇒ [4]:
⇒ [1]:
⇒ -t(1)^4*t(2)^2*t(3)^3+1
⇒ [2]:
⇒ -1
⇒ [5]:
⇒ [1]:
⇒ -t(1)^8*t(2)^4*t(3)^6+1
⇒ [2]:
⇒ 1
alexpoly;
⇒ -t(1)^36*t(2)^18*t(3)^13-t(1)^32*t(2)^16*t(3)^10-t(1)^30*t(2)^15*t(3)^11-\
t(1)^26*t(2)^13*t(3)^8+t(1)^10*t(2)^5*t(3)^5+t(1)^6*t(2)^3*t(3)^2+t(1)^4*\
t(2)^2*t(3)^3+1
zeta_monodromy;
⇒ -t^67-t^58-t^56-t^47+t^20+t^11+t^9+1
```

See also: [Section D.6.1.1 \[resolutiongraph\]](#), page 1627; [Section D.6.1.2 \[totalmultiplicities\]](#), page 1628.

### D.6.1.4 semigroup

Procedure from library `alexpoly.lib` (see [Section D.6.1 \[alexpoly-lib\]](#), page 1627).

**Usage:** `semigroup(INPUT);` INPUT poly or list

**Assume:** INPUT is either a REDUCED bivariate polynomial defining a plane curve singularity, or the output of `hnexpansion(f[, "ess"])`, or the list `hne` in the ring created by `hnexpansion(f[, "ess"])`, or the output of `develop(f)` resp. of `extdevelop(f,n)`, or a list containing the contact matrix and a list of integer vectors with the characteristic exponents of the branches of a plane curve singularity, or an integer vector containing the characteristic exponents of an irreducible plane curve singularity.

**Return:** a list with three entries. The first and the second are lists  $v_1, \dots, v_s$  and  $w_1, \dots, w_r$  respectively of integer vectors such that the semigroup of the plane curve defined by the INPUT is generated by the vectors  $v_1, \dots, v_s, w_1 + k \cdot e_1, \dots, w_r + k \cdot e_r$ , where  $e_i$  denotes the  $i$ -th standard basis vector and  $k$  runs through all non-negative integers. The third entry is the conductor of the plane curve singularity. Note that  $r$  is the number of branches of the plane curve singularity and integer vectors thus have size  $r$ .

**Note:** If the output is zero this means that the curve has one branch and is regular. In the reducible case the set of generators may not be minimal.  
If you are not sure whether the INPUT polynomial is reduced or not, use `squarefree(INPUT)` as input instead.

**Example:**

```
LIB "alexpoly.lib";
ring r=0,(x,y),ls;
// Irreducible Case
semigroup((x2-y3)^2-4x5y-x7);
⇒ [1]:
⇒   [1]:
⇒     4
⇒   [2]:
⇒     6
⇒   [3]:
⇒    17
⇒ [2]:
⇒ empty list
⇒ [3]:
⇒   20
// In the irreducible case, invariants() also calculates a minimal set of
// generators of the semigroup.
invariants((x2-y3)^2-4x5y-x7)[1][2];
⇒ 4,6,17
// Reducible Case
poly f=(y2-x3)*(y2+x3)*(y4-2x3y2-4x5y+x6-x7);
semigroup(f);
⇒ [1]:
⇒   [1]:
⇒     2,2,4
⇒   [2]:
⇒     3,3,6
⇒ [2]:
⇒   [1]:
⇒     6,6,12
⇒   [2]:
⇒     6,7,13
⇒   [3]:
```

```

    ↪      12,13,26
    ↪ [3]:
    ↪      20,21,41

```

See also: [Section D.6.1.1 \[resolutiongraph\]](#), page 1627; [Section D.6.1.2 \[totalmultiplicities\]](#), page 1628.

### D.6.1.5 proximitymatrix

Procedure from library `alexpoly.lib` (see [Section D.6.1 \[alexpoly.lib\]](#), page 1627).

**Usage:** `proximitymatrix(INPUT)`; INPUT poly or list or intmat

**Assume:** INPUT is either a REDUCED bivariate polynomial defining a plane curve singularity, or the output of `hnexpansion(f, "ess")`, or the list `hne` in the ring created by `hnexpansion(f, "ess")`, or the output of `develop(f)` resp. of `extdevelop(f, n)`, or a list containing the contact matrix and a list of integer vectors with the characteristic exponents of the branches of a plane curve singularity, or an integer vector containing the characteristic exponents of an irreducible plane curve singularity, or the resolution graph of a plane curve singularity (i.e. the output of `resolutiongraph` or the first entry in the output of `totalmultiplicities`).

**Return:** list, of three integer matrices. The first one is the proximity matrix of the plane curve defined by the INPUT, i.e. the entry  $i, j$  is 1 if the infinitely near point corresponding to row  $i$  is proximate to the infinitely near point corresponding to row  $j$ . The second integer matrix is the incidence matrix of the resolution graph of the plane curve. The entry on the diagonal in row  $i$  is  $-s-1$  if  $s$  is the number of points proximate to the infinitely near point corresponding to the  $i$ th row in the matrix. The third integer matrix is the incidence matrix of the Enriques diagram of the plane curve singularity, i.e. each row corresponds to an infinitely near point in the minimal standard resolution, including the strict transforms of the branches, the diagonal element gives the level of the point, and the entry  $i, j$  is  $-1$  if row  $i$  is proximate to row  $j$ .

**Note:** In case the Hamburger-Noether expansion of the curve  $f$  is needed for other purposes as well it is better to calculate this first with the aid of `hnexpansion` and use it as input instead of the polynomial itself.

If you are not sure whether the INPUT polynomial is reduced or not, use `squarefree(INPUT)` as input instead.

If the input is a smooth curve, then the output will consist of three one-by-one zero matrices.

For the definitions of the computed objects see e.g. the book Eduardo Casas-Alvero, *Singularities of Plane Curves*.

**Example:**

```

LIB "alexpoly.lib";
ring r=0,(x,y),ls;
poly f1=(y2-x3)^2-4x5y-x7;
poly f2=y2-x3;
poly f3=y3-x2;
list proximity=proximitymatrix(f1*f2*f3);
/// The proximity matrix P ///
print(proximity[1]);
↪      1      0      0      0      0      0      0      0      0      0
↪     -1      1      0      0      0      0      0      0      0      0
↪     -1     -1      1      0      0      0      0      0      0      0

```

```

⇒      0      0     -1      1      0      0      0      0      0      0
⇒      0      0     -1     -1      1      0      0      0      0      0
⇒      0      0      0      0     -1      1      0      0      0      0
⇒      0      0      0     -1      0      0      1      0      0      0
⇒     -1      0      0      0      0      0      0      1      0      0
⇒     -1      0      0      0      0      0      0     -1      1      0
⇒      0      0      0      0      0      0      0      0     -1      1
/// The proximity resolution graph N ///
print(proximity[2]);
⇒     -5      0      1      0      0      0      0      0      1      0
⇒      0     -2      1      0      0      0      0      0      0      0
⇒      1      1     -3      0      1      0      0      0      0      0
⇒      0      0      0     -3      1      0      1      0      0      0
⇒      0      0      1      1     -2      1      0      0      0      0
⇒      0      0      0      0      1     -1      0      0      0      0
⇒      0      0      0      1      0      0     -1      0      0      0
⇒      0      0      0      0      0      0      0     -2      1      0
⇒      1      0      0      0      0      0      0      1     -2      1
⇒      0      0      0      0      0      0      0      0      1     -1
/// They satisfy  $N = -\text{transpose}(P) * P$  ///
print(-transpose(proximity[1])*proximity[1]);
⇒     -5      0      1      0      0      0      0      0      1      0
⇒      0     -2      1      0      0      0      0      0      0      0
⇒      1      1     -3      0      1      0      0      0      0      0
⇒      0      0      0     -3      1      0      1      0      0      0
⇒      0      0      1      1     -2      1      0      0      0      0
⇒      0      0      0      0      1     -1      0      0      0      0
⇒      0      0      0      1      0      0     -1      0      0      0
⇒      0      0      0      0      0      0      0     -2      1      0
⇒      1      0      0      0      0      0      0      1     -2      1
⇒      0      0      0      0      0      0      0      0      1     -1
/// The incidence matrix of the Enriques diagram ///
print(proximity[3]);
⇒      0      0      0      0      0      0      0      0      0      0
⇒     -1      1      0      0      0      0      0      0      0      0
⇒     -1     -1      2      0      0      0      0      0      0      0
⇒      0      0     -1      3      0      0      0      0      0      0
⇒      0      0     -1     -1      4      0      0      0      0      0
⇒      0      0      0      0     -1      5      0      0      0      0
⇒      0      0      0     -1      0      0      4      0      0      0
⇒     -1      0      0      0      0      0      0      1      0      0
⇒     -1      0      0      0      0      0      0     -1      2      0
⇒      0      0      0      0      0      0      0      0     -1      3
/// If M is the matrix of multiplicities and TM the matrix of total
/// multiplicities of the singularity, then  $M = P * TM$ .
/// We therefore calculate the (total) multiplicities. Note that
/// they have to be slightly extended.
list MULT=extend_multiplicities(totalmultiplicities(f1*f2*f3));
intmat TM=MULT[1]; // Total multiplicities.
intmat M=MULT[2]; // Multiplicities.
/// Check:  $M - P * TM = 0$ .
M-proximity[1]*TM;
⇒ 0,0,0,

```

```

    ↦ 0,0,0,
    ↦ 0,0,0,
    ↦ 0,0,0,
    ↦ 0,0,0,
    ↦ 0,0,0,
    ↦ 0,0,0,
    ↦ 0,0,0,
    ↦ 0,0,0,
    ↦ 0,0,0,
    ↦ 0,0,0,
    ↦ 0,0,0
    /// Check: inverse(P)*M-TM=0.
    intmat_inverse(proximity[1])*M-TM;
    ↦ 0,0,0,
    ↦ 0,0,0,
    ↦ 0,0,0,
    ↦ 0,0,0,
    ↦ 0,0,0,
    ↦ 0,0,0,
    ↦ 0,0,0,
    ↦ 0,0,0,
    ↦ 0,0,0,
    ↦ 0,0,0,
    ↦ 0,0,0

```

See also: [Section D.6.1.3 \[alexanderpolynomial\]](#), page 1629; [Section D.6.15.2 \[develop\]](#), page 1722; [Section D.6.15.1 \[hnexpansion\]](#), page 1720; [Section D.6.1.2 \[totalmultiplicities\]](#), page 1628.

### D.6.1.6 multseq2charexp

Procedure from library `alexpoly.lib` (see [Section D.6.1 \[alexpoly.lib\]](#), page 1627).

**Assume:** The input is an intvec, which contains the multiplicity sequence of an irreducible plane curve singularity .

**Return:** An intvec, which contains the sequence of characteristic exponents of the irreducible plane curve singularity defined by v.

**Example:**

```

LIB "alexpoly.lib";
intvec v=2,1,1;
multseq2charexp(v);
↦ 2,3
intvec v1=4,2,2,1,1;
multseq2charexp(v1);
↦ 4,6,7

```

### D.6.1.7 charexp2multseq

Procedure from library `alexpoly.lib` (see [Section D.6.1 \[alexpoly.lib\]](#), page 1627).

**Usage:** `charexp2multseq(v)`, v intvec

**Assume:** v contains the characteristic exponents of an irreducible plane curve singularity

**Return:** intvec, the multiplicity sequence of the plane curve singularity

**Note:** If the curve singularity is smooth, then the multiplicity sequence is empty. This is expressed by returning zero.



**Example:**

```
LIB "alexpoly.lib";
charexp2multseq(intvec(28,64,66,77));
↦ 28,28,8,8,8,4,4,2,2,2,2,2,2,1,1
```

See also: [Section D.6.1.3 \[alexanderpolynomial\]](#), page 1629; [Section D.6.15.6 \[invariants\]](#), page 1727; [Section D.6.1.1 \[resolutiongraph\]](#), page 1627; [Section D.6.1.2 \[totalmultiplicities\]](#), page 1628.

**D.6.1.8 charexp2generators**

Procedure from library `alexpoly.lib` (see [Section D.6.1 \[alexpoly.lib\]](#), page 1627).

**Usage:** `charexp2generators(v)`, `v` intvec

**Assume:** `v` contains the characteristic exponents of an irreducible plane curve singularity

**Return:** intvec, the minimal set of generators of the semigroup of the plane curve singularity

**Example:**

```
LIB "alexpoly.lib";
charexp2generators(intvec(28,64,66,77));
↦ 28,64,450,911
```

See also: [Section D.6.1.3 \[alexanderpolynomial\]](#), page 1629; [Section D.6.15.6 \[invariants\]](#), page 1727; [Section D.6.1.1 \[resolutiongraph\]](#), page 1627; [Section D.6.1.2 \[totalmultiplicities\]](#), page 1628.

**D.6.1.9 charexp2inter**

Procedure from library `alexpoly.lib` (see [Section D.6.1 \[alexpoly.lib\]](#), page 1627).

**Usage:** `charexp2inter(contact,charexp)`, `contact` matrix, `charexp` list

**Assume:** `charexp` contains the integer vectors of characteristic exponents of the branches of a plane curve singularity, and `contact` is their contact matrix

**Return:** intmat, the matrix intersection multiplicities of the branches

**Example:**

```
LIB "alexpoly.lib";
ring r=0,(x,y),ds;
list INV=invariants((x2-y3)*(x3-y2)*((x2-y3)^2-4x5y-x7));
intmat contact=INV[4][1];
list charexp=INV[1][1],INV[2][1],INV[3][1];
// The intersection matrix is INV[4][2].
print(INV[4][2]);
↦      0      4      8
↦      4      0     17
↦      8     17      0
// And it is calculated as ...
print(charexp2inter(contact,charexp));
↦      0      4      8
↦      4      0     17
↦      8     17      0
```

See also: [Section D.6.15.6 \[invariants\]](#), page 1727; [Section D.6.1.1 \[resolutiongraph\]](#), page 1627; [Section D.6.1.4 \[semigroup\]](#), page 1630; [Section D.6.1.2 \[totalmultiplicities\]](#), page 1628.

### D.6.1.10 charexp2conductor

Procedure from library `alexpoly.lib` (see [Section D.6.1 \[alexpoly-lib\]](#), page 1627).

**Assume:** `v` contains the characteristic exponents of an irreducible plane curve singularity

**Return:** `int`, the conductor of the plane curve singularity

**Note:** If the curve singularity is smooth, the conductor is zero.

**Example:**

```
LIB "alexpoly.lib";
charexp2conductor(intvec(2,3)); // A1-Singularity
↳ 2
charexp2conductor(intvec(28,64,66,77));
↳ 1718
```

See also: [Section D.6.15.6 \[invariants\]](#), page 1727; [Section D.6.1.1 \[resolutiongraph\]](#), page 1627; [Section D.6.1.4 \[semigroup\]](#), page 1630; [Section D.6.1.2 \[totalmultiplicities\]](#), page 1628.

### D.6.1.11 charexp2poly

Procedure from library `alexpoly.lib` (see [Section D.6.1 \[alexpoly-lib\]](#), page 1627).

**Assume:** `v` an `intvec` containing the characteristic exponents of an irreducible plane curve singularity. `a` a vector containing the coefficients of a parametrization given by  $x(t)=x^v[1]$ ,  $y(t)=a(1)t^v[2]+\dots+a[n-1]t^v[n]$ , i.e. the entries of `a` are of type number.

**Return:** A polynomial `f` in the first two variables of the basering, such that `f` defines an irreducible plane curve singularity with characteristic exponents `v`.

**Note:** The entries in `a` should be of type number and the vector `v` should be the sequence of characteristic exponents of an irreducible plane curve singularity in order to get a sensible result,

**Example:**

```
LIB "alexpoly.lib";
ring r=0,(x,y),dp;
intvec v=8,12,14,17;
vector a=[1,1,1];
poly f=charexp2poly(v,a);
f;
↳ -x17+8x16-20x15+17x14-16x13y+12x12y2-2x13+32x12y-16x11y2-8x10y3+x12-8x11y\
+20x10y2-16x9y3-4x9y2+16x8y3-2x7y4-8x6y5+6x6y4-8x5y5-4x3y6+y8
invariants(f)[1][1]; // The characteristic exponents of f.
↳ 8,12,14,17
```

See also: [Section D.6.1.7 \[charexp2multseq\]](#), page 1634; [Section D.6.1.6 \[multseq2charexp\]](#), page 1634.

### D.6.1.12 tau\_es2

Procedure from library `alexpoly.lib` (see [Section D.6.1 \[alexpoly-lib\]](#), page 1627).

**Usage:** `tau_es2(INPUT)`; `INPUT` poly or list

**Assume:** `INPUT` is either a REDUCED bivariate polynomial defining a plane curve singularity, or the output of `hnexpansion(f, "ess")`, or the list `hne` in the ring created by

`hnexpansion(f[, "ess"])`, or the output of `develop(f)` resp. of `extdevelop(f,n)`, or a list containing the contact matrix and a list of integer vectors with the characteristic exponents of the branches of a plane curve singularity, or an integer vector containing the characteristic exponents of an irreducible plane curve singularity.

**Return:** int, the equisingular Tjurina number of  $f$ , i. e. the codimension of the  $\mu$ -constant stratum in the semiuniversal deformation of  $f$ , where  $\mu$  is the Milnor number of  $f$ .

**Note:** The equisingular Tjurina number is calculated with the aid of a Hamburger-Noether expansion, which is the hard part of the calculation.

In case the Hamburger-Noether expansion of the curve  $f$  is needed for other purposes as well it is better to calculate this first with the aid of `hnexpansion` and use it as input instead of the polynomial itself.

If you are not sure whether the INPUT polynomial is reduced or not, use `squarefree(INPUT)` as input instead.

**Example:**

```
LIB "alexpoly.lib";
ring r=0,(x,y),ls;
poly f1=y2-x3;
poly f2=(y2-x3)^2-4x5y-x7;
poly f3=y3-x2;
tau_es2(f1);
  ↦ 2
tau_es2(f2);
  ↦ 14
tau_es2(f1*f2*f3);
  ↦ 49
```

See also: [Section D.6.15.2 \[develop\]](#), page 1722; [Section D.6.12 \[equising\\_lib\]](#), page 1697; [Section D.6.15.1 \[hnexpansion\]](#), page 1720; [Section D.6.12.1 \[tau\\_es\]](#), page 1698; [Section D.6.1.2 \[totalmultiplicities\]](#), page 1628.

## D.6.2 arcpoint\_lib

**Library:** arcpoint.lib

**Purpose:** Truncations of arcs at a singular point

**Author:** Nadine Cremer cremer@mathematik.uni-kl.de

**Overview:** An arc is given by a power series in one variable, say  $t$ , and truncating it at a positive integer  $i$  means cutting the  $t$ -powers  $> i$ . The set of arcs truncated at order  $< \text{bound}$  is denoted  $\text{Tr}(i)$ . An algorithm for computing these sets (which happen to be constructible) is given in [Lejeune-Jalabert, M.: Courbes trac'ees sur un germe d'hypersurface, American Journal of Mathematics, 112 (1990)]. Our procedures for computing the locally closed sets contributing to the set of truncations rely on this algorithm.

**Procedures:**

### D.6.2.1 nashmult

Procedure from library `arcpoint.lib` (see [Section D.6.2 \[arcpoint\\_lib\]](#), page 1637).

**Usage:** `nashmult(f,bound)`;  $f$  polynomial,  $\text{bound}$  positive integer

**Create:**     allsteps:  
                   a list containing all relevant locally closed sets  
                   up to order <bound> and their sequences of  
                   Nash Multiplicities  
               setstep:  
                   list of relevant locally closed sets  
                   obtained from sequences of length bound+1

**Return:**     ring, original basering with additional  
                   variables t and coefficients up to  $t^{\text{<bound>}}$

**Example:**

```
LIB "arcpoint.lib";
ring r=0,(x,y,z),dp;
poly f=z4+y3-x2;
def R=nashmult(f,2);
setring R;
allsteps;
⇨ [1]:
⇨   [1]:
⇨     [1]:
⇨       2,2
⇨     [2]:
⇨       _[1]=a(1)
⇨       _[2]=b(1)
⇨     [3]:
⇨       _[1]=1
⇨ [2]:
⇨   [1]:
⇨     [1]:
⇨       2,2,1
⇨     [2]:
⇨       _[1]=a(1)
⇨       _[2]=b(1)
⇨       _[3]=c(1)^4-a(2)^2
⇨     [3]:
⇨       _[1]=a(1)
⇨       _[2]=b(1)
⇨       _[3]=c(1)
⇨       _[4]=a(2)
⇨   [2]:
⇨     [1]:
⇨       2,2,2
⇨     [2]:
⇨       _[1]=a(1)
⇨       _[2]=b(1)
⇨       _[3]=c(1)
⇨       _[4]=a(2)
⇨     [3]:
⇨       _[1]=1
```

**D.6.2.2 removepower**

Procedure from library `arcpoint.lib` (see [Section D.6.2 \[arcpoint.lib\]](#), page 1637).

**Usage:**      `removepower(I);` I ideal

**Return:**      ideal defining the same zeroset as I: if any generator of I is a power of one single variable, replace it by the respective variable

**Example:**

```
LIB "arcpoint.lib";
ring r=0,(x,y,z),dp;
ideal I = x3,y+z2-x2;
I;
↪ I[1]=x3
↪ I[2]=-x2+z2+y
removepower(I);
↪ _[1]=x
↪ _[2]=-x2+z2+y
```

See also: [Section D.6.2.3 \[idealsimplify\]](#), page 1639.

### D.6.2.3 idealsimplify

Procedure from library `arcpoint.lib` (see [Section D.6.2 \[arcpoint\\_lib\]](#), page 1637).

**Usage:**      `idealsimplify(I,m);` I ideal, m int

**Assume:**      procedure is stable for sufficiently large m

**Return:**      ideal defining the same zeroset as I: replace generators of I by the generator modulo other generating elements

**Example:**

```
LIB "arcpoint.lib";
ring r=0,(x,y,z),dp;
ideal I = x3,y+z2-x2;
I;
↪ I[1]=x3
↪ I[2]=-x2+z2+y
idealsimplify(I,10);
↪ _[1]=x
↪ _[2]=z2+y
```

### D.6.2.4 equalJinI

Procedure from library `arcpoint.lib` (see [Section D.6.2 \[arcpoint\\_lib\]](#), page 1637).

**Usage:**      `equalJinI(I,J);` (I,J ideals)

**Assume:**      J contained in I and both I and J have been processed with `idealsimplify` before

**Return:**      1, if I=J, 0 otherwise

**Example:**

```
LIB "arcpoint.lib";
ring r=0,(x,y,z),dp;
ideal I = x,y+z2;
ideal J1= x;
ideal J2= x,y+z2;
equalJinI(I,J1);
↪ 0
```

```
equalJinI(I,J2);
⇨ 1
```

See also: [Section D.6.2.3 \[idealsimplify\]](#), page 1639.

### D.6.3 arnoldclassify\_lib

**Library:** arnoldClassify.lib

**Purpose:** Arnol'd Classifier of Singularities

**Author:** Eva Maria Hemmerling, ehemmerl@rhrk.uni-kl.de

**Overview:** A library for classifying isolated hypersurface singularities from the list of V.I. Arnol'd w.r.t. right equivalence up to corank 2. The method relies on Baciuc's list of Milnor codes and Newton's rotating ruler method to distinguish the Y- and Z- singularities.

**References:**

[AVG85] Arnold, Varchenko, Gusein-Zade: Singularities of Differentiable Maps. Vol. 1: The classification of critical points caustics and wave fronts. Birkhäuser, Boston 1985

[Bac01] Corina Baciuc: The classification of Hypersurface Singularities using the Milnor Code, Diplomarbeit, Universität Kaiserslautern, 2001.

[GP12] Greuel, Pfister: A SINGULAR Introduction to Commutative Algebra, Springer Science and Business Media, 2012

[Hem17] Eva Maria Hemmerling: Algorithmic Arnol'd Classification in SINGULAR, Master Thesis, TU Kaiserslautern, 2017.

**Procedures:** See also: [Section D.6.4 \[classify\\_lib\]](#), page 1646; [Section D.6.19 \[realclassify\\_lib\]](#), page 1746.

#### D.6.3.1 arnoldListAllSeries

Procedure from library arnoldclassify.lib (see [Section D.6.3 \[arnoldclassify\\_lib\]](#), page 1640).

**Usage:** arnoldListAllSeries();

**Return:** list of names of singularity series listed by Arnol'd up to corank 2

**Example:**

```
LIB "arnoldclassify.lib";
arnoldListAllSeries();
⇨ [1]:
⇨ A[k]
⇨ [2]:
⇨ D[k]
⇨ [3]:
⇨ E[6k]
⇨ [4]:
⇨ E[6k+1]
⇨ [5]:
⇨ E[6k+2]
⇨ [6]:
⇨ J[k,0]
⇨ [7]:
⇨ J[k,r]
```

```

⇒ [8] :
⇒      W[12k]
⇒ [9] :
⇒      W[12k+1]
⇒ [10] :
⇒      W[12k+5]
⇒ [11] :
⇒      W[12k+6]
⇒ [12] :
⇒      W[k,0]
⇒ [13] :
⇒      W[k,r]
⇒ [14] :
⇒      W#[k,2r]
⇒ [15] :
⇒      W#[k,2r-1]
⇒ [16] :
⇒      X[k,0]
⇒ [17] :
⇒      X[k,r]
⇒ [18] :
⇒      Y[k,r,s]
⇒ [19] :
⇒      Z[k,r]
⇒ [20] :
⇒      Z[k,r,s]
⇒ [21] :
⇒      Z[1,6r+11]
⇒ [22] :
⇒      Z[1,6r+12]
⇒ [23] :
⇒      Z[1,6r+13]
⇒ [24] :
⇒      Z[k,12k+6r]
⇒ [25] :
⇒      Z[k,12k+6r+1]
⇒ [26] :
⇒      Z[k,12k+6r-1]

```

### D.6.3.2 arnoldShowSeries

Procedure from library `arnoldclassify.lib` (see [Section D.6.3 \[arnoldclassify.lib\]](#), page 1640).

**Usage:**      `arnoldShowSeries( S );` S string

**Assume:**    S is the name of a singularity series listed by `arnoldListAllSeries()`.

**Return:**     data of the singularity series S of type `singseries` including

- Milnor number of S,
- Corank of S,
- Milnor code of S (see [Bac01]),
- normal form of S as string with parameters k,r,s and a,b,c,d,
- restrictions on parameters in the normal form in SINGULAR syntax,
- normal form with special (valid) parameters.

**Example:**

```

LIB "arnoldclassify.lib";
arnoldShowSeries("Z[k,12k+6r]");
⇒ Series=Z[k,12k+6r]
⇒ NormalForm=(x + a(y)*y^k)*(x^3 + x*y^(2*k+2*r+1) +
⇒          b(y)* y^(3*k+3*r+2))
⇒ SpecialForm=(x + y^k)*(x^3 + x*y^(2*k+2*r+1) +y^(3*k+3*r+2))
⇒ Modality=3*k+r-2
⇒ Corank=2
⇒ MilnorNumber=12*k+6r
⇒ MilnorCode=1,1,2k-1,2k-1+2*r,2k+2*r
⇒ Restrictions= (k>1)&&(r>=0)&&(deg(a)<=(k-2))&&(jet(a,0)!=0)&&
⇒          (jet(b,0)!=0)&&(deg(b)<=(2*k+r-2))

```

**D.6.3.3 arnoldNormalForm**

Procedure from library `arnoldclassify.lib` (see [Section D.6.3 \[arnoldclassify.lib\]](#), page 1640).

**Usage:** `arnoldNormalForm( S [, l]`), `S` string or singclass, `l` list

**Assume:** If `S` is of type string, then `S` is the name of a singularity series as listed by `arnoldListAllSeries()` and `l` may contain suitable integer parameters `k,r,s`. Otherwise `S` of type singclass is a singularity class from Arnol'd's list. Optional suitable polynomial parameters `a,b,c,d` can be appended to `l`. If `a,b,c,d` are not given, valid values are chosen automatically.

**Return:** string `NF` is the normal form of the series `S` if no parameters given, or poly `NF` is the normal form of the class `S` with parameters `k,r,s`.

**Example:**

```

LIB "arnoldclassify.lib";
ring R = 0, (x,y), ds;
poly a,b,c,d;
a= 1+y2;
c= 3+y;
int k = 5;
int r = 4;
int s = 3;
arnoldNormalForm ("W[12k+1]", k,r,s,a,b,c,d);
⇒ x4+x2y11+x2y13+xy16+3y22+y23
def f = _;
def sf = arnoldClassify( f );
arnoldNormalForm(sf, a,b,c,d);
⇒ x4+x2y11+x2y13+xy16+3y22+y23
arnoldNormalForm("W[12k+1]");
⇒ x4+x*y^(3*k+1)+y^(4*k+2)
arnoldNormalForm(sf);
⇒ x4+xy16+y22

```

**D.6.3.4 arnoldClassify**

Procedure from library `arnoldclassify.lib` (see [Section D.6.3 \[arnoldclassify.lib\]](#), page 1640).

**Usage:** `arnoldClassify (f)`; `f` poly



- Assume:** The basering is local of characteristic 0 and  $f$  defines an isolated singularity from Arnol'd's list of corank at most 2.
- Compute:** singularity class with respect to right equivalence and invariants used in the process of classification
- Return:** Singularity class of  $f$  of type `singclass` containing
- name of singularity series as listed by `arnoldListAllSeries()`,
  - name of singularity class,
  - parameters  $k, r, s$  defining the singularity class, -1 if not used,
  - modality, corank, Milnor number, determinacy,
  - Tjurina number, -2 if not computed, -1 if infinite,
  - Milnor code, -1 if not computed,
  - normal form of the singularity series from Arnol'd's list,
  - restrictions on parameters as string in SINGULAR syntax.

**Example:**

```
LIB "arnoldclassify.lib";
ring r = 0,(x,y),ds;
int k = random(3,10);
poly g = x4 + x2*y^(2*k+1)+x*y^(3*k+1)+ y^(4*k +1);
arnoldClassify(g);
⇒ Series=W[12k]
⇒ Class=W[60]
⇒ k=5
⇒ r=-1
⇒ s=-1
⇒ Modality=13
⇒ Corank=2
⇒ Milnor=60
⇒ Determinacy=29
⇒ Tjurina=-2
⇒ MilnorCode=1,1,10,9,9
⇒ NormalForm=x4+a(y)*x*y^(3*k+1)+c(y)*x^2*y^(2*k+1)+y^(4*k+1)
⇒ Restrictions=(k>=1)&&(k>1||a==0)&&(deg(a)<=(k-2))&&
  (deg(c)<=(2*k-2))
map phi=r,y-x^2,x+y;
phi(g);
⇒ y4-4x2y3+6x4y2-4x6y+x8+x11y2+11x10y3+55x9y4+165x8y5+330x7y6+462x6y7+462x5\
y8+330x4y9+165x3y10+55x2y11+11xy12+y13-2x13y-22x12y2-110x11y3-330x10y4-66\
0x9y5-924x8y6-924x7y7-660x6y8-330x5y9-110x4y10-22x3y11-2x2y12+x15+11x14y+\
55x13y2+165x12y3+330x11y4+462x10y5+462x9y6+330x8y7+165x7y8+55x6y9+11x5y10\
+x4y11+x16y+16x15y2+120x14y3+560x13y4+1820x12y5+4368x11y6+8008x10y7+11440\
x9y8+12870x8y9+11440x7y10+8008x6y11+4368x5y12+1820x4y13+560x3y14+120x2y15\
+16xy16+y17-x18-16x17y-120x16y2-560x15y3-1820x14y4-4368x13y5-8008x12y6-11\
440x11y7-12870x10y8-11440x9y9-8008x8y10-4368x7y11-1820x6y12-560x5y13-120x\
4y14-16x3y15-x2y16+x21+21x20y+210x19y2+1330x18y3+5985x17y4+20349x16y5+542\
64x15y6+116280x14y7+203490x13y8+293930x12y9+352716x11y10+352716x10y11+293\
930x9y12+203490x8y13+116280x7y14+54264x6y15+20349x5y16+5985x4y17+1330x3y1\
8+210x2y19+21xy20+y21
arnoldClassify(phi(g));
⇒ Series=W[12k]
⇒ Class=W[60]
⇒ k=5
```

```

⇒ r=-1
⇒ s=-1
⇒ Modality=13
⇒ Corank=2
⇒ Milnor=60
⇒ Determinacy=29
⇒ Tjurina=-2
⇒ MilnorCode=1,1,10,9,9
⇒ NormalForm=x4+a(y)*x*y^(3*k+1)+c(y)*x^2*y^(2*k+1)+y^(4*k+1)
⇒ Restrictions=(k>=1)&&(k>1||a==0)&&(deg(a)<=(k-2))&&
    (deg(c)<=(2*k-2))
ring C = (0,i), (x,y), ds;
minpoly = i2 + 1;
poly f =(x2+y2)^2+x5;
arnoldClassify(f);
⇒ Series=Y[1,r,s]
⇒ Class=Y[1,1,1]
⇒ k=1
⇒ r=1
⇒ s=1
⇒ Modality=1
⇒ Corank=2
⇒ Milnor=11
⇒ Determinacy=5
⇒ Tjurina=-2
⇒ MilnorCode=1,1,1,2,1
⇒ NormalForm= x^(4+r)+ a(y)*x2*y2 + y^(4+s)
⇒ Restrictions=(deg(a)==0)&&(jet(a,0)!=0)&&(1<=s)&&(s<=r)

```

### D.6.3.5 arnoldClassify\_to\_string

Procedure from library `arnoldclassify.lib` (see [Section D.6.3 \[arnoldclassify.lib\]](#), page 1640).

**Usage:** `arnoldClassify_to_string (f); f poly`

**Assume:** The basering is local of characteristic 0 and  $f$  defines an isolated singularity from Arnol'd's list of corank at most 2.

**Compute:** singularity class with respect to right equivalence and invariants used in the process of classification

**Return:** string: separated by |:

- name of singularity series as listed by `arnoldListAllSeries()`,
- name of singularity class,
- parameters  $k, r, s$  defining the singularity class, -1 if not used,
- modality, corank, Milnor number, determinacy,
- Tjurina number, -2 if not computed, -1 if infinite,
- Milnor code, -1 if not computed,
- normal form of the singularity series from Arnol'd's list,
- restrictions on parameters as string in SINGULAR syntax.

**Example:**

```

LIB "arnoldclassify.lib";
ring r = 0, (x,y), ds;
int k = random(3,10);

```

```

poly g = x4 + x2*y^(2*k+1)+x*y^(3*k+1)+ y^(4*k +1);
arnoldClassify_to_string(g);
↦ (k>=1)&&(k>1||a==0)&&(deg(a)<=(k-2))&&
↦ (deg(c)<=(2*k-2))|x4+a(y)*x*y^(3*k+1)+c(y)*x^2*y^(2*k+1)+y^(4*k+1\
)|1,1,10,9,9|-2|29|60|2|13|-1|-1|W[60]|W[12k]

```

### D.6.3.6 arnoldCorank

Procedure from library `arnoldclassify.lib` (see [Section D.6.3 \[arnoldclassify.lib\], page 1640](#)).

**Usage:** `arnoldCorank(f); f poly`

**Assume:** basering is local,  $f$  in `maxideal(2)` has isolated critical point at 0

**Return:** corank of the Hessian matrix of  $f$

**Example:**

```

LIB "arnoldclassify.lib";
ring r=0,(x,y,z),ds;
poly f=(x2+3y-2z)^2+xyz-(x-y3+x2*z3)^3;
arnoldCorank(f);
↦ 2

```

### D.6.3.7 arnoldDeterminacy

Procedure from library `arnoldclassify.lib` (see [Section D.6.3 \[arnoldclassify.lib\], page 1640](#)).

**Usage:** `arnoldDeterminacy( I[, m]); I poly or ideal, m int.`

**Assume:** the basering is local,  $I$  is the Jacobian ideal of a polynomial  $f$  with isolated critical point at 0,  $m$  is the Milnor number of  $f$

**Compute:** determinacy bound  $k$  for  $f$  w.r.t. right equivalence

**Return:** integer  $k$  s.th.  $f$  is right- $k$ -determined, -1 for infinity

**Note:** uses [Cor. A.9.7,GP12]

**Example:**

```

LIB "arnoldclassify.lib";
ring r=0,(x,y),ds;
poly f=x3+xy3;
ideal I=std(jacob(f));
int k=arnoldDeterminacy(I);
print(k);
↦ 5

```

### D.6.3.8 arnoldMilnorCode

Procedure from library `arnoldclassify.lib` (see [Section D.6.3 \[arnoldclassify.lib\], page 1640](#)).

**Usage:** `arnoldMilnorCode(f[,e]); f poly, e int`

**Assume:** basering is local,  $f$  has isolated critical point at 0

**Compute:** Milnor code of  $f$  consisting of the numbers of successive repetitions of coefficients of the 2nd Hilbert series of basering/ $(\text{jacob}(f)^e)$ , see [Bac01].

**Return:** Milnor code of  $f$  as `intvec` where  $e=1$  by default

**Example:**

```
LIB "arnoldclassify.lib";
ring r=0,(x,y,z),ds;
poly f=x2y+y3+z2;
arnoldMilnorCode(f);
↦ 1,1,1
arnoldMilnorCode(f,2);
↦ 1,1,2,1
// a big second argument may result in memory overflow
```

**D.6.3.9 arnoldMorseSplit**

Procedure from library `arnoldclassify.lib` (see [Section D.6.3 \[arnoldclassify.lib\]](#), page 1640).

**Usage:**      `arnoldMorseSplit(f); f poly`

**Assume:**    base ring is local,  $f$  in `maxideal(2)` has isolated critical point at 0

**Compute:**    result of Splitting Lemma applied to  $f$

**Return:**     polynomial  $g$  in `maxideal(3)` right equivalent to  $f$

**Example:**

```
LIB "arnoldclassify.lib";
ring r=0,(x,y,z),ds;
export r;
poly f=(x2+3y-2z)2+xyz-(x-y3+x2*z3)3;
poly g=arnoldMorseSplit(f);
g;
↦ 3/2x2y-y3
```

**D.6.4 classify.lib**

**Library:**    `classify.lib`

**Purpose:**     Arnold Classifier of Singularities

**Author:**    Kai Krueger, [krueger@mathematik.uni-kl.de](mailto:krueger@mathematik.uni-kl.de)

**Overview:** A library for classifying isolated hypersurface singularities w.r.t. right equivalence, based on the determinant of singularities by V.I. Arnold.

**Procedures:** See also: [Section D.6.19 \[realclassify.lib\]](#), page 1746.

**D.6.4.1 basicinvariants**

Procedure from library `classify.lib` (see [Section D.6.4 \[classify.lib\]](#), page 1646).

**Usage:**      `basicinvariants(f); f = poly`

**Compute:**    Compute basic invariants of  $f$ : an upper bound  $d$  for the determinacy, the milnor number  $\mu$  and the corank  $c$  of  $f$

**Return:**     `intvec: d,  $\mu$ ,  $c$`

**Example:**

```
LIB "classify.lib";
ring r=0,(x,y,z),ds;
basicinvariants((x2+3y-2z)2+xyz-(x-y3+x2*z3)3);
↦ 5,4,2
```

### D.6.4.2 classify

Procedure from library `classify.lib` (see [Section D.6.4 \[classify.lib\]](#), page 1646).

**Usage:** `classify(f); f=poly`

**Compute:** normal form and singularity type of  $f$  with respect to right equivalence, as given in the book "Singularities of differentiable maps, Volume I" by V.I. Arnold, S.M. Gusein-Zade, A.N. Varchenko

**Return:** normal form of  $f$ , of type `poly`

**Remark:** This version of `classify` is only beta. Please send bugs and comments to: "Kai Krueger" <krueger@mathematik.uni-kl.de>  
Be sure to have at least Singular version 1.0.1.

**Note:** type `init_debug(n);` ( $0 \leq n \leq 10$ ) in order to get intermediate information, higher values of  $n$  give more information.  
The proc creates several global objects with names all starting with `@`, hence there should be no name conflicts.

**Example:**

```
LIB "classify.lib";
ring r=0,(x,y,z),ds;
poly f=(x2+3y-2z)^2+xyz-(x-y3+x2*z3)^3;
classify(f);
⇒ About the singularity :
⇒      Milnor number(f)    = 4
⇒      Corank(f)           = 2
⇒      Determinacy         ≤ 5
⇒ Guessing type via Milnorcode:  D[k]=D[4]
⇒
⇒ Computing normal form ...
⇒ I have to apply the splitting lemma. This will take some time....:-)
⇒      Arnold step number 4
⇒ The singularity
⇒      -x3+3/2xy2+1/2x3y-1/16x2y2+3x2y3
⇒ is R-equivalent to D[4].
⇒      Milnor number = 4
⇒      modality       = 0
⇒ 2z2+x2y+y3
init_debug(3);
⇒ Debugging level change from 0 to 3
classify(f);
⇒ Computing Basicinvariants of f ...
⇒ About the singularity :
⇒      Milnor number(f)    = 4
⇒      Corank(f)           = 2
⇒      Determinacy         ≤ 5
⇒ Hcode: 1,2,1,0,0
⇒ Milnor code : 1,1,1
⇒ Debug:(2): entering HKclass3_teil_1 1,1,1
⇒ Debug:(2): finishing HKclass3_teil_1
⇒ Guessing type via Milnorcode:  D[k]=D[4]
⇒
⇒ Computing normal form ...
```

```

⇒ I have to apply the splitting lemma. This will take some time....:-)
⇒ Debug:(3): Split the polynomial below using determinacy: 5
⇒ Debug:(3): 9y2-12yz+4z2-x3+6x2y-4x2z+xyz+x4+3x2y3
⇒ Debug:(2): Permutations: 3,2,1
⇒ Debug:(2): Permutations: 3,2,1
⇒ Debug:(2): rank determined with Morse rg= 1
⇒ Residual singularity f= -x3+3/2xy2+1/2x3y-1/16x2y2+3x2y3
⇒ Step 3
⇒ Arnold step number 4
⇒ The singularity
⇒ -x3+3/2xy2+1/2x3y-1/16x2y2+3x2y3
⇒ is R-equivalent to D[4].
⇒ Milnor number = 4
⇒ modality = 0
⇒ Debug:(2): Decode:
⇒ Debug:(2): S_in= D[4] s_in= D[4]
⇒ Debug:(2): Looking for Normalform of D[k] with (k,r,s) = ( 4 , 0 , 0 )
⇒ Debug:(2): Opening Singularity-database:
⇒ DBM: NFlist
⇒ Debug:(2): DBMread( D[k] )= x2y+y(k-1) .
⇒ Debug:(2): S= f = x2y+y(k-1); Tp= x2y+y(k-1) Key= I_D[k]
⇒ Polynom f= x2y+y3 crk= 2 Mu= 4 MlnCd= 1,1,1
⇒ Debug:(2): Info= x2y+y3
⇒ Debug:(2): Normal form NF(f)= 2*x(3)^2+x(1)^2*x(2)+x(2)^3
⇒ 2z2+x2y+y3

```

### D.6.4.3 corank

Procedure from library `classify.lib` (see [Section D.6.4 \[classify.lib\]](#), page 1646).

**Usage:** `corank(f); f=poly`

**Return:** the corank of the Hessian matrix of `f`, of type `int`

**Remark:** `corank(f)` is the number of variables occurring in the residual singularity after applying 'morsesplit' to `f`

**Example:**

```

LIB "classify.lib";
ring r=0,(x,y,z),ds;
poly f=(x2+3y-2z)2+xyz-(x-y3+x2*z3)3;
corank(f);
⇒ 2

```

### D.6.4.4 Hcode

Procedure from library `classify.lib` (see [Section D.6.4 \[classify.lib\]](#), page 1646).

**Usage:** `Hcode(v); v=intvec`

**Return:** `intvec`, coding `v` according to the number of successive repetitions of an entry

**Example:**

```

LIB "classify.lib";
intvec v1 = 1,3,5,5,2;
Hcode(v1);

```

```

⇒ 1,0,1,0,2,0,0,1,0
intvec v2 = 1,2,3,4,4,4,4,4,4,3,2,1;
Hcode(v2);
⇒ 1,1,1,7,1,1,1

```

### D.6.4.5 init\_debug

Procedure from library `classify.lib` (see [Section D.6.4 \[classify.lib\], page 1646](#)).

**Usage:** `init_debug([level]); level=int`

**Compute:** Set the global variable `@DeBug` to level. The variable `@DeBug` is used by the function `debug_log(level, list of strings)` to know when to print the list of strings. `init_debug()` reports only changes of `@DeBug`.

**Note:** The procedure `init_debug(n)`; is useful as trace-mode. `n` may range from 0 to 10, higher values of `n` give more information.

**Example:**

```

LIB "classify.lib";
init_debug();
debug_log(1,"no trace information printed");
init_debug(1);
⇒ Debugging level change from 0 to 1
debug_log(1,"some trace information");
⇒ some trace information
init_debug(2);
⇒ Debugging level change from 1 to 2
debug_log(2,"nice for debugging scripts");
⇒ Debug:(2): nice for debugging scripts
init_debug(0);
⇒ Debugging switched off.

```

### D.6.4.6 internalfunctions

Procedure from library `classify.lib` (see [Section D.6.4 \[classify.lib\], page 1646](#)).

**Usage:** `internalfunctions();`

**Return:** nothing, display names of internal procedures of `classify.lib`

**Example:**

```

LIB "classify.lib";
internalfunctions();
⇒ Internal functions for the classification using Arnold's method,
⇒ the function numbers correspond to numbers in Arnold's classifier:
⇒ Klassifiziere(poly f); //determine the type of the singularity f
⇒ Funktion1bis (poly f, list cstn)
⇒ Funktion3 (poly f, list cstn)
⇒ Funktion6 (poly f, list cstn)
⇒ Funktion13 (poly f, list cstn)
⇒ Funktion17 (poly f, list cstn)
⇒ Funktion25 (poly f, list cstn)
⇒ Funktion40 (poly f, list cstn, int k)
⇒ Funktion47 (poly f, list cstn)
⇒ Funktion50 (poly f, list cstn)

```

```

⇒ Funktion58 (poly fin, list cstn)
⇒ Funktion59 (poly f, list cstn)
⇒ Funktion66 (poly f, list cstn)
⇒ Funktion82 (poly f, list cstn)
⇒ Funktion83 (poly f, list cstn)
⇒ Funktion91 (poly f, list cstn, int k)
⇒ Funktion92 (poly f, list cstn, int k)
⇒ Funktion93 (poly f, list cstn, int k)
⇒ Funktion94 (poly f, list cstn, int k)
⇒ Funktion95 (poly f, list cstn, int k)
⇒ Funktion96 (poly f, list cstn, int k)
⇒ Funktion97 (poly f, list cstn)
⇒ Isomorphie_s82_x (poly f, poly fk, int k)
⇒ Isomorphie_s82_z (poly f, poly fk, int k)
⇒ Isomorphie_s17 (poly f, poly fk, int k, int ct)
⇒ printresult (string f,string typ,int Mu,int m,int corank,int K)
⇒
⇒   Internal functions for the classification by invariants:
⇒   Cubic (poly f)
⇒   parity (int e)           //return the parity of e
⇒   HKclass (intvec i)
⇒   HKclass3( intvec i, string SG_Typ, int cnt)
⇒   HKclass3_teil_1 (intvec i, string SG_Typ, int cnt)
⇒   HKclass5 (intvec i, string SG_Typ, int cnt)
⇒   HKclass5_teil_1 (intvec i, string SG_Typ, int cnt)
⇒   HKclass5_teil_2 (intvec i, string SG_Typ, int cnt)
⇒   HKclass7 (intvec i, string SG_Typ, int cnt)
⇒   HKclass7_teil_1 (intvec i, string SG_Typ, int cnt)
⇒
⇒   Internal functions for the Morse-splitting lemma:
⇒   Morse(poly fi, int K, int corank) //splitting lemma itself
⇒   Coeffs (list #)
⇒   Coeff
⇒
⇒   Internal functions providing tools:
⇒   ReOrder(poly f)
⇒   Singularitaet(string typ,int k,int r,int s,poly a,poly b,poly c,poly d)
⇒   RandomPolyK
⇒   Faktorisiere(poly f, poly g, int p, int k)   compute g = (ax+by^k)^p
⇒   Teile(poly f, poly g);                       //divides f by g
⇒   GetRf(poly f, int n);
⇒   Show(poly f);
⇒   checkring();
⇒   DecodeNormalFormString(string s);
⇒   Setring(int n, string ringname);
⇒

```

#### D.6.4.7 milnorcode

Procedure from library `classify.lib` (see [Section D.6.4 \[classify.lib\]](#), page 1646).

**Usage:**     `milnorcode(f[e]); f=poly, e=int`



**Return:** intvec, coding the Hilbert function of the  $e$ -th Milnor algebra of  $f$ , i.e. of  $\text{basering}/(\text{jacob}(f)^e)$  (default  $e=1$ ), according to `proc Hcode`

**Example:**

```
LIB "classify.lib";
ring r=0,(x,y,z),ds;
poly f=x2y+y3+z2;
milnorcode(f);
↪ 1,1,1
milnorcode(f,2); // a big second argument may result in memory overflow
↪ 1,0,1,0,2,0,0,1,0
```

### D.6.4.8 morsesplit

Procedure from library `classify.lib` (see [Section D.6.4 \[classify.lib\]](#), page 1646).

**Usage:** `morsesplit(f); f=poly`

**Return:** Normal form of  $f$  in  $M^3$  after application of the splitting lemma

**Compute:** apply the splitting lemma (generalized Morse lemma) to  $f$

**Example:**

```
LIB "classify.lib";
ring r=0,(x,y,z),ds;
export r;
init_debug(1);
↪ Debugging level is set to 1
poly f=(x2+3y-2z)^2+xyz-(x-y3+x2*z3)^3;
poly g=morsesplit(f);
↪ Residual singularity f= -x3+3/2xy2+1/2x3y-1/16x2y2+3x2y3
g;
↪ -x3+3/2xy2+1/2x3y-1/16x2y2+3x2y3
```

### D.6.4.9 quickclass

Procedure from library `classify.lib` (see [Section D.6.4 \[classify.lib\]](#), page 1646).

**Usage:** `quickclass(f); f=poly`

**Return:** Normal form of  $f$  in Arnold's list

**Remark:** try to determine the normal form of  $f$  by invariants, mainly by computing the Hilbert function of the Milnor algebra, no coordinate change is needed (see also `proc 'milnorcode'`).

**Example:**

```
LIB "classify.lib";
ring r=0,(x,y,z),ds;
poly f=(x2+3y-2z)^2+xyz-(x-y3+x2*z3)^3;
quickclass(f);
↪ Singularity R-equivalent to : D[k]=D[4]
↪ normal form : z2+x2y+y3
↪ z2+x2y+y3
```

### D.6.4.10 singularity

Procedure from library `classify.lib` (see [Section D.6.4 \[classify.lib\], page 1646](#)).

**Usage:**      `singularity(t, l);`  $t$ =string (name of singularity),  
                   $l$ =list of integers/polynomials (indices/parameters of singularity)

**Compute:**   get the singularity named by type  $t$  from the database. list  $l$  is as follows:  
                   $l = k [r [s [a [b [c [d]..]]]$ :  $k, r, s = \text{int}$   $a, b, c, d = \text{poly}$ .  
                  The name of the dbm-databasefile is: `NFlist.[dir,pag]`. The file is found in the current directory. If it does not exist, please run the script `MakeDBM` first.

**Return:**      Normal form and corank of the singularity named by type  $t$  and its index (indices)  $l$ .

**Example:**

```
LIB "classify.lib";
ring r=0,(x,y,z),(c,ds);
init_debug(0);
singularity("E[6k]",6);
↪ [1]:
↪   x3+xy13+y19
↪ [2]:
↪   2
singularity("T[k,r,s]", 3, 7, 5);
↪ [1]:
↪   x3+xyz+z5+y7
↪ [2]:
↪   3
poly f=y;
singularity("J[k,r]", 4, 0, 0, f);
↪ [1]:
↪   x3+x2y4+y13
↪ [2]:
↪   2
```

### D.6.4.11 A\_L

Procedure from library `classify.lib` (see [Section D.6.4 \[classify.lib\], page 1646](#)).

**Usage:**      `A_L(f);`  $f$  poly  
                  `A_L(s);`  $s$  string, the name of the singularity

**Compute:**   the normal form of  $f$  in Arnold's list of singularities in case 1, in case 2 nothing has to be computed.

**Return:**      `A_L(f)`: compute via 'milnorcode' the class of  $f$  and return the normal form of  $f$  found in the database.  
                  `A_L("name")`: get the normal form from the database for the singularity given by its name.

**Example:**

```
LIB "classify.lib";
ring r=0,(a,b,c),ds;
poly f=A_L("E[13]");
f;
↪ c2+a3+ab5+b8
```

```

A_L(f);
↳ Singularity R-equivalent to : E[6k+1]=E[13]
↳ normal form : c2+a3+ab5+b8
↳ c2+a3+ab5+b8

```

#### D.6.4.12 normalform

Procedure from library `classify.lib` (see [Section D.6.4 \[classify.lib\], page 1646](#)).

**Usage:**      `normalform(s); s=string`

**Return:**     Arnold's normal form of singularity with name `s`

**Example:**

```

LIB "classify.lib";
ring r=0,(a,b,c),ds;
normalform("E[13]");
↳ c2+a3+ab5+b8

```

#### D.6.4.13 debug\_log

Procedure from library `classify.lib` (see [Section D.6.4 \[classify.lib\], page 1646](#)).

**Usage:**      `debug_log(level,li); level=int, li=comma separated "message" list`

**Compute:**    `print "messages" if level>=@DeBug.`  
                 useful for user-defined trace messages.

**Example:**

```

LIB "classify.lib";
example init_debug;
↳ // proc init_debug from lib classify.lib
↳ EXAMPLE:
↳   init_debug();
↳   debug_log(1,"no trace information printed");
↳   init_debug(1);
↳ Debugging level change from 0 to 1
↳   debug_log(1,"some trace information");
↳ some trace information
↳   init_debug(2);
↳ Debugging level change from 1 to 2
↳   debug_log(2,"nice for debugging scripts");
↳ Debug:(2): nice for debugging scripts
↳   init_debug(0);
↳ Debugging switched off.
↳

```

See also: [Section D.6.4.5 \[init\\_debug\], page 1649](#).

#### D.6.4.14 swap

Procedure from library `classify.lib` (see [Section D.6.4 \[classify.lib\], page 1646](#)).

**Usage:**      `swap(a,b);`

**Return:**     `b,a` if `a,b` is the input (any type)

**Example:**

```
LIB "classify.lib";
swap("variable1","variable2");
⇨ variable2 variable1
```

#### D.6.4.15 modality

Procedure from library `classify.lib` (see [Section D.6.4 \[classify.lib\], page 1646](#)).

**Usage:**      `modality(f); f poly`

**Return:**    the modality of the singularity

**Example:**

```
LIB "classify.lib";
ring r = 0, (x,y,z), ds;
poly f = (x2+3y-2z)^2+xyz-(x-y3+x2z3)^3;
modality(f);
⇨ 0
```

#### D.6.4.16 complexSingType

Procedure from library `classify.lib` (see [Section D.6.4 \[classify.lib\], page 1646](#)).

**Usage:**      `complexSingType(f); f poly`

**Return:**    the type of the singularity as a string

**Example:**

```
LIB "classify.lib";
ring r = 0, (x,y,z), ds;
poly f = (x2+3y-2z)^2+xyz-(x-y3+x2z3)^3;
complexSingType(f);
⇨ D[4]
```

#### D.6.4.17 prepRealclassify

Procedure from library `classify.lib` (see [Section D.6.4 \[classify.lib\], page 1646](#)).

**Usage:**      `prepRealclassify(f); f poly`

**Return:**    a list, containing the modality of the singularity and the type of the singularity as a string  
This procedure is needed in `realclassify.lib` in order to avoid `classify()` being called more than once.

**Example:**

```
LIB "classify.lib";
ring r = 0, (x,y,z), ds;
poly f = (x2+3y-2z)^2+xyz-(x-y3+x2z3)^3;
prepRealclassify(f);
⇨ [1]:
⇨ 0
⇨ [2]:
⇨ D[4]
```

### D.6.5 classify2\_lib

**Library:** classify2.lib

**Purpose:** Classification of isolated singularities

**Authors:** Janko Boehm, email: boehm@mathematik.uni-kl.de  
 Magdaleen Marais, email: magdaleen.marais@up.ac.za  
 Gerhard Pfister, email: pfister@mathematik.uni-kl.de

**Overview:** We classify isolated singularities of corank  $\leq 2$  and modality  $\leq 2$  with respect to right-equivalence over the complex numbers according to Arnold's list. We determine the type and, for positive modality, the parameter.

V.I. Arnold has described normal forms and has developed a classifier for, in particular, all isolated hypersurface singularities over the complex numbers up to modality 2. Building on a series of 105 theorems, this classifier determines the type of the given singularity. However, for positive modality, this does not fix the right equivalence class of the singularity, since the values of the moduli parameters are not specified.

This library implements an alternative classification algorithm for isolated hypersurface singularities of corank and modality up to two. For a singularity given by a polynomial over the rationals, the algorithm determines its right equivalence class by specifying a polynomial representative in Arnold's list of normal forms. In particular, the algorithm also determines values for the moduli parameters.

The implementation is based on the paper

Janko Boehm, Magdaleen Marais, Gerhard Pfister: A Classification Algorithm for Complex Singularities of Corank and Modality up to Two, Singularities and Computer Algebra - Festschrift for Gert-Martin Greuel on the Occasion of his 70th Birthday, Springer 2017, <http://arxiv.org/abs/1604.04774>, [https://doi.org/10.1007/978-3-319-28829-1\\_2](https://doi.org/10.1007/978-3-319-28829-1_2)

There are functions for determining a normal form equation and for determining the complex type of the singularity.

Acknowledgements: This research was supported by the Staff Exchange Bursary Programme of the University of Pretoria, DFG SPP 1489, DFG TRR 195. The financial assistance of the National Research Foundation (NRF), South Africa, towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at are those of the author and are not necessarily to be attributed to the National Research Foundation, South Africa.

**Procedures:** See also: [Section D.6.4 \[classify\\_lib\]](#), page 1646; [Section D.6.19 \[realclassify\\_lib\]](#), page 1746.

#### D.6.5.1 complexClassify

Procedure from library `classify2.lib` (see [Section D.6.5 \[classify2\\_lib\]](#), page 1655).

**Usage:** `complexClassify(f1); f1 Poly`

**Assume:** `f1` is a bivariate polynomial defining a curve singularity at  $(0,0)$  of modality  $\leq 2$  and corank  $\leq 2$ . The ordering of the basering must be local.

**Return:** A normal form equation `g` for `f1` represented by an element of type `NormalFormEquation`.

Note: So far the final scaling step is not implemented, so to obtain a normal form equation it may be necessary to do a transformation of the form  $x \rightarrow \lambda_1 x$ ,  $y \rightarrow \lambda_2 y$

Note: Case X9 is not implemented yet. In this case an error is returned stating that it is case X9.

`g.normalFormEquation.in` stores the polynomial ring containing the normal form equation.

`g.normalFormEquation.value` is the normal form equation.

To access the normal form equation as a polynomial do:

```
def S = g.normalFormEquation.in;
setring S;
poly nf = g.normalFormEquation.value;
```

#### Example:

```
LIB "classify2.lib";
ring R = 0,(x,y),ds;
Poly f = -16065*y^9*x-5103*y^9-128*x^7-595*y^9*x^4+6*y^5-21*y^9*x^5-2187*y^7+4*x^2*y^9;
NormalFormEquation F = complexClassify(f);
F;
↳ Corank = 2
↳ Normalform equation of type = Y[7,8]
↳ Normalform equation = x2y2-x7+y8
↳ Milnor number = 16
↳ Modality = 1
↳ Parameter term = x2y2
↳ Determinacy <= 8
↳
def S = F.normalFormEquation.in;
setring S;
poly nf = F.normalFormEquation.value;
```

### D.6.5.2 complexType

Procedure from library `classify2.lib` (see [Section D.6.5 \[classify2.lib\]](#), page 1655).

**Usage:** `complexType(f); fl Poly`

**Assume:** `fl` is a bivariate polynomial defining a curve singularity at (0,0) of modality and corank  $\leq 2$ . The ordering of the basering must be local.

**Return:** A list with the complex type of `fl` and the modality.

#### Example:

```
LIB "classify2.lib";
ring R = 0,(x,y),ds;
Poly f = -16065*y^9*x-5103*y^9-128*x^7-595*y^9*x^4+6*y^5-21*y^9*x^5-2187*y^7+4*x^2*y^9;
complexType(f);
```

```

⇒ [1] :
⇒      Y[7,8]
⇒ [2] :
⇒      1

```

### D.6.6 classify\_aeq\_lib

**Library:** classifyAeq.lib

**Purpose:** Simple Space Curve singularities in characteristic 0

**Authors:** Faira Kanwal Janjua fairakanwaljanjua@gmail.com  
 Gerhard Pfister pfister@mathematik.uni-kl.de Khawar Mehmood  
 khawar1073@gmail.com

**Overview:** A library for classifying the simple singularities with respect to A equivalence in characteristic 0. Simple Surface singularities in characteristic 0 have been classified by Bruce and Gaffney [4] resp. Gibson and Hobbs [1] with respect to A equivalence. If the input is one of the simple singularities in [1] it returns a normal form otherwise a zero ideal (i.e not simple).

**References:**

- [1] Gibson, C.G; Hobbs, C.A.: Simple Singularities of Space Curves. Math.Proc. Comb.Phil.Soc.(1993), 113, 297.
- [2] Hefez, A; Hernandez, M.E.: Standard bases for local rings of branches and their modules of differentials. Journal of Symbolic Computation 42(2007) 178-191.
- [3] Hefez, A; Hernandez, M.E.: The Analytic Classification Of Plane Branches. Bull.Lond Math Soc. 43.(2011) 2, 289-298.
- [4] Bruce, J.W., Gaffney, T.J.: Simple singularities of mappings  $(C, 0) \rightarrow (C^2, 0)$ . J. London Math. Soc. (2) 26 (1982), 465-474.
- [5] Ishikawa, G; Janeczko, S.: The Complex Symplectic Moduli Spaces of Unimodal Parametric Plane Curve Singularities. Institute of Mathematics of the Polish Academy of Sciences, Preprint 664(2006)

**Procedures:**

#### D.6.6.1 sagbiAlg

Procedure from library `classify_aeq.lib` (see [Section D.6.6 \[classify\\_aeq.lib\]](#), page 1657).

**Return:** An ideal. The sagbi bases of I.

**Example:**

```

LIB "classify_aeq.lib";
ring R=0,t,ds;
ideal I=t8,t10+t13,t12+t15;
sagbiAlg(I);
⇒ _[1]=t8
⇒ _[2]=t10+t13
⇒ _[3]=t12+t15
⇒ _[4]=t23-t29
⇒ _[5]=t27
I=t8,t10+t13,t12+2t15;
sagbiAlg(I);
⇒ _[1]=t8

```

```

⇒ _[2]=t10+t13
⇒ _[3]=t12+2t15
⇒ _[4]=t27-3t33
⇒ _[5]=t29

```

### D.6.6.2 sagbiMod

Procedure from library `classify_aeq.lib` (see [Section D.6.6 \[classify\\_aeq.lib\]](#), page 1657).

**Return:** An ideal. the sagbi bases for the differential module.

**Example:**

```

LIB "classify_aeq.lib";
ring r=0,t,Ds;
ideal G=t8,t10+t13,t12+t15,t23-t29,t27;
ideal I=diff(G,t);
sagbiMod(I,G);
⇒ _[1]=t7
⇒ _[2]=t9+13/10t12
⇒ _[3]=t11+5/4t14
⇒ _[4]=t20
⇒ _[5]=t22-29/23t28
⇒ _[6]=t24
⇒ _[7]=t26
⇒ _[8]=t28

```

### D.6.6.3 semiGroup

Procedure from library `classify_aeq.lib` (see [Section D.6.6 \[classify\\_aeq.lib\]](#), page 1657).

**Return:** list L; list with three entries associated to the algebra generated by the sagbi basis:  
 generators of the semigroup  
 the conductor  
 the semigroup

**Example:**

```

LIB "classify_aeq.lib";
ring R=0,t,ds;
ideal I=t8,t10+t13,t12+t15,t23-t29,t27;
semiGroup(I);
⇒ [1]:
⇒ 8,10,12,23,27
⇒ [2]:
⇒ 30
⇒ [3]:
⇒ 0,8,10,12,16,18,20,22,23,24,26,27,28,30
I=t8,t10+t13,t12+2t15,t27-3t33,t29;
semiGroup(I);
⇒ [1]:
⇒ 8,10,12,27,29
⇒ [2]:
⇒ 34
⇒ [3]:
⇒ 0,8,10,12,16,18,20,22,24,26,27,28,29,30,32,34

```



### D.6.6.4 semiMod

Procedure from library `classify_aeq.lib` (see [Section D.6.6 \[classify\\_aeq.lib\], page 1657](#)).

**Return:** list K;  
           K[1]min generators of the semialgebra.  
           K[2]conductor of the algebra.  
           K[3]generators for the semialgebra.  
           K[4]min generators of the module.  
           K[5]conductor of the module.  
           K[6]semigroup of the module.

**Example:**

```
LIB "classify_aeq.lib";
ring r=0,t,Ds;
ideal G=t4,t7+t10;
ideal I=diff(G,t);
ideal k=sagbiMod(I,G);
semiMod(k,G);
⇒ [1]:
⇒ 4,7
⇒ [2]:
⇒ 18
⇒ [3]:
⇒ 0,4,7,8,11,12,14,15,16,18
⇒ [4]:
⇒ 3,6,16
⇒ [5]:
⇒ 13
⇒ [6]:
⇒ 3,6,7,10,11,13
```

### D.6.6.5 planeCur

Procedure from library `classify_aeq.lib` (see [Section D.6.6 \[classify\\_aeq.lib\], page 1657](#)).

**Return:** An ideal. Ideal is one of the singularity in the list of Bruce and Gaffney [4]

**Example:**

```
LIB "classify_aeq.lib";
ring R=0,t,Ds;
ideal I=t4+4t5+6t6+8t7+13t8+12t9+10t10+12t11+6t12+4t13+4t14+t16,t7+7t8+22t9+51t10+11t11+1828t18+2011t19+2165t20+2163t21+1982t22+1806t23+1491t24+1141t25+889t26+588t27+379t28;
planeCur(I);
⇒ // ** redefining K (intvec q=4,7;list K;) classify_aeq.lib\
b::planeCur:155
⇒ _[1]=t4
⇒ _[2]=t7+t9
```

### D.6.6.6 spaceCur

Procedure from library `classify_aeq.lib` (see [Section D.6.6 \[classify\\_aeq.lib\], page 1657](#)).

**Return:** an ideal. Ideal is one of the singularity in the list of C.G.Gibson and C.A.Hobbs.

**Example:**

```

LIB "classify_aeq.lib";
ring R=0,t,Ds;
ideal I=t3+3t4+3t5+t6,t13+14t14+92t15+377t16+1079t17+2288t18+3718t19+4719t20+4719t21;
spaceCur(I);
↪ _[1]=t3
↪ _[2]=t13+t14
↪ _[3]=t17

```

### D.6.6.7 HHnormalForm

Procedure from library `classify_aeq.lib` (see [Section D.6.6 \[classify\\_aeq.lib\], page 1657](#)).

**Usage:** `HHnormalForm(I)`,  $I=\langle x(t), y(t) \rangle$  an ideal, `#` optional, a bound for the conductor

**Compute:** computes the Hefez-Hernandez normal form of the ideal  $I=\langle x(t), y(t) \rangle$

**Return:** a list `L` with 5 entries

`L[1]` the Hefei-Hernandez normal form of the ideal  $I=\langle x(t), y(t) \rangle$  `L[2]` the semi group `Gamma`  
`L[3]` the semi module `Lambda`  
`L[4]` 0 if `Lambda=Gamma` or `lambda=min(Lambda-Gamma)` -n, the Zariski number  
`L[5]` the integers `i`,  $i > \text{lambda}$ ,  $i+n$  not in `Lambda`, 0 if this set is empty

**Example:**

```

LIB "classify_aeq.lib";
ring r=0,t,Ds;
ideal I=t6+3t7,t8+t13+t19;
HHnormalForm(I);
↪ [1]:
↪   _[1]=t6
↪   _[2]=t8-4t9+425/6t11-823/32t13
↪ [2]:
↪   0,6,8,12,14,16,18,20,22,24,25,26,28,30,31,32,33,34,36
↪ [3]:
↪   0,6,8,12,14,15,16,18,20
↪ [4]:
↪   9
↪ [5]:
↪   11,13

```

### D.6.6.8 modSagbiAlg

Procedure from library `classify_aeq.lib` (see [Section D.6.6 \[classify\\_aeq.lib\], page 1657](#)).

**Usage:** `modSagbiAlg(I,i)`;  $I$  is an ideal,  $i$  an integer (optional).

**Return:** ideal, a sagbi basis for  $I$  using modular methods.

**Purpose:** Computes a sagbi basis for the ideal given by the generators in  $I$  using modular techniques.  
 If second argument is 0 then the result is not verified.

**Example:**

```

LIB "classify_aeq.lib";
ring R=0,t,ds;
ideal I=t8,t10+t13,t12+t15;

```

```

modSagbiAlg(I,0);
↪ _[1]=t8
↪ _[2]=t10+t13
↪ _[3]=t12+t15
↪ _[4]=t23-t29
↪ _[5]=t27
modSagbiAlg(I);
↪ _[1]=t8
↪ _[2]=t10+t13
↪ _[3]=t12+t15
↪ _[4]=t23-t29
↪ _[5]=t27

```

### D.6.6.9 classSpaceCurve

Procedure from library `classify_aeq.lib` (see [Section D.6.6 \[classify\\_aeq.lib\]](#), page 1657).

**Return:** the normal form of  $I$  if  $I$  is simple, -1 if  $I$  is not simple

**Example:**

```

LIB "classify_aeq.lib";
ring R=31,t,ds;
ideal I=t3+6t4+13t5-13t6+10t7+2t8-6t9-10t10-15t11-6t12+8t13-2t14+t15+8t16,
t7+15t8+7t9-11t10-15t11-6t12+8t13-2t14+t15+8t16,
t10-11t11-6t12-t13+12t14+4t15-14t16+15t17-12t18+5t19+t20;
ideal J=classSpaceCurve(I);
J;
↪ J[1]=t3
↪ J[2]=t7+t8
↪ J[3]=t11

```

### D.6.7 classifyceq.lib

**Library:** `classifyCeq.lib`

**Purpose:** simple hypersurface singularities in characteristic  $p > 0$

**Authors:** Deeba Afzal [deebafzal@gmail.com](mailto:deebafzal@gmail.com)  
 Faira Kanwal Janjua [fairakanwaljanjua@gmail.com](mailto:fairakanwaljanjua@gmail.com)

**Overview:** A library for classifying the simple singularities with respect to contact equivalence in characteristic  $p > 0$ . Simple hypersurface singularities in characteristic  $p > 0$  were classified by Greuel and Kroening [1] with respect to contact equivalence. The classifier we use has been proposed in [2].

**References:**

- [1] Greuel, G.-M.; Kroening, H.: Simple singularities in positive characteristic. *Math.Z.* 203, 339-354 (1990).
- [2] Afzal, D.; Binyamin, M.A.; Janjua, F.K.: On the classification of simple singularities in positive characteristic.

**Procedures:**

### D.6.7.1 classifyCeq

Procedure from library `classifyceq.lib` (see [Section D.6.7 \[classifyceq.lib\]](#), page 1661).

**Usage:** `classifyCeq(f); f poly`

**Return:** string including the Tjurina number of  $f$  and its type in the classification of Greuel and Kroening

**Example:**

```
LIB "classifyceq.lib";
ring R=3,(x,y,z,u,v,w),ds;
classifyCeq(-x2+xy+y2+xz-yz-z2+w2+u3+v4);
⇒ The given function defines a simple singularity.
⇒ The tjurina number is 9.
⇒ E^0[6]:x^3+y^4
```

### D.6.8 classifyci.lib

**Library:** `classifyci.lib`

**Purpose:** Isolated complete intersection singularities in characteristic 0

**Authors:** Gerhard Pfister [pfister@mathematik.uni-kl.de](mailto:pfister@mathematik.uni-kl.de)  
Deeba Afzal [deebafzal@gmail.com](mailto:deebafzal@gmail.com)

**Overview:** A library for classifying isolated complete intersection singularities for the base field of characteristic 0 and for computing weierstrass semigroup of the space curve. Isolated complete intersection singularities were classified by M. Giusti [1] for the base field of characteristic 0. Algorithm for the semigroup of a space curve singularity is given in [2].

**References:**

- [1] Giusti, M: Classification des singularities isolees simples d'intersections completes, C.R. Acad. Sci. Paris Ser. A-B 284(1977), 167-169.
- [2] Castellanos, A., Castellanos, J., 2005: Algorithm for the semigroup of a space curve singularity. Semigroup Forum 70, 44-66.

**Procedures:**

#### D.6.8.1 classifyicis

Procedure from library `classifyci.lib` (see [Section D.6.8 \[classifyci.lib\]](#), page 1662).

**Usage:** `classifyicis(I); I ideal`

**Assume:**  $I$  is given by two generators

**Purpose:** Check whether the ideal defines a complete intersection singularity or not

**Return:** String type in the classification of Giusti, M  
or The given singularity is not simple

**Example:**

```
LIB "classifyci.lib";
ring R=0,(x,y,z),ds;
ideal I=x2+yz,xy+z4;
classifyicis(I);
```

```

⇒ // ** dim(_) may be wrong because the mixed monomial ordering
⇒ // ** dim(_) may be wrong because the mixed monomial ordering
⇒ // ** dim(I) may be wrong because the mixed monomial ordering
⇒ // ** dim(I) may be wrong because the mixed monomial ordering
⇒ U_9: (x2+yz,xy+z4)

```

### D.6.8.2 Semigroup

Procedure from library `classifysi.lib` (see [Section D.6.8 \[classifysi.lib\]](#), page 1662).

**Usage:**      `Semigroup(I)`; I is an ideal

**Purpose:**     Computes the semigroup of the ideal I corresponding to each branch

**Return:**    list of semigroup of ideal I corresponding to each branch

**Example:**

```

LIB "classifysi.lib";
ring R=0,(x,y,z),ds;
ideal I=x2+y3+z5,yz;
Semigroup(I);
⇒ [1]:
⇒    2,3
⇒ [2]:
⇒    2,5

```

### D.6.9 classifyMapGerms.lib

**Library:**    `classifyMapGerms.lib`

**Authors:**    Gerhard Pfister, pfister@mathematik.uni-kl.de  
 Deeba Afzal, deebafzal@gmail.com  
 Shamsa Kanwal, lotus\_zone16@yahoo.com

**Overview:** A library for computing the standard basis of the tangent space at the orbit of an algebraic group action. The tangent space is usually described as the sum of two modules over different rings. It computes the standard basis using modular methods and parallel modular methods. It also computes the normal form of the germ given by Riegers classification.

**References:**

- [1] Idrees N.; Pfister, G.; Steidel, S.: Parallelization of modular algorithms. J. Symbolic Comput. 46(2011), no. 6, 672-684.
- [2] Gibson, C.G; Hobbs, C.A.: Simple Singularities of Space Curves. Math.Proc. Comb.Phil.Soc.(1993), 113, 297.
- [3] Bruce, J.W., Gaffney, T.J.: Simple singularities of mappings  $(C, 0) \rightarrow (C^2, 0)$ . J. London Math. Soc. (2) 26 (1982), 465-474.
- [4] Rieger, J. H.: Families of maps from the plane to the plane. J. London Math. Soc. (2) 36(1987), no. 2. 351-369.

**Procedures:**

#### D.6.9.1 coDimMap

Procedure from library `classifyMapGerms.lib` (see [Section D.6.9 \[classifyMapGerms.lib\]](#), page 1663).

**Usage:** `coDimMap(I, #);`  $I$ =ideal,  $\#$ =list

**Compute:** a bound of the A-determinacy of the map germ defined by  $I$ .

**Return:** a list in which 1st entry gives the bound of the A-determinacy and the second entry gives the codimension of the map germ defined by  $I$ .

**Note:** if  $\#$  is empty it computes  $A^e$ -codimension(the extended codimension).

**Example:**

```
LIB "classifyMapGerms.lib";
ring R=0,(x,y),(c,ds);
poly f1=x;
poly f2=xy+y5+y7;
poly f11=f1+f2*f1;
poly f22=f2+f1^2;
map phi=basing,x+y,y+y2;
f1=phi(f11);
f2=phi(f22);
ideal I=f1,f2;
coDimMap(I);
↳ [1]:
↳ 14
↳ [2]:
↳ 2
coDimMap(I,1);
↳ [1]:
↳ 15
↳ [2]:
↳ 4
```

### D.6.9.2 coDim

Procedure from library `classifyMapGerms.lib` (see [Section D.6.9 \[classifyMapGerms.lib\]](#), [page 1663](#)).

**Usage:** `coDim(module M, module N, ideal I,int bound, list #);`  $M$  is a submodule in  $A^r$  over the basering  $=:A$ ,  $N$  is a submodule in  $R^r$  over the subring  $R$  of the basering generated by the entries of  $I$

**Compute:** computes the  $K$ -vectorspace dimension of  $A^r/M+N+\maxideal(bound)*A^r$

**Return:** an integer

**Note:** if  $\#$  is not empty the bound is corrected by computing determinacy

**Example:**

```
LIB "classifyMapGerms.lib";
ring R=0,(x,y),(c,ds);
poly f1=x;
poly f2=xy+y5+y7;
poly f11=f1+f2*f1;
poly f22=f2+f1^2;
map phi=basing,x+y,y+y2;
f1=phi(f11);
f2=phi(f22);
ideal I=f1,f2;
```

```

module M=maxideal(1)*jacob(I);
module N=I*freemodule(2);
coDim(M,N,I,15);
↦ 6

```

### D.6.9.3 vStd

Procedure from library `classifyMapGerms.lib` (see [Section D.6.9 \[classifyMapGerms.lib\]](#), [page 1663](#)).

**Usage:** `vStd(M, N, I, bound)`;  $M$  is a submodule in  $A^r$  over the basering  $=:A$ ,  $N$  is a submodule in  $R^r$  over the subring  $R$  of the basering generated by the entries of  $I$

**Compute:** a standard basis of  $M+N+\maxideal(bound)*A^r$

**Return:** a list whose  $i$ th entry gives a list where each entry is a generator of standard basis and second entry gives a list of generators after the reduced echelon form

**Example:**

```

LIB "classifyMapGerms.lib";
ring R=0,(x,y),(c,ds);
poly f1=x;
poly f2=xy+y5+y7;
poly f11=f1+f2*f1;
poly f22=f2+f1^2;
map phi=basing,x+y,y+y2;
f1=phi(f11);
f2=phi(f22);
ideal I=f1,f2;
module M=maxideal(1)*jacob(I);
module N=I*freemodule(2);
vStd(M,N,I,15);
↦ [1]:
↦ _[1]=[x+x3+4x2y+3xy2+2x3y+6x2y2+4xy3+5x2y4+6xy5+30x2y5+35xy6+77x2y6+88\
xy7+136x2y7+153xy8+234x2y8+260xy9+360x2y9+396xy10+385x2y10+420xy11+252x2y\
11+273xy12+91x2y12+98xy13+14x2y13+15xy14,3x2+4xy+2x2y+3xy2+5xy4+30xy5+77x\
y6+136xy7+234xy8+360xy9+385xy10+252xy11+91xy12+14xy13]
↦ _[2]=[y+x2y+4xy2+3y3+2x2y2+6xy3+4y4+5xy5+6y6+30xy6+35y7+77xy7+88y8+136\
xy8+153y9+234xy9+260y10+360xy10+396y11+385xy11+420y12+252xy12+273y13+91xy\
13+98y14+14xy14+15y15,3xy+4y2+2xy2+3y3+5y5+30y6+77y7+136y8+234y9+360y10+3\
85y11+252y12+91y13+14y14]
↦ _[3]=[0,36200241918436258666427374984354490488701583014810763500799640\
217463646618220452x2+8035856076792905437341008252258785899893543992700127\
4879285662359088597883320904xy+441583188494927957069827075382333685102338\
56912190511378486022141624951265100452y2-12168541735238412614834127370934\
22663372594795185815987872258114391555856973312x3-80128290556229107240472\
6095028720713615485988824254793406334032089396045463164633x2y-15945044934\
85343790469392100337122599592789362267972267053195631528816518567376530xy\
2-79443844209657655932614941804549530864067596823890328963473385755381202\
8961185209y3-724004838368725173328547499687089809774031660296215270015992\
80434927293236440904x4-30229072594889590776353928323077564115059030720231\
2591705168249748322331110851580x3y-64037256345646994117946972694997382287\
2311437350694996140858866680579218287870890x2y2-5323798636463221386247075\
73505133023277487167385104415478660852951023493207985956xy3-1218975423018\
75587875922379817225860578362871207100484041370955583839312794525742y4+24\

```

3370834704768252296682547418684532674518959037163197574451622878311171394\  
6624x5+160621637364515366859390242826872169522108976203406703477628483852\  
1966758497249202x4y+56515983032074452945623849696893455891632205234509696\  
72239871636933016905394448563x3y2+960858204298224279213338534542517287703\  
7924552091271096969608386803550323301921788x2y3+8558235582733260502366947\  
804919395067309048344239526133030288583921823321576965835xy4+303725956231\  
5639703927788489158427319647661112642428044888443421378912780120590032y5+\  
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4681938840749151389529122944x5y4+1268094120866087493081378876883927577098\  
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846402432839759113247786083098221134062815424310614057471x3y6+72310569966\  
133150794172076258509553480300124405678811107513872235037389583475861507x\  
2y7+479146133456814092788747446518150779366321978746645097706516801034958\  
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```

768286605765675723023y12+210419076104515403420153886280080566991346374983\
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y2-21303065164905409355910746821643349934000694346271714545358765232x3-12\
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73543381794238558879609013100650724098477681742811225092640516152xy2-1241\
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6949301545552350059765341231800616x5y4+4276446948871629699025913413360173\
3446463415269456118348670977957610x4y5+6488339452626821470939781957813243\
22388404747014315457004007505629741x3y6+130991807505346256819574340818231\
0230916280920024139066689143912165033x2y7+8508146842786427885159089788549\

```

```

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65227693545610182811920959256903157117y9+39881361062399044562540709872293\
7743188691892992327539959010781616x5y5+3212710065978859005115948578562282\
67991100718643383365605514894710444x4y6+178528239307575411827078182979564\
2643866951489787329489947684852740879x3y7+2664786673159464299081021373204\
820967665477618032556009617347981304943x2y8+13128671107962111960414806946\
28798927861719700512669052046486656739745xy9+1131872348818182222038173431\
09611046466620928935984686139642392021853y10-1994068053119952228127035493\
61468871594345946496163769979505390808x4y7-164194064491321758357747609358\
226636892595480471921070228562886986384x3y8-87307474644040614422647345474\
0024461613888740838954751325409404553200x2y9-1281101492267754833486875242\
467723120825769959117798884375885275683345xy10-56610481457847208304636080\
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4653259856402227059134963120205061334644400x3y10+693712669211924509432239\
69818501233031933209624614816341277674789936x2y11-21796077810439590846040\
88846772531793073092013249389053280019350781109xy12-209450712981380244886\
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298563451821524423499661653123317256140270862832x3y11-1699476738863209814\
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64203237724900368360311357267361567094896515055823975160x3y12+39561393186\
8229667642625657783981682236828187985466575178797954569236x2y13-241810069\
54550610168202557367476669472219588419182001193498265782381544xy14-226448\
69248012390569132461126480909721726229549315543455587071226705994y15]
⇒ _[5]=[0,y15]
⇒ [2]:
⇒ _[1]=[0,y14]
⇒ _[2]=[0,y13+483/16y14]
⇒ _[3]=[0,y12+4484/375y13+8614/125y14]
⇒ _[4]=[0,y11+11y12+58y13+363/2y14]
⇒ _[5]=[0,y10+10y11+48y12+287/2y13+991/4y14]
⇒ _[6]=[0,y9+9y10+389/10y11+2193/20y12+1063/5y13+3781/20y14]
⇒ _[7]=[0,y8+8y9+154/5y10+84y11+4949/25y12+10188/25y13+16034/25y14]
⇒ _[8]=[0,y7+89/2y8+321y9+1140y10+2685y11+10109/2y12+8409y13+14614y14]
⇒ _[9]=[0,y6+6y7+33/2y8+63/4y9-357/4y10-1065/2y11-1609y12-13131/4y13-807\
9/2y14]
⇒ _[10]=[0,y5+5y6+23/2y7+57/4y8-27/2y9-139y10-945/2y11-4823/4y12-5173/2y\
13-4657y14]
⇒ _[11]=[0,x+y+20y9+180y10+778y11+2318y12+5752y13+12636y14]

```

#### D.6.9.4 modVStd

Procedure from library `classifyMapGerms.lib` (see [Section D.6.9 \[classifyMapGerms.lib\]](#), [page 1663](#)).

**Usage:** `modVStd((M, N, I, bound, #); M is a submodule  
in  $A^r$  over the basering  $A$ , N is a submodule in  $R^r$  over the subring  $R$  of the  
basing generated by the entries of I`

**Compute:** a standard basis of  $A^r/M+N+\text{maxideal}(\text{bound})*A^r$  using modular version

**Return:** a list whose *ist* entry gives a list where each entry is a generator of standard basis and second entry gives a list of generators after the reduced echelon form

**Note:** if  $\#$  is not empty the bound is corrected by computing determinacy

**Example:**

```
LIB "classifyMapGerms.lib";
ring R=0,(x,y),(c,ds);
poly f1=x;
poly f2=xy+y5+y7;
poly f11=f1+f2*f1;
poly f22=f2+f1^2;
map phi=basing,x+y,y+y2;
f1=phi(f11);
f2=phi(f22);
ideal I=f1,f2;
module M=maxideal(1)*jacob(I);
module N=I*freemodule(2);
modVStd(M,N,I,15);
⇒ [1]:
⇒ _[1]=[x+x3+4x2y+3xy2+2x3y+6x2y2+4xy3+5x2y4+6xy5+30x2y5+35xy6+77x2y6+88\
xy7+136x2y7+153xy8+234x2y8+260xy9+360x2y9+396xy10+385x2y10+420xy11+252x2y\
11+273xy12+91x2y12+98xy13+14x2y13+15xy14,3x2+4xy+2x2y+3xy2+5xy4+30xy5+77x\
y6+136xy7+234xy8+360xy9+385xy10+252xy11+91xy12+14xy13]
⇒ _[2]=[y+x2y+4xy2+3y3+2x2y2+6xy3+4y4+5xy5+6y6+30xy6+35y7+77xy7+88y8+136\
xy8+153xy9+234xy9+260y10+360xy10+396y11+385xy11+420y12+252xy12+273y13+91xy\
13+98y14+14xy14+15y15,3xy+4y2+2xy2+3y3+5y5+30y6+77y7+136y8+234y9+360y10+3\
85y11+252y12+91y13+14y14]
⇒ _[3]=[0,x2+2029573260997425667835151746738174/914290180625267404551036\
030229087xy+1115283080372158263284115716509087/91429018062526740455103603\
0229087y2-30733436108312683243381117471872/914290180625267404551036030229\
087x3-8012829055622910724047260950287207136154859888242547934063340320893\
96045463164633/3620024191843625866642737498435449048870158301481076350079\
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3986133526597815764408259283688265/18100120959218129333213687492177245244\
350791507405381750399820108731823309110226xy2-794438442096576559326149418\
045495308640675968238903289634733857553812028961185209/362002419184362586\
66427374984354490488701583014810763500799640217463646618220452y3-2x4-7557\
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895/905006047960906466660684374608862262217539575370269087519991005436591\
1654555113x3y-32018628172823497058973486347498691143615571867534749807042\
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```

```

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918436258666427374984354490488701583014810763500799640217463646618220452x\
y4+7593148905789099259819471222896068299119152781606070112221108553447281\
95030147508/9050060479609064666606843746088622622175395753702690875199910\
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⇒ _[2]=[0,y13+483/16y14]
⇒ _[3]=[0,y12+4484/375y13+8614/125y14]
⇒ _[4]=[0,y11+11y12+58y13+363/2y14]
⇒ _[5]=[0,y10+10y11+48y12+287/2y13+991/4y14]
⇒ _[6]=[0,y9+9y10+389/10y11+2193/20y12+1063/5y13+3781/20y14]
⇒ _[7]=[0,y8+8y9+154/5y10+84y11+4949/25y12+10188/25y13+16034/25y14]
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⇒ _[9]=[0,y6+6y7+33/2y8+63/4y9-357/4y10-1065/2y11-1609y12-13131/4y13-807\
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⇒ _[10]=[0,y5+5y6+23/2y7+57/4y8-27/2y9-139y10-945/2y11-4823/4y12-5173/2y\
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⇒ _[11]=[0,x+y+20y9+180y10+778y11+2318y12+5752y13+12636y14]

```



### D.6.9.5 modVStd0

Procedure from library `classifyMapGerms.lib` (see [Section D.6.9 \[classifyMapGerms.lib\]](#), [page 1663](#)).

**Usage:** `modVStd0((M, N, I, bound, #));`  $M$  is a submodule in  $A^r$  over the basering  $A$ ,  $N$  is a submodule in  $R^r$  over the subring  $R$  of the basering generated by the entries of  $I$

**Compute:** a standard basis of  $M+N+\text{maxideal}(\text{bound})*A^r$  using the parallel modular version

**Return:** a list whose  $i$ th entry gives a list where each entry is a generator of standard basis and second entry gives a list of generators after the reduced echelon form

**Note:** if  $\#$  is not empty the bound is corrected by computing determinacy

**Example:**

```
LIB "classifyMapGerms.lib";
ring R=0,(x,y),(c,ds);
poly f1=x;
poly f2=xy+y5+y7;
poly f11=f1+f2*f1;
poly f22=f2+f1^2;
map phi=basing,x+y,y+y2;
f1=phi(f11);
f2=phi(f22);
ideal I=f1,f2;
module M=maxideal(1)*jacob(I);
module N=I*freemodule(2);
modVStd0(M,N,I,15);
=> [1]:
=> _[1]=[x+x3+4x2y+3xy2+2x3y+6x2y2+4xy3+5x2y4+6xy5+30x2y5+35xy6+77x2y6+88\
xy7+136x2y7+153xy8+234x2y8+260xy9+360x2y9+396xy10+385x2y10+420xy11+252x2y\
11+273xy12+91x2y12+98xy13+14x2y13+15xy14,3x2+4xy+2x2y+3xy2+5xy4+30xy5+77x\
y6+136xy7+234xy8+360xy9+385xy10+252xy11+91xy12+14xy13]
=> _[2]=[y+x2y+4xy2+3y3+2x2y2+6xy3+4y4+5xy5+6y6+30xy6+35y7+77xy7+88y8+136\
xy8+153y9+234xy9+260y10+360xy10+396y11+385xy11+420y12+252xy12+273y13+91xy\
13+98y14+14xy14+15y15,3xy+4y2+2xy2+3y3+5y5+30y6+77y7+136y8+234y9+360y10+3\
85y11+252y12+91y13+14y14]
=> _[3]=[0,x2+2029573260997425667835151746738174/914290180625267404551036\
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 091233268289758780706256250158950375616491/370530402435700552417895025263\  
 708857095416712198506291688695625x5y4+90988232954715525511189647092769645\  
 630773223977566209252491442463/296424321948560441934316020210967085676333\  
 369758805033350956500x4y5+64883394526268214709397819578132432238840474701\  
 4315457004007505629741/13931943131582340770912852949915453026787668378663\  
 8365674949555000x3y6+2787059734156303336586688102515553682800597702179019\  
 2908279657705639/29642432194856044193431602021096708567633336975880503335\  
 09565000x2y7+850814684278642788515908978854943992718583652909741589227723\  
 055849403/139319431315823407709128529499154530267876683786638365674949555\  
 000xy8+150169224672841345516835794624065227693545610182811920959256903157\  
 117/139319431315823407709128529499154530267876683786638365674949555000y9+\  
 49851701327998805703175887340367217898586486624040942494876347702/1741492\  
 8914477925963641066187394316283484585473329795709368694375x5y5+2677258388\  
 3157158375966238154685688999258393220281947133792907892537/11609952609651\  
 950642427377458262877522323056982219863806245796250x4y6+17852823930757541\  
 18270781829795642643866951489787329489947684852740879/1393194313158234077\  
 09128529499154530267876683786638365674949555000x3y7+266478667315946429908\  
 1021373204820967665477618032556009617347981304943/13931943131582340770912\  
 8529499154530267876683786638365674949555000x2y8+8752447405308074640276537\  
 9641919928524114646700844603469765777115983/92879620877215605139419019666\  
 10302017858445585775891044996637000xy9+3772907829393940740127244770320368

```

2155540309645328228713214130673951/46439810438607802569709509833051510089\
292227928879455224983185000y10-249258506639994028515879436701836089492932\
43312020471247438173851/1741492891447792596364106618739431628348458547332\
9795709368694375x4y7-6841419353805073264906150389926109870524811686330044\
592856786957766/580497630482597532121368872913143876116152849110993190312\
2898125x3y8-1455124577400676907044122424566707436023147901398257918875682\
340922/232199052193039012848547549165257550446461139644397276124915925x2y\
9-256220298453550966697375048493544624165153991823559776875177055136669/2\
7863886263164681541825705899830906053575336757327673134989911000xy10-5661\
04814578472083046360801644301134321644930132229206257890755320471/1393194\
31315823407709128529499154530267876683786638365674949555000y11+7381779437\
4442808764810880260192310734149104858463011224236020804/58049763048259753\
21213688729131438761161528491109931903122898125x3y9+267503472021284814694\
05837498027022226782299553016622581646404767237/3482985782895585192728213\
2374788632566969170946659591418737388750x2y10-364641854514542590064526034\
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154530267876683786638365674949555000xy11-43372970313948512258850140895077\
9268449447675287132498425501614757769/13931943131582340770912852949915453\
0267876683786638365674949555000y12+18020515796734728053705723266299282011\
135295674815601025306673222/696597156579117038545642647495772651339383418\
933191828374747775x3y10+8671408365149056367902996227312654128991651203076\
852042659709348742/174149289144779259636410661873943162834845854733297957\
09368694375x2y11-72653592701465302820136294892417726435769733774979635109\
3339783593703/46439810438607802569709509833051510089292227928879455224983\
185000xy12-41890142596276048977285898008212119941167680527926040983723797\
5611309/27863886263164681541825705899830906053575336757327673134989911000\
y13+2273822919264286441287320431477690552937457706640414657017533857854/1\
7414928914477925963641066187394316283484585473329795709368694375x3y11-424\
86918471580245356756691926802615376555460528621076135253239796811/3482985\
7828955851927282132374788632566969170946659591418737388750x2y12-345391753\
8547844298509454900924447229010103319065882470860159520098323/46439810438\
607802569709509833051510089292227928879455224983185000xy13-98641487846078\
64553628846890859655440498342258055057496966587459993601/1393194313158234\
07709128529499154530267876683786638365674949555000y14+5383533458551605080\
943122509209007783931684039177372412876395599379/348298578289558519272821\
3237478863256696917094665959141873738875x3y12+989034829670574169106564144\
45995420559207046996366643794699488642309/3482985782895585192728213237478\
8632566969170946659591418737388750x2y13-302262586931882627102531967093458\
3684027448552397750149187283222797693/17414928914477925963641066187394316\
283484585473329795709368694375xy14-37741448746687317615220768544134849536\
21038258219257242597845204450999/2321990521930390128485475491652575504464\
6113964439727612491592500y15]

```

⇒  $\_ [5] = [0, y15]$

⇒  $[2] :$

⇒  $\_ [1] = [0, y14]$

⇒  $\_ [2] = [0, y13 + 483/16y14]$

⇒  $\_ [3] = [0, y12 + 4484/375y13 + 8614/125y14]$

⇒  $\_ [4] = [0, y11 + 11y12 + 58y13 + 363/2y14]$

⇒  $\_ [5] = [0, y10 + 10y11 + 48y12 + 287/2y13 + 991/4y14]$

⇒  $\_ [6] = [0, y9 + 9y10 + 389/10y11 + 2193/20y12 + 1063/5y13 + 3781/20y14]$

⇒  $\_ [7] = [0, y8 + 8y9 + 154/5y10 + 84y11 + 4949/25y12 + 10188/25y13 + 16034/25y14]$

⇒  $\_ [8] = [0, y7 + 89/2y8 + 321y9 + 1140y10 + 2685y11 + 10109/2y12 + 8409y13 + 14614y14]$

⇒  $\_ [9] = [0, y6 + 6y7 + 33/2y8 + 63/4y9 - 357/4y10 - 1065/2y11 - 1609y12 - 13131/4y13 - 807\$

```

      9/2y14]
  ↪      _[10]=[0,y5+5y6+23/2y7+57/4y8-27/2y9-139y10-945/2y11-4823/4y12-5173/2y\
      13-4657y14]
  ↪      _[11]=[0,x+y+20y9+180y10+778y11+2318y12+5752y13+12636y14]

```

### D.6.9.6 classifySimpleMaps

Procedure from library `classifyMapGerms.lib` (see [Section D.6.9 \[classifyMapGerms.lib\]](#), [page 1663](#)).

**Usage:** `classifySimpleMaps(I)`; I=an ideal with 2 generators in a polynomial ring with 2 variables and local ordering defining a map germ  $C^2$  to  $C^2$

**Compute:** The normal form of the germ in Riegers classification if it is simple

**Return:** normal form of I, of type ideal

**Note:** If I is not simple it returns (0,0)

**Example:**

```

LIB "classifyMapGerms.lib";
ring R=0,(x,y),(c,ds);
poly f1=x;
poly f2=xy+y5+y7;
poly f11=f1+f2*f1;
poly f22=f2+f1^2;
map phi=basing,x+y,y+y2;
f1=phi(f11);
f2=phi(f22);
ideal I=f1,f2;
classifySimpleMaps(I);
↪ _[1]=x
↪ _[2]=xy+y5+y7

```

### D.6.9.7 classifySimpleMaps1

Procedure from library `classifyMapGerms.lib` (see [Section D.6.9 \[classifyMapGerms.lib\]](#), [page 1663](#)).

**Usage:** `classifySimpleMaps1(I)`; I=an ideal with 2 generators in a polynomial ring with 2 variables and local ordering defining a map germ  $C^2$  to  $C^2$

**Compute:** The normal form of the germ in Riegers classification if it is simple

**Return:** normal form of I, of type ideal

**Note:** If I is not simple it returns (0,0)

**Example:**

```

LIB "classifyMapGerms.lib";
ring R=0,(x,y),(c,ds);
poly f1=x;
poly f2=xy+y5+y7;
poly f11=f1+f2*f1;
poly f22=f2+f1^2;
map phi=basing,x+y,y+y2;
f1=phi(f11);

```

```

f2=phi(f22);
ideal I=f1,f2;
classifySimpleMaps1(I);
↦ _[1]=x
↦ _[2]=xy+y5+y7

```

### D.6.9.8 classifyUnimodalMaps

Procedure from library `classifyMapGerms.lib` (see [Section D.6.9 \[classifyMapGerms.lib\]](#), [page 1663](#)).

**Usage:** `classifyUnimodalMaps(I)`; I an ideal with 2 generators in a polynomial ring with 2 variables and local ordering defining a map germ  $C^2$  to  $C^2$

**Compute:** The normal form of the germ in Riegers classification if it is simple

**Return:** normal form of I, of type ideal

**Note:** If I is not unimodal it returns (0,0)

**Example:**

```

LIB "classifyMapGerms.lib";
ring R=0,(x,y),(c,ds);
poly f1=x;
poly f2=xy+y6+y9;
poly f11=f1+f2*f1;
poly f22=f2+f1^2;
map phi=basing,x+y,y+y2;
f1=phi(f11);
f2=phi(f22);
ideal I=f1,f2;
classifyUnimodalMaps(I);
↦ _[1]=x
↦ _[2]=xy+y6+y9

```

### D.6.10 curvepar.lib

**Library:** `curvepar.lib`

**Purpose:** Resolution of space curve singularities, semi-group

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Maryna Viazovska email: [viazovsk@mathematik.uni-kl.de](mailto:viazovsk@mathematik.uni-kl.de)

**Procedures:** See also: [Section D.6.21 \[spcurve.lib\]](#), [page 1768](#).

#### D.6.10.1 BlowingUp

Procedure from library `curvepar.lib` (see [Section D.6.10 \[curvepar.lib\]](#), [page 1682](#)).

**Usage:** `BlowingUp(f,I,l)`;  
f=poly  
b=ideal  
l=list



**Assume:** The basering is  $r=0, (x(1..n), a), dp$   
 $f$  is an irreducible polynomial in  $k[a]$ ,  
 $I$  is an ideal of a curve (if we consider  $a$  as a parameter)

**Compute:** Blowing-up of the curve at point 0.

**Return:** list  $C$  of charts.  
Each chart  $C[i]$  is a list of size 5 (reps. 6 in case of plane curves)  $C[i][1]$  is an integer  $j$ .  
It shows, which standard chart do we consider.  $C[i][2]$  is an irreducible poly  $g$  in  $k[a]$ .  
It is a minimal polynomial for the new parameter.  
 $C[i][3]$  is an ideal  $H$  in  $k[a]$ .  
 $c_i = F_i(a_{\text{new}})$  for  $i=1..n$ ,  
 $a_{\text{old}} = H[n+1](a_{\text{new}})$ .  
 $C[i][4]$  is a map  $\text{teta}: k[x(1)..x(n), a] \rightarrow k[x(1)..x(n), a]$  from the new curve to the old one.  
 $x(1) \rightarrow x(j) * x(1) \dots x(j) \rightarrow x(j) \dots x(n) \rightarrow x(j) * (c_{n+1} + x(n))$   
 $C[i][5]$  is an ideal  $J$  of a new curve.  $J = \text{teta}(I)$ .  
 $C[i][6]$  is the list of exceptional divisors in the chart

**Example:**

```
LIB "curvepar.lib";
ring r=0, (x(1..3), a), dp;
poly f=a^2+1;
ideal i=x(1)^2+a*x(2)^3, x(3)^2-x(2);
list l=1,3,2;
list B=BlowingUp(f,i,l);
B;
⇒ [1]:
⇒ [1]:
⇒ 3
⇒ [2]:
⇒ a^2+1
⇒ [3]:
⇒ _[1]=0
⇒ _[2]=0
⇒ _[3]=1
⇒ [4]:
⇒ _[1]=x(1)*x(3)
⇒ _[2]=x(2)*x(3)
⇒ _[3]=x(3)
⇒ _[4]=a
⇒ [5]:
⇒ _[1]=x(2)-x(3)
⇒ _[2]=x(2)^3*x(3)*a+x(1)^2
```

### D.6.10.2 CurveRes

Procedure from library `curvepar.lib` (see [Section D.6.10 \[curvepar.lib\]](#), page 1682).

**Usage:** CurveRes(I);  
 $I$  ideal

**Assume:** The basering is  $r=0, (x(1..n))$   
 $V(I)$  is a curve with a singular point 0.

**Compute:** Resolution of the curve  $V(I)$ .

**Return:** a ring  $R = \text{basering} + k[a]$   
 Ring  $R$  contains a list `Resolve`  
`Resolve` is a list of charts  
 Each `Resolve[i]` is a list of size 6  
`Resolve[i][1]` is an ideal  $J$  of a new curve.  $J = \text{teta}(I)$ . `Resolve[i][2]` ideal which represents the map  
 $\text{teta}: k[x(1)..x(n), a] \rightarrow k[x(1)..x(n), a]$  from the new curve to the old one.  
`Resolve[i][3]` is an irreducible poly  $g$  in  $k[a]$ . It is a minimal polynomial for the new parameter  $a$ .  $\deg(g)$  gives the number of branches in `Resolve[i]` `Resolve[i][4]` sequence of multiplicities (sum over all branches in `Resolve` as long as they intersect each other !)  
`Resolve[i][5]` is a list of integers  $l$ . It shows, which standard charts we considered.  
`Resolve[i][6]` HN matrix  
`Resolve[i][7]` (only for plane curves) the development of exceptional divisors the entries correspond to the  $i$ -th blowing up. The first entry is an intvec. The first negative entry gives the splitting of the (over  $\mathbb{Q}$  irreducible) branches. The second entry is a list of the exceptional divisors. If the entry is an integer  $i$ , it says that the divisor is not visible in this chart after the  $i$ -th blowing up.

**Example:**

```
LIB "curvepar.lib";
ring r=0,(x,y,z),dp;
ideal i=x2-y3,z2-y5;
def s=CurveRes(i);
setring s;
Resolve;
⇒ [1]:
⇒ [1]:
⇒ _[1]=x(1)
⇒ _[2]=x(1)*x(2)-x(3)^2+2*x(3)
⇒ [2]:
⇒ _[1]=x(1)^2*x(2)^5+2*x(1)*x(2)^4+x(2)^3
⇒ _[2]=x(1)*x(2)^3+x(2)^2
⇒ _[3]=x(1)^2*x(2)^7*x(3)-x(1)^2*x(2)^7+2*x(1)*x(2)^6*x(3)-2*x(1)*x(2)\
) ^6+x(2)^5*x(3)-x(2)^5
⇒ _[4]=a
⇒ [3]:
⇒ a
⇒ [4]:
⇒ [1]:
⇒ 4
⇒ [2]:
⇒ 2
⇒ [3]:
⇒ 2
⇒ [4]:
⇒ 2
⇒ [5]:
⇒ [1]:
⇒ 2
⇒ [2]:
```

```

↳      1
↳      [3]:
↳      2
↳      [4]:
↳      2
↳      [6]:
↳      [1]:
↳      _[1]=0
↳      _[2]=1
↳      _[3]=0
↳      [2]:
↳      _[1]=1
↳      _[2]=0
↳      _[3]=0
↳      [3]:
↳      _[1]=1
↳      _[2]=1
↳      _[3]=0
↳      [4]:
↳      _[1]=0
↳      _[2]=1
↳      _[3]=-1
↳      [2]:
↳      [1]:
↳      _[1]=x(1)
↳      _[2]=x(1)*x(2)-x(3)^2-2*x(3)
↳      [2]:
↳      _[1]=x(1)^2*x(2)^5+2*x(1)*x(2)^4+x(2)^3
↳      _[2]=x(1)*x(2)^3+x(2)^2
↳      _[3]=x(1)^2*x(2)^7*x(3)+x(1)^2*x(2)^7+2*x(1)*x(2)^6*x(3)+2*x(1)*x(2)\
) ^6+x(2)^5*x(3)+x(2)^5
↳      _[4]=a
↳      [3]:
↳      a
↳      [4]:
↳      [1]:
↳      4
↳      [2]:
↳      2
↳      [3]:
↳      2
↳      [4]:
↳      2
↳      [5]:
↳      [1]:
↳      2
↳      [2]:
↳      1
↳      [3]:
↳      2
↳      [4]:
↳      2
↳      [6]:

```

```

⇒      [1]:
⇒      _[1]=0
⇒      _[2]=1
⇒      _[3]=0
⇒      [2]:
⇒      _[1]=1
⇒      _[2]=0
⇒      _[3]=0
⇒      [3]:
⇒      _[1]=1
⇒      _[2]=1
⇒      _[3]=0
⇒      [4]:
⇒      _[1]=0
⇒      _[2]=1
⇒      _[3]=1

```

### D.6.10.3 CurveParam

Procedure from library `curvepar.lib` (see [Section D.6.10 \[curvepar.lib\]](#), page 1682).

**Usage:**     CurveParam(I);  
              I ideal

**Assume:**    I is an ideal of a curve C with a singular point 0.

**Compute:**   Parametrization for algebraic branches of the curve C.

**Return:**    list L of size 1.  
              L[1] is a ring ring rt=0,(t,a),ds;  
              Ring R contains a list Param  
              Param is a list of algebraic branches  
              Each Param[i] is a list of size 3  
              Param[i][1] is a list of polynomials  
              Param[i][2] is an irreducible polynomial  $f \in k[a]$ . It is a minimal polynomial for the  
              parameter a.  
              Param[i][3] is an integer b-upper bound for the conductor of Weierstrass semigroup

**Example:**

```

LIB "curvepar.lib";
ring r=0,(x,y,z),dp;
ideal i=x2-y3,z2-y5;
def s=CurveParam(i);
setring s;
Param;
⇒ [1]:
⇒      [1]:
⇒      t3
⇒      [2]:
⇒      t2
⇒      [3]:
⇒      -t5
⇒      [2]:
⇒      a

```

```

⇒      [3]:
⇒      38
⇒ [2]:
⇒      [1]:
⇒      [1]:
⇒      t3
⇒      [2]:
⇒      t2
⇒      [3]:
⇒      t5
⇒      [2]:
⇒      a
⇒      [3]:
⇒      38

```

#### D.6.10.4 WSemigroup

Procedure from library `curvepar.lib` (see [Section D.6.10 \[curvepar.lib\]](#), page 1682).

**Usage:**      `WSemigroup(X,b0);`  
               `X` a list of polynomials in one variable, say `t`.  
               `b0` an integer

**Compute:**    Weierstrass semigroup of space curve `C`, which is given by a parametrization `X[1](t),...,X[k](t)`, till the bound `b0`.

**Assume:**     `b0` is greater then conductor

**Return:**      list `M` of size 5.  
               `M[1]`= list of integers, which are minimal generators set of the Weierstrass semigroup.  
               `M[2]`=integer, conductor of the Weierstrass semigroup. `M[3]`=intvec, all elements of the Weierstrass semigroup till some bound `b`, which is greater than conductor.

**Warning:**     works only over the ring with one variable with ordering `ds`

**Example:**

```

LIB "curvepar.lib";
ring r=0,(t),ds;
list X=t4,t5+t11,t9+2*t7;
list L=WSemigroup(X,30);
L;
⇒ [1]:
⇒ 4,5,7
⇒ [2]:
⇒ 7
⇒ [3]:
⇒ 4,5,7,8,9,10

```

#### D.6.10.5 primparam

Procedure from library `curvepar.lib` (see [Section D.6.10 \[curvepar.lib\]](#), page 1682).

**Usage:**      `MultiplicitySequence(x,y,c);` `x` poly, `y` poly, `c` integer

**Assume:**     `x` and `y` are polynomials in  $k(a)[t]$  such that  $(x,y)$  is a primitive parametrization of a plane curve branch and  $\text{ord}(x) < \text{ord}(y)$ .

**Return:** Hamburger-Noether Matrix of the curve branch given parametrically by  $(x,y)$ .

**Example:**

```
LIB "curvepar.lib";
ring r=(0,a),t,ds;
poly x=t6;
poly y=t8+t11;
int c=15;
primparam(x,y,c);
↪ _[1,1]=0
↪ _[1,2]=t
↪ _[1,3]=0
↪ _[1,4]=0
↪ _[1,5]=0
↪ _[1,6]=0
↪ _[1,7]=0
↪ _[1,8]=0
↪ _[1,9]=0
↪ _[1,10]=0
↪ _[1,11]=0
↪ _[1,12]=0
↪ _[1,13]=0
↪ _[1,14]=0
↪ _[1,15]=0
↪ _[2,1]=0
↪ _[2,2]=0
↪ _[2,3]=1
↪ _[2,4]=0
↪ _[2,5]=t
↪ _[2,6]=0
↪ _[2,7]=0
↪ _[2,8]=0
↪ _[2,9]=0
↪ _[2,10]=0
↪ _[2,11]=0
↪ _[2,12]=0
↪ _[2,13]=0
↪ _[2,14]=0
↪ _[2,15]=0
↪ _[3,1]=0
↪ _[3,2]=1/9
↪ _[3,3]=0
↪ _[3,4]=0
↪ _[3,5]=-7/243
↪ _[3,6]=0
↪ _[3,7]=0
↪ _[3,8]=250/19683
↪ _[3,9]=0
↪ _[3,10]=0
↪ _[3,11]=-3625/531441
↪ _[3,12]=0
↪ _[3,13]=0
↪ _[3,14]=58351/14348907
```

$\mapsto \_ [3,15]=0$

### D.6.10.6 MultiplicitySequence

Procedure from library `curvepar.lib` (see [Section D.6.10 \[curvepar.lib\]](#), page 1682).

**Usage:**      `MultiplicitySequence(i); i ideal`

**Assume:**    `i` is the defining ideal of a (reducible) plane curve singularity.

**Return:**     list `X` of charts. Each chart contains the multiplicity sequence of the corresponding branch.

**Example:**

```
LIB "curvepar.lib";
ring r=0,(x,y),ds;
ideal i=x14-x4y7-y11;
MultiplicitySequence(i);
 $\mapsto$  [1]:
 $\mapsto$  1
 $\mapsto$  [2]:
 $\mapsto$  1
 $\mapsto$  [3]:
 $\mapsto$  1
 $\mapsto$  [4]:
 $\mapsto$  1
 $\mapsto$  [5]:
 $\mapsto$  7,3,3,1,1,1
```

### D.6.10.7 CharacteristicExponents

Procedure from library `curvepar.lib` (see [Section D.6.10 \[curvepar.lib\]](#), page 1682).

**Usage:**      `CharacteristicExponents(i); i ideal`

**Assume:**    `i` is the defining ideal of a (reducible) plane curve singularity.

**Return:**     list `X` of charts. Each chart contains the characteristic exponents of the corresponding branch.

**Example:**

```
LIB "curvepar.lib";
ring r=0,(x,y),ds;
ideal i=x14-x4y7-y11;
CharacteristicExponents(i);
 $\mapsto$  [1]:
 $\mapsto$  1
 $\mapsto$  [2]:
 $\mapsto$  1
 $\mapsto$  [3]:
 $\mapsto$  1
 $\mapsto$  [4]:
 $\mapsto$  1
 $\mapsto$  [5]:
 $\mapsto$  7,10
```

### D.6.10.8 IntersectionMatrix

Procedure from library `curvepar.lib` (see [Section D.6.10 \[curvepar.lib\]](#), page 1682).

**Usage:**      `IntersectionMatrix(i)`;  $i$  ideal

**Assume:**     $i$  is the defining ideal of a (reducible) plane curve singularity.

**Return:**     `intmat` of the intersection multiplicities of the branches.

**Example:**

```
LIB "curvepar.lib";
ring r=0,(x,y),ds;
ideal i=x14-x4y7-y11;
IntersectionMatrix(i);
⇒ 0,1,1,1,7,
⇒ 0,0,1,1,7,
⇒ 0,0,0,1,7,
⇒ 0,0,0,0,7,
⇒ 0,0,0,0,0
```

### D.6.10.9 ContactMatrix

Procedure from library `curvepar.lib` (see [Section D.6.10 \[curvepar.lib\]](#), page 1682).

**Usage:**      `ContactMatrix(I)`;  $I$  ideal

**Assume:**     $i$  is the defining ideal of a (reducible) plane curve singularity.

**Return:**     `intmat`  $N$  of the contact matrix of the braches of the curve.

**Example:**

```
LIB "curvepar.lib";
ring r=0,(x,y),ds;
ideal i=x14-x4y7-y11;
ContactMatrix(i);
⇒ 0,1,1,1,1,
⇒ 1,0,1,1,1,
⇒ 1,1,0,1,1,
⇒ 1,1,1,0,1,
⇒ 1,1,1,1,0
```

### D.6.10.10 plainInvariants

Procedure from library `curvepar.lib` (see [Section D.6.10 \[curvepar.lib\]](#), page 1682).

**Usage:**      `plainInvariants(i)`;  $i$  ideal

**Assume:**     $i$  is the defining ideal of a (reducible) plane curve singularity.

**Return:**     list  $L$  of charts.  $L[j]$  is the invariants of the  $j$ th branch and the last entry of  $L$  is a list containing the intersection matrix, contact matrix, resolution graph of the curve.  
 $L[j][1]$ : `intvec` (characteristic exponents of the  $j$ th branch)  $L[j][2]$ : `intvec` (generators of the semigroup of the  $j$ th branch)  $L[j][3]$ : `intvec` (first components of the puseux pairs of the  $j$ th branch)  $L[j][4]$ : `intvec` (second components of the puseux pairs of the  $j$ th branch)  $L[j][5]$ : `int` (degree of conductor of the  $j$ th branch)  $L[j][6]$ : `intvec` (multiplicity sequence of the  $j$ th branch.  $L[\text{last}][1]$ : `intmat` (intersection matrix of the branches)  $L[\text{last}][2]$ : `intmat` (contact matrix of the branches)  $L[\text{last}][3]$ : `intmat` (resolution graph of the curve)



**Example:**

```

LIB "curvepar.lib";
ring r=0,(x,y),ds;
ideal i=x14-x4y7-y11;
plainInvariants(i);
↳ [1]:
↳   [1]:
↳     1
↳   [2]:
↳     1
↳   [3]:
↳     0
↳   [4]:
↳     0
↳   [5]:
↳     0
↳   [6]:
↳     1
↳ [2]:
↳   [1]:
↳     1
↳   [2]:
↳     1
↳   [3]:
↳     0
↳   [4]:
↳     0
↳   [5]:
↳     0
↳   [6]:
↳     1
↳ [3]:
↳   [1]:
↳     1
↳   [2]:
↳     1
↳   [3]:
↳     0
↳   [4]:
↳     0
↳   [5]:
↳     0
↳   [6]:
↳     1
↳ [4]:
↳   [1]:
↳     1
↳   [2]:
↳     1
↳   [3]:
↳     0
↳   [4]:
↳     0

```

```

⇒      [5]:
⇒      0
⇒      [6]:
⇒      1
⇒ [5]:
⇒      [1]:
⇒      7,10
⇒      [2]:
⇒      7,10
⇒      [3]:
⇒      10
⇒      [4]:
⇒      7
⇒      [5]:
⇒      54
⇒      [6]:
⇒      7,3,3,1,1,1
⇒ [6]:
⇒      [1]:
⇒      0,1,1,1,7,
⇒      0,0,1,1,7,
⇒      0,0,0,1,7,
⇒      0,0,0,0,7,
⇒      0,0,0,0,0
⇒      [2]:
⇒      0,1,1,1,1,
⇒      1,0,1,1,1,
⇒      1,1,0,1,1,
⇒      1,1,1,0,1,
⇒      1,1,1,1,0
⇒      [3]:
⇒      1,1,1,1,1,0,0,1,0,0,0,
⇒      1,-1,0,0,0,0,0,0,0,0,0,
⇒      1,0,-2,0,0,0,0,0,0,0,0,
⇒      1,0,0,-3,0,0,0,0,0,0,0,
⇒      1,0,0,0,-4,0,0,0,0,0,0,
⇒      0,0,0,0,0,2,1,0,0,0,0,
⇒      0,0,0,0,0,1,3,0,0,1,0,
⇒      1,0,0,0,0,0,0,4,1,0,0,
⇒      0,0,0,0,0,0,0,1,5,1,0,
⇒      0,0,0,0,0,0,1,0,1,6,1,
⇒      0,0,0,0,0,0,0,0,0,1,-5

```

See also: [Section D.6.10.7 \[CharacteristicExponents\]](#), page 1689; [Section D.6.10.9 \[ContactMatrix\]](#), page 1690; [Section D.6.10.8 \[IntersectionMatrix\]](#), page 1690; [Section D.6.10.6 \[MultiplicitySequence\]](#), page 1689.

## D.6.11 deform.lib

**Library:** deform.lib

**Purpose:** Miniversal Deformation of Singularities and Modules

**Author:** Bernd Martin, email: martin@math.tu-cottbus.de

**Procedures:**

### D.6.11.1 versal

Procedure from library `deform.lib` (see [Section D.6.11 \[deform.lib\]](#), page 1692).

**Usage:** `versal(Fo[d,any]);` Fo=ideal, d=int, any=list

**Compute:** miniversal deformation of Fo up to degree d (default d=100),

**Return:** list L of 4 rings:

L[1] extending the basering Po by new variables given by "A,B,.. " (deformation parameters); the new variables precede the old ones, the ordering is the product of "ls" and "ord(Po)"

L[2] = L[1]/Fo extending Qo=Po/Fo,

L[3] = the embedding ring of the versal base space,

L[4] = L[1]/Js extending L[3]/Js.

In the ring L[1] the following matrices are stored:

Js = giving the versal base space (obstructions),

Fs = giving the versal family of Fo,

Rs = giving the lifting of Ro=syz(Fo).

If d is defined ( $\neq 0$ ), it computes up to degree d.

If 'any' is defined and any[1] is no string, interactive version.

Otherwise 'any' is interpreted as a list of predefined strings: "my", "param", "order", "out":

("my" internal prefix, "param" is a letter (e.g. "A") for the name of the first parameter or (e.g. "A(") for index parameter variables, "order" ordering string for ring extension), "out" name of output file).

**Note:** printlevel < 0 no additional output,  
printlevel >=0,1,2,.. informs you, what is going on;  
this proc uses 'execute'.

**Example:**

```
LIB "deform.lib";
int p          = printlevel;
printlevel     = 0;
ring r1        = 0, (x,y,z,u,v), ds;
matrix m[2][4] = x,y,z,u,y,z,u,v;
ideal Fo       = minor(m,2);
// cone over rational normal curve of degree 4
list L=versal(Fo);
⇨ // ready: T_1 and T_2
⇨ // start computation in degree 2.
⇨
⇨
⇨ // 'versal' returned a list, say L, of four rings. In L[1] are stored:
⇨ //   as matrix Fs: Equations of total space of the miniversal deformation\
,
⇨ //   as matrix Js: Equations of miniversal base space,
⇨ //   as matrix Rs: syzygies of Fs mod Js.
⇨ // To access these data, type
⇨   def Px=L[1]; setring Px; print(Fs); print(Js); print(Rs);
⇨
⇨ // L[2] = L[1]/Fo extending Qo=Po/Fo,
⇨ // L[3] = the embedding ring of the versal base space,
```

```

⇒ // L[4] = L[1]/Js extending L[3]/Js.
⇒
L;
⇒ [1]:
⇒ // coefficients: QQ
⇒ // number of vars : 9
⇒ //      block 1 : ordering ds
⇒ //      : names  A B C D
⇒ //      block 2 : ordering ds
⇒ //      : names  x y z u v
⇒ //      block 3 : ordering C
⇒ [2]:
⇒ // coefficients: QQ
⇒ // number of vars : 9
⇒ //      block 1 : ordering ds
⇒ //      : names  A B C D
⇒ //      block 2 : ordering ds
⇒ //      : names  x y z u v
⇒ //      block 3 : ordering C
⇒ // quotient ring from ideal ...
⇒ [3]:
⇒ // coefficients: QQ
⇒ // number of vars : 4
⇒ //      block 1 : ordering ds
⇒ //      : names  A B C D
⇒ //      block 2 : ordering C
⇒ [4]:
⇒ // coefficients: QQ
⇒ // number of vars : 9
⇒ //      block 1 : ordering ds
⇒ //      : names  A B C D
⇒ //      block 2 : ordering ds
⇒ //      : names  x y z u v
⇒ //      block 3 : ordering C
⇒ // quotient ring from ideal ...
def Px=L[1];
setring Px;
// ___ Equations of miniversal base space ___:
Js;"";
⇒ Js[1,1]=BD
⇒ Js[1,2]=-AD+D2
⇒ Js[1,3]=CD
⇒
// ___ Equations of miniversal total space ___:
Fs;"";
⇒ Fs[1,1]=u2-zv-Bu+Dv
⇒ Fs[1,2]=zu-yv-Au+Du
⇒ Fs[1,3]=yu-xv+Cu+Dz
⇒ Fs[1,4]=z2-yu-Az+By
⇒ Fs[1,5]=yz-xu+Bx+Cz
⇒ Fs[1,6]=y2-xz+Ax+Cy
⇒

```

### D.6.11.2 mod\_versal

Procedure from library `deform.lib` (see [Section D.6.11 \[deform.lib\]](#), page 1692).

**Usage:** `mod_versal(Mo,Io[,d,any]);` Io=ideal, Mo=module, d=int, any =list

**Compute:** miniversal deformation of `coker(Mo)` over  $Qo=Po/Io$ ,  $Po=basering$ ;

**Return:** list L of 4 rings:  
 $L[1]$  extending the basering  $Po$  by new variables given by "A,B,..", (deformation parameters); the new variables precede the old ones, the ordering is the product of "ls" and "ord( $Po$ )"  
 $L[2] = L[1]/Io$  extending  $Qo$ ,  
 $L[3]$  = the embedding ring of the versal base space,  
 $L[4] = L[1]/(Io+Js)$  ring of the versal deformation of `coker(Ms)`.  
 In the ring  $L[1]$  the following matrices are stored:  
 $Js$  = giving the versal base space (obstructions),  
 $Fs$  = giving the versal family of  $Mo$ ,  
 $Rs$  = giving the lifting of syzygies  $Lo=syz(Mo)$ . If  $d$  is defined ( $d \neq 0$ ), it computes up to degree  $d$ .  
 If 'any' is defined and `any[1]` is no string, interactive version.  
 Otherwise 'any' is interpreted as a list of predefined strings:  
 "my","param","order","out":  
 ("my" internal prefix, "param" is a letter (e.g. "A") for the name of the first parameter or (e.g. "A(") for index parameter variables, "order" ordering string for ring extension), "out" name of output file).

**Note:** `printlevel < 0` no additional output,  
`printlevel >=0,1,2,..` informs you, what is going on,  
 this proc uses 'execute'.

**Example:**

```
LIB "deform.lib";
int p = printlevel;
printlevel = 1;
ring Ro = 0,(x,y),wp(3,4);
ideal Io = x4+y3;
matrix Mo[2][2] = x2,y,-y2,x2;
list L = mod_versal(Mo,Io);
⇒ // vdim (Ext^2) = 4
⇒ // vdim (Ext^1) = 4
⇒ // ready: Ext1 and Ext2
⇒ // Ext1 is quasi-homogeneous represented: 3,6,1,4
⇒ // infinitesimal extension
⇒ x2-Ax-B, y+Cx+D,
⇒ -y2+Cxy+Dy,x2+Ax+B
⇒ // start deg = 2
⇒ // start deg = 3
⇒ // start deg = 4
⇒ // start deg = 5
⇒ // finished in degree
⇒ 5
⇒ // quasi-homogeneous weights of miniversal base
⇒ 3,
```

```

⇒ 6,
⇒ 1,
⇒ 4
⇒
⇒ // 'mod_versal' returned a list, say L, of four rings. In L[2] are stored\
:
⇒ // as matrix Ms: presentation matrix of the deformed module,
⇒ // as matrix Ls: lifted syzygies,
⇒ // as matrix Js: Equations of total space of miniversal deformation
⇒ // To access these data, type
⇒      def Qx=L[2]; setring Qx; print(Ms); print(Ls); print(Js);
⇒
def Qx=L[2]; setring Qx;
print(Ms);
⇒ x2-Ax-B+A2-C3x-3C2D+AC3,y+Cx+D,
⇒ -y2+Cxy+Dy-C2x2-2CDx-D2,x2+Ax+B
print(Ls);
⇒ -y-Cx-D,                                x2+Ax+B,
⇒ x2-Ax-B+A2-C3x-3C2D+AC3,y2-Cxy-Dy+C2x2+2CDx+D2
print(Js);
⇒ -2AB+A3+3CD2-BC3-3AC2D+A2C3,
⇒ -B2+A2B+D3-3BC2D+ABC3,
⇒ 0,
⇒ 0
printlevel = p;
if (defined(Px)) {kill Px,Qx,So;}

```

### D.6.11.3 lift\_kbase

Procedure from library `deform.lib` (see [Section D.6.11 \[deform.lib\]](#), page 1692).

**Usage:** `lift_kbase(N,M);`  $N, M = \text{poly/ideal/vector/module}$

**Return:** matrix  $A$ , coefficient matrix expressing  $N$  as linear combination of  $k$ -basis of  $M$ . Let the  $k$ -basis have  $k$  elements and  $\text{size}(N)=c$  columns. Then  $A$  satisfies:  
 $\text{matrix}(\text{reduce}(N, \text{std}(M)), k, c) = \text{matrix}(\text{kbase}(\text{std}(M))) * A$

**Assume:**  $\dim(M)=0$  and the monomial ordering is a well ordering or the last block of the ordering is  $c$  or  $C$

**Example:**

```

LIB "deform.lib";
ring R=0,(x,y),ds;
module M=[x2,xy],[y2,xy],[0,xx],[0,yy];
module N=[x3+xy,x],[x,x+y2];
print(M);
⇒ x2,y2,0, 0,
⇒ xy,xy,x2,y2
module kb=kbase(std(M));
print(kb);
⇒ y2,xy,y,x,1,0,0,0,
⇒ 0, 0, 0,0,0,y,x,1
print(N);
⇒ xy+x3,x,
⇒ x,      x+y2

```

```

matrix A=lift_kbase(N,M);
print(A);
⇒ 0,0,
⇒ 1,0,
⇒ 0,0,
⇒ 0,1,
⇒ 0,0,
⇒ 0,0,
⇒ 1,1,
⇒ 0,0
matrix(reduce(N,std(M)),nrows(kb),ncols(A)) - matrix(kbase(std(M)))*A;
⇒ _[1,1]=0
⇒ _[1,2]=0
⇒ _[2,1]=0
⇒ _[2,2]=0

```

#### D.6.11.4 lift\_rel\_kb

Procedure from library `deform.lib` (see [Section D.6.11 \[deform.lib\]](#), page 1692).

**Usage:** `lift_rel_kb(N,M[,kbaseM,p]);`

**Assume:** `[p` a monomial `]` or the product of all variables  
`N`, `M` modules of same rank, `M` depending only on variables not in `p` and `vdim(M)` is finite in this ring,  
`[ kbaseM` the kbase of `M` in the subring given by variables not in `p` `]`  
warning: these assumptions are not checked by the procedure

**Return:** matrix `A`, whose `j`-th columns present the coeff's of `N[j]` in `kbaseM`, i.e. `kbaseM*A = reduce(N,std(M))`

**Example:**

```

LIB "deform.lib";
ring r=0,(A,B,x,y),dp;
module M = [x2,xy],[xy,y3],[y2],[0,x];
module kbaseM = [1],[x],[xy],[y],[0,1],[0,y],[0,y2];
poly f=xy;
module N = [AB,BBy],[A3xy+x4,AB*(1+y2)];
matrix A = lift_rel_kb(N,M,kbaseM,f);
print(A);
⇒ AB,0,
⇒ 0, 0,
⇒ 0, A3,
⇒ 0, 0,
⇒ 0, AB,
⇒ B2,0,
⇒ 0, AB
"TEST:";
⇒ TEST:
print(matrix(kbaseM)*A-matrix(reduce(N,std(M))));
⇒ 0,0,
⇒ 0,0

```

#### D.6.12 equising\_lib

**Library:** equising.lib

**Purpose:** Equisingularity Stratum of a Family of Plane Curves

**Author:** Christoph Lossen, lossen@mathematik.uni-kl.de  
Andrea Mindnich, mindnich@mathematik.uni-kl.de

**Procedures:**

### D.6.12.1 tau\_es

Procedure from library `equising.lib` (see [Section D.6.12 \[equising\\_lib\]](#), page 1697).

**Usage:** `tau_es(f); f poly`

**Assume:** `f` is a reduced bivariate polynomial, the basering has precisely two variables, is local and no `qring`.

**Return:** `int`, the codimension of the  $\mu$ -const stratum in the semi-universal deformation base.

**Note:** `printlevel>=1` displays additional information.  
When called with any additional parameter, the computation of the Milnor number is avoided (no check for NND).

**Example:**

```
LIB "equising.lib";
ring r=32003,(x,y),ds;
poly f=(x4-y4)^2-x10;
tau_es(f);
↦ 42
```

See also: [Section D.6.12.2 \[esIdeal\]](#), page 1698; [Section D.6.15.6 \[invariants\]](#), page 1727; [Section D.6.20.15 \[tjurina\]](#), page 1764.

### D.6.12.2 esIdeal

Procedure from library `equising.lib` (see [Section D.6.12 \[equising\\_lib\]](#), page 1697).

**Usage:** `esIdeal(f[,any]); f poly`

**Assume:** `f` is a reduced bivariate polynomial, the basering has precisely two variables, is local and no `qring`, and the characteristic of the ground field does not divide `mult(f)`.

**Return:** if called with only one parameter: list of two ideals,  
     `_[1]`: equisingularity ideal of `f` (in sense of Wahl),  
     `_[2]`: ideal of equisingularity with fixed position of the singularity;  
 if called with more than one parameter: list of three ideals,  
     `_[1]`: equisingularity ideal of `f` (in sense of Wahl)  
     `_[2]`: ideal of equisingularity with fixed position of the singularity;  
     `_[3]`: ideal of all `g` such that the deformation defined by `f+eg` (`e^2=0`) is isomorphic to an equisingular deformation of `V(f)` with all equimultiple sections being trivial.

**Note:** if some of the above condition is not satisfied then return value is `list(0,0)`.

**Example:**



```

LIB "equising.lib";
ring r=0,(x,y),ds;
poly f=x7+y7+(x-y)^2*x2y2;
list K=esIdeal(f);
↳ polynomial is Newton degenerate !
↳
↳ //
↳ // versal deformation with triv. section
↳ // =====
↳ //
↳ //
↳ // Compute equisingularity Stratum over Spec(C[t]/t^2)
↳ // =====
↳ //
↳ // finished
↳ //
option(redSB);
// Wahl's equisingularity ideal:
std(K[1]);
↳ _[1]=4x4y-10x2y3+6xy4+21x6+14y6
↳ _[2]=4x3y2-6x2y3+2xy4+7x6
↳ _[3]=x2y4-xy5
↳ _[4]=x7
↳ _[5]=xy6
↳ _[6]=y7
ring rr=0,(x,y),ds;
poly f=x4+4x3y+6x2y2+4xy3+y4+2x2y15+4xy16+2y17+xy23+y24+y30+y31;
list K=esIdeal(f);
↳ polynomial is Newton degenerate !
↳
↳ //
↳ // versal deformation with triv. section
↳ // =====
↳ //
↳ //
↳ // Compute equisingularity Stratum over Spec(C[t]/t^2)
↳ // =====
↳ //
↳ // finished
↳ //
vdim(std(K[1]));
↳ 68
// the latter should be equal to:
tau_es(f);
↳ 68

```

See also: [Section D.6.12.3 \[esStratum\]](#), page 1699; [Section D.6.12.1 \[tau-es\]](#), page 1698.

### D.6.12.3 esStratum

Procedure from library `equising.lib` (see [Section D.6.12 \[equising-lib\]](#), page 1697).

**Usage:**      `esStratum(F[,m,L]);` F poly, m int, L list

- Assume:**  $F$  defines a deformation of a reduced bivariate polynomial  $f$  and the characteristic of the basering does not divide  $\text{mult}(f)$ .  
 If  $\text{nv}$  is the number of variables of the basering, then the first  $\text{nv}-2$  variables are the deformation parameters.  
 If the basering is a qring,  $\text{ideal}(\text{basing})$  must only depend on the deformation parameters.
- Compute:** equations for the stratum of equisingular deformations with fixed (trivial) section.
- Return:** list  $l$ : either consisting of a list and an integer, where  
 $l[1][1]=\text{ideal}$  defining the equisingularity stratum  
 $l[1][2]=\text{ideal}$  defining the part of the equisingularity stratum where all equimultiple sections through the non-nodes of the reduced total transform are trivial sections  
 $l[2]=1$  if some error has occurred,  $l[2]=0$  otherwise;  
 or consisting of a ring and an integer, where  
 $l[1]=\text{ESSring}$  is a ring extension of basering containing the ideal  $\text{ES}$  (describing the ES-stratum), the ideal  $\text{ES\_all\_triv}$  (describing the part with trivial equimultiple sections) and the polynomial  $p\_F=F$ ,  
 $l[2]=1$  if some error has occurred,  $l[2]=0$  otherwise.
- Note:**  $L$  is supposed to be the output of  $\text{hnexpansion}$  (with the given ordering of the variables appearing in  $f$ ).  
 If  $m$  is given, the ES Stratum over  $A/\text{maxideal}(m)$  is computed.  
 This procedure uses `execute` or calls a procedure using `execute`. `printlevel>=2` displays additional information.

**Example:**

```
LIB "equising.lib";
int p=printlevel;
printlevel=1;
ring r = 0,(a,b,c,d,e,f,g,x,y),ds;
poly F = (x2+2xy+y2+x5)+ax+by+cx2+dxy+ey2+fx3+gx4;
list M = esStratum(F);
M[1][1];
  ↳ _[1]=g
  ↳ _[2]=f
  ↳ _[3]=b
  ↳ _[4]=a
  ↳ _[5]=-4c+4d-4e+d2-4ce
printlevel=3;      // displays additional information
esStratum(F,2) ; // ES-stratum over Q[a,b,c,d,e,f,g] / <a,b,c,d,e,f,g>^2
  ↳ //
  ↳ // Compute HN expansion
  ↳ // -----
  ↳ // finished
  ↳ //
  ↳ // Blowup Step 1 completed
  ↳ // Blowup Step 2 completed
  ↳ // Blowup Step 3 completed
  ↳ // 1 branch finished
  ↳ //
  ↳ // Elimination starts:
  ↳ // -----
```

```

⇒ //
⇒ // Remove superfluous equations:
⇒ // -----
⇒ // finished
⇒ //
⇒ // output of 'esStratum' is a list consisting of:
⇒ //   _[1][1] = ideal defining the equisingularity stratum
⇒ //   _[1][2] = ideal defining the part of the equisingularity stratum
⇒ //           where all equimultiple sections are trivial
⇒ //   _[2] = 0
⇒ [1]:
⇒   [1]:
⇒     _[1]=b
⇒     _[2]=a
⇒     _[3]=c-d+e
⇒     _[4]=g
⇒     _[5]=f
⇒   [2]:
⇒     _[1]=g
⇒     _[2]=f
⇒     _[3]=d-2e
⇒     _[4]=c-e
⇒     _[5]=b
⇒     _[6]=a
⇒ [2]:
⇒   0
ideal I = f-fa,e+b;
qring q = std(I);
poly F = imap(r,F);
esStratum(F);
⇒ //
⇒ // Compute HN expansion
⇒ // -----
⇒ // finished
⇒ //
⇒ // Blowup Step 1 completed
⇒ // Blowup Step 2 completed
⇒ // Blowup Step 3 completed
⇒ // 1 branch finished
⇒ //
⇒ // Elimination starts:
⇒ // -----
⇒ //
⇒ // Remove superfluous equations:
⇒ // -----
⇒ // finished
⇒ //
⇒ // output of 'esStratum' is a list consisting of:
⇒ //   _[1][1] = ideal defining the equisingularity stratum
⇒ //   _[1][2] = ideal defining the part of the equisingularity stratum
⇒ //           where all equimultiple sections are trivial
⇒ //   _[2] = 0
⇒ [1]:

```

```

⇒      [1]:
⇒      _[1]=e
⇒      _[2]=a
⇒      _[3]=-4c+4d+d2
⇒      _[4]=g
⇒      [2]:
⇒      _[1]=g
⇒      _[2]=e
⇒      _[3]=d
⇒      _[4]=c
⇒      _[5]=a
⇒ [2]:
⇒      0
printlevel=p;

```

See also: [Section D.6.12.2 \[esIdeal\]](#), page 1698; [Section D.6.12.4 \[isEquisising\]](#), page 1702.

#### D.6.12.4 isEquisising

Procedure from library `equisising.lib` (see [Section D.6.12 \[equisising.lib\]](#), page 1697).

**Usage:** `isEquisising(F[,m,L]);` F poly, m int, L list

**Assume:** F defines a deformation of a reduced bivariate polynomial f and the characteristic of the basering does not divide `mult(f)`.

If `nv` is the number of variables of the basering, then the first `nv-2` variables are the deformation parameters.

If the basering is a qring, `ideal(basing)` must only depend on the deformation parameters.

**Compute:** tests if the given family is equisingular along the trivial section.

**Return:** int: 1 if the family is equisingular, 0 otherwise.

**Note:** L is supposed to be the output of `hnexpansion` (with the given ordering of the variables appearing in f).

If m is given, the family is considered over  $A/\text{maxideal}(m)$ .

This procedure uses `execute` or calls a procedure using `execute`. `printlevel>=2` displays additional information.

**Example:**

```

LIB "equisising.lib";
ring r = 0,(a,b,x,y),ds;
poly F = (x2+2xy+y2+x5)+ay3+bx5;
isEquisising(F);
⇒ 0
ideal I = ideal(a);
qring q = std(I);
poly F = imap(r,F);
isEquisising(F);
⇒ 1
ring rr=0,(A,B,C,x,y),ls;
poly f=x7+y7+(x-y)^2*x2y2;
poly F=f+A*y*diff(f,x)+B*x*diff(f,x);
isEquisising(F);
⇒ 0

```

```
isEquisising(F,2);    // computation over  $\mathbb{Q}[a,b] / \langle a,b \rangle^2$ 
 $\mapsto 1$ 
```

### D.6.12.5 control\_Matrix

Procedure from library `equisising.lib` (see [Section D.6.12 \[equisising\\_lib\]](#), page 1697).

**Assume:** `L` is the output of `multsequence(hnexpansion(f))`.

**Return:** list `M` of 4 intmat's:

- `M[1]` contains the multiplicities at the respective infinitely near points `p[i,j]` (`i=step of blowup+1`, `j=branch`) – if branches `j=k,...,k+m` pass through the same `p[i,j]` then the multiplicity is stored in `M[1][k,j]`, while `M[1][k+1]=...=M[1][k+m]=0`.
- `M[2]` contains the number of branches meeting at `p[i,j]` (again, the information is stored according to the above rule)
- `M[3]` contains the information about the splitting of `M[1][i,j]` with respect to different tangents of branches at `p[i,j]` (information is stored only for minimal `j>=k` corresponding to a new tangent direction). The entries are the sum of multiplicities of all branches with the respective tangent.
- `M[4]` contains the maximal sum of higher multiplicities for a branch passing through `p[i,j]` (= degree Bound for blowing up)

**Note:** the branches are ordered in such a way that only consecutive branches can meet at an infinitely near point.  
the final rows of the matrices `M[1],...,M[3]` is `(1,1,1,...,1)`, and correspond to infinitely near points such that the strict transforms of the branches are smooth and intersect the exceptional divisor transversally.

See also: [Section D.6.15.8 \[multsequence\]](#), page 1728.

### D.6.13 gmssing\_lib

**Library:** `gmssing.lib`

**Purpose:** Gauss-Manin System of Isolated Singularities

**Author:** Mathias Schulze, mschulze at mathematik.uni-kl.de

**Overview:** A library for computing invariants related to the Gauss-Manin system of an isolated hypersurface singularity.

**References:**

- [Sch01] M. Schulze: Algorithms for the Gauss-Manin connection. J. Symb. Comp. 32,5 (2001), 549-564.
- [Sch02] M. Schulze: The differential structure of the Brieskorn lattice. In: A.M. Cohen et al.: Mathematical Software - ICMS 2002. World Scientific (2002).
- [Sch03] M. Schulze: Monodromy of Hypersurface Singularities. Acta Appl. Math. 75 (2003), 3-13.
- [Sch04] M. Schulze: A normal form algorithm for the Brieskorn lattice. J. Symb. Comp. 38,4 (2004), 1207-1225.

**Procedures:** See also: [Section 7.5.4 \[dmod\\_lib\]](#), page 401; [Section D.6.14 \[gmsspoly\\_lib\]](#), page 1718; [Section D.6.17 \[monodromy\\_lib\]](#), page 1738; [Section D.6.22 \[spectrum\\_lib\]](#), page 1773.

### D.6.13.1 gmsring

Procedure from library `gmssing.lib` (see [Section D.6.13 \[gmssing-lib\]](#), page 1703).

**Usage:** `gmsring(t,s);` poly `t`, string `s`

**Assume:** characteristic 0; local degree ordering;  
isolated critical point 0 of `t`

**Return:**

```
ring G; Gauss-Manin system of t with variable s
poly gmspoly=t;
ideal gmsjacob; Jacobian ideal of t
ideal gmsstd; standard basis of Jacobian ideal
matrix gmsmatrix; matrix(gmsjacob)*gmsmatrix==matrix(gmsstd)
ideal gmsbasis; monomial vector space basis of Jacobian algebra
int Gmssing::gmsmaxdeg; maximal weight of variables
```

**Note:** `gmsbasis` is a  $C[[s]]$ -basis of  $H^*$  and  $[t,s]=s^2$

**Example:**

```
LIB "gmssing.lib";
ring @R=0,(x,y),ds;
poly t=x5+x2y2+y5;
def G=gmsring(t,"s");
setring(G);
gmspoly;
  ↳ x2y2+x5+y5
print(gmsjacob);
  ↳ 2xy2+5x4,
  ↳ 2x2y+5y4
print(gmsstd);
  ↳ 2x2y+5y4,
  ↳ 5x5-5y5,
  ↳ 2xy2+5x4,
  ↳ 10y6+25x3y4
print(gmsmatrix);
  ↳ 0,x, 1,-2xy,
  ↳ 1,-y,0,2y2+5x3
print(gmsbasis);
  ↳ y5,
  ↳ y4,
  ↳ y3,
  ↳ y2,
  ↳ xy,
  ↳ y,
  ↳ x4,
  ↳ x3,
  ↳ x2,
  ↳ x,
  ↳ 1
Gmssing::gmsmaxdeg;
  ↳ 1
```

### D.6.13.2 gmsnf

Procedure from library `gmssing.lib` (see [Section D.6.13 \[gmssing\\_lib\]](#), page 1703).

**Usage:** `gmsnf(p,K);` poly  $p$ , int  $K$

**Assume:** basering returned by `gmsring`

**Return:** list  $nf$ ;  
ideal  $nf[1]$ ; projection of  $p$  to  $\langle gmsbasis \rangle C[[s]] \bmod s^{(K+1)}$  ideal  $nf[2]$ ;  $p == nf[1] + nf[2]$

**Note:** computation can be continued by setting  $p = nf[2]$

**Example:**

```
LIB "gmssing.lib";
ring R=0,(x,y),ds;
poly t=x5+x2y2+y5;
def G=gmsring(t,"s");
setring(G);
list l0=gmsnf(gmspoly,0);
print(l0[1]);
 $\mapsto -1/2y^5$ 
list l1=gmsnf(gmspoly,1);
print(l1[1]);
 $\mapsto -1/2y^5 + 1/2s$ 
list l=gmsnf(l0[2],1);
print(l[1]);
 $\mapsto 1/2s$ 
```

### D.6.13.3 gmscoeffs

Procedure from library `gmssing.lib` (see [Section D.6.13 \[gmssing\\_lib\]](#), page 1703).

**Usage:** `gmscoeffs(p,K);` poly  $p$ , int  $K$

**Assume:** basering constructed by `gmsring`

**Return:**

list  $l$ ;  
matrix  $l[1]$ ;  $C[[s]]$ -basis representation of  $p \bmod s^{(K+1)}$   
ideal  $l[2]$ ;  $p == \text{matrix}(gmsbasis) * l[1] + l[2]$

**Note:** computation can be continued by setting  $p = l[2]$

**Example:**

```
LIB "gmssing.lib";
ring R=0,(x,y),ds;
poly t=x5+x2y2+y5;
def G=gmsring(t,"s");
setring(G);
list l0=gmscoeffs(gmspoly,0);
print(l0[1]);
 $\mapsto -1/2,$ 
 $\mapsto 0,$ 
 $\mapsto 0,$ 
 $\mapsto 0,$ 
 $\mapsto 0,$ 
```

```

↳ 0,
↳ 0,
↳ 0,
↳ 0,
↳ 0,
↳ 0
list l1=gmscoeffs(gmspoly,1);
print(l1[1]);
↳ -1/2,
↳ 0,
↳ 0,
↳ 0,
↳ 0,
↳ 0,
↳ 0,
↳ 0,
↳ 0,
↳ 0,
↳ 1/2s
list l=gmscoeffs(l0[2],1);
print(l[1]);
↳ 0,
↳ 0,
↳ 0,
↳ 0,
↳ 0,
↳ 0,
↳ 0,
↳ 0,
↳ 0,
↳ 0,
↳ 0,
↳ 0,
↳ 0,
↳ 0,
↳ 1/2s

```

#### D.6.13.4 bernstein

Procedure from library `gmssing.lib` (see [Section D.6.13 \[gmssing.lib\]](#), page 1703).

**Usage:**      `bernstein(t); poly t`

**Assume:**    `characteristic 0; local degree ordering;`  
               `isolated critical point 0 of t`

**Return:**

```

list bs; Bernstein-Sato polynomial b(s) of t
ideal bs[1];
number bs[1][i]; i-th root of b(s)
intvec bs[2];
int bs[2][i]; multiplicity of i-th root of b(s)

```

**Example:**

```

LIB "gmssing.lib";
ring R=0,(x,y),ds;
poly t=x5+x2y2+y5;
bernstein(t);
↳ [1]:

```



```

⇒  _[1] = -13/10
⇒  _[2] = -11/10
⇒  _[3] = -1
⇒  _[4] = -9/10
⇒  _[5] = -7/10
⇒  _[6] = -1/2
⇒  [2] :
⇒      1, 1, 2, 1, 1, 2

```

### D.6.13.5 monodromy

Procedure from library `gmssing.lib` (see [Section D.6.13 \[gmssing\\_lib\]](#), page 1703).

**Usage:** `monodromy(t); poly t`

**Assume:** characteristic 0; local degree ordering;  
isolated critical point 0 of  $t$

**Return:**

```

list l; Jordan data jordan(M) of monodromy matrix  $\exp(-2\pi i M)$ 
ideal l[1];
number l[1][i]; eigenvalue of i-th Jordan block of M
intvec l[2];
int l[2][i]; size of i-th Jordan block of M
intvec l[3];
int l[3][i]; multiplicity of i-th Jordan block of M

```

**Example:**

```

LIB "gmssing.lib";
ring R=0,(x,y),ds;
poly t=x5+x2y2+y5;
monodromy(t);
⇒ [1] :
⇒  _[1] = 1/2
⇒  _[2] = 7/10
⇒  _[3] = 9/10
⇒  _[4] = 1
⇒  _[5] = 11/10
⇒  _[6] = 13/10
⇒ [2] :
⇒  2, 1, 1, 1, 1, 1
⇒ [3] :
⇒  1, 2, 2, 1, 2, 2

```

See also: [Section D.3.2 \[linalg\\_lib\]](#), page 988; [Section D.6.17 \[monodromy\\_lib\]](#), page 1738.

### D.6.13.6 spectrum

Procedure from library `gmssing.lib` (see [Section D.6.13 \[gmssing\\_lib\]](#), page 1703).

**Usage:** `spectrum(t); poly t`

**Assume:** characteristic 0; local degree ordering;  
isolated critical point 0 of  $t$

**Return:**

```

list sp; singularity spectrum of t
ideal sp[1];
number sp[1][i]; i-th spectral number
intvec sp[2];
int sp[2][i]; multiplicity of i-th spectral number

```

**Example:**

```

LIB "gmssing.lib";
ring R=0,(x,y),ds;
poly t=x5+x2y2+y5;
spprint(spectrum(t));
↦ (-1/2,1),(-3/10,2),(-1/10,2),(0,1),(1/10,2),(3/10,2),(1/2,1)

```

See also: [Section D.6.22 \[spectrum\\_lib\]](#), page 1773.

**D.6.13.7 sppairs**

Procedure from library `gmssing.lib` (see [Section D.6.13 \[gmssing\\_lib\]](#), page 1703).

**Usage:** sppairs(t); poly t

**Assume:** characteristic 0; local degree ordering;  
isolated critical point 0 of t

**Return:**

```

list spp; spectral pairs of t
ideal spp[1];
number spp[1][i]; V-filtration index of i-th spectral pair
intvec spp[2];
int spp[2][i]; weight filtration index of i-th spectral pair
intvec spp[3];
int spp[3][i]; multiplicity of i-th spectral pair

```

**Example:**

```

LIB "gmssing.lib";
ring R=0,(x,y),ds;
poly t=x5+x2y2+y5;
sppprint(sppairs(t));
↦ ((-1/2,2),1),((-3/10,1),2),((-1/10,1),2),((0,1),1),((1/10,1),2),((3/10,1)\
,2),((1/2,0),1)

```

See also: [Section D.6.22 \[spectrum\\_lib\]](#), page 1773.

**D.6.13.8 vfilt**

Procedure from library `gmssing.lib` (see [Section D.6.13 \[gmssing\\_lib\]](#), page 1703).

**Usage:** vfilt(t); poly t

**Assume:** characteristic 0; local degree ordering;  
isolated critical point 0 of t

**Return:**

```

list v; V-filtration on  $H''/s^*H''$ 
ideal v[1];
number v[1][i]; V-filtration index of i-th spectral number
intvec v[2];

```

`int v[2][i];` multiplicity of  $i$ -th spectral number  
`list v[3];`  
`module v[3][i];` vector space of  $i$ -th graded part in terms of  $v[4]$   
`ideal v[4];` monomial vector space basis of  $H''/s \cdot H''$   
`ideal v[5];` standard basis of Jacobian ideal

**Example:**

```

LIB "gmssing.lib";
ring R=0,(x,y),ds;
poly t=x5+x2y2+y5;
vfilt(t);
↳ [1]:
↳   _[1]=-1/2
↳   _[2]=-3/10
↳   _[3]=-1/10
↳   _[4]=0
↳   _[5]=1/10
↳   _[6]=3/10
↳   _[7]=1/2
↳ [2]:
↳   1,2,2,1,2,2,1
↳ [3]:
↳   [1]:
↳     _[1]=gen(11)
↳   [2]:
↳     _[1]=gen(10)
↳     _[2]=gen(6)
↳   [3]:
↳     _[1]=gen(9)
↳     _[2]=gen(4)
↳   [4]:
↳     _[1]=gen(5)
↳   [5]:
↳     _[1]=gen(3)
↳     _[2]=gen(8)
↳   [6]:
↳     _[1]=gen(2)
↳     _[2]=gen(7)
↳   [7]:
↳     _[1]=gen(1)
↳ [4]:
↳   _[1]=y5
↳   _[2]=y4
↳   _[3]=y3
↳   _[4]=y2
↳   _[5]=xy
↳   _[6]=y
↳   _[7]=x4
↳   _[8]=x3
↳   _[9]=x2
↳   _[10]=x
↳   _[11]=1
↳ [5]:

```

```

⇒  _[1]=2x2y+5y4
⇒  _[2]=5x5-5y5
⇒  _[3]=2xy2+5x4
⇒  _[4]=10y6+25x3y4

```

See also: [Section D.6.22 \[spectrum.lib\]](#), page 1773.

### D.6.13.9 vwfilt

Procedure from library `gmssing.lib` (see [Section D.6.13 \[gmssing.lib\]](#), page 1703).

**Usage:** `vwfilt(t); poly t`

**Assume:** characteristic 0; local degree ordering;  
isolated critical point 0 of `t`

**Return:**

```

list vw; weighted V-filtration on  $H''/s^*H''$ 
ideal vw[1];
number vw[1][i]; V-filtration index of i-th spectral pair
intvec vw[2];
int vw[2][i]; weight filtration index of i-th spectral pair
intvec vw[3];
int vw[3][i]; multiplicity of i-th spectral pair
list vw[4];
module vw[4][i]; vector space of i-th graded part in terms of vw[5]
ideal vw[5]; monomial vector space basis of  $H''/s^*H''$ 
ideal vw[6]; standard basis of Jacobian ideal

```

**Example:**

```

LIB "gmssing.lib";
ring R=0,(x,y),ds;
poly t=x5+x2y2+y5;
vwfilt(t);
⇒ [1]:
⇒  _[1]=-1/2
⇒  _[2]=-3/10
⇒  _[3]=-1/10
⇒  _[4]=0
⇒  _[5]=1/10
⇒  _[6]=3/10
⇒  _[7]=1/2
⇒ [2]:
⇒  2,1,1,1,1,1,0
⇒ [3]:
⇒  1,2,2,1,2,2,1
⇒ [4]:
⇒  [1]:
⇒  _[1]=gen(11)
⇒  [2]:
⇒  _[1]=gen(10)
⇒  _[2]=gen(6)
⇒  [3]:
⇒  _[1]=gen(9)
⇒  _[2]=gen(4)

```

```

↳      [4]:
↳      _[1]=gen(5)
↳      [5]:
↳      _[1]=gen(3)
↳      _[2]=gen(8)
↳      [6]:
↳      _[1]=gen(2)
↳      _[2]=gen(7)
↳      [7]:
↳      _[1]=gen(1)
↳ [5]:
↳ _[1]=y5
↳ _[2]=y4
↳ _[3]=y3
↳ _[4]=y2
↳ _[5]=xy
↳ _[6]=y
↳ _[7]=x4
↳ _[8]=x3
↳ _[9]=x2
↳ _[10]=x
↳ _[11]=1
↳ [6]:
↳ _[1]=2x2y+5y4
↳ _[2]=5x5-5y5
↳ _[3]=2xy2+5x4
↳ _[4]=10y6+25x3y4

```

See also: [Section D.6.22 \[spectrum\\_lib\]](#), page 1773.

### D.6.13.10 tmatrix

Procedure from library `gmssing.lib` (see [Section D.6.13 \[gmssing\\_lib\]](#), page 1703).

**Usage:** `tmatrix(t);` poly  $t$

**Assume:** characteristic 0; local degree ordering;  
isolated critical point 0 of  $t$

**Return:**

list  $l=A0,A1,T,M$ ;  
matrix  $A0,A1$ ;  $t=A0+s*A1+s^2*(d/ds)$  on  $H''$  w.r.t.  $C[[s]]$ -basis  $M*T$   
module  $T$ ;  $C$ -basis of  $C^\mu$   
ideal  $M$ ; monomial  $C$ -basis of  $H''/sH''$

**Example:**

```

LIB "gmssing.lib";
ring R=0,(x,y),ds;
poly t=x5+x2y2+y5;
list l=tmatrix(t);
print(l[1]);
↳ 0,0,0,0,0,0,0,0,0,0,0,0,
↳ 0,0,0,0,0,0,0,0,0,0,0,0,
↳ 0,0,0,0,0,0,0,0,0,0,0,0,
↳ 0,0,0,0,0,0,0,0,0,0,0,0,

```

```

⇒ 0,0,0,0,0,0,0,0,0,0,0,0,
⇒ 0,0,0,0,0,0,0,0,0,0,0,0,
⇒ 0,0,0,0,0,0,0,0,0,0,0,0,
⇒ 0,0,0,0,0,0,0,0,0,0,0,0,
⇒ 0,0,0,0,0,0,0,0,0,0,0,0,
⇒ 0,0,0,0,0,0,0,0,0,0,0,0,
⇒ 0,0,0,0,0,0,0,0,0,0,0,0,
⇒ 1,0,0,0,0,0,0,0,0,0,0,0
print(l[2]);
⇒ 1/2,0, 0, 0, 0, 0,0, 0, 0, 0, 0,
⇒ 0, 7/10,0, 0, 0, 0,0, 0, 0, 0, 0,
⇒ 0, 0, 7/10,0, 0, 0,0, 0, 0, 0, 0,
⇒ 0, 0, 0, 9/10,0, 0,0, 0, 0, 0, 0,
⇒ 0, 0, 0, 0, 9/10,0,0, 0, 0, 0, 0,
⇒ 0, 0, 0, 0, 0, 1,0, 0, 0, 0, 0,
⇒ 0, 0, 0, 0, 0, 0,11/10,0, 0, 0, 0,
⇒ 0, 0, 0, 0, 0, 0,0, 11/10,0, 0, 0,
⇒ 0, 0, 0, 0, 0, 0,0, 0, 13/10,0, 0,
⇒ 0, 0, 0, 0, 0, 0,0, 0, 0, 13/10,0,
⇒ 0, 0, 0, 0, 0, 0,0, 0, 0, 0, 3/2
print(l[3]);
⇒ 85/4,0, 0, 0,0,85/8,0,0,0,0,1/2,
⇒ 0, 125,0, 0,0,0, 0,0,1,0,0,
⇒ 0, 0, 0, 5,0,0, 1,0,0,0,0,
⇒ 0, 0, 0, 0,4,0, 0,0,0,0,0,
⇒ 2, 0, 0, 0,0,1, 0,0,0,0,0,
⇒ 0, 0, 16, 0,0,0, 0,0,0,0,0,
⇒ 0, 0, 125,0,0,0, 0,0,0,1,0,
⇒ 0, 0, 0, 0,5,0, 0,1,0,0,0,
⇒ 0, 0, 0, 4,0,0, 0,0,0,0,0,
⇒ 0, 16, 0, 0,0,0, 0,0,0,0,0,
⇒ -1, 0, 0, 0,0,0, 0,0,0,0,0
print(l[4]);
⇒ y5,
⇒ y4,
⇒ y3,
⇒ y2,
⇒ xy,
⇒ y,
⇒ x4,
⇒ x3,
⇒ x2,
⇒ x,
⇒ 1

```

### D.6.13.11 endvfilt

Procedure from library `gmssing.lib` (see [Section D.6.13 \[gmssing\\_lib\]](#), page 1703).

**Usage:** `endvfilt(v);` list `v`

**Assume:** `v` returned by `vfilt`

**Return:**

list `ev`; V-filtration on Jacobian algebra

```

ideal ev[1];
  number ev[1][i]; i-th V-filtration index
intvec ev[2];
  int ev[2][i]; i-th multiplicity
list ev[3];
  module ev[3][i]; vector space of i-th graded part in terms of ev[4]
ideal ev[4]; monomial vector space basis of Jacobian algebra
ideal ev[5]; standard basis of Jacobian ideal

```

**Example:**

```

LIB "gmssing.lib";
ring R=0,(x,y),ds;
poly t=x5+x2y2+y5;
endvfilt(vfilt(t));
↳ [1]:
↳   _[1]=0
↳   _[2]=1/5
↳   _[3]=2/5
↳   _[4]=1/2
↳   _[5]=3/5
↳   _[6]=4/5
↳   _[7]=1
↳ [2]:
↳   1,2,2,1,2,2,1
↳ [3]:
↳   [1]:
↳     _[1]=gen(11)
↳   [2]:
↳     _[1]=gen(10)
↳     _[2]=gen(6)
↳   [3]:
↳     _[1]=gen(9)
↳     _[2]=gen(4)
↳   [4]:
↳     _[1]=gen(5)
↳   [5]:
↳     _[1]=gen(8)
↳     _[2]=gen(3)
↳   [6]:
↳     _[1]=gen(7)
↳     _[2]=gen(2)
↳   [7]:
↳     _[1]=gen(1)
↳ [4]:
↳   _[1]=y5
↳   _[2]=y4
↳   _[3]=y3
↳   _[4]=y2
↳   _[5]=xy
↳   _[6]=y
↳   _[7]=x4
↳   _[8]=x3
↳   _[9]=x2

```

```

↳   _[10]=x
↳   _[11]=1
↳   [5]:
↳   _[1]=2x2y+5y4
↳   _[2]=5x5-5y5
↳   _[3]=2xy2+5x4
↳   _[4]=10y6+25x3y4

```

### D.6.13.12 sppnf

Procedure from library `gmssing.lib` (see [Section D.6.13 \[gmssing\\_lib\]](#), page 1703).

**Usage:** `sppnf(list(a,w[m]));` ideal `a`, intvec `w`, intvec `m`

**Assume:** `ncols(a)==size(w)==size(m)`

**Return:** order  $(a[i][,w[i]])$  with multiplicity  $m[i]$  lexicographically

**Example:**

```

LIB "gmssing.lib";
ring R=0,(x,y),ds;
list sp=list(ideal(-1/2,-3/10,-3/10,-1/10,-1/10,0,1/10,1/10,3/10,3/10,1/2),
intvec(2,1,1,1,1,1,1,1,1,1,0));
sppprint(sppnf(sp));
↳ ((-1/2,2),1),((-3/10,1),2),((-1/10,1),2),((0,1),1),((1/10,1),2),((3/10,1)\
,2),((1/2,0),1)

```

### D.6.13.13 sppprint

Procedure from library `gmssing.lib` (see [Section D.6.13 \[gmssing\\_lib\]](#), page 1703).

**Usage:** `sppprint(spp);` list `spp`

**Return:** string `s`; spectral pairs `spp`

**Example:**

```

LIB "gmssing.lib";
ring R=0,(x,y),ds;
list spp=list(ideal(-1/2,-3/10,-1/10,0,1/10,3/10,1/2),intvec(2,1,1,1,1,1,0),
intvec(1,2,2,1,2,2,1));
sppprint(spp);
↳ ((-1/2,2),1),((-3/10,1),2),((-1/10,1),2),((0,1),1),((1/10,1),2),((3/10,1)\
,2),((1/2,0),1)

```

### D.6.13.14 spadd

Procedure from library `gmssing.lib` (see [Section D.6.13 \[gmssing\\_lib\]](#), page 1703).

**Usage:** `spadd(sp1,sp2);` list `sp1`, list `sp2`

**Return:** list `sp`; sum of spectra `sp1` and `sp2`

**Example:**

```

LIB "gmssing.lib";
ring R=0,(x,y),ds;
list sp1=list(ideal(-1/2,-3/10,-1/10,0,1/10,3/10,1/2),intvec(1,2,2,1,2,2,1));
spprint(sp1);

```



```

⇒ (-1/2,1), (-3/10,2), (-1/10,2), (0,1), (1/10,2), (3/10,2), (1/2,1)
list sp2=list(ideal(-1/6,1/6),intvec(1,1));
spprint(sp2);
⇒ (-1/6,1), (1/6,1)
spprint(spadd(sp1,sp2));
⇒ (-1/2,1), (-3/10,2), (-1/6,1), (-1/10,2), (0,1), (1/10,2), (1/6,1), (3/10,2), (1/2,1)

```

### D.6.13.15 spsub

Procedure from library `gmssing.lib` (see [Section D.6.13 \[gmssing.lib\]](#), page 1703).

**Usage:** `spsub(sp1,sp2);` list sp1, list sp2

**Return:** list sp; difference of spectra sp1 and sp2

**Example:**

```

LIB "gmssing.lib";
ring R=0,(x,y),ds;
list sp1=list(ideal(-1/2,-3/10,-1/10,0,1/10,3/10,1/2),intvec(1,2,2,1,2,2,1));
spprint(sp1);
⇒ (-1/2,1), (-3/10,2), (-1/10,2), (0,1), (1/10,2), (3/10,2), (1/2,1)
list sp2=list(ideal(-1/6,1/6),intvec(1,1));
spprint(sp2);
⇒ (-1/6,1), (1/6,1)
spprint(spsub(sp1,sp2));
⇒ (-1/2,1), (-3/10,2), (-1/6,-1), (-1/10,2), (0,1), (1/10,2), (1/6,-1), (3/10,2), (1/2,1)

```

### D.6.13.16 spmul

Procedure from library `gmssing.lib` (see [Section D.6.13 \[gmssing.lib\]](#), page 1703).

**Usage:** `spmul(sp0,k);` list sp0, int[vec] k

**Return:** list sp; linear combination of spectra sp0 with coefficients k

**Example:**

```

LIB "gmssing.lib";
ring R=0,(x,y),ds;
list sp=list(ideal(-1/2,-3/10,-1/10,0,1/10,3/10,1/2),intvec(1,2,2,1,2,2,1));
spprint(sp);
⇒ (-1/2,1), (-3/10,2), (-1/10,2), (0,1), (1/10,2), (3/10,2), (1/2,1)
spprint(spmul(sp,2));
⇒ (-1/2,2), (-3/10,4), (-1/10,4), (0,2), (1/10,4), (3/10,4), (1/2,2)
list sp1=list(ideal(-1/6,1/6),intvec(1,1));
spprint(sp1);
⇒ (-1/6,1), (1/6,1)
list sp2=list(ideal(-1/3,0,1/3),intvec(1,2,1));
spprint(sp2);
⇒ (-1/3,1), (0,2), (1/3,1)
spprint(spmul(list(sp1,sp2),intvec(1,2)));
⇒ (-1/3,2), (-1/6,1), (0,4), (1/6,1), (1/3,2)

```

### D.6.13.17 spissemicont

Procedure from library `gmssing.lib` (see [Section D.6.13 \[gmssing\\_lib\]](#), page 1703).

**Usage:** `spissemicont(sp[1]);` list sp, int opt

**Return:**

int k=  
 1; if sum of sp is positive on all intervals  $[a, a+1)$  [and  $(a, a+1]$   
 0; if sum of sp is negative on some interval  $[a, a+1)$  [or  $(a, a+1]$

**Example:**

```
LIB "gmssing.lib";
ring R=0,(x,y),ds;
list sp1=list(ideal(-1/2,-3/10,-1/10,0,1/10,3/10,1/2),intvec(1,2,2,1,2,2,1));
spprint(sp1);
↪ (-1/2,1),(-3/10,2),(-1/10,2),(0,1),(1/10,2),(3/10,2),(1/2,1)
list sp2=list(ideal(-1/6,1/6),intvec(1,1));
spprint(sp2);
↪ (-1/6,1),(1/6,1)
spissemicont(spsub(sp1,spmul(sp2,3)));
↪ 1
spissemicont(spsub(sp1,spmul(sp2,4)));
↪ 0
```

### D.6.13.18 spsemicont

Procedure from library `gmssing.lib` (see [Section D.6.13 \[gmssing\\_lib\]](#), page 1703).

**Usage:** `spsemicont(sp0,sp,k[1]);` list sp0, list sp

**Return:**

list l;  
 intvec l[i]; if the spectra sp0 occur with multiplicities k  
 in a deformation of a [quasihomogeneous] singularity  
 with spectrum sp then  $k \leq l[i]$

**Example:**

```
LIB "gmssing.lib";
ring R=0,(x,y),ds;
list sp0=list(ideal(-1/2,-3/10,-1/10,0,1/10,3/10,1/2),intvec(1,2,2,1,2,2,1));
spprint(sp0);
↪ (-1/2,1),(-3/10,2),(-1/10,2),(0,1),(1/10,2),(3/10,2),(1/2,1)
list sp1=list(ideal(-1/6,1/6),intvec(1,1));
spprint(sp1);
↪ (-1/6,1),(1/6,1)
list sp2=list(ideal(-1/3,0,1/3),intvec(1,2,1));
spprint(sp2);
↪ (-1/3,1),(0,2),(1/3,1)
list sp=sp1,sp2;
list l=spsemicont(sp0,sp);
l;
↪ [1]:
↪ 3
↪ [2]:
```

```

    ↪      2,1
    spissemicont(spsub(sp0,spmul(sp,l[1])));
    ↪      1
    spissemicont(spsub(sp0,spmul(sp,l[1]-1)));
    ↪      1
    spissemicont(spsub(sp0,spmul(sp,l[1]+1)));
    ↪      0

```

### D.6.13.19 spmilnor

Procedure from library `gmssing.lib` (see [Section D.6.13 \[gmssing\\_lib\]](#), page 1703).

**Usage:** `spmilnor(sp);` list `sp`

**Return:** `int mu;` Milnor number of spectrum `sp`

**Example:**

```

LIB "gmssing.lib";
ring R=0,(x,y),ds;
list sp=list(ideal(-1/2,-3/10,-1/10,0,1/10,3/10,1/2),intvec(1,2,2,1,2,2,1));
spprint(sp);
↪ (-1/2,1),(-3/10,2),(-1/10,2),(0,1),(1/10,2),(3/10,2),(1/2,1)
spmilnor(sp);
↪ 11

```

### D.6.13.20 spgeomgenus

Procedure from library `gmssing.lib` (see [Section D.6.13 \[gmssing\\_lib\]](#), page 1703).

**Usage:** `spgeomgenus(sp);` list `sp`

**Return:** `int g;` geometrical genus of spectrum `sp`

**Example:**

```

LIB "gmssing.lib";
ring R=0,(x,y),ds;
list sp=list(ideal(-1/2,-3/10,-1/10,0,1/10,3/10,1/2),intvec(1,2,2,1,2,2,1));
spprint(sp);
↪ (-1/2,1),(-3/10,2),(-1/10,2),(0,1),(1/10,2),(3/10,2),(1/2,1)
spgeomgenus(sp);
↪ 6

```

### D.6.13.21 spgamma

Procedure from library `gmssing.lib` (see [Section D.6.13 \[gmssing\\_lib\]](#), page 1703).

**Usage:** `spgamma(sp);` list `sp`

**Return:** `number gamma;` gamma invariant of spectrum `sp`

**Example:**

```

LIB "gmssing.lib";
ring R=0,(x,y),ds;
list sp=list(ideal(-1/2,-3/10,-1/10,0,1/10,3/10,1/2),intvec(1,2,2,1,2,2,1));
spprint(sp);
↪ (-1/2,1),(-3/10,2),(-1/10,2),(0,1),(1/10,2),(3/10,2),(1/2,1)
spgamma(sp);
↪ 1/240

```

### D.6.14 gmspoly\_lib

**Library:** gmspoly.lib

**Purpose:** Gauss-Manin System of Tame Polynomials

**Author:** Mathias Schulze, mschulze at mathematik.uni-kl.de

**Overview:** A library for computing the Gauss-Manin system of a cohomologically tame polynomial  $f$ . Schulze's algorithm [Sch05], based on Sabbah's theory [Sab98], is used to compute a good basis of (the Brieskorn lattice of) the Gauss-Manin system and the differential operation of  $f$  in terms of this basis. In addition, there is a test for tameness in the sense of Broughton. Tame polynomials can be considered as an affine algebraic analogue of local analytic isolated hypersurface singularities. They have only finitely many critical points, and those at infinity do not give rise to atypical values in a sense depending on the precise notion of tameness considered. Well-known notions of tameness like tameness, M-tameness, Malgrange-tameness, and cohomological tameness, and their relations, are reviewed in [Sab98,8]. For ordinary tameness, see Broughton [Bro88,3]. Sabbah [Sab98] showed that the Gauss-Manin system, the D-module direct image of the structure sheaf, of a cohomologically tame polynomial carries a similar structure as in the isolated singularity case, coming from a Mixed Hodge structure on the cohomology of the Milnor (typical) fibre (see gmssing.lib). The data computed by this library encodes the differential structure of the Gauss-Manin system, and the Mixed Hodge structure of the Milnor fibre over the complex numbers. As a consequence, it yields the Hodge numbers, spectral pairs, and monodromy at infinity.

**References:**

- [Bro88] S. Broughton: Milnor numbers and the topology of polynomial hypersurfaces. Inv. Math. 92 (1988) 217-241.
- [Sab98] C. Sabbah: Hypergeometric periods for a tame polynomial. arXiv.org math.AG/9805077.
- [Sch05] M. Schulze: Good bases for tame polynomials. J. Symb. Comp. 39,1 (2005), 103-126.

**Procedures:** See also: [Section D.6.13 \[gmssing\\_lib\]](#), page 1703.

#### D.6.14.1 isTame

Procedure from library gmspoly.lib (see [Section D.6.14 \[gmspoly\\_lib\]](#), page 1718).

**Usage:** isTame(f); poly f

**Assume:** basering has no variables named w(1),w(2),...

**Return:**

```
int k=
    1; if f is tame in the sense of Broughton [Bro88,3]
    0; if f is not tame
```

**Remarks:** procedure implements Proposition 3.1 in [Bro88]

**Example:**

```
LIB "gmspoly.lib";
ring R=0,(x,y),dp;
isTame(x2y+x);
  ↳ 0
isTame(x3+y3+xy);
  ↳ 1
```

### D.6.14.2 goodBasis

Procedure from library `gmspoly.lib` (see [Section D.6.14 \[gmspoly\\_lib\]](#), page 1718).

**Usage:** `goodBasis(f); poly f`

**Assume:** `f` is cohomologically tame in the sense of Sabbah [Sab98,8]

**Return:**

ring `R`; basering with new variable `s`  
 ideal `b`; `[matrix(b)]` is a good basis of the Brieskorn lattice  
 matrix `A`;  $A(s) = A_0 + s \cdot A_1$  and  $t[matrix(b)] = [matrix(b)](A(s) + s^2 \cdot (d/ds))$

**Remarks:** procedure implements Algorithm 6 in [Sch05]

**Example:**

```
LIB "gmspoly.lib";
ring R=0,(x,y,z),dp;
poly f=x+y+z+x2y2z2;
def Rs=goodBasis(f);
setring(Rs);
b;
⇒ b[1]=1
⇒ b[2]=s2x-3sx2+x3
⇒ b[3]=5/2x
⇒ b[4]=10s2x2-25/2sx3+5/2x4
⇒ b[5]=-25/4sx+25/4x2
print(jet(A,0));
⇒ 0,0,0,-25/8,0,
⇒ 0,0,0,0, 125/8,
⇒ 1,0,0,0, 0,
⇒ 0,1,0,0, 0,
⇒ 0,0,1,0, 0
print(jet(A/var(1),0));
⇒ 1/2,0,0, 0,0,
⇒ 0, 1,0, 0,0,
⇒ 0, 0,3/2,0,0,
⇒ 0, 0,0, 2,0,
⇒ 0, 0,0, 0,5/2
```

See also: [Section D.6.13 \[gmssing\\_lib\]](#), page 1703.

### D.6.15 hnoether\_lib

**Library:** `hnoether.lib`

**Purpose:** Hamburger-Noether (Puiseux) Expansion

**Authors:** Martin Lamm, lamm@mathematik.uni-kl.de  
 Christoph Lossen, lossen@mathematik.uni-kl.de

**Overview:** A library for computing the Hamburger-Noether expansion (analogue of Puiseux expansion over fields of arbitrary characteristic) of a reduced plane curve singularity following [Campillo, A.: Algebroid curves in positive characteristic, Springer LNM 813 (1980)].

The library contains also procedures for computing the (topological) numerical invariants of plane curve singularities.

**Procedures:**

### D.6.15.1 hnexpansion

Procedure from library `hnoether.lib` (see [Section D.6.15 \[hnoether.lib\]](#), page 1719).

**Usage:** `hnexpansion(f, "ess");` `f` poly

**Assume:** `f` is a bivariate polynomial (in the first 2 ring variables)

**Return:** list `L`, containing Hamburger-Noether data of `f`: If the computation of the HNE required no field extension, `L` is a list of lists `L[i]` (corresponding to the output of `develop`, applied to a branch of `f`, but the last entry being omitted):

`L[i][1]`; matrix:

Each row contains the coefficients of the corresponding line of the Hamburger-Noether expansion (HNE) for the `i`-th branch. The end of the line is marked in the matrix by the first ring variable (usually `x`).

`L[i][2]`; intvec:

indicating the length of lines of the HNE

`L[i][3]`; int:

0 if the 1st ring variable was transversal (with respect to the `i`-th branch),  
1 if the variables were changed at the beginning of the computation,  
-1 if an error has occurred.

`L[i][4]`; poly:

the transformed equation of the `i`-th branch to make it possible to extend the Hamburger-Noether data a posteriori without having to do all the previous calculation once again (0 if not needed).

If the computation of the HNE required a field extension, the first entry `L[1]` of the list is a ring, in which a list `hne` of lists (the HN data, as above) and a polynomial `f` (image of `f` over the new field) are stored.

If called with an additional input parameter, `hnexpansion` computes only one representative for each class of conjugate branches (over the ground field active when calling the procedure). In this case, the returned list `L` always has only two entries: `L[1]` is either a list of lists (the HN data) or a ring (as above), and `L[2]` is an integer vector (the number of branches in the respective conjugacy classes).

**Note:** If `f` is known to be irreducible as a power series, `develop(f)` could be chosen instead to avoid a change of basering during the computations.

Increasing `printlevel` leads to more and more comments.

Having defined a variable `HNDebugOn` leads to a maximum number of comments.

**Example:**

```
LIB "hnoether.lib";
ring r=0,(x,y),dp;
// First, an example which requires no field extension:
list Hne=hnexpansion(x4-y6);
⇒ // No change of ring necessary, return value is HN expansion.
size(Hne);           // number of branches
⇒ 2
displayHNE(Hne);     // HN expansion of branches
⇒ // Hamburger-Noether development of branch nr.1:
⇒ x = z(1)*y
⇒ y = z(1)^2
⇒
```

```

⇒ // Hamburger-Noether development of branch nr.2:
⇒   x = z(1)*y
⇒   y = -z(1)^2
⇒
param(Hne[1]);          // parametrization of 1st branch
⇒   _[1]=x3
⇒   _[2]=x2
param(Hne[2]);          // parametrization of 2nd branch
⇒   _[1]=-x3
⇒   _[2]=-x2
// An example which requires a field extension:
list L=hnexpansion((x4-y6)*(y2+x4));
⇒
⇒ // 'hnexpansion' created a list of one ring.
⇒ // To see the ring and the data stored in the ring, type (if you assigned
⇒ // the name L to the list):
⇒   show(L);
⇒ // To display the computed HN expansion, type
⇒   def HNring = L[1]; setring HNring; displayHNE(hne);
def R=L[1]; setring R; displayHNE(hne);
⇒ // Hamburger-Noether development of branch nr.1:
⇒   y = (a)*x^2
⇒
⇒ // Hamburger-Noether development of branch nr.2:
⇒   y = (-a)*x^2
⇒
⇒ // Hamburger-Noether development of branch nr.3:
⇒   x = z(1)*y
⇒   y = z(1)^2
⇒
⇒ // Hamburger-Noether development of branch nr.4:
⇒   x = z(1)*y
⇒   y = -z(1)^2
⇒
basing;
⇒ // coefficients: QQ[a]/(a2+1)
⇒ // number of vars : 2
⇒ //          block  1: ordering ls
⇒ //          : names  x y
⇒ //          block  2 : ordering C
setring r; kill R;
// Computing only one representative per conjugacy class:
L=hnexpansion((x4-y6)*(y2+x4),"ess");
⇒
⇒ // 'hnexpansion' created a list of one ring.
⇒ // To see the ring and the data stored in the ring, type (if you assigned
⇒ // the name L to the list):
⇒   show(L);
⇒ // To display the computed HN expansion, type
⇒   def HNring = L[1]; setring HNring; displayHNE(hne);
⇒ // As second entry of the returned list L, you obtain an integer vector,
⇒ // indicating the number of conjugates for each of the computed branches.
def R=L[1]; setring R; displayHNE(hne);

```

```

⇒ // Hamburger-Noether development of branch nr.1:
⇒ y = (a)*x^2
⇒
⇒ // Hamburger-Noether development of branch nr.2:
⇒ x = z(1)*y
⇒ y = z(1)^2
⇒
⇒ // Hamburger-Noether development of branch nr.3:
⇒ x = z(1)*y
⇒ y = -z(1)^2
⇒
L[2]; // number of branches in respective conjugacy classes
⇒ 2,1,1

```

See also: [Section D.6.15.2 \[develop\]](#), page 1722; [Section D.6.15.5 \[displayHNE\]](#), page 1726; [Section D.6.15.3 \[extdevelop\]](#), page 1723; [Section D.6.15.4 \[param\]](#), page 1724.

### D.6.15.2 develop

Procedure from library `hnoether.lib` (see [Section D.6.15 \[hnoether.lib\]](#), page 1719).

**Usage:**      `develop(f [,n]);`  $f$  poly,  $n$  int

**Assume:**     $f$  is a bivariate polynomial (in the first 2 ring variables) and irreducible as power series (for reducible  $f$  use `hnexpansion`).

**Return:**    list  $L$  with:

$L[1]$ ; matrix:

Each row contains the coefficients of the corresponding line of the Hamburger-Noether expansion (HNE). The end of the line is marked in the matrix by the first ring variable (usually  $x$ ).

$L[2]$ ; intvec:

indicating the length of lines of the HNE

$L[3]$ ; int:    0 if the 1st ring variable was transversal (with respect to  $f$ ),  
                  1 if the variables were changed at the beginning of the computation,  
                  -1 if an error has occurred.

$L[4]$ ; poly:

the transformed polynomial of  $f$  to make it possible to extend the Hamburger-Noether development a posteriori without having to do all the previous calculation once again (0 if not needed)

$L[5]$ ; int:    1 if the curve has exactly one branch (i.e., is irreducible),  
                  0 else (i.e., the curve has more than one HNE, or  $f$  is not valid).

**Display:**    The (non zero) elements of the HNE (if not called by another proc).

**Note:**        The optional parameter  $n$  affects only the computation of the LAST line of the HNE. If it is given, the HN-matrix  $L[1]$  will have at least  $n$  columns. Otherwise, the number of columns will be chosen minimal such that the matrix contains all necessary information (i.e., all lines of the HNE but the last (which is in general infinite) have place).

If  $n$  is negative, the algorithm is stopped as soon as the computed information is sufficient for `invariants(L)`, but the HN-matrix  $L[1]$  may still contain undetermined elements, which are marked with the 2nd variable (of the basering).



For time critical computations it is recommended to use `ring ..., (x,y), ls` as basering - it increases the algorithm's speed.

If `printlevel >= 0` comments are displayed (default is `printlevel=0`).

**Example:**

```
LIB "hnoether.lib";
ring exring = 7, (x,y), ds;
list Hne=develop(4x98+2x49y7+x11y14+2y14);
print(Hne[1]);
⇒ 0,0, 0,0,0,0,3,x,
⇒ 0,x, 0,0,0,0,0,0,
⇒ 0,0, 0,x,0,0,0,0,
⇒ 0,x, 0,0,0,0,0,0,
⇒ 0,-1,0,0,0,0,0,0
// therefore the HNE is:
// z(-1)= 3*z(0)^7 + z(0)^7*z(1),
// z(0) = z(1)*z(2),          (there is 1 zero in the 2nd row before x)
// z(1) = z(2)^3*z(3),        (there are 3 zeroes in the 3rd row)
// z(2) = z(3)*z(4),
// z(3) = -z(4)^2 + 0*z(4)^3 +...+ 0*z(4)^8 + ?*z(4)^9 + ...
// (the missing x in the last line indicates that it is not complete.)
Hne[2];
⇒ 7,1,3,1,-1
param(Hne);
⇒ // ** Warning: result is exact up to order 20 in x and 104 in y !
⇒ _[1]=-x14
⇒ _[2]=-3x98-x109
// parametrization:  x(t)= -t^14+0(t^21),  y(t)= -3t^98+0(t^105)
// (the term -t^109 in y may have a wrong coefficient)
displayHNE(Hne);
⇒ y = 3*x^7+z(1)*x^7
⇒ x = z(1)*z(2)
⇒ z(1) = z(2)^3*z(3)
⇒ z(2) = z(3)*z(4)
⇒ z(3) = -z(4)^2 + ..... (terms of degree >=9)
```

See also: [Section D.6.15.5 \[displayHNE\]](#), page 1726; [Section D.6.15.3 \[extdevelop\]](#), page 1723; [Section D.6.15.1 \[hnexpansion\]](#), page 1720.

### D.6.15.3 extdevelop

Procedure from library `hnoether.lib` (see [Section D.6.15 \[hnoether.lib\]](#), page 1719).

**Usage:** `extdevelop(L,N);` list L, int N

**Assume:** L is the output of `develop(f)`, or of `extdevelop(l,n)`, or one entry in the list `hne` in the ring created by `hnexpansion(f[, "ess"])`.

**Return:** an extension of the Hamburger-Noether development of `f` as a list in the same format as L has (up to the last entry in the output of `develop(f)`).  
Type `help develop`; resp. `help hnexpansion`; for more details.

**Note:** The new HN-matrix will have at least N columns (if the HNE is not finite). In particular, if `f` is irreducible then (in most cases) `extdevelop(develop(f),N)` will produce the same result as `develop(f,N)`.  
If the matrix M of L has n columns then, compared with `parametrization(L)`,

`parametrize(extdevelop(L,N))` will increase the exactness by at least  $(N-n)$  more significant monomials.

**Example:**

```
LIB "hnoether.lib";
ring exring=0,(x,y),dp;
list Hne=hnexpansion(x14-3y2x11-y3x10-y2x9+3y4x8+y5x7+3y4x6+x5*(-y6+y5)
-3y6x3-y7x2+y8);
⇒ // No change of ring necessary, return value is HN expansion.
displayHNE(Hne);    // HNE of 1st,3rd branch is finite
⇒ // Hamburger-Noether development of branch nr.1:
⇒   y = z(1)*x
⇒   x = z(1)^2
⇒
⇒ // Hamburger-Noether development of branch nr.2:
⇒   y = z(1)*x
⇒   x = z(1)^2+z(1)^2*z(2)
⇒   z(1) = z(2)^2-z(2)^3 + ..... (terms of degree >=4)
⇒
⇒ // Hamburger-Noether development of branch nr.3:
⇒   y = z(1)*x^2
⇒   x = z(1)^2
⇒
print(extdevelop(Hne[1],5)[1]);
⇒ No extension is possible
⇒ 0,x,0,
⇒ 0,1,x
list ehne=extdevelop(Hne[2],5);
displayHNE(ehne);
⇒   y = z(1)*x
⇒   x = z(1)^2+z(1)^2*z(2)
⇒   z(1) = z(2)^2-z(2)^3+z(2)^4-z(2)^5 + ..... (terms of degree >=6)
param(Hne[2]);
⇒ // ** Warning: result is exact up to order 5 in x and 7 in y !
⇒ _[1]=x7-x6-x5+x4
⇒ _[2]=-x10+2x9-2x7+x6
param(ehne);
⇒ // ** Warning: result is exact up to order 7 in x and 9 in y !
⇒ _[1]=x11-x10+x9-x8-x7+x6-x5+x4
⇒ _[2]=-x16+2x15-3x14+4x13-2x12+2x10-4x9+3x8-2x7+x6
```

See also: [Section D.6.15.2 \[develop\]](#), page 1722; [Section D.6.15.1 \[hnexpansion\]](#), page 1720; [Section D.6.15.4 \[param\]](#), page 1724.

## D.6.15.4 param

Procedure from library `hnoether.lib` (see [Section D.6.15 \[hnoether.lib\]](#), page 1719).

**Usage:** `param(L [,s]);` L list, s any type (optional)

**Assume:** L is the output of `develop(f)`, or of `extdevelop(develop(f),n)`, or (one entry in) the list of HN data created by `hnexpansion(f[,"ess"])`.

**Return:** If L are the HN data of an irreducible plane curve singularity f: a parametrization for f in the following format:

- if only the list  $L$  is given, the result is an ideal of two polynomials  $p[1], p[2]$ : if the HNE was finite then  $f(p[1], p[2])=0$ ; if not, the true parametrization will be given by two power series, and  $p[1], p[2]$  are truncations of these series.

- if the optional parameter  $s$  is given, the result is a list  $l$ :  $l[1]=\text{param}(L)$  (ideal) and  $l[2]=\text{intvec}$  with two entries indicating the highest degree up to which the coefficients of the monomials in  $l[1]$  are exact (entry -1 means that the corresponding parametrization is exact).

If  $L$  collects the HN data of a reducible plane curve singularity  $f$ , the return value is a list of parametrizations in the respective format.

**Note:** If the basering has only 2 variables, the first variable is chosen as indefinite. Otherwise, the 3rd variable is chosen.

**Example:**

```
LIB "hnoether.lib";
ring exring=0,(x,y,t),ds;
poly f=x3+2xy2+y2;
list Hne=develop(f);
list hne_extended=extdevelop(Hne,10);
// compare the HNE matrices ...
print(Hne[1]);
  => 0,x,
  => 0,-1
print(hne_extended[1]);
  => 0,x, 0,0,0,0, 0,0,0,0,
  => 0,-1,0,2,0,-4,0,8,0,-16
// ... and the resulting parametrizations:
param(Hne);
  => // ** Warning: result is exact up to order 2 in x and 3 in y !
  => _[1]=-t2
  => _[2]=-t3
param(hne_extended);
  => // ** Warning: result is exact up to order 10 in x and 11 in y !
  => _[1]=-t2+2t4-4t6+8t8-16t10
  => _[2]=-t3+2t5-4t7+8t9-16t11
param(hne_extended,0);
  => // ** Warning: result is exact up to order 10 in x and 11 in y !
  => [1]:
  =>   _[1]=-t2+2t4-4t6+8t8-16t10
  =>   _[2]=-t3+2t5-4t7+8t9-16t11
  => [2]:
  =>   10,11
// An example with more than one branch:
list L=hnexpansion(f*(x2+y4));
  =>
  => // 'hnexpansion' created a list of one ring.
  => // To see the ring and the data stored in the ring, type (if you assigned
  => // the name L to the list):
  =>   show(L);
  => // To display the computed HN expansion, type
  =>   def HNring = L[1]; setring HNring; displayHNE(hne);
def HNring = L[1]; setring HNring;
param(hne);
```

```

⇒ // Parametrization of branch number 1 computed.
⇒ // ** Warning: result is exact up to order 4 in x and 5 in y !
⇒ // Parametrization of branch number 2 computed.
⇒ // Parametrization of branch number 3 computed.
⇒ [1]:
⇒   _[1]=-x2+2*x4
⇒   _[2]=-x3+2*x5
⇒ [2]:
⇒   _[1]=(a)*x2
⇒   _[2]=x
⇒ [3]:
⇒   _[1]=(-a)*x2
⇒   _[2]=x

```

See also: [Section D.6.15.2 \[develop\], page 1722](#); [Section D.6.15.3 \[extdevelop\], page 1723](#).

### D.6.15.5 displayHNE

Procedure from library `hnoether.lib` (see [Section D.6.15 \[hnoether.lib\], page 1719](#)).

**Usage:** `displayHNE(L[,n]);` L list, n int

**Assume:** L is the output of `develop(f)`, or of `exdevelop(f,n)`, or of `hnexpansion(f[, "ess"])`, or (one entry in) the list `hne` in the ring created by `hnexpansion(f[, "ess"])`.

**Return:** - if only one argument is given and if the input are the HN data of an irreducible plane curve singularity, no return value, but display an ideal HNE of the following form:

$$\begin{aligned}
 y &= [] * x^1 + [] * x^2 + \dots + x^{<} * z(1) \\
 x &= [] * z(1)^2 + \dots + z(1)^{<} * z(2) \\
 z(1) &= [] * z(2)^2 + \dots + z(2)^{<} * z(3) \\
 &\dots\dots\dots \\
 z(r-1) &= [] * z(r)^2 + [] * z(r)^3 + \dots\dots\dots
 \end{aligned}$$

where x,y are the first 2 variables of the basering. The values of `[]` are the coefficients of the Hamburger-Noether matrix, the values of `<>` are represented by `x` in the HN matrix.

- if a second argument is given and if the input are the HN data of an irreducible plane curve singularity, return a ring containing an ideal HNE as described above.

- if L corresponds to the output of `hnexpansion(f)` or to the list of HN data computed by `hnexpansion(f[, "ess"])`, `displayHNE(L[,n])` shows the HNE's of all branches of f in the format described above. The optional parameter is then ignored.

**Note:** The 1st line of the above ideal (i.e., `HNE[1]`) means that  $y = [] * z(0)^1 + \dots$ , the 2nd line (`HNE[2]`) means that  $x = [] * z(1)^2 + \dots$ , so you can see which indeterminate corresponds to which line (it's also possible that `x` corresponds to the 1st line and `y` to the 2nd).

**Example:**

```

LIB "hnoether.lib";
ring r=0,(x,y),dp;
poly f=x3+2xy2+y2;
list hn=develop(f);
displayHNE(hn);
⇒ y = z(1)*x
⇒ x = -z(1)^2 + ..... (terms of degree >=3)

```

See also: [Section D.6.15.2 \[develop\], page 1722](#); [Section D.6.15.1 \[hnexpansion\], page 1720](#).

### D.6.15.6 invariants

Procedure from library `hnoether.lib` (see [Section D.6.15 \[hnoether.lib\]](#), page 1719).

**Usage:** `invariants(INPUT)`; INPUT list or poly

**Assume:** INPUT is the output of `develop(f)`, or of `extdevelop(develop(f),n)`, or one entry of the list of HN data computed by `hnexpansion(f[, "ess"])`.

**Return:** list INV of the following format:

```
INV[1]: intvec  (characteristic exponents)
INV[2]: intvec  (generators of the semigroup)
INV[3]: intvec  (Puisseux pairs, 1st components)
INV[4]: intvec  (Puisseux pairs, 2nd components)
INV[5]: int     (degree of the conductor)
INV[6]: intvec  (sequence of multiplicities)
```

If INPUT contains no valid HN expansion, the empty list is returned.

**Assume:** INPUT is a bivariate polynomial `f`, or the output of `hnexpansion(f)`, or the list of HN data computed by `hnexpansion(f[, "ess"])`.

**Return:** list INV, such that `INV[i]` coincides with the output of `invariants(develop(f[i]))`, where `f[i]` is the *i*-th branch of `f`, and the last entry of INV contains further invariants of `f` in the format:

```
INV[last][1] : intmat  (contact matrix of the branches)
INV[last][2] : intmat  (intersection multiplicities of the branches)
INV[last][3] : int     (delta invariant of f)
```

**Note:** In case the Hamburger-Noether expansion of the curve `f` is needed for other purposes as well it is better to calculate this first with the aid of `hnexpansion` and use it as input instead of the polynomial itself.

**Example:**

```
LIB "hnoether.lib";
ring exring=0,(x,y),dp;
list Hne=develop(y4+2x3y2+x6+x5y);
list INV=invariants(Hne);
INV[1];                      // the characteristic exponents
⇨ 4,6,7
INV[2];                      // the generators of the semigroup of values
⇨ 4,6,13
INV[3],INV[4];              // the Puisseux pairs in packed form
⇨ 3,7 2,2
INV[5] div 2;              // the delta-invariant
⇨ 8
INV[6];                    // the sequence of multiplicities
⇨ 4,2,2,1,1
// To display the invariants more 'nicely':
displayInvariants(Hne);
⇨ characteristic exponents : 4,6,7
⇨ generators of semigroup : 4,6,13
⇨ Puisseux pairs : (3,2)(7,2)
⇨ degree of the conductor : 16
⇨ delta invariant : 8
⇨ sequence of multiplicities: 4,2,2,1,1
```

```

////////////////////////////////////
INV=invariants((x2-y3)*(x3-y5));
INV[1][1];           // the characteristic exponents of the first branch
↳ 2,3
INV[2][6];           // the sequence of multiplicities of the second branch
↳ 3,2,1,1
print(INV[size(INV)][1]);           // the contact matrix of the branches
↳      0      3
↳      3      0
print(INV[size(INV)][2]);           // the intersection numbers of the branches
↳      0      9
↳      9      0
INV[size(INV)][3];           // the delta invariant of the curve
↳ 14

```

See also: [Section D.6.15.2 \[develop\]](#), page 1722; [Section D.6.15.7 \[displayInvariants\]](#), page 1728; [Section D.6.15.1 \[hnexpansion\]](#), page 1720; [Section D.6.15.10 \[intersection\]](#), page 1731; [Section D.6.15.8 \[multsequence\]](#), page 1728.

### D.6.15.7 displayInvariants

Procedure from library `hnoether.lib` (see [Section D.6.15 \[hnoether.lib\]](#), page 1719).

**Usage:** `displayInvariants(INPUT)`; INPUT list or poly

**Assume:** INPUT is a bivariate polynomial, or the output of `develop(f)`, resp. of `extdevelop(develop(f),n)`, or (one entry of) the list of HN data computed by `hnexpansion(f, "ess")`.

**Return:** none

**Display:** invariants of the corresponding branch, resp. of all branches, in a better readable form.

**Note:** If the Hamburger-Noether expansion of the curve `f` is needed for other purposes as well it is better to calculate this first with the aid of `hnexpansion` and use it as input instead of the polynomial itself.

**Example:**

```

LIB "hnoether.lib";
ring exring=0,(x,y),dp;
list Hne=develop(y4+2x3y2+x6+x5y);
displayInvariants(Hne);
↳ characteristic exponents : 4,6,7
↳ generators of semigroup : 4,6,13
↳ Puiseux pairs : (3,2)(7,2)
↳ degree of the conductor : 16
↳ delta invariant : 8
↳ sequence of multiplicities: 4,2,2,1,1

```

See also: [Section D.6.15.2 \[develop\]](#), page 1722; [Section D.6.15.1 \[hnexpansion\]](#), page 1720; [Section D.6.15.10 \[intersection\]](#), page 1731; [Section D.6.15.6 \[invariants\]](#), page 1727.

### D.6.15.8 multsequence

Procedure from library `hnoether.lib` (see [Section D.6.15 \[hnoether.lib\]](#), page 1719).

**Usage:** `multsequence(INPUT)`; INPUT list or poly

- Assume:** INPUT is the output of `develop(f)`, or of `extdevelop(develop(f),n)`, or one entry of the list of HN data computed by `hnexpansion(f[, "ess"])`.
- Return:** `intvec` corresponding to the multiplicity sequence of the irreducible plane curve singularity described by the HN data (return value coincides with `invariants(INPUT)[6]`).
- Assume:** INPUT is a bivariate polynomial `f`, or the output of `hnexpansion(f)`, or the list of HN data computed by `hnexpansion(f[, "ess"])`.
- Return:** list of two integer matrices:
- `multsequence(INPUT)[1][i,*]`  
contains the multiplicities of the branches at their infinitely near point of 0 in its (i-1) order neighbourhood (i.e., i=1: multiplicity of the branches themselves, i=2: multiplicity of their 1st quadratic transform, etc.,  
Hence, `multsequence(INPUT)[1][*,j]` is the multiplicity sequence of branch j.
- `multsequence(INPUT)[2][i,*]`  
contains the information which of these infinitely near points coincide.
- Note:** The order of the elements of the list of HN data obtained from `hnexpansion(f[, "ess"])` must not be changed (because otherwise the coincident infinitely near points couldn't be grouped together, see the meaning of the 2nd `intmat` in the example).  
Hence, it is not wise to compute the HN expansion of polynomial factors separately, put them into a list INPUT and call `multsequence(INPUT)`.  
Use `displayMultsequence` to produce a better readable output for reducible curves on the screen.  
In case the Hamburger-Noether expansion of the curve `f` is needed for other purposes as well it is better to calculate this first with the aid of `hnexpansion` and use it as input instead of the polynomial itself.

**Example:**

```
LIB "hnoether.lib";
ring r=0,(x,y),dp;
list Hne=hnexpansion((x6-y10)*(x+y2-y3)*(x+y2+y3));
⇨ // No change of ring necessary, return value is HN expansion.
multsequence(Hne[1])," | ",multsequence(Hne[2])," | ",
multsequence(Hne[3])," | ",multsequence(Hne[4]);
⇨ 3,2,1,1 | 3,2,1,1 | 1 | 1
multsequence(Hne);
⇨ [1]:
⇨ 3,3,1,1,
⇨ 2,2,1,1,
⇨ 1,1,1,1,
⇨ 1,1,1,1,
⇨ 1,1,1,1
⇨ [2]:
⇨ 4,0,0,0,
⇨ 4,0,0,0,
⇨ 2,2,0,0,
⇨ 2,1,1,0,
⇨ 1,1,1,1
// The meaning of the entries of the 2nd matrix is as follows:
displayMultsequence(Hne);
```

```

⇒ [(3,3,1,1)],
⇒ [(2,2,1,1)],
⇒ [(1,1),(1,1)],
⇒ [(1,1),(1),(1)],
⇒ [(1),(1),(1),(1)]

```

See also: [Section D.6.15.2 \[develop\]](#), page 1722; [Section D.6.15.9 \[displayMultsequence\]](#), page 1730; [Section D.6.15.1 \[hnexpansion\]](#), page 1720; [Section D.6.15.17 \[separateHNE\]](#), page 1734.

### D.6.15.9 displayMultsequence

Procedure from library `hnoether.lib` (see [Section D.6.15 \[hnoether.lib\]](#), page 1719).

**Usage:** `displayMultsequence(INPUT)`; INPUT list or poly

**Assume:** INPUT is a bivariate polynomial, or the output of `develop(f)`, resp. of `extdevelop(develop(f),n)`, or (one entry of) the list of HN data computed by `hnexpansion(f[,"ess"])`, or the output of `hnexpansion(f)`.

**Return:** nothing

**Display:** the sequence of multiplicities:

```

- if INPUT=develop(f) or INPUT=extdevelop(develop(f),n) or INPUT=hne[i]:
    a , b , c , ..... , 1
- if INPUT=f or INPUT=hnexpansion(f) or INPUT=hne:
    [(a_1, .... , b_1 , .... , c_1)],
    [(a_2, ... ), ... , (... , c_2)],
    ..... ,
    [(a_n),(b_n), ..... , (c_n)]
with:
    a_1 , ... , a_n the sequence of multiplicities of the 1st branch,
    [...] the multiplicities of the j-th transform of all branches,
    (...) indicating branches meeting in an infinitely near point.

```

**Note:** The Same restrictions as in `multsequence` apply for the input.  
In case the Hamburger-Noether expansion of the curve `f` is needed for other purposes as well it is better to calculate this first with the aid of `hnexpansion` and use it as input instead of the polynomial itself.

**Example:**

```

LIB "hnoether.lib";
ring r=0,(x,y),dp;
// Example 1: Input = output of develop
displayMultsequence(develop(x3-y5));
⇒ The sequence of multiplicities is 3,2,1,1
// Example 2: Input = bivariate polynomial
displayMultsequence((x6-y10)*(x+y2-y3)*(x+y2+y3));
⇒ [(3,3,1,1)],
⇒ [(2,2,1,1)],
⇒ [(1,1),(1,1)],
⇒ [(1,1),(1),(1)],
⇒ [(1),(1),(1),(1)]

```

See also: [Section D.6.15.2 \[develop\]](#), page 1722; [Section D.6.15.1 \[hnexpansion\]](#), page 1720; [Section D.6.15.8 \[multsequence\]](#), page 1728; [Section D.6.15.17 \[separateHNE\]](#), page 1734.



### D.6.15.10 intersection

Procedure from library `hnoether.lib` (see [Section D.6.15 \[hnoether.lib\]](#), page 1719).

**Usage:** `intersection(hne1,hne2);` hne1, hne2 lists

**Assume:** `hne1`, `hne2` represent an HN expansion of an irreducible plane curve singularity (that is, are the output of `develop(f)`, or of `extdevelop(develop(f),n)`, or one entry of the list of HN data computed by `hnexpansion(f[,"ess"])`).

**Return:** `int`, the intersection multiplicity of the irreducible plane curve singularities corresponding to `hne1` and `hne2`.

**Example:**

```
LIB "hnoether.lib";
ring r=0,(x,y),dp;
list Hne=hnexpansion((x2-y3)*(x2+y3));
⇒ // No change of ring necessary, return value is HN expansion.
intersection(Hne[1],Hne[2]);
⇒ 6
```

See also: [Section D.6.15.7 \[displayInvariants\]](#), page 1728; [Section D.6.15.1 \[hnexpansion\]](#), page 1720.

### D.6.15.11 is\_irred

Procedure from library `hnoether.lib` (see [Section D.6.15 \[hnoether.lib\]](#), page 1719).

**Usage:** `is_irred(f);` `f` poly

**Assume:** `f` is a squarefree bivariate polynomial (in the first 2 ring variables).

**Return:** `int` (0 or 1):  
 - `is_irred(f)=1` if `f` is irreducible as a formal power series over the algebraic closure of its coefficient field (`f` defines an analytically irreducible curve at zero),  
 - `is_irred(f)=0` otherwise.

**Note:** 0 and units in the ring of formal power series are considered to be not irreducible.

**Example:**

```
LIB "hnoether.lib";
ring exring=0,(x,y),ls;
is_irred(x2+y3);
⇒ 1
is_irred(x2+y2);
⇒ 0
is_irred(x2+y3+1);
⇒ 0
```

### D.6.15.12 delta

Procedure from library `hnoether.lib` (see [Section D.6.15 \[hnoether.lib\]](#), page 1719).

**Usage:** `delta(INPUT);` `INPUT` a polynomial defining an isolated plane curve singularity at 0, or the Hamburger-Noether expansion thereof, i.e. the output of `develop(f)`, or the output of `hnexpansion(f)`, or the list of HN data computed by `hnexpansion(f)`.

**Return:** `int`, the delta invariant of the singularity at 0, that is, the vector space dimension of  $\tilde{R}/R$ , ( $\tilde{R}$  the normalization of the local ring of the singularity).

**Note:** In case the Hamburger-Noether expansion of the curve  $f$  is needed for other purposes as well it is better to calculate this first with the aid of `hnexpansion` and use it as input instead of the polynomial itself.

**Example:**

```
LIB "hnoether.lib";
ring r = 32003, (x,y), ds;
poly f = x25+x24-4x23-1x22y+4x22+8x21y-2x21-12x20y-4x19y2+4x20+10x19y
+12x18y2-24x18y-20x17y2-4x16y3+x18+60x16y2+20x15y3-9x16y
-80x14y3-10x13y4+36x14y2+60x12y4+2x11y5-84x12y3-24x10y5
+126x10y4+4x8y6-126x8y5+84x6y6-36x4y7+9x2y8-1y9;
delta(f);
⇒ 96
```

See also: [Section D.4.23.11 \[deltaLoc\]](#), page 1204; [Section D.6.15.6 \[invariants\]](#), page 1727.

### D.6.15.13 newtonpoly

Procedure from library `hnoether.lib` (see [Section D.6.15 \[hnoether.lib\]](#), page 1719).

**Usage:** `newtonpoly(f); f poly`

**Assume:** basering has exactly two variables;  
 $f$  is convenient, that is,  $f(x,0) \neq 0 \neq f(0,y)$ .

**Return:** list of intvecs (= coordinates  $x,y$  of the Newton polygon of  $f$ ).

**Note:** Procedure uses `execute`; this can be avoided by calling `newtonpoly(f,1)` if the ordering of the basering is `ls`.

**Example:**

```
LIB "hnoether.lib";
ring r=0, (x,y), ls;
poly f=x5+2x3y-x2y2+3xy5+y6-y7;
newtonpoly(f);
⇒ [1]:
⇒ 0,6
⇒ [2]:
⇒ 2,2
⇒ [3]:
⇒ 3,1
⇒ [4]:
⇒ 5,0
```

### D.6.15.14 is\_NND

Procedure from library `hnoether.lib` (see [Section D.6.15 \[hnoether.lib\]](#), page 1719).

**Usage:** `is_NND(f[,mu,NP]); f poly, mu int, NP list of intvecs`

**Assume:**  $f$  is convenient, that is,  $f(x,0) \neq 0 \neq f(0,y)$ ;  
 $\mu$  (optional) is Milnor number of  $f$ .  
 $NP$  (optional) is output of `newtonpoly(f)`.

**Return:** int: 1 if  $f$  is Newton non-degenerate, 0 otherwise.

**Example:**

```

LIB "hnoether.lib";
ring r=0,(x,y),ls;
poly f=x5+y3;
is_NND(f);
↪ 1
poly g=(x-y)^5+3xy5+y6-y7;
is_NND(g);
↪ 0
// if already computed, one should give the Minor number and Newton polygon
// as second and third input:
int mu=milnor(g);
list NP=newtonpoly(g);
is_NND(g,mu,NP);
↪ 0

```

See also: [Section D.6.15.13 \[newtonpoly\]](#), page 1732.

### D.6.15.15 stripHNE

Procedure from library `hnoether.lib` (see [Section D.6.15 \[hnoether.lib\]](#), page 1719).

**Usage:** stripHNE(L); L list

**Assume:** L is the output of `develop(f)`, or of `extdevelop(develop(f),n)`, or (one entry of) the list `hne` in the ring created by `hnexpansion(f, "ess")`.

**Return:** list in the same format as L, but all polynomials `L[4]`, resp. `L[i][4]`, are set to zero.

**Note:** The purpose of this procedure is to remove huge amounts of data no longer needed. It is useful, if one or more of the polynomials in L consume much memory. It is still possible to compute invariants, parametrizations etc. with the stripped HNE(s), but it is not possible to use `extdevelop` with them.

**Example:**

```

LIB "hnoether.lib";
ring r=0,(x,y),dp;
list Hne=develop(x2+y3+y4);
Hne;
↪ [1]:
↪   _[1,1]=0
↪   _[1,2]=x
↪   _[2,1]=0
↪   _[2,2]=-1
↪ [2]:
↪   1,-1
↪ [3]:
↪   1
↪ [4]:
↪   x4-2x2y+y2+y
↪ [5]:
↪   1
stripHNE(Hne);
↪ [1]:
↪   _[1,1]=0
↪   _[1,2]=x
↪   _[2,1]=0

```

```

↳      _[2,2]==-1
↳ [2]:
↳      1,-1
↳ [3]:
↳      1
↳ [4]:
↳      0
↳ [5]:
↳      1

```

See also: [Section D.6.15.2 \[develop\]](#), page 1722; [Section D.6.15.3 \[extdevelop\]](#), page 1723; [Section D.6.15.1 \[hnexpansion\]](#), page 1720.

### D.6.15.16 `puiseux2generators`

Procedure from library `hnoether.lib` (see [Section D.6.15 \[hnoether.lib\]](#), page 1719).

**Usage:** `puiseux2generators(m,n)`; `m,n` intvec

**Assume:** `m`, resp. `n`, represent the 1st, resp. 2nd, components of Puiseux pairs (e.g., `m=invariants(L)[3]`, `n=invariants(L)[4]`).

**Return:** intvec of the generators of the semigroup of values.

**Example:**

```

LIB "hnoether.lib";
// take (3,2),(7,2),(15,2),(31,2),(63,2),(127,2) as Puiseux pairs:
puiseux2generators(intvec(3,7,15,31,63,127),intvec(2,2,2,2,2,2));
↳ 64,96,208,424,852,1706,3413

```

See also: [Section D.6.15.6 \[invariants\]](#), page 1727.

### D.6.15.17 `separateHNE`

Procedure from library `hnoether.lib` (see [Section D.6.15 \[hnoether.lib\]](#), page 1719).

**Usage:** `separateHNE(hne1,hne2)`; `hne1, hne2` lists

**Assume:** `hne1, hne2` are HNEs (=output of `develop(f)`, `extdevelop(develop(f),n)`, or one entry in the list `hne` in the ring created by `hnexpansion(f[,"ess"])`).

**Return:** number of quadratic transformations needed to separate both curves (branches).

**Example:**

```

LIB "hnoether.lib";
int p=printlevel; printlevel=-1;
ring r=0,(x,y),dp;
list hne1=develop(x);
list hne2=develop(x+y);
list hne3=develop(x+y^2);
separateHNE(hne1,hne2); // two transversal lines
↳ 1
separateHNE(hne1,hne3); // one quadratic transform. gives 1st example
↳ 2
printlevel=p;

```

See also: [Section D.6.15.2 \[develop\]](#), page 1722; [Section D.6.15.9 \[displayMultsequence\]](#), page 1730; [Section D.6.15.1 \[hnexpansion\]](#), page 1720; [Section D.6.15.8 \[multsequence\]](#), page 1728.

### D.6.15.18 squarefree

Procedure from library `hnoether.lib` (see [Section D.6.15 \[hnoether.lib\]](#), page 1719).

**Usage:**      `squarefree(f); f poly`

**Assume:**    `f` is a bivariate polynomial (in the first 2 ring variables).

**Return:**      `poly`, a squarefree divisor of `f`.

**Note:**        Usually, the return value is the greatest squarefree divisor, but there is one exception: factors with a  $p$ -th root,  $p$  the characteristic of the basering, are lost.

**Example:**

```
LIB "hnoether.lib";
ring exring=3,(x,y),dp;
squarefree((x3+y)^2);
↳ x3+y
squarefree((x+y)^3*(x-y)^2); // Warning: (x+y)^3 is lost
↳ x-y
squarefree((x+y)^4*(x-y)^2); // result is (x+y)*(x-y)
↳ x2-y2
```

See also: [Section D.6.15.19 \[allsquarefree\]](#), page 1735.

### D.6.15.19 allsquarefree

Procedure from library `hnoether.lib` (see [Section D.6.15 \[hnoether.lib\]](#), page 1719).

**Usage :**      `allsquarefree(f,g); f,g poly`

**Assume:**    `g` is the output of `squarefree(f)`.

**Return:**      the greatest squarefree divisor of `f`.

**Note :**        This proc uses `factorize` to get the missing factors of `f` not in `g` and, therefore, may be slow.

**Example:**

```
LIB "hnoether.lib";
ring exring=7,(x,y),dp;
poly f=(x+y)^7*(x-y)^8;
poly g=squarefree(f);
g; // factor x+y lost, since characteristic=7
↳ x-y
allsquarefree(f,g); // all factors (x+y)*(x-y) found
↳ x2-y2
```

See also: [Section D.6.15.18 \[squarefree\]](#), page 1735.

### D.6.15.20 further\_hn\_proc

Procedure from library `hnoether.lib` (see [Section D.6.15 \[hnoether.lib\]](#), page 1719).

**Usage:**      `further_hn_proc();`

**Note:**        The library `hnoether.lib` contains some more procedures which are not shown when typing `help hnoether.lib;`. They may be useful for interactive use (e.g. if you want to do the calculation of an HN development "by hand" to see the intermediate results), and they can be enumerated by calling `further_hn_proc()`.  
Use `help <procedure>;` for detailed information about each of them.

**Example:**

```

LIB "hnoether.lib";
further_hn_proc();
⇒
⇒ The following procedures are also part of 'hnoether.lib':
⇒
⇒ getnm(f);           intersection pts. of Newton polygon with axes
⇒ T_Transform(f,Q,N); returns f(y,xy^Q)/y^NQ (f: poly, Q,N: int)
⇒ T1_Transform(f,d,M); returns f(x,y+d*x^M) (f: poly,d:number,M:int)
⇒ T2_Transform(f,d,M,N,ref); a composition of T1 & T
⇒ koeff(f,I,J);       gets coefficient of indicated monomial of polynomial\
f
⇒ redleit(f,S,E);     restriction of monomials of f to line (S-E)
⇒ leit(f,n,m);        special case of redleit (for irred. polynomials)
⇒ testreducible(f,n,m); tests whether f is reducible
⇒ charPoly(f,M,N);    characteristic polynomial of f
⇒ find_in_list(L,p);  find int p in list L
⇒ get_last_divisor(M,N); last divisor in Euclid's algorithm
⇒ factorfirst(f,M,N); try to factor f without 'factorize'
⇒ factorlist(L);      factorize a list L of polynomials
⇒ referencepoly(D);   a polynomial f s.t. D is the Newton diagram of f

```

**D.6.16 kskernel.lib****Library:** kskernel.lib**Purpose:** procedures for computing the kernel of the kodaira-spencer map**Author:** Tetyana Povalyaeva, povalya@mathematik.uni-kl.de**Procedures:****D.6.16.1 KSkker**Procedure from library kskernel.lib (see [Section D.6.16 \[kskernel.lib\]](#), page 1736).**Usage:** KSkker(int p,q); p,q relatively prime integers

**Return:** nothing; exports ring KSring, matrix KSkkernel and list 'weights'; KSkkernel is a matrix of coefficients of the generators of the kernel of Kodaira-Spencer map, 'weights' is a list of degrees for variables T

**Example:**

```

LIB "kskernel.lib";
int p=6;
int q=7;
KSkker(p,q);
setring Kskkernel::KSring;
print(KSkkernel);
⇒ 0,      0,      0,      0,      0,      2*T(1),
⇒ 0,      0,      0,      0,      0,      3*T(2),
⇒ 0,      0,      0,      0,      0,      4*T(3),
⇒ 0,      0,      0,      2*T(1),  3*T(2),  9*T(4),
⇒ 0,      0,      0,      3*T(2),  KSkkernel[5,5], 10*T(5),
⇒ 2*T(1), 3*T(2), KSkkernel[6,3], KSkkernel[6,4], KSkkernel[6,5], 16*T(6)

```

### D.6.16.2 KSconvert

Procedure from library `kskernel.lib` (see [Section D.6.16 \[kskernel.lib\], page 1736](#)).

- Usage:** `KSconvert(matrix M);`  
`M` is a matrix of coefficients of the generators of the kernel of Kodaira-Spencer map in variables `T(i)` from the basering. To be called after the procedure `KSkker(p,q)`
- Return:** nothing; exports ring `KSring2` and matrix `KSkkernel2` within it, such that `KSring2` resp. `KSkkernel2` are in variables `T(w)` with weights `-w`. These weights are computed in the procedure `KSkker(p,q)`

**Example:**

```
LIB "kskernel.lib";
int p=6;
int q=7;
KSkker(p,q);
setring Kskkernel::KSring;
KSconvert(KSkkernel);
⇒ // ** 'KSring2' is already global
setring Kskkernel::KSring2;
print(KSkkernel2);
⇒ 0,      0,      0,      0,      0,      2*T(2),
⇒ 0,      0,      0,      0,      0,      3*T(3),
⇒ 0,      0,      0,      0,      0,      4*T(4),
⇒ 0,      0,      0,      2*T(2),  3*T(3),  9*T(9),
⇒ 0,      0,      0,      3*T(3),  KSkernel2[5,5], 10*T(10),
⇒ 2*T(2), 3*T(3), KSkernel2[6,3], KSkernel2[6,4], KSkernel2[6,5], 16*T(16)
```

### D.6.16.3 KSlinear

Procedure from library `kskernel.lib` (see [Section D.6.16 \[kskernel.lib\], page 1736](#)).

- Usage:** `KSlinear(matrix M);`  
computes matrix of linear terms of the kernel of the Kodaira-Spencer map. To be called after the procedure `KSkker(p,q)`
- Return:** nothing; but replaces elements of the matrix `KSkkernel` in the ring `KSring` with their leading monomials w.r.t. the local ordering (`ls`)

**Example:**

```
LIB "kskernel.lib";
int p=6;
int q=7;
KSkker(p,q);
setring Kskkernel::KSring;
KSlinear(KSkkernel);
print(KSkkernel);
⇒ 0,      0,      0,      0,      0,      2*T(1),
⇒ 0,      0,      0,      0,      0,      3*T(2),
```

```

⇒ 0,      0,      0,      0,      0,      4*T(3),
⇒ 0,      0,      0,      2*T(1),3*T(2), 9*T(4),
⇒ 0,      0,      0,      3*T(2),4*T(3), 10*T(5),
⇒ 2*T(1),3*T(2),4*T(3),9*T(4),10*T(5),16*T(6)

```

#### D.6.16.4 KScoef

Procedure from library `kskernel.lib` (see [Section D.6.16 \[kskernel.lib\]](#), page 1736).

**Return:** exports ring `RC` and number `C` within it. `C` is the coefficient of the word defined in the list `qq`, being a part of  $C[i,j]$  for  $x^p y^q$

**Example:**

```

LIB "kskernel.lib";
int p=5; int q=14;
int i=2; int j=9;
list L;
ring r=0,x,dp;
number c;
L[1]=3; L[2]=1; L[3]=3; L[4]=2;
KScoef(i,j,p,q,L);
c=imap(RC,C);
c;
⇒ 27/8575
L[1]=3; L[2]=1; L[3]=2; L[4]=3;
KScoef(i,j,p,q,L);
⇒ // ** redefining S ( export S;)
⇒ // ** redefining u ( export u;)
⇒ // ** redefining l ( export l;)
⇒ // ** redefining RC ( export RC;exportto(Top,RC);)
c=c+imap(RC,C);
c; // it is a coefficient of T1*T2*T3^2 in C[2,9] for x^5+y^14
⇒ 99/8575

```

#### D.6.16.5 StringF

Procedure from library `kskernel.lib` (see [Section D.6.16 \[kskernel.lib\]](#), page 1736).

**Usage:** `StringF(int i,j,p,q);`

**Return:** nothing; exports string `F` which contains an expression in variables  $T(i)$  with non-resolved brackets

**Example:**

```

LIB "kskernel.lib";
int p=5; int q=14;
int i=2; int j=9;
StringF(i,j,p,q);
F;
⇒ T(7)+T(3)*(T(4)*(T(1))+T(1)*(T(4)+T(3)*(T(2))+T(2)*(T(3)+T(1)*(T(1))))))

```

#### D.6.17 mondromy\_lib

**Library:** `mondromy.lib`



**Purpose:** Monodromy of an Isolated Hypersurface Singularity

**Author:** Mathias Schulze, email: mschulze@mathematik.uni-kl.de

**Overview:** A library to compute the monodromy of an isolated hypersurface singularity. It uses an algorithm by Brieskorn (manuscripta math. 2 (1970), 103-161) to compute a connection matrix of the meromorphic Gauss-Manin connection up to arbitrarily high order, and an algorithm of Gerard and Levelt (Ann. Inst. Fourier, Grenoble 23,1 (1973), pp. 157-195) to transform it to a simple pole.

**Procedures:** See also: [Section D.6.14 \[gmspoly\\_lib\]](#), page 1718; [Section D.6.13 \[gmssing\\_lib\]](#), page 1703.

### D.6.17.1 detadj

Procedure from library `mondromy.lib` (see [Section D.6.17 \[mondromy\\_lib\]](#), page 1738).

**Usage:** `detadj(U)`; U matrix

**Assume:** U is a square matrix with non zero determinant.

**Return:** The procedure returns a list with at most 2 entries.  
If U is not a square matrix, the list is empty.  
If U is a square matrix, then the first entry is the determinant of U. If U is a square matrix and the determinant of U not zero, then the second entry is the adjoint matrix of U.

**Display:** The procedure displays comments if `printlevel`  $\geq 1$ .

**Example:**

```
LIB "mondromy.lib";
ring R=0,x,dp;
matrix U[2][2]=1,1+x,1+x2,1+x3;
list daU=detadj(U);
daU[1];
 $\mapsto -x^2-x$ 
print(daU[2]);
 $\mapsto x^3+1, -x-1,$ 
 $\mapsto -x^2-1, 1$ 
```

### D.6.17.2 invunit

Procedure from library `mondromy.lib` (see [Section D.6.17 \[mondromy\\_lib\]](#), page 1738).

**Usage:** `invunit(u,n)`; u poly, n int

**Assume:** The polynomial u is a series unit.

**Return:** The procedure returns the series inverse of u up to order n or a zero polynomial if u is no series unit.

**Display:** The procedure displays comments if `printlevel`  $\geq 1$ .

**Example:**

```
LIB "mondromy.lib";
ring R=0,(x,y),dp;
invunit(2+x3+xy4,10);
 $\mapsto 1/8x^2y^8-1/16x^9+1/4x^4y^4+1/8x^6-1/4xy^4-1/4x^3+1/2$ 
```

### D.6.17.3 jacoblift

Procedure from library `mondromy.lib` (see [Section D.6.17 \[mondromy.lib\], page 1738](#)).

**Usage:** `jacoblift(f); f poly`

**Assume:** The polynomial  $f$  in a series ring (local ordering) defines an isolated hypersurface singularity.

**Return:** The procedure returns a list with entries  $\kappa$ ,  $\xi$ ,  $u$  of type `int`, `vector`, `poly` such that  $\kappa$  is minimal with  $f^\kappa$  in `jacob(f)`,  $u$  is a unit, and  $u \cdot f^\kappa = (\text{matrix}(\text{jacob}(f)) \cdot \xi)[1,1]$ .

**Display:** The procedure displays comments if `printlevel`  $\geq 1$ .

**Example:**

```
LIB "mondromy.lib";
ring R=0,(x,y),ds;
poly f=x2y2+x6+y6;
jacoblift(f);
⇒ [1]:
⇒ 2
⇒ [2]:
⇒ 1/2x2y3*gen(2)+1/6x7*gen(1)+5/6x6y*gen(2)-2/3xy6*gen(1)+1/6y7*gen(2)-4\
x4y5*gen(2)-3/2x9y2*gen(1)-15/2x8y3*gen(2)+9/2x3y8*gen(1)-3/2x2y9*gen(2)
⇒ [3]:
⇒ 1-9x2y2
```

### D.6.17.4 monodromyB

Procedure from library `mondromy.lib` (see [Section D.6.17 \[mondromy.lib\], page 1738](#)).

**Usage:** `monodromyB(f[,opt]); f poly, opt int`

**Assume:** The polynomial  $f$  in a series ring (local ordering) defines an isolated hypersurface singularity.

**Return:** The procedure returns a residue matrix  $M$  of the meromorphic Gauss-Manin connection of the singularity defined by  $f$  or an empty matrix if the assumptions are not fulfilled. If `opt`=0 (default),  $\exp(-2\pi i M)$  is a monodromy matrix of  $f$ , else, only the characteristic polynomial of  $\exp(-2\pi i M)$  coincides with the characteristic polynomial of the monodromy of  $f$ .

**Display:** The procedure displays more comments for higher `printlevel`.

**Example:**

```
LIB "mondromy.lib";
ring R=0,(x,y),ds;
poly f=x2y2+x6+y6;
matrix M=monodromyB(f);
print(M);
⇒ 7/6,0, 0,0, 0, 0,0, 0,-1/2,0, 0, 0, 0,
⇒ 0, 7/6,0,0, 0, 0,-1/2,0,0, 0, 0, 0, 0,
⇒ 0, 0, 1,0, 0, 0,0, 0,0, 0, 0, 0, 0,
⇒ 0, 0, 0,4/3,0, 0,0, 0,0, 0, 0, 0, 0,
⇒ 0, 0, 0,0, 4/3,0,0, 0,0, 0, 0, 0, 0,
⇒ 0, 0, 0,0, 0, 1,0, 0,0, 0, 0, 0, 0,
```

```

⇒ 0, 0, 0,0, 0, 0,5/6, 0,0, 0, 0, 0, 0,
⇒ 0, 0, 0,0, 0, 0,0, 1,0, 0, 0, 0, 0,
⇒ 0, 0, 0,0, 0, 0,0, 0,5/6, 0, 0, 0, 0,
⇒ 0, 0, 0,0, 0, 0,0, 0,0, 2/3,0, 0, 0,
⇒ 0, 0, 0,0, 0, 0,0, 0,0, 0, 2/3,0, 0,
⇒ 0, 0, 0,0, 0, 0,0, 0,0, 0, 0, 1, -1/3,
⇒ 0, 0, 0,0, 0, 0,0, 0,0, 0, 0, 3/4,0

```

### D.6.17.5 H2basis

Procedure from library `mondromy.lib` (see [Section D.6.17 \[mondromy.lib\]](#), page 1738).

**Usage:** H2basis(f); f poly

**Assume:** The polynomial  $f$  in a series ring (local ordering) defines an isolated hypersurface singularity.

**Return:** The procedure returns a list of representatives of a  $C\{f}$ -basis of the Brieskorn lattice  $H'' = \Omega^{n+1}/df \wedge d\Omega^{n-1}$ .

**Theory:**  $H''$  is a free  $C\{f}$ -module of rank  $\text{milnor}(f)$ .

**Display:** The procedure displays more comments for higher `printlevel`.

**Example:**

```

LIB "mondromy.lib";
ring R=0,(x,y),ds;
poly f=x2y2+x6+y6;
H2basis(f);
⇒ [1]:
⇒ x4
⇒ [2]:
⇒ x2y2
⇒ [3]:
⇒ y4
⇒ [4]:
⇒ x3
⇒ [5]:
⇒ x2y
⇒ [6]:
⇒ xy2
⇒ [7]:
⇒ y3
⇒ [8]:
⇒ x2
⇒ [9]:
⇒ xy
⇒ [10]:
⇒ y2
⇒ [11]:
⇒ x
⇒ [12]:
⇒ y
⇒ [13]:
⇒ 1

```

## D.6.18 qhmoduli.lib

**Library:** qhmoduli.lib

**Purpose:** Moduli Spaces of Semi-Quasihomogeneous Singularities

**Author:** Thomas Bayer, email: bayert@in.tum.de

**Procedures:**

### D.6.18.1 ArnoldAction

Procedure from library `qhmoduli.lib` (see [Section D.6.18 \[qhmoduli.lib\], page 1742](#)).

**Usage:** `ArnoldAction(f, [Gf, B]);` poly `f`; list `Gf, B`;  
'`Gf`' is a list of two rings (coming from '`StabEqn`')

**Purpose:** compute the induced action of the stabilizer `G` of `f` on `T_`, where `T_` is given by the upper monomials `B` of the Milnor algebra of `f`.

**Assume:** `f` is quasihomogeneous

**Return:** polynomial ring over the same ground field, containing the ideals '`actionid`' and '`stabid`'.  
- '`actionid`' is the ideal defining the induced action of `Gf` on `T_`  
- '`stabid`' is the ideal of the stabilizer `Gf` in the new ring

**Example:**

```
LIB "qhmoduli.lib";
ring B = 0,(x,y,z), ls;
poly f = -z5+y5+x2z+x2y;
def R = ArnoldAction(f);
⇒ // ** redefining zz (int zz = 1;) qhmoduli.lib::StabEqnId:629
setring R;
actionid;
⇒ actionid[1]=-s(2)*t(1)+s(3)*t(1)
⇒ actionid[2]=-s(2)^2*t(2)+2*s(2)^2*t(3)^2+s(3)^2*t(2)
⇒ actionid[3]=s(2)*t(3)+s(3)*t(3)
stabid;
⇒ stabid[1]=s(2)*s(3)
⇒ stabid[2]=s(1)^2*s(2)+s(1)^2*s(3)-1
⇒ stabid[3]=s(1)^2*s(3)^2-s(3)
⇒ stabid[4]=s(1)^2+s(2)^4-s(3)^4
⇒ stabid[5]=s(1)^4+s(2)^3-s(3)^3
⇒ stabid[6]=-s(1)^2*s(3)+s(3)^5
```

### D.6.18.2 ModEqn

Procedure from library `qhmoduli.lib` (see [Section D.6.18 \[qhmoduli.lib\], page 1742](#)).

**Usage:** `ModEqn(f [, opt]);` poly `f`; int `opt`;

**Purpose:** compute equations of the moduli space of semiquasihomogenous hypersurface singularity with principal part `f` w.r.t. right equivalence

**Assume:** `f` quasihomogeneous polynomial with an isolated singularity at 0

- Return:** polynomial ring, possibly a simple extension of the ground field of the basering, containing the ideal 'modid'  
 - 'modid' is the ideal of the moduli space if opt is even ( $> 0$ ). otherwise it contains generators of the coordinate ring  $R$  of the moduli space (note :  $\text{Spec}(R)$  is the moduli space)
- Options:** 1 compute equations of the mod. space,  
 2 use a primary decomposition,  
 4 compute  $E_{f0}$ , i.e., the image of  $G_{f0}$ ,  
 to combine options, add their value, default: opt = 7

**Example:**

```
LIB "qhmoduli.lib";
ring B = 0,(x,y), ls;
poly f = -x4 + xy5;
def R = ModEqn(f);
↪ // ** redefining zz (int zz = 1;) qhmoduli.lib::StabEqnId:629
setring R;
modid;
↪ modid[1]=Y(5)^2-Y(4)*Y(6)
↪ modid[2]=Y(4)*Y(5)-Y(3)*Y(6)
↪ modid[3]=Y(3)*Y(5)-Y(2)*Y(6)
↪ modid[4]=Y(2)*Y(5)-Y(1)*Y(6)
↪ modid[5]=Y(4)^2-Y(3)*Y(5)
↪ modid[6]=Y(3)*Y(4)-Y(2)*Y(5)
↪ modid[7]=Y(2)*Y(4)-Y(1)*Y(5)
↪ modid[8]=Y(3)^2-Y(2)*Y(4)
↪ modid[9]=Y(2)*Y(3)-Y(1)*Y(4)
↪ modid[10]=Y(2)^2-Y(1)*Y(3)
```

**D.6.18.3 QuotientEquations**

Procedure from library `qhmoduli.lib` (see [Section D.6.18 \[qhmoduli.lib\]](#), page 1742).

- Usage:** QuotientEquations(G,action,emb [, opt]); ideal G,action,emb;int opt
- Purpose:** compute the quotient of the variety given by the parameterization 'emb' by the linear action 'action' of the algebraic group G.
- Assume:** 'action' is linear, G must be finite if the Reynolds operator is needed (i.e., `NullCone(G,action)` returns some non-invariant polys)
- Return:** polynomial ring over a simple extension of the ground field of the basering, containing the ideals 'id' and 'embedid'.  
 - 'id' contains the equations of the quotient, if opt = 1; if opt = 0, 2, 'id' contains generators of the coordinate ring  $R$  of the quotient ( $\text{Spec}(R)$  is the quotient)  
 - 'embedid' = 0, if opt = 1;  
 if opt = 0, 2, it is the ideal defining the equivariant embedding
- Options:** 1 compute equations of the quotient,  
 2 use a primary decomposition when computing the Reynolds operator,  
 to combine options, add their value, default: opt = 3.

**D.6.18.4 StabEqn**

Procedure from library `qhmoduli.lib` (see [Section D.6.18 \[qhmoduli.lib\]](#), page 1742).

**Usage:**      `StabEqn(f); f polynomial`

**Purpose:**     compute the equations of the isometry group of  $f$ .

**Assume:**     $f$  semiquasihomogeneous polynomial with an isolated singularity at 0

**Return:**     list of two rings 'S1', 'S2'  
               - 'S1' contains the equations of the stabilizer (ideal 'stabid')  
               - 'S2' contains the action of the stabilizer (ideal 'actionid')

**Global:**     `varSubsList`, contains the index  $j$  s.t.  $x(i) \rightarrow x(i)t(j) \dots$

**Example:**

```
LIB "qhmoduli.lib";
ring B = 0,(x,y,z), ls;
poly f = -z5+y5+x2z+x2y;
list stab = StabEqn(f);
⇒ // ** redefining zz (int zz = 1;) qhmoduli.lib::StabEqnId:629
def S1 = stab[1]; setring S1; stabid;
⇒ stabid[1]=s(2)*s(3)
⇒ stabid[2]=s(1)^2*s(2)+s(1)^2*s(3)-1
⇒ stabid[3]=s(1)^2*s(3)^2-s(3)
⇒ stabid[4]=s(2)^4-s(3)^4+s(1)^2
⇒ stabid[5]=s(1)^4+s(2)^3-s(3)^3
⇒ stabid[6]=s(3)^5-s(1)^2*s(3)
def S2 = stab[2]; setring S2; actionid;
⇒ actionid[1]=s(1)*x
⇒ actionid[2]=s(3)*y+s(2)*z
⇒ actionid[3]=s(2)*y+s(3)*z
```

### D.6.18.5 StabEqnId

Procedure from library `qhmoduli.lib` (see [Section D.6.18 \[qhmoduli.lib\]](#), page 1742).

**Usage:**      `StabEqnId(I, w); I ideal, w intvec`

**Purpose:**     compute the equations of the isometry group of the ideal  $I$ , each generator of  $I$  is fixed by the stabilizer.

**Assume:**     $I$  semiquasihomogeneous ideal w.r.t. 'w' with an isolated singularity at 0

**Return:**     list of two rings 'S1', 'S2'  
               - 'S1' contains the equations of the stabilizer (ideal 'stabid')  
               - 'S2' contains the action of the stabilizer (ideal 'actionid')

**Global:**     `varSubsList`, contains the index  $j$  s.t.  $t(i) \rightarrow t(i)t(j) \dots$

**Example:**

```
LIB "qhmoduli.lib";
ring B = 0,(x,y,z), ls;
ideal I = x2,y3,z6;
intvec w = 3,2,1;
list stab = StabEqnId(I, w);
⇒ // ** redefining zz (int zz = 1;) qhmoduli.lib::StabEqnId:629
def S1 = stab[1]; setring S1; stabid;
⇒ stabid[1]=s(1)^2-1
⇒ stabid[2]=s(2)^3-1
⇒ stabid[3]=s(3)^6-1
```

```

def S2 = stab[2]; setring S2; actionid;
  ↪ actionid[1]=s(1)*x
  ↪ actionid[2]=s(2)*y
  ↪ actionid[3]=s(3)*z

```

### D.6.18.6 StabOrder

Procedure from library `qhmoduli.lib` (see [Section D.6.18 \[qhmoduli.lib\]](#), page 1742).

**Usage:**      `StabOrder(f); poly f`

**Purpose:**     compute the order of the stabilizer group of `f`.

**Assume:**    `f` quasihomogeneous polynomial with an isolated singularity at 0

**Return:**     `int`

**Global:**     `varSubsList`

### D.6.18.7 UpperMonomials

Procedure from library `qhmoduli.lib` (see [Section D.6.18 \[qhmoduli.lib\]](#), page 1742).

**Usage:**      `UpperMonomials(poly f, [intvec w])`

**Purpose:**     compute the upper monomials of the milnor algebra of `f`.

**Assume:**    `f` is quasihomogeneous (w.r.t. `w`)

**Return:**     `ideal`

**Example:**

```

LIB "qhmoduli.lib";
ring B = 0,(x,y,z), ls;
poly f = -z5+y5+x2z+x2y;
UpperMonomials(f);
  ↪ _[1]=y3z3
  ↪ _[2]=x2y3
  ↪ _[3]=x2y2

```

### D.6.18.8 Max

Procedure from library `qhmoduli.lib` (see [Section D.6.18 \[qhmoduli.lib\]](#), page 1742).

**Usage:**      `Max(data); intvec/list of integers`

**Purpose:**     find the maximal integer contained in 'data'

**Return:**     `list`

**Assume:**    'data' contains only integers and is not empty

**Example:**

```

LIB "qhmoduli.lib";
Max(list(1,2,3));
  ↪ 3

```

### D.6.18.9 Min

Procedure from library `qhmoduli.lib` (see [Section D.6.18 \[qhmoduli.lib\]](#), page 1742).

**Usage:** `Min(data);` intvec/list of integers

**Purpose:** find the minimal integer contained in 'data'

**Return:** list

**Assume:** 'data' contains only integers and is not empty

**Example:**

```
LIB "qhmoduli.lib";
Min(intvec(1,2,3));
↦ 1
```

### D.6.19 realclassify.lib

**Library:** `realclassify.lib`

**Purpose:** Classification of real singularities

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**Overview:** A library for classifying isolated hypersurface singularities over the reals w.r.t. right equivalence, based on the determinant of singularities by V.I. Arnold. This library is based on `classify2.lib` by the first and second author and G. Pfister, but handles the real case, while `classify2.lib` does the complex classification.

**References:**

Arnold, Varchenko, Gusein-Zade: Singularities of Differentiable Maps. Vol. 1: The classification of critical points caustics and wave fronts. Birkhäuser, Boston 1985

J. Boehm, M.S. Marais, A. Steenpass: The Classification of Real Singularities Using Singular. Part III: Unimodal Singularities of Corank 2, <https://arxiv.org/abs/1512.09028>

Greuel, Lossen, Shustin: Introduction to singularities and deformations. Springer, Berlin 2007

M.S. Marais, A. Steenpass: The Classification of Real Singularities Using SINGULAR. Part I: Splitting Lemma and Simple Singularities, J. Symb. Comput. 68 (2015), 61-71

M.S. Marais, A. Steenpass: The Classification of Real Singularities Using SINGULAR. Part II: The Structure of the Equivalence Classes of the Unimodal Singularities, J. Symb. Comput. 74 (2016), 346-366

Acknowledgements: This research was supported by the Staff Exchange Bursary Programme of the University of Pretoria, DFG SPP 1489, and DFG TRR 195. The financial assistance of the National Research Foundation (NRF), South Africa, towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at are those of the author and are not necessarily to be attributed to the National Research Foundation, South Africa.

**Procedures:**



### D.6.19.1 realclassify

Procedure from library `realclassify.lib` (see [Section D.6.19 \[realclassify.lib\]](#), page 1746).

**Usage:** `realclassify(f); f poly`

**Return:** A list of elements NF of type NormalFormEquation with the following keys:  
 NF.singularityType = the type of the singularity as a string  
 NF.normalFormEquation = an element F of type Poly with the normal form equation (a representative of the stable equivalence class of f, as chosen by Arnold)  
 NF.modality = the modality of f  
 NF.parameters = a list where each entry is a list consisting out of the parameter term in the normalform equation and an interval  
 NF.corank = the corank of f (of type int)  
 NF.inertiaIndex = the inertia index of f (of type int)  
 NF.milnorNumber = the milnor number of f (of type int)  
 NF.determinacy = an lower bound for the determinacy of f (of type int)

**Note:** The classification is done over the real numbers, so in contrast to `classify.lib`, the signs of coefficients of monomials where even exponents occur matter.  
 The ground field must be  $\mathbb{Q}$  (the rational numbers). No field extensions of any kind nor floating point numbers are allowed.  
 The monomial order must be local.  
 The input polynomial must be contained in `maxideal(2)` and must be an isolated singularity of modality 0 or 1. The Milnor number is checked for being finite.  
 In case of `NF.modality=1`, the parameter values are given as an element of a number field  $\mathbb{Q}\mathbb{Q}[a]/(\text{minpoly})$ , the coefficient ring of F in terms of the unique root of `minpoly` of the ring F.in in the interval specified in `NF.parameters` in conjunction with the parameter term.

**Example:**

```
LIB "realclassify.lib";
ring r = 0, (x,y,z), ds;
poly f = (x^2+3y-2z)^2+xyz-(x-y^3+x^2z^3)^3;
realclassify(f);
⇒ [1]:
⇒ Corank = 2
⇒ Inertia index = 0
⇒ Normalform equation of type = D[4]-
⇒ Normalform equation = x^2y-y^3
⇒ Milnor number = 4
⇒ Modality = 0
⇒ Determinacy <= 3
⇒
map phi = r, x+2y+y^2+x*y,x+y+y^2+x^2,z;
poly f = x^2+y^2-z^2; // A[1]
f=phi(f);
realclassify(f);
⇒ [1]:
⇒ Corank = 0
⇒ Inertia index = 1
⇒ Normalform equation of type = A[1]
⇒ Normalform equation = 0
⇒ Milnor number = 1
```

```

⇒ Modality = 0
⇒ Determinacy ≤ 2
⇒
poly f = x^3+y^2-z^2; // A[2]
f=phi(f);
realclassify(f);
⇒ [1]:
⇒ Corank = 1
⇒ Inertia index = 1
⇒ Normalform equation of type = A[2]
⇒ Normalform equation = x3
⇒ Milnor number = 2
⇒ Modality = 0
⇒ Determinacy ≤ 3
⇒
poly f = x^2*y-y^3+z^2; // D[4]-
f=phi(f);
realclassify(f);
⇒ [1]:
⇒ Corank = 2
⇒ Inertia index = 0
⇒ Normalform equation of type = D[4]-
⇒ Normalform equation = x2y-y3
⇒ Milnor number = 4
⇒ Modality = 0
⇒ Determinacy ≤ 3
⇒
poly f = x^3-y^4-z^2; // E[6]-
f=phi(f);
realclassify(f);
⇒ [1]:
⇒ Corank = 2
⇒ Inertia index = 1
⇒ Normalform equation of type = E[6]-
⇒ Normalform equation = x3-y4
⇒ Milnor number = 6
⇒ Modality = 0
⇒ Determinacy ≤ 4
⇒
poly f = x^3+x*y^3+z^2; // E[7]
f=phi(f);
realclassify(f);
⇒ [1]:
⇒ Corank = 2
⇒ Inertia index = 0
⇒ Normalform equation of type = E[7]
⇒ Normalform equation = x3+xy3
⇒ Milnor number = 7
⇒ Modality = 0
⇒ Determinacy ≤ 5
⇒
poly f = x^3+y^5-z^2; // E[8]
f=phi(f);

```

```

realclassify(f);
⇒ [1]:
⇒ Corank = 2
⇒ Inertia index = 1
⇒ Normalform equation of type = E[8]
⇒ Normalform equation =  $x^3+y^5$ 
⇒ Milnor number = 8
⇒ Modality = 0
⇒ Determinacy  $\leq 5$ 
⇒
poly f =  $x^3+3x^2y^2+xy^4-z^2$ ; //J[10]+
f=phi(f);
realclassify(f);
⇒ [1]:
⇒ Corank = 2
⇒ Inertia index = 1
⇒ Normalform equation of type = J[10]-
⇒ Normalform equation =  $x^3+(a)x^2y^2-xy^4$ 
⇒ Milnor number = 10
⇒ Modality = 1
⇒ Parameter term =  $(a)x^2y^2$ 
⇒ Minimal polynomial =  $(5a^4-81)$ 
⇒ Interval =  $[0, 731]$ 
⇒ Determinacy  $\leq 7$ 
⇒
⇒ [2]:
⇒ Corank = 2
⇒ Inertia index = 1
⇒ Normalform equation of type = J[10]+
⇒ Normalform equation =  $x^3+(a)x^2y^2+xy^4$ 
⇒ Milnor number = 10
⇒ Modality = 1
⇒ Parameter term =  $(a)x^2y^2$ 
⇒ Minimal polynomial =  $(5a^4-81)$ 
⇒ Interval =  $[-5/2, -3/2]$ 
⇒ Determinacy  $\leq 7$ 
⇒
⇒ [3]:
⇒ Corank = 2
⇒ Inertia index = 1
⇒ Normalform equation of type = J[10]+
⇒ Normalform equation =  $x^3+3x^2y^2+xy^4$ 
⇒ Milnor number = 10
⇒ Modality = 1
⇒ Parameter term =  $3x^2y^2$ 
⇒ Determinacy  $\leq 7$ 
⇒
poly f =  $x^3+x^2y^2+4y^9+z^2$ ; //J[13]+
f=phi(f);
realclassify(f);
⇒ [1]:
⇒ Corank = 2
⇒ Inertia index = 0

```

```

⇒ Normalform equation of type = J[13]+
⇒ Normalform equation = x3+x2y2+15y9
⇒ Milnor number = 13
⇒ Modality = 1
⇒ Parameter term = 15y9
⇒ Determinacy <= 10
⇒
⇒ [2]:
⇒ Corank = 2
⇒ Inertia index = 0
⇒ Normalform equation of type = J[13]+
⇒ Normalform equation = x3+x2y2-15y9
⇒ Milnor number = 13
⇒ Modality = 1
⇒ Parameter term = -15y9
⇒ Determinacy <= 10
⇒
poly f = -x^4-x^2*y^2+3y^9-z^2; //X[14]--
f=phi(f);
realclassify(f);
⇒ [1]:
⇒ Corank = 2
⇒ Inertia index = 1
⇒ Normalform equation of type = X[14]--
⇒ Normalform equation = -x4-x2y2+19683y9
⇒ Milnor number = 14
⇒ Modality = 1
⇒ Parameter term = 19683y9
⇒ Determinacy <= 9
⇒
⇒ [2]:
⇒ Corank = 2
⇒ Inertia index = 1
⇒ Normalform equation of type = X[14]--
⇒ Normalform equation = -x4-x2y2-19683y9
⇒ Milnor number = 14
⇒ Modality = 1
⇒ Parameter term = -19683y9
⇒ Determinacy <= 9
⇒
poly f = -x^2*y^2-x^7+4*y^8+z^2; //Y[7,8]--
f=phi(f);
realclassify(f);
⇒ [1]:
⇒ Corank = 2
⇒ Inertia index = 0
⇒ Normalform equation of type = Y[8,7]-+
⇒ Normalform equation = -x2y2+(a)*y7+x8
⇒ Milnor number = 16
⇒ Modality = 1
⇒ Parameter term = (a)*y7
⇒ Minimal polynomial = (a4-128)
⇒ Interval = [-130, 0]

```

```

⇒ Determinacy ≤ 8
⇒
⇒ [2]:
⇒ Corank = 2
⇒ Inertia index = 0
⇒ Normalform equation of type = Y[8,7]--+
⇒ Normalform equation = -x2y2+y7+x8
⇒ Milnor number = 16
⇒ Modality = 1
⇒ Parameter term = (a)*y7
⇒ Minimal polynomial = (a4-128)
⇒ Interval = [0, 130]
⇒ Determinacy ≤ 8
⇒
⇒ [3]:
⇒ Corank = 2
⇒ Inertia index = 0
⇒ Normalform equation of type = Y[7,8]--
⇒ Normalform equation = -x2y2-x7+4y8
⇒ Milnor number = 16
⇒ Modality = 1
⇒ Parameter term = 4y8
⇒ Determinacy ≤ 8
⇒
⇒ [4]:
⇒ Corank = 2
⇒ Inertia index = 0
⇒ Normalform equation of type = Y[7,8]--+
⇒ Normalform equation = -x2y2+x7+4y8
⇒ Milnor number = 16
⇒ Modality = 1
⇒ Parameter term = 4y8
⇒ Determinacy ≤ 8
⇒
poly f = (x2+y2)2+5*x9-z2; // tilde Y[9]
f=phi(f);
realclassify(f);
⇒ [1]:
⇒ Corank = 2
⇒ Inertia index = 1
⇒ Normalform equation of type = tilde(Y)[9]+
⇒ Normalform equation = x4+2x2y2+y4+5x9
⇒ Milnor number = 19
⇒ Modality = 1
⇒ Parameter term = 5x9
⇒ Determinacy ≤ 9
⇒
⇒ [2]:
⇒ Corank = 2
⇒ Inertia index = 1
⇒ Normalform equation of type = tilde(Y)[9]+
⇒ Normalform equation = x4+2x2y2+y4-5x9
⇒ Milnor number = 19

```

```

⇒ Modality = 1
⇒ Parameter term = -5x9
⇒ Determinacy ≤ 9
⇒
poly f = x^3+y^7+3*x*y^5+z^2; // E[12]
f=phi(f);
realclassify(f);
⇒ [1]:
⇒ Corank = 2
⇒ Inertia index = 0
⇒ Normalform equation of type = E[12]
⇒ Normalform equation = x3+3xy5+y7
⇒ Milnor number = 12
⇒ Modality = 1
⇒ Parameter term = 3xy5
⇒ Determinacy ≤ 8
⇒
poly f = x^3+x*y^5+4*y^8-z^2; // E[13]
f=phi(f);
realclassify(f);
⇒ [1]:
⇒ Corank = 2
⇒ Inertia index = 1
⇒ Normalform equation of type = E[13]
⇒ Normalform equation = x3+xy5+4y8
⇒ Milnor number = 13
⇒ Modality = 1
⇒ Parameter term = 4y8
⇒ Determinacy ≤ 9
⇒
poly f = x^3+y^8+2*x*y^6+z^2; // E[14]+
f=phi(f);
realclassify(f);
⇒ [1]:
⇒ Corank = 2
⇒ Inertia index = 0
⇒ Normalform equation of type = E[14]+
⇒ Normalform equation = x3+2xy6+y8
⇒ Milnor number = 14
⇒ Modality = 1
⇒ Parameter term = 2xy6
⇒ Determinacy ≤ 9
⇒
poly f = x^3*y+y^5+5*x*y^4-z^2; // Z[11]
f=phi(f);
realclassify(f);
⇒ [1]:
⇒ Corank = 2
⇒ Inertia index = 1
⇒ Normalform equation of type = Z[11]
⇒ Normalform equation = x3+5xy4+y5
⇒ Milnor number = 11
⇒ Modality = 1

```

```

⇒ Parameter term = 5xy4
⇒ Determinacy ≤ 5
⇒
poly f = x^3*y+x*y^4+6*x^2*y^3+z^2; // Z[12]
f=phi(f);
realclassify(f);
⇒ [1]:
⇒ Corank = 2
⇒ Inertia index = 0
⇒ Normalform equation of type = Z[12]
⇒ Normalform equation = x3y+6x2y3+xy4
⇒ Milnor number = 12
⇒ Modality = 1
⇒ Parameter term = 6x2y3
⇒ Determinacy ≤ 6
⇒
poly f = x^3*y-y^6+2*x*y^5-z^2; // Z[13]-
f=phi(f);
realclassify(f);
⇒ [1]:
⇒ Corank = 2
⇒ Inertia index = 1
⇒ Normalform equation of type = Z[13]-
⇒ Normalform equation = x3y+2xy5-y6
⇒ Milnor number = 13
⇒ Modality = 1
⇒ Parameter term = 2xy5
⇒ Determinacy ≤ 6
⇒
poly f = x^4+y^5+x^2*y^3+z^2; // W[12]+
f=phi(f);
realclassify(f);
⇒ [1]:
⇒ Corank = 2
⇒ Inertia index = 0
⇒ Normalform equation of type = W[12]+
⇒ Normalform equation = x4+x2y3+y5
⇒ Milnor number = 12
⇒ Modality = 1
⇒ Parameter term = x2y3
⇒ Determinacy ≤ 5
⇒
poly f = -x^4+x*y^4+y^6-z^2; // W[13]-
f=phi(f);
realclassify(f);
⇒ [1]:
⇒ Corank = 2
⇒ Inertia index = 1
⇒ Normalform equation of type = W[13]-
⇒ Normalform equation = -x4+xy4+y6
⇒ Milnor number = 13
⇒ Modality = 1
⇒ Parameter term = y6

```

```

⇒ Determinacy <= 6
⇒
poly p = x^4-x^2*y^2+5*y^4+x*y^3+x^3*y+z^2; //X9++
f=phi(f);
realclassify(p);
⇒ [1]:
⇒ Corank = 2
⇒ Inertia index = 0
⇒ Normalform equation of type = X[9]++
⇒ Normalform equation = x4+(a)*x2y2+y4
⇒ Milnor number = 9
⇒ Modality = 1
⇒ Parameter term = (a)*x2y2
⇒ Minimal polynomial = (18541a6-2454316a4+32984048a2-17909824)
⇒ Interval = [10829268444323/1000000000000, 433170737773/400000000000]
⇒ Determinacy <= 4
⇒
⇒ [2]:
⇒ Corank = 2
⇒ Inertia index = 0
⇒ Normalform equation of type = X[9]++
⇒ Normalform equation = x4+(a)*x2y2+y4
⇒ Milnor number = 9
⇒ Modality = 1
⇒ Parameter term = (a)*x2y2
⇒ Minimal polynomial = (18541a6-2454316a4+32984048a2-17909824)
⇒ Interval = [-94106465721/1250000000000, -752851725767/1000000000000]
⇒ Determinacy <= 4
⇒

```

See also: [Section D.6.4.2 \[classify\]](#), page 1647.

### D.6.19.2 realmorsesplit

Procedure from library `realclassify.lib` (see [Section D.6.19 \[realclassify.lib\]](#), page 1746).

**Usage:** `realmorsesplit(f[, mu]);` f poly, mu int

**Return:** a list consisting of the corank of f, the inertia index, an upper bound for the determinacy, and the residual form of f

**Note:** The characteristic of the basering must be zero, the monomial order must be local, f must be contained in `maxideal(2)` and the Milnor number of f must be finite.  
The Milnor number of f can be provided as an optional parameter in order to avoid that it is computed again.

**Example:**

```

LIB "realclassify.lib";
ring r = 0, (x,y,z), ds;
poly f = (x2+3y-2z)^2+xyz-(x-y3+x2z3)^3;
realmorsesplit(f);
⇒ [1]:
⇒ 2
⇒ [2]:
⇒ 0

```



```

⇒ [3] :
⇒      3
⇒ [4] :
⇒      -x3+2/3xy2

```

See also: [Section D.6.4.8 \[morsesplit\]](#), page 1651.

### D.6.19.3 milnornumber

Procedure from library `realclassify.lib` (see [Section D.6.19 \[realclassify\\_lib\]](#), page 1746).

**Usage:** `milnornumber(f); f poly`

**Return:** Milnor number of  $f$ , or -1 if the Milnor number is not finite

**Note:** The monomial order must be local.

**Example:**

```

LIB "realclassify.lib";
ring r = 0, (x,y), ds;
poly f = x3+y4;
milnornumber(f);
⇒ 6

```

### D.6.19.4 determinacy

Procedure from library `realclassify.lib` (see [Section D.6.19 \[realclassify\\_lib\]](#), page 1746).

**Usage:** `determinacy(f[, mu]); f poly, mu int`

**Return:** an upper bound for the determinacy of  $f$

**Note:** The characteristic of the basering must be zero, the monomial order must be local,  $f$  must be contained in `maxideal(1)` and the Milnor number of  $f$  must be finite.  
The Milnor number of  $f$  can be provided as an optional parameter in order to avoid that it is computed again.

**Example:**

```

LIB "realclassify.lib";
ring r = 0, (x,y), ds;
poly f = x3+xy3;
determinacy(f);
⇒ 5

```

See also: [Section 5.1.55 \[highcorner\]](#), page 194; [Section D.6.19.3 \[milnornumber\]](#), page 1755.

### D.6.19.5 addnondegeneratevariables

Procedure from library `realclassify.lib` (see [Section D.6.19 \[realclassify\\_lib\]](#), page 1746).

**Usage:** `addnondegeneratevariables(NFR); NFR NormalFormEquation`

**Return:** Adds squares of the non-degenerate variables (i.e. `var(cr+1)`, ..., `var(nvars(basering))` for corank  $cr$ ) to the normalform `nf`, with signs according to the inertia index.

**Example:**

```

LIB "realclassify.lib";
ring r = 0, (x,y,z), ds;
poly f = (x2+3y-2z)^2+xyz-(x-y3+x2z3)^3;
list NFR = realclassify(f);
NFR[1];
  ↳ Corank = 2
  ↳ Inertia index = 0
  ↳ Normalform equation of type = D[4]-
  ↳ Normalform equation = x2y-y3
  ↳ Milnor number = 4
  ↳ Modality = 0
  ↳ Determinacy <= 3
  ↳
addnondegeneratevariables(NFR[1]);
  ↳ Corank = 2
  ↳ Inertia index = 0
  ↳ Normalform equation of type = D[4]-
  ↳ Normalform equation = z2+x2y-y3
  ↳ Milnor number = 4
  ↳ Modality = 0
  ↳ Determinacy <= 3
  ↳

```

See also: [Section D.6.19.2 \[realmorsesplit\]](#), page 1754.

### D.6.19.6 HilbertClassPoly

Procedure from library `realclassify.lib` (see [Section D.6.19 \[realclassify.lib\]](#), page 1746).

**Return:** the monic polynomial of degree  $h(D)$  in  $\mathbb{Z}[X]$  of which  $jOft((D+\text{sqr}(D))/2)$  is a root

**Assume:**  $D$  is a negative discriminant

**Note:**  $k$  is input for the procedure "jOft",  
 $5*k$  is input for the procedure "sqr",  
 $10*k$  describes the number of decimals being calculated in the complex numbers

**Example:**

```

LIB "realclassify.lib";
ring r = 0,x,dp;
bigint D=-23;
HilbertClassPoly(D,50);
  ↳ x3+3491750x2-5151296875x+12771880859375

```

### D.6.20 sing.lib

**Library:** `sing.lib`

**Purpose:** Invariants of Singularities

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**Procedures:**

### D.6.20.1 codim

Procedure from library `sing.lib` (see [Section D.6.20 \[sing\\_lib\], page 1756](#)).

**Usage:** `codim(id1,id2)`; `id1,id2` ideal or module, both must be standard bases

**Return:** int, which is:

1. the vectorspace dimension of `id1/id2` if `id2` is contained in `id1` and if this number is finite
2. -1 if the dimension of `id1/id2` is infinite
3. -2 if `id2` is not contained in `id1`

**Compute:** consider the Hilbert series  $iv1(t)$  of `id1` and  $iv2(t)$  of `id2`. If  $\text{codim}(\text{id1},\text{id2})$  is finite,  $q(t)=(iv2(t)-iv1(t))/(1-t)^n$  is rational, and the codimension is the sum of the coefficients of  $q(t)$  ( $n$  = dimension of basering).

**Example:**

```
LIB "sing.lib";
ring r = 0,(x,y),dp;
ideal j = y6,x4;
ideal m = x,y;
attrib(m,"isSB",1); //let Singular know that ideals are a standard basis
attrib(j,"isSB",1);
codim(m,j);          // should be 23 (Milnor number -1 of y7-x5)
↦ 23
```

### D.6.20.2 deform

Procedure from library `sing.lib` (see [Section D.6.20 \[sing\\_lib\], page 1756](#)).

**Usage:** `deform(id)`; `id`=ideal or poly

**Return:** matrix, columns are kbase of infinitesimal deformations

**Example:**

```
LIB "sing.lib";
ring r = 32003,(x,y,z),ds;
ideal i = xy,xz,yz;
matrix T = deform(i);
print(T);
↦ x,0,0,
↦ 0,0,z,
↦ 0,y,0
print(deform(x3+y5+z2));
↦ xy3,y3,xy2,y2,xy,y,x,1
```

### D.6.20.3 dim\_slocus

Procedure from library `sing.lib` (see [Section D.6.20 \[sing\\_lib\], page 1756](#)).

**Usage:** `dim_slocus(i)`; `i` ideal or poly

**Return:** dimension of singular locus of `i`

**Example:**

```
LIB "sing.lib";
ring r = 32003,(x,y,z),ds;
ideal i = x5+y6+z6,x2+2y2+3z2;
dim_slocus(i);
↪ 0
```

#### D.6.20.4 is\_active

Procedure from library `sing.lib` (see [Section D.6.20 \[sing\\_lib\]](#), page 1756).

**Usage:** `is_active(f,id)`; `f` poly, `id` ideal or module

**Return:** 1 if `f` is an active element modulo `id` (i.e.  $\dim(id) = \dim(id + f \cdot R^n) + 1$ , if `id` is a submodule of  $R^n$ ) resp. 0 if `f` is not active. The basering may be a quotient ring

**Note:** regular parameters are active but not vice versa (`id` may have embedded components).  
`proc is_reg` tests whether `f` is a regular parameter

**Example:**

```
LIB "sing.lib";
ring r = 32003,(x,y,z),ds;
ideal i = yx3+y,yz3+y3z;
poly f = x;
is_active(f,i);
↪ 1
qring q = std(x4y5);
poly f = x;
module m = [yx3+x,yx3+y3x];
is_active(f,m);
↪ 0
```

#### D.6.20.5 is\_ci

Procedure from library `sing.lib` (see [Section D.6.20 \[sing\\_lib\]](#), page 1756).

**Usage:** `is_ci(i)`; `i` ideal

**Return:** `intvec` = sequence of dimensions of ideals (`j[1],...,j[k]`), for  $k=1,...,\text{size}(j)$ , where `j` is minimal base of `i`. `i` is a complete intersection if last number equals `nvars-size(i)`

**Note:**  $\dim(0\text{-ideal}) = -1$ . You may first apply `simplify(i,10)`; in order to delete zeroes and multiples from set of generators  
`printlevel >= 0`: display comments (default)

**Example:**

```
LIB "sing.lib";
int p = printlevel;
printlevel = 1; // display comments
ring r = 32003,(x,y,z),ds;
ideal i = x4+y5+z6,xyz,yx2+xz2+zy7;
is_ci(i);
↪ // complete intersection of dim 0
↪ // dim-sequence:
↪ 2,1,0
i = xy,yz;
is_ci(i);
```

```

⇒ // no complete intersection
⇒ // dim-sequence:
⇒ 2,2
printlevel = p;

```

### D.6.20.6 is\_is

Procedure from library `sing.lib` (see [Section D.6.20 \[sing\\_lib\]](#), page 1756).

**Usage:** `is_is(id)`; `id` ideal or poly

**Return:** `intvec` = sequence of dimensions of singular loci of ideals generated by `id[1]..id[i]`,  $k = 1..size(id)$ ;  
`dim(0-ideal) = -1`;  
`id` defines an isolated singularity if last number is 0

**Note:** `printlevel >= 0`: display comments (default)

**Example:**

```

LIB "sing.lib";
int p      = printlevel;
printlevel = 1;
ring r      = 32003, (x,y,z), ds;
ideal i     = x2y, x4+y5+z6, yx2+xz2+zy7;
is_is(i);
⇒ // dim of singular locus = 0
⇒ // isolated singularity if last number is 0 in dim-sequence:
⇒ 2,1,0
poly f      = xy+yz;
is_is(f);
⇒ // dim of singular locus = 1
⇒ // isolated singularity if last number is 0 in dim-sequence:
⇒ 1
printlevel = p;

```

### D.6.20.7 is\_reg

Procedure from library `sing.lib` (see [Section D.6.20 \[sing\\_lib\]](#), page 1756).

**Usage:** `is_reg(f,id)`; `f` poly, `id` ideal or module

**Return:** 1 if multiplication with `f` is injective modulo `id`, 0 otherwise

**Note:** Let  $R$  be the basering and `id` a submodule of  $R^n$ . The procedure checks injectivity of multiplication with `f` on  $R^n/id$ . The basering may be a quotient ring.

**Example:**

```

LIB "sing.lib";
ring r      = 32003, (x,y), ds;
ideal i     = x8,y8;
ideal j     = (x+y)^4;
i           = intersect(i,j);
poly f      = xy;
is_reg(f,i);
⇒ 0

```

### D.6.20.8 is\_regs

Procedure from library `sing.lib` (see [Section D.6.20 \[sing\\_lib\], page 1756](#)).

**Usage:** `is_regs(i[,id]);` `i` poly, `id` ideal or module (default: `id=0`)

**Return:** 1 if generators of `i` are a regular sequence modulo `id`, 0 otherwise

**Note:** Let  $R$  be the basering and `id` a submodule of  $R^n$ . The procedure checks injectivity of multiplication with `i[k]` on  $R^n/\text{id}+i[1..k-1]$ . The basering may be a quotient ring.  
`printlevel >=0`: display comments (default)  
`printlevel >=1`: display comments during computation

**Example:**

```
LIB "sing.lib";
int p      = printlevel;
printlevel = 1;
ring r1     = 32003,(x,y,z),ds;
ideal i     = x8,y8,(x+y)^4;
is_regs(i);
⇒ // checking whether element 1 is regular mod 1 .. 0
⇒ // checking whether element 2 is regular mod 1 .. 1
⇒ // checking whether element 3 is regular mod 1 .. 2
⇒ // elements 1..2 are regular, 3 is not regular mod 1..2
⇒ 0
module m    = [x,0,y];
i           = x8,(x+z)^4;;
is_regs(i,m);
⇒ // checking whether element 1 is regular mod 1 .. 0
⇒ // checking whether element 2 is regular mod 1 .. 1
⇒ // elements are a regular sequence of length 2
⇒ 1
printlevel = p;
```

### D.6.20.9 locstd

Procedure from library `sing.lib` (see [Section D.6.20 \[sing\\_lib\], page 1756](#)).

**Usage:** `locstd (id);` `id` = ideal

**Return:** a standard basis for a local degree ordering

**Note:** the procedure homogenizes `id` w.r.t. a new 1st variable `@t@`, computes a SB w.r.t.  $(dp(1), dp)$  and substitutes `@t@` by 1.  
Hence the result is a SB with respect to an ordering which sorts first w.r.t. the order and then refines it with `dp`. This is a local degree ordering.  
This is done in order to avoid cancellation of units and thus be able to use `option(contentSB);`

**Example:**

```
LIB "sing.lib";
ring R = 0,(x,y,z),ds;
ideal i = xyz+z5,2x2+y3+z7,3z5+y5;
locstd(i);
⇒ _[1]=y5+3z5
⇒ _[2]=3x4y3z8-4x3y3z9+6x2y4z9+3y5z10
```

```

⇒ _[3]=3x4z13-4x3z14+6x2yz14+3y2z15
⇒ _[4]=3x4yz12-4x3yz13+6x2y2z13+3y3z14
⇒ _[5]=2x2z9+x2y2z8+y3z9
⇒ _[6]=2x2y4z5+y7z5-3x2yz9
⇒ _[7]=6y2z10-3x2y3z8+4xy3z9-3y4z9
⇒ _[8]=3x2y2z8+3y3z9+2xy4z8
⇒ _[9]=18z14-4xy6z8+3y7z8-9x2yz12
⇒ _[10]=xyz+z5
⇒ _[11]=3xz6-y4z5
⇒ _[12]=3y3z6+2xy4z5-3xyz9
⇒ _[13]=y4z5-2xz9-xy2z8
⇒ _[14]=3z10+2xyz9+xy3z8
⇒ _[15]=2x2z5+y3z5-xyz8
⇒ _[16]=y4z-2xz5+yz8
⇒ _[17]=3z6+2xyz5-y2z8
⇒ _[18]=2x2+y3+z7

```

### D.6.20.10 milnor

Procedure from library `sing.lib` (see [Section D.6.20 \[sing.lib\], page 1756](#)).

**Usage:**      `milnor(i)`; *i* ideal or poly

**Return:**    Milnor number of *i*, if *i* is ICIS (isolated complete intersection singularity) in generic form, resp. -1 if not

**Note:**      use `proc nf_icis` to put generators in generic form  
               `printlevel >=1`: display comments

**Example:**

```

LIB "sing.lib";
int p      = printlevel;
printlevel = 2;
ring r     = 32003,(x,y,z),ds;
ideal j    = x5+y6+z6,x2+2y2+3z2,xyz+yx;
milnor(j);
⇒ //sequence of discriminant numbers: 100,149,70
⇒ 21
poly f     = x7+y7+(x-y)^2*x2y2+z2;
milnor(f);
⇒ 28
printlevel = p;

```

### D.6.20.11 nf\_icis

Procedure from library `sing.lib` (see [Section D.6.20 \[sing.lib\], page 1756](#)).

**Usage:**      `nf_icis(i)`; *i* ideal

**Return:**    ideal = generic linear combination of generators of *i* if *i* is an ICIS (isolated complete intersection singularity), return *i* if not

**Note:**      this proc is useful in connection with `proc milnor`  
               `printlevel >=0`: display comments (default)

**Example:**

```

LIB "sing.lib";
int p      = printlevel;
printlevel = 1;
ring r      = 32003,(x,y,z),ds;
ideal i     = x3+y4,z4+yx;
nf_icis(i);
⇨ // complete intersection of dim 1
⇨ // dim-sequence:
⇨ // dim of singular locus = 0
⇨ // isolated singularity if last number is 0 in dim-sequence:
⇨ // dim of singular locus = 0
⇨ // isolated singularity if last number is 0 in dim-sequence:
⇨ // ICIS in generic form after 1 genericity loop(s)
⇨ _[1]=2xy+x3+y4+2z4
⇨ _[2]=xy+z4
ideal j     = x3+y4,xy,yz;
nf_icis(j);
⇨ // no complete intersection
⇨ // dim-sequence:
⇨ // no complete intersection
⇨ _[1]=x3+y4
⇨ _[2]=xy
⇨ _[3]=yz
printlevel = p;

```

### D.6.20.12 slocus

Procedure from library `sing.lib` (see [Section D.6.20 \[sing\\_lib\], page 1756](#)).

**Usage:** `slocus(i); i ideal`

**Return:** ideal of singular locus of `i`. Quotient rings and rings with integer coefficients are currently not supported.

**Example:**

```

LIB "sing.lib";
ring r = 0,(u,v,w,x,y,z),dp;
ideal i = wx,wy,wz,vx,vy,vz,ux,uy,uz,y3-x2;;
slocus(i);
⇨ _[1]=x
⇨ _[2]=w
⇨ _[3]=v
⇨ _[4]=u
⇨ _[5]=y2

```

### D.6.20.13 qhspectrum

Procedure from library `sing.lib` (see [Section D.6.20 \[sing\\_lib\], page 1756](#)).

**Usage:** `qhspectrum(f,w); f=poly, w=intvec`

**Assume:** `f` is a weighted homogeneous isolated singularity w.r.t. the weights given by `w`; `w` must consist of as many positive integers as there are variables of the basering

**Compute:** the spectral numbers of the `w`-homogeneous polynomial `f`, computed in a ring of characteristic 0



**Return:** intvec  $d, s_1, \dots, s_u$  where:  
 $d = w\text{-degree}(f)$  and  $s_i/d = i\text{-th spectral-number}(f)$   
 No return value if basering has parameters or if  $f$  is no isolated singularity, displays a warning in this case.

**Example:**

```
LIB "sing.lib";
ring r;
poly f=x3+y5+z2;
intvec w=10,6,15;
qhspectrum(f,w);
↪ 30,1,7,11,13,17,19,23,29
// the spectrum numbers are:
// 1/30,7/30,11/30,13/30,17/30,19/30,23/30,29/30
```

#### D.6.20.14 Tjurina

Procedure from library `sing.lib` (see [Section D.6.20 \[sing.lib\]](#), page 1756).

**Usage:** `Tjurina(id[,<any>])`;  $\text{id}$ =ideal or poly

**Assume:**  $\text{id}$ =ICIS (isolated complete intersection singularity)

**Return:** standard basis of Tjurina-module of  $\text{id}$ ,  
 of type module if  $\text{id}$ =ideal, resp. of type ideal if  $\text{id}$ =poly. If a second argument is present (of any type) return a list:  
 $[1]$  = Tjurina number,  
 $[2]$  =  $k$ -basis of miniversal deformation,  
 $[3]$  = SB of Tjurina module,  
 $[4]$  = Tjurina module

**Display:** Tjurina number if `printlevel`  $\geq 0$  (default)

**Note:** Tjurina number = -1 implies that  $\text{id}$  is not an ICIS

**Example:**

```
LIB "sing.lib";
int p      = printlevel;
printlevel = 1;
ring r      = 0,(x,y,z),ds;
poly f      = x5+y6+z7+xyz;           // singularity T[5,6,7]
list T      = Tjurina(f,"");
↪ // Tjurina number = 16
show(T[1]);                             // Tjurina number, should be 16
↪ // int, size 1
↪ 16
show(T[2]);                             // basis of miniversal deformation
↪ // ideal, 16 generator(s)
↪ z6,
↪ z5,
↪ z4,
↪ z3,
↪ z2,
↪ z,
↪ y5,
↪ y4,
```

```

    ↪ y3,
    ↪ y2,
    ↪ y,
    ↪ x4,
    ↪ x3,
    ↪ x2,
    ↪ x,
    ↪ 1
    show(T[3]); // SB of Tjurina ideal
    ↪ // ideal, 6 generator(s)
    ↪ xy+7z6,
    ↪ xz+6y5,
    ↪ yz+5x4,
    ↪ 5x5-6y6,
    ↪ y6,
    ↪ z7
    show(T[4]); ""; // Tjurina ideal
    ↪ // ideal, 4 generator(s)
    ↪ yz+5x4,
    ↪ xz+6y5,
    ↪ xy+7z6,
    ↪ xyz+x5+y6+z7
    ↪
    ideal j = x2+y2+z2,x2+2y2+3z2;
    show(kbase(Tjurina(j))); // basis of miniversal deformation
    ↪ // Tjurina number = 5
    ↪ // module, 5 generator(s)
    ↪ [z]
    ↪ [y]
    ↪ [x]
    ↪ [1]
    ↪ [0,1]
    hilb(Tjurina(j)); // Hilbert series of Tjurina module
    ↪ // Tjurina number = 5
    ↪ (-3t4+7t3-3t2-3t+2) / (1-t)^3
    ↪ (3t+2) / (1-t)^0
    ↪ // dimension (local) = 0
    ↪ // multiplicity = 5
    printlevel = p;

```

### D.6.20.15 tjurina

Procedure from library `sing.lib` (see [Section D.6.20 \[sing.lib\]](#), page 1756).

**Usage:** `tjurina(id);` `id=ideal` or `poly`

**Assume:** `id=ICIS` (isolated complete intersection singularity)

**Return:** `int` = Tjurina number of `id`

**Note:** Tjurina number = -1 implies that `id` is not an ICIS

**Example:**

```

LIB "sing.lib";
ring r=32003,(x,y,z),(c,ds);

```

```

ideal j=x2+y2+z2,x2+2y2+3z2;
tjurina(j);
↪ 5

```

### D.6.20.16 T\_1

Procedure from library `sing.lib` (see [Section D.6.20 \[sing\\_lib\]](#), page 1756).

**Usage:** `T_1(id[,<any>]);` `id` = ideal or poly

**Return:** `T_1(id)`: of type module/ideal if `id` is of type ideal/poly. We call `T_1(id)` the `T_1`-module of `id`. It is a std basis of the presentation of 1st order deformations of  $P/id$ , if  $P$  is the basering. If a second argument is present (of any type) return a list of 3 modules:

`[1]` = `T_1(id)`

`[2]` = generators of normal bundle of `id`, lifted to  $P$

`[3]` = module of relations of `[2]`, lifted to  $P$

(note:  $\text{transpose}[3]*[2]=0 \text{ mod } id$ )

The list contains all non-easy objects which must be computed to get `T_1(id)`.

**Display:** `k`-dimension of `T_1(id)` if `printlevel`  $\geq 0$  (default)

**Note:** `T_1(id)` itself is usually of minor importance. Nevertheless, from it all relevant information can be obtained. The most important are probably `vdim(T_1(id))`; (which computes the Tjurina number), `hilb(T_1(id))`; and `kbase(T_1(id))`.

If `T_1` is called with two arguments, then `matrix([2])*(kbase([1]))` represents a basis of 1st order semiuniversal deformation of `id` (use proc 'deform', to get this in a direct way).

For a complete intersection the proc `Tjurina` is faster.

**Example:**

```

LIB "sing.lib";
int p      = printlevel;
printlevel = 1;
ring r      = 32003,(x,y,z),(c,ds);
ideal i     = xy,xz,yz;
module T    = T_1(i);
↪ // dim T_1 = 3
vdim(T);
↪ 3
// Tjurina number = dim_K(T_1), should be 3
list L=T_1(i,"");
↪ // dim T_1 = 3
module kB   = kbase(L[1]);
print(matrix(L[2])*matrix(kB)); // basis of 1st order miniversal deformation
↪ 0,0,0,
↪ z,0,0,
↪ 0,y,z
show(L[2]);
// presentation of normal bundle
↪ // module, 6 generator(s)
↪ [x]
↪ [y,z]
↪ [0,x,y]
↪ [0,z]
↪ [0,0,y]
↪ [0,0,z]

```

```

print(L[3]);                // relations of i
⇨ z, 0,
⇨ -y,y,
⇨ 0, -x
print(transpose(matrix(L[3]))*matrix(L[2])); // should be 0 (mod i)
⇨ xz,0, -xy,-yz,0, 0,
⇨ 0, yz,0, yz, -xy,-xz
printlevel = p;

```

### D.6.20.17 T\_2

Procedure from library `sing.lib` (see [Section D.6.20 \[sing\\_lib\]](#), page 1756).

**Usage:** `T_2(id[,<any>]);` `id` = ideal

**Return:** `T_2(id)`: `T_2`-module of `id`. This is a std basis of a presentation of the module of obstructions of  $R=P/id$ , if  $P$  is the basering. If a second argument is present (of any type) return a list of 4 modules and 1 ideal:

```

[1]= T_2(id)
[2]= standard basis of id (ideal)
[3]= module of relations of id (=1st syzygy module of id)
[4]= presentation of syz/kos
[5]= relations of Hom_P([3]/kos,R), lifted to P

```

The list contains all non-easy objects which must be computed to get `T_2(id)`.

**Display:** `k`-dimension of `T_2(id)` if `printlevel`  $\geq 0$  (default)

**Note:** The most important information is probably `vdim(T_2(id))`. Use `proc miniversal` to get equations of the miniversal deformation.

**Example:**

```

LIB "sing.lib";
int p      = printlevel;
printlevel = 1;
ring r     = 32003,(x,y),(c,dp);
ideal j    = x6-y4,x6y6,x2y4-x5y2;
module T   = T_2(j);
⇨ // dim T_2 = 6
vdim(T);
⇨ 6
hilb(T);"";
⇨ (t5-t3-t2+1) / (1-t)^2
⇨ (t3+2t2+2t+1) / (1-t)^0
⇨ // dimension (affine) = 0
⇨ // degree (affine) = 6
⇨
ring r1    = 0,(x,y,z),dp;
ideal id   = xy,xz,yz;
list L     = T_2(id,"");
⇨ // dim T_2 = 0
vdim(L[1]);                // vdim of T_2
⇨ 0
print(L[3]);                // syzygy module of id
⇨ -z,-z,
⇨ y, 0,

```

```

    ↪ 0, x
    printlevel = p;

```

### D.6.20.18 T\_12

Procedure from library `sing.lib` (see [Section D.6.20 \[sing\\_lib\]](#), page 1756).

**Usage:** `T_12(i[,any]);`  $i$  = ideal

**Return:** `T_12(i)`: list of 2 modules:

\* standard basis of  $T_1$ -module = `T_1(i)`, 1st order deformations

\* standard basis of  $T_2$ -module = `T_2(i)`, obstructions of  $R=P/i$

If a second argument is present (of any type) return a list of 9 modules, matrices, integers:

[1]= standard basis of  $T_1$ -module

[2]= standard basis of  $T_2$ -module

[3]= `vdim` of  $T_1$

[4]= `vdim` of  $T_2$

[5]= matrix, whose cols present infinitesimal deformations

[6]= matrix, whose cols are generators of relations of  $i(=syz(i))$

[7]= matrix, presenting  $\text{Hom}_P(syz/kos, R)$ , lifted to  $P$

[8]= presentation of  $T_1$ -module, no std basis

[9]= presentation of  $T_2$ -module, no std basis

**Display:**  $k$ -dimension of  $T_1$  and  $T_2$  if `printlevel`  $\geq 0$  (default)

**Note:** Use `proc miniversal` from `deform.lib` to get miniversal deformation of  $i$ , the list contains all objects used by `proc miniversal`.

**Example:**

```

LIB "sing.lib";
int p      = printlevel;
printlevel = 1;
ring r      = 199, (x,y,z,u,v), (c, ws(4,3,2,3,4));
ideal i     = xz-y2, yz2-xu, xv-yzu, yu-z3, z2u-yv, zv-u2;
//a cyclic quotient singularity
list L      = T_12(i,1);
↪ // dim T_1 = 5
↪ // dim T_2 = 3
print(L[5]);           //matrix of infin. deformations
↪ 0, 0, 0, 0, 0,
↪ yz, y, z2, 0, 0,
↪ -z3, -z2, -zu, yz, yu,
↪ -z2, -z, -u, 0, 0,
↪ zu, u, v, -z2, -zu,
↪ 0, 0, 0, u, v
printlevel = p;

```

### D.6.20.19 tangentcone

Procedure from library `sing.lib` (see [Section D.6.20 \[sing\\_lib\]](#), page 1756).

**Usage:** `tangentcone(id [,n]);`  $id$  = ideal,  $n$  = int

**Return:** the tangent cone of  $id$

**Note:** The procedure works for any monomial ordering.  
If  $n=0$  use `std` w.r.t. local ordering `ds`, if  $n=1$  use `locstd`.

**Example:**

```
LIB "sing.lib";
ring R = 0,(x,y,z),ds;
ideal i = 7xyz+z5,x2+y3+z7,5z5+y5;
tangentcone(i);
↪ _[1]=x2
↪ _[2]=7xyz
↪ _[3]=y5+5z5
↪ _[4]=7y4z
↪ _[5]=35z6
```

### D.6.21 spcurve.lib

**Library:** spcurve.lib

**Purpose:** Deformations and Invariants of CM-codim 2 Singularities

**Author:** Anne Fruehbis-Krueger, anne@mathematik.uni-kl.de

**Procedures:**

#### D.6.21.1 isCMcod2

Procedure from library `spcurve.lib` (see [Section D.6.21 \[spcurve.lib\]](#), page 1768).

**Usage:** `isCMcod2(i)`;  $i$  an ideal

**Return:** presentation matrix of  $i$ , if  $i$  is Cohen-Macaulay of codimension 2  
a zero matrix otherwise

**Example:**

```
LIB "spcurve.lib";
ring r=32003,(x,y,z),ds;
ideal i=xz,yz,x^3-y^4;
print(isCMcod2(i));
↪ -y, -x2,
↪ x, y3,
↪ 0, z
```

#### D.6.21.2 CMtype

Procedure from library `spcurve.lib` (see [Section D.6.21 \[spcurve.lib\]](#), page 1768).

**Usage:** `CMtype(i)`;  $i$  an ideal, CM of codimension 2

**Return:** Cohen-Macaulay type of  $i$  (integer)  
(-1, if  $i$  is not Cohen-Macaulay of codimension 2)

**Example:**

```
LIB "spcurve.lib";
ring r=32003,(x,y,z),ds;
ideal i=xy,xz,yz;
CMtype(i);
↪ 2
```

### D.6.21.3 matrixT1

Procedure from library `spcurve.lib` (see [Section D.6.21 \[spcurve.lib\]](#), page 1768).

**Usage:** `matrixT1(M,n)`;  $M$  matrix,  $n$  integer

**Assume:**  $M$  is a presentation matrix of an ideal  $i$ , CM of codimension 2; consider  $i$  as a family of ideals in a ring in the first  $n$  variables where the remaining variables are considered as parameters

**Return:** list consisting of the  $k \times (k+1)$  matrix  $M$  and a module  $K_M$  such that  $T1 = \text{Mat}(k, k+1; R)/K_M$  is the space of first order deformations of  $i$

**Example:**

```
LIB "spcurve.lib";
ring r=32003,(x(1),x(2),x(3)),ds;
ideal curve=x(1)*x(2),x(1)*x(3),x(2)*x(3);
matrix M=isCMcod2(curve);
matrixT1(M,3);
⇒ [1]:
⇒   _[1,1]=0
⇒   _[1,2]=-x(3)
⇒   _[2,1]=-x(2)
⇒   _[2,2]=x(2)
⇒   _[3,1]=x(1)
⇒   _[3,2]=0
⇒ [2]:
⇒   _[1]=gen(5)
⇒   _[2]=gen(4)-gen(3)
⇒   _[3]=-gen(2)
⇒   _[4]=x(1)*gen(5)-x(2)*gen(3)
⇒   _[5]=x(1)*gen(6)-x(2)*gen(4)
⇒   _[6]=x(2)*gen(3)-x(3)*gen(1)
⇒   _[7]=x(2)*gen(4)-x(3)*gen(2)
⇒   _[8]=-x(3)*gen(2)
⇒   _[9]=x(2)*gen(2)-x(2)*gen(1)
⇒   _[10]=x(1)*gen(1)
⇒   _[11]=-x(3)*gen(4)
⇒   _[12]=x(2)*gen(4)-x(2)*gen(3)
⇒   _[13]=x(1)*gen(3)
⇒   _[14]=-x(3)*gen(6)
⇒   _[15]=x(2)*gen(6)-x(2)*gen(5)
⇒   _[16]=x(1)*gen(5)
```

### D.6.21.4 semiCMcod2

Procedure from library `spcurve.lib` (see [Section D.6.21 \[spcurve.lib\]](#), page 1768).

**Usage:** `semiCMcod2(M,t1[,s])`;  $M$  matrix,  $t1$  module,  $s$  any

**Assume:**  $M$  is a presentation matrix of an ideal  $i$ , CM of codimension 2, and  $t1$  is a presentation of the space of first order deformations of  $i$  ( $(M,t1)$  as returned by the procedure `matrixT1`)

**Return:** new ring in which the ideal `semi` describing the semiuniversal deformation of  $i$ ; if the optional third argument is given, the perturbation matrix of the semiuniversal deformation is returned instead of the ideal.

**Note:** The current basering should not contain any variables named  $A(j)$  where  $j$  is some integer!

**Example:**

```
LIB "spcurve.lib";
ring r=32003,(x(1),x(2),x(3)),ds;
ideal curve=x(1)*x(2),x(1)*x(3),x(2)*x(3);
matrix M=isCMcod2(curve);
list l=matrixT1(M,3);
def rneu=semiCMcod2(l[1],std(l[2]));
setring rneu;
semi;
⇒ semi[1]=A(2)*A(3)-x(2)*A(3)-x(1)*x(2)
⇒ semi[2]=A(1)*A(3)+x(1)*x(3)
⇒ semi[3]=-x(2)*A(1)-x(3)*A(2)+x(2)*x(3)
```

### D.6.21.5 discr

Procedure from library `spcurve.lib` (see [Section D.6.21 \[spcurve.lib\]](#), page 1768).

**Usage:** `discr(sem,n)`; `sem` ideal, `n` integer

**Assume:** `sem` is the versal deformation of an ideal of codimension 2.  
The first `n` variables of the ring are treated as variables all the others as parameters.

**Return:** ideal describing the discriminant

**Note:** This is not a powerful algorithm!

**Example:**

```
LIB "spcurve.lib";
ring r=32003,(x(1),x(2),x(3)),ds;
ideal curve=x(1)*x(2),x(1)*x(3),x(2)*x(3);
matrix M=isCMcod2(curve);
list l=matrixT1(M,3);
def rneu=semiCMcod2(l[1],std(l[2]));
setring rneu;
discr(semi,3);
⇒ _[1]=A(1)*A(2)*A(3)
```

### D.6.21.6 qhmatrix

Procedure from library `spcurve.lib` (see [Section D.6.21 \[spcurve.lib\]](#), page 1768).

**Usage:** `qhmatrix(M)`; `M` a  $k \times (k+1)$  matrix

**Return:** list, consisting of an integer vector containing the weights of the variables of the basering and an integer matrix giving the weights of the entries of `M`, if `M` is quasihomogeneous; zero integer vector and zero integer matrix, if `M` is not quasihomogeneous, i.e. does not allow row and column weights

**Example:**

```
LIB "spcurve.lib";
ring r=0,(x,y,z),ds;
matrix M[3][2]=z,0,y,x,x^3,y;
qhmatrix(M);
```



```

↳ [1]:
↳ 1,2,1
↳ [2]:
↳ 1,0,
↳ 2,1,
↳ 3,2
pmat(M);
↳ z, 0,
↳ y, x,
↳ x3, y

```

### D.6.21.7 relweight

Procedure from library `spcurve.lib` (see [Section D.6.21 \[spcurve.lib\]](#), page 1768).

**Usage:** `relweight(N,W,a)`; N matrix, W intmat, a intvec

**Assume:** N is a non-zero matrix  
W is an integer matrix of the same size as N  
a is an integer vector giving the weights of the variables

**Return:** integer,  $\max(a\text{-weighted order}(N_{ij}) - W_{ij} \mid \text{all entries } ij)$   
string "ERROR" if sizes do not match

**Example:**

```

LIB "spcurve.lib";
ring r=32003,(x,y,z),ds;
matrix N[2][3]=z,0,y,x,x^3,y;
intmat W[2][3]=1,1,1,1,1,1;
intvec a=1,1,1;
relweight(N,W,a);
↳ 2

```

### D.6.21.8 posweight

Procedure from library `spcurve.lib` (see [Section D.6.21 \[spcurve.lib\]](#), page 1768).

**Usage:** `posweight(M,t1,n[,s])`; M matrix, t1 module, n int, s string  
n=0 : all deformations of non-negative weight  
n=1 : only non-constant deformations of non-negative weight  
n=2 : all deformations of positive weight

**Assume:** M is a presentation matrix of a Cohen-Macaulay codimension 2 ideal and t1 is its T1 space in matrix notation

**Return:** new ring containing a list posw, consisting of a presentation matrix describing the deformation given by the generators of T1 of non-negative/positive weight and the weight vector for the new variables

**Note:** The current basering should not contain any variables named T(i) where i is some integer!

**Example:**

```

LIB "spcurve.lib";
ring r=32003,(x(1),x(2),x(3)),ds;

```

```

ideal curve=(x(3)-x(1)^2)*x(3),(x(3)-x(1)^2)*x(2),x(2)^2-x(1)^7*x(3);
matrix M=isCMcod2(curve);
list l=matrixT1(M,3);
def rneu=posweight(l[1],std(l[2]),0);
setring rneu;
pmat(posw[1]);
↪ T(2)+x(1)*T(1), -x(3)+x(1)^2,
↪ -x(3),          x(2),
↪ x(2),          -x(1)^7
posw[2];
↪ 3,1

```

### D.6.21.9 KSpencerKernel

Procedure from library `spcurve.lib` (see [Section D.6.21 \[spcurve.lib\]](#), page 1768).

- Usage:** KSpencerKernel(M[,s][,v]); M matrix, s string, v intvec  
optional parameters (please specify in this order, if both are present):  
\* s = first of the names of the new rings  
e.g. "R" leads to ring names R and R1  
\* v of size n(n+1) leads to the following module ordering  
gen(v[1]) > gen(v[2]) > ... > gen(v[n(n+1)]) where the matrix entry ij corresponds to  
gen((i-1)\*n+j)
- Assume:** M is a quasihomogeneous n x (n+1) matrix where the n minors define an isolated space curve singularity
- Return:** new ring containing the coefficient matrix KS representing the kernel of the Kodaira-Spencer map of the family of non-negative deformations having the given singularity as special fibre
- Note:** \* the initial basering should not contain variables with name e(i) or T(i), since those variable names will internally be used by the script  
\* setting an intvec with 5 entries and name watchProgress shows the progress of the computations:  
watchProgress[1]>0 => option(prot) in groebner commands  
watchProgress[2]>0 => trace output for highcorner  
watchProgress[3]>0 => output of deformed matrix  
watchProgress[4]>0 => result of elimination step  
watchProgress[4]>1 => trace output of multiplications with xyz and subsequent reductions  
watchProgress[5]>0 => matrix representing the kernel using print

**Example:**

```

LIB "spcurve.lib";
ring r=0,(x,y,z),ds;
matrix M[3][2]=z-x^7,0,y^2,z,x^9,y;
def rneu=KSpencerKernel(M,"ar");
setring rneu;
basing;
↪ // coefficients: QQ
↪ // number of vars : 17
↪ //          block 1 : ordering Ws
↪ //          : names   e(1) e(2) e(3) e(4) e(5) e(6) x y z

```

```

⇒ //          : weights  -21 -10 -32 -21 -27 -16 3 16 21
⇒ //          block  2 : ordering wp
⇒ //          : names    T(1) T(2) T(3) T(4) T(5) T(6) T(7) T(8)
⇒ //          : weights   8   5   2  10   7   4   1   2
⇒ //          block  3 : ordering C
print(KS);
⇒ T(7), 0, 0, 0, 0, 0, 0, 0,
⇒ KS[2,1],6*T(3), 3*T(7), 0, 0, 0, 0, 0,
⇒ KS[3,1],KS[3,2],KS[3,3],6*T(3),3*T(7),0, 0, 0,
⇒ 10*T(4),8*T(1), 7*T(5), 5*T(2),4*T(6),2*T(8),2*T(3),T(7)

```

### D.6.22 spectrum\_lib

**Library:** spectrum.lib

**Purpose:** Singularity Spectrum for Nondegenerate Singularities

**Author:** S. Endrass

**Procedures:**

#### D.6.22.1 spectrumnd

Procedure from library `spectrum.lib` (see [Section D.6.22 \[spectrum\\_lib\]](#), page 1773).

**Usage:** spectrumnd(f[,1]); poly f

**Assume:** basering has characteristic 0 and local ordering,  
f has isolated singularity at 0 and nondegenerate principal part

**Return:**

list S:  
ideal S[1]: spectral numbers in increasing order  
intvec S[2]:  
int S[2][i]: multiplicity of spectral number S[1][i]

**Note:** if a second argument 1 is given,  
no test for a degenerate principal part will be done  
SEE\_ALSO: gmssing\_lib

**Example:**

```

LIB "spectrum.lib";
ring R=0,(x,y),ds;
poly f=x^31+x^6*y^7+x^2*y^12+x^13*y^2+y^29;
list s=spectrumnd(f);
size(s[1]);
⇒ 174
s[1][22];
⇒ -27/58
s[2][22];
⇒ 2

```

### D.6.23 surfacesignature\_lib

**Library:** surfacesignature.lib

**Purpose:** signature of surface singularity

**Authors:** Gerhard Pfister pfister@mathematik.uni-kl.de  
 Muhammad Ahsan Banyamin ahsanbanyamin@gmail.com  
 Stefan Steidel steidel@mathematik.uni-kl.de

**Overview:** A library for computing the signature of irreducible surface singularity. The signature of a surface singularity is defined in [3]. The algorithm we use has been proposed in [9].

Let  $g$  in  $C[x,y]$  define an isolated curve singularity at 0 in  $C^2$  and  $f:=z^N+g(x,y)$ . The zero-set  $V:=V(f)$  in  $C^3$  of  $f$  has an isolated singularity at 0. For a small  $e>0$  let  $V_e:=V(f-e)$  in  $C^3$  be the Milnor fibre of  $(V,0)$  and  $s: H_2(V_e, R) \times H_2(V_e, R) \rightarrow R$  be the intersection form (cf. [1],[7]).  $H_2(V_e, R)$  is an  $m$ -dimensional  $R$ -vector space,  $m$  the Milnor number of  $(V,0)$  (cf. [1],[4],[5],[6]), and  $s$  is a symmetric bilinear form. Let  $\text{sigma}(f)$  be the signature of  $s$ , called the signature of the surface singularity  $(V,0)$ . Formulae to compute the signature are given by Nemethi (cf. [8],[9]) and van Doorn, Steenbrink (cf. [2]).

We have implemented three approaches using Puiseux expansions, the resolution of singularities resp. the spectral pairs of the singularity.

**References:**

- [1] Arnold, V.I.; Gusein-Zade, S.M.; Varchenko, A.N.: Singularities of Differentiable Mappings. Vol. 1,2, Birkhäuser (1988).
- [2] van Doorn, M.G.M.; Steenbrink, J.H.M.: A supplement to the monodromy theorem. Abh. Math. Sem. Univ. Hamburg 59, 225-233 (1989).
- [3] Durfee, A.H.: The Signature of Smoothings of Complex Surface Singularities. Mathematische Annalen 232, 85-98 (1978).
- [4] de Jong, T.; Pfister, G.: Local Analytic Geometry. Vieweg (2000).
- [5] Kerner, D.; Nemethi, A.: The Milnor fibre signature is not semi-continuous. arXiv:0907.5252 (2009).
- [6] Kulikov, V.S.: Mixed Hodge Structures and Singularities. Cambridge Tracts in Mathematics 132, Cambridge University Press (1998).
- [7] Nemethi, A.: The real Seifert form and the spectral pairs of isolated hypersurface singularities. Compositio Mathematica 98, 23-41 (1995).
- [8] Nemethi, A.: Dedekind sums and the signature of  $f(x,y)+z^N$ . Selecta Mathematica, New series, Vol. 4, 361-376 (1998).
- [9] Nemethi, A.: The Signature of  $f(x,y)+z^N$ . Proceedings of Real and Complex Singularities (C.T.C. Wall's 60th birthday meeting, Liverpool (England), August 1996), London Math. Soc. Lecture Notes Series 263, 131-149 (1999).

**Procedures:**

### D.6.23.1 signatureBrieskorn

Procedure from library `surfacesignature.lib` (see [Section D.6.23 \[surfacesignature.lib\]](#), [page 1773](#)).

**Usage:** `signatureBrieskorn(a1,a2,a3);`  $a1,a2,a3 = \text{integers}$

**Return:** signature of Brieskorn singularity  $x^{a1}+y^{a2}+z^{a3}$

**Example:**

```
LIB "surfacesignature.lib";
ring R = 0,x,dp;
signatureBrieskorn(11,3,5);
↦ -48
```

### D.6.23.2 signaturePuisseux

Procedure from library `surfacesignature.lib` (see [Section D.6.23 \[surfacesignature.lib\]](#), [page 1773](#)).

**Usage:** `signaturePuisseux(N,f)`;  $N = \text{int}$ ,  $f = \text{irreducible poly in 2 variables}$

**Return:** signature of surface singularity defined by  $z^N + f(x,y) = 0$

**Example:**

```
LIB "surfacesignature.lib";
ring r = 0,(x,y),dp;
int N = 3;
poly f = x15-21x14+8x13y-6x13-16x12y+20x11y2-x12+8x11y-36x10y2
+24x9y3+4x9y2-16x8y3+26x7y4-6x6y4+8x5y5+4x3y6-y8;
signaturePuisseux(N,f);
↪ -92
```

### D.6.23.3 signatureNemethi

Procedure from library `surfacesignature.lib` (see [Section D.6.23 \[surfacesignature.lib\]](#), [page 1773](#)).

**Usage:** `signatureNemethi(N,f)`;  $N = \text{integer}$ ,  $f = \text{reduced poly in 2 variables}$ ,  $\#$  empty or 1,2,3  
 - if  $\#[1] = 1$  then resolution of singularity is used  
 - if  $\#[1] = 2$  then spectral pairs are used  
 - if  $\#$  is empty then both upper variants are used in parallel and the fastest returns the result

**Return:** signature of surface singularity defined by  $z^N + f(x,y) = 0$

**Remark:** computes the signature of some special surface singularities

**Example:**

```
LIB "surfacesignature.lib";
ring r = 0,(x,y),dp;
int N = 3;
poly f = x15-21x14+8x13y-6x13-16x12y+20x11y2-x12+8x11y-36x10y2
+24x9y3+4x9y2-16x8y3+26x7y4-6x6y4+8x5y5+4x3y6-y8;
signatureNemethi(N,f,1);
↪ -92
printlevel = 1;
signatureNemethi(N,f);
↪ Resolution of singularity has been used.
↪ -92
```

## D.7 Invariant theory

### D.7.1 finvar.lib

**Library:** `finvar.lib`

**Purpose:** Invariant Rings of Finite Groups

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**Overview:** A library for computing polynomial invariants of finite matrix groups and generators of related varieties. The algorithms are based on B. Sturmfels, G. Kemper, S. King and W. Decker et al..

**Procedures:**

### D.7.1.1 invariant\_ring

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\]](#), page 1775).

**Usage:** `invariant_ring(G1,G2,...[,flags]);`  
`G1,G2,...:` <matrices> generating a finite matrix group, `flags:` an optional <intvec> with three entries: if the first one equals 0, the program attempts to compute the Molien series and Reynolds operator, if it equals 1, the program is told that the Molien series should not be computed, if it equals -1 characteristic 0 is simulated, i.e. the Molien series is computed as if the base field were characteristic 0 (the user must choose a field of large prime characteristic, e.g. 32003) and if the first one is anything else, it means that the characteristic of the base field divides the group order (i.e. it will not even be attempted to compute the Reynolds operator or Molien series), the second component should give the size of intervals between canceling common factors in the expansion of Molien series, 0 (the default) means only once after generating all terms, in prime characteristic also a negative number can be given to indicate that common factors should always be canceled when the expansion is simple (the root of the extension field occurs not among the coefficients)

**Return:** primary and secondary invariants for any matrix representation of a finite group

**Display:** information about the various stages of the program if the third flag does not equal 0

**Theory:** Bases of homogeneous invariants are generated successively and those are chosen as primary invariants that lower the dimension of the ideal generated by the previously found invariants (see "Generating a Noetherian Normalization of the Invariant Ring of a Finite Group" by Decker, Heydtmann, Schreyer (1998)). In the non-modular case secondary invariants are calculated by finding a basis (in terms of monomials) of the basering modulo the primary invariants, mapping to invariants with the Reynolds operator and using those or their power products such that they are linearly independent modulo the primary invariants (see "Some Algorithms in Invariant Theory of Finite Groups" by Kemper and Steel (1997)). In the modular case they are generated according to "Generating Invariant Rings of Finite Groups over Arbitrary Fields" by Kemper (1996).

**Example:**

```
LIB "finvar.lib";
ring R=0,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
matrix P,S,IS=invariant_ring(A);
print(P);
  ↳ z2,x2+y2,x2y2
print(S);
  ↳ 1,xyz,x2z-y2z,x3y-xy3
print(IS);
  ↳ xyz,x2z-y2z,x3y-xy3
```

### D.7.1.2 invariant\_ring\_random

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\], page 1775](#)).

- Usage:** `invariant_ring_random(G1,G2,...,r[,flags]);`  
 $G1, G2, \dots$ : <matrices> generating a finite matrix group,  $r$ : an <int> where  $-|r|$  to  $|r|$  is the range of coefficients of random combinations of bases elements that serve as primary invariants, `flags`: an optional <intvec> with three entries: if the first equals 0, the program attempts to compute the Molien series and Reynolds operator, if it equals 1, the program is told that the Molien series should not be computed, if it equals -1 characteristic 0 is simulated, i.e. the Molien series is computed as if the base field were characteristic 0 (the user must choose a field of large prime characteristic, e.g. 32003) and if the first one is anything else, then the characteristic of the base field divides the group order (i.e. we will not even attempt to compute the Reynolds operator or Molien series), the second component should give the size of intervals between canceling common factors in the expansion of the Molien series, 0 (the default) means only once after generating all terms, in prime characteristic also a negative number can be given to indicate that common factors should always be canceled when the expansion is simple (the root of the extension field does not occur among the coefficients)
- Return:** primary and secondary invariants for any matrix representation of a finite group
- Display:** information about the various stages of the program if the third flag does not equal 0
- Theory:** is the same as for `invariant_ring` except that random combinations of basis elements are chosen as candidates for primary invariants and hopefully they lower the dimension of the previously found primary invariants by the right amount.

**Example:**

```
LIB "finvar.lib";
ring R=0,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
matrix P,S,IS=invariant_ring_random(A,1);
print(P);
  ↪ z2,x2+y2,x4+y4-z4
print(S);
  ↪ 1,xyz,x2z-y2z,x3y-xy3
print(IS);
  ↪ xyz,x2z-y2z,x3y-xy3
```

### D.7.1.3 primary\_invariants

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\], page 1775](#)).

- Usage:** `primary_invariants(G1,G2,...[,flags]);`  
 $G1, G2, \dots$ : <matrices> generating a finite matrix group, `flags`: an optional <intvec> with three entries, if the first one equals 0 (also the default), the programme attempts to compute the Molien series and Reynolds operator, if it equals 1, the programme is told that the Molien series should not be computed, if it equals -1 characteristic 0 is simulated, i.e. the Molien series is computed as if the base field were characteristic 0 (the user must choose a field of large prime characteristic, e.g. 32003) and if the first one is anything else, it means that the characteristic of the base field divides the group order, the second component should give the size of intervals between canceling common factors in the expansion of the Molien series, 0 (the default) means only once after generating all terms, in prime characteristic also a negative number can be given

to indicate that common factors should always be canceled when the expansion is simple (the root of the extension field occurs not among the coefficients)

**Display:** information about the various stages of the programme if the third flag does not equal 0

**Return:** primary invariants (type <matrix>) of the invariant ring and if computable Reynolds operator (type <matrix>) and Molien series (type <matrix>) or ring name (type string) where the Molien series can be found in the char p case; if the first flag is 1 and we are in the non-modular case then an <intvec> is returned giving some of the degrees where no non-trivial homogeneous invariants can be found

**Theory:** Bases of homogeneous invariants are generated successively and those are chosen as primary invariants that lower the dimension of the ideal generated by the previously found invariants (see paper "Generating a Noetherian Normalization of the Invariant Ring of a Finite Group" by Decker, Heydtmann, Schreyer (1998)).

**Example:**

```
LIB "finvar.lib";
ring R=0,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
list L=primary_invariants(A);
print(L[1]);
↪ z2,x2+y2,x2y2
```

#### D.7.1.4 primary\_invariants\_random

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\]](#), page 1775).

**Usage:** `primary_invariants_random(G1,G2,...,r[,flags]);`  
`G1,G2,...:` <matrices> generating a finite matrix group, `r:` an <int> where  $-|r|$  to  $|r|$  is the range of coefficients of the random combinations of bases elements, `flags:` an optional <intvec> with three entries, if the first one equals 0 (also the default), the programme attempts to compute the Molien series and Reynolds operator, if it equals 1, the programme is told that the Molien series should not be computed, if it equals -1 characteristic 0 is simulated, i.e. the Molien series is computed as if the base field were characteristic 0 (the user must choose a field of large prime characteristic, e.g. 32003) and if the first one is anything else, it means that the characteristic of the base field divides the group order, the second component should give the size of intervals between canceling common factors in the expansion of the Molien series, 0 (the default) means only once after generating all terms, in prime characteristic also a negative number can be given to indicate that common factors should always be canceled when the expansion is simple (the root of the extension field does not occur among the coefficients)

**Display:** information about the various stages of the programme if the third flag does not equal 0

**Return:** primary invariants (type <matrix>) of the invariant ring and if computable Reynolds operator (type <matrix>) and Molien series (type <matrix>), if the first flag is 1 and we are in the non-modular case then an <intvec> is returned giving some of the degrees where no non-trivial homogeneous invariants can be found

**Theory:** Bases of homogeneous invariants are generated successively and random linear combinations are chosen as primary invariants that lower the dimension of the ideal generated by the previously found invariants (see "Generating a Noetherian Normalization of the Invariant Ring of a Finite Group" by Decker, Heydtmann, Schreyer (1998)).



**Example:**

```
LIB "finvar.lib";
ring R=0,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
list L=primary_invariants_random(A,1);
print(L[1]);
↪ z2,x2+y2,x4+y4-z4
```

**D.7.1.5 invariant\_algebra\_reynolds**

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\]](#), page 1775).

- Usage:** `invariant_algebra_reynolds(REY[,v]);`  
 REY: a  $g \times n$  <matrix> representing the Reynolds operator of a finite matrix group, where  $g$  is the group order and  $n$  is the number of variables of the basering;  
 v: an optional <int>
- Return:** A minimal homogeneous generating set of the invariant ring, type <matrix>
- Assume:** We are in the non-modular case, i.e., the characteristic of the basering does not divide the group order;  
 REY is the 1st return value of `group_reynolds()`, `reynolds_molien()` or the second one of `primary_invariants()`
- Display:** Information on the progress of computations if  $v$  does not equal 0
- Theory:** We do an incremental search in increasing degree  $d$ . Generators of the invariant ring are found among the Reynolds images of monomials of degree  $d$ . The generators are chosen by Groebner basis techniques (see S. King: Minimal generating sets of non-modular invariant rings of finite groups).
- Note:** `invariant_algebra_reynolds` should not be used in rings with weighted orders.

**Example:**

```
LIB "finvar.lib";
ring R=0,(a,b,c,d),dp;
matrix A[4][4]=
0,0,1,0,
0,0,0,1,
1,0,0,0,
0,1,0,0;
list L = group_reynolds(A);
matrix G = invariant_algebra_reynolds(L[1],1);
↪ We have 4 relevant monomials in degree 1
↪ We found generator number 1 in degree 1
↪ We found generator number 2 in degree 1
↪ Computing Groebner basis up to the new degree 2
↪ We have 3 relevant monomials in degree 2
↪ We found generator number 3 in degree 2
↪ We found generator number 4 in degree 2
↪ We found generator number 5 in degree 2
↪ Computing Groebner basis up to the new degree 3
↪ We found the degree bound 2
↪ We went beyond the degree bound, so, we are done!
G;
↪ G[1,1]=b+d
```

```

⇒ G[1,2]=a+c
⇒ G[1,3]=b2+d2
⇒ G[1,4]=ab+cd
⇒ G[1,5]=a2+c2

```

See also: [Section D.7.1.6 \[invariant\\_algebra\\_perm\]](#), page 1780.

### D.7.1.6 invariant\_algebra\_perm

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar\\_lib\]](#), page 1775).

- Usage:** `invariant_algebra_perm(GEN[v]);`  
 GEN: a list of generators of a permutation group. It is given in disjoint cycle form, where trivial cycles can be omitted; e.g., the generator  $(1,2)(3,4)(5)$  is given by `<list(list(1,2),list(3,4))>`.  
 v: an optional `<int>`
- Return:** A minimal homogeneous generating set of the invariant ring of the group presented by GEN, type `<matrix>`
- Assume:** We are in the non-modular case, i.e., the characteristic of the basering does not divide the group order. Note that the function does not verify whether this assumption holds or not
- Display:** Information on the progress of computations if v does not equal 0
- Theory:** We do an incremental search in increasing degree d. Generators of the invariant ring are found among the orbit sums of degree d. The generators are chosen by Groebner basis techniques (see S. King: Minimal generating sets of non-modular invariant rings of finite groups).
- Note:** `invariant_algebra_perm` should not be used in rings with weighted orders.

**Example:**

```

LIB "finvar.lib";
ring R=0,(a,b,c,d),dp;
def GEN=list(list(list(1,3),list(2,4)));
matrix G = invariant_algebra_perm(GEN,1);
⇒ Searching generators in degree 1
⇒ We have 2 orbit sums of degree 1
⇒ We found generator number 1 in degree 1
⇒ We found generator number 2 in degree 1
⇒ Computing Groebner basis up to the new degree 2
⇒ Searching generators in degree 2
⇒ We have 3 orbit sums of degree 2
⇒ We found generator number 3 in degree 2
⇒ We found generator number 4 in degree 2
⇒ We found generator number 5 in degree 2
⇒ Computing Groebner basis up to the new degree 3
⇒ We found the degree bound 2
⇒ We went beyond the degree bound, so, we are done!
G;
⇒ G[1,1]=b+d
⇒ G[1,2]=a+c
⇒ G[1,3]=b2+d2
⇒ G[1,4]=ab+cd

```

$\mapsto G[1,5]=a2+c2$

See also: [Section D.7.1.5 \[invariant\\_algebra\\_reynolds\]](#), page 1779.

### D.7.1.7 cyclotomic

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\]](#), page 1775).

**Usage:** `cyclotomic(i); i integer > 0`

**Returns:** the  $i$ -th cyclotomic polynomial (type `<poly>`) as one in the first ring variable

**Theory:**  $x^{i-1}$  is divided by the  $j$ -th cyclotomic polynomial where  $j$  takes on the value of proper divisors of  $i$

**Example:**

```
LIB "finvar.lib";
ring R=0,(x,y,z),dp;
print(cyclotomic(25));
 $\mapsto x^{20}+x^{15}+x^{10}+x^5+1$ 
```

### D.7.1.8 group\_reynolds

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\]](#), page 1775).

**Usage:** `group_reynolds(G1,G2,...[,v]);`  
 $G1, G2, \dots$ :  $n \times n$  `<matrices>` generating a finite matrix group,  $v$ : an optional `<int>`

**Assume:**  $n$  is the number of variables of the basering,  $g$  the number of group elements

**Return:** a `<list>`, the first list element will be a  $g \times n$  `<matrix>` representing the Reynolds operator if we are in the non-modular case; if the characteristic is  $>0$ , `minpoly==0` and the finite group non-cyclic the second list element is an `<int>` giving the lowest common multiple of the matrix group elements' order (used in `molien`); in general all other list elements are  $n \times n$  `<matrices>` listing all elements of the finite group

**Display:** information if  $v$  does not equal 0

**Theory:** The entire matrix group is generated by getting all left products of generators with the new elements from the last run through the loop (or the generators themselves during the first run). All the ones that have been generated before are thrown out and the program terminates when no new elements found in one run. Additionally each time a new group element is found the corresponding ring mapping of which the Reynolds operator is made up is generated. They are stored in the rows of the first return value.

**Example:**

```
LIB "finvar.lib";
ring R=0,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
list L=group_reynolds(A);
print(L[1]);
 $\mapsto y, -x, -z,$ 
 $\mapsto -x, -y, z,$ 
 $\mapsto -y, x, -z,$ 
 $\mapsto x, y, z$ 
print(L[2..size(L)]);
 $\mapsto 0, 1, 0,$ 
```

```

⇒ -1,0,0,
⇒ 0, 0,-1
⇒ -1,0, 0,
⇒ 0, -1,0,
⇒ 0, 0, 1
⇒ 0,-1,0,
⇒ 1,0, 0,
⇒ 0,0, -1
⇒ 1,0,0,
⇒ 0,1,0,
⇒ 0,0,1

```

### D.7.1.9 molien

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\]](#), page 1775).

- Usage:** `molien(G1,G2,...[,ringname,lcm,flags]);`  
`G1,G2,...: nxn <matrices>`, all elements of a finite matrix group, `Finvar::newring:` a new ring of characteristic 0 for the Molien series in case of prime characteristic, `lcm:` an `<int>` giving the lowest common multiple of the elements' orders in case of prime characteristic, `minpoly==0` and a non-cyclic group, `flags:` an optional `<intvec>` with three components: if the first element is not equal to 0 characteristic 0 is simulated, i.e. the Molien series is computed as if the base field were characteristic 0 (the user must choose a field of large prime characteristic, e.g. 32003), the second component should give the size of intervals between canceling common factors in the expansion of the Molien series, 0 (the default) means only once after generating all terms, in prime characteristic also a negative number can be given to indicate that common factors should always be canceled when the expansion is simple (the root of the extension field does not occur among the coefficients)
- Assume:** `n` is the number of variables of the basering, `G1,G2...` are the group elements generated by `group_reynolds()`, `lcm` is the second return value of `group_reynolds()`
- Return:** in case of characteristic 0 a `1x2 <matrix>` giving enumerator and denominator of Molien series; in case of prime characteristic a ring with the name `newring` of characteristic 0 is created where the same Molien series (named `M`) is stored
- Display:** information if the third component of `flags` does not equal 0
- Theory:** In characteristic 0 the terms  $1/\det(1-xE)$  for all group elements of the Molien series are computed in a straight forward way. In prime characteristic a Brauer lift is involved. The returned matrix gives enumerator and denominator of the expanded version where common factors have been canceled.

**Example:**

```

LIB "finvar.lib";
"          note the case of prime characteristic";
⇒          note the case of prime characteristic
ring R=0,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
list L=group_reynolds(A);
matrix M=molien(L[2..size(L)]);
print(M);
⇒ x3+x2-x+1,-x7+x6+x5-x4+x3-x2-x+1

```

```

ring S=3,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
list L=group_reynolds(A);
molien(L[2..size(L)],"");
setring Finvar::newring;
print(M);
↪ x3+x2-x+1,-x7+x6+x5-x4+x3-x2-x+1
setring S;
kill Finvar::newring;

```

### D.7.1.10 reynolds\_molien

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\]](#), page 1775).

- Usage:** `reynolds_molien(G1,G2,...[,ringname,flags]);`  
`G1,G2,...: nxn <matrix>` generating a finite matrix group, `Finvar::newring`: a new ring of characteristic 0 for the Molien series in case of prime characteristic, `flags`: an optional `<intvec>` with three components: if the first element is not equal to 0 characteristic 0 is simulated, i.e. the Molien series is computed as if the base field were characteristic 0 (the user must choose a field of large prime characteristic, e.g. 32003) the second component should give the size of intervals between canceling common factors in the expansion of the Molien series, 0 (the default) means only once after generating all terms, in prime characteristic also a negative number can be given to indicate that common factors should always be canceled when the expansion is simple (the root of the extension field does not occur among the coefficients)
- Assume:** `n` is the number of variables of the basering, `G1,G2...` are the group elements generated by `group_reynolds()`, `g` is the size of the group
- Return:** a `gxn <matrix>` representing the Reynolds operator is the first return value and in case of characteristic 0 a `1x2 <matrix>` giving enumerator and denominator of Molien series is the second one; in case of prime characteristic a ring with the name `newring` of characteristic 0 is created where the same Molien series (named `M`) is stored
- Display:** information if the third component of `flags` does not equal 0
- Theory:** The entire matrix group is generated by getting all left products of the generators with new elements from the last run through the loop (or the generators themselves during the first run). All the ones that have been generated before are thrown out and the program terminates when are no new elements found in one run. Additionally each time a new group element is found the corresponding ring mapping of which the Reynolds operator is made up is generated. They are stored in the rows of the first return value. In characteristic 0 the terms  $1/\det(1-xE)$  is computed whenever a new element `E` is found. In prime characteristic a Brauer lift is involved and the terms are only computed after the entire matrix group is generated (to avoid the modular case). The returned matrix gives enumerator and denominator of the expanded version where common factors have been canceled.

**Example:**

```

LIB "finvar.lib";
"      note the case of prime characteristic";
↪      note the case of prime characteristic
ring R=0,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;

```

```

matrix REY,M=reynolds_molien(A);
print(REY);
↪ y, -x,-z,
↪ -x,-y,z,
↪ -y,x, -z,
↪ x, y, z
print(M);
↪ x3+x2-x+1,-x7+x6+x5-x4+x3-x2-x+1
ring S=3,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
matrix REY=reynolds_molien(A,"");
print(REY);
↪ y, -x,-z,
↪ -x,-y,z,
↪ -y,x, -z,
↪ x, y, z
setring Finvar::newring;
print(M);
↪ x3+x2-x+1,-x7+x6+x5-x4+x3-x2-x+1
setring S;
kill Finvar::newring;

```

### D.7.1.11 partial\_molien

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\]](#), page 1775).

- Usage:** `partial_molien(M,n[p]);`  
M: a 1x2 <matrix>, n: an <int> indicating number of terms in the expansion, p: an optional <poly>
- Assume:** M is the return value of `molien` or the second return value of `reynolds_molien`, p ought to be the second return value of a previous run of `partial_molien` and avoids recalculating known terms
- Return:** n terms (type <poly>) of the partial expansion of the Molien series (first n if there is no third parameter given, otherwise the next n terms depending on a previous calculation) and an intermediate result (type <poly>) of the calculation to be used as third parameter in a next run of `partial_molien`
- Theory:** The following calculation is implemented:  

$$\frac{(1+a_1x+a_2x^2+\dots+anx^n)/(1+b_1x+b_2x^2+\dots+bm x^m)}{(1+b_1x+b_2x^2+\dots+bm x^m)} = (1+(a_1-b_1)x+\dots$$


---


$$(a_1-b_1)x+(a_2-b_2)x^2+\dots$$

$$(a_1-b_1)x+b_1(a_1-b_1)x^2+\dots$$

**Example:**

```

LIB "finvar.lib";
ring R=0,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
matrix REY,M=reynolds_molien(A);
poly p(1..2);
p(1..2)=partial_molien(M,5);
p(1);
↪ 4x5+5x4+2x3+2x2+1

```

```
p(1..2)=partial_molien(M,5,p(2));
p(1);
↦ 18x10+12x9+13x8+8x7+8x6
```

### D.7.1.12 evaluate\_reynolds

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\], page 1775](#)).

- Usage:** `evaluate_reynolds(REY,I);`  
 REY: a <matrix> representing the Reynolds operator, I: an arbitrary <ideal>
- Assume:** REY is the first return value of `group_reynolds()` or `reynolds_molien()`
- Returns:** image of the polynomials defining I under the Reynolds operator (type <ideal>)
- Note:** the characteristic of the coefficient field of the polynomial ring should not divide the order of the finite matrix group
- Theory:** REY has been constructed in such a way that each row serves as a ring mapping of which the Reynolds operator is made up.

**Example:**

```
LIB "finvar.lib";
ring R=0,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
list L=group_reynolds(A);
ideal I=x2,y2,z2;
print(evaluate_reynolds(L[1],I));
↦ 1/2x2+1/2y2,
↦ 1/2x2+1/2y2,
↦ z2
```

### D.7.1.13 invariant\_basis

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\], page 1775](#)).

- Usage:** `invariant_basis(g,G1,G2,...);`  
 g: an <int> indicating of which degree (>0) the homogeneous basis should be, G1,G2,...: <matrices> generating a finite matrix group
- Returns:** the basis (type <ideal>) of the space of invariants of degree g
- Theory:** A general polynomial of degree g is generated and the generators of the matrix group applied. The difference ought to be 0 and this way a system of linear equations is created. It is solved by computing syzygies.

**Example:**

```
LIB "finvar.lib";
ring R=0,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
print(invariant_basis(2,A));
↦ x2+y2,
↦ z2
```

### D.7.1.14 invariant\_basis\_reynolds

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\], page 1775](#)).

- Usage:** `invariant_basis_reynolds(REY,d[,flags]);`  
 REY: a <matrix> representing the Reynolds operator, d: an <int> indicating of which degree (>0) the homogeneous basis should be, flags: an optional <intvec> with two entries: its first component gives the dimension of the space (default <0 meaning unknown) and its second component is used as the number of polynomials that should be mapped to invariants during one call of `evaluate_reynolds` if the dimension of the space is unknown or the number such that number x dimension polynomials are mapped to invariants during one call of `evaluate_reynolds`
- Assume:** REY is the first return value of `group_reynolds()` or `reynolds_molien()` and flags[1] given by `partial_molien`
- Return:** the basis (type <ideal>) of the space of invariants of degree d
- Theory:** Monomials of degree d are mapped to invariants with the Reynolds operator. A linearly independent set is generated with the help of `minbase`.

**Example:**

```
LIB "finvar.lib";
ring R=0,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
intvec flags=0,1,0;
matrix REY,M=reynolds_molien(A,flags);
flags=8,6;
print(invariant_basis_reynolds(REY,6,flags));
⇒ z6,
⇒ x2z4+y2z4,
⇒ x2y2z2,
⇒ x3yz2-xy3z2,
⇒ x4z2+y4z2,
⇒ x4y2+x2y4,
⇒ x5y-xy5,
⇒ x6+y6
```

### D.7.1.15 primary\_char0

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\], page 1775](#)).

- Usage:** `primary_char0(REY,M[,v]);`  
 REY: a <matrix> representing the Reynolds operator, M: a 1x2 <matrix> representing the Molien series, v: an optional <int>
- Assume:** REY is the first return value of `group_reynolds` or `reynolds_molien` and M the one of `molien` or the second one of `reynolds_molien`
- Display:** information about the various stages of the programme if v does not equal 0
- Return:** primary invariants (type <matrix>) of the invariant ring
- Theory:** Bases of homogeneous invariants are generated successively and those are chosen as primary invariants that lower the dimension of the ideal generated by the previously found invariants (see paper "Generating a Noetherian Normalization of the Invariant Ring of a Finite Group" by Decker, Heydtmann, Schreyer (1998)).



**Example:**

```
LIB "finvar.lib";
ring R=0,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
matrix REY,M=reynolds_molien(A);
matrix P=primary_char0(REY,M);
print(P);
↦ z2,x2+y2,x2y2
```

**D.7.1.16 primary\_charp**

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\], page 1775](#)).

- Usage:** `primary_charp(REY,ringname[,v]);`  
 REY: a <matrix> representing the Reynolds operator,  
 Finvar::newring: the ring where the Molien series is stored, v: an optional <int>
- Assume:** REY is the first return value of `group_reynolds` or `reynolds_molien` and `ringname` gives the name of a ring of characteristic 0 that has been created by `molien` or `reynolds_molien`
- Display:** information about the various stages of the programme if v does not equal 0
- Return:** primary invariants (type <matrix>) of the invariant ring
- Theory:** Bases of homogeneous invariants are generated successively and those are chosen as primary invariants that lower the dimension of the ideal generated by the previously found invariants (see paper "Generating a Noetherian Normalization of the Invariant Ring of a Finite Group" by Decker, Heydtmann, Schreyer (1998)).

**Example:**

```
LIB "finvar.lib";
ring R=3,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
list L=group_reynolds(A);
molien(L[2..size(L)], "");
matrix P=primary_charp(L[1], "");
kill Finvar::newring;
print(P);
↦ z2,x2+y2,x2y2
```

**D.7.1.17 primary\_char0\_no\_molien**

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\], page 1775](#)).

- Usage:** `primary_char0_no_molien(REY[,v]);`  
 REY: a <matrix> representing the Reynolds operator, v: an optional <int>
- Assume:** REY is the first return value of `group_reynolds` or `reynolds_molien`
- Display:** information about the various stages of the programme if v does not equal 0
- Return:** primary invariants (type <matrix>) of the invariant ring and an <intvec> listing some of the degrees where no non-trivial homogeneous invariants are to be found
- Theory:** Bases of homogeneous invariants are generated successively and those are chosen as primary invariants that lower the dimension of the ideal generated by the previously found invariants (see paper "Generating a Noetherian Normalization of the Invariant Ring of a Finite Group" by Decker, Heydtmann, Schreyer (1998)).

**Example:**

```
LIB "finvar.lib";
ring R=0,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
list L=group_reynolds(A);
list l=primary_char0_no_molien(L[1]);
print(l[1]);
↦ z2,x2+y2,x2y2
```

**D.7.1.18 primary\_charp\_no\_molien**

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\], page 1775](#)).

- Usage:** `primary_charp_no_molien(REY[,v]);`  
 REY: a <matrix> representing the Reynolds operator, v: an optional <int>
- Assume:** REY is the first return value of `group_reynolds` or `reynolds_molien`
- Display:** information about the various stages of the programme if v does not equal 0
- Return:** primary invariants (type <matrix>) of the invariant ring and an <intvec> listing some of the degrees where no non-trivial homogeneous invariants are to be found
- Theory:** Bases of homogeneous invariants are generated successively and those are chosen as primary invariants that lower the dimension of the ideal generated by the previously found invariants (see paper "Generating a Noetherian Normalization of the Invariant Ring of a Finite Group" by Decker, Heydtmann, Schreyer (1998)).

**Example:**

```
LIB "finvar.lib";
ring R=3,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
list L=group_reynolds(A);
list l=primary_charp_no_molien(L[1]);
print(l[1]);
↦ z2,x2+y2,x2y2
```

**D.7.1.19 primary\_charp\_without**

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\], page 1775](#)).

- Usage:** `primary_charp_without(G1,G2,...[,v]);`  
 G1,G2,...: <matrices> generating a finite matrix group, v: an optional <int>
- Display:** information about the various stages of the programme if v does not equal 0
- Return:** primary invariants (type <matrix>) of the invariant ring
- Theory:** Bases of homogeneous invariants are generated successively and those are chosen as primary invariants that lower the dimension of the ideal generated by the previously found invariants (see paper "Generating a Noetherian Normalization of the Invariant Ring of a Finite Group" by Decker, Heydtmann, Schreyer (1998)). No Reynolds operator or Molien series is used.

**Example:**

```
LIB "finvar.lib";
ring R=2,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
matrix P=primary_charp_without(A);
print(P);
↦ x+y,z,xy
```

### D.7.1.20 primary\_char0\_random

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\], page 1775](#)).

- Usage:** `primary_char0_random(REY,M,r[v]);`  
 REY: a <matrix> representing the Reynolds operator, M: a 1x2 <matrix> representing the Molien series, r: an <int> where  $-|r|$  to  $|r|$  is the range of coefficients of the random combinations of bases elements, v: an optional <int>
- Assume:** REY is the first return value of `group_reynolds` or `reynolds_molien` and M the one of `molien` or the second one of `reynolds_molien`
- Display:** information about the various stages of the programme if v does not equal 0
- Return:** primary invariants (type <matrix>) of the invariant ring
- Theory:** Bases of homogeneous invariants are generated successively and random linear combinations are chosen as primary invariants that lower the dimension of the ideal generated by the previously found invariants (see "Generating a Noetherian Normalization of the Invariant Ring of a Finite Group" by Decker, Heydtmann, Schreyer (1998)).

**Example:**

```
LIB "finvar.lib";
ring R=0,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
matrix REY,M=reynolds_molien(A);
matrix P=primary_char0_random(REY,M,1);
print(P);
↦ z2,x2+y2,x4+y4-z4
```

### D.7.1.21 primary\_charp\_random

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\], page 1775](#)).

- Usage:** `primary_charp_random(REY,ringname,r[v]);`  
 REY: a <matrix> representing the Reynolds operator,  
 Finvar::newring: the ring where the Molien series is stored, r: an <int> where  $-|r|$  to  $|r|$  is the range of coefficients of the random combinations of bases elements, v: an optional <int>
- Assume:** REY is the first return value of `group_reynolds` or `reynolds_molien` and ringname gives the name of a ring of characteristic 0 that has been created by `molien` or `reynolds_molien`
- Display:** information about the various stages of the programme if v does not equal 0
- Return:** primary invariants (type <matrix>) of the invariant ring
- Theory:** Bases of homogeneous invariants are generated successively and random linear combinations are chosen as primary invariants that lower the dimension of the ideal generated by the previously found invariants (see "Generating a Noetherian Normalization of the Invariant Ring of a Finite Group" by Decker, Heydtmann, Schreyer (1998)).

**Example:**

```

LIB "finvar.lib";
ring R=3,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
list L=group_reynolds(A);
molien(L[2..size(L)], "");
matrix P=primary_charp_random(L[1], "", 1);
kill Finvar::newring;
print(P);
↦ z2,x2+y2,x4+y4-z4

```

**D.7.1.22 primary\_char0\_no\_molien\_random**

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\], page 1775](#)).

- Usage:** `primary_char0_no_molien_random(REY,r[,v]);`  
 REY: a <matrix> representing the Reynolds operator, r: an <int> where  $-|r|$  to  $|r|$  is the range of coefficients of the random combinations of bases elements, v: an optional <int>
- Assume:** REY is the first return value of `group_reynolds` or `reynolds_molien`
- Display:** information about the various stages of the programme if v does not equal 0
- Return:** primary invariants (type <matrix>) of the invariant ring and an <intvec> listing some of the degrees where no non-trivial homogeneous invariants are to be found
- Theory:** Bases of homogeneous invariants are generated successively and random linear combinations are chosen as primary invariants that lower the dimension of the ideal generated by the previously found invariants (see "Generating a Noetherian Normalization of the Invariant Ring of a Finite Group" by Decker, Heydtmann, Schreyer (1998)).

**Example:**

```

LIB "finvar.lib";
ring R=0,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
list L=group_reynolds(A);
list l=primary_char0_no_molien_random(L[1],1);
print(l[1]);
↦ z2,x2+y2,x4+y4-z4

```

**D.7.1.23 primary\_charp\_no\_molien\_random**

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\], page 1775](#)).

- Usage:** `primary_charp_no_molien_random(REY,r[,v]);`  
 REY: a <matrix> representing the Reynolds operator, r: an <int> where  $-|r|$  to  $|r|$  is the range of coefficients of the random combinations of bases elements, v: an optional <int>
- Assume:** REY is the first return value of `group_reynolds` or `reynolds_molien`
- Display:** information about the various stages of the programme if v does not equal 0
- Return:** primary invariants (type <matrix>) of the invariant ring and an <intvec> listing some of the degrees where no non-trivial homogeneous invariants are to be found

**Theory:** Bases of homogeneous invariants are generated successively and random linear combinations are chosen as primary invariants that lower the dimension of the ideal generated by the previously found invariants (see "Generating a Noetherian Normalization of the Invariant Ring of a Finite Group" by Decker, Heydtmann, Schreyer (1998)).

**Example:**

```
LIB "finvar.lib";
ring R=3,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
list L=group_reynolds(A);
list l=primary_charp_no_molien_random(L[1],1);
print(l[1]);
↦ z2,x2+y2,x4+y4-z4
```

#### D.7.1.24 primary\_charp\_without\_random

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\], page 1775](#)).

**Usage:** `primary_charp_without_random(G1,G2,...,r[v]);`  
`G1,G2,...:` <matrices> generating a finite matrix group, `r:` an <int> where  $-|r|$  to  $|r|$  is the range of coefficients of the random combinations of bases elements, `v:` an optional <int>

**Display:** information about the various stages of the programme if `v` does not equal 0

**Return:** primary invariants (type <matrix>) of the invariant ring

**Theory:** Bases of homogeneous invariants are generated successively and random linear combinations are chosen as primary invariants that lower the dimension of the ideal generated by the previously found invariants (see "Generating a Noetherian Normalization of the Invariant Ring of a Finite Group" by Decker, Heydtmann, Schreyer (1998)). No Reynolds operator or Molien series is used.

**Example:**

```
LIB "finvar.lib";
ring R=2,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
matrix P=primary_charp_without_random(A,1);
print(P);
↦ x+y,z,xy
```

#### D.7.1.25 power\_products

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\], page 1775](#)).

**Usage:** `power_products(dv,d);`  
`dv:` an <intvec> giving the degrees of homogeneous polynomials, `d:` the degree of the desired power products

**Return:** a  $\text{size}(dv)*m$  <intmat> where each column ought to be interpreted as containing the exponents of the corresponding polynomials. The product of the powers is then homogeneous of degree `d`.

**Example:**

```

LIB "finvar.lib";
intvec dv=5,5,5,10,10;
print(power_products(dv,10));
⇒      2      1      1      0      0      0      0      0
⇒      0      1      0      2      1      0      0      0
⇒      0      0      1      0      1      2      0      0
⇒      0      0      0      0      0      0      1      0
⇒      0      0      0      0      0      0      0      1
print(power_products(dv,7));
⇒      0
⇒      0
⇒      0
⇒      0
⇒      0

```

### D.7.1.26 secondary\_char0

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\]](#), page 1775).

- Usage:** `secondary_char0(P,REY,M[,v][,"old"]);`  
P: a  $1 \times n$  <matrix> with homogeneous primary invariants, where  $n$  is the number of variables of the basering;  
REY: a  $g \times n$  <matrix> representing the Reynolds operator, where  $g$  the size of the corresponding group;  
M: a  $1 \times 2$  <matrix> giving numerator and denominator of the Molien series;  
v: an optional <int>;  
"old": if this string occurs as (optional) parameter, then an old version of `secondary_char0` is used (for downward compatibility)
- Assume:** The characteristic of basering is zero;  
REY is the 1st return value of `group_reynolds()`, `reynolds_molien()` or the second one of `primary_invariants()`;  
M is the return value of `molien()` or the second one of `reynolds_molien()` or the third one of `primary_invariants()`
- Return:** Homogeneous secondary invariants and irreducible secondary invariants of the invariant ring (both type <matrix>)
- Display:** Information on the progress of the computations if  $v$  is an integer different from 0.
- Theory:** The secondary invariants are calculated by finding a basis (in terms of monomials) of the basering modulo the primary invariants, mapping those to invariants with the Reynolds operator. Among these images or their power products we pick secondary invariants using Groebner basis techniques (see S. King: Fast Computation of Secondary Invariants).  
The size of this set can be read off from the Molien series.
- Note:** Secondary invariants are not uniquely determined by the given data. Specifically, the output of `secondary_char0(P,REY,M,"old")` will differ from the output of `secondary_char0(P,REY,M)`. However, the ideal generated by the irreducible homogeneous secondary invariants will be the same in both cases.  
There are three internal parameters "pieces", "MonStep" and "IrrSwitch". The default values of the parameters should be fine in most cases. However, in some cases,

different values may provide a better balance of memory consumption (smaller values) and speed (bigger values).

**Example:**

```
LIB "finvar.lib";
ring R=0,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
list L=primary_invariants(A);
matrix S,IS=secondary_char0(L[1..3],1);
↳
↳ We need to find
↳ 1 secondary invariant in degree 0
↳ 0 secondary invariants in degree 1
↳ 0 secondary invariants in degree 2
↳ 2 secondary invariants in degree 3
↳ 1 secondary invariant in degree 4
↳ In degree 0 we have: 1
↳
↳ Searching in degree 3 , we need to find 2 invariant(s)...
↳ Looking for Power Products...
↳ There are 2 irreducible secondary invariants in degree 3
↳ We found all 2 irreducibles in degree 3
↳
↳ Searching in degree 4 , we need to find 1 invariant(s)...
↳ Looking for Power Products...
↳ There are 1 irreducible secondary invariants in degree 4
↳ We found all 1 irreducibles in degree 4
↳
↳ We're done!
↳
print(S);
↳ 1,xyz,x2z-y2z,x3y-xy3
print(IS);
↳ xyz,x2z-y2z,x3y-xy3
```

See also: [Section D.7.1.27 \[irred\\_secondary\\_char0\]](#), page 1793.

### D.7.1.27 irred\_secondary\_char0

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\]](#), page 1775).

**Usage:** `irred_secondary_char0(P,REY,M[,v][,"PP"])`;  
P: a  $1 \times n$  <matrix> with homogeneous primary invariants, where  $n$  is the number of variables of the basering;  
REY: a  $g \times n$  <matrix> representing the Reynolds operator, where  $g$  the size of the corresponding group;  
M: a  $1 \times 2$  <matrix> giving numerator and denominator of the Molien series;  
v: an optional <int>;  
"PP": if this string occurs as (optional) parameter, then in all degrees power products of irr. sec. inv. will be computed.

**Return:** Irreducible homogeneous secondary invariants of the invariant ring (type <matrix>)

**Assume:** We are in the non-modular case, i.e., the characteristic of the basering does not divide the group order;





"old": if this string occurs as (optional) parameter, then an old version of secondary\_char0 is used (for downward compatibility)

- Assume:** The characteristic of basering is not zero;  
 REY is the 1st return value of group\_reynolds(), reynolds\_molien() or the second one of primary\_invariants();  
 Finvar::newring is the ring of characteristic 0 that has been created by molien() or reynolds\_molien() or primary\_invariants()
- Return:** secondary invariants of the invariant ring (type <matrix>) and irreducible secondary invariants (type <matrix>)
- Display:** information if v does not equal 0
- Theory:** The secondary invariants are calculated by finding a basis (in terms of monomials) of the basering modulo the primary invariants, mapping those to invariants with the Reynolds operator. Among these images or their power products we pick secondary invariants using Groebner basis techniques (see S. King: Fast Computation of Secondary Invariants). The size of this set can be read off from the Molien series.

**Example:**

```
LIB "finvar.lib";
ring R=3,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
list L=primary_invariants(A);
matrix S,IS=secondary_charp(L[1..size(L)],1);
⇒
⇒ We need to find
⇒ 1 secondary invariant in degree 0
⇒ 0 secondary invariants in degree 1
⇒ 0 secondary invariants in degree 2
⇒ 2 secondary invariants in degree 3
⇒ 1 secondary invariant in degree 4
⇒ In degree 0 we have: 1
⇒
⇒ Searching in degree 3 , we need to find 2 invariant(s)...
⇒ Looking for Power Products...
⇒ There are 2 irreducible secondary invariants in degree 3
⇒ We found all 2 irreducibles in degree 3
⇒
⇒ Searching in degree 4 , we need to find 1 invariant(s)...
⇒ Looking for Power Products...
⇒ There are 1 irreducible secondary invariants in degree 4
⇒ We found all 1 irreducibles in degree 4
⇒
⇒ We're done!
⇒
print(S);
⇒ 1,xyz,x2z-y2z,x3y-xy3
print(IS);
⇒ xyz,x2z-y2z,x3y-xy3
```

### D.7.1.29 secondary\_no\_molien

Procedure from library finvar.lib (see [Section D.7.1 \[finvar.lib\]](#), page 1775).

- Usage:** `secondary_no_molien(P,REY[,deg_vec,v]);`  
P: a  $1 \times n$  <matrix> with primary invariants, REY: a  $g \times n$  <matrix> representing the Reynolds operator, deg\_vec: an optional <intvec> listing some degrees where no non-trivial homogeneous invariants can be found, v: an optional <int>
- Assume:** n is the number of variables of the basering, g the size of the group, REY is the 1st return value of `group_reynolds()`, `reynolds_molien()` or the second one of `primary_invariants()`, deg\_vec is the second return value of `primary_char0_no_molien()`, `primary_charp_no_molien()`, `primary_char0_no_molien_random()` or `primary_charp_no_molien_random()`
- Return:** secondary invariants of the invariant ring (type <matrix>)
- Display:** information if v does not equal 0
- Theory:** Secondary invariants are calculated by finding a basis (in terms of monomials) of the basering modulo primary invariants, mapping those to invariants with the Reynolds operator and using these images as candidates for secondary invariants. We have the Reynolds operator, hence, we are in the non-modular case. Therefore, the invariant ring is Cohen-Macaulay, hence the number of secondary invariants is the product of the degrees of primary invariants divided by the group order.
- Note:** <secondary\_and\_irreducibles\_no\_molien> should usually be faster and of more useful functionality.

**Example:**

```
LIB "finvar.lib";
ring R=3,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
list L=primary_invariants(A,intvec(1,1,0));
// In that example, there are no secondary invariants
// in degree 1 or 2.
matrix S=secondary_no_molien(L[1..2],intvec(1,2),1);
⇒ We need to find 4 secondary invariants.
⇒
⇒ In degree 0 we have: 1
⇒
⇒ Searching in degree 3 ...
⇒ We found sec. inv. number 2 in degree 3
⇒ We found sec. inv. number 3 in degree 3
⇒ Searching in degree 4 ...
⇒ We found sec. inv. number 4 in degree 4
⇒
⇒ We're done!
⇒
print(S);
⇒ 1,xyz,x2z-y2z,x3y-xy3
```

See also: [Section D.7.1.31 \[secondary\\_and\\_irreducibles\\_no\\_molien\]](#), page 1797.

**D.7.1.30 irred\_secondary\_no\_molien**

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\]](#), page 1775).

- Usage:** `irred_secondary_no_molien(P,REY[,deg_vec,v]);`  
P: a  $1 \times n$  <matrix> with primary invariants, REY: a  $g \times n$  <matrix> representing the

Reynolds operator, deg\_vec: an optional <intvec> listing some degrees where no irreducible secondary invariants can be found, v: an optional <int>

**Assume:** n is the number of variables of the basering, g the size of the group, REY is the 1st return value of group\_reynolds(), reynolds\_molien() or the second one of primary\_invariants()

**Return:** Irreducible secondary invariants of the invariant ring (type <matrix>)

**Display:** information if v does not equal 0

**Theory:** Irred. secondary invariants are calculated by finding a basis (in terms of monomials) of the basering modulo primary and previously found secondary invariants, mapping those to invariants with the Reynolds operator. Among these images we pick secondary invariants, using Groebner basis techniques.

**Example:**

```
LIB "finvar.lib";
ring R=3,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
list L=primary_invariants(A,intvec(1,1,0));
// In that example, there are no secondary invariants
// in degree 1 or 2.
matrix IS=irred_secondary_no_molien(L[1..2],intvec(1,2),1);
⇒
⇒ Searching irred. sec. inv. in degree 3
⇒ We have 4 candidates for irred. secondaries
⇒ We found irr. sec. inv. number 1 in degree 3
⇒ We found irr. sec. inv. number 2 in degree 3
⇒ Searching irred. sec. inv. in degree 4
⇒ We have 1 candidates for irred. secondaries
⇒ We found irr. sec. inv. number 3 in degree 4
⇒ Searching irred. sec. inv. in degree 5
⇒ Searching irred. sec. inv. in degree 6
⇒ Searching irred. sec. inv. in degree 7
⇒ Searching irred. sec. inv. in degree 8
⇒ Searching irred. sec. inv. in degree 9
⇒ Searching irred. sec. inv. in degree 10
⇒ Searching irred. sec. inv. in degree 11
⇒ Searching irred. sec. inv. in degree 12
⇒ Searching irred. sec. inv. in degree 13
print(IS);
⇒ x2z-y2z,xyz,x3y-xy3
```

See also: [Section D.7.1.27 \[irred\\_secondary\\_char0\]](#), page 1793.

### D.7.1.31 secondary\_and\_irreducibles\_no\_molien

Procedure from library finvar.lib (see [Section D.7.1 \[finvar\\_lib\]](#), page 1775).

**Usage:** secondary\_and\_irreducibles\_no\_molien(P,REY[,deg\_vec,v]); P: a 1xn <matrix> with primary invariants, REY: a gxn <matrix> representing the Reynolds operator, deg\_vec: an optional <intvec> listing some degrees where no non-trivial homogeneous invariants can be found, v: an optional <int>

**Assume:** n is the number of variables of the basering, g the size of the group, REY is the 1st return value of group\_reynolds(), reynolds\_molien() or the second one of primary\_invariants()

- Return:** secondary invariants of the invariant ring (type <matrix>) and irreducible secondary invariants (type <matrix>)
- Display:** information if v does not equal 0
- Theory:** Secondary invariants are calculated by finding a basis (in terms of monomials) of the basering modulo primary invariants, mapping those to invariants with the Reynolds operator. Among these images or their power products we pick secondary invariants using Groebner basis techniques (see S. King: Fast Computation of Secondary Invariants). We have the Reynolds operator, hence, we are in the non-modular case. Therefore, the invariant ring is Cohen-Macaulay, hence the number of secondary invariants is the product of the degrees of primary invariants divided by the group order.

**Example:**

```
LIB "finvar.lib";
ring R=3,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
list L=primary_invariants(A,intvec(1,1,0));
// In that example, there are no secondary invariants
// in degree 1 or 2.
matrix S,IS=secondary_and_irreducibles_no_molien(L[1..2],intvec(1,2),1);
⇒
⇒ We need to find 4 secondary invariants.
⇒
⇒ In degree 0 we have: 1
⇒
⇒ Searching in degree 3
⇒ Looking for Power Products...
⇒ Looking for irreducible secondary invariants in degree 3
⇒ We found irreducible sec. inv. number 1 in degree 3
⇒ We found irreducible sec. inv. number 2 in degree 3
⇒
⇒ Searching in degree 4
⇒ Looking for Power Products...
⇒ Looking for irreducible secondary invariants in degree 4
⇒ We found irreducible sec. inv. number 1 in degree 4
⇒
⇒
⇒ We're done!
⇒
print(S);
⇒ 1,xyz,x2z-y2z,x3y-xy3
print(IS);
⇒ xyz,x2z-y2z,x3y-xy3
```

See also: [Section D.7.1.29 \[secondary\\_no\\_molien\]](#), page 1795.

**D.7.1.32 secondary\_not\_cohen\_macaulay**

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar\\_lib\]](#), page 1775).

- Usage:** `secondary_not_cohen_macaulay(P,G1,G2,...[,v]);`  
P: a 1xn <matrix> with primary invariants, G1,G2,...: nxn <matrices> generating a finite matrix group, v: optional <int>
- Assume:** n is the number of variables of the basering

- Return:** secondary invariants of the invariant ring (type <matrix>)
- Display:** information on the progress of computation if v does not equal 0
- Theory:** Secondary invariants are generated following "Generating Invariant Rings of Finite Groups over Arbitrary Fields" by Kemper (1996).

**Example:**

```
LIB "finvar.lib";
ring R=2,(x,y,z),dp;
matrix A[3][3]=0,1,0,-1,0,0,0,0,-1;
list L=primary_invariants(A);
matrix S=secondary_not_cohen_macaulay(L[1],A);
print(S);
⇒ 1
```

**D.7.1.33 orbit\_variety**

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\], page 1775](#)).

- Usage:** `orbit_variety(F,s);`  
F: a 1xm <matrix> defining an invariant ring, s: a <string> giving the name for a new ring
- Return:** a Groebner basis (type <ideal>, named G) for the ideal defining the orbit variety (i.e. the syzygy ideal) in the new ring (named newring)
- Theory:** The ideal of algebraic relations of the invariant ring generators is calculated, then the variables of the original ring are eliminated and the polynomials that are left over define the orbit variety

**Example:**

```
LIB "finvar.lib";
ring R=0,(x,y,z),dp;
matrix F[1][7]=x2+y2,z2,x4+y4,1,x2z-1y2z,xyz,x3y-1xy3;
orbit_variety(F,"");
print(G);
⇒ y(4)-1,
⇒ y(5)*y(6)-y(2)*y(7),
⇒ y(2)*y(3)-y(5)^2-2*y(6)^2,
⇒ y(1)^2*y(6)-2*y(3)*y(6)+y(5)*y(7),
⇒ y(1)^2*y(5)-y(3)*y(5)-2*y(6)*y(7),
⇒ y(1)^2*y(2)-y(2)*y(3)-2*y(6)^2,
⇒ y(1)^4-3*y(1)^2*y(3)+2*y(3)^2+2*y(7)^2
basing;
⇒ // coefficients: QQ
⇒ // number of vars : 7
⇒ //          block 1 : ordering dp
⇒ //          : names y(1) y(2) y(3) y(4) y(5) y(6) y(7)
⇒ //          block 2 : ordering C
```

**D.7.1.34 rel\_orbit\_variety**

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\], page 1775](#)).

- Usage:** `rel_orbit_variety(I,F[,s]);`  
 I: an <ideal> invariant under the action of a group,  
 F: a 1xm <matrix> defining the invariant ring of this group.  
 s: optional <string>; if s is present then (for downward compatibility) the old procedure <relative\_orbit\_variety> is called, and in this case s gives the name of a new <ring>.
- Return:** Without optional string s, a list L of two rings is returned.  
 The ring L[1] carries a weighted degree order with variables  $y(1..m)$ , the weight of  $y(k)$  equal to the degree of the k-th generators  $F[1,k]$  of the invariant ring.  
 L[1] contains a Groebner basis (type <ideal>, named G) of the ideal defining the relative orbit variety with respect to I.  
 The ring L[2] has the variables of the basering together with  $y(1..m)$  and carries a block order: The first block is the order of the basering, the second is the weighted degree order occurring in L[1]. L[2] contains G and a Groebner basis (type <ideal>, named Conv) such that if p is any invariant polynomial expressed in the variables of the basering then `reduce(p,Conv)` is a polynomial in the new variables  $y(1..m)$  such that evaluation at the generators of the invariant ring yields p. This can be used to avoid the application of <algebra\_containment> (see [Section D.4.2.1 \[algebra\\_containment\]](#), page 1005).  
 For the case of optional string s, the function is equivalent to [Section D.7.1.35 \[relative\\_orbit\\_variety\]](#), page 1801.
- Theory:** A Groebner basis of the ideal of algebraic relations of the invariant ring generators is calculated, then one of the basis elements plus the ideal generators. The variables of the original ring are eliminated and the polynomials that are left define the relative orbit variety with respect to I. The elimination is done by a weighted blockorder that has the advantage of dealing with quasi-homogeneous ideals.
- Note:** We provide the ring L[1] for the sake of downward compatibility, since it is closer to the ring returned by `relative_orbit_variety` than L[2]. However, L[1] carries a weighted degree order, whereas the ring returned by `relative_orbit_variety` is lexicographically ordered.

**Example:**

```
LIB "finvar.lib";
ring R=0,(x,y,z),dp;
matrix F[1][3]=x+y+z,xy+xz+yz,xyz;
ideal I=x2+y2+z2-1,x2y+y2z+z2x-2x-2y-2z,xy2+yz2+zx2-2x-2y-2z;
list L = rel_orbit_variety(I,F);
↳
↳ // 'rel_orbit_variety' created a list of two rings.
↳ // If L is the name of that list, you can access
↳ // the first ring by
↳ //   def R = L[1]; setring R;
↳ // (similarly for the second ring)
def AllR = L[2];
setring(AllR);
print(G);
↳ y(1)^2-2*y(2)-1,
↳ y(1)*y(2)-3*y(3)-4*y(1),
↳ 2*y(2)^2-3*y(1)*y(3)-7*y(2)-4,
↳ 6*y(3)^2-15*y(1)*y(3)+25*y(2)+12
print(Conv);
↳ x+y+z-y(1),
```

```

⇒ y^2+y*z+z^2-y*y(1)-z*y(1)+y(2),
⇒ z^3-z^2*y(1)+z*y(2)-y(3)
basing;
⇒ // coefficients: QQ
⇒ // number of vars : 6
⇒ //      block 1 : ordering dp
⇒ //      : names  x y z
⇒ //      block 2 : ordering wp
⇒ //      : names  y(1) y(2) y(3)
⇒ //      : weights 1 2 3
⇒ //      block 3 : ordering C

```

See also: [Section D.7.1.35 \[relative\\_orbit\\_variety\]](#), page 1801.

### D.7.1.35 relative\_orbit\_variety

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\]](#), page 1775).

- Usage:** `relative_orbit_variety(I,F,s);`  
 I: an <ideal> invariant under the action of a group,  
 F: a 1xm <matrix> defining the invariant ring of this group,  
 Finvar::newring: the new ring
- Return:** The procedure ends with a new ring named `newring`.  
 It contains a Groebner basis  
 (type <ideal>, named G) for the ideal defining the  
 relative orbit variety with respect to I in the new ring.
- Theory:** A Groebner basis of the ideal of algebraic relations of the invariant ring generators is calculated, then one of the basis elements plus the ideal generators. The variables of the original ring are eliminated and the polynomials that are left define the relative orbit variety with respect to I.
- Note:** This procedure is now replaced by `rel_orbit_variety` (see [Section D.7.1.34 \[rel\\_orbit\\_variety\]](#), page 1799), which uses a different elimination order that should usually allow faster computations.

**Example:**

```

LIB "finvar.lib";
ring R=0,(x,y,z),dp;
matrix F[1][3]=x+y+z,xy+xz+yz,xyz;
ideal I=x2+y2+z2-1,x2y+y2z+z2x-2x-2y-2z,xy2+yz2+zx2-2x-2y-2z;
string newring="E";
relative_orbit_variety(I,F,newring);
print(G);
⇒ 27*y(3)^6-513*y(3)^4+33849*y(3)^2-784,
⇒ 1475*y(2)+9*y(3)^4-264*y(3)^2+736,
⇒ 8260*y(1)+9*y(3)^5-87*y(3)^3+5515*y(3)
basing;
⇒ // coefficients: QQ
⇒ // number of vars : 3
⇒ //      block 1 : ordering lp
⇒ //      : names  y(1) y(2) y(3)
⇒ //      block 2 : ordering C

```

See also: [Section D.7.1.34 \[rel\\_orbit\\_variety\]](#), page 1799.

### D.7.1.36 image\_of\_variety

Procedure from library `finvar.lib` (see [Section D.7.1 \[finvar.lib\]](#), page 1775).

- Usage:** `image_of_variety(I,F);`  
 I: an arbitrary <ideal>,  
 F: a 1xm <matrix> defining an invariant ring of some matrix group
- Return:** The <ideal> defining the image under that group of the variety defined by I
- Theory:** `rel_orbit_variety(I,F)` is called and the newly introduced variables in the output are replaced by the generators of the invariant ring. This ideal in the original variables defines the image of the variety defined by I

**Example:**

```
LIB "finvar.lib";
ring R=0,(x,y,z),dp;
matrix F[1][3]=x+y+z,xy+xz+yz,xyz;
ideal I=xy;
print(image_of_variety(I,F));
↦ xyz
```

## D.7.2 ainvar.lib

- Library:** `ainvar.lib`
- Purpose:** Invariant Rings of the Additive Group
- Authors:** Gerhard Pfister (email: [pfister@mathematik.uni-kl.de](mailto:pfister@mathematik.uni-kl.de)), Gert-Martin Greuel (email: [greuel@mathematik.uni-kl.de](mailto:greuel@mathematik.uni-kl.de))

**Procedures:**

### D.7.2.1 invariantRing

Procedure from library `ainvar.lib` (see [Section D.7.2 \[ainvar.lib\]](#), page 1802).

- Usage:** `invariantRing(m,p,q,b[r,pa]);` m matrix, p,q poly, b,r int, pa string
- Assume:** p,q variables with  $m(p)=q$  and q invariant under m  
 i.e. if  $p=x(i)$  and  $q=x(j)$  then  $m[j,1]=0$  and  $m[i,1]=x(j)$
- Return:** ideal, containing generators of the ring of invariants of the additive group  $(K,+)$  given by the vector field
- $$m = m[1,1]*d/dx(1) + \dots + m[n,1]*d/dx(n).$$
- If  $b>0$  the computation stops after all invariants of degree  $\leq b$  (and at least one of higher degree) are found or when all invariants are computed.  
 If  $b\leq 0$ , the computation continues until all generators of the ring of invariants are computed (should be used only if the ring of invariants is known to be finitely generated, otherwise the algorithm might not stop).  
 If  $r=1$  a different reduction is used which is sometimes faster (default  $r=0$ ).
- Display:** if pa is given (any string as 5th or 6th argument), the computation pauses whenever new invariants are found and displays them



**Theory:** The algorithm for computing the ring of invariants works in char 0 or sufficiently large characteristic.  
 $(K,+)$  acts as the exponential of the vector field defined by the matrix  $m$ .  
 For background see G.-M. Greuel, G. Pfister,  
 Geometric quotients of unipotent group actions, Proc. London Math. Soc. (3) 67,  
 75-105 (1993).

**Example:**

```
LIB "ainvar.lib";
//Winkelmann: free action but Spec(k[x(1),...,x(5)]) --> Spec(invariant ring)
//is not surjective
ring rw=0,(x(1..5)),dp;
matrix m[5][1];
m[3,1]=x(1);
m[4,1]=x(2);
m[5,1]=1+x(1)*x(4)+x(2)*x(3);
ideal in=invariantRing(m,x(3),x(1),0);      //compute full invariant ring
in;
in[1]=x(1)
in[2]=x(2)
in[3]=x(2)*x(3)*x(4)-x(2)*x(5)+x(4)
in[4]=x(1)*x(3)*x(4)-x(1)*x(5)+x(3)
//Deveney/Finston: The ring of invariants is not finitely generated
ring rf=0,(x(1..7)),dp;
matrix m[7][1];
m[4,1]=x(1)^3;
m[5,1]=x(2)^3;
m[6,1]=x(3)^3;
m[7,1]=(x(1)*x(2)*x(3))^2;
ideal in=invariantRing(m,x(4),x(1),6);      //all invariants up to degree 6
in;
in[1]=x(1)
in[2]=x(3)
in[3]=x(2)
in[4]=x(3)^3*x(4)-x(1)^3*x(6)
in[5]=x(2)^3*x(4)-x(1)^3*x(5)
in[6]=x(2)^2*x(3)^2*x(4)-x(1)*x(7)
in[7]=x(1)^2*x(2)^2*x(6)-x(3)*x(7)
in[8]=x(1)^2*x(3)^2*x(5)-x(2)*x(7)
in[9]=x(1)^2*x(2)*x(3)^4*x(4)*x(5)+x(1)^2*x(2)^4*x(3)*x(4)*x(6)-x(1)^5*x(
2)*x(3)*x(5)*x(6)-2*x(2)^2*x(3)^2*x(4)*x(7)+x(1)*x(7)^2
```

### D.7.2.2 derivate

Procedure from library `ainvar.lib` (see [Section D.7.2 \[ainvar.lib\]](#), page 1802).

**Usage:** `derivate(m,id)`;  $m$  matrix,  $id$  poly/vector/ideal

**Assume:**  $m$  is an  $n \times 1$  matrix, where  $n$  = number of variables of the basering

**Return:** poly/vector/ideal (same type as input), result of applying the vector field by the matrix  $m$  componentwise to  $id$ ;

**Note:** the vector field is  $m[1,1]*d/dx(1) + \dots + m[1,n]*d/dx(n)$

**Example:**

```

LIB "ainvar.lib";
ring q=0,(x,y,z,u,v,w),dp;
poly f=2xz-y2;
matrix m[6][1] =x,y,0,u,v;
derivate(m,f);
  ↪ -2y2+2xz
vector v = [2xz-y2,u6-3];
derivate(m,v);
  ↪ 6u6*gen(2)-2y2*gen(1)+2xz*gen(1)
derivate(m,ideal(2xz-y2,u6-3));
  ↪ _[1]=-2y2+2xz
  ↪ _[2]=6u6

```

### D.7.2.3 actionIsProper

Procedure from library `ainvar.lib` (see [Section D.7.2 \[ainvar.lib\], page 1802](#)).

**Usage:**      `actionIsProper(m)`; `m` matrix

**Assume:**    `m` is a `nx1` matrix, where `n` = number of variables of the basering

**Return:**     `int` = 1, if the action defined by `m` is proper, 0 if not

**Note:**       `m` defines a group action which is the exponential of the vector field  $m[1,1]*d/dx(1) + \dots + m[1,n]*d/dx(n)$

**Example:**

```

LIB "ainvar.lib";
ring rf=0,x(1..7),dp;
matrix m[7][1];
m[4,1]=x(1)^3;
m[5,1]=x(2)^3;
m[6,1]=x(3)^3;
m[7,1]=(x(1)*x(2)*x(3))^2;
actionIsProper(m);
  ↪ 0
ring rd=0,x(1..5),dp;
matrix m[5][1];
m[3,1]=x(1);
m[4,1]=x(2);
m[5,1]=1+x(1)*x(4)^2;
actionIsProper(m);
  ↪ 1

```

### D.7.2.4 reduction

Procedure from library `ainvar.lib` (see [Section D.7.2 \[ainvar.lib\], page 1802](#)).

**Usage:**      `reduction(p,I[,q,n])`; `p` poly, `I` ideal, [`q` monomial, `n` int (optional)]

**Return:**     a polynomial equal to  $p-H(f_1,\dots,f_r)$ , in case the leading term  $LT(p)$  of  $p$  is of the form  $H(LT(f_1),\dots,LT(f_r))$  for some polynomial  $H$  in  $r$  variables over the base field,  $I=f_1,\dots,f_r$ ; if  $q$  is given, a maximal power  $a$  is computed such that  $q^a$  divides  $p-H(f_1,\dots,f_r)$ , and then  $(p-H(f_1,\dots,f_r))/q^a$  is returned; return  $p$  if no  $H$  is found  
if  $n=1$ , a different algorithm is chosen which is sometimes faster (default:  $n=0$ ;  $q$  and  $n$  can be given (or not) in any order)

**Note:** this is a kind of SAGBI reduction in the subalgebra  $K[f_1, \dots, f_r]$  of the basering

**Example:**

```
LIB "ainvar.lib";
ring q=0,(x,y,z,u,v,w),dp;
poly p=x2yz-x2v;
ideal dom =x-w,u2w+1,yz-v;
reduction(p,dom);
↦ 2xyzw-yzw2-2xvw+vw2
reduction(p,dom,w);
↦ 2xyz-yzw-2xv+vw
```

### D.7.2.5 completeReduction

Procedure from library `ainvar.lib` (see [Section D.7.2 \[ainvar.lib\], page 1802](#)).

**Usage:** `completeReduction(p,I[,q,n]);` p poly, I ideal, [q monomial, n int]

**Return:** a polynomial, the SAGBI reduction of the polynomial p with respect to I via the procedure 'reduction' as long as possible  
if n=1, a different algorithm is chosen which is sometimes faster (default: n=0; q and n can be given (or not) in any order)

**Note:** `help reduction;` shows an explanation of SAGBI reduction

**Example:**

```
LIB "ainvar.lib";
ring q=0,(x,y,z,u,v,w),dp;
poly p=x2yz-x2v;
ideal dom =x-w,u2w+1,yz-v;
completeReduction(p,dom);
↦ 2xyzw-yzw2-2xvw+vw2
completeReduction(p,dom,w);
↦ 0
```

### D.7.2.6 localInvar

Procedure from library `ainvar.lib` (see [Section D.7.2 \[ainvar.lib\], page 1802](#)).

**Usage:** `localInvar(m,p,q,h);` m matrix, p,q,h polynomials

**Assume:**  $m(q)$  and  $h$  are invariant under the vector field  $m$ , i.e.  $m(m(q))=m(h)=0$   $h$  must be a ring variable

**Return:** a polynomial, the invariant polynomial of the vector field

$$m = m[1,1]*d/dx(1) + \dots + m[n,1]*d/dx(n)$$

with respect to  $p,q,h$ . It is defined as follows: set  $inv = p$  if  $p$  is invariant, and else set  $inv = m(q)^N * \sum_{i=1..N-1} \{ (-1)^i * (1/i!) * m^i(p) * (q/m(q))^i \}$  where  $m^N(p) = 0$ ,  $m^{(N-1)}(p) \neq 0$ ; the result is  $inv$  divided by  $h$  as often as possible

**Example:**

```
LIB "ainvar.lib";
ring q=0,(x,y,z),dp;
matrix m[3][1];
m[2,1]=x;
m[3,1]=y;
```

```

poly in=localInvar(m,z,y,x);
in;
 $\mapsto -1/2y^2+xz$ 

```

### D.7.2.7 furtherInvar

Procedure from library `ainvar.lib` (see [Section D.7.2 \[ainvar.lib\]](#), page 1802).

**Usage:** `furtherInvar(m,id,karl,q);` m matrix, id,karl ideals, q poly, n int

**Assume:** karl,id,q are invariant under the vector field m,  
moreover, q must be a variable

**Return:** list of two ideals, the first ideal contains further invariants of the vector field  
 $m = \sum m[i,1]*d/dx(i)$  with respect to id,p,q,  
 i.e. we compute elements in the (invariant) subring generated by id which are divisible by q and divide them by q as often as possible. The second ideal contains all invariants given before. If n=1, a different algorithm is chosen which is sometimes faster (default: n=0)

**Example:**

```

LIB "ainvar.lib";
ring r=0,(x,y,z,u),dp;
matrix m[4][1];
m[2,1]=x;
m[3,1]=y;
m[4,1]=z;
ideal id=localInvar(m,z,y,x),localInvar(m,u,y,x);
ideal karl=id,x;
list in=furtherInvar(m,id,karl,x);
in;
 $\mapsto$  [1]:
 $\mapsto$   $_{[1]}=y^2z^2-8/3xz^3-2y^3u+6xyzu-3x^2u^2$ 
 $\mapsto$  [2]:
 $\mapsto$   $_{[1]}=-1/2y^2+xz$ 
 $\mapsto$   $_{[2]}=1/3y^3-xyz+x^2u$ 
 $\mapsto$   $_{[3]}=x$ 

```

### D.7.2.8 sortier

Procedure from library `ainvar.lib` (see [Section D.7.2 \[ainvar.lib\]](#), page 1802).

**Usage:** `sortier(id);` id ideal/module

**Return:** the same ideal/module but with generators ordered by their leading terms, starting with the smallest

**Example:**

```

LIB "ainvar.lib";
ring q=0,(x,y,z,u,v,w),dp;
ideal i=w,x,z,y,v;
sortier(i);
 $\mapsto$   $_{[1]}=w$ 
 $\mapsto$   $_{[2]}=v$ 
 $\mapsto$   $_{[3]}=z$ 
 $\mapsto$   $_{[4]}=y$ 
 $\mapsto$   $_{[5]}=x$ 

```

### D.7.3 rinvar.lib

**Library:** rinvar.lib

**Purpose:** Invariant Rings of Reductive Groups

**Author:** Thomas Bayer, tbayer@in.tum.de  
<http://wwwmayr.informatik.tu-muenchen.de/personen/bayer/> Current Address: Institut fuer Informatik, TU Muenchen

**Overview:** Implementation based on Derksen's algorithm. Written in the scope of the diploma thesis (advisor: Prof. Gert-Martin Greuel) 'Computations of moduli spaces of semi-quasihomogenous singularities and an implementation in Singular'

**Procedures:** See also: [Section D.6.18 \[qhmoduli.lib\]](#), page 1742; [Section D.8.10 \[zeroset.lib\]](#), page 1899.

#### D.7.3.1 HilbertSeries

Procedure from library `rinvar.lib` (see [Section D.7.3 \[rinvar.lib\]](#), page 1807).

**Usage:** HilbertSeries(I, w); ideal I, intvec w

**Purpose:** compute the polynomial  $p$  of the Hilbert Series, represented by  $p/q$ , of the ring  $K[t_1, \dots, t_m, y_1, \dots, y_r]/I$  where 'w' are the weights of the variables, computed, e.g., by 'HilbertWeights', 'I' is of the form  $I[1] - y_1, \dots, I[r] - y_r$  and is quasihomogenous w.r.t. 'w'

**Return:** intvec

**Note:** the leading 0 of the result does not belong to  $p$ , but is needed in the Hilbert driven 'std'.

#### D.7.3.2 HilbertWeights

Procedure from library `rinvar.lib` (see [Section D.7.3 \[rinvar.lib\]](#), page 1807).

**Purpose:** compute the weights of the "slack" variables needed for the computation of the algebraic relations of the generators of 'I' s.t. the Hilbert driven 'std' can be used.

**Return:** intvec

**Assume:** basering =  $K[t_1, \dots, t_m, \dots]$ , 'I' is quasihomogenous w.r.t. 'w' and contains only polynomials in  $t_1, \dots, t_m$

#### D.7.3.3 ImageVariety

Procedure from library `rinvar.lib` (see [Section D.7.3 \[rinvar.lib\]](#), page 1807).

**Usage:** ImageVariety(ideal I, F [, w]); ideal I; F is a list/ideal, intvec w.

**Purpose:** compute the Zariski closure of the image of the variety of I under the morphism F.

**Note:** if 'I' and 'F' are quasihomogenous w.r.t. 'w' then the Hilbert-driven 'std' is used.

**Return:** polynomial ring over the same ground field, containing the ideal 'imageid'. The variables are  $Y(1), \dots, Y(k)$  where  $k = \text{size}(F) - \text{'imageid'}$  is the ideal of the Zariski closure of  $F(X)$  where  $X$  is the variety of I.

**Example:**

```

LIB "rinvar.lib";
ring B = 0,(x,y),dp;
ideal I = x4 - y4;
ideal F = x2, y2, x*y;
def R = ImageVariety(I, F);
↳
↳ // 'ImageVariety' created a new ring.
↳ // To see the ring, type (if the name 'R' was assigned to the return value):
↳      show(R);
↳ // To access the ideal of the image variety, type
↳      setring R; imageid;
↳
setring R;
imageid;
↳ imageid[1]=Y(1)*Y(2)-Y(3)^2
↳ imageid[2]=Y(1)^2-Y(2)^2
↳ imageid[3]=Y(2)^3-Y(1)*Y(3)^2

```

### D.7.3.4 ImageGroup

Procedure from library `rinvar.lib` (see [Section D.7.3 \[rinvar.lib\]](#), page 1807).

**Usage:** `ImageGroup(G, action);` ideal `G`, action;

**Purpose:** compute the ideal of the image of `G` in  $GL(m, K)$  induced by the linear action 'action', where `G` is an algebraic group and 'action' defines an action of `G` on  $K^m$  ( $\text{size}(\text{action}) = m$ ).

**Return:** ring, a polynomial ring over the same ground field as the basering, containing the ideals 'groupid' and 'actionid'.  
 - 'groupid' is the ideal of the image of `G` (order  $\leq$  order of `G`) - 'actionid' defines the linear action of 'groupid' on  $K^m$ .

**Note:** 'action' and 'actionid' have the same orbits  
 all variables which give only rise to 0's in the  $m \times m$  matrices of `G` have been omitted.

**Assume:** basering  $K[s(1..r), t(1..m)]$  has  $r + m$  variables, `G` is the ideal of an algebraic group and `F` is an action of `G` on  $K^m$ . `G` contains only the variables  $s(1) \dots s(r)$ . The action 'action' is given by polynomials  $f_1, \dots, f_m$  in basering, s.t. on the ring level we have  $K[t_1, \dots, t_m] \rightarrow K[s_1, \dots, s_r, t_1, \dots, t_m]/G$   
 $t_i \rightarrow f_i(s_1, \dots, s_r, t_1, \dots, t_m)$

**Example:**

```

LIB "rinvar.lib";
ring B = 0,(s(1..2), t(1..2)),dp;
ideal G = s(1)^3-1, s(2)^10-1;
ideal action = s(1)*s(2)^8*t(1), s(1)*s(2)^7*t(2);
def R = ImageGroup(G, action);
↳
↳ // 'ImageGroup' created a new ring.
↳ // To see the ring, type (if the name 'R' was assigned to the return value):
↳      show(R);
↳ // To access the ideal of the image of the input group and to access the \

```

```

new
⇒ // action of the group, type
⇒      setring R;  groupid; actionid;
⇒
setring R;
groupid;
⇒ groupid[1]=-s(1)+s(2)^4
⇒ groupid[2]=s(1)^8-s(2)^2
⇒ groupid[3]=s(1)^7*s(2)^2-1
actionid;
⇒ actionid[1]=s(1)*t(1)
⇒ actionid[2]=s(2)*t(2)

```

### D.7.3.5 InvariantRing

Procedure from library `rinvar.lib` (see [Section D.7.3 \[rinvar.lib\]](#), page 1807).

**Usage:**     InvariantRing(G, Gact [, opt]); ideal G, Gact; int opt

**Purpose:**     compute generators of the invariant ring of G w.r.t. the action 'Gact'

**Assume:**    G is a finite group and 'Gact' is a linear action.

**Return:**     ring R; this ring comes with the ideals 'invars' and 'groupid' and with the poly 'newA':  
               - 'invars' contains the algebra generators of the invariant ring - 'groupid' is the ideal of  
               G in the new ring  
               - 'newA' is the new representation of the primitive root of the minimal polynomial of  
               the ring which was active when calling the procedure (if the minpoly did not change,  
               'newA' is set to 'a').

**Note:**       the minimal polynomial of the output ring depends on some random choices

**Example:**

```

LIB "rinvar.lib";
ring B = 0, (s(1..2), t(1..2)), dp;
ideal G = -s(1)+s(2)^3, s(1)^4-1;
ideal action = s(1)*t(1), s(2)*t(2);
def R = InvariantRing(std(G), action);
⇒
⇒ // 'InvariantRing' created a new ring.
⇒ // To see the ring, type (if the name 'R' was assigned to the return valu\
e):
⇒      show(R);
⇒ // To access the generators of the invariant ring type
⇒      setring R; invars;
⇒ // Note that the input group G is stored in R as the ideal 'groupid'; to
⇒ // see it, type
⇒      groupid;
⇒ // Note that 'InvariantRing' might change the minimal polynomial
⇒ // The representation of the algebraic number is given by 'newA'
⇒
setring R;
invars;
⇒ invars[1]=t(1)^4
⇒ invars[2]=t(1)^3*t(2)^3
⇒ invars[3]=t(1)^2*t(2)^6

```

```

↳ invars[4]=t(1)*t(2)^9
↳ invars[5]=t(2)^12

```

### D.7.3.6 InvariantQ

Procedure from library `rinvar.lib` (see [Section D.7.3 \[rinvar.lib\], page 1807](#)).

**Usage:** InvariantQ(f, G, action); poly f; ideal G, action

**Purpose:** check whether the polynomial f is invariant w.r.t. G, where G acts via 'action' on  $K^m$ .

**Assume:** basering =  $K[s_1, \dots, s_m, t_1, \dots, t_m]$  where  $K = \mathbb{Q}$  or  $K = \mathbb{Q}(a)$  and minpoly  $\neq 0$ , f contains only  $t_1, \dots, t_m$ , G is the ideal of an algebraic group and a standardbasis.

**Return:** int;  
0 if f is not invariant,  
1 if f is invariant

**Note:** G need not be finite

### D.7.3.7 LinearizeAction

Procedure from library `rinvar.lib` (see [Section D.7.3 \[rinvar.lib\], page 1807](#)).

**Usage:** LinearizeAction(G, action, r); ideal G, action; int r

**Purpose:** linearize the group action 'action' and find an equivariant embedding of  $K^m$  where  $m = \text{size}(\text{action})$ .

**Assume:** G contains only variables `var(1..r)` ( $r = \text{nrs}$ )  
basing =  $K[s(1..r), t(1..m)]$ ,  $K = \mathbb{Q}$  or  $K = \mathbb{Q}(a)$  and minpoly  $\neq 0$ .

**Return:** polynomial ring containing the ideals 'actionid', 'embedid', 'groupid' - 'actionid' is the ideal defining the linearized action of G - 'embedid' is a parameterization of an equivariant embedding (closed) - 'groupid' is the ideal of G in the new ring

**Note:** set `printlevel > 0` to see a trace

**Example:**

```

LIB "rinvar.lib";
ring B = 0, (s(1..5), t(1..3)), dp;
ideal G = s(3)-s(4), s(2)-s(5), s(4)*s(5), s(1)^2*s(4)+s(1)^2*s(5)-1, s(1)^2*s(5)^2-1;
ideal action = -s(4)*t(1)+s(5)*t(1), -s(4)^2*t(2)+2*s(4)^2*t(3)^2+s(5)^2*t(2), s(4)*t(3)-s(5)*t(3);
LinearActionQ(action, 5);
↳ 0
def R = LinearizeAction(G, action, 5);
↳
↳ // 'LinearizeAction' created a new ring.
↳ // To see the ring, type (if the name 'R' was assigned to the return value):
↳ show(R);
↳ // To access the new action and the equivariant embedding, type
↳ setring R; actionid; embedid; groupid
↳
setring R;
R;
↳ // coefficients: QQ
↳ // number of vars : 9

```



```

⇒ //          block   1 : ordering dp
⇒ //          : names   s(1) s(2) s(3) s(4) s(5) t(1) t(2) t(3) t(\
  4)
⇒ //          block   2 : ordering C
actionid;
⇒ actionid[1]=-s(4)*t(1)+s(5)*t(1)
⇒ actionid[2]=-s(4)^2*t(2)+s(5)^2*t(2)+2*s(4)^2*t(4)
⇒ actionid[3]=s(4)*t(3)+s(5)*t(3)
⇒ actionid[4]=s(4)^2*t(4)+s(5)^2*t(4)
embedid;
⇒ embedid[1]=t(1)
⇒ embedid[2]=t(2)
⇒ embedid[3]=t(3)
⇒ embedid[4]=t(3)^2
groupid;
⇒ groupid[1]=s(3)-s(4)
⇒ groupid[2]=s(2)-s(5)
⇒ groupid[3]=s(4)*s(5)
⇒ groupid[4]=s(1)^2*s(4)+s(1)^2*s(5)-1
⇒ groupid[5]=s(1)^2*s(5)^2-s(5)
⇒ groupid[6]=s(4)^4-s(5)^4+s(1)^2
⇒ groupid[7]=s(1)^4+s(4)^3-s(5)^3
⇒ groupid[8]=s(5)^5-s(1)^2*s(5)
LinearActionQ(actionid, 5);
⇒ 1

```

### D.7.3.8 LinearActionQ

Procedure from library `rinvar.lib` (see [Section D.7.3 \[rinvar.lib\]](#), page 1807).

**Usage:**     LinearActionQ(action,nrs); ideal action, int nrs

**Purpose:**    check whether the action defined by 'action' is linear w.r.t. the variables `var(nrs + 1...nvars(basering))`.

**Return:**    0 action not linear  
               1 action is linear

**Example:**

```

LIB "rinvar.lib";
ring R = 0,(s(1..5), t(1..3)),dp;
ideal G = s(3)-s(4), s(2)-s(5), s(4)*s(5), s(1)^2*s(4)+s(1)^2*s(5)-1,
s(1)^2*s(5)^2-s(5), s(4)^4-s(5)^4+s(1)^2, s(1)^4+s(4)^3-s(5)^3,
s(5)^5-s(1)^2*s(5);
ideal Gaction = -s(4)*t(1)+s(5)*t(1),
-s(4)^2*t(2)+2*s(4)^2*t(3)^2+s(5)^2*t(2),
s(4)*t(3)+s(5)*t(3);
LinearActionQ(Gaction, 5);
⇒ 0
LinearActionQ(Gaction, 8);
⇒ 1

```

### D.7.3.9 LinearCombinationQ

Procedure from library `rinvar.lib` (see [Section D.7.3 \[rinvar.lib\]](#), page 1807).

**Usage:** LinearCombination(I, f); ideal I, poly f

**Purpose:** test whether f can be written as a linear combination of the generators of I.

**Return:** 0 f is not a linear combination  
1 f is a linear combination

### D.7.3.10 MinimalDecomposition

Procedure from library `rinvar.lib` (see [Section D.7.3 \[rinvar.lib\], page 1807](#)).

**Usage:** MinimalDecomposition(f,a,b); poly f; int a, b.

**Purpose:** decompose f as a sum  $M[1,1]*M[2,1] + \dots + M[1,r]*M[2,r]$  where  $M[1,i]$  contains only  $s(1..a)$ ,  $M[2,i]$  contains only  $t(1..b)$  s.t. r is minimal

**Assume:** f polynomial in  $K[s(1..a), t(1..b)]$ ,  $K = \mathbb{Q}$  or  $K = \mathbb{Q}(a)$  and `minpoly` != 0

**Return:** 2 x r matrix M s.t.  $f = M[1,1]*M[2,1] + \dots + M[1,r]*M[2,r]$

**Example:**

```
LIB "rinvar.lib";
ring R = 0, (s(1..2), t(1..2)), dp;
poly h = s(1)*(t(1) + t(1)^2) + (t(2) + t(2)^2)*(s(1)^2 + s(2));
matrix M = MinimalDecomposition(h, 2, 2);
M;
↪ M[1,1]=s(1)^2+s(2)
↪ M[1,2]=s(1)
↪ M[2,1]=t(2)^2+t(2)
↪ M[2,2]=t(1)^2+t(1)
M[1,1]*M[2,1] + M[1,2]*M[2,2] - h;
↪ 0
```

### D.7.3.11 NullCone

Procedure from library `rinvar.lib` (see [Section D.7.3 \[rinvar.lib\], page 1807](#)).

**Usage:** NullCone(G, action); ideal G, action

**Purpose:** compute the ideal of the nullcone of the linear action of G on  $K^n$ , given by 'action', by means of Deksen's algorithm

**Assume:** `basing` =  $K[s(1..r), t(1..n)]$ ,  $K = \mathbb{Q}$  or  $K = \mathbb{Q}(a)$  and `minpoly` != 0, G is an ideal of a reductive algebraic group in  $K[s(1..r)]$ , 'action' is a linear group action of G on  $K^n$  (`n` = `ncols(action)`)

**Return:** ideal of the nullcone of G.

**Note:** the generators of the nullcone are homogeneous, but in general not invariant

**Example:**

```
LIB "rinvar.lib";
ring R = 0, (s(1..2), x, y), dp;
ideal G = -s(1)+s(2)^3, s(1)^4-1;
ideal action = s(1)*x, s(2)*y;
ideal inv = NullCone(G, action);
inv;
↪ inv[1]=x^4
↪ inv[2]=x^3*y^3
```

```

 $\mapsto \text{inv}[3]=x^2*y^6$ 
 $\mapsto \text{inv}[4]=x*y^9$ 
 $\mapsto \text{inv}[5]=y^{12}$ 

```

### D.7.3.12 ReynoldsImage

Procedure from library `rinvar.lib` (see [Section D.7.3 \[rinvar.lib\], page 1807](#)).

**Usage:** `ReynoldsImage(RO, f);` list RO, poly f

**Purpose:** compute the Reynolds image of the polynomial f, where RO represents the Reynolds operator

**Return:** poly

### D.7.3.13 ReynoldsOperator

Procedure from library `rinvar.lib` (see [Section D.7.3 \[rinvar.lib\], page 1807](#)).

**Usage:** `ReynoldsOperator(G, action [, opt]);` ideal G, action; int opt

**Purpose:** compute the Reynolds operator of the group G which acts via 'action'

**Return:** polynomial ring R over a simple extension of the ground field of the basering (the extension might be trivial), containing a list 'ROelements', the ideals 'id', 'actionid' and the polynomial 'newA'.  $R = K(a)[s(1..r), t(1..n)]$ .  
 - 'ROelements' is a list of ideals, each ideal represents a substitution map  $F : R \rightarrow R$  according to the zero-set of G - 'id' is the ideal of G in the new ring  
 - 'newA' is the new representation of a' in terms of a. If the basering does not contain a parameter then 'newA' = 'a'.

**Assume:**  $\text{basing} = K[s(1..r), t(1..n)]$ ,  $K = \mathbb{Q}$  or  $K = \mathbb{Q}(a')$  and  $\text{minpoly} \neq 0$ , G is the ideal of a finite group in  $K[s(1..r)]$ , 'action' is a linear group action of G

### D.7.3.14 SimplifyIdeal

Procedure from library `rinvar.lib` (see [Section D.7.3 \[rinvar.lib\], page 1807](#)).

**Purpose:** simplify ideal I to the ideal I', do not change the names of the first m variables, new ideal I' might contain less variables. I' contains variables `var(1..m)`

**Return:** list  
 -[1] ideal I'  
 -[2] ideal representing a map phi to a ring with probably less vars. s.th.  $\text{phi}(I) = I'$   
 -[3] list of variables  
 -[4] list from 'elimpart'

## D.7.4 invar.lib

**Library:** `invar.lib`

**Purpose:** Procedures to compute invariant rings of  $SL(n)$  and torus groups

**Author:** Harm Derksen, `hderksen@math.unibas.ch`

**Procedures:**

### D.7.4.1 SL

Procedure from library `invar.lib` (see [Section D.7.4 \[invar.lib\], page 1813](#)).

**Usage:** `SL(<int>)`

**Returns:** `SL(n)` sets the current group to `SL_n`. The following global variables will be set:  
 group of type `<ring>`  
 groupideal of type `<ideal>`  
 SLrep of type `<matrix>`  
 reynolds of type `<proc>`  
 The quotient of `'group'` and `'groupideal'` is the coordinate ring of `SL_n`. The matrix `'SLrep'` will be set to the standard representation of `SL_n`. The basering will be set to `'group'`.

**Example:**

```
LIB "invar.lib";
SL(3);
Invar::group;
⇨ // coefficients: QQ
⇨ // number of vars : 9
⇨ //          block 1 : ordering dp
⇨ //          : names  g(1) g(2) g(3) g(4) g(5) g(6) g(7) g(8) g(\
9)
⇨ //          block 2 : ordering C
groupideal;
⇨ groupideal[1]=g(3)*g(5)*g(7)-g(2)*g(6)*g(7)-g(3)*g(4)*g(8)+g(1)*g(6)*g(8)\
+g(2)*g(4)*g(9)-g(1)*g(5)*g(9)+1
print(SLrep);
⇨ g(1),g(2),g(3),
⇨ g(4),g(5),g(6),
⇨ g(7),g(8),g(9)
```

### D.7.4.2 torus

Procedure from library `invar.lib` (see [Section D.7.4 \[invar.lib\], page 1813](#)).

**Usage:** `torus(<int>)`

**Returns:** `torus(n)` sets the current group to an `n`-dimensional torus. The following global variables will be changed:  
 group of type `<ring>`  
 groupideal of type `<ideal>`  
 reynolds of type `<proc>`  
 The quotient of `'group'` and `'groupideal'` is the coordinate ring of an `n`-dimensional torus. The basering will be set to `'group'`.

**Example:**

```
LIB "invar.lib";
torus(3);
Invar::group;
⇨ // coefficients: QQ
⇨ // number of vars : 4
⇨ //          block 1 : ordering dp
⇨ //          : names  g(1) g(2) g(3) g(4)
```

```

⇒ //          block    2 : ordering C
groupideal;
⇒ groupideal[1]=g(1)*g(2)*g(3)*g(4)-1

```

### D.7.4.3 torusrep

Procedure from library `invar.lib` (see [Section D.7.4 \[invar.lib\]](#), page 1813).

**Usage:** `torusrep(<list>)`, `<list>` must be a list of integer vectors of length `n`, where `n` is the dimension of the current torusgroup.

**Returns:** `torusrep(m)` gives a matrix with entries in 'group'. This matrix represents the action of the torus with weights  
`m[1],m[2],...,m[size(m)]`

**Example:**

```

LIB "invar.lib";
torus(1);           // Take the 1-dimensional torus, the multiplicative group
list weights=-2,-3,7; // 3-dimensional action with weights -2,-3,7
matrix m=torusrep(weights); // compute the matrix of the representation
invar(m);           // compute the invariant ring
⇒
⇒ Ideal B:
⇒ x(1)^2*x(2)*x(3)-y(1)^2*y(2)*y(3),
⇒ x(2)^2*y(1)^3-x(1)^3*y(2)^2,
⇒ x(2)^3*x(3)*y(1)-x(1)*y(2)^3*y(3),
⇒ x(1)^5*x(3)*y(2)-x(2)*y(1)^5*y(3),
⇒ x(1)*x(2)^4*x(3)^2-y(1)*y(2)^4*y(3)^2,
⇒ x(1)^7*x(3)^2-y(1)^7*y(3)^2,
⇒ x(2)^7*x(3)^3-y(2)^7*y(3)^3
⇒
⇒ Zero Fiber Ideal:
⇒ x(1)^2*x(2)*x(3),
⇒ x(1)*x(2)^4*x(3)^2,
⇒ x(1)^7*x(3)^2,
⇒ x(2)^7*x(3)^3
⇒
⇒ Generating Invariants:
⇒ x(1)^2*x(2)*x(3),
⇒ x(1)*x(2)^4*x(3)^2,
⇒ x(1)^7*x(3)^2,
⇒ x(2)^7*x(3)^3

```

### D.7.4.4 finiterep

Procedure from library `invar.lib` (see [Section D.7.4 \[invar.lib\]](#), page 1813).

**Usage:** `finiterep(<list>)`, `<list>` must be a list of matrices

**Returns:** `finiterep(m)` gives a matrix with coefficients in the ring 'group' which represents the action of the finite group where the elements of the finite group act as  
`m[1],m[2],...,m[size(m)]`.

**Example:**

```

LIB "invar.lib";
finite(6);                                // The symmetric group S_3
matrix id=unitmat(3);                     // identity matrix
matrix m3[3][3]=0,1,0,0,0,1,1,0,0;        // corresponds with (1 2 3)
matrix m2[3][3]=0,1,0,1,0,0,0,0,1;        // corresponds with (1 2)
list a=id,m3,m3*m3,m2,m2*m3,m2*m3*m3;    // all elements of S_3
matrix rep=finiterep(a);                  // compute matrix of standard repr.
invar(rep);                               // compute the invariant ring
↳
↳ Ideal B:
↳ x(1)+x(2)+x(3)-y(1)-y(2)-y(3),
↳ 36*x(2)^2-115*x(1)*x(3)-79*x(2)*x(3)-79*x(3)^2+24*x(1)*y(1)-12*x(2)*y(1)+\
  103*x(3)*y(1)-24*y(1)^2-36*x(2)*y(2)+79*x(3)*y(2)+12*y(1)*y(2)+310*x(1)*y\
  (3)+274*x(2)*y(3)+389*x(3)*y(3)-298*y(1)*y(3)-274*y(2)*y(3)-310*y(3)^2,
↳ 1296*x(3)^3+1584*x(1)*x(3)*y(1)+1584*x(2)*x(3)*y(1)+288*x(3)^2*y(1)-1584*\
  x(3)*y(1)^2+2112*x(1)^2*y(2)+4224*x(1)*x(2)*y(2)+2112*x(2)^2*y(2)+6648*x(\
  1)*x(3)*y(2)+6648*x(2)*x(3)*y(2)+3240*x(3)^2*y(2)-2112*x(1)*y(1)*y(2)-211\
  2*x(2)*y(1)*y(2)-4824*x(3)*y(1)*y(2)-2112*x(1)*y(2)^2-2112*x(2)*y(2)^2-45\
  36*x(3)*y(2)^2+5248*x(1)^2*y(3)+13645*x(1)*x(2)*y(3)+8397*x(2)^2*y(3)+448\
  95*x(1)*x(3)*y(3)+48044*x(2)*x(3)*y(3)+38351*x(3)^2*y(3)-6832*x(1)*y(1)*y\
  (3)-9981*x(2)*y(1)*y(3)-41519*x(3)*y(1)*y(3)+1584*y(1)^2*y(3)-16120*x(1)*\
  y(2)*y(3)-19269*x(2)*y(2)*y(3)-51647*x(3)*y(2)*y(3)+9048*y(1)*y(2)*y(3)+8\
  760*y(2)^2*y(3)-26722*x(1)*y(3)^2-29871*x(2)*y(3)^2-61121*x(3)*y(3)^2+230\
  58*y(1)*y(3)^2+30234*y(2)*y(3)^2+21474*y(3)^3
↳
↳ Zero Fiber Ideal:
↳ x(1)+x(2)+x(3),
↳ 36*x(2)^2-115*x(1)*x(3)-79*x(2)*x(3)-79*x(3)^2,
↳ x(3)^3
↳
↳ Generating Invariants:
↳ x(1)+x(2)+x(3),
↳ -43/3*x(1)^2-194/3*x(1)*x(2)-43/3*x(2)^2-194/3*x(1)*x(3)-194/3*x(2)*x(3)-\
  43/3*x(3)^2,
↳ 1/3*x(1)^3+1/3*x(2)^3+1/3*x(3)^3

```

### D.7.4.5 sympower

Procedure from library `invar.lib` (see [Section D.7.4 \[invar.lib\]](#), page 1813).

**Usage:** `sympower(<matrix>,<int>)`

**Returns:** If `m` is a matrix with coefficients in the ring 'group', representing the action on some vector space `V`, then `sympower(m,n)` gives the matrix of the representation of the group on the `n`-th symmetric power of `V`.

**Example:**

```

LIB "invar.lib";
SL(2);
print(SLrep);
↳ g(1),g(2),
↳ g(3),g(4)
print(sympower(SLrep,3));
↳ g(1)^3,      3*g(1)^2*g(2),      3*g(1)*g(2)^2,      g(2)^3,

```

$$\begin{aligned} \mapsto & g(1)^2g(3), 3g(1)^2g(4)-2g(1), 3g(1)g(2)g(4)-g(2), g(2)^2g(4), \\ \mapsto & g(1)g(3)^2, 3g(1)g(3)g(4)-g(3), 3g(1)g(4)^2-2g(4), g(2)g(4)^2, \\ \mapsto & g(3)^3, 3g(3)^2g(4), 3g(3)g(4)^2, g(4)^3 \end{aligned}$$

### D.7.4.6 invar

Procedure from library `invar.lib` (see [Section D.7.4 \[invar.lib\]](#), page 1813).

**Usage:** `invar(<matrix>)`

**Returns:** If  $m$  is a  $n \times n$  matrix with coefficients in the ring 'group', representing the action on some vector space  $V$ , then `invar(m)`; gives polynomials in  $x(1), x(2), \dots, x(n)$  who generate the invariant ring. The following global variables will be set:  
 polyring of type <ring> polynomial ring in  $x(1), \dots, x(n)$  invring of type <ideal> entries generate the inv. ring representation of type <matrix>  
 The base ring will be set to 'polyring' which is a global variable representing the polynomial ring on which the group acts. The variable 'representation' will be set to the input  $m$ .

**Example:**

```
LIB "invar.lib";
SL(2);                                // Take the group SL_2
matrix m=dsum(SLrep,SLrep,SLrep,SLrep);
// 4 copies of the standard representation
invar(m);                             // empirical evidence for FFT
 $\mapsto$ 
 $\mapsto$  Ideal B:
 $\mapsto x(6)*x(7)-x(5)*x(8)-y(6)*y(7)+y(5)*y(8),$ 
 $\mapsto x(4)*x(7)-x(3)*x(8)-y(4)*y(7)+y(3)*y(8),$ 
 $\mapsto x(2)*x(7)-x(1)*x(8)-y(2)*y(7)+y(1)*y(8),$ 
 $\mapsto x(4)*x(5)-x(3)*x(6)-y(4)*y(5)+y(3)*y(6),$ 
 $\mapsto x(2)*x(5)-x(1)*x(6)-y(2)*y(5)+y(1)*y(6),$ 
 $\mapsto x(2)*x(3)-x(1)*x(4)-y(2)*y(3)+y(1)*y(4),$ 
 $\mapsto x(8)*y(4)*y(5)-x(8)*y(3)*y(6)-x(6)*y(4)*y(7)+x(4)*y(6)*y(7)+x(6)*y(3)*y(8\backslash$ 
 $\mapsto -x(4)*y(5)*y(8),$ 
 $\mapsto x(7)*y(4)*y(5)-x(7)*y(3)*y(6)-x(5)*y(4)*y(7)+x(3)*y(6)*y(7)+x(5)*y(3)*y(8\backslash$ 
 $\mapsto -x(3)*y(5)*y(8),$ 
 $\mapsto x(8)*y(2)*y(5)-x(8)*y(1)*y(6)-x(6)*y(2)*y(7)+x(2)*y(6)*y(7)+x(6)*y(1)*y(8\backslash$ 
 $\mapsto -x(2)*y(5)*y(8),$ 
 $\mapsto x(7)*y(2)*y(5)-x(7)*y(1)*y(6)-x(5)*y(2)*y(7)+x(1)*y(6)*y(7)+x(5)*y(1)*y(8\backslash$ 
 $\mapsto -x(1)*y(5)*y(8),$ 
 $\mapsto x(8)*y(2)*y(3)-x(8)*y(1)*y(4)-x(4)*y(2)*y(7)+x(2)*y(4)*y(7)+x(4)*y(1)*y(8\backslash$ 
 $\mapsto -x(2)*y(3)*y(8),$ 
 $\mapsto x(7)*y(2)*y(3)-x(7)*y(1)*y(4)-x(3)*y(2)*y(7)+x(1)*y(4)*y(7)+x(3)*y(1)*y(8\backslash$ 
 $\mapsto -x(1)*y(3)*y(8),$ 
 $\mapsto x(6)*y(2)*y(3)-x(6)*y(1)*y(4)-x(4)*y(2)*y(5)+x(2)*y(4)*y(5)+x(4)*y(1)*y(6\backslash$ 
 $\mapsto -x(2)*y(3)*y(6),$ 
 $\mapsto x(5)*y(2)*y(3)-x(5)*y(1)*y(4)-x(3)*y(2)*y(5)+x(1)*y(4)*y(5)+x(3)*y(1)*y(6\backslash$ 
 $\mapsto -x(1)*y(3)*y(6)$ 
 $\mapsto$ 
 $\mapsto$  Zero Fiber Ideal:
 $\mapsto x(6)*x(7)-x(5)*x(8),$ 
 $\mapsto x(4)*x(7)-x(3)*x(8),$ 
 $\mapsto x(2)*x(7)-x(1)*x(8),$ 
```

```

    ↪ x(4)*x(5)-x(3)*x(6),
    ↪ x(2)*x(5)-x(1)*x(6),
    ↪ x(2)*x(3)-x(1)*x(4)
    ↪
    ↪ Generating Invariants:
    ↪ x(6)*x(7)-x(5)*x(8),
    ↪ x(4)*x(7)-x(3)*x(8),
    ↪ x(2)*x(7)-x(1)*x(8),
    ↪ x(4)*x(5)-x(3)*x(6),
    ↪ x(2)*x(5)-x(1)*x(6),
    ↪ x(2)*x(3)-x(1)*x(4)
    setring Invar::polyring;
    Invar::reynolds(x(1)*x(4));           // The reynolds operator is computed using
    ↪ -1/2*x(2)*x(3)+1/2*x(1)*x(4)
    // the Omega process.

```

#### D.7.4.7 SLreynolds

Procedure from library `invar.lib` (see [Section D.7.4 \[invar.lib\]](#), page 1813).

**Usage:** SLreynolds(f) poly f  
**Returns:** the reynolds operator applied to f  
**Note:** if the group is  $SL_n$  the omega-process is used

#### D.7.4.8 torusreynolds

Procedure from library `invar.lib` (see [Section D.7.4 \[invar.lib\]](#), page 1813).

### D.7.5 stratify\_lib

**Library:** stratify.lib  
**Purpose:** Algorithmic Stratification for Unipotent Group-Actions  
**Author:** Anne Fruehbis-Krueger, anne@mathematik.uni-kl.de  
**Procedures:**

#### D.7.5.1 prepMat

Procedure from library `stratify.lib` (see [Section D.7.5 \[stratify\\_lib\]](#), page 1818).

**Usage:** prepMat(M,wr,ws,step);  
 where M is a matrix, wr is an intvec of size ncols(M), ws an intvec of size nrows(M) and step is an integer  
**Return:** 2 lists of submatrices corresponding to the filtrations specified by wr and ws:  
 the first list corresponds to the list for the filtration of  $AdA$ , i.e. the ranks of these matrices will be the  $r_i$ , the second one to the list for the filtration of  $L$ , i.e. the ranks of these matrices will be the  $s_i$   
**Note:** \* the entries of the matrix M are  $M_{ij}=\delta_i(x_j)$ ,  
 \* wr is used to determine what subset of the set of all  $dx_i$  is generating  $AdF^1(A)$ :  
 if  $(k-1)*step \leq wr[i] < k*step$ , then  $dx_i$  is in the set of generators of  $AdF^1(A)$  for all  $k \geq 1$  and the i-th column of M appears in each submatrix starting from the k-th



\* ws is used to determine what subset of the set of all  $\delta_{i,j}$  is generating  $Z_1(L)$ :  
 if  $(k-1)*step \leq ws[i] < k*step$ , then  $\delta_{i,j}$  is in the set of generators of  $Z_1(A)$  for  $1 \leq j \leq k$  and the  $i$ -th row of  $M$  appears in each submatrix up to the  $(k-1)$ th  
 \* the entries of  $wr$  and  $ws$  as well as  $step$  should be positive integers

**Example:**

```
LIB "stratify.lib";
ring r=0,(t(1..3)),dp;
matrix M[2][3]=0,t(1),3*t(2),0,0,t(1);
print(M);
  ↳ 0,t(1),3*t(2),
  ↳ 0,0,    t(1)
intvec wr=1,3,5;
intvec ws=2,4;
int step=2;
prepMat(M,wr,ws,step);
  ↳ [1]:
  ↳   [1]:
  ↳     _[1,1]=0
  ↳     _[2,1]=0
  ↳   [2]:
  ↳     _[1,1]=0
  ↳     _[1,2]=t(1)
  ↳     _[2,1]=0
  ↳     _[2,2]=0
  ↳   [3]:
  ↳     _[1,1]=0
  ↳     _[1,2]=t(1)
  ↳     _[1,3]=3*t(2)
  ↳     _[2,1]=0
  ↳     _[2,2]=0
  ↳     _[2,3]=t(1)
  ↳ [2]:
  ↳   [1]:
  ↳     _[1,1]=0
  ↳     _[1,2]=t(1)
  ↳     _[1,3]=3*t(2)
  ↳     _[2,1]=0
  ↳     _[2,2]=0
  ↳     _[2,3]=t(1)
  ↳   [2]:
  ↳     _[1,1]=0
  ↳     _[1,2]=0
  ↳     _[1,3]=t(1)
```

**D.7.5.2 stratify**

Procedure from library `stratify.lib` (see [Section D.7.5 \[stratify.lib\]](#), page 1818).

**Usage:**     `stratify(M,wr,ws,step);`  
               where  $M$  is a matrix,  $wr$  is an intvec of size  $\text{ncols}(M)$ ,  $ws$  an intvec of size  $\text{nrows}(M)$   
               and  $step$  is an integer

**Return:**    list of lists, each entry of the big list corresponds to one locally closed set and has the following entries:

- 1) intvec giving the corresponding rs-vector
- 2) ideal determining the closed set
- 3) list d of polynomials determining the open set  $D(d[1])$  empty list if there is more than one open set
- 4-n) lists of polynomials determining open sets which all lead to the same rs-vector

**Note:**

- \* ring ordering should be global, i.e. the ring should be a polynomial ring
- \* the entries of the matrix M are  $M_{ij} = \delta_{ij}(x_j)$ ,
- \* wr is used to determine what subset of the set of all  $dx_i$  is generating  $\text{AdF}^l(A)$ :  
if  $(k-1)*\text{step} < wr[i] \leq k*\text{step}$ , then  $dx_i$  is in the set of generators of  $\text{AdF}^l(A)$  for all  $l \geq k$
- \* ws is used to determine what subset of the set of all  $\delta_{ij}$  is generating  $Z_l(L)$ :  
if  $(k-1)*\text{step} \leq ws[i] < k*\text{step}$ , then  $\delta_{ij}$  is in the set of generators of  $Z_l(A)$  for  $l < k$
- \* the entries of wr and ws as well as step should be positive integers
- \* the filtrations have to be known, no sanity checks concerning the filtrations are performed !!!

**Example:**

```
LIB "stratify.lib";
ring r=0,(t(1..3)),dp;
matrix M[2][3]=0,t(1),3*t(2),0,0,t(1);
intvec wr=1,3,5;
intvec ws=2,4;
int step=2;
stratify(M,wr,ws,step);
⇒ [1]:
⇒   [1]:
⇒     0,0,0,0
⇒   [2]:
⇒     _[1]=t(2)
⇒     _[2]=t(1)
⇒   [3]:
⇒     [1]:
⇒       1
⇒ [2]:
⇒   [1]:
⇒     0,1,0,1
⇒   [2]:
⇒     _[1]=t(1)
⇒   [3]:
⇒     [1]:
⇒       t(2)
⇒     [2]:
⇒       t(2)
⇒ [3]:
⇒   [1]:
⇒     1,2,1,2
⇒   [2]:
⇒     _[1]=0
⇒   [3]:
⇒     [1]:
⇒       t(1)
```

```

      ↪      [2]:
      ↪      t(1)

```

## D.8 Symbolic-numerical solving

### D.8.1 ffsolve\_lib

**Library:** ffsolve.lib

**Purpose:** multivariate equation solving over finite fields

**Author:** Gergo Gyula Borus, borisz@borisz.net

**Procedures:**

#### D.8.1.1 ffsolve

Procedure from library `ffsolve.lib` (see [Section D.8.1 \[ffsolve.lib\], page 1821](#)).

**Usage:** ffsolve(I[, L]); I ideal, L list of strings

**Return:** list L, the common roots of I as ideal

**Assume:** basering is a finite field of type  $(p^n, a)$

**Example:**

```

LIB "ffsolve.lib";
ring R = (2,a),x(1..3),lp;
minpoly=a2+a+1;
ideal I;
I[1]=x(1)^2*x(2)+(a)*x(1)*x(2)^2+(a+1);
I[2]=x(1)^2*x(2)*x(3)^2+(a)*x(1);
I[3]=(a+1)*x(1)*x(3)+(a+1)*x(1);
ffsolve(I);
↪ [1]:
↪   _[1]=x(3)+1
↪   _[2]=x(2)+(a)
↪   _[3]=x(1)+1
↪ [2]:
↪   _[1]=x(3)+1
↪   _[2]=x(2)+(a+1)
↪   _[3]=x(1)+(a+1)

```

#### D.8.1.2 PEsolve

Procedure from library `ffsolve.lib` (see [Section D.8.1 \[ffsolve.lib\], page 1821](#)).

**Usage:** PEsolve(I[, i]); I ideal, i optional integer  
 solve I (system of multivariate equations) over a  
 finite field using an equivalence property when i is  
 not given or set to 2, otherwise if i is set to 0  
 then check whether common roots exists

**Return:** list if optional parameter is not given or set to 2,  
 integer if optional is set to 0

**Assume:** basering is a finite field of type  $(p^n, a)$

**Note:** When the optional parameter is set to 0, speoff only checks if I has common roots, then return 1, otherwise return 0.

**Example:**

```
LIB "ffsolve.lib";
ring R = (2,a),x(1..3),lp;
minpoly=a2+a+1;
ideal I;
I[1]=x(1)^2*x(2)+(a)*x(1)*x(2)^2+(a+1);
I[2]=x(1)^2*x(2)*x(3)^2+(a)*x(1);
I[3]=(a+1)*x(1)*x(3)+(a+1)*x(1);
PEsolve(I);
⇒ // ** redefining I ( ideal I = defaultIdeal();) ffsolve.lib::productOfEq\
s:1001
⇒ // ** redefining new ( list new = increment(start,i);) ffsolve.lib::\
melyseg:969
⇒ // ** redefining l (int l=1;) ffsolve.lib::melyseg:970
⇒ // ** redefining new ( list new = increment(start,i);) ffsolve.lib::\
melyseg:969
⇒ // ** redefining l (int l=1;) ffsolve.lib::melyseg:970
⇒ // ** redefining new ( list new = increment(start,i);) ffsolve.lib::\
melyseg:969
⇒ // ** redefining l (int l=1;) ffsolve.lib::melyseg:970
⇒ // ** redefining new ( list new = increment(start,i);) ffsolve.lib::\
melyseg:969
⇒ // ** redefining l (int l=1;) ffsolve.lib::melyseg:970
⇒ // ** redefining new ( list new = increment(start,i);) ffsolve.lib::\
melyseg:969
⇒ // ** redefining l (int l=1;) ffsolve.lib::melyseg:970
⇒ // ** redefining new ( list new = increment(start,i);) ffsolve.lib::\
melyseg:969
⇒ // ** redefining l (int l=1;) ffsolve.lib::melyseg:970
⇒ // ** redefining new ( list new = increment(start,i);) ffsolve.lib::\
melyseg:969
⇒ // ** redefining l (int l=1;) ffsolve.lib::melyseg:970
⇒ // ** redefining new ( list new = increment(start,i);) ffsolve.lib::\
melyseg:969
⇒ // ** redefining l (int l=1;) ffsolve.lib::melyseg:970
⇒ // ** redefining res ( ideal res;) ffsolve.lib::PEsolve:273
⇒ // ** redefining new ( list new = increment(start,i);) ffsolve.lib::\
melyseg:969
⇒ // ** redefining l (int l=1;) ffsolve.lib::melyseg:970
⇒ [1]:
⇒ _[1]=x(3)+1
⇒ _[2]=x(2)+(a)
```

```

↳      _[3]=x(1)+1
↳ [2] :
↳      _[1]=x(3)+1
↳      _[2]=x(2)+(a+1)
↳      _[3]=x(1)+(a+1)

```

### D.8.1.3 simplesolver

Procedure from library `ffsolve.lib` (see [Section D.8.1 \[ffsolve.lib\]](#), page 1821).

**Usage:**        `simplesolver(I)`; I ideal  
                  solve I (system of multivariate equations) over a  
                  finite field by exhausting search

**Return:**      list L, the common roots of I as ideal

**Assume:**      basering is a finite field of type  $(p^n, a)$

**Example:**

```

LIB "ffsolve.lib";
ring R = (2,a),x(1..3),lp;
minpoly=a2+a+1;
ideal I;
I[1]=x(1)^2*x(2)+(a)*x(1)*x(2)^2+(a+1);
I[2]=x(1)^2*x(2)*x(3)^2+(a)*x(1);
I[3]=(a+1)*x(1)*x(3)+(a+1)*x(1);
simplesolver(I);
↳ [1] :
↳      _[1]=x(3)+1
↳      _[2]=x(2)+(a)
↳      _[3]=x(1)+1
↳ [2] :
↳      _[1]=x(3)+1
↳      _[2]=x(2)+(a+1)
↳      _[3]=x(1)+(a+1)

```

### D.8.1.4 GBsolve

Procedure from library `ffsolve.lib` (see [Section D.8.1 \[ffsolve.lib\]](#), page 1821).

**Usage:**        `GBsolve(I)`; I ideal  
                  solve I (system of multivariate equations) over an  
                  extension of  $\mathbb{Z}/p$  by Groebner basis methods

**Return:**      list L, the common roots of I as ideal

**Assume:**      basering is a finite field of type  $(p^n, a)$

**Example:**

```

LIB "ffsolve.lib";
ring R = (2,a),x(1..3),lp;
minpoly=a2+a+1;
ideal I;
I[1]=x(1)^2*x(2)+(a)*x(1)*x(2)^2+(a+1);
I[2]=x(1)^2*x(2)*x(3)^2+(a)*x(1);
I[3]=(a+1)*x(1)*x(3)+(a+1)*x(1);

```

```

GBsolve(I);
↳ [1]:
↳   _[1]=x(3)+1
↳   _[2]=x(2)+(a)
↳   _[3]=x(1)+1
↳ [2]:
↳   _[1]=x(3)+1
↳   _[2]=x(2)+(a+1)
↳   _[3]=x(1)+(a+1)

```

### D.8.1.5 XLsolve

Procedure from library `ffsolve.lib` (see [Section D.8.1 \[ffsolve.lib\]](#), page 1821).

**Usage:** `XLsolve(I[, d]);` I ideal, d optional integer  
 solve I (system of multivariate polynomials) with a  
 variant of the linearization technique, multiplying  
 the polynomials with monomials of degree at most d  
 (default is 2)

**Return:** list L of the common roots of I as ideals

**Assume:** basering is a finite field of type  $(p^n, a)$

**Example:**

```

LIB "ffsolve.lib";
ring R = (2,a),x(1..3),lp;
minpoly=a2+a+1;
ideal I;
I[1]=(a)*x(1)^2+x(2)^2+(a+1);
I[2]=(a)*x(1)^2+(a)*x(1)*x(3)+(a)*x(2)^2+1;
I[3]=(a)*x(1)*x(3)+1;
I[4]=x(1)^2+x(1)*x(3)+(a);
XLsolve(I, 3);
↳ [1]:
↳   _[1]=x(3)+(a+1)
↳   _[2]=x(2)+1
↳   _[3]=x(1)+1

```

### D.8.1.6 ZZsolve

Procedure from library `ffsolve.lib` (see [Section D.8.1 \[ffsolve.lib\]](#), page 1821).

**Usage:** `ZZsolve(I);` I ideal  
 solve I (system of multivariate equations) over a  
 finite field by mapping the polynomials to a single  
 univariate polynomial over extension of the basering

**Return:** list, the common roots of I as ideal

**Assume:** basering is a finite field of type  $(p^n, a)$

**Example:**

```

LIB "ffsolve.lib";
ring R = (2,a),x(1..3),lp;
minpoly=a2+a+1;

```

```

ideal I;
I[1]=x(1)^2*x(2)+(a)*x(1)*x(2)^2+(a+1);
I[2]=x(1)^2*x(2)*x(3)^2+(a)*x(1);
I[3]=(a+1)*x(1)*x(3)+(a+1)*x(1);
ZZsolve(I);
⇒ [1]:
⇒   _[1]=x(3)+1
⇒   _[2]=x(2)+(a)
⇒   _[3]=x(1)+1
⇒ [2]:
⇒   _[1]=x(3)+1
⇒   _[2]=x(2)+(a+1)
⇒   _[3]=x(1)+(a+1)

```

### D.8.2 interval\_lib

**Library:** interval.lib

**Purpose:** implements interval arithmetic on polynomials

**Authors:** Dominik Bendle  
Clara Petroll

**Overloads:**

```

// intervals
[ intervalGet indexing
= intervalAssign assigning
== intervalEqual equality
print intervalPrint pretty print
+ intervalAdd addition
- intervalNegate negation (unary)
- intervalSubtract subtraction
* intervalMultiply multiplication
/ intervalDivide division
^ intervalPotentiate potentiation

// boxes
= boxSet assigning
[ boxGet indexing
== boxEqual equality
print boxPrint printing
- boxSubtract subtraction
intersect boxIntersect intersection

// intervalmatrices
[ ivmatGet indexing
print ivmatPrint printing
nrows ivmatNrows number of rows
ncols ivmatNcols number of columns
det determinant determinant
* ivmatMultiply matrix multiplication

```

**Procedures:** See also: [Section D.8.8 \[rootisolation\\_lib\]](#), page 1890.

### D.8.2.1 length2

Procedure from library `interval.lib` (see [Section D.8.2 \[interval.lib\], page 1825](#)).

**Usage:**      `length2(I)`, `I` interval

**Return:**    `length/size` in measure sense

**Example:**

```
LIB "interval.lib";
ring R = 0,x,lp;
interval I = bounds2(0, 1);    I;
⇒ [0, 1]
⇒
length2(I);
⇒ 1
I = bounds2(-1/2, 3/7);      I;
⇒ [-1/2, 3/7]
⇒
length2(I);
⇒ 13/14
```

### D.8.2.2 bounds2

Procedure from library `interval.lib` (see [Section D.8.2 \[interval.lib\], page 1825](#)).

**Usage:**      `bounds2(a, b)`, `a`, `b` number

**Return:**    interval `[a, b]`.

### D.8.2.3 intervalmatrixInit

Procedure from library `interval.lib` (see [Section D.8.2 \[interval.lib\], page 1825](#)).

**Usage:**      `intervalmatrixInit(m, n)` `m`, `n` int

**Return:**    `mxn` matrix of `[0,0]`-intervals

**Example:**

```
LIB "interval.lib";
ring R = 0,x(1..5),lp;
ivmat A = intervalmatrixInit(4, 5); A;
⇒ [0, 0],[0, 0],[0, 0],[0, 0],[0, 0]
⇒ [0, 0],[0, 0],[0, 0],[0, 0],[0, 0]
⇒ [0, 0],[0, 0],[0, 0],[0, 0],[0, 0]
⇒ [0, 0],[0, 0],[0, 0],[0, 0],[0, 0]
⇒
```

### D.8.2.4 unitMatrix2

Procedure from library `interval.lib` (see [Section D.8.2 \[interval.lib\], page 1825](#)).

**Usage:**      `unitMatrix2(n)`

**Return:**    `nxn` unit matrix

**Example:**



```

LIB "interval.lib";
ring R = 0,x,lp;
ivmat E = unitMatrix2(4); E;
↳ [1, 1],[0, 0],[0, 0],[0, 0]
↳ [0, 0],[1, 1],[0, 0],[0, 0]
↳ [0, 0],[0, 0],[1, 1],[0, 0]
↳ [0, 0],[0, 0],[0, 0],[1, 1]
↳

```

### D.8.2.5 applyMatrix

Procedure from library `interval.lib` (see [Section D.8.2 \[interval.lib\]](#), page 1825).

**Usage:** `A * b`, `A ivmat`, `b box`

**Return:** `A*b`

**Example:**

```

LIB "interval.lib";
ring R = 0,(x,y,z),lp;
ideal I = xyz3+z2y2+x,x4+y3+2z+3,xyz+1/2;
interval J = bounds2(1/2, 3/2);
box B = list(J,J,J);
ivmat A = evalJacobianAtBox(I, B); A;
↳ [17/16, 97/16],[5/16, 189/16],[7/16, 351/16]
↳ [1/2, 27/2],[3/4, 27/4],[2, 2]
↳ [1/4, 9/4],[1/4, 9/4],[1/4, 9/4]
↳
A*B;
↳ [29/32, 1911/32],[13/8, 267/8],[3/8, 81/8]
↳
unitMatrix2(3) * B;
↳ [1/2, 3/2],[1/2, 3/2],[1/2, 3/2]
↳
diagMatrix(3, bounds2(0, 1)) * B;
↳ [0, 3/2],[0, 3/2],[0, 3/2]
↳

```

### D.8.2.6 ivmatGaussian2

Procedure from library `interval.lib` (see [Section D.8.2 \[interval.lib\]](#), page 1825).

**Usage:** `ivmatGaussian2(A)` `A ivmat`

**Return:** 0 if `A` not invertible, 1, `Ainv` if `A` invertible

**Example:**

```

LIB "interval.lib";
ring R = 0,(x,y),lp;
ideal I = 2x2-xy+2y2-2,2x2-3xy+3y2-2;
box B = list(bounds2(7/8, 9/8), bounds2(-1/10, 1/20));
ivmat J = evalJacobianAtBox (I, B); J;
↳ [69/20, 23/5],[-61/40, -27/40]
↳ [67/20, 24/5],[-159/40, -93/40]
↳
list result = ivmatGaussian2(J);

```

```

ivmat Jinv = result[2];
Jinv;
⇒ [1060/4273, 620/187],[-1220/561, -180/4273]
⇒ [2680/12819, 1280/187],[-920/187, -3680/12819]
⇒
Jinv * J;
⇒ [-7657037/799051, 12073835/799051],[-3961862/799051, 6773506/799051]
⇒ [-18293234/799051, 73172936/2397153],[-7807572/799051, 15513600/799051]
⇒
ivmat Jadj = diagMatrix(2, 1/det(J)) * adjunct(J);
Jadj;
⇒ [620/4273, 1060/187],[-1220/561, -180/4273]
⇒ [2680/12819, 1280/187],[-3680/561, -920/4273]
⇒
Jadj * J;
⇒ [-7940903/799051, 20722387/799051],[-6829045/799051, 6829045/799051]
⇒ [-24583090/799051, 24583090/799051],[-7940903/799051, 20722387/799051]
⇒

```

### D.8.2.7 evalPolyAtBox2

Procedure from library `interval.lib` (see [Section D.8.2 \[interval.lib\]](#), page 1825).

**Usage:**      `evalPolyAtBox2(f, B)`, `f` poly, `B` box

**Return:**    interval extension `ff(intervals)`

**Example:**

```

LIB "interval.lib";
ring R = 0,x,lp;
interval I1 = bounds2(0, 1); I1;
⇒ [0, 1]
⇒
poly f = x3 + 4x + 3;
evalPolyAtBox2(f, list(I1));
⇒ [3, 8]
⇒
ring S = 0,(x,y,z),lp;
interval I2 = bounds2(0, 1);
box B = list(I2, I2, I2);
poly f = xyz2 + 2x2 + (3/2)*y3x + z + 1;
evalPolyAtBox2(f, B);
⇒ [1, 13/2]
⇒

```

### D.8.2.8 exclusionTest

Procedure from library `interval.lib` (see [Section D.8.2 \[interval.lib\]](#), page 1825).

**Usage:**      exclusion test for roots with interval arithmetic

**Return:**    list of boxes

**Example:**

```

LIB "interval.lib";
ring R = 0,(x,y),lp;
ideal I = 2x2-xy+2y2-2,2x2-3xy+3y2-2; // V(I) has four elements
interval i = bounds2(-3/2,3/2);
box B = list(i, i);
list result = exclusionTest(I, B, 1/512);
size(result[1]);
↪ 0
size(result[2]);
↪ 4

```

### D.8.3 presolve\_lib

**Library:** presolve.lib

**Purpose:** Pre-Solving of Polynomial Equations

**Author:** Gert-Martin Greuel, email: greuel@mathematik.uni-kl.de,

**Procedures:**

#### D.8.3.1 degreepart

Procedure from library `presolve.lib` (see [Section D.8.3 \[presolve\\_lib\]](#), page 1829).

**Usage:** `degreepart(id,d1,d2[v]);` id=ideal/module, d1,d1=integers, v=intvec

**Return:** list of size 2,  
 \_[1]: generators of id of [v-weighted] total degree  $\geq d1$  and  $\leq d2$  (default:  $v = 1, \dots, 1$ )  
 \_[2]: remaining generators of id

**Note:** if id is of type int/number/poly it is converted to ideal, if id is of type int-mat/matrix/vector to module and then the corresponding generators are computed

**Example:**

```

LIB "presolve.lib";
ring r=0,(x,y,z),dp;
ideal i=1+x+x2+x3+x4,3,xz+y3+z8;
degreepart(i,0,4);
↪ [1]:
↪   _[1]=x4+x3+x2+x+1
↪   _[2]=3
↪ [2]:
↪   _[1]=z8+y3+xz
module m=[x,y,z],x*[x3,y2,z],[1,x2,z3,0,1];
intvec v=2,3,6;
show(degreepart(m,8,8,v));
↪ // list, 2 element(s):
↪ [1]:
↪   // module, 1 generator(s)
↪ [x4,xy2,xz]
↪ [2]:
↪   // module, 2 generator(s)
↪ [x,y,z]
↪ [1,x2,z3,0,1]

```

### D.8.3.2 elimlinearpart

Procedure from library `presolve.lib` (see [Section D.8.3 \[presolve.lib\], page 1829](#)).

**Usage:** `elimlinearpart(i[,n]);`  $i$ =ideal,  $n$ =integer,  
default:  $n$ =`nvars(basing)`

**Return:** list  $L$  with 5 entries:  
 $L[1]$ : ideal obtained from  $i$  by substituting from the first  $n$  variables those which appear in a linear part of  $i$ , by putting this part into triangular form  
 $L[2]$ : ideal of variables which have been substituted  
 $L[3]$ : ideal,  $j$ -th element defines substitution of  $j$ -th var in  $[2]$   
 $L[4]$ : ideal of variables of basering, eliminated ones are set to 0  
 $L[5]$ : ideal, describing the map from the basering to itself such that  $L[1]$  is the image of  $i$

**Note:** the procedure always interreduces the ideal  $i$  internally w.r.t. ordering `dp`.

**Example:**

```
LIB "presolve.lib";
ring s=0,(u,x,y,z),dp;
ideal i = u3+y3+z-x,x2y2+z3,y+z+1,y+u;
elimlinearpart(i);
⇒ [1]:
⇒      _[1]=z4+3z3+z2
⇒ [2]:
⇒      _[1]=u
⇒      _[2]=x
⇒      _[3]=y
⇒ [3]:
⇒      _[1]=u-z-1
⇒      _[2]=x-z
⇒      _[3]=y+z+1
⇒ [4]:
⇒      _[1]=0
⇒      _[2]=0
⇒      _[3]=0
⇒      _[4]=z
⇒ [5]:
⇒      _[1]=z+1
⇒      _[2]=z
⇒      _[3]=-z-1
⇒      _[4]=z
```

### D.8.3.3 elimpart

Procedure from library `presolve.lib` (see [Section D.8.3 \[presolve.lib\], page 1829](#)).

**Usage:** `elimpart(i [,n,e] );`  $i$ =ideal,  $n,e$ =integers  
 $n$  : only the first  $n$  vars are considered for substitution,  
 $e=0$ : substitute from linear part of  $i$  (same as `elimlinearpart`)  
 $e!=0$ : eliminate also by direct substitution  
(default:  $n = \text{nvars(basing)}$ ,  $e = 1$ )

**Return:** list of 5 objects:

- [1]: ideal obtained by substituting from the first  $n$  variables those from  $i$ , which appear in the linear part of  $i$  (or, if  $e!=0$ , which can be expressed directly in the remaining vars)
- [2]: ideal, variables which have been substituted
- [3]: ideal,  $i$ -th element defines substitution of  $i$ -th var in [2]
- [4]: ideal of variables of basering, substituted ones are set to 0
- [5]: ideal, describing the map from the basering, say  $k[x(1..m)]$ , to itself onto  $k[..\text{variables from [4]}..]$  and [1] is the image of  $i$

The ideal  $i$  is generated by [1] and [3] in  $k[x(1..m)]$ , the map [5] maps [3] to 0, hence induces an isomorphism

$$k[x(1..m)]/i \rightarrow k[..\text{variables from [4]}..]/[1]$$

**Note:** Applying `elimpart` to `interred(i)` may result in more substitutions. However, `interred` may be more expansive than `elimpart` for big ideals

**Example:**

```
LIB "presolve.lib";
ring s=0,(u,x,y,z),dp;
ideal i = xy2-xu4-x+y2,x2y2+z3+zy,y+z2+1,y+u2;
elimpart(i);
↳ [1]:
↳   _[1]=u2-z2-1
↳   _[2]=u12-u2z+z3
↳ [2]:
↳   _[1]=y
↳   _[2]=x
↳ [3]:
↳   _[1]=u2+y
↳   _[2]=-u4+x
↳ [4]:
↳   _[1]=u
↳   _[2]=0
↳   _[3]=0
↳   _[4]=z
↳ [5]:
↳   _[1]=u
↳   _[2]=u4
↳   _[3]=-u2
↳   _[4]=z
i = interred(i); i;
↳ i[1]=z2+y+1
↳ i[2]=y2-x
↳ i[3]=u2+y
↳ i[4]=x3+z3+yz
elimpart(i);
↳ [1]:
↳   _[1]=u2-z2-1
↳   _[2]=u12-u2z+z3
↳ [2]:
↳   _[1]=x
```

```

↳   _[2]=y
↳ [3]:
↳   _[1]=-y2+x
↳   _[2]=u2+y
↳ [4]:
↳   _[1]=u
↳   _[2]=0
↳   _[3]=0
↳   _[4]=z
↳ [5]:
↳   _[1]=u
↳   _[2]=u4
↳   _[3]=-u2
↳   _[4]=z
elimpart(i,2);
↳ [1]:
↳   _[1]=z2+y+1
↳   _[2]=u2+y
↳   _[3]=y6+z3+yz
↳ [2]:
↳   _[1]=x
↳ [3]:
↳   _[1]=-y2+x
↳ [4]:
↳   _[1]=u
↳   _[2]=0
↳   _[3]=y
↳   _[4]=z
↳ [5]:
↳   _[1]=u
↳   _[2]=y2
↳   _[3]=y
↳   _[4]=z

```

#### D.8.3.4 elimpartanyr

Procedure from library `presolve.lib` (see [Section D.8.3 \[presolve.lib\]](#), page 1829).

**Usage:** `elimpartanyr(i [,p,e] );` `i`=ideal, `p`=polynomial, `e`=integer  
`p`: product of vars to be eliminated,  
`e` =0: substitute from linear part of `i` (same as `elimlinearpart`)  
`e`!=0: eliminate also by direct substitution  
(default: `p`=product of all vars, `e`=1)

**Return:** list of 6 objects:

- [1]: (interreduced) ideal obtained by substituting from `i` those vars appearing in `p`, which occur in the linear part of `i` (or which can be expressed directly in the remaining variables, if `e`!=0)
- [2]: ideal, variables which have been substituted
- [3]: ideal, `i`-th element defines substitution of `i`-th var in [2]
- [4]: ideal of variables of basering, substituted ones are set to 0
- [5]: ideal, describing the map from the basering, say  $k[x(1..m)]$ , to itself onto  $k[..\text{variables from [4]}..]$  and [1] is the image of `i`
- [6]: int, # of vars considered for substitution (= # of factors of `p`)

The ideal  $i$  is generated by [1] and [3] in  $k[x(1..m)]$ , the map [5] maps [3] to 0, hence induces an isomorphism

$$k[x(1..m)]/i \rightarrow k[\text{..variables from [4]..}]/[1]$$

**Note:** the procedure creates a ring with ordering `dp` and vars placed correctly and then applies `elimpart`.

**Example:**

```
LIB "presolve.lib";
ring s=0,(x,y,z),dp;
ideal i = x3+y2+z,x2y2+z3,y+z+1;
elimpartanyr(i,z);
⇒ [1]:
⇒   _[1]=x3+y2-y-1
⇒   _[2]=x2y2-y3-3y2-3y-1
⇒ [2]:
⇒   _[1]=z
⇒ [3]:
⇒   _[1]=y+z+1
⇒ [4]:
⇒   _[1]=0
⇒   _[2]=x
⇒   _[3]=y
⇒ [5]:
⇒   _[1]=-y-1
⇒   _[2]=x
⇒   _[3]=y
⇒ [6]:
⇒   1
```

### D.8.3.5 fastelim

Procedure from library `presolve.lib` (see [Section D.8.3 \[presolve.lib\]](#), page 1829).

**Usage:** `fastelim(i,p[h,o,a,b,e,m]);`  $i$ =ideal,  $p$ =polynomial;  $h,o,a,b,e$ =integers

$p$ : product of variables to be eliminated;

Optional parameters:

- $h \neq 0$ : use Hilbert-series driven std-basis computation
- $o \neq 0$ : use proc `valvars` for a - hopefully - optimal ordering of vars
- $a \neq 0$ : order vars to be eliminated w.r.t. increasing complexity
- $b \neq 0$ : order vars not to be eliminated w.r.t. increasing complexity
- $e \neq 0$ : use `elimpart` first to eliminate easy part
- $m \neq 0$ : compute a minimal system of generators

(default:  $h,o,a,b,e,m = 0,1,0,0,0,0$ )

**Return:** ideal obtained from  $i$  by eliminating those variables, which occur in  $p$

**Example:**

```
LIB "presolve.lib";
ring s=31991,(e,f,x,y,z,t,u,v,w,a,b,c,d),dp;
ideal i = w2+f2-1, x2+t2+a2-1, y2+u2+b2-1, z2+v2+c2-1,
```

```

d2+e2-1, f4+2u, wa+tf, xy+tu+ab;
fastelim(i,xytua,1,1);          //with hilb,valvars
↳ _[1]=z2+v2+c2-1
↳ _[2]=f2+w2-1
↳ _[3]=e2+d2-1
fastelim(i,xytua,1,0,1);       //with hilb,minbase
↳ _[1]=z2+v2+c2-1
↳ _[2]=f2+w2-1
↳ _[3]=e2+d2-1

```

### D.8.3.6 findvars

Procedure from library `presolve.lib` (see [Section D.8.3 \[presolve.lib\]](#), page 1829).

**Usage:** `findvars(id);` `id=poly/ideal/vector/module/matrix`

**Return:** list L with 4 entries:

- L[1]: ideal of variables occurring in id
- L[2]: intvec of variables occurring in id
- L[3]: ideal of variables not occurring in id
- L[4]: intvec of variables not occurring in id

**Example:**

```

LIB "presolve.lib";
ring s = 0,(e,f,x,y,t,u,v,w,a,d),dp;
ideal i = w2+f2-1, x2+t2+a2-1;
findvars(i);
↳ [1]:
↳ _[1]=f
↳ _[2]=x
↳ _[3]=t
↳ _[4]=w
↳ _[5]=a
↳ [2]:
↳ 2,3,5,8,9
↳ [3]:
↳ _[1]=e
↳ _[2]=y
↳ _[3]=u
↳ _[4]=v
↳ _[5]=d
↳ [4]:
↳ 1,4,6,7,10

```

See also: [Section 5.1.166 \[variables\]](#), page 285.

### D.8.3.7 hilbvec

Procedure from library `presolve.lib` (see [Section D.8.3 \[presolve.lib\]](#), page 1829).

**Usage:** `hilbvec(id[,c,o]);` `id=poly/ideal/vector/module/matrix`, `c,o=strings`,  
`c=char`, `o=ordering` used by `hilb` (default: `c="32003"`, `o="dp"`)

**Return:** bigintvec of 1st Hilbert-series of id, computed in char c and ordering o



**Note:** id must be homogeneous (i.e. all vars have weight 1)

**Example:**

```
LIB "presolve.lib";
ring s = 0, (e,f,x,y,z,t,u,v,w,a,b,c,d,H), dp;
ideal id = w2+f2-1, x2+t2+a2-1, y2+u2+b2-1, z2+v2+c2-1,
d2+e2-1, f4+2u, wa+tf, xy+tu+ab;
id = homog(id,H);
hilbvec(id);
↦ 1,0,-7,0,20,0,-28,0,14,0,14,0,-28,0,20,0,-7,0,1,0
```

### D.8.3.8 linearpart

Procedure from library `presolve.lib` (see [Section D.8.3 \[presolve.lib\]](#), page 1829).

**Usage:** linearpart(id); id=ideal/module

**Return:** list of size 2,  
 -[1]: generators of id of total degree  $\leq 1$   
 -[2]: remaining generators of id

**Note:** all variables have degree 1 (independent of ordering of basering)

**Example:**

```
LIB "presolve.lib";
ring r=0,(x,y,z),dp;
ideal i=1+x+x2+x3,3,x+3y+5z;
linearpart(i);
↦ [1]:
↦ _[1]=3
↦ _[2]=x+3y+5z
↦ [2]:
↦ _[1]=x3+x2+x+1
module m=[x,y,z],x*[x3,y2,z],[1,x2,z3,0,1];
show(linearpart(m));
↦ // list, 2 element(s):
↦ [1]:
↦ // module, 1 generator(s)
↦ [x,y,z]
↦ [2]:
↦ // module, 2 generator(s)
↦ [x4,xy2,xz]
↦ [1,x2,z3,0,1]
```

### D.8.3.9 tolessvars

Procedure from library `presolve.lib` (see [Section D.8.3 \[presolve.lib\]](#), page 1829).

**Usage:** tolessvars(id [,s1,s2] ); id poly/ideal/vector/module/matrix, s1=string (new ordering) [default: s1="dp" or "ds" depending on whether the first block of the old ordering is a p- or an s-ordering, respectively]

**Return:** If id contains all vars of the basering: empty list.  
 Else: ring R with the same char as the basering, but possibly less variables (only those variables which actually occur in id). In R an object IMAG (image of id under imap) is stored.

**Display:** If `printlevel >= 0`, display ideal of vars, which have been omitted from the old ring.

**Example:**

```
LIB "presolve.lib";
ring r = 0,(x,y,z),dp;
ideal i = y^2-x^3,x-3,y-2x;
def R_r = tolessvars(i,"lp");
↳
↳ // variables which did not occur:
↳ z
↳
↳ // 'tolessvars' created a ring, in which an object IMAG is stored.
↳ // To access the object, type (if the name R was assigned to the return value):
↳      setring R; IMAG;
setring R_r;
show(basing);
↳ // ring: (QQ),(x,y),(lp(2),C);
↳ // minpoly = 0
↳ // objects belonging to this ring:
↳ // IMAG [0] ideal, 3 generator(s)
IMAG;
↳ IMAG[1]=-x^3+y^2
↳ IMAG[2]=x-3
↳ IMAG[3]=-2x+y
kill R_r;
```

### D.8.3.10 solvelinearpart

Procedure from library `presolve.lib` (see [Section D.8.3 \[presolve.lib\]](#), page 1829).

**Usage:** `solverlinearpart(id [,n] );` `id`=ideal/module, `n`=integer (default: `n=0`)

**Return:** (interreduced) generators of `id` of degree  $\leq 1$  in reduced triangular form if `n=0` [non-reduced triangular form if `n!=0`]

**Assume:** monomial ordering is a global ordering (p-ordering)

**Note:** may be used to solve a system of linear equations, see `gauss_row` from 'matrix.lib' for a different method

**Warning:** the result is very likely to be false for 'real' coefficients, use `char 0` instead!

**Example:**

```
LIB "presolve.lib";
// Solve the system of linear equations:
//      3x +  y +  z -  u = 2
//      3x + 8y + 6z - 7u = 1
//      14x + 10y + 6z - 7u = 0
//      7x + 4y + 3z - 3u = 3
ring r = 0,(x,y,z,u),lp;
ideal i= 3x +  y +  z -  u,
13x + 8y + 6z - 7u,
14x + 10y + 6z - 7u,
7x + 4y + 3z - 3u;
ideal j= 2,1,0,3;
```

```

j = matrix(i)-matrix(j);          // difference of 1x4 matrices
// compute reduced triangular form, setting
solvelinearpart(j);              // the RHS equal 0 gives the solutions!
⇨ _[1]=u-4
⇨ _[2]=z-4
⇨ _[3]=y+1
⇨ _[4]=x-1
solvelinearpart(j,1); "";        // triangular form, not reduced
⇨ _[1]=u-4
⇨ _[2]=2z-u-4
⇨ _[3]=11y+5z-8u+23
⇨ _[4]=3x+y+z-u-2
⇨

```

### D.8.3.11 sortandmap

Procedure from library `presolve.lib` (see [Section D.8.3 \[presolve.lib\]](#), page 1829).

**Usage:** `sortandmap(id [,n1,p1,n2,p2...,o1,m1,o2,m2...]);`  
`id=poly/ideal/vector/module,`  
`p1,p2,...= polynomials (product of variables),`  
`n1,n2,...= integers,`  
`o1,o2,...= strings,`  
`m1,m2,...= integers`  
(default: `p1=product of all vars, n1=0, o1="dp",m1=0`)  
the last `pi` (containing the remaining vars) may be omitted

**Return:** a ring `R`, in which a `poly/ideal/vector/module` `IMAG` is stored:  
- the ring `R` differs from the active basering only in the choice of monomial ordering and in the sorting of the variables.  
- `IMAG` is the image (under `imap`) of the input `ideal/module id`  
The new monomial ordering and sorting of vars is as follows:  
- each block of vars occurring in `pi` is sorted w.r.t. its complexity in `id`,  
- `ni` controls the sorting in `i`-th block (= vars occurring in `pi`):  
`ni=0` (resp. `ni!=0`) means that least complex (resp. most complex) vars come first  
- `oi` and `mi` define the monomial ordering of the `i`-th block:  
if `mi =0`, `oi=ordstr(i-th block)`  
if `mi!=0`, the ordering of the `i`-th block itself is a blockordering,  
each subblock having `ordstr=oi`, such that vars of same complexity are in one block

Note that only simple ordstrings `oi` are allowed: `"lp","dp","Dp", "ls","ds","Ds"`.

**Note:** We define a variable `x` to be more complex than `y` (with respect to `id`) if `val(x) > val(y)` lexicographically, where `val(x)` denotes the valuation vector of `x`:  
consider `id` as list of polynomials in `x` with coefficients in the remaining variables. Then:  
`val(x) = (maximal occurring power of x, # of all monomials in leading coefficient, # of all monomials in coefficient of next smaller power of x,...)`.

**Example:**

```
LIB "presolve.lib";
```

```

ring s = 32003,(x,y,z),dp;
ideal i=x3+y2,xz+z2;
def R_r=sortandmap(i);
↳
↳ // 'sortandmap' created a ring, in which an object IMAG is stored.
↳ // To access the object, type (if the name R was assigned to the return v\
    alue):
↳      setring R; IMAG;
show(R_r);
↳ // ring: (ZZ/32003),(x,y,z),(dp(3),C);
↳ // minpoly = 0
↳ // objects belonging to this ring:
↳ // IMAG [0] ideal, 2 generator(s)
setring R_r; IMAG;
↳ IMAG[1]=x3+y2
↳ IMAG[2]=xz+z2
kill R_r; setring s;
def R_r=sortandmap(i,1,xy,0,z,0,"ds",0,"lp",0);
↳
↳ // 'sortandmap' created a ring, in which an object IMAG is stored.
↳ // To access the object, type (if the name R was assigned to the return v\
    alue):
↳      setring R; IMAG;
show(R_r);
↳ // ring: (ZZ/32003),(x,y,z),(ds(2),lp(1),C);
↳ // minpoly = 0
↳ // objects belonging to this ring:
↳ // IMAG [0] ideal, 2 generator(s)
setring R_r; IMAG;
↳ IMAG[1]=y2+x3
↳ IMAG[2]=z2+xz
kill R_r;

```

### D.8.3.12 sortvars

Procedure from library `presolve.lib` (see [Section D.8.3 \[presolve.lib\]](#), page 1829).

**Usage:**     `sortvars(id[,n1,p1,n2,p2,...]);`  
               `id=poly/ideal/vector/module,`  
               `p1,p2,...= polynomials (product of vars),`  
               `n1,n2,...= integers`  
               (default: `p1=product of all vars, n1=0`)  
               the last `pi` (containing the remaining vars) may be omitted

**Compute:** sort variables with respect to their complexity in `id`

**Return:** list of two elements, an ideal and a list:

- [1]: ideal, variables of basering sorted w.r.t their complexity in `id`  
       `ni` controls the ordering in `i`-th block (= vars occurring in `pi`):  
       `ni=0` (resp. `ni!=0`) means that less (resp. more) complex vars come first
- [2]: a list with 4 entries for each `pi`:
  - [1]: ideal `ai` : vars of `pi` in correct order,
  - [2]: intvec `vi`: permutation vector describing the ordering in `ai`,
  - [3]: intmat `Mi`: valuation matrix of `ai`, the columns of `Mi` being the

valuation vectors of the vars in  $a_i$   
 $_{-}[4]$ : intvec  $w_i$ : size of 1-st, 2-nd,... block of identical columns of  $M_i$   
 (vars with same valuation)

**Note:** We define a variable  $x$  to be more complex than  $y$  (with respect to  $id$ ) if  $\text{val}(x) > \text{val}(y)$  lexicographically, where  $\text{val}(x)$  denotes the valuation vector of  $x$ :  
 consider  $id$  as list of polynomials in  $x$  with coefficients in the remaining variables. Then:  
 $\text{val}(x) = (\text{maximal occurring power of } x, \# \text{ of all monomials in leading coefficient, } \# \text{ of all monomials in coefficient of next smaller power of } x, \dots)$ .

**Example:**

```
LIB "presolve.lib";
ring s=0,(x,y,z,w),dp;
ideal i = x3+y2+yw2,xz+z2,xyz-w2;
sortvars(i,0,xy,1,zw);
↳ [1]:
↳   _[1]=x
↳   _[2]=y
↳   _[3]=z
↳   _[4]=w
↳ [2]:
↳   [1]:
↳     _[1]=y
↳     _[2]=x
↳   [2]:
↳     2,1
↳   [3]:
↳     2,3,
↳     1,1,
↳     2,0,
↳     0,2
↳   [4]:
↳     1,1
↳   [5]:
↳     _[1]=w
↳     _[2]=z
↳   [6]:
↳     2,1
↳   [7]:
↳     2,2,
↳     2,1,
↳     0,2
↳   [8]:
↳     1,1
```

### D.8.3.13 valvars

Procedure from library `presolve.lib` (see [Section D.8.3 \[presolve.lib\]](#), page 1829).

**Usage:** `valvars(id[,n1,p1,n2,p2,...]);`  
 $id$ =poly/ideal/vector/module,  
 $p1,p2,\dots$ = polynomials (product of vars),  
 $n1,n2,\dots$ = integers,

$ni$  controls the ordering of vars occurring in  $pi$ :  $ni=0$  (resp.  $ni!=0$ ) means that less (resp. more) complex vars come first (default:  $p1=$ product of all vars,  $n1=0$ ), the last  $pi$  (containing the remaining vars) may be omitted

**Compute:** valuation (complexity) of variables with respect to  $id$ .  
 $ni$  controls the ordering of vars occurring in  $pi$ :  
 $ni=0$  (resp.  $ni!=0$ ) means that less (resp. more) complex vars come first.

**Return:** list with 3 entries:  
 [1]: intvec, say  $v$ , describing the permutation such that the permuted ring variables are ordered with respect to their complexity in  $id$   
 [2]: list of intvecs,  $i$ -th intvec, say  $v(i)$  describing permutation of vars in  $a(i)$  such that  $v=v(1), v(2), \dots$   
 [3]: list of ideals and intmat's, say  $a(i)$  and  $M(i)$ , where  
 $a(i)$ : factors of  $pi$ ,  
 $M(i)$ : valuation matrix of  $a(i)$ , such that the  $j$ -th column of  $M(i)$  is the valuation vector of  $j$ -th generator of  $a(i)$

**Note:** Use `sortvars` in order to actually sort the variables! We define a variable  $x$  to be more complex than  $y$  (with respect to  $id$ ) if  $val(x) > val(y)$  lexicographically, where  $val(x)$  denotes the valuation vector of  $x$ :  
 consider  $id$  as list of polynomials in  $x$  with coefficients in the remaining variables. Then:  
 $val(x) = (\text{maximal occurring power of } x, \# \text{ of all monomials in leading coefficient, } \# \text{ of all monomials in coefficient of next smaller power of } x, \dots)$ .

**Example:**

```
LIB "presolve.lib";
ring s=0,(x,y,z,a,b),dp;
ideal i=ax2+ay3-b2x,abz+by2;
valvars (i,0,xyz);
⇒ [1]:
⇒ 1,2,3,4,5
⇒ [2]:
⇒ [1]:
⇒ 3,1,2
⇒ [2]:
⇒ 1,2
⇒ [3]:
⇒ [1]:
⇒ _[1]=x
⇒ _[2]=y
⇒ _[3]=z
⇒ [2]:
⇒ 2,3,1,
⇒ 1,1,1,
⇒ 1,1,0
⇒ [3]:
⇒ _[1]=a
⇒ _[2]=b
⇒ [4]:
⇒ 1,2,
⇒ 3,1,
⇒ 0,2
```

### D.8.3.14 idealSplit

Procedure from library `presolve.lib` (see [Section D.8.3 \[presolve.lib\]](#), page 1829).

**Usage:** `idealSplit(id,timeF,timeS);` `id` ideal and optional  
`timeF`, `timeS` integers to bound the time which can be used for factorization resp.  
 standard basis computation

**Return:** a list of ideals such that their intersection  
 has the same radical as `id`

**Example:**

```
LIB "presolve.lib";
ring r=32003,(b,s,t,u,v,w,x,y,z),dp;
ideal i=
bv+su,
bw+tu,
sw+tv,
by+sx,
bz+tx,
sz+ty,
uy+vx,
uz+wx,
vz+wy,
bvz;
idealSplit(i);
↳ [1]:
↳ _[1]=x
↳ _[2]=u
↳ _[3]=t
↳ _[4]=s
↳ _[5]=b
↳ _[6]=wy+vz
↳ [2]:
↳ _[1]=z
↳ _[2]=w
↳ _[3]=t
↳ _[4]=s
↳ _[5]=b
↳ _[6]=vx+uy
↳ [3]:
↳ _[1]=z
↳ _[2]=x
↳ _[3]=w
↳ _[4]=u
↳ _[5]=t
↳ _[6]=b
↳ [4]:
↳ _[1]=z
↳ _[2]=y
↳ _[3]=x
↳ _[4]=t
↳ _[5]=s
↳ _[6]=b
```

```

↳ [5] :
↳   _[1]=z
↳   _[2]=y
↳   _[3]=x
↳   _[4]=u
↳   _[5]=b
↳   _[6]=tv+sw
↳ [6] :
↳   _[1]=z
↳   _[2]=y
↳   _[3]=x
↳   _[4]=w
↳   _[5]=t
↳   _[6]=su+bv
↳ [7] :
↳   _[1]=w
↳   _[2]=v
↳   _[3]=u
↳   _[4]=t
↳   _[5]=s
↳   _[6]=b
↳ [8] :
↳   _[1]=x
↳   _[2]=w
↳   _[3]=v
↳   _[4]=u
↳   _[5]=b
↳   _[6]=ty+sz
↳ [9] :
↳   _[1]=z
↳   _[2]=w
↳   _[3]=v
↳   _[4]=u
↳   _[5]=t
↳   _[6]=sx+by
↳ [10] :
↳   _[1]=z
↳   _[2]=y
↳   _[3]=x
↳   _[4]=w
↳   _[5]=v
↳   _[6]=u
↳ [11] :
↳   _[1]=y
↳   _[2]=v
↳   _[3]=t
↳   _[4]=s
↳   _[5]=b
↳   _[6]=wx+uz
↳ [12] :
↳   _[1]=y
↳   _[2]=x
↳   _[3]=v

```



```

↳   _[4]=u
↳   _[5]=s
↳   _[6]=b
↳ [13]:
↳   _[1]=z
↳   _[2]=y
↳   _[3]=x
↳   _[4]=v
↳   _[5]=s
↳   _[6]=tu+bw
↳ [14]:
↳   _[1]=z
↳   _[2]=y
↳   _[3]=w
↳   _[4]=v
↳   _[5]=t
↳   _[6]=s
↳ [15]:
↳   _[1]=y
↳   _[2]=w
↳   _[3]=v
↳   _[4]=u
↳   _[5]=s
↳   _[6]=tx+bz

```

## D.8.4 solve\_lib

**Library:** solve.lib

**Purpose:** Complex Solving of Polynomial Systems

**Author:** Moritz Wenk, email: wenk@mathematik.uni-kl.de  
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**Procedures:**

### D.8.4.1 laguerre\_solve

Procedure from library `solve.lib` (see [Section D.8.4 \[solve.lib\]](#), page 1843).

**Usage:** laguerre\_solve(f [, m, l, n, s] ); f = polynomial,  
 m, l, n, s = integers (control parameters of the method)  
 m: precision of output in digits ( 4 <= m), if basering is not ring of complex numbers;  
 l: precision of internal computation in decimal digits ( l >= 8 ) only if the basering is  
 not complex or complex with smaller precision;  
 n: control of multiplicity of roots or of splitting of f into squarefree factors  
 n < 0, no split of f (good, if all roots are simple)  
 n >= 0, try to split  
 n = 0, return only different roots  
 n > 0, find all roots (with multiplicity)  
 s: s != 0, returns ERROR if | f(root) | > 0.1^m (when computing in the current ring)  
 ( default: m, l, n, s = 8, 30, 1, 0 )

**Assume:** f is a univariate polynomial;  
 basering has characteristic 0 and is either complex or without parameters.

**Return:** list of (complex) roots of the polynomial  $f$ , depending on  $n$ . The entries of the result are of type  
 string: if the basering is not complex,  
 number: otherwise.

**Note:** If `printlevel > 0`: displays comments ( default = 0 ).  
 If `s != 0` and if the procedure stops with ERROR, try a higher internal precision  $m$ .

**Example:**

```
LIB "solve.lib";
// Find all roots of an univariate polynomial using Laguerre's method:
ring rs1= 0,(x,y),lp;
poly f = 15x5 + x3 + x2 - 10;
// 10 digits precision
laguerre_solve(f,10);
⇒ [1]:
⇒ 0.8924637479
⇒ [2]:
⇒ (-0.7392783383+i*0.5355190078)
⇒ [3]:
⇒ (-0.7392783383-i*0.5355190078)
⇒ [4]:
⇒ (0.2930464644-i*0.9003002396)
⇒ [5]:
⇒ (0.2930464644+i*0.9003002396)
// Now with complex coefficients,
// internal precision is 30 digits (default)
printlevel=2;
ring rsc= (real,10,i),x,lp;
poly f = (15.4+i*5)*x^5 + (25.0e-2+i*2)*x^3 + x2 - 10*i;
list l = laguerre_solve(f);
⇒ //BEGIN laguerre_solve
⇒ //control: complex ring with precision 30
⇒ //working in: complex,10,10,i
⇒ // polynomial has complex coefficients
⇒ //split in working ring:
⇒ //split without result
⇒ //END laguerre_solve
l;
⇒ [1]:
⇒ (-0.8557376852+i*0.3557664188)
⇒ [2]:
⇒ (-0.5462895588-i*0.6796668873)
⇒ [3]:
⇒ (0.04588498039+i*0.9133296179)
⇒ [4]:
⇒ (0.5037408279-i*0.8058051828)
⇒ [5]:
⇒ (0.8524014357+i*0.2163760334)
// check result, value of substituted polynomial should be near to zero
// remember that l contains a list of strings
// in the case of a different ring
subst(f,x,l[1]);
⇒ 0
```

```
subst(f,x,l[2]);
 $\mapsto 0$ 
```

### D.8.4.2 solve

Procedure from library `solve.lib` (see [Section D.8.4 \[solve.lib\], page 1843](#)).

**Usage:** `solve(G [, m, n [, l]] [, "oldring"] [, "nodisplay"] );`  $G$  = ideal,  $m, n, l$  = integers (control parameters of the method), `outR` ring,  
 $m$ : precision of output in digits (  $4 \leq m$  ) and of the generated ring of complex numbers;  
 $n$ : control of multiplicity  
 $n = 0$ , return all different roots  
 $n \neq 0$ , find all roots (with multiplicity)  
 $l$ : precision of internal computation in decimal digits (  $l \geq 8$  ) only if the basering is not complex or complex with smaller precision,  
[default:  $(m,n,l) = (8,0,30)$ , or if only  $(m,n)$  are set explicitly with  $n \neq 0$ , then  $(m,n,l) = (m,n,60)$  ]

**Assume:** the ideal is 0-dimensional;  
basing has characteristic 0 and is either complex or without parameters;

**Return:** (1) If called without the additional parameter `"oldring"`:  
ring  $R$  with the same number of variables but with complex coefficients (and precision  $m$ ).  $R$  comes with a list `SOL` of numbers, in which complex roots of  $G$  are stored:  
\* If  $n = 0$ , `SOL` is the list of all different solutions, each of them being represented by a list of numbers.  
\* If  $n \neq 0$ , `SOL` is a list of two list: `SOL[i][1]` is the list of all different solutions with the multiplicity `SOL[i][2]`.  
`SOL` is ordered w.r.t. multiplicity (the smallest first).  
(2) If called with the additional parameter `"oldring"`, the procedure looks for an appropriate ring (at top level) in which the solutions can be stored (interactive).  
The user may then select an appropriate ring and choose a name for the output list in this ring. The list is exported directly to the selected ring and the return value is a string "result exported to" + name of the selected ring.

**Note:** If the problem is not 0-dim. the procedure stops with ERROR. If the ideal  $G$  is not a lexicographic Groebner basis, the lexicographic Groebner basis is computed internally (Hilbert driven).  
The computed solutions are displayed, unless `solve` is called with the additional parameter `"nodisplay"`.

**Example:**

```
LIB "solve.lib";
// Find all roots of a multivariate ideal using triangular sets:
int d,t,s = 4,3,2 ;
int i;
ring A=0,x(1..d),dp;
poly p=-1;
for (i=d; i>0; i--) { p=p+x(i)^s; }
ideal I = x(d)^t-x(d)^s+p;
for (i=d-1; i>0; i--) { I=x(i)^t-x(i)^s+p,I; }
I;
 $\mapsto I[1]=x(1)^3+x(2)^2+x(3)^2+x(4)^2-1$ 
```

```

⇒ I[2]=x(2)^3+x(1)^2+x(3)^2+x(4)^2-1
⇒ I[3]=x(3)^3+x(1)^2+x(2)^2+x(4)^2-1
⇒ I[4]=x(4)^3+x(1)^2+x(2)^2+x(3)^2-1
// the multiplicity is
vdim(std(I));
⇒ 81
def AC=solve(I,6,0,"nodisplay"); // solutions should not be displayed
⇒
⇒ // 'solve' created a ring, in which a list SOL of numbers (the complex so\
  lutions)
⇒ // is stored.
⇒ // To access the list of complex solutions, type (if the name R was assign\
  ed
⇒ // to the return value):
⇒      setring R; SOL;
// list of solutions is stored in AC as the list SOL (default name)
setring AC;
size(SOL); // number of different solutions
⇒ 37
SOL[5]; // the 5th solution
⇒ [1]:
⇒ 0.587401
⇒ [2]:
⇒ -0.32748
⇒ [3]:
⇒ 0.587401
⇒ [4]:
⇒ 0.587401
// you must start with char. 0
setring A;
def AC1=solve(I,6,1,"nodisplay");
⇒
⇒ // 'solve' created a ring, in which a list SOL of numbers (the complex so\
  lutions)
⇒ // is stored.
⇒ // To access the list of complex solutions, type (if the name R was assign\
  ed
⇒ // to the return value):
⇒      setring R; SOL;
setring AC1;
size(SOL); // number of different multiplicities
⇒ 2
SOL[1][1][1]; // a solution with
⇒ [1]:
⇒ (0.766044+i*0.477895)
⇒ [2]:
⇒ (0.766044+i*0.477895)
⇒ [3]:
⇒ (0.766044-i*0.477895)
⇒ [4]:
⇒ (0.766044-i*0.477895)
SOL[1][2]; // multiplicity 1
⇒ 1

```

```

SOL[2][1][1];          // a solution with
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ 1
⇒ [4]:
⇒ 0
SOL[2][2];             // multiplicity 12
⇒ 12
// the number of different solutions is equal to
size(SOL[1][1])+size(SOL[2][1]);
⇒ 37
// the number of complex solutions (counted with multiplicities) is
size(SOL[1][1])*SOL[1][2]+size(SOL[2][1])*SOL[2][2];
⇒ 81

```

### D.8.4.3 ures\_solve

Procedure from library `solve.lib` (see [Section D.8.4 \[solve.lib\]](#), page 1843).

- Usage:** `ures_solve(i [, k, p] );`  $i$  = ideal,  $k, p$  = integers  
 $k=0$ : use sparse resultant matrix of Gelfand, Kapranov and Zelevinsky,  
 $k=1$ : use resultant matrix of Macaulay which works only for homogeneous ideals,  
 $p>0$ : defines precision of the long floats for internal computation if the basering is not complex (in decimal digits),  
(default:  $k=0, p=30$ )
- Assume:**  $i$  is a zerodimensional ideal given by a quadratic system, that is,  
 $\text{nvars}(\text{basing}) = \text{ncols}(i) =$  number of vars actually occurring in  $i$ ,
- Return:** If the ground field is the field of complex numbers: list of numbers (the complex roots of the polynomial system  $i=0$ ).  
Otherwise: ring  $R$  with the same number of variables but with complex coefficients (and precision  $p$ ).  $R$  comes with a list `SOL` of numbers, in which complex roots of the polynomial system  $i$  are stored:

**Example:**

```

LIB "solve.lib";
// compute the intersection points of two curves
ring rsq = 0,(x,y),lp;
ideal gls= x2 + y2 - 10, x2 + xy + 2y2 - 16;
def R=ures_solve(gls,0,16);
⇒
⇒ // 'ures_solve' created a ring, in which a list SOL of numbers (the compl\
ex
⇒ // solutions) is stored.
⇒ // To access the list of complex solutions, type (if the name R was assign\
ed
⇒ // to the return value):
⇒      setring R; SOL;
setring R; SOL;

```

```

⇒ [1]:
⇒ [1]:
⇒ -2.82842712474619
⇒ [2]:
⇒ -1.414213562373095
⇒ [2]:
⇒ [1]:
⇒ -1
⇒ [2]:
⇒ 3
⇒ [3]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ -3
⇒ [4]:
⇒ [1]:
⇒ 2.82842712474619
⇒ [2]:
⇒ 1.414213562373095

```

#### D.8.4.4 mp\_res\_mat

Procedure from library `solve.lib` (see [Section D.8.4 \[solve.lib\]](#), page 1843).

**Usage:** `mp_res_mat(i [, k] );` `i` ideal, `k` integer,  
`k=0`: sparse resultant matrix of Gelfand, Kapranov and Zelevinsky,  
`k=1`: resultant matrix of Macaulay (`k=0` is default)

**Assume:** The number of elements in the input system must be the number of variables in the basering plus one;  
if `k=1` then `i` must be homogeneous.

**Return:** module representing the multipolynomial resultant matrix

**Example:**

```

LIB "solve.lib";
// compute resultant matrix in ring with parameters (sparse resultant matrix)
ring rsq= (0,u0,u1,u2),(x1,x2),lp;
ideal i= u0+u1*x1+u2*x2,x1^2 + x2^2 - 10,x1^2 + x1*x2 + 2*x2^2 - 16;
module m = mp_res_mat(i);
print(m);
⇒ -16,0, -10,0, (u0),0, 0, 0, 0, 0,
⇒ 0, -16,0, -10,(u2),(u0),0, 0, 0, 0,
⇒ 2, 0, 1, 0, 0, (u2),0, 0, 0, 0,
⇒ 0, 2, 0, 1, 0, 0, 0, 0, 0, 0,
⇒ 0, 0, 0, 0, (u1),0, -10,(u0),0, -16,
⇒ 1, 0, 0, 0, 0, (u1),0, (u2),(u0),0,
⇒ 0, 1, 0, 0, 0, 0, 1, 0, (u2),2,
⇒ 1, 0, 1, 0, 0, 0, 0, (u1),0, 0,
⇒ 0, 1, 0, 1, 0, 0, 0, 0, (u1),1,
⇒ 0, 0, 0, 0, 0, 0, 1, 0, 0, 1
// computing sparse resultant
det(m);
⇒ (-2*u0^4+18*u0^2*u1^2+4*u0^2*u1*u2+22*u0^2*u2^2-16*u1^4+80*u1^3*u2-52*u1^2

```

```

2*u2^2-120*u1*u2^3-36*u2^4)
// compute resultant matrix (Macaulay resultant matrix)
ring rdq= (0,u0,u1,u2),(x0,x1,x2),lp;
ideal h= homog(imap(rsq,i),x0);
h;
⇒ h[1]=(u0)*x0+(u1)*x1+(u2)*x2
⇒ h[2]=-10*x0^2+x1^2+x2^2
⇒ h[3]=-16*x0^2+x1^2+x1*x2+2*x2^2
module m = mp_res_mat(h,1);
print(m);
⇒ x0, x1, x2, 0, 0, 0, 0,0, 0, 0,
⇒ 0, x0, 0, x1,x2,0, 0,0, 0, 0,
⇒ 0, 0, x0, 0, x1,x2,0,0, 0, 0,
⇒ -10,0, 0, 1, 0, 1, 0,0, 0, 0,
⇒ 0, 0, 0, 0, x0,0, 0,x1,x2,0,
⇒ -16,0, 0, 1, 1, 2, 0,0, 0, 0,
⇒ 0, -10,0, 0, 0, 0, 1,0, 1, 0,
⇒ 0, 0, -10,0, 0, 0, 0,1, 0, 1,
⇒ 0, -16,0, 0, 0, 0, 1,1, 2, 0,
⇒ 0, 0, -16,0, 0, 0, 0,1, 1, 2
// computing Macaulay resultant (should be the same as above!)
det(m);
⇒ 2*x0^4-18*x0^2*x1^2-4*x0^2*x1*x2-22*x0^2*x2^2+16*x1^4-80*x1^3*x2+52*x1^2*\
x2^2+120*x1*x2^3+36*x2^4
// compute numerical sparse resultant matrix
setring rsq;
ideal ir= 15+2*x1+5*x2,x1^2 + x2^2 - 10,x1^2 + x1*x2 + 2*x2^2 - 16;
module mn = mp_res_mat(ir);
print(mn);
⇒ 15,0, -10,0, 0, 0, 0, -16,0, 0,
⇒ 5, 15,0, -10,0, 0, 0, 0, -16,0,
⇒ 0, 5, 1, 0, 0, 0, 0, 2, 0, 0,
⇒ 0, 0, 0, 1, 0, 0, 0, 0, 2, 0,
⇒ 2, 0, 0, 0, 15,0, -10,0, 0, -16,
⇒ 0, 2, 0, 0, 5, 15,0, 1, 0, 0,
⇒ 0, 0, 0, 0, 0, 5, 1, 0, 1, 2,
⇒ 0, 0, 1, 0, 2, 0, 0, 1, 0, 0,
⇒ 0, 0, 0, 1, 0, 2,0, 0, 1, 1,
⇒ 0, 0, 0, 0, 0, 0, 1, 0, 0, 1
// computing sparse resultant
det(mn);
⇒ -7056

```

### D.8.4.5 interpolate

Procedure from library `solve.lib` (see [Section D.8.4 \[solve.lib\]](#), page 1843).

**Usage:** `interpolate(p,v,d)`;  $p,v$ =ideals of numbers,  $d$ =integer

**Assume:** Ground field  $K$  is the field of rational numbers,  $p$  and  $v$  are lists of elements of the ground field  $K$  with  $p[j] \neq -1,0,1$ ,  $\text{size}(p) = n$  (= number of vars) and  $\text{size}(v)=N=(d+1)^n$ .

**Return:** poly  $f$ , the unique polynomial  $f$  of degree  $n*d$  with prescribed values  $v[i]$  at the points  $p(i)=(p[1]^{(i-1)},\dots,p[n]^{(i-1)})$ ,  $i=1,\dots,N$ .

**Note:** mainly useful when  $n=1$ , i.e.  $f$  is satisfying  $f(p^{(i-1)}) = v[i]$ ,  $i=1..d+1$ .

**Example:**

```
LIB "solve.lib";
ring r1 = 0,(x),lp;
// determine f with deg(f) = 4 and
// v = values of f at points 3^0, 3^1, 3^2, 3^3, 3^4
ideal v=16,0,11376,1046880,85949136;
interpolate( 3, v, 4 );
↪ 2x4-22x2+36
```

See also: [Section 5.1.164 \[vandermonde\]](#), page 284.

#### D.8.4.6 fglm\_solve

Procedure from library `solve.lib` (see [Section D.8.4 \[solve.lib\]](#), page 1843).

**Usage:** `fglm_solve(i [, p] );`  $i$  ideal,  $p$  integer

**Assume:** the ground field has char 0.

**Return:** ring  $R$  with the same number of variables but with complex coefficients (and precision  $p$ ).  $R$  comes with a list `rlist` of numbers, in which the complex roots of  $i$  are stored.  
 $p>0$ : gives precision of complex numbers in decimal digits [default:  $p=30$ ].

**Note:** The procedure uses a standard basis of  $i$  to determine all complex roots of  $i$ .

**Example:**

```
LIB "solve.lib";
ring r = 0,(x,y),dp;
// compute the intersection points of two curves
ideal s = x2 + y2 - 10, x2 + xy + 2y2 - 16;
def R = fglm_solve(s,10);
↪
↪ // 'fglm_solve' created a ring, in which a list rlist of numbers (the
↪ // complex solutions) is stored.
↪ // To access the list of complex solutions, type (if the name R was assign\
    ned
↪ // to the return value):
↪      setring R; rlist;
setring R; rlist;
↪ [1]:
↪      [1]:
↪      1
↪      [2]:
↪      -3
↪      [2]:
↪      [1]:
↪      -2.828427125
↪      [2]:
↪      -1.414213562
↪      [3]:
↪      [1]:
↪      2.828427125
↪      [2]:
↪      1.414213562
```



```

⇒ [4]:
⇒ [1]:
⇒ -1
⇒ [2]:
⇒ 3

```

#### D.8.4.7 lex\_solve

Procedure from library `solve.lib` (see [Section D.8.4 \[solve.lib\]](#), page 1843).

**Usage:** `lex_solve( i[p] );`  $i$ =ideal,  $p$ =integer,  
 $p > 0$ : gives precision of complex numbers in decimal digits (default:  $p=30$ ).

**Assume:**  $i$  is a reduced lexicographical Groebner bases of a zero-dimensional ideal, sorted by increasing leading terms.

**Return:** ring  $R$  with the same number of variables but with complex coefficients (and precision  $p$ ).  $R$  comes with a list `rlist` of numbers, in which the complex roots of  $i$  are stored.

**Example:**

```

LIB "solve.lib";
ring r = 0,(x,y),lp;
// compute the intersection points of two curves
ideal s = x2 + y2 - 10, x2 + xy + 2y2 - 16;
def R = lex_solve(stdfglm(s),10);
⇒
⇒ // 'lex_solve' created a ring, in which a list rlist of numbers (the
⇒ // complex solutions) is stored.
⇒ // To access the list of complex solutions, type (if the name R was assign\
    ned
⇒ // to the return value):
⇒      setring R; rlist;
setring R; rlist;
⇒ [1]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ -3
⇒ [2]:
⇒ [1]:
⇒ -2.828427125
⇒ [2]:
⇒ -1.414213562
⇒ [3]:
⇒ [1]:
⇒ 2.828427125
⇒ [2]:
⇒ 1.414213562
⇒ [4]:
⇒ [1]:
⇒ -1
⇒ [2]:
⇒ 3

```

### D.8.4.8 simplexOut

Procedure from library `solve.lib` (see [Section D.8.4 \[solve.lib\]](#), page 1843).

**Usage:** `simplexOut(l); l list`

**Assume:** `l` is the output of `simplex`.

**Return:** Nothing. The procedure prints the computed solution of `simplex` (as strings) in a nice format.

**Example:**

```
LIB "solve.lib";
ring r = (real,10),(x),lp;
// consider the max. problem:
//
//      maximize  x(1) + x(2) + 3*x(3) - 0.5*x(4)
//
// with constraints:  x(1) +          2*x(3)          <= 740
//                   2*x(2)          - 7*x(4) <= 0
//                   x(2) - x(3) + 2*x(4) >= 0.5
//                   x(1) + x(2) + x(3) + x(4) = 9
//
matrix sm[5][5]= 0, 1, 1, 3,-0.5,
740,-1, 0,-2, 0,
0, 0,-2, 0, 7,
0.5, 0,-1, 1,-2,
9,-1,-1,-1,-1;
int n = 4; // number of constraints
int m = 4; // number of variables
int m1= 2; // number of <= constraints
int m2= 1; // number of >= constraints
int m3= 1; // number of == constraints
list sol=simplex(sm, n, m, m1, m2, m3);
simplexOut(sol);
↪ z = 17.025
↪ x2 = 3.325
↪ x4 = 0.95
↪ x3 = 4.725
```

See also: [Section 5.1.142 \[simplex\]](#), page 261.

### D.8.4.9 triangLf\_solve

Procedure from library `solve.lib` (see [Section D.8.4 \[solve.lib\]](#), page 1843).

**Usage:** `triangLf_solve(i [, p] ); i ideal, p integer,  
p>0: gives precision of complex numbers in digits (default: p=30).`

**Assume:** the ground field has char 0; `i` is a zero-dimensional ideal

**Return:** ring `R` with the same number of variables but with complex coefficients (and precision `p`). `R` comes with a list `rlist` of numbers, in which the complex roots of `i` are stored.

**Note:** The procedure uses a triangular system (Lazard's Algorithm with factorization) computed from a standard basis to determine recursively all complex roots of the input ideal `i` with Laguerre's algorithm.

**Example:**

```

LIB "solve.lib";
ring r = 0,(x,y),lp;
// compute the intersection points of two curves
ideal s = x2 + y2 - 10, x2 + xy + 2y2 - 16;
def R = triangLf_solve(s,10);
⇒
⇒ // 'triangLf_solve' created a ring, in which a list rlist of numbers (the
⇒ // complex solutions) is stored.
⇒ // To access the list of complex solutions, type (if the name R was assign\
    ned
⇒ // to the return value):
⇒      setring R; rlist;
setring R; rlist;
⇒ [1]:
⇒      [1]:
⇒      -1
⇒      [2]:
⇒      3
⇒ [2]:
⇒      [1]:
⇒      1
⇒      [2]:
⇒      -3
⇒ [3]:
⇒      [1]:
⇒      -2.828427125
⇒      [2]:
⇒      -1.414213562
⇒ [4]:
⇒      [1]:
⇒      2.828427125
⇒      [2]:
⇒      1.414213562

```

**D.8.4.10 triangM\_solve**

Procedure from library `solve.lib` (see [Section D.8.4 \[solve.lib\]](#), page 1843).

**Usage:**     `triangM_solve(i [, p ] )`; `i`=ideal, `p`=integer,  
               `p>0`: gives precision of complex numbers in digits (default: `p=30`).

**Assume:**   the ground field has char 0;  
               `i` zero-dimensional ideal

**Return:**    ring `R` with the same number of variables but with complex coefficients (and precision `p`). `R` comes with a list `rlist` of numbers, in which the complex roots of `i` are stored.

**Note:**       The procedure uses a triangular system (Moellers Algorithm) computed from a standard basis of input ideal `i` to determine recursively all complex roots with Laguerre's algorithm.

**Example:**

```

LIB "solve.lib";
ring r = 0,(x,y),lp;

```

```

// compute the intersection points of two curves
ideal s = x2 + y2 - 10, x2 + xy + 2y2 - 16;
def R = triangM_solve(s,10);
↳
↳ // 'triangM_solve' created a ring, in which a list rlist of numbers (the
↳ // complex solutions) is stored.
↳ // To access the list of complex solutions, type (if the name R was assign\
    ned
↳ // to the return value):
↳      setring R; rlist;
setring R; rlist;
↳ [1]:
↳      [1]:
↳      1
↳      [2]:
↳      -3
↳ [2]:
↳      [1]:
↳      -2.828427125
↳      [2]:
↳      -1.414213562
↳ [3]:
↳      [1]:
↳      2.828427125
↳      [2]:
↳      1.414213562
↳ [4]:
↳      [1]:
↳      -1
↳      [2]:
↳      3

```

#### D.8.4.11 triangL\_solve

Procedure from library `solve.lib` (see [Section D.8.4 \[solve.lib\]](#), page 1843).

**Usage:** `triangL_solve(i [, p] );`  $i$ =ideal,  $p$ =integer,  
 $p > 0$ : gives precision of complex numbers in digits (default:  $p=30$ ).

**Assume:** the ground field has char 0;  $i$  is a zero-dimensional ideal.

**Return:** ring  $R$  with the same number of variables, but with complex coefficients (and precision  $p$ ).  $R$  comes with a list `rlist` of numbers, in which the complex roots of  $i$  are stored.

**Note:** The procedure uses a triangular system (Lazard's Algorithm) computed from a standard basis of input ideal  $i$  to determine recursively all complex roots with Laguerre's algorithm.

**Example:**

```

LIB "solve.lib";
ring r = 0,(x,y),lp;
// compute the intersection points of two curves
ideal s = x2 + y2 - 10, x2 + xy + 2y2 - 16;
def R = triangL_solve(s,10);
↳

```

```

⇒ // 'triangL_solve' created a ring, in which a list rlist of numbers (the
⇒ // complex solutions) is stored.
⇒ // To access the list of complex solutions, type (if the name R was assign\
    ned
⇒ // to the return value):
⇒      setring R; rlist;
setring R; rlist;
⇒ [1]:
⇒      [1]:
⇒      1
⇒      [2]:
⇒      -3
⇒ [2]:
⇒      [1]:
⇒      -2.828427125
⇒      [2]:
⇒      -1.414213562
⇒ [3]:
⇒      [1]:
⇒      2.828427125
⇒      [2]:
⇒      1.414213562
⇒ [4]:
⇒      [1]:
⇒      -1
⇒      [2]:
⇒      3

```

#### D.8.4.12 triang\_solve

Procedure from library `solve.lib` (see [Section D.8.4 \[solve.lib\]](#), page 1843).

- Usage:**     `triang_solve(l,p [,d] );`  $l$ =list,  $p,d$ =integers  
 $l$  is a list of finitely many triangular systems, such that the union of their varieties equals the variety of the initial ideal.  
 $p>0$ : gives precision of complex numbers in digits,  
 $d>0$ : gives precision ( $1<d<p$ ) for near-zero-determination,  
(default:  $d=1/2*p$ ).
- Assume:**   the ground field has char 0;  
 $l$  was computed using the algorithm of Lazard or the algorithm of Moeller (see `triang.lib`).
- Return:**    ring  $R$  with the same number of variables, but with complex coefficients (and precision  $p$ ).  $R$  comes with a list `rlist` of numbers, in which the complex roots of  $l$  are stored.

**Example:**

```

LIB "solve.lib";
ring r = 0,(x,y),lp;
// compute the intersection points of two curves
ideal s= x2 + y2 - 10, x2 + xy + 2y2 - 16;
def R=triang_solve(triangLfak(stdfglm(s)),10);
⇒

```

```

⇒ // 'triang_solve' created a ring, in which a list rlist of numbers (the
⇒ // complex solutions) is stored.
⇒ // To access the list of complex solutions, type (if the name R was assign\
    ned
⇒ // to the return value):
⇒      setring R; rlist;
setring R; rlist;
⇒ [1]:
⇒      [1]:
⇒      -1
⇒      [2]:
⇒      3
⇒ [2]:
⇒      [1]:
⇒      1
⇒      [2]:
⇒      -3
⇒ [3]:
⇒      [1]:
⇒      -2.828427125
⇒      [2]:
⇒      -1.414213562
⇒ [4]:
⇒      [1]:
⇒      2.828427125
⇒      [2]:
⇒      1.414213562

```

### D.8.5 triang\_lib

**Library:** triang.lib

**Purpose:** Decompose Zero-dimensional Ideals into Triangular Sets

**Author:** D. Hillebrand

**Procedures:**

#### D.8.5.1 triangL

Procedure from library `triang.lib` (see [Section D.8.5 \[triang\\_lib\]](#), page 1856).

**Usage:** triangL(G); G=ideal

**Assume:** G is the reduced lexicographical Groebner basis of the zero-dimensional ideal (G), sorted by increasing leading terms.

**Return:** a list of finitely many triangular systems, such that the union of their varieties equals the variety of (G).

**Note:** Algorithm of Lazard (see: Lazard, D.: Solving zero-dimensional algebraic systems, J. Symb. Comp. 13, 117 - 132, 1992).

**Example:**

```

LIB "triang.lib";
ring rC5 = 0,(e,d,c,b,a),lp;

```

```

triangL(stdfglm(cyclic(5)));
⇒ [1]:
⇒   _[1]=a5-1
⇒   _[2]=b-a
⇒   _[3]=c2+3ca+a2
⇒   _[4]=d+c+3a
⇒   _[5]=e-a
⇒ [2]:
⇒   _[1]=a5-1
⇒   _[2]=b-a
⇒   _[3]=c-a
⇒   _[4]=d2+3da+a2
⇒   _[5]=e+d+3a
⇒ [3]:
⇒   _[1]=a5-1
⇒   _[2]=b6+4b5a+5b4a2+5b3a3+5b2a4+4b+a
⇒   _[3]=5c+8b5a+30b4a2+30b3a3+25b2a4+30b+22a
⇒   _[4]=5d-2b5a-10b4a2-15b3a3-10b2a4-10b-8a
⇒   _[5]=5e-6b5a-20b4a2-15b3a3-15b2a4-15b-9a
⇒ [4]:
⇒   _[1]=a10+123a5+1
⇒   _[2]=55b2-2ba6-233ba-8a7-987a2
⇒   _[3]=55c+a6+144a
⇒   _[4]=55d+a6+144a
⇒   _[5]=55e+55b-2a6-233a

```

### D.8.5.2 triangLfak

Procedure from library `triang.lib` (see [Section D.8.5 \[triang.lib\]](#), page 1856).

**Usage:**     `triangLfak(G); G=ideal`

**Assume:**    `G` is the reduced lexicographical Groebner basis of the zero-dimensional ideal (`G`), sorted by increasing leading terms.

**Return:**     a list of finitely many triangular systems, such that the union of their varieties equals the variety of (`G`).

**Note:**       Algorithm of Lazard with factorization (see: Lazard, D.: Solving zero-dimensional algebraic systems, J. Symb. Comp. 13, 117 - 132, 1992).

**Remark:**    each polynomial of the triangular systems is factorized.

**Example:**

```

LIB "triang.lib";
ring rC5 = 0,(e,d,c,b,a),lp;
triangLfak(stdfglm(cyclic(5)));
⇒ [1]:
⇒   _[1]=a-1
⇒   _[2]=b-1
⇒   _[3]=c-1
⇒   _[4]=d2+3d+1
⇒   _[5]=e+d+3
⇒ [2]:
⇒   _[1]=a-1
⇒   _[2]=b-1

```

```

⇒  _[3]=c2+3c+1
⇒  _[4]=d+c+3
⇒  _[5]=e-1
⇒  [3]:
⇒  _[1]=a-1
⇒  _[2]=b2+3b+1
⇒  _[3]=c+b+3
⇒  _[4]=d-1
⇒  _[5]=e-1
⇒  [4]:
⇒  _[1]=a-1
⇒  _[2]=b4+b3+b2+b+1
⇒  _[3]=-c+b2
⇒  _[4]=-d+b3
⇒  _[5]=e+b3+b2+b+1
⇒  [5]:
⇒  _[1]=a2+3a+1
⇒  _[2]=b-1
⇒  _[3]=c-1
⇒  _[4]=d-1
⇒  _[5]=e+a+3
⇒  [6]:
⇒  _[1]=a2+3a+1
⇒  _[2]=b+a+3
⇒  _[3]=c-1
⇒  _[4]=d-1
⇒  _[5]=e-1
⇒  [7]:
⇒  _[1]=a4-4a3+6a2+a+1
⇒  _[2]=-11b2+6ba3-26ba2+41ba-4b-8a3+31a2-40a-24
⇒  _[3]=11c+3a3-13a2+26a-2
⇒  _[4]=11d+3a3-13a2+26a-2
⇒  _[5]=-11e-11b+6a3-26a2+41a-4
⇒  [8]:
⇒  _[1]=a4+a3+a2+a+1
⇒  _[2]=b-1
⇒  _[3]=c+a3+a2+a+1
⇒  _[4]=-d+a3
⇒  _[5]=-e+a2
⇒  [9]:
⇒  _[1]=a4+a3+a2+a+1
⇒  _[2]=b-a
⇒  _[3]=c-a
⇒  _[4]=d2+3da+a2
⇒  _[5]=e+d+3a
⇒  [10]:
⇒  _[1]=a4+a3+a2+a+1
⇒  _[2]=b-a
⇒  _[3]=c2+3ca+a2
⇒  _[4]=d+c+3a
⇒  _[5]=e-a
⇒  [11]:
⇒  _[1]=a4+a3+a2+a+1

```



```

⇒  _[2]=b3+b2a+b2+ba2+ba+b+a3+a2+a+1
⇒  _[3]=c+b2a3+b2a2+b2a+b2
⇒  _[4]=-d+b2a2+b2a+b2+ba2+ba+a2
⇒  _[5]=-e+b2a3-ba2-ba-b-a2-a
⇒  [12]:
⇒  _[1]=a4+a3+a2+a+1
⇒  _[2]=b2+3ba+a2
⇒  _[3]=c+b+3a
⇒  _[4]=d-a
⇒  _[5]=e-a
⇒  [13]:
⇒  _[1]=a4+a3+6a2-4a+1
⇒  _[2]=-11b2+6ba3+10ba2+39ba+2b+16a3+23a2+104a-24
⇒  _[3]=11c+3a3+5a2+25a+1
⇒  _[4]=11d+3a3+5a2+25a+1
⇒  _[5]=-11e-11b+6a3+10a2+39a+2

```

### D.8.5.3 triangM

Procedure from library `triang.lib` (see [Section D.8.5 \[triang.lib\]](#), page 1856).

**Usage:**      `triangM(G[i]);`  $G$ =ideal,  $i$ =integer,

**Assume:**     $G$  is the reduced lexicographical Groebner basis of the zero-dimensional ideal  $(G)$ , sorted by increasing leading terms.

**Return:**     a list of finitely many triangular systems, such that the union of their varieties equals the variety of  $(G)$ . If  $i = 2$ , then each polynomial of the triangular systems is factorized.

**Note:**       Algorithm of Moeller (see: Moeller, H.M.: On decomposing systems of polynomial equations with finitely many solutions, Appl. Algebra Eng. Commun. Comput. 4, 217 - 230, 1993).

**Example:**

```

LIB "triang.lib";
ring rC5 = 0,(e,d,c,b,a),lp;
triangM(stdfglm(cyclic(5))); //oder: triangM(stdfglm(cyclic(5)),2);
⇒  [1]:
⇒  _[1]=a5-1
⇒  _[2]=b-a
⇒  _[3]=c-a
⇒  _[4]=d2+3da+a2
⇒  _[5]=e+d+3a
⇒  [2]:
⇒  _[1]=a10+123a5+1
⇒  _[2]=55b2-2ba6-233ba-8a7-987a2
⇒  _[3]=55c+a6+144a
⇒  _[4]=55d+a6+144a
⇒  _[5]=55e+55b-2a6-233a
⇒  [3]:
⇒  _[1]=a5-1
⇒  _[2]=b6+4b5a+5b4a2+5b3a3+5b2a4+4b+a
⇒  _[3]=5c+8b5a+30b4a2+30b3a3+25b2a4+30b+22a
⇒  _[4]=5d-2b5a-10b4a2-15b3a3-10b2a4-10b-8a

```

```

⇒      _[5]=5e-6b5a-20b4a2-15b3a3-15b2a4-15b-9a
⇒ [4]:
⇒      _[1]=a5-1
⇒      _[2]=b-a
⇒      _[3]=c2+3ca+a2
⇒      _[4]=d+c+3a
⇒      _[5]=e-a

```

#### D.8.5.4 triangMH

Procedure from library `triang.lib` (see [Section D.8.5 \[triang.lib\]](#), page 1856).

**Usage:** `triangMH(G[i]);`  $G$ =ideal,  $i$ =integer

**Assume:**  $G$  is the reduced lexicographical Groebner basis of the zero-dimensional ideal  $(G)$ , sorted by increasing leading terms.

**Return:** a list of finitely many triangular systems, such that the disjoint union of their varieties equals the variety of  $(G)$ . If  $i = 2$ , then each polynomial of the triangular systems is factorized.

**Note:** Algorithm of Moeller and Hillebrand (see: Moeller, H.M.: On decomposing systems of polynomial equations with finitely many solutions, Appl. Algebra Eng. Commun. Comput. 4, 217 - 230, 1993 and Hillebrand, D.: Triangulierung nulldimensionaler Ideale - Implementierung und Vergleich zweier Algorithmen, master thesis, Universitaet Dortmund, Fachbereich Mathematik, Prof. Dr. H.M. Moeller, 1999).

**Example:**

```

LIB "triang.lib";
ring rC5 = 0, (e,d,c,b,a), lp;
triangMH(stdfglm(cyclic(5)));
⇒ [1]:
⇒      _[1]=a5-1
⇒      _[2]=b-a
⇒      _[3]=c-a
⇒      _[4]=d2+3da+a2
⇒      _[5]=e+d+3a
⇒ [2]:
⇒      _[1]=a10+123a5+1
⇒      _[2]=55b2-2ba6-233ba-8a7-987a2
⇒      _[3]=55c+a6+144a
⇒      _[4]=55d+a6+144a
⇒      _[5]=55e+55b-2a6-233a
⇒ [3]:
⇒      _[1]=a5-1
⇒      _[2]=b6+4b5a+5b4a2+5b3a3+5b2a4+4b+a
⇒      _[3]=5c+8b5a+30b4a2+30b3a3+25b2a4+30b+22a
⇒      _[4]=5d-2b5a-10b4a2-15b3a3-10b2a4-10b-8a
⇒      _[5]=5e-6b5a-20b4a2-15b3a3-15b2a4-15b-9a
⇒ [4]:
⇒      _[1]=a5-1
⇒      _[2]=b-a
⇒      _[3]=c2+3ca+a2
⇒      _[4]=d+c+3a
⇒      _[5]=e-a

```

## D.8.6 ntsolve.lib

**Library:** ntsolve.lib

**Purpose:** Real Newton Solving of Polynomial Systems

**Authors:** Wilfred Pohl, email: pohl@mathematik.uni-kl.de  
Dietmar Hillebrand

**Procedures:**

### D.8.6.1 nt\_solve

Procedure from library `ntsolve.lib` (see [Section D.8.6 \[ntsolve.lib\], page 1861](#)).

**Usage:** `nt_solve(gls,ini[,ipar]);` `gl`s, `ini`= ideals, `ipar`=list/intvec,  
`gl`s: contains the equations, for which a solution will be computed  
`ini`: ideal of initial values (approximate solutions to start with),  
`ipar`: control integers (default: `ipar = [100, 10]`)  
`ipar[1]`: max. number of iterations  
`ipar[2]`: accuracy (we have the  $l_2$ -norm  $||.||$ ): accepts solution `sol`  
if  $||gl(sol)|| < eps0 \cdot (0.1^{ipar[2]})$   
where  $eps0 = ||gl(ini)||$  is the initial error

**Assume:** `gl`s is a zerodimensional ideal with `nvars(basing) = size(gls) (>1)`

**Return:** ideal, coordinates of one solution (if found), 0 else

**Note:** if `printlevel > 0`: displays comments (default = 0)

**Example:**

```
LIB "ntsolve.lib";
ring rsq = (real,40),(x,y,z,w),lp;
ideal gls = x2+y2+z2-10, y2+z3+w-8, xy+yz+xz+w5 - 1,w3+y;
ideal ini = 3.1,2.9,1.1,0.5;
intvec ipar = 200,0;
ideal sol = nt_solve(gls,ini,ipar);
sol;
⇒ sol[1]=0.8698104581550055082008024755236550190005
⇒ sol[2]=2.821577445750324600849626251551515976097
⇒ sol[3]=1.132312008466417990006094015504381806805
⇒ sol[4]=-1.413071026406678849397999475515923744295
```

### D.8.6.2 triMNewton

Procedure from library `ntsolve.lib` (see [Section D.8.6 \[ntsolve.lib\], page 1861](#)).

**Usage:** `triMNewton(G,a[,ipar]);` `G`, `a`= ideals, `ipar`=list/intvec

**Assume:** `G`:  $g_1, \dots, g_n$ , a triangular system of  $n$  equations in  $n$  vars, i.e.  $g_i = g_i(\text{var}(n-i+1), \dots, \text{var}(n))$ ,  
`a`: ideal of numbers, coordinates of an approximation of a common zero of `G` to start  
with (with `a[i]` to be substituted in `var(i)`),  
`ipar`: control integer vector (default: `ipar = [100, 10]`)  
`ipar[1]`: max. number of iterations  
`ipar[2]`: accuracy (we have as norm  $||.||$  absolute value ):  
accepts solution `sol` if  $|G(sol)| < |G(a)| \cdot (0.1^{ipar[2]})$ .

**Return:** an ideal, coordinates of a better approximation of a zero of  $G$

**Example:**

```
LIB "ntsolve.lib";
ring r = (real,30),(z,y,x),(lp);
ideal i = x^2-1,y^2+x^4-3,z^2-y^4+x-1;
ideal a = 2,3,4;
intvec e = 20,10;
ideal l = trimNewton(i,a,e);
l;
⇒ l[1]==-2.000000000004226573888027914342
⇒ l[2]=1.41421356237309504880168872421
⇒ l[3]=1
```

### D.8.7 recover.lib

**Library:** recover.lib

**Purpose:** Hybrid numerical/symbolical algorithms for algebraic geometry

**Author:** Adrian Koch (kocha at rhrk.uni-kl.de)

**Overview:** In this library you'll find implementations of some of the algorithms presented in the paper listed below: Bertini is used to compute a witness set of a given ideal  $I$ . Then a lattice basis reduction algorithm is used to recover exact results from the inexact numerical data. More precisely, we obtain elements of prime components of  $I$ , the radical of  $I$ , or an elimination ideal of  $I$ .

NOTE that Bertini may create quite a lot of files in the current directory (or overwrite files which have the same names as the files it wants to create). It also prints information to the screen.

The usefulness of the results of the exactness recovery algorithms heavily depends on the quality of the witness set and the quality of the lattice basis reduction algorithm. The procedures requiring a witness set as part of their input use a simple, unsophisticated version of the LLL algorithm.

**References:**

Daniel Bates, Jonathan Hauenstein, Timothy McCoy, Chris Peterson, and Andrew Sommese; Recovering exact results from inexact numerical data in algebraic geometry; Published in Experimental Mathematics 22(1) on pages 38-50 in 2013

**Procedures:**

#### D.8.7.1 substAll

Procedure from library `recover.lib` (see [Section D.8.7 \[recover.lib\]](#), page 1862).

**Usage:** `substAll(v,p);` poly  $v$ , list  $p$

**Return:** poly: the polynomial obtained from  $v$  by substituting the elements of  $p$  for the ring variables

**Note:** The list  $p$  should have as many elements as there are ring variables.

**Example:**

```

LIB "recover.lib";
ring r=0,(x,y,z),dp;
poly v=x+y+z;
list p=7/11,5/11,-1/11;
poly f=substAll(v,p);
f;
↪ 1

```

### D.8.7.2 veronese

Procedure from library `recover.lib` (see [Section D.8.7 \[recover.lib\]](#), page 1862).

**Usage:**      `veronese(d,p);` int d, list p

**Return:**     ideal: the image of the point p under the degree d Veronese embedding

**Note:**        The list p should have as many elements as there are ring variables. The order of the points in the returned ideal corresponds to the order of the monomials in `maxideal(d)`.

**Example:**

```

LIB "recover.lib";
ring R=0,(x,y,z),dp;
list p=2,3,5;
ideal V=veronese(1,p);
V;
↪ V[1]=2
↪ V[2]=3
↪ V[3]=5
V=veronese(2,p);
V;
↪ V[1]=25
↪ V[2]=15
↪ V[3]=9
↪ V[4]=10
↪ V[5]=6
↪ V[6]=4

```

See also: [Section 5.1.88 \[maxideal\]](#), page 219.

### D.8.7.3 getRelations

Procedure from library `recover.lib` (see [Section D.8.7 \[recover.lib\]](#), page 1862).

**Usage:**      `getRelations(p,D,C);` list p, int D, bigint C

**Return:**     list K: a list of ideals; the ideals contain homogeneous polynomial relations of degree  $\leq D$  between the components of the point p

**Note:**        This procedure uses only the images of the one point p under the Veronese embeddings to find homogeneous polynomial relations.

**Example:**

```

LIB "recover.lib";
ring r=(complex,50),(x,y,z),dp;
list p=1,-1,0.5;
getRelations(p,2,10000);
↪ [1]:

```

```

⇒  _[1]=x+y
⇒  _[2]=x-2*z
⇒  [2]:
⇒  _[1]=xz+yz
⇒  _[2]=xy+y2
⇒  _[3]=x2-y2
⇒  _[4]=-xy-xz+yz
⇒  _[5]=-xz+2*z2

```

See also: [Section D.8.7.4 \[getRelationsRadical\]](#), page 1864.

### D.8.7.4 getRelationsRadical

Procedure from library `recover.lib` (see [Section D.8.7 \[recover\\_lib\]](#), page 1862).

**Usage:** `getRelationsRadical(P,D,C)`; list P, int D, bigint C

**Return:** list K: a list of ideals; the ideals contain homogeneous polynomial relations of degree  $\leq D$  between the components of the points in P

**Note:** This procedure uses random linear combination of the Veronese embeddings of all points in P to find homogeneous polynomial relations.

**Example:**

```

LIB "recover.lib";
ring r=(complex,50),(x,y,z),dp;
list p1=1,-1,0.5;
list p2=1,0,-1;
list P=list(p1)+list(p2);
getRelationsRadical(P,2,10**5);
⇒ [1]:
⇒  _[1]=-2*x2+2*y2+yz+2*z2

```

See also: [Section D.8.7.3 \[getRelations\]](#), page 1863.

### D.8.7.5 gaussRowWithoutPerm

Procedure from library `recover.lib` (see [Section D.8.7 \[recover\\_lib\]](#), page 1862).

**Usage:** `gaussRowWithoutPerm(M)`; M a matrix of constant polynomials

**Return:** matrix: basic Gaussian row reduction of M, just without permuting the rows

**Example:**

```

LIB "recover.lib";
ring r=0,x,dp;
matrix M[5][4]=0,0,2,1,4,5,1,3,0,9,2,0,8,1,0,6,0,9,4,1;
print(M);
⇒ 0,0,2,1,
⇒ 4,5,1,3,
⇒ 0,9,2,0,
⇒ 8,1,0,6,
⇒ 0,9,4,1
print(gaussRowWithoutPerm(M));
⇒ 0,0,2,1,
⇒ 4,5,1,3,
⇒ 0,9,2,0,
⇒ 0,0,0,0,
⇒ 0,0,0,0

```

### D.8.7.6 gaussColWithoutPerm

Procedure from library `recover.lib` (see [Section D.8.7 \[recover.lib\], page 1862](#)).

**Usage:** `gaussColWithoutPerm(M)`; M a matrix of constant polynomials

**Return:** matrix: basic Gaussian column reduction of M, just without permuting the columns

**Example:**

```
LIB "recover.lib";
ring r=0,x,dp;
matrix M[3][4]=0,1,0,2,1,2,3,4,1,0,5,0;
print(M);
  ↳ 0,1,0,2,
  ↳ 1,2,3,4,
  ↳ 1,0,5,0
print(gaussColWithoutPerm(M));
  ↳ 0,1,0,0,
  ↳ 1,2,0,0,
  ↳ 1,0,2,0
```

### D.8.7.7 getWitnessSet

Procedure from library `recover.lib` (see [Section D.8.7 \[recover.lib\], page 1862](#)).

**Usage:** `getWitnessSet()`;

**Assume:** There is a text-document "main\_data" in the current directory which was produced by Bertini.  
The basefield is the field of real numbers or the field of complex numbers.

**Return:** list; a list P of lists p.i of numbers: P a set of witness points

**Note:** Reads the file "main\_data", searches the strings containing the witness points, and converts them into floating point numbers.

**Example:**

```
LIB "recover.lib";
//First, we write the input file for bertini, then run bertini
ring r=0,(x,y,z),dp;
ideal I=(x-y)*(y-z)*(x-z);
writeBertiniInput(I,40);
system("sh","bertini input");
  ↳
  ↳ Bertini(TM) v1.6
  ↳ (May 22, 2018)
  ↳
  ↳ D.J. Bates, J.D. Hauenstein,
  ↳ A.J. Sommese, C.W. Wampler
  ↳
  ↳ (using GMP v6.0.0, MPFR v3.1.2)
  ↳
  ↳
  ↳
  ↳ NOTE: You have requested to use adaptive path tracking. Please make sure\
    that you have
  ↳ setup the following tolerances appropriately:
```

```

⇒ CoeffBound: 6.000000000000e+00, DegreeBound: 3.000000000000e+00
⇒ AMPSafetyDigits1: 1, AMPSafetyDigits2: 1, AMPMaxPrec: 160
⇒
⇒
⇒ Tracking regeneration codim 1 of 1: 3 paths to track.
⇒ Tracking path 0 of 3
⇒ Tracking path 1 of 3
⇒ Tracking path 2 of 3
⇒
⇒ Sorting codimension 1 of 1: 3 paths to sort.
⇒ Sorting 0 of 3
⇒ Sorting 1 of 3
⇒ Sorting 2 of 3
⇒
⇒
⇒ ***** Regenerative Cascade Summary *****
⇒
⇒ NOTE: nonsingular vs singular is based on rank deficiency and identical e\
ndpoints
⇒
⇒ |codim|   paths   |witness superset| nonsingular | singular |nonsolutions\
  | inf endpoints | other bad endpoints
⇒ -----
⇒ | 1   |   3     | 3           | 3           | 0         | 0         | \
  | 0     |   0     |
⇒ -----
⇒ |total|   3
⇒
⇒ *****
⇒
⇒
⇒ ***** Witness Set Summary *****
⇒
⇒ NOTE: nonsingular vs singular is based on rank deficiency and identical e\
ndpoints
⇒
⇒ |codim| witness points | nonsingular | singular
⇒ -----
⇒ | 1   |   3           | 3           | 0
⇒ -----
⇒
⇒ *****
⇒
⇒
⇒ Calculating traces for codimension 1.
⇒ Calculating 0 of 3
⇒ Calculating 1 of 3
⇒ Calculating 2 of 3
⇒
⇒ Using combinatorial trace test to decompose codimension 1.
⇒
⇒

```



```

⇒ ***** Witness Set Decomposition *****
⇒
⇒ | dimension | components | classified | unclassified
⇒ -----
⇒ |    1      |    0      |    0      |    3
⇒ -----
⇒
⇒ ***** Decomposition by Degree *****
⇒
⇒ *****
⇒
⇒ 0
//Then we change the ring and extract the witness set from main_data
ring R=(complex,40,i),(x,y,z),dp;
list P=getWitnessSet();
P;
⇒ [1]:
⇒   [1]:
⇒ (-0.6057243693763992319979984307996615526734-i*1.709982652953457070889449\
922086668170706)
⇒   [2]:
⇒ (-1.269607021784058783195617483929885915314-i*0.5475380639917566253239678\
662144421820642)
⇒   [3]:
⇒ (-0.6057243693763992319979984307996615526734-i*1.709982652953457070889449\
922086668170706)
⇒ [2]:
⇒   [1]:
⇒ (-0.638848264983581466632640880998032211225-i*0.4061431632210439417975087\
265709479968559)
⇒   [2]:
⇒ (-0.638848264983581466632640880998032211225-i*0.4061431632210439417975087\
265709479968559)
⇒   [3]:
⇒ (-0.387249893894885084292184854385615253034-i*0.8722070287327304669824940\
317116046090152)
⇒ [3]:
⇒   [1]:
⇒ (-0.6267800014172102445712156474049935144712+i*0.447359782835199644529317\
2364490495294033)
⇒   [2]:
⇒ (-0.2227056608625241971680237404179802544887-i*0.329995341310052437519469\
1540457834984593)
⇒   [3]:
⇒ (-0.2227056608625241971680237404179802544887-i*0.329995341310052437519469\
1540457834984593)

```

### D.8.7.8 writeBertiniInput

Procedure from library `recover.lib` (see [Section D.8.7 \[recover.lib\]](#), page 1862).

**Usage:**     `writeBertiniInput(J);` ideal J

**Return:**    none; writes the input-file for bertini using the polynomials given by J as functions

**Note:** Either creates a file named `input` in the current directory or overwrites the existing one.  
If you want to pass different parameters to `bertini`, you can edit the produced `input` file or redefine this procedure.

**Example:**

```
LIB "recover.lib";
ring r=0,(x,y,z),dp;
poly f1=x+y+z;
poly f2=x2+xy+y2;
ideal I=f1,f2;
writeBertiniInput(I,300);
```

### D.8.7.9 num\_prime\_decom

Procedure from library `recover.lib` (see [Section D.8.7 \[recover.lib\]](#), page 1862).

**Usage:** `num_prime_decom(I,D);` ideal `I`, int `D`  
`D` a bound to the degree of the elements of the components of a prime decomposition of `I`.

**Return:** list of ideals: each of the ideals a prime component of the radical of `I`

**Remarks:** Uses Bertini.

**Note:** Should only be called from a ring over the rational numbers.

**Example:**

```
LIB "recover.lib";
ring R=0,(x,y,z),dp;
ideal I=(x+y)*(y+2z), (x+y)*(x-3z);
int D=2;
int Prec=300;
num_prime_decom(I,D,Prec);
⇒
⇒ Bertini(TM) v1.6
⇒ (May 22, 2018)
⇒
⇒ D.J. Bates, J.D. Hauenstein,
⇒ A.J. Sommese, C.W. Wampler
⇒
⇒ (using GMP v6.0.0, MPFR v3.1.2)
⇒
⇒
⇒ NOTE: You have requested to use adaptive path tracking. Please make sure\
that you have
⇒ setup the following tolerances appropriately:
⇒ CoeffBound: 8.000000000000e+00, DegreeBound: 2.000000000000e+00
⇒ AMPSafetyDigits1: 1, AMPSafetyDigits2: 1, AMPMaxPrec: 1024
⇒
⇒
⇒ Tracking regeneration codim 1 of 2: 2 paths to track.
⇒ Tracking path 0 of 2
⇒ Tracking path 1 of 2
```

```

⇒
⇒ Sorting codimension 1 of 2: 2 paths to sort.
⇒ Sorting 0 of 2
⇒ Sorting 1 of 2
⇒
⇒ Preparing regeneration codim 2 of 2: 1 witness point to move.
⇒ Moving 0 of 1
⇒
⇒ Tracking regeneration codim 2 of 2: 2 paths to track.
⇒ Tracking path 0 of 2
⇒ Tracking path 1 of 2
⇒
⇒ Sorting codimension 2 of 2: 2 paths to sort.
⇒ Sorting 0 of 2
⇒ Sorting 1 of 2
⇒
⇒
⇒ ***** Regenerative Cascade Summary *****
⇒
⇒ NOTE: nonsingular vs singular is based on rank deficiency and identical e\
ndpoints
⇒
⇒ |codim|   paths   |witness superset| nonsingular | singular |nonsolutions\
  | inf endpoints | other bad endpoints
⇒ -----
⇒ | 1 |   2   | 1           | 1           | 0           | 1           | \
  | 0           | 0
⇒ | 2 |   2   | 2           | 1           | 1           | 0           | \
  | 0           | 0
⇒ -----
⇒ |total|   4
⇒
⇒ *****
⇒
⇒
⇒ Removing junk points from codimension 2: 1 endpoints to check.
⇒ Checking 0 of 1
⇒
⇒
⇒ ***** Witness Set Summary *****
⇒
⇒ NOTE: nonsingular vs singular is based on rank deficiency and identical e\
ndpoints
⇒
⇒ |codim| witness points | nonsingular | singular
⇒ -----
⇒ | 1 |   1   | 1           | 0
⇒ | 2 |   1   | 1           | 0
⇒ -----
⇒
⇒ *****
⇒
⇒

```

```

⇒ Calculating traces for codimension 1.
⇒ Calculating 0 of 1
⇒
⇒ Using combinatorial trace test to decompose codimension 1.
⇒
⇒ Calculating traces for codimension 2.
⇒ Calculating 0 of 1
⇒
⇒
⇒ ***** Witness Set Decomposition *****
⇒
⇒ | dimension | components | classified | unclassified
⇒ -----
⇒ |    1      |    0      |    0      |    1
⇒ |    0      |    1      |    1      |    0
⇒ -----
⇒
⇒ ***** Decomposition by Degree *****
⇒
⇒ Dimension 0: 1 classified component
⇒ -----
⇒     degree 1: 1 component
⇒
⇒ *****
⇒
⇒ 0
⇒ empty list
//Let us compare that to the result of primdecSY:
primdecSY(I);
⇒ [1]:
⇒   [1]:
⇒   _[1]=x+y
⇒   [2]:
⇒   _[1]=x+y
⇒ [2]:
⇒   [1]:
⇒   _[1]=y+2z
⇒   _[2]=x-3z
⇒   [2]:
⇒   _[1]=y+2z
⇒   _[2]=x-3z

```

#### D.8.7.10 num\_prime\_decom1

Procedure from library `recover.lib` (see [Section D.8.7 \[recover.lib\]](#), page 1862).

**Usage:**     `num_prime_decom1(P,D,C)`; list P, int D, bigint C  
P a list of lists representing a witness point set representing an ideal I D should be a bound to the degree of the elements of the components of the prime decomposition of I  
C the number with which the images of the Veronese embeddings are multiplied

**Return:**    list of ideals: each of the ideals a prime component of the radical of I

**Note:**     Should only be called from a ring over the complex numbers.

**Example:**

```

LIB "recover.lib";
//First, we compute a prime decomposition of the ideal I=x+y;
ring R1=(complex,300,IUnit),(x,y),dp;
list p1=1,-1;
list P=list(p1);
int D=2;
bigint C=bigint(10)**300;
num_prime_decom1(P,D,C);
⇒ [1]:
⇒   _[1]=x+y
//Now, we try to obtain a prime decomposition of the ideal I=(x+y)*(y+2z), (x+y)*(x-3z);
ring R2=(complex,20,IUnit),(x,y,z),dp;
p1=1.7381623928,-1.7381623928,0.2819238763;
list p2=-3.578512854,2.385675236,-1.192837618;
P=p1,p2;
num_prime_decom1(P,D,10000);
⇒ [1]:
⇒   _[1]=x+y
⇒ [2]:
⇒   _[1]=-2*x-3*y
⇒   _[2]=-y^2+x*z+z^2
⇒   _[3]=x*z+y*z-z^2
//Now, we look at the result of a purely symbolic algorithm
ring r2=0,(x,y,z),dp;
ideal I=(x+y)*(y+2z), (x+y)*(x-3z);
primdecSY(I);
⇒ [1]:
⇒   [1]:
⇒   _[1]=x+y
⇒   [2]:
⇒   _[1]=x+y
⇒ [2]:
⇒   [1]:
⇒   _[1]=y+2z
⇒   _[2]=x-3z
⇒   [2]:
⇒   _[1]=y+2z
⇒   _[2]=x-3z
//If you compare the results, you may find that they don't match.
//Most likely, the hybrid algorithm got the second component wrong. This is due to the
//way the algorithm looks for homogeneous polynomial relations, and the specific version
//of the LLL algorithm used here (an implementation into Singular of a rather simple
//version which allows real input). It looks in degree 1, finds one relation and is
//thereafter unable to see a second one. Then it moves on to degree 2 and finds
//relations containing degree-1 relations as a factor.

```

**D.8.7.11 num\_radical\_via\_decom**

Procedure from library `recover.lib` (see [Section D.8.7 \[recover.lib\]](#), page 1862).

**Usage:**     `num_radical_via_decom(I,D);` ideal `I`, int `D`  
              `D` a bound to the degree of the elements of the components.

**Return:** ideal: the radical of I

**Remarks:** Uses Bertini.

This procedure merely calls num\_prime\_decom with the same input and then intersects the returned components.

**Note:** Should only be called from a ring over the rational numbers.

**Example:**

```
LIB "recover.lib";
//First, we attempt to compute the radical via the hybrid algorithm.
ring R=0,(x,y,z),dp;
ideal I=(x+y)^2*(y+2z)^3, (x+y)^3*(x-3z)^2;
int D=2;
int Prec=300;
ideal numRad=num_radical_via_decom(I,D,Prec);
⇒
⇒ Bertini(TM) v1.6
⇒ (May 22, 2018)
⇒
⇒ D.J. Bates, J.D. Hauenstein,
⇒ A.J. Sommese, C.W. Wampler
⇒
⇒ (using GMP v6.0.0, MPFR v3.1.2)
⇒
⇒
⇒ NOTE: You have requested to use adaptive path tracking. Please make sure\
that you have
⇒ setup the following tolerances appropriately:
⇒ CoeffBound: 1.280000000000e+02, DegreeBound: 5.000000000000e+00
⇒ AMPSafetyDigits1: 1, AMPSafetyDigits2: 1, AMPMaxPrec: 1024
⇒
⇒
⇒ Tracking regeneration codim 1 of 2: 5 paths to track.
⇒ Tracking path 0 of 5
⇒ Tracking path 1 of 5
⇒ Tracking path 2 of 5
⇒ Tracking path 3 of 5
⇒ Tracking path 4 of 5
⇒
⇒ Sorting codimension 1 of 2: 5 paths to sort.
⇒ Sorting 0 of 5
⇒ Sorting 1 of 5
⇒ Sorting 2 of 5
⇒ Sorting 3 of 5
⇒ Sorting 4 of 5
⇒
⇒ Preparing regeneration codim 2 of 2: 12 witness points to move.
⇒ Moving 0 of 12
⇒ Moving 1 of 12
⇒ Moving 2 of 12
⇒ Moving 3 of 12
⇒ Moving 4 of 12
```

```

⇒ Moving 5 of 12
⇒ Moving 6 of 12
⇒ Moving 7 of 12
⇒ Moving 8 of 12
⇒ Moving 9 of 12
⇒ Moving 10 of 12
⇒ Moving 11 of 12
⇒
⇒ Tracking regeneration codim 2 of 2: 15 paths to track.
⇒ Tracking path 0 of 15
⇒ Tracking path 1 of 15
⇒ Tracking path 2 of 15
⇒ Tracking path 3 of 15
⇒ Tracking path 4 of 15
⇒ Tracking path 5 of 15
⇒ Tracking path 6 of 15
⇒ Tracking path 7 of 15
⇒ Tracking path 8 of 15
⇒ Tracking path 9 of 15
⇒ Tracking path 10 of 15
⇒ Tracking path 11 of 15
⇒ Tracking path 12 of 15
⇒ Tracking path 13 of 15
⇒ Tracking path 14 of 15
⇒
⇒ Sorting codimension 2 of 2: 15 paths to sort.
⇒ Sorting 0 of 15
⇒ Sorting 1 of 15
⇒ Sorting 2 of 15
⇒ Sorting 3 of 15
⇒ Sorting 4 of 15
⇒ Sorting 5 of 15
⇒ Sorting 6 of 15
⇒ Sorting 7 of 15
⇒ Sorting 8 of 15
⇒ Sorting 9 of 15
⇒ Sorting 10 of 15
⇒ Sorting 11 of 15
⇒ Sorting 12 of 15
⇒ Sorting 13 of 15
⇒ Sorting 14 of 15
⇒
⇒
⇒ ***** Regenerative Cascade Summary *****
⇒
⇒ NOTE: nonsingular vs singular is based on rank deficiency and identical e\
    ndpoints
⇒
⇒ |codim|   paths   |witness superset| nonsingular | singular |nonsolutions\
    | inf endpoints | other bad endpoints
⇒ -----
⇒ | 1   |   5   | 2           | 0           | 2           | 3           | \
    | 0           | 0

```

```

⇒ | 2 | 15 | 15 | 0 | 15 | 0 \
  | 0 | 0 |
⇒ -----
⇒ |total| 20
⇒
⇒ *****
⇒
⇒
⇒ Removing junk points from codimension 2: 2 endpoints to check.
⇒ Checking 0 of 2
⇒ Checking 1 of 2
⇒
⇒
⇒ ***** Witness Set Summary *****
⇒
⇒ NOTE: nonsingular vs singular is based on rank deficiency and identical e\
  ndpoints
⇒
⇒ |codim| witness points | nonsingular | singular
⇒ -----
⇒ | 1 | 1 | 0 | 1
⇒ | 2 | 1 | 0 | 1
⇒ -----
⇒
⇒ *****
⇒
⇒
⇒ Calculating traces for codimension 1.
⇒ Calculating 0 of 1
⇒
⇒ Using combinatorial trace test to decompose codimension 1.
⇒
⇒ Calculating traces for codimension 2.
⇒ Calculating 0 of 1
⇒
⇒
⇒ ***** Witness Set Decomposition *****
⇒
⇒ | dimension | components | classified | unclassified
⇒ -----
⇒ | 1 | 0 | 0 | 1
⇒ | 0 | 1 | 1 | 0
⇒ -----
⇒
⇒ ***** Decomposition by Degree *****
⇒
⇒ Dimension 0: 1 classified component
⇒ -----
⇒ degree 1: 1 component
⇒
⇒ *****
⇒
⇒ Witness Points Deflated: 2

```



```

⇒ 0
numRad;
⇒ numRad[1]=1
//Then we compute the radical symbolically and compare the results.
ideal Rad=radical(I);
Rad;
⇒ Rad[1]=xy+y2+2xz+2yz
⇒ Rad[2]=x2-y2-5xz-5yz
reduce(Rad,std(numRad));
⇒ _[1]=0
⇒ _[2]=0
reduce(numRad,std(Rad));
⇒ _[1]=1

```

See also: [Section D.8.7.9 \[num\\_prime\\_decom\]](#), page 1868; [Section D.8.7.12 \[num\\_radical\\_via\\_randlincom\]](#), page 1875.

### D.8.7.12 num\_radical\_via\_randlincom

Procedure from library `recover.lib` (see [Section D.8.7 \[recover\\_lib\]](#), page 1862).

**Usage:**     `num_radical_via_randlincom(I,D)`; ideal `I`, int `D`  
               `D` a bound to the degree of the elements of the components.

**Return:**    ideal: the radical of `I`

**Remarks:**   Uses Bertini.  
               Instead of using the images of the Veronese embeddings of each individual witness point, this procedure first computes a random linear combination of those images and searches for homogeneous polynomial relations for this linear combination.

**Note:**       Should only be called from a ring over the rational numbers.

**Example:**

```

LIB "recover.lib";
//First, we attempt to compute the radical via the hybrid algorithm.
ring R=0,(x,y,z),dp;
ideal I=(x+y)^2*(y+2z)^3, (x+y)^3*(x-3z)^2;
int D=2;
int Prec=300;
ideal numRad=num_radical_via_randlincom(I,D,Prec);
⇒
⇒ Bertini(TM) v1.6
⇒ (May 22, 2018)
⇒
⇒ D.J. Bates, J.D. Hauenstein,
⇒ A.J. Sommese, C.W. Wampler
⇒
⇒ (using GMP v6.0.0, MPFR v3.1.2)
⇒
⇒
⇒ NOTE: You have requested to use adaptive path tracking. Please make sure\
that you have
⇒ setup the following tolerances appropriately:
⇒ CoeffBound: 1.2800000000000e+02, DegreeBound: 5.0000000000000e+00

```

```

⇒ AMPSafetyDigits1: 1, AMPSafetyDigits2: 1, AMPMaxPrec: 1024
⇒
⇒
⇒ Tracking regeneration codim 1 of 2: 5 paths to track.
⇒ Tracking path 0 of 5
⇒ Tracking path 1 of 5
⇒ Tracking path 2 of 5
⇒ Tracking path 3 of 5
⇒ Tracking path 4 of 5
⇒
⇒ Sorting codimension 1 of 2: 5 paths to sort.
⇒ Sorting 0 of 5
⇒ Sorting 1 of 5
⇒ Sorting 2 of 5
⇒ Sorting 3 of 5
⇒ Sorting 4 of 5
⇒
⇒ Preparing regeneration codim 2 of 2: 12 witness points to move.
⇒ Moving 0 of 12
⇒ Moving 1 of 12
⇒ Moving 2 of 12
⇒ Moving 3 of 12
⇒ Moving 4 of 12
⇒ Moving 5 of 12
⇒ Moving 6 of 12
⇒ Moving 7 of 12
⇒ Moving 8 of 12
⇒ Moving 9 of 12
⇒ Moving 10 of 12
⇒ Moving 11 of 12
⇒
⇒ Tracking regeneration codim 2 of 2: 15 paths to track.
⇒ Tracking path 0 of 15
⇒ Tracking path 1 of 15
⇒ Tracking path 2 of 15
⇒ Tracking path 3 of 15
⇒ Tracking path 4 of 15
⇒ Tracking path 5 of 15
⇒ Tracking path 6 of 15
⇒ Tracking path 7 of 15
⇒ Tracking path 8 of 15
⇒ Tracking path 9 of 15
⇒ Tracking path 10 of 15
⇒ Tracking path 11 of 15
⇒ Tracking path 12 of 15
⇒ Tracking path 13 of 15
⇒ Tracking path 14 of 15
⇒
⇒ Sorting codimension 2 of 2: 15 paths to sort.
⇒ Sorting 0 of 15
⇒ Sorting 1 of 15
⇒ Sorting 2 of 15
⇒ Sorting 3 of 15

```

```

⇒ Sorting 4 of 15
⇒ Sorting 5 of 15
⇒ Sorting 6 of 15
⇒ Sorting 7 of 15
⇒ Sorting 8 of 15
⇒ Sorting 9 of 15
⇒ Sorting 10 of 15
⇒ Sorting 11 of 15
⇒ Sorting 12 of 15
⇒ Sorting 13 of 15
⇒ Sorting 14 of 15
⇒
⇒
⇒ ***** Regenerative Cascade Summary *****
⇒
⇒ NOTE: nonsingular vs singular is based on rank deficiency and identical e\
  ndpoints
⇒
⇒ |codim|   paths   |witness superset| nonsingular | singular |nonsolutions\
  | inf endpoints | other bad endpoints
⇒ -----
⇒ | 1 | | 5 | | 2 | | 0 | | 2 | | 3 | | \
  | 0 | | | | 0 | | | | | | |
⇒ | 2 | | 15 | | 15 | | 0 | | 15 | | 0 | | \
  | 0 | | | | 0 | | | | | | |
⇒ -----
⇒ |total|   20
⇒
⇒ *****
⇒
⇒
⇒ Removing junk points from codimension 2: 2 endpoints to check.
⇒ Checking 0 of 2
⇒ Checking 1 of 2
⇒
⇒
⇒ ***** Witness Set Summary *****
⇒
⇒ NOTE: nonsingular vs singular is based on rank deficiency and identical e\
  ndpoints
⇒
⇒ |codim| witness points | nonsingular | singular
⇒ -----
⇒ | 1 | | 1 | | 0 | | 1 |
⇒ | 2 | | 1 | | 0 | | 1 |
⇒ -----
⇒
⇒ *****
⇒
⇒
⇒ Calculating traces for codimension 1.
⇒ Calculating 0 of 1
⇒

```

```

⇒ Calculating traces for codimension 2.
⇒ Calculating 0 of 1
⇒
⇒
⇒ ***** Witness Set Decomposition *****
⇒
⇒ | dimension | components | classified | unclassified
⇒ -----
⇒ |    1      |    1      |    1      |    0
⇒ |    0      |    1      |    1      |    0
⇒ -----
⇒
⇒ ***** Decomposition by Degree *****
⇒
⇒ Dimension 1: 1 classified component
⇒ -----
⇒     degree 1: 1 component
⇒
⇒ Dimension 0: 1 classified component
⇒ -----
⇒     degree 1: 1 component
⇒
⇒ *****
⇒
⇒ Witness Points Deflated: 2
⇒ 0
numRad;
⇒ numRad[1]=0
//Then we compute the radical symbolically and compare the results.
ideal Rad=radical(I);
Rad;
⇒ Rad[1]=xy+y2+2xz+2yz
⇒ Rad[2]=x2-y2-5xz-5yz
reduce(Rad,std(numRad));
⇒ _[1]=xy+y2+2xz+2yz
⇒ _[2]=x2-y2-5xz-5yz
reduce(numRad,std(Rad));
⇒ _[1]=0

```

See also: [Section D.8.7.11 \[num\\_radical\\_via\\_decom\]](#), page 1871.

### D.8.7.13 num\_radical1

Procedure from library `recover.lib` (see [Section D.8.7 \[recover\\_lib\]](#), page 1862).

- Usage:** `num_radical1(P,D,C)`; list P, int D, bigint C  
P a list of lists representing a witness point set representing an ideal I D should be a bound to the degree of the elements of the components C the number with which the images of the Veronese embeddings are multiplied
- Return:** list of ideals: each of the ideals a prime component of the radical of I
- Remarks:** This procedure merely calls `num_prime_decom1` with the same input and then intersects the returned components.
- Note:** Should only be called from a ring over the complex numbers.

**Example:**

```

LIB "recover.lib";
//First, we write the input file for bertini and compute the radical symbolically.
ring r=0,(x,y,z),dp;
ideal I=4xy2-4z3, -2x2y+5xz2;
ideal Rad=radical(I);
writeBertiniInput(I,100);
//Then we attempt to compute the radical via the hybrid algorithm.
ring R=(complex,100,i),(x,y,z),dp;
system("sh","bertini input");
↳
↳ Bertini(TM) v1.6
↳ (May 22, 2018)
↳
↳ D.J. Bates, J.D. Hauenstein,
↳ A.J. Sommese, C.W. Wampler
↳
↳ (using GMP v6.0.0, MPFR v3.1.2)
↳
↳
↳ NOTE: You have requested to use adaptive path tracking. Please make sure\
that you have
↳ setup the following tolerances appropriately:
↳ CoeffBound: 8.000000000000e+00, DegreeBound: 3.000000000000e+00
↳ AMPSafetyDigits1: 1, AMPSafetyDigits2: 1, AMPMaxPrec: 352
↳
↳
↳ Tracking regeneration codim 1 of 2: 3 paths to track.
↳ Tracking path 0 of 3
↳ Tracking path 1 of 3
↳ Tracking path 2 of 3
↳
↳ Sorting codimension 1 of 2: 3 paths to sort.
↳ Sorting 0 of 3
↳ Sorting 1 of 3
↳ Sorting 2 of 3
↳
↳ Preparing regeneration codim 2 of 2: 6 witness points to move.
↳ Moving 0 of 6
↳ Moving 1 of 6
↳ Moving 2 of 6
↳ Moving 3 of 6
↳ Moving 4 of 6
↳ Moving 5 of 6
↳
↳ Tracking regeneration codim 2 of 2: 9 paths to track.
↳ Tracking path 0 of 9
↳ Tracking path 1 of 9
↳ Tracking path 2 of 9
↳ Tracking path 3 of 9
↳ Tracking path 4 of 9
↳ Tracking path 5 of 9

```

```

⇒ Tracking path 6 of 9
⇒ Tracking path 7 of 9
⇒ Tracking path 8 of 9
⇒
⇒ Sorting codimension 2 of 2: 9 paths to sort.
⇒ Sorting 0 of 9
⇒ Sorting 1 of 9
⇒ Sorting 2 of 9
⇒ Sorting 3 of 9
⇒ Sorting 4 of 9
⇒ Sorting 5 of 9
⇒ Sorting 6 of 9
⇒ Sorting 7 of 9
⇒ Sorting 8 of 9
⇒
⇒
⇒ ***** Regenerative Cascade Summary *****
⇒
⇒ NOTE: nonsingular vs singular is based on rank deficiency and identical e\
ndpoints
⇒
⇒ |codim|   paths   |witness superset| nonsingular | singular |nonsolutions\
   | inf endpoints | other bad endpoints
⇒ -----
⇒ | 1 |   3   | 0           | 0           | 0           | 3           | \
   | 0           | 0
⇒ | 2 |   9   | 9           | 1           | 8           | 0           | \
   | 0           | 0
⇒ -----
⇒ |total|   12
⇒
⇒ *****
⇒
⇒
⇒ ***** Witness Set Summary *****
⇒
⇒ NOTE: nonsingular vs singular is based on rank deficiency and identical e\
ndpoints
⇒
⇒ |codim| witness points | nonsingular | singular
⇒ -----
⇒ | 2 |   3   | 1           | 2
⇒ -----
⇒
⇒ *****
⇒
⇒ Calculating traces for codimension 2.
⇒ Calculating 0 of 3
⇒ Calculating 1 of 3
⇒ Calculating 2 of 3
⇒

```

```

↳
↳ ***** Witness Set Decomposition *****
↳
↳ | dimension | components | classified | unclassified
↳ -----
↳ |    0      |    3      |    3      |    0
↳ -----
↳
↳ ***** Decomposition by Degree *****
↳
↳ Dimension 0: 3 classified components
↳ -----
↳      degree 1: 3 components
↳
↳ *****
↳
↳ Witness Points Deflated: 2
↳ 0
list P=getWitnessSet();
int D=2;
bigint C=bigint(10)**30;
ideal Rad1=num_radical1(P,D,C);
//Lastly, we compare the results.
Rad1;
↳ Rad1[1]=yz-0.4*z2
↳ Rad1[2]=xz-6.25*z2
↳ Rad1[3]=xy-2.5*z2
ideal Rad=fetch(r,Rad);
Rad;
↳ Rad[1]=5*yz-2*z2
↳ Rad[2]=4*xz-25*z2
↳ Rad[3]=2*xy-5*z2
reduce(Rad,std(Rad1));
↳ // ** groebner base computations with inexact coefficients can not be tru\
   sted due to rounding errors
↳ _[1]=0
↳ _[2]=0
↳ _[3]=0
reduce(Rad1,std(Rad));
↳ // ** groebner base computations with inexact coefficients can not be tru\
   sted due to rounding errors
↳ _[1]=0
↳ _[2]=0
↳ _[3]=0

```

See also: [Section D.8.7.10 \[num\\_prime\\_decom1\]](#), page 1870; [Section D.8.7.14 \[num\\_radical2\]](#), page 1881.

### D.8.7.14 num\_radical2

Procedure from library `recover.lib` (see [Section D.8.7 \[recover\\_lib\]](#), page 1862).

**Usage:**      `num_radical2(P,D,C);` list P, int D, bigint C  
               P a list of lists representing a witness point set representing an ideal I D should be a

bound to the degree of the elements of the components  $C$  the number with which the images of the Veronese embeddings are multiplied

**Return:** list of ideals: each of the ideals a prime component of the radical of  $I$

**Remarks:** Instead of using the images of the Veronese embeddings of each individual witness point, this procedure first computes a random linear combination of those images and searches for homogeneous polynomial relations for this linear combination.

**Note:** Should only be called from a ring over the complex numbers.

**Example:**

```
LIB "recover.lib";
//First, we write the input file for bertini and compute the radical symbolically.
ring r=0,(x,y,z),dp;
ideal I=4xy2-4z3, -2x2y+5xz2;
ideal Rad=radical(I);
writeBertiniInput(I,100);
//Then we attempt to compute the radical via the hybrid algorithm.
ring R=(complex,100,i),(x,y,z),dp;
system("sh","bertini input");
↳
↳ Bertini(TM) v1.6
↳ (May 22, 2018)
↳
↳ D.J. Bates, J.D. Hauenstein,
↳ A.J. Sommese, C.W. Wampler
↳
↳ (using GMP v6.0.0, MPFR v3.1.2)
↳
↳
↳
↳ NOTE: You have requested to use adaptive path tracking. Please make sure\
that you have
↳ setup the following tolerances appropriately:
↳ CoeffBound: 8.000000000000e+00, DegreeBound: 3.000000000000e+00
↳ AMPSafetyDigits1: 1, AMPSafetyDigits2: 1, AMPMaxPrec: 352
↳
↳
↳ Tracking regeneration codim 1 of 2: 3 paths to track.
↳ Tracking path 0 of 3
↳ Tracking path 1 of 3
↳ Tracking path 2 of 3
↳
↳ Sorting codimension 1 of 2: 3 paths to sort.
↳ Sorting 0 of 3
↳ Sorting 1 of 3
↳ Sorting 2 of 3
↳
↳ Preparing regeneration codim 2 of 2: 6 witness points to move.
↳ Moving 0 of 6
↳ Moving 1 of 6
↳ Moving 2 of 6
↳ Moving 3 of 6
↳ Moving 4 of 6
```



```

⇒ Moving 5 of 6
⇒
⇒ Tracking regeneration codim 2 of 2: 9 paths to track.
⇒ Tracking path 0 of 9
⇒ Tracking path 1 of 9
⇒ Tracking path 2 of 9
⇒ Tracking path 3 of 9
⇒ Tracking path 4 of 9
⇒ Tracking path 5 of 9
⇒ Tracking path 6 of 9
⇒ Tracking path 7 of 9
⇒ Tracking path 8 of 9
⇒
⇒ Sorting codimension 2 of 2: 9 paths to sort.
⇒ Sorting 0 of 9
⇒ Sorting 1 of 9
⇒ Sorting 2 of 9
⇒ Sorting 3 of 9
⇒ Sorting 4 of 9
⇒ Sorting 5 of 9
⇒ Sorting 6 of 9
⇒ Sorting 7 of 9
⇒ Sorting 8 of 9
⇒
⇒
⇒ ***** Regenerative Cascade Summary *****
⇒
⇒ NOTE: nonsingular vs singular is based on rank deficiency and identical e\
  ndpoints
⇒
⇒ |codim|   paths   |witness superset| nonsingular | singular |nonsolutions\
  | inf endpoints | other bad endpoints
⇒ -----
⇒ | 1 |   3   | 0           | 0           | 0           | 3           | \
  | 0           | 0           |
⇒ | 2 |   9   | 9           | 1           | 8           | 0           | \
  | 0           | 0           |
⇒ -----
⇒ |total|   12
⇒
⇒ *****
⇒
⇒
⇒ ***** Witness Set Summary *****
⇒
⇒ NOTE: nonsingular vs singular is based on rank deficiency and identical e\
  ndpoints
⇒
⇒ |codim| witness points | nonsingular | singular
⇒ -----
⇒ | 2 |   3   | 1           | 2
⇒ -----

```

```

⇒
⇒ *****
⇒
⇒
⇒ Calculating traces for codimension 2.
⇒ Calculating 0 of 3
⇒ Calculating 1 of 3
⇒ Calculating 2 of 3
⇒
⇒
⇒ ***** Witness Set Decomposition *****
⇒
⇒ | dimension | components | classified | unclassified
⇒ -----
⇒ |    0      |    3      |    3      |    0
⇒ -----
⇒
⇒ ***** Decomposition by Degree *****
⇒
⇒ Dimension 0: 3 classified components
⇒ -----
⇒     degree 1: 3 components
⇒
⇒ *****
⇒
⇒ Witness Points Deflated: 2
⇒ 0
list P=getWitnessSet();
int D=2;
bigint C=bigint(10)**30;
ideal Rad2=num_radical2(P,D,C);
⇒ [1]:
⇒   _[1]=5*yz-2*z2
⇒   _[2]=-2*xy+5*z2
//Lastly, we compare the results.
Rad2;
⇒ Rad2[1]=5*yz-2*z2
⇒ Rad2[2]=-2*xy+5*z2
ideal Rad=fetch(r,Rad);
Rad;
⇒ Rad[1]=5*yz-2*z2
⇒ Rad[2]=4*xz-25*z2
⇒ Rad[3]=2*xy-5*z2
reduce(Rad,std(Rad2));
⇒ // ** groebner base computations with inexact coefficients can not be tru\
    sted due to rounding errors
⇒ _[1]=0
⇒ _[2]=4*xz-25*z2
⇒ _[3]=0
reduce(Rad2,std(Rad));
⇒ // ** groebner base computations with inexact coefficients can not be tru\
    sted due to rounding errors
⇒ _[1]=0

```

⇒ `_ [2]=0`

See also: [Section D.8.7.13 \[num\\_radical1\]](#), page 1878.

### D.8.7.15 num\_elim

Procedure from library `recover.lib` (see [Section D.8.7 \[recover\\_lib\]](#), page 1862).

**Usage:**      `num_elim(I,f,D)`; ideal I, poly f, int D  
                 f the product of the ring variables you want to eliminate D a bound to the degree of  
                 the elements of the components

**Return:**     ideal: the ideal obtained from I by eliminating the variables specified in f

**Remarks:**   This procedure uses Bertini to compute a set of witness points for I, projects them  
                   onto the components corresponding to the variables specified in f and then proceeds as  
                   `num_radical_via_randlincom`.

**Note:**        Should only be called from a ring over the rational numbers.

**Example:**

```
LIB "recover.lib";
ring r=0,(x,y,z),dp;
poly f1=x-y;
poly f2=z*(x+3y);
poly f3=z*(x2+y2);
ideal I=f1,f2,f3;
//First, we attempt to compute the elimination ideal with the hybrid algorithm.
ideal E1=num_elim(I,z,3,200);
⇒
⇒ Bertini(TM) v1.6
⇒ (May 22, 2018)
⇒
⇒ D.J. Bates, J.D. Hauenstein,
⇒ A.J. Sommese, C.W. Wampler
⇒
⇒ (using GMP v6.0.0, MPFR v3.1.2)
⇒
⇒
⇒
⇒ NOTE: You have requested to use adaptive path tracking. Please make sure\
that you have
⇒ setup the following tolerances appropriately:
⇒ CoeffBound: 4.000000000000e+00, DegreeBound: 3.000000000000e+00
⇒ AMPSafetyDigits1: 1, AMPSafetyDigits2: 1, AMPMaxPrec: 672
⇒
⇒
⇒ Tracking regeneration codim 1 of 2: 3 paths to track.
⇒ Tracking path 0 of 3
⇒ Tracking path 1 of 3
⇒ Tracking path 2 of 3
⇒
⇒ Sorting codimension 1 of 2: 3 paths to sort.
⇒ Sorting 0 of 3
⇒ Sorting 1 of 3
⇒ Sorting 2 of 3
```

```

↳
↳ Preparing regeneration codim 2 of 2: 3 witness points to move.
↳ Moving 0 of 3
↳ Moving 1 of 3
↳ Moving 2 of 3
↳
↳ Tracking regeneration codim 2 of 2: 6 paths to track.
↳ Tracking path 0 of 6
↳ Tracking path 1 of 6
↳ Tracking path 2 of 6
↳ Tracking path 3 of 6
↳ Tracking path 4 of 6
↳ Tracking path 5 of 6
↳
↳ Sorting codimension 2 of 2: 6 paths to sort.
↳ Sorting 0 of 6
↳ Sorting 1 of 6
↳ Sorting 2 of 6
↳ Sorting 3 of 6
↳ Sorting 4 of 6
↳ Sorting 5 of 6
↳
↳
↳ ***** Regenerative Cascade Summary *****
↳
↳ NOTE: nonsingular vs singular is based on rank deficiency and identical e\
  ndpoints
↳
↳ |codim|   paths   |witness superset| nonsingular | singular |nonsolutions\
  | inf endpoints | other bad endpoints
↳ -----
↳ | 1 |   3   | 0           | 0           | 0           | 3           | \
  | 0           | 0           |
↳ | 2 |   6   | 2           | 2           | 0           | 3           | \
  | 1           | 0           |
↳ -----
↳ |total|   9
↳
↳ *****
↳
↳
↳ ***** Witness Set Summary *****
↳
↳ NOTE: nonsingular vs singular is based on rank deficiency and identical e\
  ndpoints
↳
↳ |codim| witness points | nonsingular | singular
↳ -----
↳ | 2 |   2           | 2           | 0
↳ -----
↳
↳ *****

```

```

⇒
⇒
⇒ Calculating traces for codimension 2.
⇒ Calculating 0 of 2
⇒ Calculating 1 of 2
⇒
⇒
⇒ ***** Witness Set Decomposition *****
⇒
⇒ | dimension | components | classified | unclassified
⇒ -----
⇒ |    0      |    2      |    2      |    0
⇒ -----
⇒
⇒ ***** Decomposition by Degree *****
⇒
⇒ Dimension 0: 2 classified components
⇒ -----
⇒      degree 1: 2 components
⇒
⇒ *****
⇒
⇒ 0
//Now, we compute the elimination ideal symbolically.
ideal E2=elim(I,z);
//Lastly, we compare the results.
E1;
⇒ E1[1]=0
E2;
⇒ E2[1]=x-y

```

### D.8.7.16 num\_elim1

Procedure from library `recover.lib` (see [Section D.8.7 \[recover.lib\]](#), page 1862).

**Usage:**     `num_elim1(P,D,C,v);` list P, int D, bigint C, intvec v  
P a list of lists representing a witness point set representing an ideal J D should be a bound to the degree of the elements of the components C the number with which the images of the Veronese embeddings are multiplied v an intvec specifying the numbers/positions of the variables to be eliminated

**Return:**    ideal: the ideal obtained from J by eliminating the variables specified in v

**Remarks:** This procedure just canonically projects the witness points onto the components specified in the intvec v and then applies `num_radical1` to the resulting points.

**Note:**     Should only be called from a ring over the complex numbers.

**Example:**

```

LIB "recover.lib";
//First, we write the input file for bertini and compute the elimination ideal
//symbolically.
ring r=0,(x,y,z),dp;
poly f1=x-y;
poly f2=z*(x+3y);

```

```

poly f3=z*(x2+y2);
ideal J=f1,f2,f3;
ideal E2=elim(J,z);
writeBertiniInput(J,100);
//Then we attempt to compute the elimination ideal via the hybrid algorithm.
ring R=(complex,100,i),(x,y,z),dp;
system("sh","bertini input");
↳
↳ Bertini(TM) v1.6
↳ (May 22, 2018)
↳
↳ D.J. Bates, J.D. Hauenstein,
↳ A.J. Sommese, C.W. Wampler
↳
↳ (using GMP v6.0.0, MPFR v3.1.2)
↳
↳
↳
↳ NOTE: You have requested to use adaptive path tracking. Please make sure\
    that you have
↳ setup the following tolerances appropriately:
↳ CoeffBound: 4.000000000000e+00, DegreeBound: 3.000000000000e+00
↳ AMPSafetyDigits1: 1, AMPSafetyDigits2: 1, AMPMaxPrec: 352
↳
↳
↳ Tracking regeneration codim 1 of 2: 3 paths to track.
↳ Tracking path 0 of 3
↳ Tracking path 1 of 3
↳ Tracking path 2 of 3
↳
↳ Sorting codimension 1 of 2: 3 paths to sort.
↳ Sorting 0 of 3
↳ Sorting 1 of 3
↳ Sorting 2 of 3
↳
↳ Preparing regeneration codim 2 of 2: 3 witness points to move.
↳ Moving 0 of 3
↳ Moving 1 of 3
↳ Moving 2 of 3
↳
↳ Tracking regeneration codim 2 of 2: 6 paths to track.
↳ Tracking path 0 of 6
↳ Tracking path 1 of 6
↳ Tracking path 2 of 6
↳ Tracking path 3 of 6
↳ Tracking path 4 of 6
↳ Tracking path 5 of 6
↳
↳ Sorting codimension 2 of 2: 6 paths to sort.
↳ Sorting 0 of 6
↳ Sorting 1 of 6
↳ Sorting 2 of 6
↳ Sorting 3 of 6

```

```

⇒ Sorting 4 of 6
⇒ Sorting 5 of 6
⇒
⇒
⇒ ***** Regenerative Cascade Summary *****
⇒
⇒ NOTE: nonsingular vs singular is based on rank deficiency and identical e\
  ndpoints
⇒
⇒ |codim|   paths   |witness superset| nonsingular | singular |nonsolutions\
  | inf endpoints | other bad endpoints
⇒ -----
⇒ | 1 |   3   | 0 | 0 | 0 | 3 | \
  | 0 |       | 0 |   |   |   |
⇒ | 2 |   6   | 2 | 2 | 0 | 3 | \
  | 1 |       | 0 |   |   |   |
⇒ -----
⇒ |total|   9
⇒
⇒ *****
⇒
⇒
⇒ ***** Witness Set Summary *****
⇒
⇒ NOTE: nonsingular vs singular is based on rank deficiency and identical e\
  ndpoints
⇒
⇒ |codim| witness points | nonsingular | singular
⇒ -----
⇒ | 2 |   2   | 2 | 0
⇒ -----
⇒ *****
⇒
⇒
⇒ Calculating traces for codimension 2.
⇒ Calculating 0 of 2
⇒ Calculating 1 of 2
⇒
⇒
⇒ ***** Witness Set Decomposition *****
⇒
⇒ | dimension | components | classified | unclassified
⇒ -----
⇒ | 0 |   2   | 2 | 0
⇒ -----
⇒
⇒ ***** Decomposition by Degree *****
⇒
⇒ Dimension 0: 2 classified components
⇒ -----
⇒ degree 1: 2 components

```

```

↳
↳ *****
↳
↳ 0
list P=getWitnessSet();
intvec v=3;
bigint C=bigint(10)**25;
ideal E1=num_elim1(P,2,C,v);
//Lastly, we compare the results.
E1;
↳ E1[1]=x-y
setring r;
E2;
↳ E2[1]=x-y

```

### D.8.7.17 realLLL

Procedure from library `recover.lib` (see [Section D.8.7 \[recover.lib\]](#), page 1862).

**Usage:** `realLLL(M);` matrix `M`

**Assume:** The columns of `M` represent a basis of a lattice.  
The groundfield is the field of real number or the field of complex numbers, the elements of `M` are real numbers.

**Return:** matrix: the columns representing an LLL-reduced basis of the lattice given by `M`

**Example:**

```

LIB "recover.lib";
ring r=(real,50),x,dp;
matrix M[5][4]=
1,0,0,0,
0,1,0,0,
0,0,1,0,
0,0,0,1,
5*81726716.91827716, 817267.1691827716, poly(10)**30, 13*81726716.91827716;
matrix L=realLLL(M);
print(L);
↳ 1,      0,      -3,      -315358105194840436449,
↳ -500,1,      0,      -630716210389680833,
↳ 0,      0,      0,      1,
↳ 0,      0,      1,      -819931073506585134768,
↳ 0,      817267.1691827716, -163453433.83655432, 771738001439.2007536172

```

### D.8.8 rootisolation.lib

**Library:** `rootisolation.lib`

**Purpose:** implements an algorithm for real root isolation using interval arithmetic

**Authors:** Dominik Bendle (bendle@rhrk.uni-kl.de)  
Janko Boehm (boehm@mathematik.uni-lk.de), supervisor Fachpraktikum  
Clara Petroll (petroll@rhrk.uni-kl.de)



**Overview:** In this library the interval arithmetic from `interval.so` is used. The new type `ivmat`, a matrix consisting of intervals, is implemented as `newstruct`. There are various functions for computations with interval matrices implemented, such as Gaussian elimination for interval matrices.

Interval arithmetic, the interval Newton Step and exclusion methods are used to implement the procedure `rootIsolation`, an algorithm which finds boxes containing elements of the vanishing locus of an ideal. This algorithm is specialised for zero-dimensional radical ideals. The theory about the interval Newton Step is detailed in [2].

Note that interval arithmetic and the aforementioned procedures are intended for rational or real polynomial rings.

**References:**

- [1] Cloud, Kearfott, Moore: Introduction to Interval Analysis, Society for Industrial and Applied Mathematics, 2009
- [2] Eisenbud, Grayson, Herzog, Stillman, Vasconcelos: Computational Methods in Commutative Algebra and Algebraic Geometry, Springer Verlag Berlin-Heidelberg, 3. edition 2004
- [3] Andrew J. Sommese and Charles W. Wampler: The Numerical Solution of Systems of Polynomials - Arising in Engineering and Science, World Scientific Publishing Co. Pte. Ltd., 2005

**Overloads:**

```
[ ivmatGet indexing
 print ivmatPrint printing
 nrows ivmatNrows number of rows
 ncols ivmatNcols number of columns
 * ivmatMultiplyGeneral matrix multiplication
```

**Procedures:** See also: [Section D.8.2 \[interval\\_lib\]](#), page 1825.

### D.8.8.1 bounds

Procedure from library `rootisolation.lib` (see [Section D.8.8 \[rootisolation\\_lib\]](#), page 1890).

**Usage:**      `bounds(a); a number;`  
               `bounds(a, b); a, b number;`

**Return:**    interval: if `size(#)==0` it returns the interval `[a, a]`, else the interval `[a,#[1]]`

**Example:**

```
LIB "rootisolation.lib";
ring R = 0,x,dp;
interval I = bounds(1);
I;
↪ [1, 1]
interval J = bounds(2/3,3);
J;
↪ [2/3, 3]
```

### D.8.8.2 length

Procedure from library `rootisolation.lib` (see [Section D.8.8 \[rootisolation.lib\]](#), page 1890).

**Usage:**      `length(I); I interval`

**Return:**     number, the Euclidean length of the interval

**Example:**

```
LIB "rootisolation.lib";
ring R = 0,x,dp;
interval I = -1,3;
length(I);
↦ 4
I = 1/5,1/3;
length(I);
↦ 2/15
```

### D.8.8.3 boxSet

Procedure from library `rootisolation.lib` (see [Section D.8.8 \[rootisolation.lib\]](#), page 1890).

**Usage:**      `boxSet(B, i, I); B box, i int, I interval`

**Return:**     new box C where  $C[i] = I$

**Purpose:**     modifies a single entry of a box

**Example:**

```
LIB "rootisolation.lib";
ring R = 0,(x,y,z),dp;
box B; B;
↦ [0, 0] x [0, 0] x [0, 0]
B = boxSet(B, 2, bounds(-1,1)); B;
↦ [0, 0] x [-1, 1] x [0, 0]
B = boxSet(B, 1, B[2]); B;
↦ [-1, 1] x [-1, 1] x [0, 0]
```

### D.8.8.4 ivmatInit

Procedure from library `rootisolation.lib` (see [Section D.8.8 \[rootisolation.lib\]](#), page 1890).

**Usage:**      `ivmatInit(m, n); m, n int`

**Return:**      $m \times n$  matrix of  $[0,0]$ -intervals

**Purpose:**     initialises an interval matrix with  $[0,0]$  intervals to ensure the proper structure of the `ivmat` type

**Example:**

```
LIB "rootisolation.lib";
ring R = 0,x(1..5),dp;
ivmat A = ivmatInit(4, 5); A;
↦ [0, 0],[0, 0],[0, 0],[0, 0],[0, 0]
↦ [0, 0],[0, 0],[0, 0],[0, 0],[0, 0]
↦ [0, 0],[0, 0],[0, 0],[0, 0],[0, 0]
↦ [0, 0],[0, 0],[0, 0],[0, 0],[0, 0]
↦
```

### D.8.8.5 ivmatSet

Procedure from library `rootisolation.lib` (see [Section D.8.8 \[rootisolation.lib\]](#), page 1890).

**Usage:** `ivmatSet(A, i, j, I);` A ivmat, i, j, int, I interval

**Return:** interval matrix A where  $A[i][j] == I$

**Purpose:** modify a single entry of an ivmat

**Example:**

```
LIB "rootisolation.lib";
ring R = 0,x,dp;
ivmat A = ivmatInit(2,2);           A;
⇒ [0, 0],[0, 0]
⇒ [0, 0],[0, 0]
⇒
A = ivmatSet(A, 1, 2, bounds(1, 2)); A;
⇒ [0, 0],[1, 2]
⇒ [0, 0],[0, 0]
⇒
```

### D.8.8.6 unitMatrix

Procedure from library `rootisolation.lib` (see [Section D.8.8 \[rootisolation.lib\]](#), page 1890).

**Usage:** `unitMatrix(n);` n int

**Return:** nxn unit interval matrix

**Example:**

```
LIB "rootisolation.lib";
ring R = 0,(x,y),dp;
unitMatrix(2);
⇒ [1, 1],[0, 0]
⇒ [0, 0],[1, 1]
⇒
unitMatrix(3);
⇒ [1, 1],[0, 0],[0, 0]
⇒ [0, 0],[1, 1],[0, 0]
⇒ [0, 0],[0, 0],[1, 1]
⇒
```

### D.8.8.7 ivmatGaussian

Procedure from library `rootisolation.lib` (see [Section D.8.8 \[rootisolation.lib\]](#), page 1890).

**Usage:** `ivmatGaussian(A);` A ivmat

**Return:** 0, if A not invertible, (1, Ainv) if A invertible where Ainv is the inverse matrix

**Purpose:** Inverts an interval matrix using Gaussian elimination in the setting of interval arithmetic. Pivoting is handled as a special case as  $I/I \neq [1,1]$  and  $I-I \neq [0,0]$  in general.

**Example:**

```

LIB "rootisolation.lib";
ring R = 0,(x,y),dp;
ideal I = 2x2-xy+2y2-2, 2x2-3xy+3y2-2;
box B = list(bounds(7/8, 9/8), bounds(-1/10, 1/20));
ivmat J = evalJacobianAtBox (I, B); J;
⇒ [69/20, 23/5],[-61/40, -27/40]
⇒ [67/20, 24/5],[-159/40, -93/40]
⇒
list result = ivmatGaussian(J);
ivmat Jinv = result[2];
Jinv;
⇒ [1060/4273, 620/187],[-1220/561, -180/4273]
⇒ [2680/12819, 1280/187],[-920/187, -3680/12819]
⇒
Jinv * J;
⇒ [-7657037/799051, 12073835/799051],[-3961862/799051, 6773506/799051]
⇒ [-18293234/799051, 73172936/2397153],[-7807572/799051, 15513600/799051]
⇒

```

### D.8.8.8 evalPolyAtBox

Procedure from library `rootisolation.lib` (see [Section D.8.8 \[rootisolation.lib\], page 1890](#)).

**Usage:** `evalPolyAtBox(f, B);` *f* poly, *B* box

**Return:** interval, evaluation of *f* at *B* using interval arithmetic

**Purpose:** computes an interval extension of the polynomial

**Example:**

```

LIB "rootisolation.lib";
ring R = 0,(x,y),dp;
poly f = x2+y-1;
box B = list(bounds(-1,1), bounds(1,3)/2);
interval I = evalPolyAtBox(f, B); I;
⇒ [-1/2, 3/2]

```

### D.8.8.9 evalJacobianAtBox

Procedure from library `rootisolation.lib` (see [Section D.8.8 \[rootisolation.lib\], page 1890](#)).

**Usage:** `evalJacobianAtBox(I, B);` *I* ideal, *B* box

**Return:** Jacobian matrix of *I* where polynomials are evaluated at *B*

**Purpose:** evaluates each polynomial of the Jacobian matrix of *I* using interval arithmetic

**Example:**

```

LIB "rootisolation.lib";
ring R = 0,(x,y),dp;
ideal I = 2x2-xy+2y2-2, 2x2-3xy+3y2-2;
interval J = bounds(-1,1);
evalJacobianAtBox(I, list(J,J));
⇒ [-5, 5],[-5, 5]
⇒ [-7, 7],[-9, 9]
⇒

```

### D.8.8.10 rootIsolationNoPreprocessing

Procedure from library `rootisolation.lib` (see [Section D.8.8 \[rootisolation.lib\]](#), page 1890).

- Usage:** `rootIsolationNoPreprocessing(I, B, eps);`      I ideal, B list of boxes, eps number;
- Assume:** I is a zero-dimensional radical ideal
- Return:** (L1, L2), where L1 contains boxes smaller than eps which may contain an element of  $V(I)$ , i.e. a root and L2 contains boxes which contain exactly one element of  $V(I)$
- Purpose:** Given input box(es) `start` we try to find all roots of I lying in `start` by computing boxes that contain exactly one root. If `eps` > 0 then boxes that become smaller than `eps` will be returned.
- Theory:** We first check for every box if it contains no roots by interval arithmetic. If this is inconclusive we apply the Newton step which, as outlined in [2] and [3], converges to a root lying in the starting box. If the result of the Newton step is already contained in the interior of the starting box, it contains a unique root.
- Note:** While `rootIsolation` does, `rootIsolationNoPreprocessing` does not check input ideal for necessary conditions.

**Example:**

```
LIB "rootisolation.lib";
ring R = 0,(x,y),dp;
ideal I = 2x2-xy+2y2-2,2x2-3xy+3y2-2; // V(I) has four elements
interval i = bounds(-3/2,3/2);
box B = list(i, i);
list result = rootIsolationNoPreprocessing(I, list(B), 1/512);
size(result[1]);
⇒ 0
size(result[2]);
⇒ 4
result;
⇒ [1]:
⇒ empty list
⇒ [2]:
⇒ [1]:
⇒ [-9/16, -15/32] x [-129/128, -127/128]
⇒ [2]:
⇒ [15/32, 33/64] x [31/32, 129/128]
⇒ [3]:
⇒ [63/64, 33/32] x [-1/256, 1/32]
⇒ [4]:
⇒ [-129/128, -63/64] x [-1/256, 1/32]
```

### D.8.8.11 rootIsolation

Procedure from library `rootisolation.lib` (see [Section D.8.8 \[rootisolation.lib\]](#), page 1890).

- Usage:** `rootIsolation(I, [start, eps]);` I ideal, start box, eps number
- Assume:** I is a zero-dimensional radical ideal
- Return:** (L1, L2), where L1 contains boxes smaller than eps which may contain an element of  $V(I)$ , i.e. a root and L2 contains boxes which contain exactly one element of  $V(I)$

- Purpose:** same as `rootIsolationNoPreprocessing`, but speeds up computation by preprocessing starting box
- Theory:** As every root of  $I$  is a root of the polynomials  $I[i]$ , we use Groebner elimination to find univariate polynomials for every variable which have these roots as well. Using that  $I$  is zero-dimensional these Groebner bases may be quickly computed using FGLM. Applying root isolation to these univariate polynomials then provides smaller starting boxes which speed up computations in the multivariate case.
- Note:** This algorithm and some procedures used therein perform Groebner basis computations in `basing`. It is thus advised to define  $I$  w.r.t. a fast monomial ordering. The algorithm performs checks on  $I$  to prevent errors. If  $I$  does not have the right number of generators, we first try to find a suitable Groebner basis. If this fails we apply the algorithm to the triangular decomposition of  $I$ .

**Example:**

```
LIB "rootisolation.lib";
ring R = 0,(x,y),dp;
ideal I = 2x2-xy+2y2-2,2x2-3xy+3y2-2; // V(I) has four elements
interval i = bounds(-3/2,3/2);
box B = list(i, i);
list result = rootIsolation(I, B);
result;
⇒ [1]:
⇒ empty list
⇒ [2]:
⇒ [1]:
⇒ [-1, -1] x [0, 0]
⇒ [2]:
⇒ [-1/2, -1/2] x [-1, -1]
⇒ [3]:
⇒ [1/2, 1/2] x [1, 1]
⇒ [4]:
⇒ [1, 1] x [0, 0]
```

**D.8.8.12 rootIsolationPrimdec**

Procedure from library `rootisolation.lib` (see [Section D.8.8 \[rootisolation.lib\]](#), page 1890).

- Usage:** `rootIsolationPrimdec(I); I ideal`
- Assume:**  $I$  is a zero-dimensional radical ideal
- Return:**  $L$ , where  $L$  contains boxes which contain exactly one element of  $V(I)$
- Purpose:** same as `rootIsolation`, but speeds up computation and improves output by doing a primary decomposition before doing the root isolation
- Theory:** For the primary decomposition we use the algorithm of Gianni-Traeger-Zarcharias.
- Note:** This algorithm and some procedures used therein perform Groebner basis computations in `basing`. It is thus advised to define  $I$  w.r.t. a fast monomial ordering.

**Example:**

```
LIB "rootisolation.lib";
ring R = 0,(x,y),dp;
```

```

ideal I = 2x2-xy+2y2-2,2x2-3xy+3y2-2; // V(I) has four elements
list result = rootIsolationPrimdec(I);
result;
⇒ [1]:
⇒ [1, 1] x [0, 0]
⇒ [2]:
⇒ [1/2, 1/2] x [1, 1]
⇒ [3]:
⇒ [-1/2, -1/2] x [-1, -1]
⇒ [4]:
⇒ [-1, -1] x [0, 0]

```

### D.8.9 signcond\_lib

**Library:** signcond.lib

**Purpose:** Routines for computing realizable sign conditions

**Author:** Enrique A. Tobis, etobis@dc.uba.ar

**Overview:** Routines to determine the number of solutions of a multivariate polynomial system which satisfy a given sign configuration.

**References:**

Basu, Pollack, Roy, "Algorithms in Real Algebraic Geometry", Springer, 2003.

**Procedures:**

#### D.8.9.1 signcnd

Procedure from library `signcond.lib` (see [Section D.8.9 \[signcond\\_lib\]](#), page 1897).

**Usage:** signcnd(P,I); ideal P,I

**Return:** list: the sign conditions realized by the polynomials of P on V(I). The output of signcnd is a list of two lists. Both lists have the same length. This length is the number of sign conditions realized by the polynomials of P on the set V(i).  
 Each element of the first list indicates a sign condition of the polynomials of P.  
 Each element of the second list indicates how many elements of V(I) give rise to the sign condition expressed by the same position on the first list.  
 See the example for further explanations of the output.

**Assume:** I is a Groebner basis.

**Note:** The procedure psigncnd performs some pretty printing of this output.

**Example:**

```

LIB "signcond.lib";
ring r = 0,(x,y),dp;
ideal i = (x-2)*(x+3)*x,y*(y-1);
ideal P = x,y;
list l = signcnd(P,i);
size(l[1]); // = the number of sign conditions of P on V(i)
⇒ 6
//Each element of l[1] indicates a sign condition of the polynomials of P.
//The following means P[1] > 0, P[2] = 0:
l[1][2];

```

```

    ↪ [1]:
    ↪ 1
    ↪ [2]:
    ↪ 0
    //Each element of l[2] indicates how many elements of V(I) give rise to
    //the sign condition expressed by the same position on the first list.
    //The following means that exactly 1 element of V(I) gives rise to the
    //condition  $P[1] > 0, P[2] = 0$ :
    l[2][2];
    ↪ 1

```

See also: [Section D.8.9.3 \[firstoct\]](#), page 1898; [Section D.8.9.2 \[psigncnd\]](#), page 1898.

### D.8.9.2 psigncnd

Procedure from library `signcond.lib` (see [Section D.8.9 \[signcond.lib\]](#), page 1897).

**Usage:** `psigncnd(P,l)`; ideal  $P$ , list  $l$

**Return:** list: a formatted version of  $l$

**Example:**

```

LIB "signcond.lib";
ring r = 0,(x,y),dp;
ideal i = (x-2)*(x+3)*x,(y-1)*(y+2)*(y+4);
ideal P = x,y;
list l = signcnd(P,i);
psigncnd(P,l);
↪ 1 elements of V(I) satisfy {P[1] = 0,P[2] > 0}
↪ 1 elements of V(I) satisfy {P[1] > 0,P[2] > 0}
↪ 1 elements of V(I) satisfy {P[1] < 0,P[2] > 0}
↪ 2 elements of V(I) satisfy {P[1] = 0,P[2] < 0}
↪ 2 elements of V(I) satisfy {P[1] > 0,P[2] < 0}
↪ 2 elements of V(I) satisfy {P[1] < 0,P[2] < 0}
↪

```

See also: [Section D.8.9.1 \[signcnd\]](#), page 1897.

### D.8.9.3 firstoct

Procedure from library `signcond.lib` (see [Section D.8.9 \[signcond.lib\]](#), page 1897).

**Usage:** `firstoct(I)`;  $I$  ideal

**Return:** number: the number of points of  $V(I)$  lying in the first octant

**Assume:**  $I$  is given by a Groebner basis.

**Example:**

```

LIB "signcond.lib";
ring r = 0,(x,y),dp;
ideal i = (x-2)*(x+3)*x,y*(y-1);
firstoct(i);
↪ 1

```

See also: [Section D.8.9.1 \[signcnd\]](#), page 1897.



### D.8.10 zeroset.lib

**Library:** zeroset.lib

**Purpose:** Procedures for roots and factorization

**Author:** Thomas Bayer, email: tbayer@mathematik.uni-kl.de,  
<http://wwwmayr.informatik.tu-muenchen.de/personen/bayert/>  
 Current address: Hochschule Ravensburg-Weingarten

**Overview:** Algorithms for finding the zero-set of a zero-dim. ideal in  $\mathbb{Q}(a)[x_1, \dots, x_n]$ , roots and factorization of univariate polynomials over  $\mathbb{Q}(a)[t]$  where  $a$  is an algebraic number. Written in the scope of the diploma thesis (advisor: Prof. Gert-Martin Greuel) 'Computations of moduli spaces of semiquasihomogeneous singularities and an implementation in Singular'. This library is meant as a preliminary extension of the functionality of Singular for univariate factorization of polynomials over simple algebraic extensions in characteristic 0.

**Note:** Subprocedures with postfix 'Main' require that the ring contains a variable 'a' and no parameters, and the ideal 'mpoly', where 'minpoly' from the basering is stored.

**Procedures:**

#### D.8.10.1 Quotient

Procedure from library `zeroset.lib` (see [Section D.8.10 \[zeroset.lib\]](#), page 1899).

**Usage:** Quotient(f, g); where f,g are polynomials

**Purpose:** compute the quotient q and remainder r s.t.  $f = g \cdot q + r$ ,  $\deg(r) < \deg(g)$

**Return:** list of polynomials  
     `_[1]` = quotient q  
     `_[2]` = remainder r

**Assume:** basering =  $\mathbb{Q}[x]$  or  $\mathbb{Q}(a)[x]$

**Note:** This procedure is outdated, and should no longer be used. Use `div` and `mod` instead.

**Example:**

```
LIB "zeroset.lib";
ring R = (0,a), x, lp;
minpoly = a2+1;
poly f = x4 - 2;
poly g = x - a;
list qr = Quotient(f, g);
qr;
⇒ [1]:
⇒ x3+(a)*x2-x+(-a)
⇒ [2]:
⇒ 0
qr[1]*g + qr[2] - f;
⇒ 1
```

### D.8.10.2 remainder

Procedure from library `zeroset.lib` (see [Section D.8.10 \[zeroset.lib\], page 1899](#)).

**Usage:** `remainder(f, g);` where  $f, g$  are polynomials

**Purpose:** compute the remainder of the division of  $f$  by  $g$ , i.e. a polynomial  $r$  s.t.  $f = g \cdot q + r$ ,  $\deg(r) < \deg(g)$ .

**Return:** `poly`

**Assume:** `basing = Q[x]` or `Q(a)[x]`

**Note:** outdated, use `mod/reduce` instead

**Example:**

```
LIB "zeroset.lib";
ring R = (0,a), x, lp;
minpoly = a2+1;
poly f = x4 - 1;
poly g = x3 - 1;
remainder(f, g);
⇒ x-1
```

### D.8.10.3 roots

Procedure from library `zeroset.lib` (see [Section D.8.10 \[zeroset.lib\], page 1899](#)).

**Usage:** `roots(f);` where  $f$  is a polynomial

**Purpose:** compute all roots of  $f$  in a finite extension of  $\mathbb{Q}\mathbb{Q}$  without multiplicities.

**Return:** `ring`, a polynomial ring over an extension field of  $\mathbb{Q}\mathbb{Q}$ , containing a list `'theRoots'` and polynomials `'newA'` and `'f'`:

- `'theRoots'` is the list of roots of the polynomial  $f$  (no multiplicities)
- if the ground field is  $\mathbb{Q}(a')$  and the extension field is  $\mathbb{Q}(a)$ , then `'newA'` is the representation of  $a'$  in  $\mathbb{Q}(a)$ .  
If the basering contains a parameter `'a'` and the `minpoly` remains unchanged then `'newA' = 'a'`.  
If the basering does not contain a parameter then `'newA' = 'a'` (default).
- `'f'` is the polynomial  $f$  in  $\mathbb{Q}(a)$  ( $a'$  being substituted by `'newA'`)

**Assume:** ground field to be  $\mathbb{Q}$  or a simple extension of  $\mathbb{Q}$  given by a `minpoly`

**Example:**

```
LIB "zeroset.lib";
ring R = (0,a), x, lp;
minpoly = a2+1;
poly f = x3 - a;
def R1 = roots(f);
⇒
⇒ // 'roots' created a new ring which contains the list 'theRoots' and
⇒ // the polynomials 'f' and 'newA'
⇒ // To access the roots, newA and the new representation of f, type
⇒ def R = roots(f); setring R; theRoots; newA; f;
⇒
```

```

setring R1;
minpoly;
↪ (a4-a2+1)
newA;
↪ (a3)
f;
↪ x3+(-a3)
theRoots;
↪ [1]:
↪      (-a3)
↪ [2]:
↪      (a)
↪ [3]:
↪      (a3-a)
map F;
F[1] = theRoots[1];
F(f);
↪ 0

```

#### D.8.10.4 sqfrNorm

Procedure from library `zeroset.lib` (see [Section D.8.10 \[zeroset.lib\], page 1899](#)).

**Usage:** `sqfrNorm(f)`; where  $f$  is a polynomial in  $\mathbb{Q}(a)[x]$

**Purpose:** compute the norm of the squarefree polynomial  $f$  in  $\mathbb{Q}(a)[x]$ .

**Return:** list with 3 entries  
`_[1]` = squarefree norm of  $g$  (poly)  
`_[2]` =  $g$  (=  $f(x - s*a)$ ) (poly)  
`_[3]` =  $s$  (int)

**Assume:**  $f$  must be squarefree,  $\text{basing} = \mathbb{Q}(a)[x]$  and  $\text{minpoly} \neq 0$ .

**Note:** the norm is an element of  $\mathbb{Q}[x]$

**Example:**

```

LIB "zeroset.lib";
ring R = (0,a), x, lp;
minpoly = a2+1;
poly f = x4 - 2*x + 1;
sqfrNorm(f);
↪ [1]:
↪      x8+4*x6-4*x5+8*x4+8*x3-4*x2+8*x+8
↪ [2]:
↪      x4+(-4a)*x3-6*x2+(4a-2)*x+(2a+2)
↪ [3]:
↪      1

```

#### D.8.10.5 zeroSet

Procedure from library `zeroset.lib` (see [Section D.8.10 \[zeroset.lib\], page 1899](#)).

**Usage:** `zeroSet(I [,opt] )`;  $I$ =ideal,  $\text{opt}$ =integer

**Purpose:** compute the zero-set of the zero-dim. ideal  $I$ , in a finite extension of  $\mathbb{Q}\mathbb{Q}$ .

**Return:** ring, a polynomial ring over an extension field of  $\mathbb{Q}\mathbb{Q}$ , containing a list 'theZeroSet', a polynomial 'newA', and an ideal 'id':

- 'theZeroSet' is the list of the zeros of the ideal I, each zero is an ideal.
- if the ground field is  $\mathbb{Q}(b)$  and the extension field is  $\mathbb{Q}(a)$ , then 'newA' is the representation of  $b$  in  $\mathbb{Q}(a)$ .  
If the basering contains a parameter 'a' and the minpoly remains unchanged then 'newA' = 'a'.
- If the basering does not contain a parameter then 'newA' = 'a' (default).
- 'id' is the ideal I in  $\mathbb{Q}(a)[x_1, \dots]$  (a' substituted by 'newA')

**Assume:**  $\dim(I) = 0$ , and ground field to be  $\mathbb{Q}$  or a simple extension of  $\mathbb{Q}$  given by a minpoly.

**Options:** opt = 0: no primary decomposition (default)  
opt > 0: primary decomposition

**Note:** If I contains an algebraic number (parameter) then I must be transformed w.r.t. 'newA' in the new ring.

**Example:**

```
LIB "zeroset.lib";
ring R = (0,a), (x,y,z), lp;
minpoly = a2 + 1;
ideal I = x2 - 1/2, a*z - 1, y - 2;
def T = zeroSet(I);
setring T;
minpoly;
 $\mapsto (4a^4+4a^2+9)$ 
newA;
 $\mapsto (1/3a^3+5/6a)$ 
id;
 $\mapsto \text{id}[1]=(1/3a^3+5/6a)*z-1$ 
 $\mapsto \text{id}[2]=y-2$ 
 $\mapsto \text{id}[3]=2*x^2-1$ 
theZeroSet;
 $\mapsto [1]:$ 
 $\mapsto \_ [1]=(-1/3a^3+1/6a)$ 
 $\mapsto \_ [2]=2$ 
 $\mapsto \_ [3]=(-1/3a^3-5/6a)$ 
 $\mapsto [2]:$ 
 $\mapsto \_ [1]=(1/3a^3-1/6a)$ 
 $\mapsto \_ [2]=2$ 
 $\mapsto \_ [3]=(-1/3a^3-5/6a)$ 
map F1 = basering, theZeroSet[1];
map F2 = basering, theZeroSet[2];
F1(id);
 $\mapsto \_ [1]=0$ 
 $\mapsto \_ [2]=0$ 
 $\mapsto \_ [3]=0$ 
F2(id);
 $\mapsto \_ [1]=0$ 
 $\mapsto \_ [2]=0$ 
 $\mapsto \_ [3]=0$ 
```

### D.8.10.6 egcdMain

Procedure from library `zeroset.lib` (see [Section D.8.10 \[zeroset.lib\], page 1899](#)).

**Usage:** `egcdMain(f, g)`; where  $f, g$  are polynomials in  $\mathbb{Q}[a, x]$

**Purpose:** compute the polynomial gcd of  $f$  and  $g$  over  $\mathbb{Q}(a)[x]$

**Return:** `poly`

**Assume:** `basing =  $\mathbb{Q}[x, a]$`  and ideal `mpoly` is defined (it might be 0), this represents the ring  $\mathbb{Q}(a)[x]$  together with its minimal polynomial.

**Note:** outdated, use `gcd` instead

### D.8.10.7 factorMain

Procedure from library `zeroset.lib` (see [Section D.8.10 \[zeroset.lib\], page 1899](#)).

**Usage:** `factorMain(f)`; where  $f$  is a polynomial

**Purpose:** compute the factorization of the squarefree polynomial  $f$  over  $\mathbb{Q}(a)[t]$ , `minpoly =  $p(a)$` .

**Return:** list with 2 entries

`_[1]` = factors, first is a constant

`_[2]` = multiplicities (not yet implemented)

**Assume:** `basing =  $\mathbb{Q}[x, a]$` , representing  $\mathbb{Q}(a)[x]$ . An ideal `mpoly` must be defined, representing the minimal polynomial (it might be 0!).

**Note:** outdated, use `factorize` instead

### D.8.10.8 invertNumberMain

Procedure from library `zeroset.lib` (see [Section D.8.10 \[zeroset.lib\], page 1899](#)).

**Usage:** `invertNumberMain(f)`; where  $f$  is a polynomial in  $\mathbb{Q}[a]$

**Purpose:** compute  $1/f$  if  $f$  is a number in  $\mathbb{Q}(a)$ , i.e.,  $f$  is represented by a polynomial in  $\mathbb{Q}[a]$ .

**Return:** `poly`  $1/f$

**Assume:** `basing =  $\mathbb{Q}[x_1, \dots, x_n, a]$` , ideal `mpoly` must be defined and `!= 0` !

**Note:** outdated, use `/` instead

### D.8.10.9 quotientMain

Procedure from library `zeroset.lib` (see [Section D.8.10 \[zeroset.lib\], page 1899](#)).

**Usage:** `quotientMain(f, g)`; where  $f, g$  are polynomials in  $\mathbb{Q}(a)[x]$

**Purpose:** compute the quotient  $q$  and remainder  $r$  s.th.  $f = g \cdot q + r$ ,  $\deg(r) < \deg(g)$

**Return:** list of polynomials

`_[1]` = quotient  $q$

`_[2]` = remainder  $r$

**Assume:** `basing =  $\mathbb{Q}[x, a]$`  and ideal `mpoly` is defined (it might be 0), this represents the ring  $\mathbb{Q}(a)[x]$  together with its minimal polynomial.

**Note:** outdated, use `div/mod` instead

### D.8.10.10 remainderMain

Procedure from library `zeroset.lib` (see [Section D.8.10 \[zeroset.lib\], page 1899](#)).

**Usage:** `remainderMain(f, g);` where  $f, g$  are polynomials in  $\mathbb{Q}[a, x]$

**Purpose:** compute the remainder  $r$  s.t.  $f = g \cdot q + r$ ,  $\deg(r) < \deg(g)$

**Return:** `poly`

**Assume:** `basing =  $\mathbb{Q}[x, a]$`  and ideal `mpoly` is defined (it might be 0), this represents the ring  $\mathbb{Q}(a)[x]$  together with its minimal polynomial.

**Note:** outdated, use `mod/reduce` instead

### D.8.10.11 rootsMain

Procedure from library `zeroset.lib` (see [Section D.8.10 \[zeroset.lib\], page 1899](#)).

**Usage:** `rootsMain(f);` where  $f$  is a polynomial in  $\mathbb{Q}[a, x]$

**Purpose:** compute all roots of  $f$  in a finite extension of the  $\mathbb{Q}\mathbb{Q}$  without multiplicities.

**Return:** list, all entries are polynomials

- `_[1]` = roots of  $f$ , each entry is a polynomial
- `_[2]` = 'newA' - if the ground field is  $\mathbb{Q}(b)$  and the extension field is  $\mathbb{Q}(a)$ , then 'newA' is the representation of  $b$  in  $\mathbb{Q}(a)$
- `_[3]` = minpoly of the algebraic extension of the ground field

**Assume:** `basing =  $\mathbb{Q}[x, a]$`  ideal `mpoly` must be defined, it might be 0!

**Note:** might change the ideal `mpoly`!!

### D.8.10.12 sqfrNormMain

Procedure from library `zeroset.lib` (see [Section D.8.10 \[zeroset.lib\], page 1899](#)).

**Usage:** `sqfrNorm(f);` where  $f$  is a polynomial in  $\mathbb{Q}(a)[x]$

**Purpose:** compute the norm of the squarefree polynomial  $f$  in  $\mathbb{Q}(a)[x]$ .

**Return:** list with 3 entries

- `_[1]` = squarefree norm of  $g$  (poly)
- `_[2]` =  $g (= f(x - s \cdot a))$  (poly)
- `_[3]` =  $s$  (int)

**Assume:**  $f$  must be squarefree, `basing =  $\mathbb{Q}[x, a]$`  and ideal `mpoly` is equal to 'minpoly', this represents the ring  $\mathbb{Q}(a)[x]$  together with 'minpoly'.

**Note:** the norm is an element of  $\mathbb{Q}[x]$

### D.8.10.13 containedQ

Procedure from library `zeroset.lib` (see [Section D.8.10 \[zeroset.lib\], page 1899](#)).

**Usage:** `containedQ(data, f [, opt]);` `data`=list; `f`=any type; `opt`=integer

**Purpose:** test if  $f$  is an element of `data`.

**Return:** int  
 0 if f not contained in data  
 1 if f contained in data

**Options:** opt = 0 : use '==' for comparing f with elements from data  
 opt = 1 : use `sameQ` for comparing f with elements from data

### D.8.10.14 sameQ

Procedure from library `zeroset.lib` (see [Section D.8.10 \[zeroset.lib\], page 1899](#)).

**Usage:** `sameQ(a, b);` a,b=list/intvec

**Purpose:** test  $a == b$  elementwise, i.e.,  $a[i] = b[i]$ .

**Return:** int  
 0 if  $a \neq b$   
 1 if  $a == b$

## D.9 Visualization

### D.9.1 graphics\_lib

**Library:** `graphics.lib`

**Purpose:** Procedures to use Graphics with Mathematica

**Author:** Christian Gorzel, gorzelc@math.uni-muenster.de

**Procedures:**

#### D.9.1.1 staircase

Procedure from library `graphics.lib` (see [Section D.9.1 \[graphics.lib\], page 1905](#)).

**Usage:** `staircase(s,I);` s a string, I ideal in two variables

**Return:** string with Mathematica input for displaying staircase diagrams of an ideal I, i.e. exponent vectors of the initial ideal of I

**Note:** ideal I should be given by a standard basis. Let `s=""` and copy and paste the result into a Mathematica notebook.

**Example:**

```
LIB "graphics.lib";
ring r0 = 0,(x,y),ls;
ideal I = -1x2y6-1x4y2, 7x6y5+1/2x7y4+6x4y6;
staircase("",std(I));
⇒
⇒ Show[Graphics[{
⇒ {GrayLevel[0.5],Map[Rectangle[#, {9,9}] &, {{2,6},{6,2}}]},
⇒ {PointSize[0.03], Map[Point,{{2,6},{6,2}}]},
⇒ Table[Circle[{i,j},0.1],{i,0,9},{j,0,9}]],
⇒ Axes->True,AspectRatio->Automatic]]
⇒
ring r1 = 0,(x,y),dp;
ideal I = fetch(r0,I);
```

```

staircase("",std(I));
⇒
⇒ Show[Graphics[{
⇒ {GrayLevel[0.5],Map[Rectangle[#{12,9}] &, {{2,6},{7,4},{9,2}}]},
⇒ {PointSize[0.03], Map[Point,{{2,6},{7,4},{9,2}}]},
⇒ Table[Circle[{i,j},0.1],{i,0,12},{j,0,9}]},
⇒ Axes->True,AspectRatio->Automatic]]
⇒
ring r2 = 0,(x,y),wp(2,3);
ideal I = fetch(r0,I);
staircase("",std(I));
⇒
⇒ Show[Graphics[{
⇒ {GrayLevel[0.5],Map[Rectangle[#{13,9}] &, {{2,6},{8,3},{10,2},{6,5}}]},
⇒ {PointSize[0.03], Map[Point,{{2,6},{8,3},{10,2},{6,5}}]},
⇒ Table[Circle[{i,j},0.1],{i,0,13},{j,0,9}]},
⇒ Axes->True,AspectRatio->Automatic]]
⇒
// Paste the output into a Mathematica notebook
// active evaluation of the cell with SHIFT RETURN

```

### D.9.1.2 mathinit

Procedure from library `graphics.lib` (see [Section D.9.1 \[graphics.lib\]](#), page 1905).

**Usage:** `mathinit();`

**Return:** initializing string for loading Mathematica's `ImplicitPlot`

**Example:**

```

LIB "graphics.lib";
mathinit();
⇒ << Graphics`ImplicitPlot`
// Paste the output into a Mathematica notebook
// active evaluation of the cell with SHIFT RETURN

```

### D.9.1.3 mplot

Procedure from library `graphics.lib` (see [Section D.9.1 \[graphics.lib\]](#), page 1905).

**Usage:** `mplot(fname, I [,I1,I2,..,s] );` `fname=string`; `I,I1,I2,..=ideals`, `s=string` representing the plot region.  
Use the ideals `I1,I2,..` in order to produce multiple plots (they need to have the same number of entries as `I!`).

**Return:** string, text with Mathematica commands to display a plot

**Note:** The `plotregion` is defaulted to `-1,1` around zero.  
For implicit given curves enter first the string returned by procedure `mathinit` into Mathematica in order to load `ImplicitPlot`. The following conventions for `I` are used:

- ideal with 2 entries in one variable means a parametrised plane curve,
- ideal with 3 entries in one variable means a parametrised space curve,
- ideal with 3 entries in two variables means a parametrised surface,
- ideal with 2 entries in two variables means an implicit curve  
given as `I[1]==I[2]`,



- ideal with 1 entry (or one polynomial) in two variables means  
an implicit curve given as  $f == 0$ ,

**Example:**

```
LIB "graphics.lib";
// ----- plane curves -----
ring rr0 = 0,x,dp; export rr0;
ideal I = x3 + x, x2;
ideal J = x2, -x+x3;
mplot("",I,J,"-2,2");
↳
↳ ParametricPlot[{{s^3+s,s^2},{s^2,s^3-s}},{s,-2,2},
↳ AspectRatio->Automatic];
↳
// Paste the output into a Mathematica notebook
// active evaluation of the cell with SHIFT RETURN
// ----- space curves -----
I = x3,-1/10x3+x2,x2;
mplot("",I);
↳
↳ ParametricPlot3D[{{s^3,-1/10*s^3+s^2,s^2}},{s,-1,1},
↳ ViewPoint->{1.3,-2.4,2}];
↳
// Paste the output into a Mathematica notebook
// active evaluation of the cell with SHIFT RETURN
// ----- surfaces -----
ring rr1 = 0,(x,y),dp; export rr1;
ideal J = xy,y,x2;
mplot("",J,"-2,1","1,2");
↳
↳ ParametricPlot3D[{{s*t,t,s^2}},{s,-2,1},{t,1,2},
↳ Boxed->True, Axes->True, ViewPoint->{1.3,-2.4,2}];
↳
// Paste the output into a Mathematica notebook
// active evaluation of the cell with SHIFT RETURN
kill rr0,rr1;
```

## D.9.2 latex\_lib

**Library:** latex.lib

**Purpose:** Typesetting of Singular-Objects in LaTeX2e

**Author:** Christian Gorzel, gorzelc@math.uni-muenster.de

**Global variables:**

TeXwidth, TeXnofrac, TeXbrack, TeXproj, TeXaligned, TeXreplace, NoDollars  
are used to control the typesetting. Call `texdemo()`; to obtain a LaTeX2e file  
`texlibdemo.tex` explaining the features of `latex.lib` and its global variables.

TeXwidth (int) -1, 0, 1..9, >9: controls breaking of long polynomials

TeXnofrac (int) flag: write  $1/2$  instead of  $\frac{1}{2}$

TeXbrack (string) "{", "(", "<", "|", empty string:

controls brackets around ideals and matrices

TeXproj (int) flag: write ":" instead of "," in vectors  
 TeXaligned (int) flag: write maps (and ideals) aligned  
 TeXreplace (list) list entries = 2 strings: replacing symbols  
 NoDollars (int) flag: suppresses surrounding \$ signs

**Procedures:****D.9.2.1 closetex**

Procedure from library `latex.lib` (see [Section D.9.2 \[latex.lib\], page 1907](#)).

**Usage:** `closetex(fname)`; `fname` string

**Return:** nothing; writes a LaTeX2e closing line into file `<fname>`.

**Note:** preceding ">>" are deleted and suffix ".tex" (if not given) is added to `fname`.

**Example:**

```
LIB "latex.lib";
opentex("exmpl");
texobj("exmpl", "{\\large \\bf hello}");
closetex("exmpl");
```

**D.9.2.2 opentex**

Procedure from library `latex.lib` (see [Section D.9.2 \[latex.lib\], page 1907](#)).

**Usage:** `opentex(fname)`; `fname` string

**Return:** nothing; writes a LaTeX2e header into a new file `<fname>`.

**Note:** preceding ">>" are deleted and suffix ".tex" (if not given) is added to `fname`.

**Example:**

```
LIB "latex.lib";
opentex("exmpl");
texobj("exmpl", "hello");
closetex("exmpl");
```

**D.9.2.3 tex**

Procedure from library `latex.lib` (see [Section D.9.2 \[latex.lib\], page 1907](#)).

**Usage:** `tex(fname)`; `fname` string

**Return:** nothing; calls latex (LaTeX2e) for compiling the file `fname`

**Note:** preceding ">>" are deleted and suffix ".tex" (if not given) is added to `fname`.

**Example:**

```
LIB "latex.lib";
ring r;
ideal I = maxideal(7);
opentex("exp001");           // open latex2e document
texobj("exp001", "An ideal ", I);
closetex("exp001");
tex("exp001");
↳ calling latex2e for: exp001.tex
↳
```

```

⇒ This is pdfTeX, Version 3.14159265-2.6-1.40.18 (TeX Live 2017) (preloaded\
   format=latex)
⇒ restricted \write18 enabled.
⇒ entering extended mode
⇒ (./exp001.tex
⇒ LaTeX2e <2017/01/01> patch level 3
⇒ Babel <3.10> and hyphenation patterns for 3 language(s) loaded.
⇒ (/usr/share/texmf-dist/tex/latex/base/article.cls
⇒ Document Class: article 2014/09/29 v1.4h Standard LaTeX document class
⇒ (/usr/share/texmf-dist/tex/latex/base/size10.clo))
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsmath.sty
⇒ For additional information on amsmath, use the '?' option.
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amstext.sty
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsgen.sty))
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsbsy.sty)
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsopn.sty))
⇒ (/usr/share/texmf-dist/tex/latex/amsfonts/amssymb.sty
⇒ (/usr/share/texmf-dist/tex/latex/amsfonts/amsfonts.sty))
⇒ No file exp001.aux.
⇒ (/usr/share/texmf-dist/tex/latex/amsfonts/umsa.fd)
⇒ (/usr/share/texmf-dist/tex/latex/amsfonts/umsb.fd) [1] (./exp001.aux) )
⇒ Output written on exp001.dvi (1 page, 2912 bytes).
⇒ Transcript written on exp001.log.
system("sh","rm exp001.*");
⇒ 0

```

#### D.9.2.4 texdemo

Procedure from library `latex.lib` (see [Section D.9.2 \[latex.lib\]](#), page 1907).

**Usage:** `texdemo();`

**Return:** nothing; generates a LaTeX2e file called `texlibdemo.tex` explaining the features of `latex.lib` and its global variables.

#### D.9.2.5 texfactorize

Procedure from library `latex.lib` (see [Section D.9.2 \[latex.lib\]](#), page 1907).

**Usage:** `texfactorize(fname,f);` `fname` string, `f` poly

**Return:** if `fname=""`: string, `f` as a product of its irreducible factors  
otherwise: append this string to the file `<fname>`, and return nothing.

**Note:** preceding `>>>` are deleted and suffix `.tex` (if not given) is added to `fname`.

**Example:**

```

LIB "latex.lib";
ring r2 = 13,(x,y),dp;
poly f = (x+1+y)^2*x^3y*(2x-2y)*y^12;
texfactorize("",f);
⇒ $-2\cdot (-x+y)\cdot (x+y+1)^2\cdot x^3\cdot y^{13}$
ring R49 = (7,a),x,dp;
minpoly = a^2+a+3;
poly f = (a^24x^5+x^3)*a^2x^6*(x+1)^2;
f;

```

```

⇒ (a+3)*x13+(2a-1)*x12+(-2a+1)*x10+(-a-3)*x9
texfactorize("",f);
⇒ $(a+3)\cdot (x-1)\cdot (x+1)^3\cdot x^9$

```

### D.9.2.6 texmap

Procedure from library `latex.lib` (see [Section D.9.2 \[latex.lib\], page 1907](#)).

- Usage:** `texmap(fname,m,@r1,@r2)`; `fname` string, `m` string/map, `@r1,@r2` rings
- Return:** if `fname=""`: string, the map `m` from `@r1` to `@r2` (preceded by its name if `m = string`) in TeX-typesetting;  
otherwise: append this string to the file `<fname>`, and return nothing.
- Note:** preceding `>>>` are deleted in `fname`, and suffix `.tex` (if not given) is added to `fname`.  
If `m` is a string then it has to be the name of an existing map from `@r1` to `@r2`.

**Example:**

```

LIB "latex.lib";
// ----- prepare for example -----
if (defined(TeXaligned)) {int Teali=TeXaligned; kill TeXaligned;}
if (defined(TeXreplace)) {list Terep=TeXreplace; kill TeXreplace;}
// ----- the example starts here -----
//
string fname = "tldemo";
ring @r1=0,(x,y,z),dp;
export @r1;
ring r2=0,(u,v),dp;
map @phi =(@r1,u2,uv -v,v2); export @phi;
list TeXreplace;
TeXreplace[1] = list("@phi","\\phi"); // @phi --> \phi
export TeXreplace;
texmap("", "@phi",@r1,r2); // standard form
⇒ $$
⇒ \begin{array}{rcc}
⇒ \phi: \mathbb{Q}[x,y,z] & \xrightarrow{\quad} & \mathbb{Q}[u,v] \\
⇒ \left(x,y,z\right) & \longmapsto & \left(u\right. \\
⇒ \left( & & \\
⇒ \begin{array}{c}
⇒ u^2 \\
⇒ uv-v \\
⇒ v^2 \end{array} \\
⇒ \end{array} \\
⇒ \right) \\
⇒ \end{array} \\
⇒ $$
//
int TeXaligned; export TeXaligned; // map in one line
texmap("",@phi,@r1,r2);
⇒ $\mathbb{Q}[x,y,z]\xrightarrow{\quad}\mathbb{Q}[u,v], \quad \left(x,y,z\right)\longmapsto \left(u\right. \\
\quad \left.^2,uv-v,v^2\right)$
//
kill @r1,TeXreplace,TeXaligned;
//
// --- restore global variables if previously defined ---

```

```

if (defined(Teali)) {int TeXaligned=Teali; export TeXaligned; kill Teali;}
if (defined(Terep)) {list TeXreplace=Terep; export TeXreplace; kill Terep;}

```

### D.9.2.7 texname

Procedure from library `latex.lib` (see [Section D.9.2 \[latex.lib\], page 1907](#)).

**Usage:** `texname(fname,s);` `fname,s` strings

**Return:** if `fname=""`: the transformed string `s`, for which the following rules apply:

<code>s' + "~"</code>	<code>--&gt; "\\tilde{" + s' + "}"</code>
<code>"_" + int</code>	<code>--&gt; "_" + int + "}"</code>
<code>"[" + s' + "]"</code>	<code>--&gt; "_" + s' + "}"</code>
<code>"A..Z" + int</code>	<code>--&gt; "A..Z" + "^{" + int + "}"</code>
<code>"a..z" + int</code>	<code>--&gt; "a..z" + "_" + int + "}"</code>
<code>"(" + int + "," + s' + ")"</code>	<code>--&gt; "_" + int + "}" + "^{" + s' + "}"</code>

Furthermore, strings which begin with a left brace are modified by deleting the first and the last character (which is then assumed to be a right brace).

if `fname!=""`: append the transformed string `s` to the file `<fname>`, and return nothing.

**Note:** preceding `>>` are deleted in `fname`, and suffix `.tex` (if not given) is added to `fname`.

**Example:**

```

LIB "latex.lib";
ring r = 0,(x,y),lp;
poly f = 3xy4 + 2xy2 + x5y3 + x + y6;
texname("", "{f(10)}");
⇒ f(10)
texname("", "f(10) =");
⇒ f_{10} =
texname("", "n1");
⇒ n_{1}
texname("", "T1_12");
⇒ T^{1}_{12}
texname("", "g'_{11}");
⇒ g'_{11}
texname("", "f23");
⇒ f_{23}
texname("", "M[2,3]");
⇒ M_{2,3}
texname("", "A(0,3);");
⇒ A_{0}^{3};
texname("", "E~(3)");
⇒ \tilde{E}_{3}

```

### D.9.2.8 texobj

Procedure from library `latex.lib` (see [Section D.9.2 \[latex.lib\], page 1907](#)).

**Usage:** `texobj(fname,l);` `fname` string, `l` list

**Return:** if `fname=""`: string, the entries of `l` in LaTeX-typesetting;  
otherwise: append this string to the file `<fname>`, and return nothing.

**Note:** preceding `>>` are deleted in `fname`, and suffix `.tex` (if not given) is added to `fname`.

**Example:**

```

LIB "latex.lib";
// ----- prepare for example -----
if (defined(TeXaligned)) {int Teali=TeXaligned; kill TeXaligned;}
if (defined(TeXbrack)){string Tebra=TeXbrack; kill TeXbrack;}
//
// ----- typesetting for polynomials -----
ring r = 0,(x,y),lp;
poly f = x5y3 + 3xy4 + 2xy2 + y6;
f;
⇒ x5y3+3xy4+2xy2+y6
texobj("",f);
⇒ $$
⇒ \begin{array}{rl}
⇒ & x^5y^3+3xy^4+2xy^2+y^6\\
⇒ & \end{array}
⇒ $$
// ----- typesetting for ideals -----
ideal G = jacob(f);
G;
⇒ G[1]=5x4y3+3y4+2y2
⇒ G[2]=3x5y2+12xy3+4xy+6y5
texobj("",G);
⇒ $$
⇒ \left(
⇒ \begin{array}{c}
⇒ 5x^4y^3+3y^4+2y^2, \\
⇒ 3x^5y^2+12xy^3+4xy+6y^5
⇒ \end{array}
⇒ \right)$$
// ----- variation of typesetting for ideals -----
int TeXaligned = 1; export TeXaligned;
string TeXbrack = "<"; export TeXbrack;
texobj("",G);
⇒ $\left<5x^4y^3+3y^4+2y^2,3x^5y^2+12xy^3+4xy+6y^5\right>$
kill TeXaligned, TeXbrack;
// ----- typesetting for matrices -----
matrix J = jacob(G);
texobj("",J);
⇒ $$
⇒ \left(
⇒ \begin{array}{*{2}{c}}
⇒ 20x^3y^3 & 15x^4y^2+12y^3+4y \\
⇒ 15x^4y^2+12y^3+4y & 6x^5y+36xy^2+4x+30y^4
⇒ \end{array}
⇒ \right)
⇒ $$
// ----- typesetting for intmats -----
intmat m[3][4] = 9,2,4,5,2,5,-2,4,-6,10,-1,2,7;
texobj("",m);
⇒ $$
⇒ \left(
⇒ \begin{array}{*{4}{r}}

```

```

    ↪ 9 & 2 & 4 & 5\\
    ↪ 2 & 5 & -2 & 4\\
    ↪ -6 & 10 & -1 & 2
    ↪ \end{array}
    ↪ \right)
    ↪ $$
    //
    // --- restore global variables if previously defined ---
    if (defined(Teali)){int TeXaligned=Teali; export TeXaligned; kill Teali;}
    if (defined(Tebra)){string TeXbrack=Tebra; export TeXbrack; kill Tebra;}

```

### D.9.2.9 texpoly

Procedure from library `latex.lib` (see [Section D.9.2 \[latex.lib\], page 1907](#)).

**Usage:** `texpoly(fname,p);` fname string, p poly

**Return:** if `fname=""`: string, the polynomial p in LaTeX-typesetting;  
otherwise: append this string to the file `<fname>`, and return nothing.

**Note:** preceding `>>>` are deleted in `fname`, and suffix `.tex` (if not given) is added to `fname`.

**Example:**

```

LIB "latex.lib";
ring r0=0,(x,y,z),dp;
poly f = -1x^2 + 2;
texpoly("",f);
↪ $-x^{2}+2$
ring rr= real,(x,y,z),dp;
texpoly("",2x2y23z);
↪ $2.000x^{2}y^{23}z$
ring r7= 7,(x,y,z),dp;
poly f = 2x2y23z;
texpoly("",f);
↪ $2x^{2}y^{23}z$
ring rab =(0,a,b),(x,y,z),dp;
poly f = (-2a2 +b3 -2)/a * x2y4z5 + (a2+1)*x + a+1;
f;
↪ (-2a2+b3-2)/(a)*x2y4z5+(a2+1)*x+(a+1)
texpoly("",f);
↪ $-\frac{2a^{2}-b^{3}+2}{a}\cdot x^{2}y^{4}z^{5}+(a^{2}+1)\cdot x+(a+1)$
texpoly("",1/(a2+2)*x2+2/b*x+(a+1)/3);
↪ $\frac{1}{a^{2}+2}\cdot x^{2}+\frac{2}{b}\cdot x+\frac{a+1}{3}$

```

### D.9.2.10 texproc

Procedure from library `latex.lib` (see [Section D.9.2 \[latex.lib\], page 1907](#)).

**Usage:** `texproc(fname,pname);` fname,pname strings

**Assume:** ‘pname’ is a procedure.

**Return:** if `fname=""`: string, the proc ‘pname’ in a verbatim environment in LaTeX-typesetting;  
otherwise: append this string to the file `<fname>`, and return nothing.

**Note:** preceding `>>>` are deleted in `fname`, and suffix `.tex` (if not given) is added to `fname`.

**Example:**

```

LIB "latex.lib";
texproc("", "texproc");
⇨ \begin{verbatim}
⇨ proc texproc(string fname, string pname)
⇨ {
⇨   int i, j, k, nl;
⇨   string @p, s, t;
⇨
⇨   j = 1;
⇨
⇨   if (defined('pname'))
⇨   { if (typeof('pname')=="proc")
⇨     { @p = string('pname');
⇨       nl = find(@p, newline);
⇨       s = "\\begin{verbatim}" + newline;
⇨       s = s + "proc " + pname + "(";
⇨       i = find(@p, "parameter");          // collecting the parameters
⇨       k = find(@p, "alias");              // and the alias arguments
⇨       while((i and i < nl) or (k and k < nl))
⇨       {
⇨         if (i and (k==0 or i<k))
⇨         {
⇨           j=find(@p, ";", i);
⇨           t = @p[i+10, j-i-10];
⇨           if(i>1){s = s + ",";};
⇨           s = s + t;
⇨         }
⇨         if (k and (i==0 or k<i))
⇨         {
⇨           j=find(@p, ";", k);
⇨           t = @p[k, j-k];
⇨           if(k>1){s = s + ",";};
⇨           s = s + t;
⇨         }
⇨         i = find(@p, "parameter", j);
⇨         k = find(@p, "alias", j);
⇨       }
⇨       s = s + ")" + newline;
⇨       j++;                          // skip one for the newline
⇨       i = find(@p, ";"+"return();" + newline, j);
⇨       if (!(i))
⇨       { i = find(@p, ";"+"RETURN();" + newline, j); } // j kann hier weg
⇨       s = s + "{" + @p[j, i-j-1] + "}" + newline;
⇨       s = s + "\\end{verbatim}" + newline;
⇨     }
⇨   }
⇨   else
⇨   { print(" // -- Error: No such proc defined");
⇨     return();
⇨   }
⇨   if(size(fname))
⇨   { i=1;

```



```

⇒ while (fname[i]==">"){i++;}
⇒ fname = fname[i,size(fname)-i+1];
⇒ if (size(fname)>=4) // check if filename is ending with ".tex"
⇒ { if(fname[size(fname)-3,4]!=".tex") {fname = fname + ".tex"; }
⇒ }
⇒ else {fname = fname + ".tex";}
⇒ write(fname,s);
⇒ }
⇒ else{return(s);}
⇒ }
⇒ \end{verbatim}
⇒

```

### D.9.2.11 texring

Procedure from library `latex.lib` (see [Section D.9.2 \[latex.lib\]](#), page 1907).

**Usage:** `texring(fname, r[L]);` fname string, r ring, L list

**Return:** if `fname=""`: string, the ring in TeX-typesetting;  
otherwise: append this string to the file `<fname>` and return nothing.

**Note:** preceding `>>` are deleted and suffix `".tex"` (if not given) is added to `fname`.  
The optional list L is assumed to be a list of strings which control, for instance the symbol for the field of coefficients.  
For more details call `texdemo()`; (generates a LaTeX2e file called `texlibdemo.tex` which explains all features of `texring`).

**Example:**

```

LIB "latex.lib";
ring r0 = 0,(x,y),dp; // char = 0, polynomial ordering
texring("",r0);
⇒ $\mathbb{Q}[x,y]$
//
ring r7 = 7,(x(0..2)),ds; // char = 7, local ordering
texring("",r7);
⇒ $\mathbb{Z}_7[[x_0,x_1,x_2]]$
//
ring r1 = 0,(x1,x2,y1,y2),wp(1,2,3,4);
texring("",r1);
⇒ $\mathbb{Q}[x_1,x_2,y_1,y_2]$
//
ring rr = real,(x),dp; // real numbers
texring("",rr);
⇒ $\mathbb{R}[x]$
//
ring rC = complex,x,dp; // complex coefficients
texring("",rC);
⇒ $\mathbb{C}[x]$
//
ring rabc =(0,t1,t2,t3),(x,y),dp; // ring with parameters
texring("",rabc);
⇒ $\mathbb{Q}(t_1,t_2,t_3)[x,y]$
//
ring ralg = (7,a),(x1,x2),ds; // algebraic extension

```

```

minpoly = a2-a+3;
texring("",ralg);
 $\mapsto \mathbb{Z}_7(a)[[x_1, x_2]]$ 
texring("",ralg,"mipo");
 $\mapsto \mathbb{Z}_7(a)/(a^2-a+3)[[x_1, x_2]]$ 
//
ring r49=(49,a),x,dp;                // Galois field
texring("",r49);
 $\mapsto \mathbb{F}_{49}[x]$ 
//
setring r0;                          // quotient ring
ideal i = x2-y3;
qring q = std(i);
texring("",q);
 $\mapsto \mathbb{Q}[x,y]/\langle y^3-x^2 \rangle$ 
//
// ----- additional features -----
ring r9 =0,(x(0..9)),ds;
texring("",r9,1);
 $\mapsto \mathbb{Q}[[x_0, \ldots, x_9]]$ 
texring("",r9,"C","{","^G");
 $\mapsto \mathbb{C}[[x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9]]^G$ 
//
ring rxy = 0,(x(1..5),y(1..6)),ds;
intvec v = 5,6;
texring("",rxy,v);
 $\mapsto \mathbb{Q}[[x_1, \ldots, x_5, y_1, \ldots, y_6]]$ 

```

### D.9.2.12 rmx

Procedure from library `latex.lib` (see [Section D.9.2 \[latex.lib\]](#), page 1907).

**Usage:** `rmx(fname)`; fname string

**Return:** nothing; removes the `.log` and `.aux` files associated to the LaTeX file `<fname>`.

**Note:** If `fname` ends by `".dvi"` or `".tex"`, the `.dvi` or `.tex` file will be deleted, too.

**Example:**

```

LIB "latex.lib";
ring r;
poly f = x+y+z;
opentex("exp001");                // defaulted latex2e document
texobj("exp001","A polynom",f);
closetex("exp001");
tex("exp001");
 $\mapsto$  calling latex2e for: exp001.tex
 $\mapsto$ 
 $\mapsto$  This is pdfTeX, Version 3.14159265-2.6-1.40.18 (TeX Live 2017) (preloaded\
format=latex)
 $\mapsto$  restricted \write18 enabled.
 $\mapsto$  entering extended mode
 $\mapsto$  (./exp001.tex
 $\mapsto$  LaTeX2e <2017/01/01> patch level 3

```

```

⇒ Babel <3.10> and hyphenation patterns for 3 language(s) loaded.
⇒ (/usr/share/texmf-dist/tex/latex/base/article.cls
⇒ Document Class: article 2014/09/29 v1.4h Standard LaTeX document class
⇒ (/usr/share/texmf-dist/tex/latex/base/size10.clo))
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsmath.sty
⇒ For additional information on amsmath, use the '?' option.
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amstext.sty
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsgen.sty))
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsbsy.sty)
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsopn.sty))
⇒ (/usr/share/texmf-dist/tex/latex/amsfonts/amssymb.sty
⇒ (/usr/share/texmf-dist/tex/latex/amsfonts/amsfonts.sty))
⇒ No file exp001.aux.
⇒ (/usr/share/texmf-dist/tex/latex/amsfonts/umsa.fd)
⇒ (/usr/share/texmf-dist/tex/latex/amsfonts/umsb.fd) [1] (./exp001.aux) )
⇒ Output written on exp001.dvi (1 page, 308 bytes).
⇒ Transcript written on exp001.log.
rmx("exp001"); // removes aux and log file of exp001
system("sh","rm exp001.*");
⇒ 0

```

### D.9.2.13 xdvi

Procedure from library `latex.lib` (see [Section D.9.2 \[latex.lib\]](#), page 1907).

**Usage:** `xdvi(fname[,style]);` `fname,style = string`

**Return:** nothing; displays dvi-file `fname.dvi` with previewer `xdvi`

**Note:** suffix `.dvi` may be omitted in `fname`  
`style` captures the program that will be called instead of the default (`xdvi`)

**Example:**

```

LIB "latex.lib";
intmat m[3][4] = 9,2,4,5,2,5,-2,4,-6,10,-1,2,7;
opentex("exp001");
texobj("exp001","An intmat: ",m);
closetex("exp001");
tex("exp001");
⇒ calling latex2e for: exp001.tex
⇒
⇒ This is pdfTeX, Version 3.14159265-2.6-1.40.18 (TeX Live 2017) (preloaded\
format=latex)
⇒ restricted \write18 enabled.
⇒ entering extended mode
⇒ (./exp001.tex
⇒ LaTeX2e <2017/01/01> patch level 3
⇒ Babel <3.10> and hyphenation patterns for 3 language(s) loaded.
⇒ (/usr/share/texmf-dist/tex/latex/base/article.cls
⇒ Document Class: article 2014/09/29 v1.4h Standard LaTeX document class
⇒ (/usr/share/texmf-dist/tex/latex/base/size10.clo))
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsmath.sty
⇒ For additional information on amsmath, use the '?' option.
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amstext.sty
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsgen.sty))

```

```

⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsbsy.sty)
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsopn.sty)
⇒ (/usr/share/texmf-dist/tex/latex/amsfonts/amssymb.sty)
⇒ (/usr/share/texmf-dist/tex/latex/amsfonts/amsfonts.sty)
⇒ No file exp001.aux.
⇒ (/usr/share/texmf-dist/tex/latex/amsfonts/umsa.fd)
⇒ (/usr/share/texmf-dist/tex/latex/amsfonts/umsb.fd) [1] (./exp001.aux) )
⇒ Output written on exp001.dvi (1 page, 524 bytes).
⇒ Transcript written on exp001.log.
xdvi("exp001");
⇒ calling xdvi for : exp001
⇒
system("sh","rm exp001.*");
⇒ 0

```

### D.9.3 surf\_lib

**Library:** surf.lib

**Purpose:** Procedures for Graphics with Surf

**Author:** Hans Schoenemann, Frank Seelisch

**Note:** Using this library requires the program `surf` to be installed. You can download `surf` either from <http://sourceforge.net/projects/surf> or from <ftp://jim.mathematik.uni-kl.de/pub/Math/Singular/utils/>. The procedure `surfer` requires the program `surfer` (version 1.4.1 or newer) to be installed. You can download `surfer` from <http://imaginary.org/program/surfer>

**Procedures:** See also: [Section D.9.4 \[surfex\\_lib\]](#), page 1919.

#### D.9.3.1 plot

Procedure from library `surf.lib` (see [Section D.9.3 \[surf\\_lib\]](#), page 1918).

**Usage:** `plot(I)`; `I` ideal or poly

**Assume:** `I` defines a plane curve or a surface given by one equation

**Return:** nothing

**Note:** requires the external program ‘`surf`’ to be installed,  
to close the graphical interface just press ‘`Q`’

**Example:**

```

LIB "surf.lib";
// ----- plane curves -----
ring rr0 = 0, (x1, x2), dp;
ideal I = x1^3 - x2^2;
plot(I);
ring rr1 = 0, (x, y, z), dp;
ideal I(1) = 2x^2 - 1/2x^3 + 1 - y + 1;
plot(I(1));
// ---- Singular Logo -----
poly logo = ((x+3)^3 + 2*(x+3)^2 - y^2)*(x^3 - y^2)*((x-3)^3 - 2*(x-3)^2 - y^2);
plot(logo);
// Steiner surface

```

```

ideal J(2) = x^2*y^2+x^2*z^2+y^2*z^2-17*x*y*z;
plot(J(2));
// -----
plot(x*(x2-y2)+z2);
// E7
plot(x^3-x*y^3+z^2);
// Whitney umbrella
plot(z^2-x^2*y);

```

### D.9.3.2 surfer

Procedure from library `surf.lib` (see [Section D.9.3 \[surf.lib\]](#), page 1918).

**Usage:** `surfer(f); f poly`

**Assume:** `f` defines a surface given by one equation

**Return:** nothing

**Note:** requires the external program ‘surfer’ to be installed, to close the graphical interface just close the window of surfer

**Example:**

```

LIB "surf.lib";
ring rr1 = 0,(x,y,z),dp;
// Steiner surface
ideal J(2) = x^2*y^2+x^2*z^2+y^2*z^2-17*x*y*z;
surfer(J(2));
// -----
surfer(x*(x2-y2)+z2);
// E7
surfer(x^3-x*y^3+z^2);
// Whitney umbrella
surfer(z^2-x^2*y);

```

### D.9.4 surfex\_lib

**Library:** `surfex.lib`

**Purpose:** Procedures for visualizing and rotating surfaces.

**Author:** Oliver Labs  
 This library uses the program `surf`  
 (written by Stefan Endrass and others)  
 and `surfex` (written by Oliver Labs and others, mainly Stephan Holzer).

**Note:** It is still an alpha version (see <http://www.AlgebraicSurface.net>)  
 This library requires the program `surfex`, `surf` and `java` to be installed. The software is used for producing raytraced images of surfaces. You can download `surfex` from <http://www.surfex.AlgebraicSurface.net>  
`surfex` is a front-end for `surf` which aims to be easier to use than the original tool.

**Procedures:** See also: [Section D.9.3 \[surf.lib\]](#), page 1918.

### D.9.4.1 plotRotated

Procedure from library `surfex.lib` (see [Section D.9.4 \[surfex.lib\]](#), page 1919).

**Usage:** `plotRotated(poly p, list coords, list #)`  
 This opens the external program `surfex` for drawing the surface given by `p`, seen as a surface in the real affine space with coordinates `coords`. The optional `int` parameter can be used to set plotting quality.

**Assume:** `coords` is a list of three variables.  
 The basering is of characteristic zero and without parameters.

**Example:**

```
LIB "surfex.lib";
"Example:";
// An easy example: a surface with four conical nodes.
ring r = 0, (x,y,z), dp;
poly cayley_cubic = x^3+y^3+z^3+1^3-1/4*(x+y+z+1)^3;
// plotRotated(cayley_cubic, list(x,y,z));
// A difficult example: a surface with a one-dimensional real component!
poly whitney_umbrella = x^2*z-y^2;
// The Whitney Umbrella without its handle:
plotRotated(whitney_umbrella, list(x,y,z));
// The Whitney Umbrella together with its handle:
plotRotated(whitney_umbrella, list(x,y,z), 2);
```

### D.9.4.2 plotRot

Procedure from library `surfex.lib` (see [Section D.9.4 \[surfex.lib\]](#), page 1919).

**Usage:** `plotRot(poly p, list #)`  
 Similar to `plotRotated`, but guesses automatically which coordinates should be used. The optional `int` parameter can be used to set plotting quality.  
 It opens the external program `surfex` for drawing the surface given by `p`, seen as a surface in the real affine space with coordinates `coords`.

**Assume:** The basering is of characteristic zero and without parameters.

**Example:**

```
LIB "surfex.lib";
"Example:";
// More variables in the basering, but only 3 variables in the polynomial:
ring r1 = 0, (w,x,y,z), dp;
poly cayley_cubic = x^3+y^3+z^3+1^3-1/4*(x+y+z+1)^3;
plotRot(cayley_cubic);
// Three variables in the basering, but fewer variables in the polynomial:
ring r2 = 0, (x,y,z), dp;
plotRot(x^2+y^2-1);
plotRot(y^2+z^2-1);
// A cubic surface with a solitary point:
// Use the additional parameter 3 to ask singular
// to compute the singular locus before calling surfex.
ring r3 = 0, (x,y,z), dp;
poly kn_10 = x^3-3*x*y^2+z^3+3*x^2+3*y^2+z^2;
plotRot(kn_10, 3);
```

```
// The swallowtail:
// a surface with a real solitary curve sticking out of the surface.
// Use the additional parameter 3 to ask singular
// to compute the singular locus before calling surfex.
poly swallowtail = -4*y^2*z^3-16*x*z^4+27*y^4+144*x*y^2*z+128*x^2*z^2-256*x^3;
```

### D.9.4.3 plotRotatedList

Procedure from library `surfex.lib` (see [Section D.9.4 \[surfex.lib\], page 1919](#)).

**Usage:** `plotRotatedList(list varieties, list coords, list #)`  
This opens the external program `surfex` for drawing the surfaces given by varieties, seen as a surface in the real affine space with coordinates `coords`. The optional `int` parameter can be used to set plotting quality.

**Assume:** `coords` is a list of three variables, `varieties` is a list of ideals describing the varieties to be shown.  
The basering is of characteristic zero and without parameters.

**Example:**

```
LIB "surfex.lib";
"Example:";
// A cubic surface together with a tritangent plane
// (i.e. a plane which cuts out three lines).
ring r = 0, (x,y,z), dp;
poly cayley_cubic = x^3+y^3+z^3+1^3-1/4*(x+y+z+1)^3;
poly plane = 1-x-y-z;
plotRotatedList(list(cayley_cubic, plane), list(x,y,z));
// The same cubic and plane.
// The plane is not shown but only its intersection with the surface.
plotRotatedList(list(cayley_cubic, ideal(cayley_cubic, plane)), list(x,y,z));
```

### D.9.4.4 plotRotatedDirect

Procedure from library `surfex.lib` (see [Section D.9.4 \[surfex.lib\], page 1919](#)).

**Usage:** `plotRotatedDirect(list varieties, list #)`  
This opens the external program `surfex` for drawing the surfaces given by varieties, seen as a surface in the real affine space with coordinates `x,y,z`. The format for the list varieties is not fully documented yet; please, see the examples below and try to adjust the examples to suit your needs.  
The optional `int` parameter can be used to set plotting quality.

**Assume:** Passes the equations directly to `surfex`, i.e., the variable names should be `x,y,z`.  
The advantage is that one can use parameters `p1, p2, ...`; these will be passed to `surfex`.

**Example:**

```
LIB "surfex.lib";
"Example:";
// A cubic surface depending on a parameter:
ring r = (0,p1), (x,y,z), dp;
poly cayley_cubic = x^3+y^3+z^3+1^3-p1*(x+y+z+1)^3;
// The entries of the list of varieties can either be polynomials
plotRotatedDirect(list(list(list(cayley_cubic)),
list(),
```

```

list(list(1,"0.0","1.0","500","0.25+0.25*sin(PI*p1)"))
));
// or strings which represent surfex-readable polynomials
plotRotatedDirect(list(list(list("x^3+y^3+z^3+1^3-p1*(x+y+z+1)^3")),
list(),
list(list("1","0.0","1.0","500","0.25+0.25*sin(PI*p1)"))
));
// More complicated varieties
plotRotatedDirect(list(list(list("x^2+y^2-z^2-3^2"),
list("x*sin(p1)+y*cos(p1)-3")),
list(list(list(1,2))),
list(list("1","0.0","1.0","500","2*PI*p1"))
));

```

### D.9.4.5 plotRotatedListFromSpecifyList

Procedure from library `surfex.lib` (see [Section D.9.4 \[surfex.lib\]](#), page 1919).

**Usage:** `plotRotatedListFromSpecifyList(list varietiesList, list #);` varietiesList has a complicated format (not documented yet); see the example.

The optional int parameter can be used to set plotting quality.

**Assume:** The basering is of characteristic zero.

**Example:**

```

LIB "surfex.lib";
"Example:";
// A cubic surface depending on a parameter:
ring r = (0,p1), (x,y,z), dp;
poly cayley_cubic = x^3+y^3+z^3+1^3-p1*(x+y+z+1)^3;
poly plane = 1-x-y-z;
plotRotatedListFromSpecifyList(list(list(list(list("eqno:", "1"),
list("equation:",
string(cayley_cubic))
)
),
list(),
list(list(1,"0.0","1.0","500","0.25+0.25*sin(PI*p1)")),
list()
));

```

## D.10 Coding theory

### D.10.1 brnoeth.lib

**Library:** `brnoeth.lib`

**Purpose:** Brill-Noether Algorithm, Weierstrass-SG and AG-codes

**Authors:** Jose Ignacio Farran Martin, ignfar@eis.uva.es  
 Christoph Lossen, lossen@mathematik.uni-kl.de

**Overview:** Implementation of the Brill-Noether algorithm for solving the Riemann-Roch problem and applications to Algebraic Geometry codes. The computation of Weierstrass semi-groups is also implemented.



The procedures are intended only for plane (singular) curves defined over a prime field of positive characteristic.

For more information about the library see the end of the file `brnoeth.lib`.

#### Procedures:

##### D.10.1.1 Adj\_div

Procedure from library `brnoeth.lib` (see [Section D.10.1 \[brnoeth.lib\]](#), page 1922).

**Usage:** `Adj_div( f [,l] );` `f` a poly, `[l]` a list

**Return:** list `L` with the computed data:

- `L[1]` a list of rings: `L[1][1]=aff_r` (affine), `L[1][2]=Proj-R` (projective),
- `L[2]` an intvec with 2 entries (degree, genus),
- `L[3]` a list of intvec (closed places),
- `L[4]` an intvec (conductor),
- `L[5]` a list of lists:
  - `L[5][d][1]` a (local) ring over an extension of degree `d`,
  - `L[5][d][2]` an intvec (degrees of base points of places of degree `d`)

**Note:** `Adj_div(f)`; computes and stores the fundamental data of the plane curve defined by `f` as needed for AG codes.

In the affine ring you can find the following data:

- poly `CHI`: affine equation of the curve,
- ideal `Aff_SLocus`: affine singular locus (std),
- list `Inf_Points`: points at infinity
  - `Inf_Points[1]`: singular points
  - `Inf_Points[2]`: non-singular points,
- list `Aff_SPoints`: affine singular points (if not empty).

In the projective ring you can find the projective equation `CHI` of the curve (poly).

In the local rings `L[5][d][1]` you find:

- list `POINTS`: base points of the places of degree `d`,
- list `LOC_EQS`: local equations of the curve at the base points,
- list `BRANCHES`: Hamburger-Noether developments of the places,
- list `PARAMETRIZATIONS`: local parametrizations of the places,

Each entry of the list `L[3]` corresponds to one closed place (i.e., a place and all its conjugates) which is represented by an intvec of size two, the first entry is the degree of the place (in particular, it tells the local ring where to find the data describing one representative of the closed place), and the second one is the position of those data in the lists `POINTS`, etc., inside this local ring.

In the intvec `L[4]` (conductor) the `i`-th entry corresponds to the `i`-th entry in the list of places `L[3]`.

With no optional arguments, the conductor is computed by local invariants of the singularities; otherwise it is computed by the Dedekind formula.

An affine point is represented by a list `P` where `P[1]` is std of a prime ideal and `P[2]` is an intvec containing the position of the places above `P` in the list of closed places `L[3]`.

If the point is at infinity, `P[1]` is a homogeneous irreducible polynomial in two variables.

If `printlevel`  $\geq 0$  additional comments are displayed (default: `printlevel=0`).

**Warning:** The parameter of the needed field extensions is called 'a'. Thus, there should be no global object named 'a' when executing Adj\_div.

**Example:**

```
LIB "brnoeth.lib";
int plevel=printlevel;
printlevel=-1;
ring s=2,(x,y),lp;
list C=Adj_div(y9+y8+xy6+x2y3+y2+x3);
↳ The genus of the curve is 3
def aff_R=C[1][1];      // the affine ring
setring aff_R;
listvar(aff_R);        // data in the affine ring
↳ // aff_R                [0] *ring
↳ // Inf_Points           [0] list, size: 2
↳ // Aff_SPoints          [0] list, size: 3
↳ // Aff_SLocus           [0] ideal (SB), 2 generator(s)
↳ // CHI                  [0] poly
CHI;                   // affine equation of the curve
↳ x3+x2y3+xy6+y9+y8+y2
Aff_SLocus;            // ideal of the affine singular locus
↳ Aff_SLocus[1]=y8+y2
↳ Aff_SLocus[2]=x2+y6
Aff_SPoints[1];        // 1st affine singular point: (1:1:1), no.1
↳ [1]:
↳   _[1]=y
↳   _[2]=x
↳ [2]:
↳   1
Inf_Points[1];         // singular point(s) at infinity: (1:0:0), no.4
↳ [1]:
↳   [1]:
↳     y
↳   [2]:
↳     4
Inf_Points[2];         // list of non-singular points at infinity
↳ empty list
//
def proj_R=C[1][2];    // the projective ring
setring proj_R;
CHI;                   // projective equation of the curve
↳ x3z6+x2y3z4+xy6z2+y9+y8z+y2z7
C[2][1];               // degree of the curve
↳ 9
C[2][2];               // genus of the curve
↳ 3
C[3];                  // list of computed places
↳ [1]:
↳   1,1
↳ [2]:
↳   1,2
↳ [3]:
↳   2,1
```

```

⇒ [4]:
⇒ 1,3
C[4]; // adjunction divisor (all points are singular!)
⇒ 2,2,2,42
//
// we look at the place(s) of degree 2 by changing to the ring
C[5][2][1];
⇒ // coefficients: ZZ/2[a]/(...)
⇒ // number of vars : 3
⇒ // block 1 : ordering ls
⇒ // : names x y t
⇒ // block 2 : ordering C
def S(2)=C[5][2][1];
setring S(2);
POINTS; // base point(s) of place(s) of degree 2: (1:a:1)
⇒ [1]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ (a)
⇒ [3]:
⇒ 1
LOC_EQS; // local equation(s)
⇒ [1]:
⇒ y2+y3+(a+1)*y4+y6+(a+1)*y8+y9+(a)*xy2+(a+1)*xy4+xy6+(a+1)*x2y+(a)*x2y2\
+x2y3+x3
PARAMETRIZATIONS; // parametrization(s) and exactness
⇒ [1]:
⇒ [1]:
⇒ _[1]=t2+(a+1)*t3
⇒ _[2]=t3+(a+1)*t4
⇒ [2]:
⇒ 3,4
BRANCHES; // Hamburger-Noether development
⇒ [1]:
⇒ [1]:
⇒ _[1,1]=0
⇒ _[1,2]=x
⇒ _[1,3]=0
⇒ _[2,1]=0
⇒ _[2,2]=1
⇒ _[2,3]=(a+1)
⇒ [2]:
⇒ 1,-4
⇒ [3]:
⇒ 0
⇒ [4]:
⇒ y+(a+1)*xy+(a)*x2y+(a)*x2y2+(a+1)*x3+x3y+x3y3+(a)*x4+(a+1)*x4y2+(a\
1)*x4y3+x5+x5y2+(a)*x6+(a+1)*x6y2+x6y4+x6y5+x7y+(a+1)*x8+(a+1)*x8y+x8y4+(\
a+1)*x8y6+x9+x9y7+(a+1)*x10+x11y6+(a+1)*x12y4+x13y5+x14+x14y+x15y4+x16+(a\
+1)*x16y2+x17y3+x19y2+(a+1)*x20+x21y+x23
printlevel=plevel;

```

See also: [Section D.10.1.2 \[NSplaces\]](#), page 1926; [Section D.10.1.10 \[closed\\_points\]](#), page 1935.

### D.10.1.2 NSplaces

Procedure from library `brnoeth.lib` (see [Section D.10.1 \[brnoeth.lib\]](#), page 1922).

**Usage:** `NSplaces( h, CURVE )`, where `h` is an intvec and `CURVE` is a list

**Return:** list `L` with updated data of `CURVE` after computing all non-singular affine closed places whose degrees are in the intvec `h`:

in `L[1][1]`: (affine ring) lists `Aff_Points(d)` with affine non-singular  
(closed) points of degree `d` (if non-empty),  
in `L[3]`: the newly computed closed places are added,  
in `L[5]`: local rings created/updated to store (repres. of) new places.

See [Section D.10.1.1 \[Adj\\_div\]](#), page 1923 for a description of the entries in `L`.

**Note:** The `list_expression` should be the output of the procedure `Adj_div`.  
Raising `printlevel`, additional comments are displayed (default: `printlevel=0`).

**Warning:** The parameter of the needed field extensions is called `'a'`. Thus, there should be no global object named `'a'` when executing `NSplaces`.

**Example:**

```
LIB "brnoeth.lib";
int plevel=printlevel;
printlevel=-1;
ring s=2,(x,y),lp;
list C=Adj_div(x3y+y3+x);
⇒ The genus of the curve is 3
// The list of computed places:
C[3];
⇒ [1]:
⇒ 1,1
⇒ [2]:
⇒ 1,2
// create places up to degree 4
list L=NSplaces(1..4,C);
// The list of computed places is now:
L[3];
⇒ [1]:
⇒ 1,1
⇒ [2]:
⇒ 1,2
⇒ [3]:
⇒ 1,3
⇒ [4]:
⇒ 2,1
⇒ [5]:
⇒ 3,1
⇒ [6]:
⇒ 3,2
⇒ [7]:
⇒ 3,3
⇒ [8]:
⇒ 3,4
```

```

⇒ [9]:
⇒ 3,5
⇒ [10]:
⇒ 3,6
⇒ [11]:
⇒ 3,7
⇒ [12]:
⇒ 4,1
⇒ [13]:
⇒ 4,2
⇒ [14]:
⇒ 4,3
// e.g., affine non-singular points of degree 4 :
def aff_r=L[1][1];
setring aff_r;
Aff_Points(4);
⇒ [1]:
⇒ [1]:
⇒ _[1]=y2+y+1
⇒ _[2]=x2+xy+x+1
⇒ [2]:
⇒ 12
⇒ [2]:
⇒ [1]:
⇒ _[1]=y4+y3+1
⇒ _[2]=x+y3+y
⇒ [2]:
⇒ 13
⇒ [3]:
⇒ [1]:
⇒ _[1]=y4+y3+y2+y+1
⇒ _[2]=x+y2+y+1
⇒ [2]:
⇒ 14
// e.g., base point of the 1st place of degree 4 :
def S(4)=L[5][4][1];
setring S(4);
POINTS[1];
⇒ [1]:
⇒ (a3)
⇒ [2]:
⇒ (a2+a)
⇒ [3]:
⇒ 1
printlevel=plevel;

```

See also: [Section D.10.1.1 \[Adj\\_div\]](#), page 1923; [Section D.10.1.10 \[closed\\_points\]](#), page 1935.

### D.10.1.3 BrillNoether

Procedure from library `brnoeth.lib` (see [Section D.10.1 \[brnoeth\\_lib\]](#), page 1922).

**Usage:**      BrillNoether(G,CURVE); G an intvec, CURVE a list

**Return:** list of ideals (each of them with two homogeneous generators, which represent the numerator, resp. denominator, of a rational function).  
The corresponding rational functions form a vector space basis of the linear system  $L(G)$ ,  $G$  a rational divisor over a non-singular curve.

**Note:** The procedure must be called from the ring `CURVE[1][2]`, where `CURVE` is the output of the procedure `NSplaces`.  
The intvec  $G$  represents a rational divisor supported on the closed places of `CURVE[3]` (e.g.  $G=2,0,-1$ ; means 2 times the closed place 1 minus 1 times the closed place 3).

**Example:**

```
LIB "brnoeth.lib";
int plevel=printlevel;
printlevel=-1;
ring s=2,(x,y),lp;
list C=Adj_div(x3y+y3+x);
⇒ The genus of the curve is 3
C=NSplaces(1..4,C);
// the first 3 Places in C[3] are of degree 1.
// we define the rational divisor G = 4*C[3][1]+4*C[3][3] (of degree 8):
intvec G=4,0,4;
def R=C[1][2];
setring R;
list LG=BrillNoether(G,C);
⇒ Vector basis successfully computed
// here is the vector basis of L(G):
LG;
⇒ [1]:
⇒   _[1]=1
⇒   _[2]=1
⇒ [2]:
⇒   _[1]=y
⇒   _[2]=x
⇒ [3]:
⇒   _[1]=z
⇒   _[2]=x
⇒ [4]:
⇒   _[1]=y2
⇒   _[2]=x2
⇒ [5]:
⇒   _[1]=xz2+y3
⇒   _[2]=x3
⇒ [6]:
⇒   _[1]=xyz2+y4
⇒   _[2]=x4
printlevel=plevel;
```

See also: [Section D.10.1.1 \[Adj\\_div\]](#), page 1923; [Section D.10.1.2 \[NSplaces\]](#), page 1926; [Section D.10.1.4 \[Weierstrass\]](#), page 1928.

#### D.10.1.4 Weierstrass

Procedure from library `brnoeth.lib` (see [Section D.10.1 \[brnoeth.lib\]](#), page 1922).

**Usage:** `Weierstrass( i, m, CURVE );`  $i, m$  integers and `CURVE` a list

**Return:** list WS of two lists:  
     WS[1] list of integers (Weierstr. semigroup of the curve at place i up to m)  
     WS[2] list of ideals (the associated rational functions)

**Note:** The procedure must be called from the ring CURVE[1][2], where CURVE is the output of the procedure NSplaces.  
     i represents the place CURVE[3][i].  
     Rational functions are represented by numerator/denominator in form of ideals with two homogeneous generators.

**Warning:** The place must be rational, i.e., necessarily CURVE[3][i][1]=1.

**Example:**

```
LIB "brnoeth.lib";
int plevel=printlevel;
printlevel=-1;
ring s=2,(x,y),lp;
list C=Adj_div(x3y+y3+x);
⇒ The genus of the curve is 3
C=NSplaces(1..4,C);
def R=C[1][2];
setring R;
// Place C[3][1] has degree 1 (i.e it is rational);
list WS=Weierstrass(1,7,C);
⇒ Vector basis successfully computed
// the first part of the list is the Weierstrass semigroup up to 7 :
WS[1];
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 3
⇒ [3]:
⇒ 5
⇒ [4]:
⇒ 6
⇒ [5]:
⇒ 7
// and the second part are the corresponding functions :
WS[2];
⇒ [1]:
⇒ _[1]=1
⇒ _[2]=1
⇒ [2]:
⇒ _[1]=y
⇒ _[2]=z
⇒ [3]:
⇒ _[1]=xy
⇒ _[2]=z2
⇒ [4]:
⇒ _[1]=y2
⇒ _[2]=z2
⇒ [5]:
```

```

↳      _[1]=y3
↳      _[2]=xz2
printlevel=plevel;

```

See also: [Section D.10.1.1 \[Adj\\_div\]](#), page 1923; [Section D.10.1.3 \[BrillNoether\]](#), page 1927; [Section D.10.1.2 \[NSplaces\]](#), page 1926.

### D.10.1.5 extcurve

Procedure from library `brnoeth.lib` (see [Section D.10.1 \[brnoeth.lib\]](#), page 1922).

**Usage:** `extcurve( d, CURVE );` `d` an integer, `CURVE` a list

**Return:** list `L` which is the update of the list `CURVE` with additional entries

```

L[1][3]: ring (p,a),(x,y),lp (affine),
L[1][4]: ring (p,a),(x,y,z),lp (projective),
L[1][5]: ring (p,a),(x,y,t),ls (local),
L[2][3]: int (the number of rational places),

```

the rings being defined over a field extension of degree `d`.

If `d < 2` then `extcurve(d, CURVE)`; creates a list `L` which is the update of the list `CURVE` with additional entries

```

L[1][5]: ring p,(x,y,t),ls,
L[2][3]: int (the number of computed places over the base field).

```

In both cases, in the ring `L[1][5]` lists with the data for all the computed rational places (after a field extension of degree `d`) are created (see [Section D.10.1.1 \[Adj\\_div\]](#), page 1923):

lists `POINTS`, `LOC_EQS`, `BRANCHES`, `PARAMETRIZATIONS`.

**Note:** The list `CURVE` should be the output of `NSplaces`, and must contain (at least) one place of degree `d`.

You actually need all the places with degree dividing `d`. Otherwise, not all the places are computed, but only part of them.

This procedure must be executed before constructing AG codes, even if no extension is needed. The ring `L[1][4]` must be active when constructing codes over the field extension.

#### Example:

```

LIB "brnoeth.lib";
int plevel=printlevel;
printlevel=-1;
ring s=2,(x,y),lp;
list C=Adj_div(x5+y2+y);
↳ The genus of the curve is 2
C=NSplaces(1..4,C);
// since we have all points up to degree 4, we can extend the curve
// to that extension, in order to get rational points over F_16;
C=extcurve(4,C);
↳ Total number of rational places : NrRatPl = 33
// e.g., display the basepoint of place no. 32:
def R=C[1][5];

```



```

setring R;
POINTS[32];
↪ [1]:
↪ (a3+a+1)
↪ [2]:
↪ (a+1)
↪ [3]:
↪ 1
printlevel=plevel;

```

See also: [Section D.10.1.6 \[AGcode\\_L\]](#), page 1931; [Section D.10.1.7 \[AGcode\\_Omega\]](#), page 1932; [Section D.10.1.1 \[Adj\\_div\]](#), page 1923; [Section D.10.1.2 \[NSplaces\]](#), page 1926; [Section D.10.1.10 \[closed\\_points\]](#), page 1935.

### D.10.1.6 AGcode\_L

Procedure from library `brnoeth.lib` (see [Section D.10.1 \[brnoeth\\_lib\]](#), page 1922).

**Usage:** `AGcode_L( G, D, EC );` G,D intvec, EC a list

**Return:** a generator matrix for the evaluation AG code defined by the divisors G and D.

**Note:** The procedure must be called within the ring `EC[1][4]`, where EC is the output of `extcurve(d)` (or within the ring `EC[1][2]` if `d=1`).

The entry `i` in the intvec `D` refers to the `i`-th rational place in `EC[1][5]` (i.e., to `POINTS[i]`, etc., see [Section D.10.1.5 \[extcurve\]](#), page 1930).

The intvec `G` represents a rational divisor (see [Section D.10.1.3 \[BrillNoether\]](#), page 1927 for more details).

The code evaluates the vector space basis of  $L(G)$  at the rational places given by `D`.

**Warnings:** `G` should satisfy  $2 * \text{genus} - 2 < \text{deg}(G) < \text{size}(D)$ , which is not checked by the algorithm.

`G` and `D` should have disjoint supports (checked by the algorithm).

**Example:**

```

LIB "brnoeth.lib";
int plevel=printlevel;
printlevel=-1;
ring s=2,(x,y),lp;
list HC=Adj_div(x3+y2+y);
↪ The genus of the curve is 1
HC=NSplaces(1..2,HC);
HC=extcurve(2,HC);
↪ Total number of rational places : NrRatPl = 9
def ER=HC[1][4];
setring ER;
intvec G=5;      // the rational divisor G = 5*HC[3][1]
intvec D=2..9;  // D = sum of the rational places no. 2..9 over F_4
// let us construct the corresponding evaluation AG code :
matrix C=AGcode_L(G,D,HC);
↪ Vector basis successfully computed
// here is a linear code of type [8,5,>=3] over F_4
print(C);
↪ 0,0,1, 1,    (a),  (a+1),(a+1),(a),
↪ 0,1,(a),(a+1),(a),  (a+1),(a),  (a+1),

```

```

    ↪ 1,1,1, 1, 1, 1, 1, 1,
    ↪ 0,0,1, 1, (a+1),(a), (a), (a+1),
    ↪ 0,0,(a),(a+1),(a+1),(a), 1, 1
    printlevel=plevel;

```

See also: [Section D.10.1.7 \[AGcode\\_Omega\]](#), page 1932; [Section D.10.1.1 \[Adj\\_div\]](#), page 1923; [Section D.10.1.3 \[BrillNoether\]](#), page 1927; [Section D.10.1.5 \[extcurve\]](#), page 1930.

### D.10.1.7 AGcode\_Omega

Procedure from library `brnoeth.lib` (see [Section D.10.1 \[brnoeth\\_lib\]](#), page 1922).

**Usage:** `AGcode_Omega( G, D, EC );`  $G, D$  intvec,  $EC$  a list

**Return:** a generator matrix for the residual AG code defined by the divisors  $G$  and  $D$ .

**Note:** The procedure must be called within the ring  $EC[1][4]$ , where  $EC$  is the output of `extcurve(d)` (or within the ring  $EC[1][2]$  if  $d=1$ ).

The entry  $i$  in the intvec  $D$  refers to the  $i$ -th rational place in  $EC[1][5]$  (i.e., to `POINTS[i]`, etc., see [Section D.10.1.5 \[extcurve\]](#), page 1930).

The intvec  $G$  represents a rational divisor (see [Section D.10.1.3 \[BrillNoether\]](#), page 1927 for more details).

The code computes the residues of a vector space basis of

$\Omega(G - D)$  at the rational places given by  $D$ .

**Warnings:**  $G$  should satisfy  $2 * genus - 2 < deg(G) < size(D)$ , which is not checked by the algorithm.

$G$  and  $D$  should have disjoint supports (checked by the algorithm).

**Example:**

```

LIB "brnoeth.lib";
int plevel=printlevel;
printlevel=-1;
ring s=2,(x,y),lp;
list HC=Adj_div(x3+y2+y);
↪ The genus of the curve is 1
HC=NSplaces(1..2,HC);
HC=extcurve(2,HC);
↪ Total number of rational places : NrRatPl = 9
def ER=HC[1][4];
setring ER;
intvec G=5; // the rational divisor G = 5*HC[3][1]
intvec D=2..9; // D = sum of the rational places no. 2..9 over F_4
// let us construct the corresponding residual AG code :
matrix C=AGcode_Omega(G,D,HC);
↪ Vector basis successfully computed
// here is a linear code of type [8,3,>=5] over F_4
print(C);
↪ 1, 0, (a),(a+1),1,1,0,0,
↪ (a+1),(a), 1, 0, 1,0,1,0,
↪ (a+1),(a+1),(a),(a), 1,0,0,1
printlevel=plevel;

```

See also: [Section D.10.1.6 \[AGcode\\_L\]](#), page 1931; [Section D.10.1.1 \[Adj\\_div\]](#), page 1923; [Section D.10.1.3 \[BrillNoether\]](#), page 1927; [Section D.10.1.5 \[extcurve\]](#), page 1930.

### D.10.1.8 prepSV

Procedure from library `brnoeth.lib` (see [Section D.10.1 \[brnoeth.lib\]](#), page 1922).

**Usage:** `prepSV( G, D, F, EC );` G,D,F intvecs and EC a list

**Return:** list E of size n+3, where n=size(D). All its entries but E[n+3] are matrices:

E[1]: parity check matrix for the current AG code  
 E[2] ... E[n+2]: matrices used in the procedure `decodeSV`  
 E[n+3]: intvec with  
     E[n+3][1]: correction capacity

*epsilon*  
 of the algorithm  
     E[n+3][2]: designed Goppa distance

*delta*  
 of the current AG code

**Note:** Computes the preprocessing for the basic (Skorobogatov-Vladut) decoding algorithm. The procedure must be called within the ring `EC[1][4]`, where EC is the output of `extcurve(d)` (or in the ring `EC[1][2]` if d=1). The intvec G and F represent rational divisors (see [Section D.10.1.3 \[BrillNoether\]](#), page 1927 for more details). The intvec D refers to rational places (see [Section D.10.1.7 \[AGcode\\_Omega\]](#), page 1932 for more details). The current AG code is `AGcode_Omega(G,D,EC)`. If you know the exact minimum distance d and you want to use it in `decodeSV` instead of *delta*, you can change the value of E[n+3][2] to d before applying `decodeSV`. If you have a systematic encoding for the current code and want to keep it during the decoding, you must previously permute D (using `permute_L(D,P)`), e.g., according to the permutation  $P=L[3]$ , L being the output of `sys_code`.

**Warnings:** F must be a divisor with support disjoint from the support of D and with degree *epsilon* + *genus*, where

$$epsilon := [(deg(G) - 3 * genus + 1)/2] .$$

G should satisfy  $2 * genus - 2 < deg(G) < size(D)$ , which is not checked by the algorithm.

G and D should also have disjoint supports (checked by the algorithm).

**Example:**

```
LIB "brnoeth.lib";
int plevel=printlevel;
printlevel=-1;
ring s=2,(x,y),lp;
list HC=Adj_div(x^3+y^2+y);
--> The genus of the curve is 1
HC=NSplaces(1..2,HC);
HC=extcurve(2,HC);
--> Total number of rational places : NrRatPl = 9
def ER=HC[1][4];
setring ER;
```

```

intvec G=5;      // the rational divisor G = 5*HC[3][1]
intvec D=2..9;   // D = sum of the rational places no. 2..9 over F_4
// construct the corresp. residual AG code of type [8,3,>=5] over F_4:
matrix C=AGcode_Omega(G,D,HC);
↳ Vector basis successfully computed
// we can correct 1 error and the genus is 1, thus F must have degree 2
// and support disjoint from that of D;
intvec F=2;
list SV=prepSV(G,D,F,HC);
↳ Vector basis successfully computed
↳ Vector basis successfully computed
↳ Vector basis successfully computed
// now everything is prepared to decode with the basic algorithm;
// for example, here is a parity check matrix to compute the syndrome :
print(SV[1]);
↳ 0,0,1, 1, (a), (a+1),(a+1),(a),
↳ 0,1,(a),(a+1),(a), (a+1),(a), (a+1),
↳ 1,1,1, 1, 1, 1, 1, 1,
↳ 0,0,1, 1, (a+1),(a), (a), (a+1),
↳ 0,0,(a),(a+1),(a+1),(a), 1, 1
// and here you have the correction capacity of the algorithm :
int epsilon=SV[size(D)+3][1];
epsilon;
↳ 1
printlevel=plevel;

```

See also: [Section D.10.1.7 \[AGcode\\_Omega\]](#), page 1932; [Section D.10.1.9 \[decodeSV\]](#), page 1934; [Section D.10.1.5 \[extcurve\]](#), page 1930; [Section D.10.1.13 \[permute\\_L\]](#), page 1937; [Section D.10.1.12 \[sys\\_code\]](#), page 1936.

### D.10.1.9 decodeSV

Procedure from library `brnoeth.lib` (see [Section D.10.1 \[brnoeth.lib\]](#), page 1922).

**Usage:** `decodeSV( y, K );` `y` a row-matrix and `K` a list

**Return:** a codeword (row-matrix) if possible, resp. the 0-matrix (of size 1) if decoding is impossible.

For decoding the basic (Skorobogatov-Vladut) decoding algorithm is applied.

**Note:** The `list-expression` should be the output `K` of the procedure `prepSV`.

The `matrix-expression` should be a  $(1 \times n)$ -matrix, where  $n = \text{ncols}(K[1])$ .

The decoding may fail if the number of errors is greater than the correction capacity of the algorithm.

**Example:**

```

LIB "brnoeth.lib";
int plevel=printlevel;
printlevel=-1;
ring s=2,(x,y),lp;
list HC=Adj_div(x3+y2+y);
↳ The genus of the curve is 1
HC=NSplaces(1..2,HC);
HC=extcurve(2,HC);
↳ Total number of rational places : NrRatPl = 9

```

```

def ER=HC[1][4];
setring ER;
intvec G=5;      // the rational divisor G = 5*HC[3][1]
intvec D=2..9;   // D = sum of the rational places no. 2..9 over F_4
// construct the corresp. residual AG code of type [8,3,>=5] over F_4:
matrix C=AGcode_Omega(G,D,HC);
⇒ Vector basis successfully computed
// we can correct 1 error and the genus is 1, thus F must have degree 2
// and support disjoint from that of D
intvec F=2;
list SV=prepSV(G,D,F,HC);
⇒ Vector basis successfully computed
⇒ Vector basis successfully computed
⇒ Vector basis successfully computed
// now we produce 1 error on the zero-codeword :
matrix y[1][8];
y[1,3]=a;
// and then we decode :
print(decodeSV(y,SV));
⇒ 0,0,0,0,0,0,0,0
printlevel=plevel;

```

See also: [Section D.10.1.7 \[AGcode\\_Omega\]](#), page 1932; [Section D.10.1.5 \[extcurve\]](#), page 1930; [Section D.10.1.8 \[prepSV\]](#), page 1933.

### D.10.1.10 closed\_points

Procedure from library `brnoeth.lib` (see [Section D.10.1 \[brnoeth.lib\]](#), page 1922).

**Usage:**      `closed_points(I)`; I an ideal

**Return:**     list of prime ideals (each a Groebner basis), corresponding to the (distinct affine closed) points of  $V(I)$

**Note:**        The ideal must have dimension 0, the basering must have 2 variables, the ordering must be `lp`, and the base field must be finite and prime.  
It might be convenient to set the option(`redSB`) in advance.

**Example:**

```

LIB "brnoeth.lib";
ring s=2,(x,y),lp;
// this is just the affine plane over F_4 :
ideal I=x4+x,y4+y;
list L=closed_points(I);
// and here you have all the points :
L;
⇒ [1]:
⇒   _[1]=y
⇒   _[2]=x
⇒ [2]:
⇒   _[1]=y
⇒   _[2]=x+1
⇒ [3]:
⇒   _[1]=y
⇒   _[2]=x2+x+1

```

```

⇒ [4] :
⇒   _[1]=y+1
⇒   _[2]=x
⇒ [5] :
⇒   _[1]=y+1
⇒   _[2]=x+1
⇒ [6] :
⇒   _[1]=y+1
⇒   _[2]=x2+x+1
⇒ [7] :
⇒   _[1]=y2+y+1
⇒   _[2]=x
⇒ [8] :
⇒   _[1]=y2+y+1
⇒   _[2]=x+1
⇒ [9] :
⇒   _[1]=y2+y+1
⇒   _[2]=x+y
⇒ [10] :
⇒   _[1]=y2+y+1
⇒   _[2]=x+y+1

```

See also: [Section D.8.5 \[triang\\_lib\], page 1856](#).

#### D.10.1.11 dual\_code

Procedure from library `brnoeth.lib` (see [Section D.10.1 \[brnoeth\\_lib\], page 1922](#)).

**Usage:** `dual_code(G)`; G a matrix of numbers

**Return:** a generator matrix of the dual code generated by G

**Note:** The input should be a matrix G of numbers.  
The output is also a parity check matrix for the code defined by G

**Example:**

```

LIB "brnoeth.lib";
ring s=2,T,lp;
// here is the Hamming code of length 7 and dimension 3
matrix G[3][7]=1,0,1,0,1,0,1,0,1,1,0,0,1,1,0,0,0,1,1,1,1;
print(G);
⇒ 1,0,1,0,1,0,1,
⇒ 0,1,1,0,0,1,1,
⇒ 0,0,0,1,1,1,1
matrix H=dual_code(G);
print(H);
⇒ 1,1,1,0,0,0,0,
⇒ 1,0,0,1,1,0,0,
⇒ 0,1,0,1,0,1,0,
⇒ 1,1,0,1,0,0,1

```

#### D.10.1.12 sys\_code

Procedure from library `brnoeth.lib` (see [Section D.10.1 \[brnoeth\\_lib\], page 1922](#)).

**Usage:** `sys_code(C)`; C is a matrix of constants

**Return:** list L with:

L[1] is the generator matrix in standard form of an equivalent code,  
 L[2] is the parity check matrix in standard form of such code,  
 L[3] is an intvec which represents the needed permutation.

**Note:** Computes a systematic code which is equivalent to the given one.  
 The input should be a matrix of numbers.

The output has to be interpreted as follows: if the input was the generator matrix of an AG code then one should apply the permutation L[3] to the divisor D of rational points by means of `permute_L(D,L[3])`; before continuing to work with the code (for instance, if you want to use the systematic encoding together with a decoding algorithm).

**Example:**

```
LIB "brnoeth.lib";
ring s=3,T,lp;
matrix C[2][5]=0,1,0,1,1,0,1,0,0,1;
print(C);
  ⇒ 0,1,0,1,1,
  ⇒ 0,1,0,0,1
list L=sys_code(C);
L[3];
  ⇒ 2,4,3,1,5
// here is the generator matrix in standard form
print(L[1]);
  ⇒ 1,0,0,0,1,
  ⇒ 0,1,0,0,0
// here is the control matrix in standard form
print(L[2]);
  ⇒ 0, 0,1,0,0,
  ⇒ 0, 0,0,1,0,
  ⇒ -1,0,0,0,1
// we can check that both codes are dual to each other
print(L[1]*transpose(L[2]));
  ⇒ 0,0,0,
  ⇒ 0,0,0
```

See also: [Section D.10.1.7 \[AGcode\\_Omega\]](#), page 1932; [Section D.10.1.13 \[permute\\_L\]](#), page 1937; [Section D.10.1.8 \[prepSV\]](#), page 1933.

### D.10.1.13 permute\_L

Procedure from library `brnoeth.lib` (see [Section D.10.1 \[brnoeth.lib\]](#), page 1922).

**Usage:** `permute_L( L, P );` L,P either intvecs or lists

**Return:** list obtained from L by applying the permutation given by P.

**Note:** If P is a list, all entries must be integers.

**Example:**

```
LIB "brnoeth.lib";
list L=list();
L[1]="a";
L[2]="b";
```

```

L[3]="c";
L[4]="d";
intvec P=1,3,4,2;
// the list L is permuted according to P :
permute_L(L,P);
⇨ [1]:
⇨    a
⇨ [2]:
⇨    c
⇨ [3]:
⇨    d
⇨ [4]:
⇨    b

```

See also: [Section D.10.1.7 \[AGcode-Omega\]](#), page 1932; [Section D.10.1.8 \[prepSV\]](#), page 1933; [Section D.10.1.12 \[sys\\_code\]](#), page 1936.

## D.10.2 decodegb\_lib

**Library:** decodegb.lib

**Purpose:** Decoding and min distance of linear codes with GB

**Author:** Stanislav Bulygin, [bulygin@mathematik.uni-kl.de](mailto:bulygin@mathematik.uni-kl.de)

**Overview:** In this library we generate several systems used for decoding cyclic codes and finding their minimum distance. Namely, we work with the Cooper's philosophy and generalized Newton identities. The origideal method of quadratic equations is worked out here as well. We also (for comparison) enable to work with the system of Fitzgerald-Lax. We provide some auxiliary functions for further manipulations and decoding. For an overview of the methods mentioned above [Section C.8 \[Decoding codes with Groebner bases\]](#), page 785. For the vanishing ideal computation the algorithm of Farr and Gao is implemented.

**Procedures:**

### D.10.2.1 sysCRHT

Procedure from library `decodegb.lib` (see [Section D.10.2 \[decodegb\\_lib\]](#), page 1938).

**Usage:** `sysCRHT(n,defset,e,q,m,[k]);` `n,e,q,m,k` are int, `defset` list of int's

- `n` length of the cyclic code,
- `defset` is a list representing the defining set,
- `e` the error-correcting capacity,
- `q` field size
- `m` degree extension of the splitting field,
- if `k>0` additional equations representing the fact that every two error positions are either different or at least one of them is zero

**Return:** the ring to work with the CRHT-ideal (with Sala's additions), containing an ideal with name 'crht'

**Theory:** Based on 'defset' of the given cyclic code, the procedure constructs the corresponding Cooper-Reed-Heleseth-Truong ideal 'crht'. With its help one can solve the decoding problem. For basics of the method [Section C.8.2 \[Cooper philosophy\]](#), page 787.

**Example:**



```

LIB "decodegb.lib";
// binary cyclic [15,7,5] code with defining set (1,3)
intvec v = option(get);
list defset=1,3;          // defining set
int n=15;                 // length
int e=2;                  // error-correcting capacity
int q=2;                  // basefield size
int m=4;                  // degree extension of the splitting field
int sala=1;               // indicator to add additional equations
def A=sysCRHT(n,defset,e,q,m);
setring A;
A;                        // shows the ring we are working in
⇒ // coefficients: ZZ/2(a)
⇒ // number of vars : 6
⇒ //          block 1 : ordering lp
⇒ //          : names  Y(2) Y(1) Z(1) Z(2) X(2) X(1)
⇒ //          block 2 : ordering C
print(crht);              // the CRHT-ideal
⇒ Y(2)*Z(2)+Y(1)*Z(1)+X(1),
⇒ Y(2)*Z(2)^3+Y(1)*Z(1)^3+X(2),
⇒ X(1)^16+X(1),
⇒ X(2)^16+X(2),
⇒ Z(1)^16+Z(1),
⇒ Z(2)^16+Z(2),
⇒ Y(1)+1,
⇒ Y(2)+1
option(redSB);
ideal red_crht=std(crht); // reduced Groebner basis
print(red_crht);
⇒ X(1)^16+X(1),
⇒ X(2)*X(1)^15+X(2),
⇒ X(2)^8+X(2)^4*X(1)^12+X(2)^2*X(1)^3+X(2)*X(1)^6,
⇒ Z(2)^2*X(1)+Z(2)*X(1)^2+X(2)+X(1)^3,
⇒ Z(2)^2*X(2)+Z(2)*X(2)*X(1)+X(2)^2*X(1)^14+X(2)*X(1)^2,
⇒ Z(2)^16+Z(2),
⇒ Z(1)+Z(2)+X(1),
⇒ Y(1)+1,
⇒ Y(2)+1
//=====
A=sysCRHT(n,defset,e,q,m,sala);
setring A;
print(crht);              // CRHT-ideal with additional equations from Sala
⇒ Y(2)*Z(2)+Y(1)*Z(1)+X(1),
⇒ Y(2)*Z(2)^3+Y(1)*Z(1)^3+X(2),
⇒ X(1)^16+X(1),
⇒ X(2)^16+X(2),
⇒ Z(1)^16+Z(1),
⇒ Z(2)^16+Z(2),
⇒ Y(1)+1,
⇒ Y(2)+1,
⇒ Z(1)^15*Z(2)+Z(1)^14*Z(2)^2+Z(1)^13*Z(2)^3+Z(1)^12*Z(2)^4+Z(1)^11*Z(2)^5+
  Z(1)^10*Z(2)^6+Z(1)^9*Z(2)^7+Z(1)^8*Z(2)^8+Z(1)^7*Z(2)^9+Z(1)^6*Z(2)^10+Z(1)^5*Z(2)^11+
  Z(1)^4*Z(2)^12+Z(1)^3*Z(2)^13+Z(1)^2*Z(2)^14+Z(1)*Z(2)^15

```

```

option(redSB);
ideal red_crht=std(crht);    // reduced Groebner basis
print(red_crht);
⇒ X(1)^16+X(1),
⇒ X(2)*X(1)^15+X(2),
⇒ X(2)^8+X(2)^4*X(1)^12+X(2)^2*X(1)^3+X(2)*X(1)^6,
⇒ Z(2)*X(1)^15+Z(2),
⇒ Z(2)^2+Z(2)*X(1)+X(2)*X(1)^14+X(1)^2,
⇒ Z(1)+Z(2)+X(1),
⇒ Y(1)+1,
⇒ Y(2)+1
red_crht[5];                // general error-locator polynomial for this code
⇒ Z(2)^2+Z(2)*X(1)+X(2)*X(1)^14+X(1)^2
option(set,v);

```

See also: [Section D.10.2.4 \[sysBin\], page 1945](#); [Section D.10.2.3 \[sysNewton\], page 1942](#).

### D.10.2.2 sysCRHTMindist

Procedure from library `decodegb.lib` (see [Section D.10.2 \[decodegb.lib\], page 1938](#)).

**Usage:** `sysCRHTMindist(n,defset,w)`;  $n, w$  are int, `defset` is list of int's

- $n$  length of the cyclic code,
- `defset` is a list representing the defining set,
- $w$  is a candidate for the minimum distance

**Return:** the ring to work with the Sala's ideal for the minimum distance containing the ideal with name 'crht\_md'

**Theory:** Based on 'defset' of the given cyclic code, the procedure constructs the corresponding Cooper-Reed-Heleseth-Truong ideal 'crht\_md'. With its help one can find minimum distance of the code in the binary case. For basics of the method [Section C.8.2 \[Cooper philosophy\], page 787](#).

**Example:**

```

LIB "decodegb.lib";
intvec v = option(get);
// binary cyclic [15,7,5] code with defining set (1,3)
list defset=1,3;           // defining set
int n=15;                  // length
int d=5;                   // candidate for the minimum distance
def A=sysCRHTMindist(n,defset,d);
setring A;
A;                          // shows the ring we are working in
⇒ // coefficients: ZZ/2
⇒ // number of vars : 5
⇒ //          block 1 : ordering lp
⇒ //          : names  Z(1) Z(2) Z(3) Z(4) Z(5)
⇒ //          block 2 : ordering C
print(crht_md);            // the Sala's ideal for mindist
⇒ Z(1)+Z(2)+Z(3)+Z(4)+Z(5),
⇒ Z(1)^3+Z(2)^3+Z(3)^3+Z(4)^3+Z(5)^3,
⇒ Z(1)^15+1,
⇒ Z(2)^15+1,
⇒ Z(3)^15+1,

```

```

⇒ Z(4)^15+1,
⇒ Z(5)^15+1,
⇒ Z(1)^15*Z(2)+Z(1)^14*Z(2)^2+Z(1)^13*Z(2)^3+Z(1)^12*Z(2)^4+Z(1)^11*Z(2)^5+\
  Z(1)^10*Z(2)^6+Z(1)^9*Z(2)^7+Z(1)^8*Z(2)^8+Z(1)^7*Z(2)^9+Z(1)^6*Z(2)^10+Z\
  (1)^5*Z(2)^11+Z(1)^4*Z(2)^12+Z(1)^3*Z(2)^13+Z(1)^2*Z(2)^14+Z(1)*Z(2)^15,
⇒ Z(1)^15*Z(3)+Z(1)^14*Z(3)^2+Z(1)^13*Z(3)^3+Z(1)^12*Z(3)^4+Z(1)^11*Z(3)^5+\
  Z(1)^10*Z(3)^6+Z(1)^9*Z(3)^7+Z(1)^8*Z(3)^8+Z(1)^7*Z(3)^9+Z(1)^6*Z(3)^10+Z\
  (1)^5*Z(3)^11+Z(1)^4*Z(3)^12+Z(1)^3*Z(3)^13+Z(1)^2*Z(3)^14+Z(1)*Z(3)^15,
⇒ Z(1)^15*Z(4)+Z(1)^14*Z(4)^2+Z(1)^13*Z(4)^3+Z(1)^12*Z(4)^4+Z(1)^11*Z(4)^5+\
  Z(1)^10*Z(4)^6+Z(1)^9*Z(4)^7+Z(1)^8*Z(4)^8+Z(1)^7*Z(4)^9+Z(1)^6*Z(4)^10+Z\
  (1)^5*Z(4)^11+Z(1)^4*Z(4)^12+Z(1)^3*Z(4)^13+Z(1)^2*Z(4)^14+Z(1)*Z(4)^15,
⇒ Z(1)^15*Z(5)+Z(1)^14*Z(5)^2+Z(1)^13*Z(5)^3+Z(1)^12*Z(5)^4+Z(1)^11*Z(5)^5+\
  Z(1)^10*Z(5)^6+Z(1)^9*Z(5)^7+Z(1)^8*Z(5)^8+Z(1)^7*Z(5)^9+Z(1)^6*Z(5)^10+Z\
  (1)^5*Z(5)^11+Z(1)^4*Z(5)^12+Z(1)^3*Z(5)^13+Z(1)^2*Z(5)^14+Z(1)*Z(5)^15,
⇒ Z(2)^15*Z(3)+Z(2)^14*Z(3)^2+Z(2)^13*Z(3)^3+Z(2)^12*Z(3)^4+Z(2)^11*Z(3)^5+\
  Z(2)^10*Z(3)^6+Z(2)^9*Z(3)^7+Z(2)^8*Z(3)^8+Z(2)^7*Z(3)^9+Z(2)^6*Z(3)^10+Z\
  (2)^5*Z(3)^11+Z(2)^4*Z(3)^12+Z(2)^3*Z(3)^13+Z(2)^2*Z(3)^14+Z(2)*Z(3)^15,
⇒ Z(2)^15*Z(4)+Z(2)^14*Z(4)^2+Z(2)^13*Z(4)^3+Z(2)^12*Z(4)^4+Z(2)^11*Z(4)^5+\
  Z(2)^10*Z(4)^6+Z(2)^9*Z(4)^7+Z(2)^8*Z(4)^8+Z(2)^7*Z(4)^9+Z(2)^6*Z(4)^10+Z\
  (2)^5*Z(4)^11+Z(2)^4*Z(4)^12+Z(2)^3*Z(4)^13+Z(2)^2*Z(4)^14+Z(2)*Z(4)^15,
⇒ Z(2)^15*Z(5)+Z(2)^14*Z(5)^2+Z(2)^13*Z(5)^3+Z(2)^12*Z(5)^4+Z(2)^11*Z(5)^5+\
  Z(2)^10*Z(5)^6+Z(2)^9*Z(5)^7+Z(2)^8*Z(5)^8+Z(2)^7*Z(5)^9+Z(2)^6*Z(5)^10+Z\
  (2)^5*Z(5)^11+Z(2)^4*Z(5)^12+Z(2)^3*Z(5)^13+Z(2)^2*Z(5)^14+Z(2)*Z(5)^15,
⇒ Z(3)^15*Z(4)+Z(3)^14*Z(4)^2+Z(3)^13*Z(4)^3+Z(3)^12*Z(4)^4+Z(3)^11*Z(4)^5+\
  Z(3)^10*Z(4)^6+Z(3)^9*Z(4)^7+Z(3)^8*Z(4)^8+Z(3)^7*Z(4)^9+Z(3)^6*Z(4)^10+Z\
  (3)^5*Z(4)^11+Z(3)^4*Z(4)^12+Z(3)^3*Z(4)^13+Z(3)^2*Z(4)^14+Z(3)*Z(4)^15,
⇒ Z(3)^15*Z(5)+Z(3)^14*Z(5)^2+Z(3)^13*Z(5)^3+Z(3)^12*Z(5)^4+Z(3)^11*Z(5)^5+\
  Z(3)^10*Z(5)^6+Z(3)^9*Z(5)^7+Z(3)^8*Z(5)^8+Z(3)^7*Z(5)^9+Z(3)^6*Z(5)^10+Z\
  (3)^5*Z(5)^11+Z(3)^4*Z(5)^12+Z(3)^3*Z(5)^13+Z(3)^2*Z(5)^14+Z(3)*Z(5)^15,
⇒ Z(4)^15*Z(5)+Z(4)^14*Z(5)^2+Z(4)^13*Z(5)^3+Z(4)^12*Z(5)^4+Z(4)^11*Z(5)^5+\
  Z(4)^10*Z(5)^6+Z(4)^9*Z(5)^7+Z(4)^8*Z(5)^8+Z(4)^7*Z(5)^9+Z(4)^6*Z(5)^10+Z\
  (4)^5*Z(5)^11+Z(4)^4*Z(5)^12+Z(4)^3*Z(5)^13+Z(4)^2*Z(5)^14+Z(4)*Z(5)^15
option(redSB);
ideal red_crht_md=std(crht_md);
print(red_crht_md);          // reduced Groebner basis
⇒ Z(5)^15+1,
⇒ Z(4)^12+Z(4)^9*Z(5)^3+Z(4)^6*Z(5)^6+Z(4)^3*Z(5)^9+Z(5)^12,
⇒ Z(3)^6+Z(3)^4*Z(4)*Z(5)+Z(3)^2*Z(4)^2*Z(5)^2+Z(3)*Z(4)^4*Z(5)+Z(3)*Z(4)*Z\
  (5)^4+Z(4)^6+Z(5)^6,
⇒ Z(2)^2+Z(2)*Z(3)+Z(2)*Z(4)+Z(2)*Z(5)+Z(3)^5*Z(4)^10*Z(5)^2+Z(3)^5*Z(4)^9*\
  Z(5)^3+Z(3)^5*Z(4)^8*Z(5)^4+Z(3)^5*Z(4)^4*Z(5)^8+Z(3)^5*Z(4)^3*Z(5)^9+Z(3)\
  ^5*Z(4)^2*Z(5)^10+Z(3)^4*Z(4)^11*Z(5)^2+Z(3)^4*Z(4)^8*Z(5)^5+Z(3)^4*Z(4)\
  ^5*Z(5)^8+Z(3)^4*Z(4)^2*Z(5)^11+Z(3)^3*Z(4)^10*Z(5)^4+Z(3)^3*Z(4)^9*Z(5)\
  ^5+Z(3)^3*Z(4)^8*Z(5)^6+Z(3)^3*Z(4)^4*Z(5)^10+Z(3)^3*Z(4)^3*Z(5)^11+Z(3)^3\
  *Z(4)^2*Z(5)^12+Z(3)^3*Z(5)^14+Z(3)^2*Z(4)^11*Z(5)^4+Z(3)^2*Z(4)^8*Z(5)^7\
  +Z(3)^2*Z(4)^5*Z(5)^10+Z(3)^2*Z(4)^2*Z(5)^13+Z(3)^2*Z(4)*Z(5)^14+Z(3)^2*Z\
  (3)*Z(4)^10*Z(5)^6+Z(3)*Z(4)^9*Z(5)^7+Z(3)*Z(4)^8*Z(5)^8+Z(3)*Z(4)^4*Z(5)\
  ^12+Z(3)*Z(4)^3*Z(5)^13+Z(3)*Z(4)+Z(4)^11*Z(5)^6+Z(4)^8*Z(5)^9+Z(4)^5*Z(5)\
  ^12+Z(4)^3*Z(5)^14+Z(4)^2,
⇒ Z(1)+Z(2)+Z(3)+Z(4)+Z(5)
option(set,v);

```

### D.10.2.3 sysNewton

Procedure from library `decodegb.lib` (see [Section D.10.2 \[decodegb.lib\]](#), page 1938).

**Usage:** `sysNewton (n,defset,t,q,m,[tr]);`  $n, t, q, m, tr$  int, `defset` is list int's

- $n$  is length,
- `defset` is the defining set,
- $t$  is the number of errors,
- $q$  is basefield size,
- $m$  is degree extension of the splitting field,
- if  $tr > 0$  it indicates that Newton identities in triangular form should be constructed

**Return:** the ring to work with the generalized Newton identities (in triangular form if applicable) containing the ideal with name 'newton'

**Theory:** Based on 'defset' of the given cyclic code, the procedure constructs the corresponding ideal 'newton' with the generalized Newton identities. With its help one can solve the decoding problem. For basics of the method [Section C.8.3 \[Generalized Newton identities\]](#), page 788.

**Example:**

```
LIB "decodegb.lib";
// Newton identities for a binary 3-error-correcting cyclic code of
//length 31 with defining set (1,5,7)
int n=31;           // length
list defset=1,5,7; //defining set
int t=3;           // number of errors
int q=2;           // basefield size
int m=5;           // degree extension of the splitting field
int tr=1;          // indicator of triangular form of Newton identities
def A=sysNewton(n,defset,t,q,m);
setring A;
A;                // shows the ring we are working in
⇒ // coefficients: ZZ/2(a)
⇒ // number of vars : 34
⇒ //      block 1 : ordering lp
⇒ //      : names      S(31) S(30) S(29) S(28) S(27) S(26) S(25) \
S(24) S(23) S(22) S(21) S(20) S(19) S(18) S(17) S(16) S(15) S(14) S(13) S\
(12) S(11) S(10) S(9) S(8) S(6) S(4) S(3) S(2) sigma(1) sigma(2) sigma(3)\
S(7) S(5) S(1)
⇒ //      block 2 : ordering C
print(newton);    // generalized Newton identities
⇒ S(31)*sigma(1)+S(30)*sigma(2)+S(29)*sigma(3)+S(1),
⇒ S(31)*sigma(2)+S(30)*sigma(3)+S(2)+sigma(1)*S(1),
⇒ S(31)*sigma(3)+S(3)+S(2)*sigma(1)+sigma(2)*S(1),
⇒ S(4)+S(3)*sigma(1)+S(2)*sigma(2)+sigma(3)*S(1),
⇒ S(4)*sigma(1)+S(3)*sigma(2)+S(2)*sigma(3)+S(5),
⇒ S(6)+S(4)*sigma(2)+S(3)*sigma(3)+sigma(1)*S(5),
⇒ S(6)*sigma(1)+S(4)*sigma(3)+sigma(2)*S(5)+S(7),
⇒ S(8)+S(6)*sigma(2)+sigma(1)*S(7)+sigma(3)*S(5),
⇒ S(9)+S(8)*sigma(1)+S(6)*sigma(3)+sigma(2)*S(7),
⇒ S(10)+S(9)*sigma(1)+S(8)*sigma(2)+sigma(3)*S(7),
⇒ S(11)+S(10)*sigma(1)+S(9)*sigma(2)+S(8)*sigma(3),
⇒ S(12)+S(11)*sigma(1)+S(10)*sigma(2)+S(9)*sigma(3),
```

```

⇒ S(13)+S(12)*sigma(1)+S(11)*sigma(2)+S(10)*sigma(3),
⇒ S(14)+S(13)*sigma(1)+S(12)*sigma(2)+S(11)*sigma(3),
⇒ S(15)+S(14)*sigma(1)+S(13)*sigma(2)+S(12)*sigma(3),
⇒ S(16)+S(15)*sigma(1)+S(14)*sigma(2)+S(13)*sigma(3),
⇒ S(17)+S(16)*sigma(1)+S(15)*sigma(2)+S(14)*sigma(3),
⇒ S(18)+S(17)*sigma(1)+S(16)*sigma(2)+S(15)*sigma(3),
⇒ S(19)+S(18)*sigma(1)+S(17)*sigma(2)+S(16)*sigma(3),
⇒ S(20)+S(19)*sigma(1)+S(18)*sigma(2)+S(17)*sigma(3),
⇒ S(21)+S(20)*sigma(1)+S(19)*sigma(2)+S(18)*sigma(3),
⇒ S(22)+S(21)*sigma(1)+S(20)*sigma(2)+S(19)*sigma(3),
⇒ S(23)+S(22)*sigma(1)+S(21)*sigma(2)+S(20)*sigma(3),
⇒ S(24)+S(23)*sigma(1)+S(22)*sigma(2)+S(21)*sigma(3),
⇒ S(25)+S(24)*sigma(1)+S(23)*sigma(2)+S(22)*sigma(3),
⇒ S(26)+S(25)*sigma(1)+S(24)*sigma(2)+S(23)*sigma(3),
⇒ S(27)+S(26)*sigma(1)+S(25)*sigma(2)+S(24)*sigma(3),
⇒ S(28)+S(27)*sigma(1)+S(26)*sigma(2)+S(25)*sigma(3),
⇒ S(29)+S(28)*sigma(1)+S(27)*sigma(2)+S(26)*sigma(3),
⇒ S(30)+S(29)*sigma(1)+S(28)*sigma(2)+S(27)*sigma(3),
⇒ S(31)+S(30)*sigma(1)+S(29)*sigma(2)+S(28)*sigma(3),
⇒ sigma(1)^32+sigma(1),
⇒ sigma(2)^32+sigma(2),
⇒ sigma(3)^32+sigma(3),
⇒ S(2)+S(1)^2,
⇒ S(4)+S(2)^2,
⇒ S(6)+S(3)^2,
⇒ S(8)+S(4)^2,
⇒ S(10)+S(5)^2,
⇒ S(12)+S(6)^2,
⇒ S(14)+S(7)^2,
⇒ S(16)+S(8)^2,
⇒ S(18)+S(9)^2,
⇒ S(20)+S(10)^2,
⇒ S(22)+S(11)^2,
⇒ S(24)+S(12)^2,
⇒ S(26)+S(13)^2,
⇒ S(28)+S(14)^2,
⇒ S(30)+S(15)^2,
⇒ S(16)^2+S(1),
⇒ S(17)^2+S(3),
⇒ S(18)^2+S(5),
⇒ S(19)^2+S(7),
⇒ S(20)^2+S(9),
⇒ S(21)^2+S(11),
⇒ S(22)^2+S(13),
⇒ S(23)^2+S(15),
⇒ S(24)^2+S(17),
⇒ S(25)^2+S(19),
⇒ S(26)^2+S(21),
⇒ S(27)^2+S(23),
⇒ S(28)^2+S(25),
⇒ S(29)^2+S(27),
⇒ S(30)^2+S(29),
⇒ S(31)^2+S(31)

```

```
//=====
A=sysNewton(n,defset,t,q,m,tr);
setring A;
print(newton);      // generalized Newton identities in triangular form
⇒ sigma(1)+S(1),
⇒ S(2)+sigma(1)*S(1),
⇒ S(3)+S(2)*sigma(1)+sigma(2)*S(1)+sigma(3),
⇒ S(4)+S(3)*sigma(1)+S(2)*sigma(2)+sigma(3)*S(1),
⇒ S(4)*sigma(1)+S(3)*sigma(2)+S(2)*sigma(3)+S(5),
⇒ S(6)+S(4)*sigma(2)+S(3)*sigma(3)+sigma(1)*S(5),
⇒ S(6)*sigma(1)+S(4)*sigma(3)+sigma(2)*S(5)+S(7),
⇒ S(8)+S(6)*sigma(2)+sigma(1)*S(7)+sigma(3)*S(5),
⇒ S(9)+S(8)*sigma(1)+S(6)*sigma(3)+sigma(2)*S(7),
⇒ S(10)+S(9)*sigma(1)+S(8)*sigma(2)+sigma(3)*S(7),
⇒ S(11)+S(10)*sigma(1)+S(9)*sigma(2)+S(8)*sigma(3),
⇒ S(12)+S(11)*sigma(1)+S(10)*sigma(2)+S(9)*sigma(3),
⇒ S(13)+S(12)*sigma(1)+S(11)*sigma(2)+S(10)*sigma(3),
⇒ S(14)+S(13)*sigma(1)+S(12)*sigma(2)+S(11)*sigma(3),
⇒ S(15)+S(14)*sigma(1)+S(13)*sigma(2)+S(12)*sigma(3),
⇒ S(16)+S(15)*sigma(1)+S(14)*sigma(2)+S(13)*sigma(3),
⇒ S(17)+S(16)*sigma(1)+S(15)*sigma(2)+S(14)*sigma(3),
⇒ S(18)+S(17)*sigma(1)+S(16)*sigma(2)+S(15)*sigma(3),
⇒ S(19)+S(18)*sigma(1)+S(17)*sigma(2)+S(16)*sigma(3),
⇒ S(20)+S(19)*sigma(1)+S(18)*sigma(2)+S(17)*sigma(3),
⇒ S(21)+S(20)*sigma(1)+S(19)*sigma(2)+S(18)*sigma(3),
⇒ S(22)+S(21)*sigma(1)+S(20)*sigma(2)+S(19)*sigma(3),
⇒ S(23)+S(22)*sigma(1)+S(21)*sigma(2)+S(20)*sigma(3),
⇒ S(24)+S(23)*sigma(1)+S(22)*sigma(2)+S(21)*sigma(3),
⇒ S(25)+S(24)*sigma(1)+S(23)*sigma(2)+S(22)*sigma(3),
⇒ S(26)+S(25)*sigma(1)+S(24)*sigma(2)+S(23)*sigma(3),
⇒ S(27)+S(26)*sigma(1)+S(25)*sigma(2)+S(24)*sigma(3),
⇒ S(28)+S(27)*sigma(1)+S(26)*sigma(2)+S(25)*sigma(3),
⇒ S(29)+S(28)*sigma(1)+S(27)*sigma(2)+S(26)*sigma(3),
⇒ S(30)+S(29)*sigma(1)+S(28)*sigma(2)+S(27)*sigma(3),
⇒ S(31)+S(30)*sigma(1)+S(29)*sigma(2)+S(28)*sigma(3),
⇒ sigma(1)^32+sigma(1),
⇒ sigma(2)^32+sigma(2),
⇒ sigma(3)^32+sigma(3),
⇒ S(2)+S(1)^2,
⇒ S(4)+S(2)^2,
⇒ S(6)+S(3)^2,
⇒ S(8)+S(4)^2,
⇒ S(10)+S(5)^2,
⇒ S(12)+S(6)^2,
⇒ S(14)+S(7)^2,
⇒ S(16)+S(8)^2,
⇒ S(18)+S(9)^2,
⇒ S(20)+S(10)^2,
⇒ S(22)+S(11)^2,
⇒ S(24)+S(12)^2,
⇒ S(26)+S(13)^2,
⇒ S(28)+S(14)^2,
⇒ S(30)+S(15)^2,
```

```

⇒ S(16)^2+S(1),
⇒ S(17)^2+S(3),
⇒ S(18)^2+S(5),
⇒ S(19)^2+S(7),
⇒ S(20)^2+S(9),
⇒ S(21)^2+S(11),
⇒ S(22)^2+S(13),
⇒ S(23)^2+S(15),
⇒ S(24)^2+S(17),
⇒ S(25)^2+S(19),
⇒ S(26)^2+S(21),
⇒ S(27)^2+S(23),
⇒ S(28)^2+S(25),
⇒ S(29)^2+S(27),
⇒ S(30)^2+S(29),
⇒ S(31)^2+S(31)

```

See also: [Section D.10.2.4 \[sysBin\]](#), page 1945; [Section D.10.2.1 \[sysCRHT\]](#), page 1938.

### D.10.2.4 sysBin

Procedure from library `decodegb.lib` (see [Section D.10.2 \[decodegb.lib\]](#), page 1938).

**Usage:**     `sysBin (v,Q,n,[odd]);` `v,n,odd` are int, `Q` is list of int's

- `v` a number if errors,
- `Q` is a defining set of the code,
- `n` the length,
- `odd` is an additional parameter: if  
     set to 1, then the defining set is enlarged by odd elements,  
     which are  $2^{(\text{some power})} * (\text{some element in the def.set}) \bmod n$

**Return:**     the ring with the resulting system called 'bin'

**Theory:**     Based on `Q` of the given cyclic code, the procedure constructs the corresponding ideal 'bin' with the use of the Waring function. With its help one can solve the decoding problem. For basics of the method [Section C.8.3 \[Generalized Newton identities\]](#), page 788.

**Example:**

```

LIB "decodegb.lib";
// [31,16,7] quadratic residue code
list l=1,5,7,9,19,25;
// we do not need even synromes here
def A=sysBin(3,l,31);
setring A;
print(bin);
⇒ S(1)+sigma(1),
⇒ S(5)+sigma(1)^5+sigma(1)^3*sigma(2)+sigma(1)^2*sigma(3)+sigma(1)*sigma(2)\
  ^2+sigma(2)*sigma(3),
⇒ S(7)+sigma(1)^7+sigma(1)^5*sigma(2)+sigma(1)^4*sigma(3)+sigma(1)^2*sigma(\
  2)*sigma(3)+sigma(1)*sigma(2)^3+sigma(1)*sigma(3)^2+sigma(2)^2*sigma(3),
⇒ S(9)+sigma(1)^9+sigma(1)^7*sigma(2)+sigma(1)^6*sigma(3)+sigma(1)^5*sigma(\
  2)^2+sigma(1)^4*sigma(2)*sigma(3)+sigma(1)*sigma(2)^4+sigma(1)*sigma(2)*s\
  igma(3)^2+sigma(2)^3*sigma(3)+sigma(3)^3,
⇒ S(19)+sigma(1)^19+sigma(1)^17*sigma(2)+sigma(1)^16*sigma(3)+sigma(1)^14*s\

```

```

sigma(2)*sigma(3)+sigma(1)^13*sigma(2)^3+sigma(1)^13*sigma(3)^2+sigma(1)^1\
2*sigma(2)^2*sigma(3)+sigma(1)^11*sigma(2)^4+sigma(1)^9*sigma(2)^5+sigma(\
1)^8*sigma(2)^4*sigma(3)+sigma(1)^8*sigma(2)*sigma(3)^3+sigma(1)^3*sigma(\
2)^8+sigma(1)^2*sigma(2)*sigma(3)^5+sigma(1)*sigma(2)^9+sigma(1)*sigma(2)\
^3*sigma(3)^4+sigma(1)*sigma(3)^6+sigma(2)^8*sigma(3)+sigma(2)^5*sigma(3)\
^3+sigma(2)^2*sigma(3)^5,
↦ S(25)+sigma(1)^25+sigma(1)^23*sigma(2)+sigma(1)^22*sigma(3)+sigma(1)^21*s\
igma(2)^2+sigma(1)^20*sigma(2)*sigma(3)+sigma(1)^17*sigma(2)^4+sigma(1)^1\
7*sigma(2)*sigma(3)^2+sigma(1)^16*sigma(2)^3*sigma(3)+sigma(1)^16*sigma(3\
)^3+sigma(1)^11*sigma(2)*sigma(3)^4+sigma(1)^10*sigma(3)^5+sigma(1)^9*sig\
ma(2)^5*sigma(3)^2+sigma(1)^9*sigma(2)^2*sigma(3)^4+sigma(1)^8*sigma(2)^7\
*sigma(3)+sigma(1)^8*sigma(2)^4*sigma(3)^3+sigma(1)^8*sigma(2)*sigma(3)^5\
+sigma(1)^7*sigma(2)^9+sigma(1)^6*sigma(2)^8*sigma(3)+sigma(1)^5*sigma(2)\
^10+sigma(1)^4*sigma(2)^9*sigma(3)+sigma(1)*sigma(2)^12+sigma(1)*sigma(2)\
^9*sigma(3)^2+sigma(1)*sigma(3)^8+sigma(2)^11*sigma(3)+sigma(2)^8*sigma(3\
)^3

```

See also: [Section D.10.2.1 \[sysCRHT\], page 1938](#); [Section D.10.2.3 \[sysNewton\], page 1942](#).

### D.10.2.5 encode

Procedure from library `decodegb.lib` (see [Section D.10.2 \[decodegb.lib\], page 1938](#)).

**Usage:**        `encode (x, g);` x a row vector (message), and g a generator matrix

**Return:**        corresponding codeword

**Example:**

```

LIB "decodegb.lib";
ring r=2,x,dp;
matrix x[1][4]=1,0,1,0;
matrix g[4][7]=1,0,0,0,0,1,1,
0,1,0,0,1,0,1,
0,0,1,0,1,1,1,
0,0,0,1,1,1,0;
//encode x with the generator matrix g
print(encode(x,g));
↦ 1,0,1,0,1,0,0

```

### D.10.2.6 syndrome

Procedure from library `decodegb.lib` (see [Section D.10.2 \[decodegb.lib\], page 1938](#)).

**Usage:**        `syndrome (h, c);` h a check matrix, c a row vector (codeword)

**Return:**        corresponding syndrome

**Example:**

```

LIB "decodegb.lib";
ring r=2,x,dp;
matrix x[1][4]=1,0,1,0;
matrix g[4][7]=1,0,0,0,0,1,1,
0,1,0,0,1,0,1,
0,0,1,0,1,1,1,
0,0,0,1,1,1,0;
//encode x with the generator matrix g

```



```

matrix c=encode(x,g);
// disturb
c[1,3]=0;
//compute syndrome
//corresponding check matrix
matrix check[3][7]=1,0,0,1,1,0,1,0,1,0,1,0,1,1,0,0,1,0,1,1,1;
print(syndrome(check,c));
⇒ 0,
⇒ 0,
⇒ 1
c[1,3]=1;
//now c is a codeword
print(syndrome(check,c));
⇒ 0,
⇒ 0,
⇒ 0

```

### D.10.2.7 sysQE

Procedure from library `decodegb.lib` (see [Section D.10.2 \[decodegb.lib\]](#), page 1938).

**Usage:** `sysQE(check,y,t,[fieldeq,formal]);` `check,y` matrix; `t,fieldeq,formal` int

- `check` is a parity check matrix of the code
- `y` is a received word,
- `t` the number of errors to be corrected,
- if `fieldeq=1`, then field equations are added,
- if `formal=0`, field equations on (known) syndrome variables are not added, in order to add them (note that the exponent should be equal to the number of elements in the INITIAL alphabet) one needs to set `formal>0` for the exponent

**Return:** the ring to work with together with the resulting system called 'qe'

**Theory:** Based on 'check' of the given linear code, the procedure constructs the corresponding ideal that gives an opportunity to compute unknown syndrome of the received word `y`. After computing the unknown syndromes one is able to solve the decoding problem. For basics of the method [Section C.8.5 \[Decoding method based on quadratic equations\]](#), page 790.

**Example:**

```

LIB "decodegb.lib";
intvec v = option(get);
//correct 2 errors in [7,3] 8-ary code RS code
int t=2; int q=8; int n=7; int redun=4;
ring r=(q,a),x,dp;
matrix h_full=genMDSMat(n,a);
matrix h=submat(h_full,1..redun,1..n);
matrix g=dual_code(h);
matrix x[1][3]=0,0,1,0;
matrix y[1][7]=encode(x,g);
//disturb with 2 errors
matrix rec[1][7]=errorInsert(y,list(2,4),list(1,a));
//generate the system
def A=sysQE(h,rec,t);

```

```

setring A;
print(qe);
⇨ U(1)+a^3,
⇨ U(2)+a^2,
⇨ U(3)+a^6,
⇨ U(4),
⇨ V(1)*U(1)+V(2)*U(2)+U(3),
⇨ V(1)*U(2)+V(2)*U(3)+U(4),
⇨ V(1)*U(3)+V(2)*U(4)+U(5),
⇨ V(1)*U(4)+V(2)*U(5)+U(6),
⇨ V(1)*U(5)+V(2)*U(6)+U(7),
⇨ V(1)*U(6)+V(2)*U(7)+U(1),
⇨ V(2)*U(1)+V(1)*U(7)+U(2)
//let us decode
option(redSB);
ideal sys_qe=std(qe);
print(sys_qe);
⇨ U(7)+a,
⇨ U(6)+a^3,
⇨ U(5)+a^3,
⇨ U(4),
⇨ U(3)+a^6,
⇨ U(2)+a^2,
⇨ U(1)+a^3,
⇨ V(2)+1,
⇨ V(1)+a^4
option(set,v);

```

See also: [Section D.10.2.17 \[sysFL\]](#), page 1960.

### D.10.2.8 errorInsert

Procedure from library `decodegb.lib` (see [Section D.10.2 \[decodegb.lib\]](#), page 1938).

**Usage:**      `errorInsert(y,pos,val)`; `y` is matrix, `pos,val` are list of int's

- `y` is a (code) word,
- `pos` = positions where errors occurred,
- `val` = their corresponding values

**Return:**      corresponding received word

**Example:**

```

LIB "decodegb.lib";
//correct 2 errors in [7,3] 8-ary code RS code
int t=2; int q=8; int n=7; int redun=4;
ring r=(q,a),x,dp;
matrix h_full=genMDSMat(n,a);
matrix h=submat(h_full,1..redun,1..n);
matrix g=dual_code(h);
matrix x[1][3]=0,0,1,0;
matrix y[1][7]=encode(x,g);
print(y);
⇨ a6,a6,a3,a,0,0,1
//disturb with 2 errors
matrix rec[1][7]=errorInsert(y,list(2,4),list(1,a));

```

```

print(rec);
↪ a6,a2,a3,0,0,0,1
print(rec-y);
↪ 0,1,0,a,0,0,0

```

### D.10.2.9 errorRand

Procedure from library `decodegb.lib` (see [Section D.10.2 \[decodegb.lib\]](#), page 1938).

**Usage:**      `errorRand(y, num, e)`; `y` is matrix, `num`, `e` are int

- `y` is a (code) word,
- `num` is the number of errors,
- `e` is an extension degree (if one wants values to be from  $\text{GF}(p^e)$ )

**Return:**      corresponding received word

**Example:**

```

LIB "decodegb.lib";
//correct 2 errors in [7,3] 8-ary code RS code
int t=2; int q=8; int n=7; int redun=4;
ring r=(q,a),x,dp;
matrix h_full=genMDSMat(n,a);
matrix h=submat(h_full,1..redun,1..n);
matrix g=dual_code(h);
matrix x[1][3]=0,0,1,0;
matrix y[1][7]=encode(x,g);
//disturb with 2 random errors
matrix rec[1][7]=errorRand(y,2,3);
print(rec);
↪ a3,a6,a2,a,0,0,1
print(rec-y);
↪ a4,0,a5,0,0,0,0

```

### D.10.2.10 randomCheck

Procedure from library `decodegb.lib` (see [Section D.10.2 \[decodegb.lib\]](#), page 1938).

**Usage:**      `randomCheck(m, n, e)`; `m`, `n`, `e` are int

- `m` x `n` are dimensions of the matrix,
- `e` is an extension degree (if one wants values to be from  $\text{GF}(p^e)$ )

**Return:**      random check matrix

**Example:**

```

LIB "decodegb.lib";
int redun=5; int n=15;
ring r=2,x,dp;
//generate random check matrix for a [15,5] binary code
matrix h=randomCheck(redun,n,1);
print(h);
↪ 0,1,0,0,0,1,1,1,0,1,1,0,0,0,0,
↪ 1,1,0,0,0,0,0,1,0,0,0,1,0,0,0,
↪ 1,0,1,1,1,1,0,0,0,1,0,0,1,0,0,
↪ 1,1,0,1,1,0,0,0,0,1,0,0,0,1,0,
↪ 0,1,0,0,0,0,0,1,1,0,0,0,0,0,1

```

```
//corresponding generator matrix
matrix g=dual_code(h);
print(g);
⇒ 0,0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,
⇒ 0,0,1,0,0,1,1,0,0,0,0,0,0,0,0,0,
⇒ 0,1,1,1,0,0,0,1,0,0,0,0,0,0,0,0,
⇒ 1,1,0,0,0,1,0,0,1,0,0,0,0,0,0,0,
⇒ 0,0,1,1,0,1,0,0,0,1,0,0,0,0,0,0,
⇒ 0,0,1,0,0,1,0,0,0,0,1,0,0,0,0,0,
⇒ 1,0,0,1,0,0,0,0,0,0,0,1,0,0,0,0,
⇒ 0,0,1,0,0,0,0,0,0,0,0,0,1,0,0,0,
⇒ 0,0,1,1,0,0,0,0,0,0,0,0,0,1,0,0,
⇒ 1,1,0,0,0,1,0,0,0,0,0,0,0,0,0,1
```

### D.10.2.11 genMDSMat

Procedure from library `decodegb.lib` (see [Section D.10.2 \[decodegb.lib\]](#), page 1938).

**Usage:**      `genMDSMat(n, a);`  $n$  is int,  $a$  is number  
                   -  $n \times n$  are dimensions of the MDS matrix,  
                   -  $a$  is a primitive element of the field.

**Note:**      An MDS matrix is constructed in the following way. We take ' $a$ ' to be a generator of the multiplicative group of the field. Then we construct the Vandermonde matrix with this ' $a$ '.

**Assume:**    extension field should already be defined

**Return:**    a matrix with the MDS property.

**Example:**

```
LIB "decodegb.lib";
int q=16; int n=15;
ring r=(q,a),x,dp;
//generate an MDS (Vandermonde) matrix
matrix h_full=genMDSMat(n,a);
print(h_full);
⇒ 1,1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
⇒ 1,a, a2, a3, a4, a5, a6, a7, a8, a9, a10,a11,a12,a13,a14,
⇒ 1,a2, a4, a6, a8, a10,a12,a14,a, a3, a5, a7, a9, a11,a13,
⇒ 1,a3, a6, a9, a12,1, a3, a6, a9, a12,1, a3, a6, a9, a12,
⇒ 1,a4, a8, a12,a, a5, a9, a13,a2, a6, a10,a14,a3, a7, a11,
⇒ 1,a5, a10,1, a5, a10,1, a5, a10,1, a5, a10,1, a5, a10,
⇒ 1,a6, a12,a3, a9, 1, a6, a12,a3, a9, 1, a6, a12,a3, a9,
⇒ 1,a7, a14,a6, a13,a5, a12,a4, a11,a3, a10,a2, a9, a, a8,
⇒ 1,a8, a, a9, a2, a10,a3, a11,a4, a12,a5, a13,a6, a14,a7,
⇒ 1,a9, a3, a12,a6, 1, a9, a3, a12,a6, 1, a9, a3, a12,a6,
⇒ 1,a10,a5, 1, a10,a5, 1, a10,a5, 1, a10,a5, 1, a10,a5,
⇒ 1,a11,a7, a3, a14,a10,a6, a2, a13,a9, a5, a, a12,a8, a4,
⇒ 1,a12,a9, a6, a3, 1, a12,a9, a6, a3, 1, a12,a9, a6, a3,
⇒ 1,a13,a11,a9, a7, a5, a3, a, a14,a12,a10,a8, a6, a4, a2,
⇒ 1,a14,a13,a12,a11,a10,a9, a8, a7, a6, a5, a4, a3, a2, a
```

See also: [Section C.8.5 \[Decoding method based on quadratic equations\]](#), page 790.

### D.10.2.12 mindist

Procedure from library `decodegb.lib` (see [Section D.10.2 \[decodegb.lib\], page 1938](#)).

**Usage:**      `mindist (check, q);` check matrix, q int  
                  - check is a check matrix,  
                  - q is the field size

**Return:**     minimum distance of the code

**Example:**

```
LIB "decodegb.lib";
//determine a minimum distance for a [7,3] binary code
int q=8; int n=7; int redun=4; int t=redun+1;
ring r=(q,a),x,dp;
//generate random check matrix
matrix h=randomCheck(redun,n,1);
print(h);
↪ 0,1,0,1,0,0,0,
↪ 0,0,1,0,1,0,0,
↪ 1,1,0,0,0,1,0,
↪ 1,1,1,0,0,0,1
int l=mindist(h);
1;
↪ 3
```

### D.10.2.13 decode

Procedure from library `decodegb.lib` (see [Section D.10.2 \[decodegb.lib\], page 1938](#)).

**Usage:**      `decode(check, rec, t);` check, rec matrix, t int  
                  - check is the check matrix of the code,  
                  - rec is a received word,  
                  - t is an upper bound for the number of errors one wants to correct

**Note:**        The method described in [Section C.8.5 \[Decoding method based on quadratic equations\], page 790](#) is used for decoding.

**Assume:**     Errors in rec should be correctable, otherwise the output is unpredictable

**Return:**      a codeword that is closest to rec

**Example:**

```
LIB "decodegb.lib";
//correct 1 error in [15,7] binary code
int t=1; int q=16; int n=15; int redun=10;
ring r=(q,a),x,dp;
//generate random check matrix
matrix h=randomCheck(redun,n,1);
matrix g=dual_code(h);
matrix x[1][n-redun]=0,0,1,0,1,0,1;
matrix y[1][n]=encode(x,g);
print(y);
↪ 1,0,1,0,1,0,1,1,1,1,0,0,1,0,1
// find out the minimum distance of the code
list l=mindist(h);
```

#### D.10.2.14 decodeRandom

**Usage:**      decodeRandom(redun,q,ncodes,ntrials,[e]); all parameters int

- redun is a redundancy of a (random) code,
- q is the field size,
- ncodes is the number of random codes to be processed,
- ntrials is the number of received vectors per code to be corrected
- If e is given it sets the correction capacity explicitly. It should be used in case one expects some lower bound, otherwise the procedure tries to compute the real minimum distance to find out the error-correction capacity

**Example:**

```
LIB "decodegb.lib";  
int q=32; int n=25; int redun=n-11; int t=redun+1;  
ring r=(q,a),x,dp;  
// correct 2 errors in 2 random binary codes, 3 trials each  
decodeRandom(n,redun,2,3,2);  
⇒ check matrix:  
⇒ 0,1,0,0,0,1,1,1,0,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,  
⇒ 1,0,0,0,0,0,0,1,0,0,1,0,0,1,0,0,0,0,0,0,0,0,0,0,  
⇒ 1,1,1,1,0,0,0,1,1,1,0,0,0,1,0,0,0,0,0,0,0,0,0,0,  
⇒ 1,1,0,0,0,0,1,0,1,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,  
⇒ 0,0,0,1,1,0,1,0,0,1,1,0,0,0,0,1,0,0,0,0,0,0,0,0,  
⇒ 0,0,1,0,1,1,1,0,1,1,1,0,0,0,0,0,1,0,0,0,0,0,0,0,  
⇒ 0,1,0,0,0,1,0,1,0,0,1,0,0,0,0,0,0,1,0,0,0,0,0,0,  
⇒ 0,1,1,1,0,1,1,1,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,  
⇒ 1,0,0,0,1,1,1,1,0,1,0,0,0,0,0,0,0,0,0,1,0,0,0,0,  
⇒ 0,1,1,1,0,1,0,0,1,1,1,0,0,0,0,0,0,0,0,0,1,0,0,0,  
⇒ 0,0,1,0,0,1,0,1,1,0,1,0,0,0,0,0,0,0,0,0,0,1,0,0,  
⇒ 0,0,1,0,0,0,1,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,1,0,  
⇒ 0,0,0,0,1,0,0,0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,1,  
⇒ 1,0,1,1,1,1,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,1  
⇒ The system is generated  
⇒ Codeword:  
⇒ 1,0,0,1,0,1,0,1,1,1,0,1,0,1,0,0,1,0,1,0,0,1,1,1,0  
⇒ Received word:  
⇒ 1,0,0,1,0,1,0,0,1,1,0,1,0,1,0,0,1,1,1,0,0,1,1,1,0  
⇒ The Groebner basis of the QE system:
```

```

⇒ U(25)+a25,
⇒ U(24)+a20,
⇒ U(23)+a28,
⇒ U(22)+a7,
⇒ U(21)+a29,
⇒ U(20)+a19,
⇒ U(19)+a23,
⇒ U(18)+a19,
⇒ U(17)+a21,
⇒ U(16)+a9,
⇒ U(15)+a14,
⇒ U(14)+a25,
⇒ U(13)+a28,
⇒ U(12)+a14,
⇒ U(11)+a30,
⇒ U(10)+a27,
⇒ U(9)+a26,
⇒ U(8)+a7,
⇒ U(7)+a14,
⇒ U(6)+a15,
⇒ U(5)+a13,
⇒ U(4)+a7,
⇒ U(3)+a22,
⇒ U(2)+a11,
⇒ U(1),
⇒ V(2)+a11,
⇒ V(1)+a24
⇒ Codeword:
⇒ 0,0,1,1,1,1,1,0,0,1,0,1,0,1,1,0,1,1,0,0,0,0,1,0,0
⇒ Received word:
⇒ 0,0,0,1,1,1,1,0,0,1,0,1,0,1,1,0,1,1,0,0,0,0,1,1,0
⇒ The Groebner basis of the QE system:
⇒ U(25)+a6,
⇒ U(24)+a16,
⇒ U(23)+a8,
⇒ U(22)+a2,
⇒ U(21)+a8,
⇒ U(20)+a13,
⇒ U(19)+a,
⇒ U(18)+a12,
⇒ U(17)+a29,
⇒ U(16)+a,
⇒ U(15)+a21,
⇒ U(14)+a16,
⇒ U(13)+a3,
⇒ U(12)+a4,
⇒ U(11)+a4,
⇒ U(10)+a16,
⇒ U(9)+a30,
⇒ U(8)+a26,
⇒ U(7)+a17,
⇒ U(6)+a2,
⇒ U(5)+a15,

```

```

⇒ U(4)+a^24,
⇒ U(3)+a^23,
⇒ U(2)+a^27,
⇒ U(1),
⇒ V(2)+a^27,
⇒ V(1)+a^25
⇒ Codeword:
⇒ 0,0,0,1,0,1,0,0,0,1,1,1,1,0,0,1,1,0,0,0,0,0,1,0,1
⇒ Received word:
⇒ 0,0,0,1,1,1,0,1,0,1,1,1,1,0,0,1,1,0,0,0,0,0,1,0,1
⇒ The Groebner basis of the QE system:
⇒ U(25)+a^7,
⇒ U(24)+a^21,
⇒ U(23)+a^5,
⇒ U(22)+a^9,
⇒ U(21)+a^21,
⇒ U(20)+a^11,
⇒ U(19)+a^22,
⇒ U(18)+a^14,
⇒ U(17)+a,
⇒ U(16)+a^11,
⇒ U(15)+a^13,
⇒ U(14)+a^10,
⇒ U(13)+a^19,
⇒ U(12)+a^18,
⇒ U(11)+a^26,
⇒ U(10)+a^11,
⇒ U(9)+a^16,
⇒ U(8)+a^22,
⇒ U(7)+a^25,
⇒ U(6)+a^13,
⇒ U(5)+a^8,
⇒ U(4)+a^28,
⇒ U(3)+a^4,
⇒ U(2)+a^2,
⇒ U(1),
⇒ V(2)+a^2,
⇒ V(1)+a^11
⇒ check matrix:
⇒ 0,1,0,1,1,0,1,1,1,1,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,
⇒ 0,1,1,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,
⇒ 0,0,0,0,0,0,1,0,1,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,
⇒ 0,0,1,0,0,1,1,1,0,1,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,
⇒ 1,1,0,0,1,1,0,1,1,1,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,
⇒ 1,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,0,0,0,0,0,
⇒ 1,1,1,0,1,1,0,1,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,
⇒ 0,1,1,0,0,0,0,1,0,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,
⇒ 1,0,0,1,1,0,1,0,1,0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0,
⇒ 0,0,0,0,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,
⇒ 1,0,1,0,0,1,0,1,1,1,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,
⇒ 1,0,0,1,1,1,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,
⇒ 1,0,0,1,0,0,1,1,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,
⇒ 1,0,0,0,1,0,1,1,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,1

```



```

⇒ The system is generated
⇒ Codeword:
⇒ 0,0,1,0,1,0,1,0,1,0,0,1,1,0,0,0,0,0,1,1,0,0,0,1,0
⇒ Received word:
⇒ 0,0,1,0,1,1,1,0,1,0,0,0,1,0,0,0,0,0,1,1,0,0,0,1,0
⇒ The Groebner basis of the QE system:
⇒ U(25)+a^4,
⇒ U(24)+a^4,
⇒ U(23)+a^6,
⇒ U(22)+a^17,
⇒ U(21)+a^13,
⇒ U(20)+a^27,
⇒ U(19)+a^21,
⇒ U(18)+a^8,
⇒ U(17)+a^16,
⇒ U(16)+a^8,
⇒ U(15)+a^15,
⇒ U(14)+a^12,
⇒ U(13)+a^2,
⇒ U(12)+a^3,
⇒ U(11)+a^22,
⇒ U(10)+a^26,
⇒ U(9)+a^8,
⇒ U(8)+a^23,
⇒ U(7)+a,
⇒ U(6)+a^11,
⇒ U(5)+a^4,
⇒ U(4)+a^16,
⇒ U(3)+a^2,
⇒ U(2)+a,
⇒ U(1),
⇒ V(2)+a,
⇒ V(1)+a^16
⇒ Codeword:
⇒ 1,0,0,1,1,1,1,0,0,0,0,0,1,1,0,1,0,1,0,1,1,0,0,0
⇒ Received word:
⇒ 0,0,0,1,1,0,1,1,0,0,0,0,0,1,1,0,1,0,1,0,1,1,0,0,0
⇒ The Groebner basis of the QE system:
⇒ U(25)+a^6,
⇒ U(24)+a^7,
⇒ U(23)+a^30,
⇒ U(22)+a^23,
⇒ U(21)+a^22,
⇒ U(20)+a^5,
⇒ U(19)+a^26,
⇒ U(18)+a^12,
⇒ U(17)+a,
⇒ U(16)+a^14,
⇒ U(15)+a^20,
⇒ U(14)+a^29,
⇒ U(13)+a^3,
⇒ U(12)+a^15,
⇒ U(11)+a^11,

```

```

⇒ U(10)+a^13,
⇒ U(9)+a^16,
⇒ U(8)+a^10,
⇒ U(7)+a^17,
⇒ U(6)+a^21,
⇒ U(5)+a^8,
⇒ U(4)+a^24,
⇒ U(3)+a^4,
⇒ U(2)+a^2,
⇒ U(1),
⇒ V(2)+a^2,
⇒ V(1)+a^5
⇒ Codeword:
⇒ 1,0,1,1,1,1,0,0,0,1,1,1,1,0,1,0,0,1,0,0,0,0,0,1,1
⇒ Received word:
⇒ 1,0,1,1,0,1,0,0,0,1,1,1,0,0,1,0,0,1,0,0,0,0,0,1,1
⇒ The Groebner basis of the QE system:
⇒ U(25)+a^30,
⇒ U(24)+a^2,
⇒ U(23)+a^20,
⇒ U(22)+a^5,
⇒ U(21)+a^20,
⇒ U(20)+a^9,
⇒ U(19)+a^18,
⇒ U(18)+a^29,
⇒ U(17)+a^12,
⇒ U(16)+a^4,
⇒ U(15)+a^5,
⇒ U(14)+a^9,
⇒ U(13)+a^15,
⇒ U(12)+a^10,
⇒ U(11)+a^10,
⇒ U(10)+a^9,
⇒ U(9)+a^6,
⇒ U(8)+a^18,
⇒ U(7)+a^23,
⇒ U(6)+a^5,
⇒ U(5)+a^3,
⇒ U(4)+a^27,
⇒ U(3)+a^17,
⇒ U(2)+a^24,
⇒ U(1),
⇒ V(2)+a^24,
⇒ V(1)+a^16

```

### D.10.2.15 decodeCode

Procedure from library `decodegb.lib` (see [Section D.10.2 \[decodegb.lib\]](#), page 1938).

**Usage:**      `decodeCode(check, ntrials, [e]);` check matrix, ntrials, e int

- check is a parity check matrix for the code,
- ntrials is the number of received vectors per code to be corrected.

- If  $e$  is given it sets the correction capacity explicitly. It should be used in case one expects some lower bound, otherwise the procedure tries to compute the real minimum distance to find out the error-correction capacity

**Return:** nothing;

**Example:**

```
LIB "decodegb.lib";
int q=32; int n=25; int redun=n-11; int t=redun+1;
ring r=(q,a),x,dp;
matrix check=randomCheck(redun,n,1);
// correct 2 errors in using the code above, 3 trials
decodeCode(check,3,2);
⇒ check matrix:
⇒ 0,1,0,0,0,1,1,1,0,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
⇒ 1,0,0,0,0,0,1,0,0,1,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
⇒ 1,1,1,1,0,0,0,1,1,1,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,
⇒ 1,1,0,0,0,0,1,0,1,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,
⇒ 0,0,0,1,1,0,1,0,0,1,1,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,
⇒ 0,0,1,0,1,1,1,0,1,1,1,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,
⇒ 0,1,0,0,0,1,0,1,0,0,1,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,
⇒ 0,1,1,0,1,1,1,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,
⇒ 1,0,0,0,1,1,1,1,0,1,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,
⇒ 0,1,1,1,0,1,0,0,1,1,1,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,
⇒ 0,0,1,0,0,1,0,1,1,0,1,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,
⇒ 0,0,1,0,0,0,1,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,
⇒ 0,0,0,0,1,0,0,0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,
⇒ 1,0,1,1,1,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1
⇒ The system is generated
⇒ Codeword:
⇒ 1,0,0,1,0,1,0,1,1,1,0,1,0,1,0,0,1,0,1,0,0,1,1,1,0
⇒ Received word:
⇒ 1,0,0,1,0,1,0,0,1,1,0,1,0,1,0,0,1,1,1,0,0,1,1,1,0
⇒ Groebner basis of the QE system:
⇒ U(25)+a^25,
⇒ U(24)+a^20,
⇒ U(23)+a^28,
⇒ U(22)+a^7,
⇒ U(21)+a^29,
⇒ U(20)+a^19,
⇒ U(19)+a^23,
⇒ U(18)+a^19,
⇒ U(17)+a^21,
⇒ U(16)+a^9,
⇒ U(15)+a^14,
⇒ U(14)+a^25,
⇒ U(13)+a^28,
⇒ U(12)+a^14,
⇒ U(11)+a^30,
⇒ U(10)+a^27,
⇒ U(9)+a^26,
⇒ U(8)+a^7,
⇒ U(7)+a^14,
```

```

⇒ U(6)+a15,
⇒ U(5)+a13,
⇒ U(4)+a7,
⇒ U(3)+a22,
⇒ U(2)+a11,
⇒ U(1),
⇒ V(2)+a11,
⇒ V(1)+a24
⇒ Codeword:
⇒ 0,0,1,1,1,1,0,0,1,0,1,0,1,1,0,1,1,0,0,0,0,1,0,0
⇒ Received word:
⇒ 0,0,0,1,1,1,1,0,0,1,0,1,0,1,1,0,1,1,0,0,0,0,1,1,0
⇒ Groebner basis of the QE system:
⇒ U(25)+a6,
⇒ U(24)+a16,
⇒ U(23)+a8,
⇒ U(22)+a2,
⇒ U(21)+a8,
⇒ U(20)+a13,
⇒ U(19)+a,
⇒ U(18)+a12,
⇒ U(17)+a29,
⇒ U(16)+a,
⇒ U(15)+a21,
⇒ U(14)+a16,
⇒ U(13)+a3,
⇒ U(12)+a4,
⇒ U(11)+a4,
⇒ U(10)+a16,
⇒ U(9)+a30,
⇒ U(8)+a26,
⇒ U(7)+a17,
⇒ U(6)+a2,
⇒ U(5)+a15,
⇒ U(4)+a24,
⇒ U(3)+a23,
⇒ U(2)+a27,
⇒ U(1),
⇒ V(2)+a27,
⇒ V(1)+a25
⇒ Codeword:
⇒ 0,0,0,1,0,1,0,0,0,1,1,1,1,0,0,1,1,0,0,0,0,0,1,0,1
⇒ Received word:
⇒ 0,0,0,1,1,1,0,1,0,1,1,1,1,0,0,1,1,0,0,0,0,0,1,0,1
⇒ Groebner basis of the QE system:
⇒ U(25)+a7,
⇒ U(24)+a21,
⇒ U(23)+a5,
⇒ U(22)+a9,
⇒ U(21)+a21,
⇒ U(20)+a11,
⇒ U(19)+a22,
⇒ U(18)+a14,

```

```

⇒ U(17)+a,
⇒ U(16)+a^11,
⇒ U(15)+a^13,
⇒ U(14)+a^10,
⇒ U(13)+a^19,
⇒ U(12)+a^18,
⇒ U(11)+a^26,
⇒ U(10)+a^11,
⇒ U(9)+a^16,
⇒ U(8)+a^22,
⇒ U(7)+a^25,
⇒ U(6)+a^13,
⇒ U(5)+a^8,
⇒ U(4)+a^28,
⇒ U(3)+a^4,
⇒ U(2)+a^2,
⇒ U(1),
⇒ V(2)+a^2,
⇒ V(1)+a^11

```

### D.10.2.16 vanishId

Procedure from library `decodegb.lib` (see [Section D.10.2 \[decodegb.lib\]](#), page 1938).

**Usage:**      `vanishId (points);` point is a list of matrices  
                  'points' is a list of points for which the vanishing ideal is to be constructed

**Return:**      Vanishing ideal corresponding to the given set of points

**Example:**

```

LIB "decodegb.lib";
ring r=3,(x(1..3)),dp;
//generate all 3-vectors over GF(3)
list points=pointsGen(3,1);
list points2=convPoints(points);
//grasps the first 11 points
list p=graspList(points2,1,11);
print(p);
⇒ [1]:
⇒   _[1,1]=0
⇒   _[2,1]=0
⇒   _[3,1]=0
⇒ [2]:
⇒   _[1,1]=0
⇒   _[2,1]=0
⇒   _[3,1]=1
⇒ [3]:
⇒   _[1,1]=0
⇒   _[2,1]=0
⇒   _[3,1]=-1
⇒ [4]:
⇒   _[1,1]=0
⇒   _[2,1]=1
⇒   _[3,1]=0

```

```

↳ [5] :
↳   _[1,1]=0
↳   _[2,1]=1
↳   _[3,1]=1
↳ [6] :
↳   _[1,1]=0
↳   _[2,1]=1
↳   _[3,1]=-1
↳ [7] :
↳   _[1,1]=0
↳   _[2,1]=-1
↳   _[3,1]=0
↳ [8] :
↳   _[1,1]=0
↳   _[2,1]=-1
↳   _[3,1]=1
↳ [9] :
↳   _[1,1]=0
↳   _[2,1]=-1
↳   _[3,1]=-1
↳ [10] :
↳   _[1,1]=1
↳   _[2,1]=0
↳   _[3,1]=0
↳ [11] :
↳   _[1,1]=1
↳   _[2,1]=0
↳   _[3,1]=1
//construct the vanishing ideal
ideal id=vanishId(p);
print(id);
↳ x(1)*x(2),
↳ x(1)^2-x(1),
↳ x(3)^3-x(3),
↳ x(1)*x(3)^2-x(1)*x(3),
↳ x(2)^3-x(2)

```

### D.10.2.17 sysFL

Procedure from library `decodegb.lib` (see [Section D.10.2 \[decodegb.lib\]](#), page 1938).

**Usage:**     `sysFL (check,y,t,e,s);` `check,y` matrix, `t,e,s` int

- `check` is a parity check matrix of the code,
- `y` is a received word,
- `t` the number of errors to correct,
- `e` is the extension degree,
- `s` is the dimension of the point for the vanishing ideal

**Return:**    the system of Fitzgerald-Lax for the given decoding problem

**Theory:**    Based on 'check' of the given linear code, the procedure constructs the corresponding ideal constructed with a generalization of Cooper's philosophy. For basics of the method [Section C.8.4 \[Fitzgerald-Lax method\]](#), page 789.

**Example:**

```

LIB "decodegb.lib";
intvec vopt = option(get);
list l=FLpreprocess(3,1,11,2,"");
def r=l[1];
setring r;
int s_work=l[2];
//the check matrix of [11,6,5] ternary code
matrix h[5][11]=1,0,0,0,0,1,1,1,-1,-1,0,
0,1,0,0,0,1,1,-1,1,0,-1,
0,0,1,0,0,1,-1,1,0,1,-1,
0,0,0,1,0,1,-1,0,1,-1,1,
0,0,0,0,1,1,0,-1,-1,1,1;
matrix g=dual_code(h);
matrix x[1][6];
matrix y[1][11]=encode(x,g);
//disturb with 2 errors
matrix rec[1][11]=errorInsert(y,list(2,4),list(1,-1));
//the Fitzgerald-Lax system
ideal sys=sysFL(h,rec,2,1,s_work);
print(sys);
⇨ x1(3)^3-x1(3),
⇨ x1(2)^3-x1(2),
⇨ x1(3)^2*x1(1)-x1(3)*x1(1),
⇨ x1(2)*x1(1),
⇨ x1(1)^2-x1(1),
⇨ x1(6)^3-x1(6),
⇨ x1(5)^3-x1(5),
⇨ x1(6)^2*x1(4)-x1(6)*x1(4),
⇨ x1(5)*x1(4),
⇨ x1(4)^2-x1(4),
⇨ x1(1)^3-x1(1),
⇨ x1(4)^3-x1(4),
⇨ e(1)^2-1,
⇨ e(2)^2-1,
⇨ -e(2)*x1(6)^2+e(2)*x1(6)*x1(5)^2-e(2)*x1(6)*x1(4)+e(2)*x1(5)^2+e(2)*x1(5)\
+e(2)*x1(4)+e(2)-e(1)*x1(3)^2+e(1)*x1(3)*x1(2)^2-e(1)*x1(3)*x1(1)+e(1)*x1\
(2)^2+e(1)*x1(2)+e(1)*x1(1)+e(1),
⇨ -e(2)*x1(6)^2+e(2)*x1(6)*x1(5)^2+e(2)*x1(6)*x1(5)+e(2)*x1(6)*x1(4)-e(2)*x\
1(6)-e(2)*x1(5)^2+e(2)*x1(5)-e(1)*x1(3)^2+e(1)*x1(3)*x1(2)^2+e(1)*x1(3)*x\
1(2)+e(1)*x1(3)*x1(1)-e(1)*x1(3)-e(1)*x1(2)^2+e(1)*x1(2)-1,
⇨ -e(2)*x1(6)^2*x1(5)^2+e(2)*x1(6)^2*x1(5)-e(2)*x1(6)^2-e(2)*x1(6)*x1(5)^2+\
e(2)*x1(6)*x1(5)+e(2)*x1(6)*x1(4)+e(2)*x1(6)+e(2)*x1(5)^2-e(2)*x1(5)+e(2)\
*x1(4)-e(1)*x1(3)^2*x1(2)^2+e(1)*x1(3)^2*x1(2)-e(1)*x1(3)^2-e(1)*x1(3)*x1\
(2)^2+e(1)*x1(3)*x1(2)+e(1)*x1(3)*x1(1)+e(1)*x1(3)+e(1)*x1(2)^2-e(1)*x1(2)\
)+e(1)*x1(1),
⇨ -e(2)*x1(6)^2*x1(5)^2-e(2)*x1(6)^2*x1(5)+e(2)*x1(6)*x1(5)^2-e(2)*x1(6)*x1\
(4)+e(2)*x1(5)-e(2)*x1(4)-e(1)*x1(3)^2*x1(2)^2-e(1)*x1(3)^2*x1(2)+e(1)*x1\
(3)*x1(2)^2-e(1)*x1(3)*x1(1)+e(1)*x1(2)-e(1)*x1(1)+1,
⇨ e(2)*x1(6)^2*x1(5)+e(2)*x1(4)+e(1)*x1(3)^2*x1(2)+e(1)*x1(1)
option(redSB);
ideal red_sys=std(sys);
red_sys;
⇨ red_sys[1]=x1(1)

```

```

⇒ red_sys[2]=x1(2)^2-x1(2)
⇒ red_sys[3]=x1(3)+x1(2)-1
⇒ red_sys[4]=e(1)-x1(2)-1
⇒ red_sys[5]=x1(4)
⇒ red_sys[6]=x1(5)+x1(2)-1
⇒ red_sys[7]=x1(6)-x1(2)
⇒ red_sys[8]=e(2)+x1(2)+1
// read the solutions from this redGB
// the points are (0,0,1) and (0,1,0) with error values 1 and -1 resp.
// use list points to find error positions;
points;
⇒ [1]:
⇒   _[1,1]=0
⇒   _[2,1]=0
⇒   _[3,1]=0
⇒ [2]:
⇒   _[1,1]=0
⇒   _[2,1]=0
⇒   _[3,1]=1
⇒ [3]:
⇒   _[1,1]=0
⇒   _[2,1]=0
⇒   _[3,1]=-1
⇒ [4]:
⇒   _[1,1]=0
⇒   _[2,1]=1
⇒   _[3,1]=0
⇒ [5]:
⇒   _[1,1]=0
⇒   _[2,1]=1
⇒   _[3,1]=1
⇒ [6]:
⇒   _[1,1]=0
⇒   _[2,1]=1
⇒   _[3,1]=-1
⇒ [7]:
⇒   _[1,1]=0
⇒   _[2,1]=-1
⇒   _[3,1]=0
⇒ [8]:
⇒   _[1,1]=0
⇒   _[2,1]=-1
⇒   _[3,1]=1
⇒ [9]:
⇒   _[1,1]=0
⇒   _[2,1]=-1
⇒   _[3,1]=-1
⇒ [10]:
⇒   _[1,1]=1
⇒   _[2,1]=0
⇒   _[3,1]=0
⇒ [11]:
⇒   _[1,1]=1

```



```

⇒    _[2,1]=0
⇒    _[3,1]=1
⇒ [12]:
⇒    _[1,1]=1
⇒    _[2,1]=0
⇒    _[3,1]=-1
⇒ [13]:
⇒    _[1,1]=1
⇒    _[2,1]=1
⇒    _[3,1]=0
⇒ [14]:
⇒    _[1,1]=1
⇒    _[2,1]=1
⇒    _[3,1]=1
⇒ [15]:
⇒    _[1,1]=1
⇒    _[2,1]=1
⇒    _[3,1]=-1
⇒ [16]:
⇒    _[1,1]=1
⇒    _[2,1]=-1
⇒    _[3,1]=0
⇒ [17]:
⇒    _[1,1]=1
⇒    _[2,1]=-1
⇒    _[3,1]=1
⇒ [18]:
⇒    _[1,1]=1
⇒    _[2,1]=-1
⇒    _[3,1]=-1
⇒ [19]:
⇒    _[1,1]=-1
⇒    _[2,1]=0
⇒    _[3,1]=0
⇒ [20]:
⇒    _[1,1]=-1
⇒    _[2,1]=0
⇒    _[3,1]=1
⇒ [21]:
⇒    _[1,1]=-1
⇒    _[2,1]=0
⇒    _[3,1]=-1
⇒ [22]:
⇒    _[1,1]=-1
⇒    _[2,1]=1
⇒    _[3,1]=0
⇒ [23]:
⇒    _[1,1]=-1
⇒    _[2,1]=1
⇒    _[3,1]=1
⇒ [24]:
⇒    _[1,1]=-1
⇒    _[2,1]=1

```

```

⇒      _[3,1]==-1
⇒ [25]:
⇒      _[1,1]==-1
⇒      _[2,1]==-1
⇒      _[3,1]=0
⇒ [26]:
⇒      _[1,1]==-1
⇒      _[2,1]==-1
⇒      _[3,1]=1
⇒ [27]:
⇒      _[1,1]==-1
⇒      _[2,1]==-1
⇒      _[3,1]==-1
option(set,vopt);

```

See also: [Section D.10.2.7 \[sysQE\]](#), page 1947.

### D.10.2.18 decodeRandomFL

Procedure from library `decodegb.lib` (see [Section D.10.2 \[decodegb.lib\]](#), page 1938).

**Usage:**      `decodeRandomFL(redun,p,e,n,t,ncodes,ntrials,minpol);`

- n is length of codes generated,
- redun = redundancy of codes generated,
- p is the characteristic,
- e is the extension degree,
- t is the number of errors to correct,
- ncodes is the number of random codes to be processed,
- ntrials is the number of received vectors per code to be corrected,
- minpol: due to some peculiarities of SINGULAR one needs to provide minimal polynomial for the extension explicitly

**Return:**    nothing

**Example:**

```

LIB "decodegb.lib";
// correcting one error for one random binary code of length 25,
// redundancy 14; 10 words are processed
decodeRandomFL(25,14,2,1,1,1,10,"");
⇒ Codeword:
⇒ 1,0,0,1,0,1,0,1,1,1,0,1,0,1,0,0,1,0,1,0,0,1,1,1,0
⇒ Received word
⇒ 1,0,0,1,0,1,0,1,1,1,0,1,0,1,0,0,1,1,1,0,0,1,1,1,0
⇒ Groebner basis of the FL system:
⇒ x1(1)+1,
⇒ x1(2),
⇒ x1(3),
⇒ x1(4),
⇒ x1(5)+1,
⇒ e(1)+1
⇒ Codeword:
⇒ 1,0,0,1,1,1,1,0,0,1,0,1,1,1,0,0,0,1,1,1,1,1,0,0,0
⇒ Received word
⇒ 1,0,0,1,1,1,1,0,0,1,0,1,0,1,0,0,0,1,1,1,1,1,0,0,0
⇒ Groebner basis of the FL system:

```

```

⇒ x1(1),
⇒ x1(2)+1,
⇒ x1(3)+1,
⇒ x1(4),
⇒ x1(5),
⇒ e(1)+1
⇒ Codeword:
⇒ 0,0,1,1,1,0,1,1,1,0,1,1,1,0,0,0,1,0,0,1,0,0,0,0,0
⇒ Received word
⇒ 0,0,1,1,1,0,1,1,0,0,1,1,1,0,0,0,1,0,0,1,0,0,0,0,0
⇒ Groebner basis of the FL system:
⇒ x1(1),
⇒ x1(2)+1,
⇒ x1(3),
⇒ x1(4),
⇒ x1(5),
⇒ e(1)+1
⇒ Codeword:
⇒ 0,0,0,1,0,1,0,0,0,1,1,1,1,0,0,1,1,0,0,0,0,0,1,0,1
⇒ Received word
⇒ 0,0,0,1,1,1,0,0,0,1,1,1,1,0,0,1,1,0,0,0,0,0,1,0,1
⇒ Groebner basis of the FL system:
⇒ x1(1),
⇒ x1(2),
⇒ x1(3)+1,
⇒ x1(4),
⇒ x1(5),
⇒ e(1)+1
⇒ Codeword:
⇒ 1,1,0,0,1,0,1,0,0,0,0,0,0,0,1,0,0,1,0,1,1,0,1,1,1
⇒ Received word
⇒ 1,1,0,0,1,0,1,0,0,0,0,0,0,0,1,1,0,0,1,0,1,1,0,1,1,1
⇒ Groebner basis of the FL system:
⇒ x1(1),
⇒ x1(2)+1,
⇒ x1(3)+1,
⇒ x1(4),
⇒ x1(5)+1,
⇒ e(1)+1
⇒ Codeword:
⇒ 0,1,1,1,1,0,1,0,0,0,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0
⇒ Received word
⇒ 0,1,1,1,1,0,1,0,0,0,1,1,1,1,0,0,0,0,0,0,1,0,0,0,0
⇒ Groebner basis of the FL system:
⇒ x1(1)+1,
⇒ x1(2),
⇒ x1(3)+1,
⇒ x1(4),
⇒ x1(5),
⇒ e(1)+1
⇒ Codeword:
⇒ 1,0,0,1,0,1,0,0,0,0,1,0,1,0,1,0,0,0,0,0,1,0,0,1,1
⇒ Received word

```

```

⇒ 1,0,0,1,0,1,0,1,0,0,1,0,1,0,1,0,0,0,0,0,1,0,0,1,1
⇒ Groebner basis of the FL system:
⇒ x1(1),
⇒ x1(2),
⇒ x1(3)+1,
⇒ x1(4)+1,
⇒ x1(5)+1,
⇒ e(1)+1
⇒ Codeword:
⇒ 0,1,0,1,1,0,1,1,1,0,0,1,1,0,1,1,1,0,0,1,1,0,1,1,1
⇒ Received word
⇒ 0,1,0,1,1,0,1,1,0,0,0,1,1,0,1,1,1,0,0,1,1,0,1,1,1
⇒ Groebner basis of the FL system:
⇒ x1(1),
⇒ x1(2)+1,
⇒ x1(3),
⇒ x1(4),
⇒ x1(5),
⇒ e(1)+1
⇒ Codeword:
⇒ 0,1,0,0,1,0,0,1,1,0,0,0,0,1,0,1,0,0,0,0,0,0,0,1,1
⇒ Received word
⇒ 0,1,0,0,1,0,0,1,1,0,0,0,0,1,0,1,0,0,0,0,0,0,0,1,0
⇒ Groebner basis of the FL system:
⇒ x1(1)+1,
⇒ x1(2)+1,
⇒ x1(3),
⇒ x1(4),
⇒ x1(5),
⇒ e(1)+1
⇒ Codeword:
⇒ 1,0,0,1,1,0,0,0,0,0,0,0,1,0,1,0,1,0,1,0,0,1,1
⇒ Received word
⇒ 1,0,0,1,1,0,0,0,0,0,0,0,1,0,1,0,1,0,1,0,1,1,0,1,1
⇒ Groebner basis of the FL system:
⇒ x1(1)+1,
⇒ x1(2),
⇒ x1(3)+1,
⇒ x1(4),
⇒ x1(5)+1,
⇒ e(1)+1

```

## D.11 System and Control theory

### D.11.1 Control theory background

Control systems are usually described by differential (or difference) equations, but their properties of interest are most naturally expressed in terms of the system trajectories (the set of all solutions to the equations). This is formalized by the notion of the system *behavior*. On the other hand, the manipulation of linear system equations can be formalized using algebra, more precisely module theory. The relationship between modules and behaviors is very rich and leads to deep results on system structure.

The key to the module-behavior correspondence is a property of some signal spaces that are modules over the ring of differential (or difference) operators, namely, *the injective cogenerator property*. This property makes it possible to translate any statement on the solution spaces that can be expressed in terms of images and kernels, to an equivalent statement on the modules. Thus analytic properties can be identified with algebraic properties, and conversely, the results of manipulating the modules using computer algebra can be re-translated and interpreted using the language of systems theory. This duality (*algebraic analysis*) is widely used in behavioral systems and control theory today.

For instance, a system is **controllable** (a fundamental property for any control system) if and only if the associated module is torsion-free. This concept can be refined by the so-called controllability degrees. The strongest form of controllability (*flatness*) corresponds to a projective (or even free) module.

Controllability means that one can switch from one system trajectory to another without violating the system law (concatenation of trajectories). For one-dimensional systems (ODE) that evolve in time, this is usually interpreted as switching from a given past trajectory to a desired future trajectory. Thus the system can be forced to behave in an arbitrarily prescribed way.

The extreme case opposed to controllability is **autonomy**: autonomous systems evolve independently according to their law, without being influenceable from the outside. Again, the property can be refined in terms of autonomy degrees.

## D.11.2 control.lib

**Library:** control.lib

**Purpose:** Algebraic analysis tools for System and Control Theory

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**Support:** Forschungsschwerpunkt 'Mathematik und Praxis' (Project of Dr. E. Zerz and V. Levandovskyy), University of Kaiserslautern

**Procedures:**

### D.11.2.1 control

Procedure from library `control.lib` (see [Section D.11.2 \[control.lib\]](#), page 1967).

**Usage:** control(R); R a module (R is the matrix of the system of equations to be investigated)

**Return:** list

**Purpose:** compute the list of all the properties concerning controllability of the system (behavior), represented by the matrix R

**Note:** the properties and corresponding data like controllability, flatness, dimension of the system, degree of controllability, kernel and image representations, genericity of parameters, obstructions to controllability, annihilator of torsion submodule and left inverse are investigated

**Example:**

```
LIB "control.lib";
// a WindTunnel example
ring A = (0,a, omega, zeta, k),(D1, delta),dp;
module R;
```

```

R = [D1+a, -k*a*delta, 0, 0],
[0, D1, -1, 0],
[0, omega^2, D1+2*zeta*omega, -omega^2];
R=transpose(R);
view(R);
⇒ D1+(a), (-a*k)*delta, 0, 0,
⇒ 0, D1, -1, 0,
⇒ 0, (omega^2), D1+(2*omega*zeta), (-omega^2)
view(control(R));
⇒ number of first nonzero Ext:
⇒
⇒ 2
⇒
⇒ controllable, not reflexive, image representation:
⇒
⇒ (a*omega^2*k)*delta,
⇒ (omega^2)*D1+(a*omega^2),
⇒ (omega^2)*D1^2+(a*omega^2)*D1,
⇒ D1^3+(a+2*omega*zeta)*D1^2+(2*a*omega*zeta+omega^2)*D1+(a*omega^2)
⇒
⇒ dimension of the system:
⇒
⇒ 2
⇒
⇒ Parameter constellations which might lead to a non-controllable system:
⇒
⇒ a,k,omega
⇒
⇒

```

### D.11.2.2 controlDim

Procedure from library `control.lib` (see [Section D.11.2 \[control.lib\]](#), page 1967).

**Usage:** `controlDim(R)`; R a module (R is the matrix of the system of equations to be investigated)

**Return:** list

**Purpose:** computes list of all the properties concerning controllability of the system (behavior), represented by the matrix R

**Note:** the properties and corresponding data like controllability, flatness, dimension of the system, degree of controllability, kernel and image representations, genericity of parameters, obstructions to controllability, annihilator of torsion submodule and left inverse are investigated.

This procedure is analogous to 'control' but uses dimension calculations.

The implemented approach works for full row rank matrices only (the check is done automatically).

**Example:**

```

LIB "control.lib";
//a WindTunnel example
ring A = (0,a, omega, zeta, k),(D1, delta),dp;
module R;

```

```

R = [D1+a, -k*a*delta, 0, 0],
[0, D1, -1, 0],
[0, omega^2, D1+2*zeta*omega, -omega^2];
R=transpose(R);
view(R);
⇒ D1+(a), (-a*k)*delta, 0, 0,
⇒ 0, D1, -1, 0,
⇒ 0, (omega^2), D1+(2*omega*zeta), (-omega^2)
view(controlDim(R));
⇒ number of first nonzero Ext:
⇒
⇒ 2
⇒
⇒ controllable, not reflexive, image representation:
⇒
⇒ (a*omega^2*k)*delta,
⇒ (omega^2)*D1+(a*omega^2),
⇒ (omega^2)*D1^2+(a*omega^2)*D1,
⇒ D1^3+(a+2*omega*zeta)*D1^2+(2*a*omega*zeta+omega^2)*D1+(a*omega^2)
⇒
⇒ dimension of the system:
⇒
⇒ 2
⇒
⇒ Parameter constellations which might lead to a non-controllable system:
⇒
⇒ a,k,omega
⇒
⇒

```

### D.11.2.3 autonom

Procedure from library `control.lib` (see [Section D.11.2 \[control.lib\]](#), page 1967).

**Usage:** `autonom(R)`; R a module (R is a matrix of the system of equations which is to be investigated)

**Return:** list

**Purpose:** find all the properties concerning autonomy of the system (behavior) represented by the matrix R

**Note:** the properties and corresponding data like autonomy resp. strong autonomy, dimension of the system, autonomy degree, kernel representation and (over)determinacy are investigated

**Example:**

```

LIB "control.lib";
// Cauchy
ring r=0,(s1,s2,s3,s4),dp;
module R= [s1,-s2],
[s2, s1],
[s3,-s4],
[s4, s3];
R=transpose(R);

```

```

view( R );
⇒ s1,-s2,
⇒ s2,s1 ,
⇒ s3,-s4,
⇒ s4,s3
view( autonom(R) );
⇒ number of first nonzero Ext:
⇒
⇒ 2
⇒
⇒ overdetermined, not strongly autonomous
⇒
⇒ dimension of the system:
⇒
⇒ 2
⇒

```

#### D.11.2.4 autonomDim

Procedure from library `control.lib` (see [Section D.11.2 \[control.lib\], page 1967](#)).

**Usage:**      `autonomDim(R)`; R a module (R is a matrix of the system of equations which is to be investigated)

**Return:**    list

**Purpose:**    computes the list of all the properties concerning autonomy of the system (behavior), represented by the matrix R

**Note:**      the properties and corresponding data like autonomy resp. strong autonomy, dimension of the system, autonomy degree, kernel representation and (over)determinacy are investigated.

This procedure is analogous to 'autonom' but uses dimension calculations

**Example:**

```

LIB "control.lib";
// Cauchy1 example
ring r=0,(s1,s2,s3,s4),dp;
module R= [s1,-s2],
[s2, s1],
[s3,-s4],
[s4, s3];
R=transpose(R);
view( R );
⇒ s1,-s2,
⇒ s2,s1 ,
⇒ s3,-s4,
⇒ s4,s3
view( autonomDim(R) );
⇒ number of first nonzero Ext:
⇒
⇒ 2
⇒
⇒ overdetermined, not strongly autonomous
⇒

```



```

↳ dimension of the system:
↳
↳ 2
↳

```

### D.11.2.5 leftKernel

Procedure from library `control.lib` (see [Section D.11.2 \[control.lib\]](#), page 1967).

**Usage:**     `leftKernel(M)`; M a matrix

**Return:**    module

**Purpose:**    computes left kernel of matrix M (a module of all elements  $v$  such that  $vM=0$ )

**Example:**

```

LIB "control.lib";
ring r= 0,(x,y,z),dp;
matrix M[3][1] = x,y,z;
print(M);
↳ x,
↳ y,
↳ z
matrix L = leftKernel(M);
print(L);
↳ 0, -z,y,
↳ -y,x, 0,
↳ -z,0, x
// check:
print(L*M);
↳ 0,
↳ 0,
↳ 0

```

### D.11.2.6 rightKernel

Procedure from library `control.lib` (see [Section D.11.2 \[control.lib\]](#), page 1967).

**Usage:**     `rightKernel(M)`; M a matrix

**Return:**    module

**Purpose:**    computes the right kernel of matrix M (a module of all elements  $v$  such that  $Mv=0$ )

**Example:**

```

LIB "control.lib";
ring r = 0,(x,y,z),dp;
matrix M[1][3] = x,y,z;
print(M);
↳ x,y,z
matrix R = rightKernel(M);
print(R);
↳ 0, -y,-z,
↳ -z,x, 0,
↳ y, 0, x
// check:
print(M*R);
↳ 0,0,0

```

### D.11.2.7 leftInverse

Procedure from library `control.lib` (see [Section D.11.2 \[control.lib\], page 1967](#)).

**Usage:** `leftInverse(M)`; M a module

**Return:** module

**Purpose:** computes such a matrix L, that  $LM = \text{Id}$ ;

**Note:** exists only in the case when M is free submodule

**Example:**

```
LIB "control.lib";
// a trivial example:
ring r = 0,(x,z),dp;
matrix M[2][1] = 1,x2z;
print(M);
↳ 1,
↳ x2z
print( leftInverse(M) );
↳ 1,0
kill r;
// derived from the example TwoPendula:
ring r=(0,m1,m2,M,g,L1,L2),Dt,dp;
matrix U[3][1];
U[1,1]=(-L2)*Dt^4+(g)*Dt^2;
U[2,1]=(-L1)*Dt^4+(g)*Dt^2;
U[3,1]=(L1*L2)*Dt^4+(-g*L1-g*L2)*Dt^2+(g^2);
module M = module(U);
module L = leftInverse(M);
print(L);
↳ (L1^2)/(g^2*L1-g^2*L2), (-L2^2)/(g^2*L1-g^2*L2), 1/(g^2)
// check
print(matrix(L)*matrix(M));
↳ 1
```

### D.11.2.8 rightInverse

Procedure from library `control.lib` (see [Section D.11.2 \[control.lib\], page 1967](#)).

**Usage:** `rightInverse(M)`; M a module

**Return:** module

**Purpose:** computes such a matrix L, that  $ML = \text{Id}$

**Note:** exists only in the case when M is free submodule

**Example:**

```
LIB "control.lib";
// a trivial example:
ring r = 0,(x,z),dp;
matrix M[1][2] = 1,x2+z;
print(M);
↳ 1,x2+z
print( rightInverse(M) );
↳ 1,
```

```

↳ 0
kill r;
// derived from the TwoPendula example:
ring r=(0,m1,m2,M,g,L1,L2),Dt,dp;
matrix U[1][3];
U[1,1]=(-L2)*Dt^4+(g)*Dt^2;
U[1,2]=(-L1)*Dt^4+(g)*Dt^2;
U[1,3]=(L1*L2)*Dt^4+(-g*L1-g*L2)*Dt^2+(g^2);
module M = module(U);
module L = rightInverse(M);
print(L);
↳ (L1^2)/(g^2*L1-g^2*L2),
↳ (-L2^2)/(g^2*L1-g^2*L2),
↳ 1/(g^2)
// check
print(matrix(M)*matrix(L));
↳ 1

```

### D.11.2.9 colrank

Procedure from library `control.lib` (see [Section D.11.2 \[control.lib\], page 1967](#)).

**Usage:** `colrank(M)`; M a matrix/module

**Return:** `int`

**Purpose:** compute the column rank of M as of matrix

**Note:** this procedure uses Bareiss algorithm

**Example:**

```

LIB "control.lib";
// de Rham complex
ring r=0,(D(1..3)),dp;
module R;
R=[0,-D(3),D(2)],
[D(3),0,-D(1)],
[-D(2),D(1),0];
R=transpose(R);
colrank(R);
↳ 2

```

### D.11.2.10 genericity

Procedure from library `control.lib` (see [Section D.11.2 \[control.lib\], page 1967](#)).

**Usage:** `genericity(M)`; M is a matrix/module

**Return:** `list` (of strings)

**Purpose:** determine parametric expressions which have been assumed to be non-zero in the process of computing the Groebner basis

**Note:** the output list consists of strings. The first string contains the variables only, whereas each further string contains a single polynomial in parameters.

We strongly recommend to switch on the `redSB` and `redTail` options.

The procedure is effective with the lift procedure for modules with parameters

**Example:**

```

LIB "control.lib";
// TwoPendula
ring r=(0,m1,m2,M,g,L1,L2),Dt,dp;
module RR =
[m1*L1*Dt^2, m2*L2*Dt^2, -1, (M+m1+m2)*Dt^2],
[m1*L1^2*Dt^2-m1*L1*g, 0, 0, m1*L1*Dt^2],
[0, m2*L2^2*Dt^2-m2*L2*g, 0, m2*L2*Dt^2];
module R = transpose(RR);
module SR = std(R);
matrix T = lift(R,SR);
genericity(T);
↳ [1]:
↳ m1,g,L1,L2
↳ [2]:
↳ L1-L2
/-- The result might be different when computing reduced bases:
matrix T2;
option(redSB);
option(redTail);
module SR2 = std(R);
T2 = lift(R,SR2);
genericity(T2);
↳ [1]:
↳ m1,g,L1,m2,L2
↳ [2]:
↳ L1-L2

```

**D.11.2.11 canonize**

Procedure from library `control.lib` (see [Section D.11.2 \[control.lib\]](#), page 1967).

**Usage:** `canonize(L)`; L a list

**Return:** list

**Purpose:** modules in the list are canonized by computing their reduced minimal (= unique up to constant factor w.r.t. the given ordering) Groebner bases

**Assume:** L is the output of `control/autonomy` procedures

**Example:**

```

LIB "control.lib";
// TwoPendula with L1=L2=L
ring r=(0,m1,m2,M,g,L),Dt,dp;
module RR =
[m1*L*Dt^2, m2*L*Dt^2, -1, (M+m1+m2)*Dt^2],
[m1*L^2*Dt^2-m1*L*g, 0, 0, m1*L*Dt^2],
[0, m2*L^2*Dt^2-m2*L*g, 0, m2*L*Dt^2];
module R = transpose(RR);
list C = control(R);
list CC = canonize(C);
view(CC);
↳ number of first nonzero Ext:
↳

```

```

⇒ 1
⇒
⇒ not controllable , image representation for controllable part:
⇒
⇒ -Dt^2
⇒ -Dt^2
⇒ (M*L)*Dt^4+(-m1*g-m2*g-M*g)*Dt^2,
⇒ (L)*Dt^2+(-g)
⇒
⇒ kernel representation for controllable part:
⇒
⇒ 1,0,0,
⇒ 0,1,0,
⇒ 0,0,1
⇒
⇒ obstruction to controllability
⇒
⇒ 1,0,0
⇒ 0,1,0
⇒ 0,0,(L)*Dt^2+(-g)
⇒
⇒ annihilator of torsion module (of obstruction to controllability)
⇒
⇒ (L)*Dt^2+(-g)
⇒
⇒ dimension of the system:
⇒
⇒ 1
⇒

```

### D.11.2.12 iostruct

Procedure from library `control.lib` (see [Section D.11.2 \[control.lib\]](#), page 1967).

**Usage:** `iostruct( R );` R a module

**Return:** list L with entries: string s, intvec v, module P and module Q

**Purpose:** if R is the kernel-representation-matrix of some system, then we output a input-ouput representation  $Py=Qu$  of the system, the components that have been chosen as outputs(intvec v) and a comment s

**Note:** the procedure uses Bareiss algorithm

**Example:**

```

LIB "control.lib";
//Example Antenna
ring r = (0, K1, K2, Te, Kp, Kc),(Dt, delta), (c,dp);
module RR;
RR =
[Dt, -K1, 0, 0, 0, 0, 0, 0, 0],
[0, Dt+K2/Te, 0, 0, 0, 0, -Kp/Te*delta, -Kc/Te*delta, -Kc/Te*delta],
[0, 0, Dt, -K1, 0, 0, 0, 0, 0],
[0, 0, 0, Dt+K2/Te, 0, 0, -Kc/Te*delta, -Kp/Te*delta, -Kc/Te*delta],
[0, 0, 0, 0, Dt, -K1, 0, 0, 0],

```

```

[0, 0, 0, 0, 0, Dt+K2/Te, -Kc/Te*delta, -Kc/Te*delta, -Kp/Te*delta];
module R = transpose(RR);
view(iostruct(R));
⇒ The following components have been chosen as outputs:
⇒
⇒ 1,
⇒ 2,
⇒ 3,
⇒ 4,
⇒ 5,
⇒ 6
⇒
⇒ Dt, (-K1)      , 0 , 0      , 0 , 0      ,
⇒ 0 , Dt+(K2)/(Te), 0 , 0      , 0 , 0      ,
⇒ 0 , 0      , Dt, (-K1)      , 0 , 0      ,
⇒ 0 , 0      , 0 , Dt+(K2)/(Te), 0 , 0      ,
⇒ 0 , 0      , 0 , 0      , Dt, (-K1)      ,
⇒ 0 , 0      , 0 , 0      , 0 , Dt+(K2)/(Te)
⇒
⇒ 0      , 0      , 0      ,
⇒ (-Kp)/(Te)*delta, (-Kc)/(Te)*delta, (-Kc)/(Te)*delta,
⇒ 0      , 0      , 0      ,
⇒ (-Kc)/(Te)*delta, (-Kp)/(Te)*delta, (-Kc)/(Te)*delta,
⇒ 0      , 0      , 0      ,
⇒ (-Kc)/(Te)*delta, (-Kc)/(Te)*delta, (-Kp)/(Te)*delta
⇒

```

### D.11.2.13 findTorsion

Procedure from library `control.lib` (see [Section D.11.2 \[control.lib\]](#), page 1967).

**Usage:** `findTorsion(R, I)`; R an ideal/matrix/module, I an ideal

**Return:** module

**Purpose:** computes the Groebner basis of the submodule of R, annihilated by I

**Note:** especially helpful, when I is the annihilator of the  $t(R)$  - the torsion submodule of R.  
In this case, the result is the explicit presentation of  $t(R)$  as the submodule of R

**Example:**

```

LIB "control.lib";
// Flexible Rod
ring A = 0, (D1, D2), (c, dp);
module R = [D1, -D1*D2, -1], [2*D1*D2, -D1-D1*D2^2, 0];
module RR = transpose(R);
list L = control(RR);
// here, we have the annihilator:
ideal LAnn = D1; // = L[10]
module Tr = findTorsion(RR, LAnn);
print(RR); // the module itself
⇒ D1,      -D1*D2,      -1,
⇒ 2*D1*D2, -D1*D2^2-D1, 0
print(Tr); // generators of the torsion submodule
⇒ 0,
⇒ 1

```

### D.11.2.14 controlExample

Procedure from library `control.lib` (see [Section D.11.2 \[control.lib\], page 1967](#)).

**Usage:** `controlExample(s)`; `s` a string

**Return:** ring

**Purpose:** set up an example from the mini database by initializing a ring and a module in a ring

**Note:** in order to see the list of available examples, execute `controlExample("show")`;  
To use an example, one has to do the following. Suppose one calls the ring, where the example will be activated, `A`. Then, by executing  
`def A = controlExample("Antenna");` and `setring A;`,  
`A` will become a basering from the example "Antenna" with the predefined system module `R` (transposed). After that one can just execute `control(R)`; respectively `autonom(R)`; to perform the control resp. autonomy analysis of `R`.

**Example:**

```
LIB "control.lib";
controlExample("show"); // let us see all available examples:
⇒ The list of examples:
⇒ name: Cauchy1, desc: 1-dimensional Cauchy equation
⇒ name: Cauchy2, desc: 2-dimensional Cauchy equation
⇒ name: Control1, desc: example of a simple noncontrollable system
⇒ name: Control2, desc: example of a simple controllable system
⇒ name: Antenna, desc: antenna
⇒ name: Einstein, desc: Einstein equations in vacuum
⇒ name: FlexibleRod, desc: flexible rod
⇒ name: TwoPendula, desc: two pendula mounted on a cart
⇒ name: WindTunnel, desc: wind tunnel
⇒ name: Zerz1, desc: example from the lecture of Eva Zerz
def B = controlExample("TwoPendula"); // let us set up a particular example
setring B;
print(R);
⇒ (m1*L1)*Dt^2, (m2*L2)*Dt^2, -1,(m1+m2+M)*Dt^2,
⇒ (m1*L1^2)*Dt^2+(-m1*g*L1),0, 0, (m1*L1)*Dt^2,
⇒ 0, (m2*L2^2)*Dt^2+(-m2*g*L2),0, (m2*L2)*Dt^2
```

### D.11.2.15 view

Procedure from library `control.lib` (see [Section D.11.2 \[control.lib\], page 1967](#)).

**Usage:** `view(M)`; `M` is of any type

**Return:** no return value

**Purpose:** procedure for (well-) formatted output of modules, matrices, lists of modules, matrices; shows everything even if entries are long

**Note:** in case of other types( not 'module', 'matrix', 'list') works just as standard 'print' procedure

**Example:**

```
LIB "control.lib";
ring r;
list L;
```

```

matrix M[1][3] = x2+x,y3-y,z5-4z+7;
L[1] = "a matrix:";
L[2] = M;
L[3] = "an ideal:";
L[4] = ideal(M);
view(L);
↳ a matrix:
↳
↳ x2+x,y3-y,z5-4z+7
↳
↳ an ideal:
↳
↳ x2+x,
↳ y3-y,
↳ z5-4z+7
↳

```

### D.11.3 jacobson\_lib

**Library:** jacobson.lib

**Purpose:** Algorithms for Smith and Jacobson Normal Form

**Author:** Kristina Schindelar, Kristina.Schindelar@math.rwth-aachen.de,  
Viktor Levandovskyy, levandov@math.rwth-aachen.de

**Overview:** We work over a ring  $R$ , that is an Euclidean principal ideal domain. If  $R$  is commutative, we suppose  $R$  to be a polynomial ring in one variable. If  $R$  is non-commutative, we suppose  $R$  to have two variables, say  $x$  and  $d$ . We treat then the basering as the Ore localization of  $R$  with respect to the mult. closed set  $S = K[x]$  without 0. Thus, we treat basering as principal ideal ring with  $d$  a polynomial variable and  $x$  an invertible one.

Note, that in computations no division by  $x$  will actually happen.

Given a rectangular matrix  $M$  over  $R$ , one can compute unimodular (that is invertible) square matrices  $U$  and  $V$ , such that  $U*M*V=D$  is a diagonal matrix. Depending on the ring, the diagonal entries of  $D$  have certain properties.

We call a square matrix  $D$  as before 'a weak Jacobson normal form of  $M$ '. It is known, that over the first rational Weyl algebra  $K(x)\langle d \rangle$ ,  $D$  can be further transformed into a diagonal matrix  $(1,1,\dots,1,f,0,\dots,0)$ , where  $f$  is in  $K(x)\langle d \rangle$ . We call such a form of  $D$  the strong Jacobson normal form. The existence of strong form is not guaranteed if one works with algebra, which is not rational Weyl algebra.

#### References:

- [1] N. Jacobson, 'The theory of rings', AMS, 1943.
- [2] Manuel Avelino Insua Herno, 'Varias perspectivas sobre las bases de Groebner : Forma normal de Smith, Algorithme de Berlekamp y algebras de Leibniz'. PhD thesis, Universidad de Santiago de Compostela, 2005.
- [3] V. Levandovskyy, K. Schindelar 'Computing Jacobson normal form using Groebner bases', to appear in Journal of Symbolic Computation, 2010.

**Procedures:** See also: [Section D.11.2 \[control\\_lib\]](#), page 1967.



### D.11.3.1 smith

Procedure from library `jacobson.lib` (see [Section D.11.3 \[jacobson.lib\]](#), page 1978).

**Usage:** `smith(M[, eng1, eng2]);` M matrix, eng1 and eng2 are optional integers

**Return:** matrix or list of matrices, depending on arguments

**Assume:** Basing is a commutative polynomial ring in one variable

**Purpose:** compute the Smith Normal Form of M with (optionally) transformation matrices

**Theory:** Groebner bases are used for the Smith form like in [2] and [3].

**Note:** By default, just the Smith normal form of M is returned.  
 If the optional integer **eng1** is non-zero, the list {U,D,V} is returned where  $U \cdot M \cdot V = D$  and the diagonal field entries of D are not normalized. The normalization of the latter can be done with the 'divideUnits' procedure. U and V above are square unimodular (invertible) matrices.  
 Note, that the procedure works for a rectangular matrix M.

The optional integer **eng2** determines the Groebner basis engine:  
 0 (default) ensures the use of 'slimgb', otherwise 'std' is used.

**Display:** If `printlevel=1`, progress debug messages will be printed,  
 if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "jacobson.lib";
ring r = 0,x,Dp;
matrix m[3][2]=x, x^4+x^2+21, x^4+x^2+x, x^3+x, 4*x^2+x, x;
list s=smith(m,1);
print(s[2]); // non-normalized Smith form of m
  => 21,0,
  => 0, x,
  => 0, 0
print(s[1]*m*s[3] - s[2]); // check U*M*V = D
  => 0,0,
  => 0,0,
  => 0,0
list t = divideUnits(s);
print(t[2]); // the Smith form of m
  => 1,0,
  => 0,x,
  => 0,0
```

See also: [Section D.11.3.3 \[divideUnits\]](#), page 1980; [Section D.11.3.2 \[jacobson\]](#), page 1979.

### D.11.3.2 jacobson

Procedure from library `jacobson.lib` (see [Section D.11.3 \[jacobson.lib\]](#), page 1978).

**Usage:** `jacobson(M, eng);` M matrix, eng an optional int

**Return:** list

**Assume:** Basing is a (non-commutative) ring in two variables.

**Purpose:** compute a weak Jacobson normal form of M over the basering

**Theory:** Groebner bases and involutions are used, following [3]

**Note:** A list  $L$  of matrices  $\{U, D, V\}$  is returned. That is  $L[1]*M*L[3]=L[2]$ , where  $L[2]$  is a diagonal matrix and  $L[1]$ ,  $L[3]$  are square invertible polynomial (unimodular) matrices. Note, that  $M$  can be rectangular. The optional integer `eng2` determines the Groebner basis engine: 0 (default) ensures the use of 'slimgb', otherwise 'std' is used.

**Display:** If `printlevel=1`, progress debug messages will be printed, if `printlevel>=2`, all the debug messages will be printed.

**Example:**

```
LIB "jacobson.lib";
ring r = 0,(x,d),Dp;
def R = nc_algebra(1,1); setring R; // the 1st Weyl algebra
matrix m[2][2] = d,x,0,d; print(m);
  ↪ d,x,
  ↪ 0,d
list J = jacobson(m); // returns a list with 3 entries
print(J[2]); // a Jacobson Form D for m
  ↪ xd2-d,0,
  ↪ 0, 1
print(J[1]*m*J[3] - J[2]); // check that U*M*V = D
  ↪ 0,0,
  ↪ 0,0
/* now, let us do the same for the shift algebra */
ring r2 = 0,(x,s),Dp;
def R2 = nc_algebra(1,s); setring R2; // the 1st shift algebra
matrix m[2][2] = s,x,0,s; print(m); // matrix of the same for as above
  ↪ s,x,
  ↪ 0,s
list J = jacobson(m);
print(J[2]); // a Jacobson Form D, quite different from above
  ↪ xs2+s2,0,
  ↪ 0, x
print(J[1]*m*J[3] - J[2]); // check that U*M*V = D
  ↪ 0,0,
  ↪ 0,0
```

See also: [Section D.11.3.3 \[divideUnits\]](#), page 1980; [Section D.11.3.1 \[smith\]](#), page 1979.

### D.11.3.3 divideUnits

Procedure from library `jacobson.lib` (see [Section D.11.3 \[jacobson.lib\]](#), page 1978).

**Usage:** `divideUnits(L)`; list  $L$

**Return:** matrix or list of matrices

**Assume:**  $L$  is an output of `smith` or a `jacobson` procedures, that is either  $L$  contains one rectangular matrix with elements only on the main diagonal or  $L$  consists of three matrices, where  $L[1]$  and  $L[3]$  are square invertible matrices while  $L[2]$  is a rectangular matrix with elements only on the main diagonal

**Purpose:** divide out units from the diagonal and reflect this in transformation matrices

**Example:**

```

LIB "jacobson.lib";
ring R=(0,m,M,L1,L2,m1,m2,g), D, lp; // two pendula example
matrix P[3][4]=m1*L1*D^2,m2*L2*D^2,(M+m1+m2)*D^2,-1,
m1*L1^2*D^2-m1*L1*g,0,m1*L1*D^2,0,0,
m2*L2^2*D^2-m2*L2*g,m2*L2*D^2,0;
list s=smith(P,1); // returns a list with 3 entries
print(s[2]); // a diagonal form, close to the Smith form
↪ (L1*L2*m2*g^2-L2^2*m2*g^2),0, 0, 0,
↪ 0, (L2),0, 0,
↪ 0, 0, (g^2),0
print(s[1]); // U, left transformation matrix
↪ 0, (-L2*m2)/(L1*m1), 1,
↪ (-L2),(M*L2+L2*m1)/(L1*m1),1,
↪ 0, 1/(L1*m1), 0
list t = divideUnits(s);
print(t[2]); // the Smith form of the matrix P
↪ 1,0,0,0,
↪ 0,1,0,0,
↪ 0,0,1,0
print(t[1]); // U', modified left transformation matrix
↪ 0, -1/(L1^2*m1*g^2-L1*L2*m1*g^2),1/(L1*L2*m2*g^2-L2^2*m2*g^2),
↪ -1,(M+m1)/(L1*m1), 1/(L2),
↪ 0, 1/(L1*m1*g^2), 0

```

**D.11.4 findifs\_lib****Library:** findifs.lib**Purpose:** Tools for the finite difference schemes**Authors:** Viktor Levandovskyy, levandov@math.rwth-aachen.de**Overview:** We provide the presentation of difference operators in a polynomial, semi-factorized and a nodal form. Running `findifs_example()`; will demonstrate, how we generate finite difference schemes of linear PDEs from given approximations.

Theory: The method we use have been developed by V. Levandovskyy and Bernd Martin. The computation of a finite difference scheme of a given single linear partial differential equation with constant coefficients with a given approximation rules boils down to the computation of a Groebner basis of a submodule of a free module with respect to the ordering, eliminating module components.

Support: SpezialForschungsBereich F1301 of the Austrian FWF

**Procedures:** See also: [Section D.15.4 \[finitediff\\_lib\]](#), page 2330; [Section D.9.2 \[latex\\_lib\]](#), page 1907.**D.11.4.1 findifs\_example**Procedure from library `findifs.lib` (see [Section D.11.4 \[findifs\\_lib\]](#), page 1981).**Usage:** `findifs_example()`;**Return:** nothing (demo)**Purpose:** demonstration of our approach and this library**Example:**

```

LIB "findifs.lib";
findifs_example();
⇒ * Equation: u_tt - A^2 u_xx - B^2 u_yy = 0; A,B are constants
⇒ * we employ three central differences
⇒ * the vector we act on is (u_xx, u_yy, u_tt, u)^T
⇒ * Set up the ring:
⇒ ring r = (0,A,B,dt,dx,dy),(Tx,Ty,Tt),(c,dp);
⇒ * Set up the matrix with equation and approximations:
⇒ matrix M[4][4] =
⇒ // direct equation:
⇒ -A^2, -B^2, 1, 0,
⇒ // central difference u_tt
⇒ 0, 0, -dt^2*Tt, (Tt-1)^2,
⇒ // central difference u_xx
⇒ -dx^2*Tx, 0, 0, (Tx-1)^2,
⇒ // central difference u_yy
⇒ 0, -dy^2*Ty, 0, (Ty-1)^2;
⇒ * Print the differential form of equations:
⇒ (-A^2)*Uxx+(-B^2)*Uyy+Utt,
⇒ (-dt^2*Tt)*Utt+(Tt^2-2*Tt+1)*U,
⇒ (-dx^2*Tx)*Uxx+(Tx^2-2*Tx+1)*U,
⇒ (-dy^2*Ty)*Uyy+(Ty^2-2*Ty+1)*U
⇒ * Perform the elimination of module components:
⇒ module R = transpose(M);
⇒ module S = std(R);
⇒ * See the result of Groebner bases: generators are columns
⇒ print(S);
⇒ 0, 0, 0, 0, 0, (A^2),
⇒ 0, 0, 0, (dy^2)*Ty, S[2,5], (B^2),
⇒ 0, (dt^2)*Tt, S[3,3], 0, (-dx^2)*Tx, -1,
⇒ S[4,1], -Tt^2+2*Tt-1, S[4,3], -Ty^2+2*Ty-1, S[4,5], 0
⇒ * So, only the first column has its nonzero element in the last componen\
t
⇒ * Hence, this polynomial is the scheme
⇒ poly p = S[4,1];
⇒ print(p);
⇒ (A^2*dt^2*dy^2)*Tx^2*Ty*Tt+(B^2*dt^2*dx^2)*Tx*Ty^2*Tt+(-dx^2*dy^2)*Tx*Ty*\
Tt^2+(-2*A^2*dt^2*dy^2-2*B^2*dt^2*dx^2+2*dx^2*dy^2)*Tx*Ty*Tt+(-dx^2*dy^2)\
*Tx*Ty+(B^2*dt^2*dx^2)*Tx*Tt+(A^2*dt^2*dy^2)*Ty*Tt
⇒ * Create the nodal of the scheme in TeX format:
⇒ ideal I = decoef(p,dt);
⇒ difpoly2tex(I,L);
⇒ \frac{1}{\tri t^2}\cdot (u^{\{n+2\}}_{j+1,k+1}+(-2) u^{\{n+1\}}_{j+1,k+1}+u^{\{n\}}_{\
_{j+1,k+1}}+ \frac{-\lambda^{\{2\}}}{\tri x^{\{2\}}}\cdot (u^{\{n+1\}}_{j+2,k+1}+\fra\
c{B^{\{2\}}}{\tri x^{\{2\}}}\{\lambda^{\{2\}}\tri y^{\{2\}} u^{\{n+1\}}_{j+1,k+2}+\frac{-(2\lam\
bda^{\{2\}}}{\tri y^{\{2\}}+2B^{\{2\}}}{\tri x^{\{2\}}}\{\lambda^{\{2\}}\tri y^{\{2\}} u^{\{n+1\}}_{j+1,k\
+1}+\frac{B^{\{2\}}}{\tri x^{\{2\}}}\{\lambda^{\{2\}}\tri y^{\{2\}} u^{\{n+1\}}_{j+1}+u^{\{n+1\}}_{\
j,k+1})
⇒ * Preparations for the semi-factorized form:
⇒ poly pi1 = subst(I[2],B,0);
⇒ poly pi2 = I[2] - pi1;
⇒ * Show the semi-factorized form of the scheme: 1st summand
⇒ factorize(I[1]);

```

```

⇒ [1]:
⇒   _[1]=(-dx^2*dy^2)
⇒   _[2]=Tx
⇒   _[3]=Ty
⇒   _[4]=Tt-1
⇒ [2]:
⇒   1,1,1,2
⇒ * Show the semi-factorized form of the scheme: 2nd summand
⇒   factorize(pi1);
⇒ [1]:
⇒   _[1]=(A^2*dt^2*dy^2)
⇒   _[2]=Ty
⇒   _[3]=Tt
⇒   _[4]=Tx-1
⇒ [2]:
⇒   1,1,1,2
⇒ * Show the semi-factorized form of the scheme: 3rd summand
⇒   factorize(pi1);
⇒ [1]:
⇒   _[1]=(B^2*dt^2*dx^2)
⇒   _[2]=Tx
⇒   _[3]=Tt
⇒   _[4]=Ty-1
⇒ [2]:
⇒   1,1,1,2

```

#### D.11.4.2 decoef

Procedure from library `findifs.lib` (see [Section D.11.4 \[findifs.lib\]](#), page 1981).

**Usage:**      `decoef(P,n)`;  $P$  a poly,  $n$  a number

**Return:**    ideal

**Purpose:**    decompose polynomial  $P$  into summands with respect to the presence of the number  $n$  in the coefficients

**Note:**       $n$  is usually a parameter with no power

**Example:**

```

LIB "findifs.lib";
" EXAMPLE:";
⇒ EXAMPLE:
ring r = (0,dh,dt),(Tx,Tt),dp;
poly P = (4*dh^2-dt)*Tx^3*Tt + dt*dh*Tt^2 + dh*Tt;
decoef(P,dt);
⇒ _[1]=(4*dh^2)*Tx^3*Tt+(dh)*Tt
⇒ _[2]=(-dt)*Tx^3*Tt+(dh*dt)*Tt^2
decoef(P,dh);
⇒ _[1]=(-dt)*Tx^3*Tt
⇒ _[2]=(4*dh^2)*Tx^3*Tt+(dh*dt)*Tt^2+(dh)*Tt

```

#### D.11.4.3 difpoly2tex

Procedure from library `findifs.lib` (see [Section D.11.4 \[findifs.lib\]](#), page 1981).

- Usage:** `difpoly2tex(S,P[,Q]);` S an ideal, P and optional Q are lists
- Return:** string
- Purpose:** present the difference scheme in the nodal form
- Assume:** ideal S is the result of `decoef` procedure
- Note:** a list P may be empty or may contain parameters, which will not appear in denominators  
an optional list Q represents the part of the scheme, depending on other function, than the major part

**Example:**

```
LIB "findifs.lib";
ring r = (0,dh,dt,V),(Tx,Tt),dp;
poly M = (4*dh*Tx+dt)^2*(Tt-1) + V*Tt*Tx;
ideal I = decoef(M,dt);
list L; L[1] = V;
difpoly2tex(I,L);
⇒ \frac{1}{8\tri t}\cdot (u^{\{n+1\}}_{\{j+2\}}-u^{\{n\}}_{\{j+2\}}+\frac{\nu}{16\tri h^{\{2\}}}
   \cdot u^{\{n+1\}}_{\{j+1\}})+ \frac{1}{16\tri h}\cdot (u^{\{n+1\}}_{\{j+1\}}-u^{\{n\}}_{\{j+1\}}+\frac{1}{8\tri t}
   \cdot u^{\{n+1\}}_{\{j\}}+\frac{-1}{8\tri h}\cdot u^{\{n\}}_{\{j\}})
poly G = V*dh^2*(Tt-Tx)^2;
difpoly2tex(I,L,G);
⇒ \frac{1}{8\tri t}\cdot (u^{\{n+1\}}_{\{j+2\}}-u^{\{n\}}_{\{j+2\}}+\frac{\nu}{16\tri h^{\{2\}}}
   \cdot u^{\{n+1\}}_{\{j+1\}})+ \frac{1}{16\tri h}\cdot (u^{\{n+1\}}_{\{j+1\}}-u^{\{n\}}_{\{j+1\}}+\frac{1}{8\tri t}
   \cdot u^{\{n+1\}}_{\{j\}}+\frac{-1}{8\tri h}\cdot u^{\{n\}}_{\{j\}})+ \frac{\nu}{128\tri t}\cdot (p^{\{n\}}_{\{j+2\}}+(-2)
   \cdot p^{\{n+1\}}_{\{j+1\}}+p^{\{n+2\}}_{\{j\}})
```

**D.11.4.4 exp2pt**

Procedure from library `findifs.lib` (see [Section D.11.4 \[findifs.lib\]](#), page 1981).

- Usage:** `exp2pt(P[,L]);` P poly, L an optional list of strings
- Return:** string
- Purpose:** convert a polynomial M into the TeX format, in nodal form
- Assume:** coefficients must not be fractional
- Note:** an optional list L contains a string, which will replace the default value 'u' for the discretized function

**Example:**

```
LIB "findifs.lib";
" EXAMPLE:";
⇒ EXAMPLE:
ring r = (0,dh,dt),(Tx,Tt),dp;
poly M = (4*dh*Tx^2+1)*(Tt-1)^2;
print(exp2pt(M));
⇒ 4\tri h u^{\{n+2\}}_{\{j+2\}}+(-8\tri h) u^{\{n+1\}}_{\{j+2\}}+4\tri h u^{\{n\}}_{\{j+2\}}+u^{\{n+2\}}_{\{j\}}
   +(-2) u^{\{n+1\}}_{\{j\}}+u^{\{n\}}_{\{j\}}
print(exp2pt(M,"F"));
⇒ 4\tri h F^{\{n+2\}}_{\{j+2\}}+(-8\tri h) F^{\{n+1\}}_{\{j+2\}}+4\tri h F^{\{n\}}_{\{j+2\}}+F^{\{n+2\}}_{\{j\}}
   +(-2) F^{\{n+1\}}_{\{j\}}+F^{\{n\}}_{\{j\}}
```

### D.11.4.5 texcoef

Procedure from library `findifs.lib` (see [Section D.11.4 \[findifs.lib\], page 1981](#)).

**Usage:** `texcoef(n)`;  $n$  a number

**Return:** string

**Purpose:** converts the number  $n$  into TeX format

**Note:** if  $n$  is a polynomial, `texcoef` adds extra brackets and performs some space substitutions

**Example:**

```
LIB "findifs.lib";
" EXAMPLE:";
⇨ EXAMPLE:
ring r = (0,dh,dt),(Tx,Tt),dp;
number n1,n2,n3 = dt/(4*dh^2-dt),(dt+dh)^2, 1/dh;
n1; texcoef(n1);
⇨ (dt)/(4*dh^2-dt)
⇨ \frac{\tri t}{4\tri h^2-\tri t}
n2; texcoef(n2);
⇨ (dh^2+2*dh*dt+dt^2)
⇨ (\tri h^2+2\tri h\tri t+\tri t^2)
n3; texcoef(n3);
⇨ 1/(dh)
⇨ \frac{1}{\tri h}
```

### D.11.4.6 npar

Procedure from library `findifs.lib` (see [Section D.11.4 \[findifs.lib\], page 1981](#)).

**Usage:** `npar(n)`;  $n$  a number

**Return:** int

**Purpose:** searches for 'n' among the parameters and returns its number

**Example:**

```
LIB "findifs.lib";
ring r = (0,dh,dt,theta,A),t,dp;
npar(dh);
⇨ 1
number T = theta;
npar(T);
⇨ 3
npar(dh^2);
⇨ Incorrect parameter
⇨ 0
```

### D.11.4.7 magnitude

Procedure from library `findifs.lib` (see [Section D.11.4 \[findifs.lib\], page 1981](#)).

**Usage:** `magnitude(P)`;  $P$  a poly

**Return:** poly

**Purpose:** compute the square of the magnitude of a complex expression

**Assume:**  $i$  is the variable of a basering

**Example:**

```
LIB "findifs.lib";
ring r = (0,d),(g,i,sin,cos),dp;
poly P = d*i*sin - g*cos +d^2*i;
NF( magnitude(P), std(i^2+1) );
 $\mapsto g^2 \cos^2 + (d^2) \sin^2 + (2d^3) \sin + (d^4)$ 
```

#### D.11.4.8 replace

Procedure from library `findifs.lib` (see [Section D.11.4 \[findifs.lib\], page 1981](#)).

**Usage:** `replace(s,what,with);` s,what,with strings

**Return:** string

**Purpose:** replaces in 's' all the substrings 'what' with substring 'with'

**Note:**

**Example:**

```
LIB "findifs.lib";
" EXAMPLE:";
 $\mapsto$  EXAMPLE:
ring r = (0,dt,theta),Tt,dp;
poly p = (Tt*dt+theta+1)^2+2;
string s = texfactorize("",p);
s;
 $\mapsto \$(dt^2 \cdot Tt^2 + (2dt\theta + 2dt) \cdot Tt + (\theta^2 + 2\theta + 3))\$$ 
s = replace(s,"Tt","T_t"); s;
 $\mapsto \$(dt^2 \cdot T_t^2 + (2dt\theta + 2dt) \cdot T_t + (\theta^2 + 2\theta + 3))\$$ 
s = replace(s,"dt","\tri t"); s;
 $\mapsto \$(\tri t^2 \cdot T_t^2 + (2\tri t\theta + 2\tri t) \cdot T_t + (\theta^2 + 2\theta + 3))\$$ 
s = replace(s,"theta","\theta"); s;
 $\mapsto \$(\tri t^2 \cdot T_t^2 + (2\tri t\theta + 2\tri t) \cdot T_t + (\theta^2 + 2\theta + 3))\$$ 
```

#### D.11.4.9 xchange

Procedure from library `findifs.lib` (see [Section D.11.4 \[findifs.lib\], page 1981](#)).

**Usage:** `xchange(w,a,b);` w,a,b strings

**Return:** string

**Purpose:** exchanges substring 'a' with a substring 'b' in the string w

**Note:**

**Example:**

```
LIB "findifs.lib";
" EXAMPLE:";
 $\mapsto$  EXAMPLE:
ring r = (0,dt,dh,A),Tt,dp;
poly p = (Tt*dt+dh+1)^2+2*A;
string s = texpoly("",p);
```



```

s;
 $\mapsto dt^2 \cdot Tt^2 + (2dtdh + 2dt) \cdot Tt + (dh^2 + 2dh + 2A + 1)$ 
string t = xchange(s,"dh","dt");
t;
 $\mapsto dh^2 \cdot Tt^2 + (2dhdt + 2dh) \cdot Tt + (dt^2 + 2dt + 2A + 1)$ 

```

## D.12 Teaching

The libraries in this section are intended to be used for teaching purposes but not for serious computations.

### D.12.1 aksaka\_lib

**Library:** aksaka.lib

**Purpose:** Procedures for primality testing after Agrawal, Saxena, Kayal

**Authors:** Christoph Mang

**Overview:** Algorithms for primality testing in polynomial time based on the ideas of Agrawal, Saxena and Kayal.

**Procedures:**

#### D.12.1.1 fastExpt

Procedure from library `aksaka.lib` (see [Section D.12.1 \[aksaka\\_lib\]](#), page 1987).

**Usage:** `fastExpt(a,m,n)`;  $a, m, n = \text{number}$ ;

**Return:** the  $m$ -th power of  $a$ ; if  $a^m > n$  the procedure returns  $n+1$

**Note:** uses fast exponentiation

**Example:**

```

LIB "aksaka.lib";
fastExpt(2,10,1022);
 $\mapsto 1023$ 

```

#### D.12.1.2 log2

Procedure from library `aksaka.lib` (see [Section D.12.1 \[aksaka\\_lib\]](#), page 1987).

**Usage:** `log2(x)`;

**Return:** logarithm to basis 2 of  $x$

**Note:** calculates the natural logarithm of  $x$  with a power-series of the  $\ln$ , then the basis is changed to 2

**Example:**

```

LIB "aksaka.lib";
log2(1024);
 $\mapsto 10$ 

```

### D.12.1.3 PerfectPowerTest

Procedure from library `aksaka.lib` (see [Section D.12.1 \[aksaka.lib\]](#), page 1987).

**Usage:**      `PerfectPowerTest(n);`

**Return:**     0 if there are numbers  $a, b > 1$  with  $a^b = n$ ;  
                  1 if there are no numbers  $a, b > 1$  with  $a^b = n$ ;  
                  if `printlevel`  $\geq 1$  and there are  $a, b > 1$  with  $a^b = n$ ,  
                  then  $a, b$  are printed

**Example:**

```
LIB "aksaka.lib";
PerfectPowerTest(887503681);
↦ 0
```

### D.12.1.4 wurzel

Procedure from library `aksaka.lib` (see [Section D.12.1 \[aksaka.lib\]](#), page 1987).

**Usage:**      `wurzel(r);`

**Assume:**    characteristic of basering is 0,  $r \geq 0$

**Return:**     number, square root of  $r$

**Example:**

```
LIB "aksaka.lib";
ring R = 0,x,dp;
wurzel(7629412809180100);
↦ 3441627728253432342460123298897595164661535507549430119821503875373993001\
950914732024064729050404203240063089015829737039655/394020061963944792122\
7904010014361380507973927046544666794829340424572177149721061141426625488\
4915640806627990306816
```

### D.12.1.5 euler

Procedure from library `aksaka.lib` (see [Section D.12.1 \[aksaka.lib\]](#), page 1987).

**Usage:**      `euler(r);`

**Return:**     bigint  $\phi(r)$ , where  $\phi$  is Eulers phi-function

**Note:**       first  $r$  is factorized with `proc PollardRho`, then  $\phi(r)$  is calculated with the help of  $\phi(p)$  of every factor  $p$ ;

**Example:**

```
LIB "aksaka.lib";
euler(99991);
↦ 99990
```

### D.12.1.6 coeffmod

Procedure from library `aksaka.lib` (see [Section D.12.1 \[aksaka.lib\]](#), page 1987).

**Usage:**      `coeffmod(f,n);`

**Assume:**    poly  $f$  depends on at most `var(1)` of the basering

**Return:** poly f modulo number n

**Note:** at first the coefficients of the monomials of the polynomial f are determined, then their remainder modulo n is calculated, after that the polynomial 'is put together' again

**Example:**

```
LIB "aksaka.lib";
ring R = 0,x,dp;
poly f=2457*x4+52345*x3-98*x2+5;
bigint n=3;
coeffmod(f,n);
↦ x3+x2+2
```

### D.12.1.7 powerpolyX

Procedure from library `aksaka.lib` (see [Section D.12.1 \[aksaka.lib\]](#), page 1987).

**Usage:** powerpolyX(q,n,a,r);

**Return:** the q-th power of poly a modulo poly r and number n

**Example:**

```
LIB "aksaka.lib";
ring R=0,x,dp;
poly a=3*x3-x2+5;
poly r=x7-1;
bigint q=123;
bigint n=5;
powerpolyX(q,n,a,r);
↦ 3x5+2x4+x3+x2+1
```

### D.12.1.8 ask

Procedure from library `aksaka.lib` (see [Section D.12.1 \[aksaka.lib\]](#), page 1987).

**Usage:** ask(n);

**Assume:** n>1

**Return:** 0 if n is composite;  
1 if n is prime;  
if printlevel>=1, you are informed what the procedure will do or has calculated

**Note:** ASK-algorithm; uses proc powerpolyX for step 5

**Example:**

```
LIB "aksaka.lib";
//ask(100003);
ask(32003);
↦ 1
```

## D.12.2 crypto\_lib

**Library:** crypto.lib

**Purpose:** Procedures for teaching cryptography

**Authors:** Gerhard Pfister, pfister@mathematik.uni-kl.de  
David Brittinger, dativ@gmx.net

**Overview:** The library contains procedures to compute the discrete logarithm, primality-tests, factorization included elliptic curves. The library is intended to be used for teaching purposes but not for serious computations. Sufficiently high printlevel allows to control each step, thus illustrating the algorithms at work.

**Procedures:**

### D.12.2.1 round

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `round(r);`

**Return:** the nearest number to  $r$  out of  $\mathbb{Z}$

**Assume:**  $r$  should be a rational or a real number

**Example:**

```
LIB "crypto.lib";
ring R = (real,50),x,dp;
number r=7357683445788723456321.6788643224;
round(r);
⇨ 7357683445788723456322
```

### D.12.2.2 bubblesort

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `bubblesort(L);`

**Return:** list  $L$ , sort in decreasing order

**Example:**

```
LIB "crypto.lib";
ring r = 0,x,dp;
list L=-567,-233,446,12,-34,8907;
bubblesort(L);
⇨ [1]:
⇨ 8907
⇨ [2]:
⇨ 446
⇨ [3]:
⇨ 12
⇨ [4]:
⇨ -34
⇨ [5]:
⇨ -233
⇨ [6]:
⇨ -567
```

### D.12.2.3 decimal

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:**      `decimal(s); s = string`

**Return:**    the (decimal) number corresponding to the hexadecimal number `s`

**Example:**

```
LIB "crypto.lib";
string s = "8edfe37dae96cfd2466d77d3884d4196";
decimal(s);
↦ 189912871665444375716340628395668619670
```

### D.12.2.4 eexgcdN

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:**      `eexgcdN(L);`

**Return:**    list `T` such that  $\sum_i L[i] \cdot T[i] = T[n+1] = \gcd(L[1], \dots, L[n])$

**Example:**

```
LIB "crypto.lib";
eexgcdN(list(24,15,21));
↦ [1]:
↦    2
↦ [2]:
↦   -3
↦ [3]:
↦    0
↦ [4]:
↦    3
```

### D.12.2.5 lcmN

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:**      `lcmN(a,b);`

**Return:**    `lcm(a,b);`

**Example:**

```
LIB "crypto.lib";
lcmN(24,15);
↦ 120
```

### D.12.2.6 powerN

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:**      `powerN(m,d,n);`

**Return:**     $m^d \bmod n$

**Example:**

```
LIB "crypto.lib";
powerN(24,15,7);
↦ 6
```

### D.12.2.7 chineseRem

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `chineseRem(T,L);`

**Return:**  $x$  such that  $x = T[i] \bmod L[i]$

**Note:** chinese remainder theorem

**Example:**

```
LIB "crypto.lib";
chineseRem(list(24,15,7),list(2,3,5));
↦ 12
```

### D.12.2.8 Jacobi

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `Jacobi(a,n);`

**Return:** the generalized Legendre symbol

**Note:** if  $n$  is an odd prime then  $Jacobi(a,n)=0,1,-1$  if  $n|a$ ,  $a=x^2 \bmod n$ , else

**Example:**

```
LIB "crypto.lib";
Jacobi(13580555397810650806,5792543);
↦ 1
```

### D.12.2.9 primList

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `primList(n);`

**Return:** the list of all primes  $\leq n$

**Example:**

```
LIB "crypto.lib";
list L=primList(100);
size(L);
↦ 25
L[size(L)];
↦ 97
```

### D.12.2.10 primL

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `primL(q);`

**Return:** list of the first primes  $p_1, \dots, p_r$  such that  $q > p_1 \cdot \dots \cdot p_{(r-1)}$  and  $q < p_1 \cdot \dots \cdot p_r$

**Example:**

```
LIB "crypto.lib";
primL(20);
↦ [1]:
↦ 2
```

```

↦ [2] :
↦      3
↦ [3] :
↦      5

```

#### D.12.2.11 intPart

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:**     `intPart(x);`

**Return:**    the integral part of a rational number

**Example:**

```

LIB "crypto.lib";
ring r=0,x,dp;
intPart(7/3);
↦ 2

```

#### D.12.2.12 intRoot

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:**     `intRoot(m);`

**Return:**    the integral part of the square root of m

**Example:**

```

LIB "crypto.lib";
intRoot(20);
↦ 4

```

#### D.12.2.13 squareRoot

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:**     `squareRoot(a,p);`

**Return:**    the square root of a in  $\mathbb{Z}/p$ , p prime

**Note:**     assumes the Jacobi symbol is 1 or p=2.

**Example:**

```

LIB "crypto.lib";
squareRoot(8315890421938608,32003);
↦ 18784

```

#### D.12.2.14 solutionsMod2

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:**     `solutionsMod2(M);`

**Return:**    an intmat containing a basis of the vector space of solutions of the linear system of equations defined by M over the prime field of characteristic 2

**Example:**

```

LIB "crypto.lib";
bigintmat M[3][3]=1,2,3,4,5,6,7,6,5;
solutionsMod2(M);
↦ 1,0,1

```

### D.12.2.15 powerX

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:**      `powerX(q,i,I);`

**Return:**     the  $q$ -th power of the  $i$ -th variable modulo  $I$

**Assume:**     $I$  is a standard basis

**Example:**

```
LIB "crypto.lib";
ring R = 0,(x,y),dp;
powerX(100,2,std(ideal(x3-1,y2-x)));
↦ x2
```

### D.12.2.16 babyGiant

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:**      `babyGiant(b,y,p);`

**Return:**     the discrete logarithm  $x$ :  $b^x = y \bmod p$

**Note:**        This procedure works based on Shank's baby step - giant step method.

**Example:**

```
LIB "crypto.lib";
bigint b=2;
bigint y=10;
bigint p=101;
babyGiant(b,y,p);
↦ 25
```

### D.12.2.17 rho

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:**      `rho(b,y,p);`

**Return:**     the discrete logarithm  $x = \log_b(y)$ :  $b^x = y \bmod p$

**Note:**        Pollard's rho:  
choose random  $f_0$  in  $0, \dots, p-2$ ,  $e_0 = 0$ , define  $x_0 = b^{f_0}$ , define  $x_i = y^{e_i} b^{f_i}$  as below.  
For  $i$  large enough there is  $i$  with  $x_{(i/2)} = x_i$ . Let  $s := e_{(i/2)} - e_i \bmod p-1$  and  $t := f_i - f_{(i/2)} \bmod p-1$ ,  $d = \gcd(s, p-1) = u*s + v*(p-1)$  then  $x = tu/d + j*(p-1)/d$  for some  $j$  (to be found by trying)

**Example:**

```
LIB "crypto.lib";
bigint b=2;
bigint y=10;
bigint p=101;
rho(b,y,p);
↦ 25
```



### D.12.2.18 MillerRabin

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** MillerRabin(n,k);

**Return:** 1 if n is prime, 0 else

**Note:** probabilistic test of Miller-Rabin with k loops to test if n is prime. Using the theorem: If n is prime,  $n-1=2^s \cdot r$ , r odd, then  $\text{powerN}(a,r,n)=1$  or  $\text{powerN}(a,r \cdot 2^i,n)=-1$  for some i

**Example:**

```
LIB "crypto.lib";
bigint x=2;
x=x^787-1;
MillerRabin(x,3);
↦ 0
```

### D.12.2.19 SolowayStrassen

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** SolowayStrassen(n,k);

**Return:** 1 if n is prime, 0 else

**Note:** probabilistic test of Soloway-Strassen with k loops to test if n is prime using the theorem: If n is prime then  $\text{powerN}(a,(n-1)/2,n)=\text{Jacobi}(a,n) \bmod n$

**Example:**

```
LIB "crypto.lib";
bigint h=10;
bigint p=h^100+267;
//p=h^100+43723;
//p=h^200+632347;
SolowayStrassen(h,3);
↦ 0
```

### D.12.2.20 PocklingtonLehmer

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** PocklingtonLehmer(N); optional: PocklingtonLehmer(N,L); L a list of the first k primes

**Return:** message N is not prime or  $\{A,\{p\},\{a_p\}\}$  as certificate for N being prime

**Note:** assumes that it is possible to factorize  $N-1=A \cdot B$  such that  $\gcd(A,B)=1$ , the factorization of A is completely known and  $A^2 > N$ .  
N is prime if and only if for each prime factor p of A we can find  $a_p$  such that  $a_p^{(N-1)}=1 \bmod N$  and  $\gcd(a_p^{((N-1)/p)-1},N)=1$

**Example:**

```
LIB "crypto.lib";
bigint N=105554676553297;
PocklingtonLehmer(N);
↦ [1]:
```

```

⇒      6442503168
⇒ [2]:
⇒      [1]:
⇒      [1]:
⇒      2
⇒      [2]:
⇒      2
⇒      [2]:
⇒      [1]:
⇒      3
⇒      [2]:
⇒      2
⇒      [3]:
⇒      [1]:
⇒      2097169
⇒      [2]:
⇒      2
list L=primList(1000);
PocklingtonLehmer(N,L);
⇒ [1]:
⇒      3221246976
⇒ [2]:
⇒      [1]:
⇒      [1]:
⇒      2
⇒      [2]:
⇒      2
⇒      [2]:
⇒      [1]:
⇒      3
⇒      [2]:
⇒      2
⇒      [3]:
⇒      [1]:
⇒      1048583
⇒      [2]:
⇒      2

```

### D.12.2.21 PollardRho

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\]](#), page 1989).

**Usage:** PollardRho(n,k,allFactors); optional: PollardRho(n,k,allFactors,L); L a list of the first k primes

**Return:** a list of factors of n (which could be just n), if allFactors=0  
a list of all factors of n, if allFactors=1

**Note:** probabilistic rho-algorithm of Pollard to find a factor of n in k loops. Creates a sequence  $x_i$  such that  $(x_i)^2 = (x_{2i})^2 \pmod n$  for some i, computes  $\gcd(x_i - x_{2i}, n)$  to find a divisor. To define the sequence choose x, a and define  $x_{n+1} = x_n^2 + a \pmod n$ ,  $x_1 = x$ . If allFactors is 1, it tries to find recursively all prime factors using the Soloway-Strassen test.

**Example:**

```

LIB "crypto.lib";
bigint h=10;
bigint p=h^30+4;
PollardRho(p,5000,0);
⇒ [1]:
⇒ 2
⇒ [2]:
⇒ 157
⇒ [3]:
⇒ 18737561
⇒ [4]:
⇒ 84982068258408294013

```

See also: [Section 5.1.119 \[primefactors\]](#), page 241.

### D.12.2.22 pFactor

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\]](#), page 1989).

**Usage:** `pFactor(n,B,P)`;  $n$  to be factorized,  $B$  a bound,  $P$  a list of primes

**Return:** a list of factors of  $n$  or  $n$  if no factor found

**Note:** Pollard's  $p$ -factorization creates the product  $k$  of powers of primes (bounded by  $B$ ) from the list  $P$  with the idea that for a prime divisor  $p$  of  $n$  we have  $p-1 \mid k$ , and then  $p$  divides  $\gcd(a^{k-1}, n)$  for some random  $a$

**Example:**

```

LIB "crypto.lib";
list L=primList(1000);
pFactor(1241143,13,L);
⇒ 547
bigint h=10;
h=h^30+25;
pFactor(h,20,L);
⇒ 325

```

### D.12.2.23 quadraticSieve

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\]](#), page 1989).

**Usage:** `quadraticSieve(n,c,B,k)`;  $n$  to be factorized,  $[-c,c]$  the sieve-intervall,  $B$  a list of primes,  $k$  for using the first  $k$  elements in  $B$

**Return:** a list of factors of  $n$  or the message: no divisor found

**Note:** The idea being used is to find  $x,y$  such that  $x^2=y^2 \pmod n$  then  $\gcd(x-y,n)$  can be a proper divisor of  $n$

**Example:**

```

LIB "crypto.lib";
list L=primList(5000);
quadraticSieve(7429,3,L,4);
⇒ 17
quadraticSieve(1241143,100,L,50);
⇒ 547

```

**D.12.2.24 isOnCurve**

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `isOnCurve(N,a,b,P);`

**Return:** 1 or 0 (depending on whether P is on the curve or not)

**Note:** checks whether  $P=(P[1]:P[2]:P[3])$  is a point on the elliptic curve defined by  $y^2z=x^3+a*xz^2+b*z^3$  over  $\mathbb{Z}/N$

**Example:**

```
LIB "crypto.lib";
isOnCurve(32003,5,7,list(10,16,1));
↦ 0
```

**D.12.2.25 ellipticAdd**

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `ellipticAdd(N,a,b,P,Q);`

**Return:** list L, representing the point  $P+Q$

**Note:**  $P=(P[1]:P[2]:P[3])$ ,  $Q=(Q[1]:Q[2]:Q[3])$  points on the elliptic curve defined by  $y^2z=x^3+a*xz^2+b*z^3$  over  $\mathbb{Z}/N$

**Example:**

```
LIB "crypto.lib";
bigint N=11;
bigint a=1;
bigint b=6;
list P,Q;
P[1]=2;
P[2]=4;
P[3]=1;
Q[1]=3;
Q[2]=5;
Q[3]=1;
ellipticAdd(N,a,b,P,Q);
↦ [1]:
↦ 7
↦ [2]:
↦ 2
↦ [3]:
↦ 1
```

**D.12.2.26 ellipticMult**

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `ellipticMult(N,a,b,P,k);`

**Return:** a list L representing the point  $k*P$

**Note:**  $P=(P[1]:P[2]:P[3])$  a point on the elliptic curve defined by  $y^2z=x^3+a*xz^2+b*z^3$  over  $\mathbb{Z}/N$

**Example:**

```

LIB "crypto.lib";
bigint N=11;
bigint a=1;
bigint b=6;
list P;
P[1]=2;
P[2]=4;
P[3]=1;
ellipticMult(N,a,b,P,3);
⇒ [1]:
⇒ 8
⇒ [2]:
⇒ 8
⇒ [3]:
⇒ 1

```

### D.12.2.27 ellipticRandomCurve

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `ellipticRandomCurve(N);`

**Return:** a list of two random numbers  $a, b$  and  $4a^3 + 27b^2 \bmod N$

**Note:**  $y^2z = x^3 + a*xz^2 + b*z^3$  defines an elliptic curve over  $\mathbb{Z}/N$

**Example:**

```

LIB "crypto.lib";
ellipticRandomCurve(32003);
⇒ [1]:
⇒ 22857
⇒ [2]:
⇒ 24963
⇒ [3]:
⇒ 1

```

### D.12.2.28 ellipticRandomPoint

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `ellipticRandomPoint(N,a,b);`

**Return:** a list representing a random point  $(x:y:z)$  of the elliptic curve defined by  $y^2z = x^3 + a*xz^2 + b*z^3$  over  $\mathbb{Z}/N$

**Example:**

```

LIB "crypto.lib";
ellipticRandomPoint(32003,3,181);
⇒ [1]:
⇒ 22857
⇒ [2]:
⇒ 17476
⇒ [3]:
⇒ 1

```

**D.12.2.29 countPoints**

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `countPoints(N,a,b);`

**Return:** the number of points of the elliptic curve defined by  $y^2 = x^3 + a*x + b$  over  $\mathbb{Z}/N$

**Note:** trivial approach

**Example:**

```
LIB "crypto.lib";
countPoints(181,71,150);
↪ 198
```

**D.12.2.30 ellipticAllPoints**

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `ellipticAllPoints(N,a,b);`

**Return:** list of points  $(x:y:z)$  of the elliptic curve defined by  $y^2z = x^3 + a*xz^2 + b*z^3$  over  $\mathbb{Z}/N$

**Example:**

```
LIB "crypto.lib";
list L=ellipticAllPoints(181,71,150);
size(L);
↪ 198
L[size(L)];
↪ [1]:
↪ 179
↪ [2]:
↪ 0
↪ [3]:
↪ 1
```

**D.12.2.31 ShanksMestre**

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `ShanksMestre(q,a,b);` optional: `ShanksMestre(q,a,b,s);` s the number of loops in the algorithm (default  $s=1$ )

**Return:** the number of points of the elliptic curve defined by  $y^2 = x^3 + a*x + b$  over  $\mathbb{Z}/N$

**Note:** algorithm of Shanks and Mestre (baby-step-giant-step)

**Example:**

```
LIB "crypto.lib";
ShanksMestre(32003,71,602);
↪ 32021
```

**D.12.2.32 Schoof**

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `Schoof(N,a,b);`

**Return:** the number of points of the elliptic curve defined by  $y^2 = x^3 + a*x + b$  over  $\mathbb{Z}/N$

**Note:** algorithm of Schoof

**Example:**

```
LIB "crypto.lib";
Schoof(32003,71,602);
↦ 32021
```

### D.12.2.33 generateG

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `generateG(a,b,m);`

**Return:** m-th division polynomial

**Note:** generate the so-called division polynomials, i.e., the recursively defined polynomials  $p_m = \text{generateG}(a,b,m)$  in  $\mathbb{Z}[x, y]$  such that, for a point  $(x:y:1)$  on the elliptic curve defined by  $y^2 = x^3 + ax + b$  over  $\mathbb{Z}/N$  the point  $m*P = (x - (p_{-(m-1)} * p_{-(m+1)}) / p_m^2, (p_{-(m+2)} * p_{-(m-1)}^2 - p_{-(m-2)} * p_{-(m+1)}^2) / 4y * p_m^3 : 1)$  and  $m*P = 0$  if and only if  $p_m(P) = 0$

**Example:**

```
LIB "crypto.lib";
ring R = 0, (x,y), dp;
generateG(7,15,4);
↦ 4xy6+140xy4+1200xy3-980xy2-1680xy-8572x
```

### D.12.2.34 factorLenstraECM

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `factorLenstraECM(N,S,B);` optional: `factorLenstraECM(N,S,B,d);` d+1 the number of loops in the algorithm (default d=0)

**Return:** a factor of N or the message no factor found

**Note:**

- computes a factor of N using Lenstra's ECM factorization
- the idea is that the fact that N is not prime is detected using the operations on the elliptic curve
- is similarly to Pollard's p-1-factorization

**Example:**

```
LIB "crypto.lib";
list L=primList(1000);
factorLenstraECM(181*32003,L,10,5);
↦ 181
bigint h=10;
h=h^30+25;
factorLenstraECM(h,L,4,3);
↦ 13
```

### D.12.2.35 ECPP

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `ECPP(N);`

**Return:** message:  $N$  is not prime or  $\{L, P, m, q\}$  as certificate for  $N$  being prime  
 $L$  a list ( $y^2 = x^3 + L[1]*x + L[2]$  defines an elliptic curve  $C$ )  
 $P$  a list  $((P[1]:P[2]:P[3]))$  is a point of  $C$   
 $m, q$  integers

**Assume:**  $\gcd(N, 6) = 1$

**Note:** The basis of the algorithm is the following theorem:  
 Given  $C$ , an elliptic curve over  $\mathbb{Z}/N$ ,  $P$  a point of  $C(\mathbb{Z}/N)$ ,  $m$  an integer,  $q$  a prime with the following properties:  
 -  $q \mid m$   
 -  $q > (4\text{-th root}(N) + 1)^2$   
 -  $m * P = 0 = (0:1:0)$   
 -  $(m/q) * P = (x:y:z)$  and  $z$  a unit in  $\mathbb{Z}/N$   
 Then  $N$  is prime.

**Example:**

```
LIB "crypto.lib";
bigint N=1267985441;
ECP(N);
⇨ P= [1]:
⇨ 780306204
⇨ [2]:
⇨ 1106324420
⇨ [3]:
⇨ 1
⇨ [1]:
⇨ [1]:
⇨ 67394594
⇨ [2]:
⇨ 380636642
⇨ [2]:
⇨ [1]:
⇨ 780306204
⇨ [2]:
⇨ 1106324420
⇨ [3]:
⇨ 1
⇨ [3]:
⇨ 1267993236
⇨ [4]:
⇨ 105666103
```

**D.12.2.36 calculate\_ordering**

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\]](#), page 1989).

**Usage:** `calculate_ordering(num1, primitive, mod1)`

**Return:**  $x$  so that  $\text{primitive}^x == \text{num1} \bmod \text{mod1}$

**Example:**

```
LIB "crypto.lib";
bigint mod1 = 33;
bigint primitive = 14;
```



```

bigint num1 = 5;
calculate_ordering(num1,primitive,mod1);
⇒ 3

```

### D.12.2.37 is\_primitive\_root

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\]](#), page 1989).

**Usage:** `is_primitive_root(primitive, mod1)`

**Return:** 1 if primitive is a primitive root modulo mod1, 0 otherwise

**Example:**

```

LIB "crypto.lib";
is_primitive_root(3,7);
⇒ 1
is_primitive_root(2,7);
⇒ 0

```

### D.12.2.38 find\_first\_primitive\_root

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\]](#), page 1989).

**Usage:** `find_first_primitive_root(mod1)`

**Return:** First primitive root modulo mod1, 0 if no root can be found.

**Example:**

```

LIB "crypto.lib";
ring r = 0,x,lp;
find_first_primitive_root(7);
⇒ 3
find_first_primitive_root(557);
⇒ 2

```

### D.12.2.39 binary\_add

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\]](#), page 1989).

**Usage:** `binary_add(binary_list)`

**Return:** binary encoded list, increased by 1

**Example:**

```

LIB "crypto.lib";
ring r = 0,x,lp;
list binary_list = 1,0,1,1,1;
binary_add(binary_list);
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ 0
⇒ [4]:
⇒ 0
⇒ [5]:
⇒ 0

```

**D.12.2.40 inverse\_modulus**

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:**      `inverse_modulus(num, mod1)`

**Return:**     inverse element of num modulo mod1

**Example:**

```
LIB "crypto.lib";
ring r = 0,x,lp;
int mod1 = 13;
int num = 5;
inverse_modulus(num,mod1);
↦ 8
```

**D.12.2.41 is\_prime**

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:**      `is_prime(n)`

**Return:**     1 if n is prime, 0 otherwise

**Example:**

```
LIB "crypto.lib";
ring r = 0,x,lp;
is_prime(10);
↦ 0
is_prime(7);
↦ 1
```

**D.12.2.42 find\_index**

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:**      `find_index(a, e)`

**Return:**     Returns the list index of element e in list a. Returns 0 if e is not in a

**Example:**

```
LIB "crypto.lib";
list a = 1,5,20,6,37;
find_index(a,20);
↦ 3
find_index(a,6);
↦ 4
find_index(a,100);
↦ 0
```

**D.12.2.43 subset\_sum01**

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:**      `subset_sum01(knapsack,solution)`

**Return:**     binary list of the positions of the elements included in the subset sum or 0 if no solution exists

**Note:** This will return the first solution of the ssk-problem, given by the smallest binary encoding. It won't return several solutions if they exist

**Example:**

```
LIB "crypto.lib";
list h=1,4,7,32;
subset_sum01(h,20);
⇒ 0
subset_sum01(h,11);
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ 1
⇒ [4]:
⇒ 0
subset_sum01(h,33);
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ 0
⇒ [4]:
⇒ 1
```

#### D.12.2.44 subset\_sum02

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\]](#), page 1989).

**Usage:** subset\_sum02(knapsack,sol)

**Return:** binary list of the positions of the elements included in the subset sum or 0 if no solution exists

**Example:**

```
LIB "crypto.lib";
list h=1,4,7,32;
subset_sum02(h,20);
⇒ 0
subset_sum02(h,11);
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ 1
⇒ [4]:
⇒ 0
subset_sum02(h,33);
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 0
```

```

⇒ [3]:
⇒ 0
⇒ [4]:
⇒ 1

```

### D.12.2.45 unbounded\_knapsack

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `unbounded_knapsack(knapsack,profit,capacity)`

**Return:** list of maximum profit of each iteration. For example, `output_list[2]` contains the maximum profit that can be achieved if the knapsack has capacity 2.

**Example:**

```

LIB "crypto.lib";
list h=1,4,7,32;
list knapsack = 5,2;
list profit = 10,3;
int capacity = 5;
unbounded_knapsack(knapsack,profit,capacity);
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ 3
⇒ [4]:
⇒ 3
⇒ [5]:
⇒ 6
⇒ [6]:
⇒ 10

```

### D.12.2.46 multidimensional\_knapsack

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `multidimensional_knapsack(m,capacities,profits)`

**Return:** binary list of the positions of the elements included in the optimal selection

**Example:**

```

LIB "crypto.lib";
ring r = 0,x,lp;
matrix m[3][3] = 1,4,10,7,8,3,1,9,7;
list c = 12,17,10;
list p = 3,2,5;
multidimensional_knapsack(m,c,p);
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ 1

```

**D.12.2.47 naccache\_stern\_generation**

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `naccache_stern_generation(key, primenum)`

**Return:** a hard knapsack list

**Example:**

```
LIB "crypto.lib";
naccache_stern_generation(5,292);
⇒ 0
naccache_stern_generation(5,293);
⇒ [1]:
⇒ 85
⇒ [2]:
⇒ 164
⇒ [3]:
⇒ 117
⇒ [4]:
⇒ 44
```

**D.12.2.48 naccache\_stern\_encryption**

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `naccache_stern_encryption(knapsack, message, primenum)`

**Return:** an encrypted message as integer

**Example:**

```
LIB "crypto.lib";
//Please note that the values for primenum and hardknapsack have been obtained from t
list hardknapsack = 85,164,117,44;
int primenum = 293;
list message = 1,0,1,0;
naccache_stern_encryption(hardknapsack,message,primenum);
⇒ 9945
```

**D.12.2.49 naccache\_stern\_decryption**

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `naccache_stern_decryption(knapsack, key, primenum, message)`

**Return:** decrypted binary list

**Example:**

```
LIB "crypto.lib";
//Please note that the values have been obtained from the example of naccache_stern_g
int primenum = 293;
int message = 9945;
int key = 5;
list hardknapsack = 85,164,117,44;
naccache_stern_decryption(hardknapsack,key,primenum,message);
⇒ [1]:
⇒ 1
```

```

⇒ [2]:
⇒ 0
⇒ [3]:
⇒ 1
⇒ [4]:
⇒ 0

```

### D.12.2.50 m\_merkle\_hellman\_transformation

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\]](#), page 1989).

**Usage:** `m_merkle_hellman_transformation(knapsack, primitive, mod1)`

**Return:** list containing a hard knapsack

**Example:**

```

LIB "crypto.lib";
//Please note that the values for primenum and hardknapsack have been obtained from t
list knapsack = 2,3,5,7;
int mod1 = 211;
int primitive = 2;
m_merkle_hellman_transformation(knapsack,primitive,mod1);
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 43
⇒ [3]:
⇒ 132
⇒ [4]:
⇒ 139

```

### D.12.2.51 m\_merkle\_hellman\_encryption

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\]](#), page 1989).

**Usage:** `m_merkle_hellman_encryption(knapsack, message)`

**Return:** an encrypted message as integer

**Note:** This works in the same way as `merkle_hellman_encryption`. The additional function is created to keep consistency with the needed functions for every kryptosystem.

**Example:**

```

LIB "crypto.lib";
//Please note that the values for primenum and hardknapsack have been obtained from t
list knapsack = 1,43,132,139;
list message = 1,0,0,1;
m_merkle_hellman_encryption(knapsack,message);
⇒ 140

```

### D.12.2.52 m\_merkle\_hellman\_decryption

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\]](#), page 1989).

**Usage:** `m_merkle_hellman_decryption(knapsack, primitive, mod1, message)`

**Return:** decrypted binary list

**Example:**

```

LIB "crypto.lib";
//Please note that the values have been obtained from the example of m_merkle_hellman
list knapsack = 2,3,5,7;
int message = 140;
bigint primitive = 2;
bigint mod1 = 211;
m_merkle_hellman_decryption(knapsack,primitive,mod1,message);
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 0
⇒ [3]:
⇒ 0
⇒ [4]:
⇒ 1

```

**D.12.2.53 merkle\_hellman\_encryption**

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\]](#), page 1989).

**Usage:**      `merkle_hellman_encryption(knapsack, message)`

**Return:**    encrypted integer

**Example:**

```

LIB "crypto.lib";
//Please note that the values have been obtained from the example of merkle_hellman
list hardknapsack =3,9,15,13;
list message = 0,1,0,1;
merkle_hellman_encryption(hardknapsack,message);
⇒ 22

```

**D.12.2.54 merkle\_hellman\_decryption**

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\]](#), page 1989).

**Usage:**      `merkle_hellman_decryption(knapsack, key, mod1, message)`

**Return:**    decrypted binary list

**Example:**

```

LIB "crypto.lib";
//Please note that the values have been obtained from the example of merkle_hellman
list hardknapsack =3,9,15,13;
int key = 3;
int message = 22;
int mod1 = 23;
merkle_hellman_decryption(hardknapsack, key, mod1, message);
⇒ 15
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 1
⇒ [3]:

```

```

⇒      0
⇒ [4] :
⇒      1

```

### D.12.2.55 `super_increasing_knapsack`

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `super_increasing_knapsack(ksize)`

**Return:** super-increasing knapsack list

**Example:**

```

LIB "crypto.lib";
super_increasing_knapsack(10);
⇒ [1] :
⇒      512
⇒ [2] :
⇒      256
⇒ [3] :
⇒      128
⇒ [4] :
⇒      64
⇒ [5] :
⇒      32
⇒ [6] :
⇒      16
⇒ [7] :
⇒      8
⇒ [8] :
⇒      4
⇒ [9] :
⇒      2
⇒ [10] :
⇒      1

```

### D.12.2.56 `h_increasing_knapsack`

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `h_increasing_knapsack(ksize, h)`

**Return:** h-increasing knapsack list

**Example:**

```

LIB "crypto.lib";
h_increasing_knapsack(10,5);
⇒ [1] :
⇒      1
⇒ [2] :
⇒      2
⇒ [3] :
⇒      4
⇒ [4] :
⇒      8
⇒ [5] :

```



```

↳      16
↳ [6]:
↳      32
↳ [7]:
↳      63
↳ [8]:
↳     124
↳ [9]:
↳     244
↳ [10]:
↳     480

```

### D.12.2.57 injective\_knapsack

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:**      `injective_knapsack(ksize, kmaxelement)`

**Return:**     list of injective knapsacks with maximal element `kmaxelement` and size `ksize`

**Example:**

```

LIB "crypto.lib";
injective_knapsack(3,9);
↳ [1]:
↳   [1]:
↳     9
↳   [2]:
↳     1
↳   [3]:
↳     3
↳ [2]:
↳   [1]:
↳     9
↳   [2]:
↳     1
↳   [3]:
↳     4
↳ [3]:
↳   [1]:
↳     9
↳   [2]:
↳     1
↳   [3]:
↳     5
↳ [4]:
↳   [1]:
↳     9
↳   [2]:
↳     1
↳   [3]:
↳     6
↳ [5]:
↳   [1]:
↳     9

```

```

↳ [2]:
↳ 1
↳ [3]:
↳ 7
↳ [6]:
↳ [1]:
↳ 9
↳ [2]:
↳ 2
↳ [3]:
↳ 1
↳ [7]:
↳ [1]:
↳ 9
↳ [2]:
↳ 2
↳ [3]:
↳ 3
↳ [8]:
↳ [1]:
↳ 9
↳ [2]:
↳ 2
↳ [3]:
↳ 4
↳ [9]:
↳ [1]:
↳ 9
↳ [2]:
↳ 2
↳ [3]:
↳ 5
↳ [10]:
↳ [1]:
↳ 9
↳ [2]:
↳ 2
↳ [3]:
↳ 6
↳ [11]:
↳ [1]:
↳ 9
↳ [2]:
↳ 3
↳ [3]:
↳ 4
↳ [12]:
↳ [1]:
↳ 9
↳ [2]:
↳ 3
↳ [3]:
↳ 5

```

```

⇒ [13]:
⇒   [1]:
⇒     9
⇒   [2]:
⇒     3
⇒   [3]:
⇒     7
⇒ [14]:
⇒   [1]:
⇒     9
⇒   [2]:
⇒     4
⇒   [3]:
⇒     6
⇒ [15]:
⇒   [1]:
⇒     9
⇒   [2]:
⇒     4
⇒   [3]:
⇒     7
⇒ [16]:
⇒   [1]:
⇒     9
⇒   [2]:
⇒     5
⇒   [3]:
⇒     6
⇒ [17]:
⇒   [1]:
⇒     9
⇒   [2]:
⇒     5
⇒   [3]:
⇒     7
⇒ [18]:
⇒   [1]:
⇒     9
⇒   [2]:
⇒     6
⇒   [3]:
⇒     7
⇒ [19]:
⇒   [1]:
⇒     9
⇒   [2]:
⇒     8
⇒   [3]:
⇒     2
⇒ [20]:
⇒   [1]:
⇒     9
⇒   [2]:

```

```

⇒      8
⇒      [3]:
⇒      3
⇒ [21]:
⇒      [1]:
⇒      9
⇒      [2]:
⇒      8
⇒      [3]:
⇒      4
⇒ [22]:
⇒      [1]:
⇒      9
⇒      [2]:
⇒      8
⇒      [3]:
⇒      5
⇒ [23]:
⇒      [1]:
⇒      9
⇒      [2]:
⇒      8
⇒      [3]:
⇒      6
⇒ [24]:
⇒      [1]:
⇒      9
⇒      [2]:
⇒      8
⇒      [3]:
⇒      7

```

### D.12.2.58 `calculate_max_sum`

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:**      `calculate_max_sum(a)`

**Return:**     sum of all elements in `a`

**Example:**

```

LIB "crypto.lib";
list a = 1,5,3,2,12;
calculate_max_sum(a);
⇒ 23

```

### D.12.2.59 `set_is_injective`

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:**      `set_is_injective(a)`

**Return:**     1 if `a` is injective, 0 otherwise

**Example:**

```

LIB "crypto.lib";
list inj = 1,5,7,41;
list non_inj = 1,2,3,4;
set_is_injective(inj);
 $\mapsto$  1
set_is_injective(non_inj);
 $\mapsto$  0

```

### D.12.2.60 is\_h\_injective

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `is_h_injective(a, h)`

**Return:** 1 if `a` is `h`-injective, 0 otherwise

**Example:**

```

LIB "crypto.lib";
list h_inj = 1,2,4,10,17;
is_h_injective(h_inj,3);
 $\mapsto$  1
//1+2+4+10=17
is_h_injective(h_inj,4);
 $\mapsto$  0

```

### D.12.2.61 is\_fix\_injective

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `is_fix_injective(a)`

**Return:** 1 if `a` is fix-injective, 0 otherwise

**Example:**

```

LIB "crypto.lib";
//this is fix-injective because 17=10+2+4+1 with different numbers of addends.
list fix_inj = 1,2,4,10,17;
//this is not fix-injective because 4+1=2+3.
list not_fix_inj = 1,2,3,4;
is_fix_injective(fix_inj);
 $\mapsto$  1
is_fix_injective(not_fix_inj);
 $\mapsto$  0

```

### D.12.2.62 three\_elements

Procedure from library `crypto.lib` (see [Section D.12.2 \[crypto.lib\], page 1989](#)).

**Usage:** `three_elements(out, iterations)`

**Return:** `Injective_knapsack` created with the three elements method

**Example:**

```

LIB "crypto.lib";
//this is fix-injective because 17=10+2+4+1 with different numbers of addends.
list super_increasing = 1,2,4,10,20;
list a = three_elements(super_increasing,2);

```

```

a;
↳ [1]:
↳ 1529
↳ [2]:
↳ 1147
↳ [3]:
↳ 764
↳ [4]:
↳ 153
↳ [5]:
↳ 115
↳ [6]:
↳ 76
↳ [7]:
↳ 20
↳ [8]:
↳ 10
↳ [9]:
↳ 4
↳ [10]:
↳ 2
↳ [11]:
↳ 1
set_is_injective(a);
↳ 1

```

### D.12.3 hyperel\_lib

**Library:** hyperel.lib

**Author:** Markus Hochstetter, markushochstetter@gmx.de

**Note:** The library provides procedures for computing with divisors in the jacobian of hyperelliptic curves. In addition procedures are available for computing the rational representation of divisors and vice versa. The library is intended to be used for teaching and demonstrating purposes but not for efficient computations.

**Procedures:**

#### D.12.3.1 ishyper

Procedure from library `hyperel.lib` (see [Section D.12.3 \[hyperel\\_lib\]](#), page 2016).

**Usage:** ishyper(h,f); h,f=poly

**Return:** 1 if  $y^2+h(x)y=f(x)$  is hyperelliptic, 0 otherwise

**Note:** Tests, if  $y^2+h(x)y=f(x)$  is a hyperelliptic curve.  
Curve is defined over basering. Additionally shows error-messages.

**Example:**

```

LIB "hyperel.lib";
ring R=7,x,dp;
// hyperelliptic curve  $y^2 + h*y = f$ 
poly h=x;
poly f=x5+5x4+6x2+x+3;

```

```
ishyper(h,f);
↦ 1
```

### D.12.3.2 isoncurve

Procedure from library `hyperel.lib` (see [Section D.12.3 \[hyperel.lib\]](#), page 2016).

**Usage:** `isoncurve(P,h,f);`  $h,f$ =poly;  $P$ =list

**Return:** 1 or 0 (if  $P$  is on curve or not)

**Note:** Tests, if  $P=(P[1],P[2])$  is on the hyperelliptic curve  $y^2+h(x)y=f(x)$ . Curve is defined over basering.

**Example:**

```
LIB "hyperel.lib";
ring R=7,x,dp;
// hyperelliptic curve y^2 + h*y = f
poly h=x;
poly f=x5+5x4+6x2+x+3;
list P=2,3;
isoncurve(P,h,f);
↦ 1
```

### D.12.3.3 chinrestp

Procedure from library `hyperel.lib` (see [Section D.12.3 \[hyperel.lib\]](#), page 2016).

**Usage:** `chinrestp(b,moduli);`  $b$ ,  $moduli$ =list of polynomials

**Return:** poly  $x$ , s.t.  $x \equiv b[i] \pmod{moduli[i]}$

**Note:** chinese remainder theorem for polynomials

**Example:**

```
LIB "hyperel.lib";
ring R=7,x,dp;
list b=3x-4, -3x2+1, 1, 4;
list moduli=(x-2)^2, (x-5)^3, x-1, x-6;
chinrestp(b,moduli);
↦ -x6-3x5-x4+2x3-2x2+3x+3
```

### D.12.3.4 norm

Procedure from library `hyperel.lib` (see [Section D.12.3 \[hyperel.lib\]](#), page 2016).

**Usage:** `norm(a,b,h,f);`

**Return:** norm of  $a(x)-b(x)y$  in  $\mathbb{F}[C]$

**Note:** The norm is a polynomial in just one variable.  
Curve  $C: y^2+h(x)y=f(x)$  is defined over basering.

**Example:**

```
LIB "hyperel.lib";
ring R=7,x,dp;
// hyperelliptic curve y^2 + h*y = f
poly h=x;
```

```

poly f=x5+5x4+6x2+x+3;
poly a=x2+1;
poly b=x;
norm(a,b,h,f);
↦ -x7+2x6+3x4-x3+1

```

### D.12.3.5 multi

Procedure from library `hyperel.lib` (see [Section D.12.3 \[hyperel.lib\]](#), page 2016).

**Usage:** `multi(a,b,c,d,h,f);`

**Return:** list L with  $L[1]-L[2]y=(a(x)-b(x)y)*(c(x)-d(x)y)$  in  $\text{IF}[C]$

**Note:** Curve C:  $y^2+h(x)y=f(x)$  is defined over basering.

**Example:**

```

LIB "hyperel.lib";
ring R=7,x,dp;
poly h=x;
poly f=x5+5x4+6x2+x+3;
// hyperelliptic curve y^2 + h*y = f
poly a=x2+1;
poly b=x;
poly c=5;
poly d=-x;
multi(a,b,c,d,h,f);
↦ [1]:
↦ -x7+2x6+x4-x3+2x2-2
↦ [2]:
↦ -2x3-3x

```

### D.12.3.6 divisor

Procedure from library `hyperel.lib` (see [Section D.12.3 \[hyperel.lib\]](#), page 2016).

**Usage:** `divisor(a,b,h,f);` optional: `divisor(a,b,h,f,s);`  $s=0,1$

**Return:** list P

**Note:**  $P[1][3]*(P[1][1], P[1][2]) + \dots + P[\text{size}(P)][3]*$   
 $(P[\text{size}(P)][1], P[\text{size}(P)][2]) - (*)$   $\text{infty} = \text{div}(a(x)-b(x)y)$  if there is an optional parameter  $s!=0$ , then divisor additionally returns a parameter, which says, whether irreducible polynomials occurred during computations or not. Otherwise only warnings are displayed on the monitor. For  $s=0$  nothing happens.  
 Curve C:  $y^2+h(x)y=f(x)$  is defined over basering.

**Example:**

```

LIB "hyperel.lib";
ring R=7,x,dp;
// hyperelliptic curve y^2 + h*y = f
poly h=x;
poly f=x5+5x4+6x2+x+3;
poly a=(x-1)^2*(x-6);
poly b=0;
divisor(a,b,h,f,1);

```



```

⇒ [1]:
⇒ [1]:
⇒ -1
⇒ [2]:
⇒ -3
⇒ [3]:
⇒ 2
⇒ [2]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ -2
⇒ [3]:
⇒ 2
⇒ [3]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ 2
⇒ 0

```

### D.12.3.7 gcddivisor

Procedure from library `hyperel.lib` (see [Section D.12.3 \[hyperel.lib\]](#), page 2016).

**Usage:** `gcddivisor(p,q);`

**Return:** list P

**Note:** gcd of two divisors

**Example:**

```

LIB "hyperel.lib";
ring R=7,x,dp;
// hyperelliptic curve y^2 + h*y = f
poly h=x;
poly f=x5+5x4+6x2+x+3;
// two divisors
list p=list(-1,-3,1),list(1,1,2);
list q=list(1,1,1),list(2,2,1);
gcddivisor(p,q);
⇒ [1]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ 1

```

### D.12.3.8 semidiv

Procedure from library `hyperel.lib` (see [Section D.12.3 \[hyperel.lib\]](#), page 2016).

**Usage:** `semidiv(D,h,f);`

**Return:** list P

**Note:** important: Divisor D has to be semireduced!  
 Computes semireduced divisor  $P[1][3]*(P[1][1], P[1][2]) + \dots + P[\text{size}(P)][3]*$   
 $*(P[\text{size}(P)][1], P[\text{size}(P)][2]) - (*)_{\text{infty}} = \text{div}(D[1], D[2])$   
 Curve  $C: y^2 + h(x)y = f(x)$  is defined over basering.

**Example:**

```
LIB "hyperel.lib";
ring R=7,x,dp;
// hyperelliptic curve y^2 + h*y = f
poly h=x;
poly f=x5+5x4+6x2+x+3;
// Divisor
list D=x2-1,2x-1;
semidiv(D,h,f);
↳ [1]:
↳ [1]:
↳ -1
↳ [2]:
↳ -3
↳ [3]:
↳ 1
↳ [2]:
↳ [1]:
↳ 1
↳ [2]:
↳ 1
↳ [3]:
↳ 1
```

### D.12.3.9 cantoradd

Procedure from library `hyperel.lib` (see [Section D.12.3 \[hyperel.lib\]](#), page 2016).

**Usage:** cantoradd(D,Q,h,f);

**Return:** list P

**Note:** Cantor's Algorithm - composition  
 important: D and Q have to be semireduced!  
 Computes semireduced divisor  $\text{div}(P[1], P[2]) = \text{div}(D[1], D[2]) + \text{div}(Q[1], Q[2])$  The di-  
 visors are defined over the basering.  
 Curve  $C: y^2 + h(x)y = f(x)$  is defined over the basering.

**Example:**

```
LIB "hyperel.lib";
ring R=7,x,dp;
// hyperelliptic curve y^2 + h*y = f
poly h=x;
poly f=x5+5x4+6x2+x+3;
// two divisors in rational representation
list D=x2-1,2x-1;
list Q=x2-3x+2,-3x+1;
cantoradd(D,Q,h,f);
```

```

↪ [1]:
↪      x2-x-2
↪ [2]:
↪      -3x+1

```

### D.12.3.10 cantorred

Procedure from library `hyperel.lib` (see [Section D.12.3 \[hyperel.lib\]](#), page 2016).

**Usage:** `cantorred(D,h,f);`

**Return:** list N

**Note:** Cantor's algorithm - reduction.  
 important: Divisor D has to be semireduced!  
 Computes reduced divisor  $\text{div}(N[1], N[2]) = \text{div}(D[1], D[2])$ .  
 The divisors are defined over the basering.  
 Curve C:  $y^2 + h(x)y = f(x)$  is defined over the basering.

**Example:**

```

LIB "hyperel.lib";
ring R=7,x,dp;
// hyperelliptic curve y^2 + h*y = f
poly h=x;
poly f=x5+5x4+6x2+x+3;
// semireduced divisor
list D=2x4+3x3-3x-2, -x3-2x2+3x+1;
cantorred(D,h,f);
↪ [1]:
↪      x2-2x+2
↪ [2]:
↪      2x-2

```

### D.12.3.11 double

Procedure from library `hyperel.lib` (see [Section D.12.3 \[hyperel.lib\]](#), page 2016).

**Usage:** `double(D,h,f);`

**Return:** list  $Q=2*D$

**Note:** important: Divisor D has to be semireduced!  
 Special case of Cantor's algorithm.  
 Computes reduced divisor  $\text{div}(Q[1], Q[2]) = 2 * \text{div}(D[1], D[2])$ .  
 The divisors are defined over the basering.  
 Curve C:  $y^2 + h(x)y = f(x)$  is defined over the basering.

**Example:**

```

LIB "hyperel.lib";
ring R=7,x,dp;
// hyperelliptic curve y^2 + h*y = f
poly h=x;
poly f=x5+5x4+6x2+x+3;
// reduced divisor
list D=x2-1, 2x-1;
double(D,h,f);

```

```

⇒ [1]:
⇒      x2-2x+1
⇒ [2]:
⇒      3x-2

```

### D.12.3.12 cantormult

Procedure from library `hyperel.lib` (see [Section D.12.3 \[hyperel.lib\]](#), page 2016).

**Usage:** `cantormult(m,D,h,f);`

**Return:** list `res=m*D`

**Note:** important: Divisor `D` has to be semireduced!  
 Uses repeated doublings for a faster computation of the reduced divisor `m*D`.  
 Attention: Factor `m=int`, this means bounded.  
 For `m<0` the inverse of `m*D` is returned.  
 The divisors are defined over the basering.  
 Curve `C: y2+h(x)y=f(x)` is defined over the basering.

**Example:**

```

LIB "hyperel.lib";
ring R=7,x,dp;
// hyperelliptic curve y2 + h*y = f
poly h=x;
poly f=x5+5x4+6x2+x+3;
// reduced divisor
list D=x2-1,2x-1;
cantormult(34,D,h,f);
⇒ [1]:
⇒      x2-3x-3
⇒ [2]:
⇒      x+1

```

### D.12.4 teachstd.lib

**Library:** `teachstd.lib`

**Purpose:** Procedures for teaching standard bases

**Author:** G.-M. Greuel, greuel@mathematik.uni-kl.de

**Note:** The library is intended to be used for teaching purposes, but not for serious computations. Sufficiently high `printlevel` allows to control each step, thus illustrating the algorithms at work. The procedures are implemented exactly as described in the book 'A SINGULAR Introduction to Commutative Algebra' by G.-M. Greuel and G. Pfister (Springer 2002).

**Procedures:**

#### D.12.4.1 ecart

Procedure from library `teachstd.lib` (see [Section D.12.4 \[teachstd.lib\]](#), page 2022).

**Usage:** `ecart(f); f poly or vector`

**Return:** the ecart e of f of type int

**Example:**

```
LIB "teachstd.lib";
ring r=0,(x,y,z),ls;
ecart((y+z+x+xyz)**2);
  ↦ 4
ring s=0,(x,y,z),dp;
ecart((y+z+x+xyz)**2);
  ↦ 0
```

#### D.12.4.2 tail

Procedure from library `teachstd.lib` (see [Section D.12.4 \[teachstd.lib\], page 2022](#)).

**Usage:** `tail(f)`; f poly or vector

**Return:** `f-lead(f)`, the tail of f of type poly

**Example:**

```
LIB "teachstd.lib";
ring r=0,(x,y,z),ls;
tail((y+z+x+xyz)**2);
  ↦ 2yz+y2+2xz+2xy+2xyz2+2xy2z+x2+2x2yz+x2y2z2
ring s=0,(x,y,z),dp;
tail((y+z+x+xyz)**2);
  ↦ 2x2yz+2xy2z+2xyz2+x2+2xy+y2+2xz+2yz+z2
```

#### D.12.4.3 sameComponent

Procedure from library `teachstd.lib` (see [Section D.12.4 \[teachstd.lib\], page 2022](#)).

**Usage:** `sameComponent(f,g)`; f,g poly or vector

**Return:** 1 if f and g are of type poly or if f and g are of type vector and their leading monomials involve the same module component, 0 if not

**Example:**

```
LIB "teachstd.lib";
ring r=0,(x,y,z),dp;
sameComponent([y+z+x,xyz],[z2,xyz]);
  ↦ 1
sameComponent([y+z+x,xyz],[z4,xyz]);
  ↦ 0
sameComponent(y+z+x+xyz, xy+z5);
  ↦ 1
```

#### D.12.4.4 leadmonomial

Procedure from library `teachstd.lib` (see [Section D.12.4 \[teachstd.lib\], page 2022](#)).

**Usage:** `leadmonomial(f)`; f poly or vector

**Return:** the leading monomial of f of type poly

**Note:** if f is of type poly, `leadmonomial(f)=leadmonom(f)`, if f is of type vector and if `leadmonom(f)=m*gen(i)` then `leadmonomial(f)=m`

**Example:**

```
LIB "teachstd.lib";
ring s=0,(x,y,z),(c,dp);
leadmonomial((y+z+x+xyz)^2);
↦ x2y2z2
leadmonomial([(y+z+x+xyz)^2,xyz5]);
↦ x2y2z2
```

**D.12.4.5 monomialLcm**

Procedure from library `teachstd.lib` (see [Section D.12.4 \[teachstd.lib\], page 2022](#)).

**Usage:** `monomialLcm(m,n)`;  $m,n$  of type `poly` or `vector`

**Return:** least common multiple of leading monomials of  $m$  and  $n$ , of type `poly`

**Note:** if  $m = (x_1 \dots x_r)^{(a_1, \dots, a_r)} \cdot \text{gen}(i)$  ( $\text{gen}(i)=1$  if  $m$  is of type `poly`) and  $n = (x_1 \dots x_r)^{(b_1, \dots, b_r)} \cdot \text{gen}(j)$ , then the proc returns  $(x_1 \dots x_r)^{(\max(a_1, b_1), \dots, \max(a_r, b_r))}$  if  $i=j$  and 0 if  $i \neq j$ .

**Example:**

```
LIB "teachstd.lib";
ring r=0,(x,y,z),ds;
monomialLcm(xy2,yz3);
↦ xy2z3
monomialLcm([xy2,xz],[yz3]);
↦ 0
monomialLcm([xy2,xz3],[yz3]);
↦ xy2z3
```

**D.12.4.6 spoly**

Procedure from library `teachstd.lib` (see [Section D.12.4 \[teachstd.lib\], page 2022](#)).

**Usage:** `spoly(f,g[s])`;  $f,g$  `poly` or `vector`,  $s$  `int`

**Return:** the  $s$ -polynomial of  $f$  and  $g$ , of type `poly` or `vector`  
if  $s \neq 0$  the symmetric  $s$ -polynomial (without division) is returned

**Example:**

```
LIB "teachstd.lib";
ring r=0,(x,y,z),ls;
spoly(2x2+x2y,3y3+xyz);
↦ x2y4-2/3x3yz
ring s=0,(x,y,z),(c,dp);
spoly(2x2+x2y,3y3+xyz);
↦ -1/3x3yz+2x2y2
spoly(2x2+x2y,3y3+xyz,1); //symmetric s-poly without division
↦ -x3yz+6x2y2
spoly([5x2+x2y,z5],[x2,y3,y4]); //s-poly for vectors
↦ [5x2,z5-y4,-y5]
```

### D.12.4.7 minEcart

Procedure from library `teachstd.lib` (see [Section D.12.4 \[teachstd.lib\]](#), page 2022).

**Usage:** `minEcart(T,h)`; T ideal or module, h poly or vector

**Return:** element g from T such that `leadmonom(g)` divides `leadmonom(h)`  
`ecart(g)` is minimal with this property (if  $T \neq 0$ );  
 return 0 if T is 0 or  $h = 0$

**Example:**

```
LIB "teachstd.lib";
ring R=0,(x,y,z),dp;
ideal T = x2y+x2,y3+xyz,xyz2+z4;
poly h = x2y2z2+x5+yx3+z6;
minEcart(T,h);"";
↳ x2y+x2
↳
ring S=0,(x,y,z),(c,ds);
module T = [x2+x2y,y2],[y3+xyz,x3-z3],[x3y+z4,0,x2];
vector h = [x3y+x5+x2y2z2+z6,x3];
minEcart(T,h);
↳ [x3y+z4,0,x2]
```

### D.12.4.8 NFMora

Procedure from library `teachstd.lib` (see [Section D.12.4 \[teachstd.lib\]](#), page 2022).

**Usage:** `NFMora(f,G[,s])`; f poly or vector, G ideal or module, s int

**Return:** the Mora normal form of f w.r.t. G, same type as f  
 if  $s \neq 0$  the symmetric s-polynomial (without division) is used

**Note:** Show comments if `printlevel > 0`, pauses computation if `printlevel > 1`

**Example:**

```
LIB "teachstd.lib";
ring r=0,(x,y,z),dp;
poly f = x2y2z2+x5+yx3+z6-3y3;
ideal G = x2y+x2,y3+xyz,xyz2+z6;
NFMora(f,G);"";
↳ x5-x2yz2+x3y-xyz2-3y3
↳
ring s=0,(x,y,z),ds;
poly f = x3y+x5+x2y2z2+z6;
ideal G = x2+x2y,y3+xyz,x3y2+z4;
NFMora(f,G);"";
↳ 0
↳
vector v = [f,x2+x2y];
module M = [x2+x2y,f],[y3+xyz,y3],[x3y2+z4,z2];
NFMora(v,M);
↳ x2*gen(2)+x2y*gen(2)+x3y*gen(1)+x5*gen(1)+x2y2z2*gen(1)+z6*gen(1)
```

### D.12.4.9 prodcrit

Procedure from library `teachstd.lib` (see [Section D.12.4 \[teachstd.lib\], page 2022](#)).

**Usage:** `prodcrit(f,g[o]);`  $f, g$  poly or vector, and optional int argument  $o$

**Return:** 1 if product criterion applies in the same module component, 2 if `lead(f)` and `lead(g)` involve different components, 0 else

**Note:** if product criterion applies we can delete  $(f, g)$  from pairset. This procedure returns 0 if  $o$  is given and is a positive integer, or you may set the attribute "default\_arg" for `prodcrit` to 1.

**Example:**

```
LIB "teachstd.lib";
ring r=0,(x,y,z),dp;
poly f = y3z3+x5+yx3+z6;
poly g = x5+yx3;
prodcrit(f,g);
⇒ 1
vector v = x3z2*gen(1)+x3y*gen(1)+x2y*gen(2);
vector w = y4*gen(1)+y3*gen(2)+xyz*gen(1);
prodcrit(v,w);
⇒ 0
```

### D.12.4.10 chaincrit

Procedure from library `teachstd.lib` (see [Section D.12.4 \[teachstd.lib\], page 2022](#)).

**Usage:** `chaincrit(f,g,h);`  $f, g, h$  poly or module

**Return:** 1 if chain criterion applies, 0 else

**Note:** if chain criterion applies to  $f, g, h$  we can delete  $(g, h)$  from pairset

**Example:**

```
LIB "teachstd.lib";
ring r=0,(x,y,z),dp;
poly f = x2y2z2+x5+yx3+z6;
poly g = x5+yx3;
poly h = y2z5+x5+yx3;
chaincrit(f,g,h);
⇒ 1
vector u = [x2y3-z2,x2y];
vector v = [x2y2+z2,x2-y2];
vector w = [x2y4+z3,x2+y2];
chaincrit(u,v,w);
⇒ 1
```

### D.12.4.11 pairset

Procedure from library `teachstd.lib` (see [Section D.12.4 \[teachstd.lib\], page 2022](#)).

**Usage:** `pairset(G);`  $G$  ideal or module

**Return:** list  $L$ ,  
 $L[1]$  = the pairset of  $G$  as list (not containing pairs for which the product or the chain criterion applies),  
 $L[2]$  = intvec  $v$ ,  $v[1]$  = # product criterion,  $v[2]$  = # chain criterion



**Example:**

```

LIB "teachstd.lib";
ring r=0,(x,y,z),dp;
ideal G = x2y+x2,y3+xyz,xyz2+z4;
pairset(G);"";
↳ [1]:
↳ [1]:
↳ _[1]=x2y+x2
↳ _[2]=y3+xyz
↳ [2]:
↳ _[1]=x2y+x2
↳ _[2]=xyz2+z4
↳ [3]:
↳ _[1]=y3+xyz
↳ _[2]=xyz2+z4
↳ [2]:
↳ 0,0
↳
module T = [x2y3-z2,x2y],[x2y2+z2,x2-y2],[x2y4+z3,x2+y2];
pairset(T);
↳ [1]:
↳ [1]:
↳ _[1]=x2y3*gen(1)+x2y*gen(2)-z2*gen(1)
↳ _[2]=x2y2*gen(1)+x2*gen(2)-y2*gen(2)+z2*gen(1)
↳ [2]:
↳ _[1]=x2y3*gen(1)+x2y*gen(2)-z2*gen(1)
↳ _[2]=x2y4*gen(1)+z3*gen(1)+x2*gen(2)+y2*gen(2)
↳ [2]:
↳ 0,1

```

**D.12.4.12 updatePairs**

Procedure from library `teachstd.lib` (see [Section D.12.4 \[teachstd.lib\]](#), page 2022).

**Usage:**     `updatePairs(P,S,h)`; P list, S ideal or module, h poly or vector  
               P a list of pairs of polys or vectors (obtained from `pairset`)

**Return:**    list Q,  
                $Q[1]$  = the pairset P enlarged by all pairs (f,h), f from S, without pairs for which the  
                   product or the chain criterion applies  
                $Q[2]$  = intvec v,  $v[1]$  = # product criterion,  $v[2]$  = # chain criterion

**Example:**

```

LIB "teachstd.lib";
ring R1=0,(x,y,z),(c,dp);
ideal S = x2y+x2,y3+xyz;
poly h = x2y+xyz;
list P = pairset(S)[1];
P;"";
↳ [1]:
↳ _[1]=x2y+x2
↳ _[2]=y3+xyz
↳
updatePairs(P,S,h);"";

```

```

⇒ [1]:
⇒ [1]:
⇒ _[1]=x2y+x2
⇒ _[2]=y3+xyz
⇒ [2]:
⇒ _[1]=x2y+x2
⇒ _[2]=x2y+xyz
⇒ [2]:
⇒ 0,1
⇒
module T = [x2y3-z2,x2y],[x2y4+z3,x2+y2];
P = pairset(T)[1];
P;"";
⇒ [1]:
⇒ _[1]=[x2y3-z2,x2y]
⇒ _[2]=[x2y4+z3,x2+y2]
⇒
updatePairs(P,T,[x2+x2y,y3+xyz]);
⇒ [1]:
⇒ [1]:
⇒ _[1]=[x2y3-z2,x2y]
⇒ _[2]=[x2y4+z3,x2+y2]
⇒ [2]:
⇒ _[1]=[x2y3-z2,x2y]
⇒ _[2]=[x2y+x2,y3+xyz]
⇒ [2]:
⇒ 0,1

```

#### D.12.4.13 standard

Procedure from library `teachstd.lib` (see [Section D.12.4 \[teachstd.lib\]](#), page 2022).

**Usage:** `standard(i[s]);` id ideal or module, s int

**Return:** a standard basis of id, using generalized Mora's algorithm which is Buchberger's algorithm for global monomial orderings. If  $s!=0$  the symmetric s-polynomial (without division) is used

**Note:** Show comments if `printlevel > 0`, pauses computation if `printlevel > 1`

**Example:**

```

LIB "teachstd.lib";
ring r=0,(x,y,z),dp;
ideal G = x2y+x2,y3+xyz,xyz2+z4;
standard(G);"";
⇒ _[1]=x2y+x2
⇒ _[2]=y3+xyz
⇒ _[3]=xyz2+z4
⇒ _[4]=x3z-x2y
⇒ _[5]=-xz4+x2z2
⇒ _[6]=-y2z4-x2z3
⇒ _[7]=z6+x2yz2
⇒ _[8]=x2z3-x3z
⇒ _[9]=-x2z2+x3
⇒ _[10]=x4-x2yz

```

```

↳
ring s=0,(x,y,z),(c,ds);
ideal G = 2x2+x2y,y3+xyz,3x3y+z4;
standard(G);"";
↳ _[1]=2x2+x2y
↳ _[2]=y3+xyz
↳ _[3]=3x3y+z4
↳ _[4]=-2/3z4+x3y2
↳
standard(G,1);"";          //use symmetric s-poly without division
↳ _[1]=2x2+x2y
↳ _[2]=y3+xyz
↳ _[3]=3x3y+z4
↳ _[4]=-2z4+3x3y2
↳
module M = [2x2,x3y+z4],[3y3+xyz,y3],[5z4,z2];
standard(M);
↳ _[1]=[2x2,x3y+z4]
↳ _[2]=[3y3+xyz,y3]
↳ _[3]=[5z4,z2]
↳ _[4]=[0,-2/3x2y3+x3y4+1/3x4y2z+y3z4+1/3xyz5]
↳ _[5]=[0,-2/5x2z2+x3yz4+z8]
↳ _[6]=[0,-3/5y3z2-1/5xyz3+y3z4]

```

#### D.12.4.14 localstd

Procedure from library `teachstd.lib` (see [Section D.12.4 \[teachstd.lib\], page 2022](#)).

**Usage:** `localstd(id);` `id` = ideal

**Return:** A standard basis for a local degree ordering, using Lazard's method.

**Note:** The procedure homogenizes `id` w.r.t. a new 1st variable `local@t`, computes a SB w.r.t.  $(dp(1), dp)$  and substitutes `local@t` by 1. Hence the result is a SB with respect to an ordering which sorts first w.r.t. the subdegree of the original variables and then refines it with `dp`. This is the local degree ordering `ds`.  
`localstd` may be used in order to avoid cancellation of units and thus to be able to use `option(contentSB)` also for local orderings.

**Example:**

```

LIB "teachstd.lib";
ring R = 0,(x,y,z),ds;
ideal i = xyz+z5,2x2+y3+z7,3z5+y5;
localstd(i);
↳ _[1]=y5+3z5
↳ _[2]=3x4y3z8-4x3y3z9+6x2y4z9+3y5z10
↳ _[3]=3x4z13-4x3z14+6x2yz14+3y2z15
↳ _[4]=3x4yz12-4x3yz13+6x2y2z13+3y3z14
↳ _[5]=2x2z9+x2y2z8+y3z9
↳ _[6]=2x2y4z5+y7z5-3x2yz9
↳ _[7]=6y2z10-3x2y3z8+4xy3z9-3y4z9
↳ _[8]=3x2y2z8+3y3z9+2xy4z8
↳ _[9]=18z14-4xy6z8+3y7z8-9x2yz12
↳ _[10]=xyz+z5
↳ _[11]=3xz6-y4z5

```

```

⇒ _[12]=3y3z6+2xy4z5-3xyz9
⇒ _[13]=y4z5-2xz9-xy2z8
⇒ _[14]=3z10+2xyz9+xy3z8
⇒ _[15]=2x2z5+y3z5-xyz8
⇒ _[16]=y4z-2xz5+yz8
⇒ _[17]=3z6+2xyz5-y2z8
⇒ _[18]=2x2+y3+z7

```

### D.12.5 weierstr.lib

**Library:** weierstr.lib

**Purpose:** Procedures for the Weierstrass Theorems

**Author:** G.-M. Greuel, greuel@mathematik.uni-kl.de

**Procedures:**

#### D.12.5.1 weierstrDiv

Procedure from library `weierstr.lib` (see [Section D.12.5 \[weierstr.lib\]](#), page 2030).

**Usage:** weierstrDiv(g,f,d); g,f=poly, d=integer

**Assume:** f must be general of finite order, say b, in the last ring variable, say T; if not use the procedure `lastvarGeneral` first

**Purpose:** perform the Weierstrass division of g by f up to order d

**Return:** - a list, say l, of two polynomials and an integer, such that  
 $g = l[1]*f + l[2]$ ,  $\deg_T(l[2]) < b$ , up to (including) total degree d  
 - l[3] is the number of iterations used  
 - if f is not T-general, return (0,g)

**Note:** the procedure works for any monomial ordering

**Theory:** the proof of Grauert-Remmert (Analytische Stellenalgebren) is used for the algorithm

**Example:**

```

LIB "weierstr.lib";
ring R = 0,(x,y),ds;
poly f = y - xy2 + x2;
poly g = y;
list l = weierstrDiv(g,f,10); l;"";
⇒ [1]:
⇒ 1+xy-x3+x2y2-2x4y+2x6+x3y3-3x5y2+5x7y+x4y4-5x9-4x6y3+9x8y2+x5y5
⇒ [2]:
⇒ -x2+x5-2x8
⇒ [3]:
⇒ 5
⇒
l[1]*f + l[2]; //g = l[1]*f+l[2] up to degree 10
⇒ y-5x11+14x10y2+5x7y5-9x9y4-x6y7

```

### D.12.5.2 weierstrPrep

Procedure from library `weierstr.lib` (see [Section D.12.5 \[weierstr.lib\], page 2030](#)).

**Usage:** `weierstrPrep(f,d); f=poly, d=integer`

**Assume:** `f` must be general of finite order, say `b`, in the last ring variable, say `T`; if not apply the procedure `lastvarGeneral` first

**Purpose:** perform the Weierstrass preparation of `f` up to order `d`

**Return:** - a list, say `l`, of two polynomials and one integer,  
`l[1]` a unit, `l[2]` a Weierstrass polynomial, `l[3]` an integer such that  $l[1]*f = l[2]$ , where `l[2]` is a Weierstrass polynomial, (i.e.  $l[2] = T^b + \text{lower terms in } T$ ) up to (including) total degree `d` `l[3]` is the number of iterations used  
 - if `f` is not `T`-general, return (0,0)

**Note:** the procedure works for any monomial ordering

**Theory:** the proof of Grauert-Remmert (Analytische Stellenalgebren) is used for the algorithm

**Example:**

```
LIB "weierstr.lib";
ring R = 0,(x,y),ds;
poly f = xy+y2+y4;
list l = weierstrPrep(f,5); l; ""
⇒ [1]:
⇒ 1-x2+xy-y2+3x4-3x3y+3x2y2-2xy3+y4
⇒ [2]:
⇒ xy+y2-x3y
⇒ [3]:
⇒ 6
⇒
f*l[1]-l[2]; // = 0 up to degree 5
⇒ 3x5y+3x4y4-3x3y5+3x2y6-2xy7+y8
```

### D.12.5.3 lastvarGeneral

Procedure from library `weierstr.lib` (see [Section D.12.5 \[weierstr.lib\], page 2030](#)).

**Usage:** `lastvarGeneral(f,d); f=poly`

**Return:** poly, say `g`, obtained from `f` by a generic change of variables, s.t. `g` is general of finite order `b` w.r.t. the last ring variable, say `T` (i.e.  $g(0,\dots,0,T) = c*T^b + \text{higher terms, } c \neq 0$ )

**Note:** the procedure works for any monomial ordering

**Example:**

```
LIB "weierstr.lib";
ring R = 2,(x,y,z),ls;
poly f = xyz;
lastvarGeneral(f);
⇒ z24+yz19+xz6+xyz
```

### D.12.5.4 generalOrder

Procedure from library `weierstr.lib` (see [Section D.12.5 \[weierstr.lib\]](#), page 2030).

**Usage:** `generalOrder(f); f=poly`

**Return:** integer `b` if `f` is general of order `b` w.r.t. the last variable, say `T`, resp. `-1` if not (i.e.  $f(0, \dots, 0, T)$  is of order `b`, resp.  $f(0, \dots, 0, T) = 0$ )

**Note:** the procedure works for any monomial ordering

**Example:**

```
LIB "weierstr.lib";
ring R = 0,(x,y),ds;
poly f = x2-4xy+4y2-2xy2+4y3+y4;
generalOrder(f);
↪ 2
```

### D.12.6 rootsmr.lib

**Library:** `rootsmr.lib`

**Purpose:** Counting the number of real roots of polynomial systems

**Author:** Enrique A. Tobis, `etobis@dc.uba.ar`

**Overview:** Routines for counting the number of real roots of a multivariate polynomial system. Two methods are implemented: deterministic computation of the number of roots, via the signature of a certain bilinear form (`nrRootsDeterm`); and a rational univariate projection, using a pseudorandom polynomial (`nrRootsProbab`). It also includes a command to verify the correctness of the pseudorandom answer.

**References:**

Basu, Pollack, Roy, "Algorithms in Real Algebraic Geometry", Springer, 2003.

**Procedures:**

#### D.12.6.1 nrRootsProbab

Procedure from library `rootsmr.lib` (see [Section D.12.6 \[rootsmr.lib\]](#), page 2032).

**Return:** `int`: the number of real roots of the ideal `I` by a probabilistic algorithm

**Assume:** If `I` is not a Groebner basis, then a Groebner basis will be computed by using `std`. If `I` is already a Groebner basis (i.e. if `attrib(I,"isSB");` returns 1) then this Groebner basis will be used, hence it must be one w.r.t. (any) global ordering. This may be useful if the ideal is known to be a Groebner basis or if it can be computed faster by a different method.

**Note:** If `n<10` is given, `n` is the number of digits being used for constructing a random characteristic polynomial, a bigger `n` is more safe but slower (default: `n=5`).  
If `printlevel>0` the number of complex solutions is displayed (default: `printlevel=0`).

**Example:**

```
LIB "rootsmr.lib";
ring r = 0,(x,y,z),lp;
ideal i = (x-1)*(x-2),(y-1)^3*(x-y),(z-1)*(z-2)*(z-3)^2;
nrRootsProbab(i);          //no of real roots (using internally std)
```

```

⇒ 9
i = groebner(i);          //using the hilbert driven GB computation
int pr = printlevel;
printlevel = 2;
nrRootsProbab(i);
⇒ //ideal has 32 complex solutions, counted with multiplicity
⇒ *****
⇒ * WARNING: This polynomial was obtained using pseudorandom numbers.*
⇒ * If you want to verify the result, please use the command          *
⇒ *                                                                    *
⇒ * verify(p,b,i)                                                    *
⇒ *                                                                    *
⇒ * where p is the polynomial I returned, b is the monomial basis    *
⇒ * used, and i the Groebner basis of the ideal                      *
⇒ *****
⇒ 9
printlevel = pr;

```

See also: [Section D.12.6.2 \[nrRootsDeterm\]](#), page 2033; [Section D.12.7.15 \[nrroots\]](#), page 2047; [Section D.12.6.9 \[randcharpoly\]](#), page 2037; [Section D.8.4.2 \[solve\]](#), page 1845.

### D.12.6.2 nrRootsDeterm

Procedure from library `rootsmr.lib` (see [Section D.12.6 \[rootsmr.lib\]](#), page 2032).

**Return:** int: the number of real roots of the ideal I by a deterministic algorithm

**Assume:** If I is not a Groebner basis, then a Groebner basis will be computed by using `std`. If I is already a Groebner basis (i.e. if `attrib(I,"isSB")` returns 1) then this Groebner basis will be used, hence it must be one w.r.t. (any) global ordering. This may be useful if the ideal is known to be a Groebner basis or if it can be computed faster by a different method.

**Note:** If `printlevel>0` the number of complex solutions is displayed (default: `printlevel=0`). The procedure `nrRootsProbab` is usually faster.

**Example:**

```

LIB "rootsmr.lib";
ring r = 0,(x,y,z),lp;
ideal I = (x-1)*(x-2),(y-1),(z-1)*(z-2)*(z-3)^2;
nrRootsDeterm(I);          //no of real roots (using internally std)
⇒ 6
I = groebner(I);          //using the hilbert driven GB computation
int pr = printlevel;
printlevel = 2;
nrRootsDeterm(I);
⇒ //ideal has 8 complex solutions, counted with multiplicity
⇒ 6
printlevel = pr;

```

See also: [Section D.12.6.1 \[nrRootsProbab\]](#), page 2032; [Section D.12.7.15 \[nrroots\]](#), page 2047; [Section D.8.4.2 \[solve\]](#), page 1845; [Section D.12.6.4 \[sturmquery\]](#), page 2034.

### D.12.6.3 symsignature

Procedure from library `rootsmr.lib` (see [Section D.12.6 \[rootsmr.lib\]](#), page 2032).

**Usage:**      `symsignature(m)`; `m` matrix. `m` must be symmetric.

**Return:**    `int`: the signature of `m`

**Example:**

```
LIB "rootsmr.lib";
ring r = 0,(x,y),dp;
ideal i = x4-y2x,y2-13;
i = std(i);
ideal b = qbase(i);
matrix m = matbil(1,b,i);
symsignature(m);
↪ 4
```

See also: [Section D.12.6.5 \[matbil\]](#), page 2034; [Section D.12.6.4 \[sturmquery\]](#), page 2034.

### D.12.6.4 sturmquery

Procedure from library `rootsmr.lib` (see [Section D.12.6 \[rootsmr.lib\]](#), page 2032).

**Usage:**      `sturmquery(h,b,i)`; `h` poly, `b,i` ideal

**Return:**    `int`: the Sturm query of `h` in  $V(i)$

**Assume:**    `i` is a Groebner basis, `b` is an ordered monomial basis of  $r/i$ , `r` = basering.

**Example:**

```
LIB "rootsmr.lib";
ring r = 0,(x,y),dp;
ideal i = x4-y2x,y2-13;
i = std(i);
ideal b = qbase(i);
sturmquery(1,b,i);
↪ 4
```

See also: [Section D.12.6.5 \[matbil\]](#), page 2034; [Section D.12.6.3 \[symsignature\]](#), page 2033.

### D.12.6.5 matbil

Procedure from library `rootsmr.lib` (see [Section D.12.6 \[rootsmr.lib\]](#), page 2032).

**Usage:**      `matbil(h,b,i)`; `h` poly, `b,i` ideal

**Return:**    `matrix`: the matrix of the bilinear form  $(f,g) \mapsto \text{trace}(m_{fhg})$ ,  $m_{fhg}$  = multiplication with `fhg` on  $r/i$

**Assume:**    `i` is a Groebner basis and `b` is an ordered monomial basis of  $r/i$ , `r` = basering

**Example:**

```
LIB "rootsmr.lib";
ring r = 0,(x,y),dp;
ideal i = x4-y2x,y2-13;
i = std(i);
ideal b = qbase(i);
poly f = x3-xy+y-13+x4-y2x;
matrix m = matbil(f,b,i);
print(m);
↪ 0,      13182, 0,      -13182,0,      0,      0,      1014,
↪ 13182, 0,      -13182,0,      0,      0,      1014, 0,
```



```

⇒ 0,      -13182,0,      0,      0,      1014, 0,      -1014,
⇒ -13182,0,      0,      0,      1014, 0,      -1014,0,
⇒ 0,      0,      0,      1014, 0,      -1014,0,      0,
⇒ 0,      0,      1014, 0,      -1014,0,      0,      0,
⇒ 0,      1014, 0,      -1014, 0,      0,      -338, 104,
⇒ 1014, 0,      -1014, 0,      0,      0,      104, -26

```

See also: [Section D.12.6.6 \[matmult\]](#), page 2035; [Section D.12.6.7 \[tracemult\]](#), page 2035.

### D.12.6.6 matmult

Procedure from library `rootsmr.lib` (see [Section D.12.6 \[rootsmr.lib\]](#), page 2032).

**Usage:** `matmult(f,b,i)`; `f` poly, `b,i` ideal

**Return:** matrix: the matrix of the multiplication map by `f` (`m.f`) on `r/i` w.r.t. to the monomial basis `b` of `r/i` (`r = basering`)

**Assume:** `i` is a Groebner basis and `b` is an ordered monomial basis of `r/i`, as given by `qbase(i)`

**Example:**

```

LIB "rootsmr.lib";
ring r = 0,(x,y),dp;
ideal i = x4-y2x,y2-13;
i = std(i);
ideal b = qbase(i);
poly f = x3-xy+y-13+x4-y2x;
matrix m = matmult(f,b,i);
print(m);
⇒ 0,  1,  0, -1,0,  0, 1,  0,
⇒ 13, 0, -13,0, 0, 0, 0, 1,
⇒ 0,  0,  0, 1, 0, -1,0,  0,
⇒ 0,  0, 13, 0, -13,0, 0,  0,
⇒ 0, -13,0, 0, 0, 1, 0, -1,
⇒ -169,0, 0, 0, 13, 0, -13,0,
⇒ 0,  0,  0, 0, 0, 0, -13,1,
⇒ 0,  0,  0, 0, 0, 0, 13, -13

```

See also: [Section D.12.6.8 \[coords\]](#), page 2036; [Section D.12.6.5 \[matbil\]](#), page 2034.

### D.12.6.7 tracemult

Procedure from library `rootsmr.lib` (see [Section D.12.6 \[rootsmr.lib\]](#), page 2032).

**Usage:** `tracemult(f,B,I)`; `f` poly, `B,I` ideal

**Return:** number: the trace of the multiplication by `f` (`m.f`) on `r/I`, written in the monomial basis `B` of `r/I`, `r = basering` (faster than `matmult + trace`)

**Assume:** `I` is given by a Groebner basis and `B` is an ordered monomial basis of `r/I`

**Example:**

```

LIB "rootsmr.lib";
ring r = 0,(x,y),dp;
ideal i = x4-y2x,y2-13;
i = std(i);
ideal b = qbase(i);
poly f = x3-xy+y-13+x4-y2x;

```

```

matrix m = matmult(f,b,i);
print(m);
⇒ 0, 1, 0, -1,0, 0, 1, 0,
⇒ 13, 0, -13,0, 0, 0, 0, 1,
⇒ 0, 0, 0, 1, 0, -1,0, 0,
⇒ 0, 0, 13, 0, -13,0, 0, 0,
⇒ 0, -13,0, 0, 0, 1, 0, -1,
⇒ -169,0, 0, 0, 13, 0, -13,0,
⇒ 0, 0, 0, 0, 0, 0, -13,1,
⇒ 0, 0, 0, 0, 0, 0, 13, -13
tracemult(f,b,i);          //the trace of m
⇒ -26

```

See also: [Section D.12.6.6 \[matmult\], page 2035](#); [Section 5.1.158 \[trace\], page 281](#).

### D.12.6.8 coords

Procedure from library `rootsmr.lib` (see [Section D.12.6 \[rootsmr.lib\], page 2032](#)).

**Usage:** `coords(f,b,i)`, `f` poly, `b,i` ideal

**Return:** list of numbers: the coordinates of the class of `f (mod i)` in the monomial basis `b`

**Assume:** `i` is a Groebner basis and `b` is an ordered monomial basis of `r/i`, `r = basering`

**Example:**

```

LIB "rootsmr.lib";
ring r = 0,(x,y),dp;
ideal i = x4-y2x,y2-13;
poly f = x3-xy+y-13+x4-y2x;
i = std(i);
ideal b = qbase(i);
b;
⇒ b[1]=x3y
⇒ b[2]=x3
⇒ b[3]=x2y
⇒ b[4]=x2
⇒ b[5]=xy
⇒ b[6]=x
⇒ b[7]=y
⇒ b[8]=1
coords(f,b,i);
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 1
⇒ [3]:
⇒ 0
⇒ [4]:
⇒ 0
⇒ [5]:
⇒ -1
⇒ [6]:
⇒ 0
⇒ [7]:
⇒ 1

```

$\mapsto$  [8]:  
 $\mapsto$  -13

See also: [Section D.12.6.5 \[matbil\]](#), page 2034; [Section D.12.6.6 \[matmult\]](#), page 2035.

### D.12.6.9 randcharpoly

Procedure from library `rootsmr.lib` (see [Section D.12.6 \[rootsmr.lib\]](#), page 2032).

**Usage:** `randcharpoly(b,i); randcharpoly(b,i,n);` `b,i` ideal; `n` int

**Return:** `poly`: the characteristic polynomial of a pseudorandom rational univariate projection having one zero per zero of `i`. If `n<10` is given, it is the number of digits being used for the pseudorandom coefficients (default: `n=5`)

**Assume:** `i` is a Groebner basis and `b` is an ordered monomial basis of `r/i`, `r = basering`

**Note:** shows a warning if `printlevel>0` (default: `printlevel=0`)

**Example:**

```

LIB "rootsmr.lib";
ring r = 0,(x,y,z),dp;
ideal i = (x-1)*(x-2),(y-1),(z-1)*(z-2)*(z-3)^2;
i = std(i);
ideal b = qbase(i);
poly p = randcharpoly(b,i);
p;
 $\mapsto$  z8-1989306z7+1720335326522z6-844575738768293508z5+25739857498506957597394\
5z4-49855396253842786126599566442z3+5991506449298102407845582886576172z2-\
408335865183407651473343362162998177144z+12078261759575784323866334900781\
464660123776
nrroots(p); // See nrroots in urrcount.lib
 $\mapsto$  6
int pr = printlevel;
printlevel = pr+2;
p = randcharpoly(b,i,5);
 $\mapsto$  // poly, 9 monomial(s)
 $\mapsto$  z8-2923964*z7+3712323518934*z6-2671920147197312780*z5+1191863249059288760\
005489*z4-337242235263204293461543939056*z3+59079952041382728808956425746\
100736*z2-5855367303472484622963975143953858560000*z+25120629313761950033\
6395930918610534400000000
 $\mapsto$  *****
 $\mapsto$  * WARNING: This polynomial was obtained using pseudorandom numbers.*
 $\mapsto$  * If you want to verify the result, please use the command *
 $\mapsto$  *
 $\mapsto$  *
 $\mapsto$  * verify(p,b,i) *
 $\mapsto$  *
 $\mapsto$  * where p is the polynomial I returned, b is the monomial basis *
 $\mapsto$  * used, and i the Groebner basis of the ideal *
 $\mapsto$  *****
nrroots(p);
 $\mapsto$  6
printlevel = pr;

```

### D.12.6.10 verify

Procedure from library `rootsmr.lib` (see [Section D.12.6 \[rootsmr.lib\]](#), page 2032).

**Usage:**      `verify(p,B,I);` p poly, B,I,ideal  
**Return:**      integer: 1 if and only if the polynomial p splits the points of  $V(I)$ . It's used to check the result of `randcharpoly`  
**Assume:**      I is given by a Groebner basis and B is an ordered monomial basis of  $r/I$ ,  $r = \text{basing}$   
**Note:**          comments the result if `printlevel>0` (default: `printlevel=0`)

**Example:**

```
LIB "rootsmr.lib";
ring r = 0,(x,y),dp;
poly f = x3-xy+y-13+x4-y2x;
ideal i = x4-y2x,y2-13;
i = std(i);
ideal b = qbase(i);
poly p = randcharpoly(b,i);
verify(p,b,i);
⇒ 1
```

See also: [Section D.12.6.9 \[randcharpoly\]](#), page 2037.

**D.12.6.11 randlinpoly**

Procedure from library `rootsmr.lib` (see [Section D.12.6 \[rootsmr.lib\]](#), page 2032).

**Usage:**      `randlinpoly();` `randlinpoly(n);` n int  
**Return:**      poly: linear combination of the variables of the ring, with pseudorandom coefficients.  
 If `n<10` is given, it is the number of digits being used for the range of the coefficients  
 (default: `n=5`)

**Example:**

```
LIB "rootsmr.lib";
ring r = 0,(x,y,z,w),dp;
poly p = randlinpoly();
p;
⇒ 80035x+36642y+40875z+54263w
randlinpoly(5);
⇒ 68857x+95664y+28174z+34170w
```

See also: [Section D.12.6.9 \[randcharpoly\]](#), page 2037.

**D.12.6.12 powersums**

Procedure from library `rootsmr.lib` (see [Section D.12.6 \[rootsmr.lib\]](#), page 2032).

**Usage:**      `powersums(f,b,i);` f poly; b,i ideal  
**Return:**      list: the powersums of the results of evaluating f at the zeros of I  
**Assume:**      i is a Groebner basis and b is an ordered monomial basis of  $r/i$ ,  $r = \text{basing}$

**Example:**

```
LIB "rootsmr.lib";
ring r = 0,(x,y,z),dp;
ideal i = (x-1)*(x-2),(y-1),(z+5); // V(I) = {(1,1,-5),(2,1,-5)}
i = std(i);
ideal b = qbase(i);
```

```

poly f = x+y+z;
list psums = list(-2-3,4+9); // f evaluated at V(I) gives {-3,-2}
list l = powersums(f,b,i);
psums;
↳ [1]:
↳ -5
↳ [2]:
↳ 13
l;
↳ [1]:
↳ -5
↳ [2]:
↳ 13

```

See also: [Section D.12.6.13 \[symmfunc\]](#), page 2039.

### D.12.6.13 symmfunc

Procedure from library `rootsmr.lib` (see [Section D.12.6 \[rootsmr.lib\]](#), page 2032).

**Usage:**      `symmfunc(s); s list`

**Return:**    `list`: the symmetric functions of the roots of a polynomial, given the power sums of those roots.

**Example:**

```

LIB "rootsmr.lib";
ring r = 0,x,dp;
poly p = (x-1)*(x-2)*(x-3);
list psums = list(1+2+3,1+4+9,1+8+27);
list l = symmfunc(psums);
l;
↳ [1]:
↳ 1
↳ [2]:
↳ -6
↳ [3]:
↳ 11
↳ [4]:
↳ -6
p; // Compare p with the elements of l
↳ x3-6x2+11x-6

```

See also: [Section D.12.6.12 \[powersums\]](#), page 2038.

### D.12.6.14 univarpoly

Procedure from library `rootsmr.lib` (see [Section D.12.6 \[rootsmr.lib\]](#), page 2032).

**Usage:**      `univarpoly(l); l list`

**Return:**    `poly`: a polynomial `p` on the first variable of basering, say `x`, with  $p = l[1] + l[2]*x + l[3]*x^2 + \dots$

**Example:**

```

LIB "rootsmr.lib";
ring r = 0,x,dp;

```

```
list l = list(1,2,3,4,5);
poly p = univarpoly(1);
p;
↪ x4+2x3+3x2+4x+5
```

### D.12.6.15 qbase

Procedure from library `rootsmr.lib` (see [Section D.12.6 \[rootsmr.lib\]](#), page 2032).

**Usage:** `qbase(I)`; I zero-dimensional ideal

**Return:** ideal: A monomial basis of the quotient between the basering and the ideal I, sorted according to the basering order.

**Example:**

```
LIB "rootsmr.lib";
ring r = 0, (x,y,z), dp;
ideal i = 2x2,-y2,z3;
i = std(i);
ideal b = qbase(i);
b;
↪ b[1]=xyz2
↪ b[2]=xyz
↪ b[3]=xz2
↪ b[4]=yz2
↪ b[5]=xy
↪ b[6]=xz
↪ b[7]=yz
↪ b[8]=z2
↪ b[9]=x
↪ b[10]=y
↪ b[11]=z
↪ b[12]=1
b = kbase(i);
b; // Compare this with the result of qbase
↪ b[1]=xyz2
↪ b[2]=yz2
↪ b[3]=xz2
↪ b[4]=z2
↪ b[5]=xyz
↪ b[6]=yz
↪ b[7]=xz
↪ b[8]=z
↪ b[9]=xy
↪ b[10]=y
↪ b[11]=x
↪ b[12]=1
```

See also: [Section 5.1.69 \[kbase\]](#), page 205.

### D.12.7 rootsur\_lib

**Library:** `rootsur.lib`

**Purpose:** Counting number of real roots of univariate polynomial

**Author:** Enrique A. Tobis, etobis@dc.uba.ar

**Overview:** Routines for bounding and counting the number of real roots of a univariate polynomial, by means of several different methods, namely Descartes' rule of signs, the Budan-Fourier theorem, Sturm sequences and Sturm-Habicht sequences. The first two give bounds on the number of roots. The other two compute the actual number of roots of the polynomial. There are several wrapper functions, to simplify the application of the aforesaid theorems and some functions to determine whether a given polynomial is univariate.

**References:**

Basu, Pollack, Roy, "Algorithms in Real Algebraic Geometry", Springer, 2003.

**Procedures:**

### D.12.7.1 isuni

Procedure from library `rootsur.lib` (see [Section D.12.7 \[rootsur.lib\]](#), page 2040).

**Usage:** `isuni(p); poly p;`

**Return:** `poly`: if `p` is a univariate polynomial, it returns the variable. If not, zero.

**Example:**

```
LIB "rootsur.lib";
ring r = 0,(x,y),dp;
poly p = 6x7-3x2+2x-15/7;
isuni(p);
  ↳ x
isuni(p*y);
  ↳ 0
```

See also: [Section D.12.7.2 \[whichvariable\]](#), page 2041.

### D.12.7.2 whichvariable

Procedure from library `rootsur.lib` (see [Section D.12.7 \[rootsur.lib\]](#), page 2040).

**Usage:** `whichvariable(p); poly p`

**Return:** `poly`: if `p` is a univariate monomial, the variable. Otherwise 0.

**Assume:** `p` is a monomial

**Example:**

```
LIB "rootsur.lib";
ring r = 0,(x,y),dp;
whichvariable(x5);
  ↳ x
whichvariable(x3y);
  ↳ 0
```

See also: [Section D.12.7.1 \[isuni\]](#), page 2041.

### D.12.7.3 varsigns

Procedure from library `rootsur.lib` (see [Section D.12.7 \[rootsur.lib\]](#), page 2040).

**Usage:**      `varsigns(l)`; list `l`.

**Return:**     `int`: the number of sign changes in the list `l`

**Example:**

```
LIB "rootsur.lib";
ring r = 0,x,dp;
list l = 1,2,3;
varsigns(l);
↪ 0
l = 1,-1,2,-2,3,-3;
varsigns(l);
↪ 5
```

See also: [Section D.12.7.5 \[boundposDes\]](#), page 2042.

### D.12.7.4 boundBuFou

Procedure from library `rootsur.lib` (see [Section D.12.7 \[rootsur.lib\]](#), page 2040).

**Usage:**      `boundBuFou(p,a,b)`; `p` poly, `a,b` number

**Return:**     `int`: an upper bound for the number of real roots of `p` in  $(a,b]$ , with the same parity as the actual number of roots (using the Budan-Fourier Theorem)

**Assume:**    - `p` is a univariate polynomial with rational coefficients  
               - `a`, `b` are rational numbers with  $a < b$

**Example:**

```
LIB "rootsur.lib";
ring r = 0,x,dp;
poly p = (x+2)*(x-1)*(x-5);
boundBuFou(p,-3,5);
↪ 3
boundBuFou(p,-2,5);
↪ 2
```

See also: [Section D.12.7.5 \[boundposDes\]](#), page 2042; [Section D.12.7.3 \[varsigns\]](#), page 2042.

### D.12.7.5 boundposDes

Procedure from library `rootsur.lib` (see [Section D.12.7 \[rootsur.lib\]](#), page 2040).

**Usage:**      `boundposDes(p)`; poly `p`

**Return:**     `int`: an upper bound for the number of positive roots of `p`, with the same parity as the actual number of positive roots of `p`.

**Assume:**    `p` is a univariate polynomial with rational coefficients

**Example:**

```
LIB "rootsur.lib";
ring r = 0,x,dp;
poly p = (x+2)*(x-1)*(x-5);
boundposDes(p);
```



```

    ↪ 2
    p = p*(x2+1);
    boundposDes(p);
    ↪ 4

```

See also: [Section D.12.7.4 \[boundBuFou\]](#), page 2042.

### D.12.7.6 boundDes

Procedure from library `rootsur.lib` (see [Section D.12.7 \[rootsur.lib\]](#), page 2040).

**Usage:**      `boundDes(p); poly p`

**Return:**    `int`: an upper bound for the number of real roots of `p`, with the same parity as the actual number of real roots of `p`.

**Assume:**    `p` is a univariate polynomial with rational coefficients

**Example:**

```

LIB "rootsur.lib";
ring r = 0,x,dp;
poly p = (x+2)*(x-1)*(x-5);
boundDes(p);
↪ 3
p = p*(x2+1);
boundDes(p);
↪ 5

```

See also: [Section D.12.7.4 \[boundBuFou\]](#), page 2042.

### D.12.7.7 allrealst

Procedure from library `rootsur.lib` (see [Section D.12.7 \[rootsur.lib\]](#), page 2040).

**Usage:**      `allrealst(p); poly p`

**Return:**    `int`: 1 if and only if all the roots of `p` are real, 0 otherwise. Checks by using Sturm's Theorem whether all the roots of `p` are real

**Assume:**    `p` is a univariate polynomial with rational coefficients

**Example:**

```

LIB "rootsur.lib";
ring r = 0,x,dp;
poly p = (x+2)*(x-1)*(x-5);
allrealst(p);
↪ 1
p = p*(x2+1);
allrealst(p);
↪ 0

```

See also: [Section D.12.7.9 \[allreal\]](#), page 2044; [Section D.12.7.10 \[sturm\]](#), page 2044; [Section D.12.7.12 \[sturmha\]](#), page 2045.

### D.12.7.8 maxabs

Procedure from library `rootsur.lib` (see [Section D.12.7 \[rootsur.lib\]](#), page 2040).

**Usage:**      `maxabs(p);` poly `p`

**Return:**     number: an upper bound for the largest absolute value of a root of `p`

**Assume:**    `p` is a univariate polynomial with rational coefficients

**Example:**

```
LIB "rootsur.lib";
ring r = 0,x,dp;
poly p = (x+2)*(x-1)*(x-5);
maxabs(p);
↪ 11
```

See also: [Section D.12.7.10 \[sturm\]](#), page 2044.

### D.12.7.9 allreal

Procedure from library `rootsur.lib` (see [Section D.12.7 \[rootsur.lib\]](#), page 2040).

**Usage:**      `allreal(p);`

**Return:**     int: 1 if and only if all the roots of `p` are real, 0 otherwise

**Example:**

```
LIB "rootsur.lib";
ring r = 0,x,dp;
poly p = (x+2)*(x-1)*(x-5);
allreal(p);
↪ 1
p = p*(x2+1);
allreal(p);
↪ 0
```

See also: [Section D.12.7.7 \[allrealst\]](#), page 2043.

### D.12.7.10 sturm

Procedure from library `rootsur.lib` (see [Section D.12.7 \[rootsur.lib\]](#), page 2040).

**Usage:**      `sturm(p,a,b);` poly `p`, number `a,b`

**Return:**     int: the number of real roots of `p` in  $(a,b]$

**Assume:**    `p` is a univariate polynomial with rational coefficients,  
              `a, b` are rational numbers with  $a < b$

**Example:**

```
LIB "rootsur.lib";
ring r = 0,x,dp;
poly p = (x+2)*(x-1)*(x-5);
sturm(p,-3,6);
↪ 3
p = p*(x2+1);
sturm(p,-3,6);
↪ 3
```

```

p = p*(x+2);
sturm(p,-3,6);
↦ 3

```

See also: [Section D.12.7.9 \[allreal\]](#), page 2044; [Section D.12.7.7 \[allrealst\]](#), page 2043; [Section D.12.7.12 \[sturmha\]](#), page 2045.

### D.12.7.11 sturmseq

Procedure from library `rootsur.lib` (see [Section D.12.7 \[rootsur.lib\]](#), page 2040).

**Usage:** `sturmseq(p);` p poly

**Return:** list: a Sturm sequence of p

**Assume:** p is a univariate polynomial with rational coefficients

**Theory:** The Sturm sequence of p (also called remainder sequence) is the sequence beginning with p, p' and goes on with the negative part of the remainder of the two previous polynomials, until the remainder is zero.

See: Basu, Pollack, Roy, Algorithms in Real Algebraic Geometry, Springer, 2003.

**Example:**

```

LIB "rootsur.lib";
ring r = 0,(z,x),dp;
poly p = x5-3x4+12x3+7x-153;
sturmseq(p);
↦ [1]:
↦ x5-3x4+12x3+7x-153
↦ [2]:
↦ x4-12/5x3+36/5x2+7/5
↦ [3]:
↦ -x3-9/7x2-5/3x+317/7
↦ [4]:
↦ -x2-756/151x+2433/151
↦ [5]:
↦ x-514191/177889
↦ [6]:
↦ 1

```

See also: [Section D.12.7.10 \[sturm\]](#), page 2044; [Section D.12.7.13 \[sturmhaseq\]](#), page 2046.

### D.12.7.12 sturmha

Procedure from library `rootsur.lib` (see [Section D.12.7 \[rootsur.lib\]](#), page 2040).

**Usage:** `sturmha(p,a,b);` poly p, number a,b

**Return:** int: the number of real roots of p in (a,b) (using a Sturm-Habicht sequence)

**Example:**

```

LIB "rootsur.lib";
ring r = 0,x,dp;
poly p = (x+2)*(x-1)*(x-5);
sturmha(p,-3,6);
↦ 3
p = p*(x2+1);
sturmha(p,-3,6);

```

↦ 3

See also: [Section D.12.7.9 \[allreal\]](#), page 2044; [Section D.12.7.10 \[sturm\]](#), page 2044.

### D.12.7.13 sturmhaseq

Procedure from library `rootsur.lib` (see [Section D.12.7 \[rootsur.lib\]](#), page 2040).

**Usage:**      `sturmhaseq(P)`; P poly.

**Return:**    list: the non-zero polynomials of the Sturm-Habicht sequence of P

**Assume:**    P is a univariate polynomial.

**Theory:**    The Sturm-Habicht sequence (also subresultant sequence) is closely related to the Sturm sequence, but behaves better with respect to the size of the coefficients. It is defined via subresultants. See: Basu, Pollack, Roy, *Algorithms in Real Algebraic Geometry*, Springer, 2003.

**Example:**

```
LIB "rootsur.lib";
ring r = 0,x,dp;
poly p = x5-x4+x-3/2;
list l = sturmhaseq(p);
l;
↦ [1]:
↦ 132949/16
↦ [2]:
↦ -25x-332
↦ [3]:
↦ -16x2+42x-24
↦ [4]:
↦ 4x3-20x+73/2
↦ [5]:
↦ 5x4-4x3+1
↦ [6]:
↦ x5-x4+x-3/2
```

See also: [Section D.12.7.10 \[sturm\]](#), page 2044; [Section D.12.7.12 \[sturmha\]](#), page 2045; [Section D.12.7.11 \[sturmseq\]](#), page 2045.

### D.12.7.14 reverse

Procedure from library `rootsur.lib` (see [Section D.12.7 \[rootsur.lib\]](#), page 2040).

**Usage:**      `reverse(l)`; l list

**Return:**    list: l reversed.

**Example:**

```
LIB "rootsur.lib";
ring r = 0,x,dp;
list l = 1,2,3,4,5;
list rev = reverse(l);
l;
↦ [1]:
↦ 1
↦ [2]:
```

```

    ↪      2
    ↪ [3] :
    ↪      3
    ↪ [4] :
    ↪      4
    ↪ [5] :
    ↪      5
    rev;
    ↪ [1] :
    ↪      5
    ↪ [2] :
    ↪      4
    ↪ [3] :
    ↪      3
    ↪ [4] :
    ↪      2
    ↪ [5] :
    ↪      1

```

### D.12.7.15 nrroots

Procedure from library `rootsur.lib` (see [Section D.12.7 \[rootsur.lib\]](#), page 2040).

**Usage:**      `nrroots(p); poly p`

**Return:**    `int`: the number of real roots of `p`

**Example:**

```

LIB "rootsur.lib";
ring r = 0,x,dp;
poly p = (x+2)*(x-1)*(x-5);
nrroots(p);
↪ 3
p = p*(x2+1);
nrroots(p);
↪ 3

```

See also: [Section D.12.7.5 \[boundposDes\]](#), page 2042; [Section D.12.7.10 \[sturm\]](#), page 2044; [Section D.12.7.12 \[sturmha\]](#), page 2045.

### D.12.7.16 isparam

Procedure from library `rootsur.lib` (see [Section D.12.7 \[rootsur.lib\]](#), page 2040).

**Usage:**      `isparam(ideal/module/poly/list);`

**Return:**    `int`: 0 if the argument has non-parametric coefficients and 1 if it has parametric coefficients

**Example:**

```

LIB "rootsur.lib";
ring r = 0,x,dp;
isparam(2x3-56x+2);
↪ 0
ring s = (0,a,b,c),x,dp;
isparam(2x3-56x+2);

```

```

    ↦ 0
    isparam(2x3-56x+abc);
    ↦ 1

```

## D.13 Tropical Geometry

### D.13.1 cimonom\_lib

**Library:** cimonom.lib

**Purpose:** Determines if the toric ideal of an affine monomial curve is a complete intersection

**Authors:** I.Bermejo, ibermejo@ull.es  
 I.Garcia-Marco, iggarcia@ull.es  
 J.-J.Salazar-Gonzalez, jjsalaza@ull.es

**Overview:** A library for determining if the toric ideal of an affine monomial curve is a complete intersection with NO NEED of computing explicitly a system of generators of such ideal. It also contains procedures to obtain the minimum positive multiple of an integer which is in a semigroup of positive integers. The procedures are based on a paper by Isabel Bermejo, Ignacio Garcia and Juan Jose Salazar-Gonzalez: 'An algorithm to check whether the toric ideal of an affine monomial curve is a complete intersection', Preprint.

**Procedures:** See also: [Section C.6.4 \[Integer programming\]](#), page 784.

#### D.13.1.1 BelongSemig

Procedure from library cimonom.lib (see [Section D.13.1 \[cimonom.lib\]](#), page 2048).

**Usage:** BelongSemig (n,v[,sup]); n bigint, v and sup intvec

**Return:** In the default form, it returns 1 if n is in the semigroup generated by the elements of v or 0 otherwise. If the argument sup is added and in case n belongs to the semigroup generated by the elements of v, it returns a monomial in the variables {x(i) | i in sup} of degree n if we set  $\deg(x(\sup[j])) = v[j]$ .

**Assume:** v and sup positive integer vectors of same size, sup has no repeated entries, x(i) has to be an indeterminate in the current ring for all i in sup.

**Example:**

```

LIB "cimonom.lib";
ring r=0,x(1..5),dp;
int a = 125;
intvec v = 13,17,51;
intvec sup = 2,4,1;
BelongSemig(a,v,sup);
↦ x(2)^7*x(4)^2
BelongSemig(a,v);
↦ 1

```

### D.13.1.2 MinMult

Procedure from library `cimonom.lib` (see [Section D.13.1 \[cimonom.lib\], page 2048](#)).

**Usage:** MinMult (a, b); a integer, b integer vector.

**Return:** an integer k, the minimum positive integer such that  $ka$  belongs to the semigroup generated by the integers in b.

**Assume:** a is a positive integer, b is a positive integers vector.

**Example:**

```
LIB "cimonom.lib";
"int a = 46;";
↪ int a = 46;
"intvec b = 13,17,59;";
↪ intvec b = 13,17,59;
"MinMult(a,b);";
↪ MinMult(a,b);
int a = 46;
intvec b = 13,17,59;
MinMult(a,b);
↪ 3
"// 3*a = 8*b[1] + 2*b[2]"
```

### D.13.1.3 CompInt

Procedure from library `cimonom.lib` (see [Section D.13.1 \[cimonom.lib\], page 2048](#)).

**Usage:** CompInt(d); d intvec.

**Return:** 1 if the toric ideal  $I(d)$  is a complete intersection or 0 otherwise.

**Assume:** d is a vector of positive integers.

**Note:** If `printlevel > 0`, additional info is displayed in case  $I(d)$  is a complete intersection: if `printlevel >= 1`, it displays a minimal set of generators of the toric ideal formed by quasihomogeneous binomials. Moreover, if `printlevel >= 2` and  $\gcd(d) = 1$ , it also shows the Frobenius number of the semigroup generated by the elements in d.

**Example:**

```
LIB "cimonom.lib";
printlevel = 0;
intvec d = 14,15,10,21;
CompInt(d);
↪ 1
printlevel = 3;
d = 36,54,125,150,225;
CompInt(d);
↪ // Toric ideal:
↪ id[1]=-x(1)^3+x(2)^2
↪ id[2]=-x(4)^3+x(5)^2
↪ id[3]=-x(3)^3+x(4)*x(5)
↪ id[4]=-x(1)^11*x(2)+x(4)^3
↪ // Frobenius number of the numerical semigroup:
↪ 793
↪ 1
```

```

d = 45,70,75,98,147;
CompInt(d);
↦ 0

```

### D.13.2 gfan.lib

**Library:** gfan.lib

**Purpose:** Interface to gfan and gfanlib for computations in convex geometry

**Authors:** Anders N. Jensen, email: jensen@imf.au.dk  
 Yue Ren, email: ren@mathematik.uni-kl.de  
 Frank Seelisch

**Procedures:**

#### D.13.2.1 fullSpace

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\], page 2050](#)).

**Usage:** fullSpace(n); n int

**Return:** cone, the ambient space of dimension n

**Example:**

```

LIB "gfan.lib";
cone c = fullSpace(2);
generatorsOfLinealitySpace(c);
↦ -1, 0,
↦ 0, -1

```

#### D.13.2.2 origin

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\], page 2050](#)).

**Usage:** origin(n); n int

**Return:** cone, the origin in an ambient space of dimension n

**Example:**

```

LIB "gfan.lib";
cone c = origin(2);
equations(c);
↦ 1,0,
↦ 0,1

```

#### D.13.2.3 positiveOrthant

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\], page 2050](#)).

**Usage:** positiveOrthant(n); n int

**Return:** cone, the positive orthant of dimension n

**Example:**

```

LIB "gfan.lib";
cone c = positiveOrthant(2);
rays(c);
↦ 1,0,
↦ 0,1

```



### D.13.2.4 ambientDimension

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:**      `ambientDimension(c)`; `c` cone  
               `ambientDimension(f)`; `f` fan  
               `ambientDimension(p)`; `p` polytope

**Return:**    `int`, the dimension of the ambient space the input lives in

**Example:**

```
LIB "gfan.lib";
intmat M1[2][2]=
1,0,
0,1;
cone c1=coneViaPoints(M1);
ambientDimension(c1);
⇒ 2
intmat M2[2][3]=
1,0,0,
0,1,0;
cone c2=coneViaPoints(M2);
ambientDimension(c2);
⇒ 3
fan f = emptyFan(3);
ambientDimension(f);
⇒ 3
```

### D.13.2.5 canonicalizeCone

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:**      `canonicalizeCone(c)`; `c` cone

**Return:**    `cone`, a unique representation of the cone `c`

**Example:**

```
LIB "gfan.lib";
intmat M[5][3]=
8,1,9,
9,2,4,
0,6,2,
8,8,8,
0,9,5;
cone c=coneViaInequalities(M);
c;
⇒ AMBIENT_DIM
⇒ 3
⇒ INEQUALITIES
⇒ 0,3,1,
⇒ 0,9,5,
⇒ 1,1,1,
⇒ 8,1,9,
⇒ 9,2,4
⇒ LINEAR_SPAN
⇒
```

```

↳
cone cc=canonicalizeCone(c);
cc;
↳ AMBIENT_DIM
↳ 3
↳ FACETS
↳ 0,3,1,
↳ 0,9,5,
↳ 8,1,9,
↳ 9,2,4
↳ LINEAR_SPAN
↳
↳
// computes a unique representation of c
c == cc;
↳ 1
// some procedures work with the known inequalities and equations
// in order to obtain a unique output,
// bring the cone in canonical form beforehand
relativeInteriorPoint(c);
↳ 7,4,-6
relativeInteriorPoint(cc);
↳ 7,4,-6

```

### D.13.2.6 codimension

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:**      `codimension(c); c cone`  
               `codimension(f); f fan`  
               `codimension(p); p polytope`

**Return:**    `int`, the codimension of the input

**Example:**

```

LIB "gfan.lib";
intmat M1[1][2]=
1,0;
cone c1=coneViaPoints(M1);
codimension(c1);
↳ 1
intmat M2[2][2]=
1,0,
0,1;
cone c2=coneViaPoints(M2);
codimension(c2);
↳ 0
fan f = emptyFan(2);
codimension(f);
↳ -1
insertCone(f,c1);
codimension(f);
↳ 1
insertCone(f,c2);

```

```
codimension(f);
↳ 0
```

### D.13.2.7 coneViaPoints

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:**      `coneViaPoints(HL);` intmat HL  
                  `coneViaPoints(HL,L);` intmat HL, intmat L  
                  `coneViaPoints(HL,L,flags);` intmat HL, intmat L, int flags

**Return:**     cone

**Purpose:**     cone generated by half lines generated by the row vectors of HL and (if stated) by lines generated by the row vectors of L; flags may range between 0,...,3 defining an upper and lower bit ( $0=0*2+0$ ,  $1=0*2+1$ ,  $2=1*2+0$ ,  $3=1*2+1$ ),  
                  if upper bit is 1, then program assumes that each row vector in HL generates a ray of the cone,  
                  if lower bit is 1, then program assumes that the span of the row vectors of L is the lineality space of the cone,  
                  if either bit is 0, then program computes the information itself.

**Example:**

```
LIB "gfan.lib";
// Let's define a cone in R^3 generated by the following half lines:
intmat HL[5][3]=
1,0, 0,
-1,0, 0,
0,1, 1,
0,1,-1,
0,0, 1;
cone c=coneViaPoints(HL);
c;
↳ AMBIENT_DIM
↳ 3
↳ FACETS
↳ 0,1,0,
↳ 0,1,1
↳ LINEAR_SPAN
↳
↳
kill HL,c;
// Note that (1,0,0) and (-1,0,0) form a line, hence also possible:
intmat HL[3][3]=
0,1, 1,
0,1,-1,
0,0, 1;
intmat L[1][3]=
1,0,0;
cone c=coneViaPoints(HL,L);
c;
↳ AMBIENT_DIM
↳ 3
↳ FACETS
```

```

⇒ 0,1,0,
⇒ 0,1,1
⇒ LINEAR_SPAN
⇒
⇒
kill HL,L,c;
// lineality space is exactly Lin(1,0,0)
intmat HL[3][3]=
0,1, 1,
0,1,-1,
0,0, 1;
intmat L[1][3]=
1,0,0;
cone c=coneViaPoints(HL,L,1);
c;
⇒ AMBIENT_DIM
⇒ 3
⇒ FACETS
⇒ 0,1,0,
⇒ 0,1,1
⇒ LINEAR_SPAN
⇒
⇒
kill HL,L,c;
// and that (0,1,-1), (0,1,1) generate rays
intmat HL[3][3]=
0,1, 1,
0,1,-1;
intmat L[1][3]=
1,0,0;
cone c=coneViaPoints(HL,L,1);
c;
⇒ AMBIENT_DIM
⇒ 3
⇒ FACETS
⇒ 0,1,-1,
⇒ 0,1, 1
⇒ LINEAR_SPAN
⇒
⇒
kill HL,L,c;
// and that (0,1,-1), (0,1,1) generate rays
intmat HL[3][3]=
0,1, 1,
0,1,-1;
intmat L[1][3]=
1,0,0;
cone c=coneViaPoints(HL,L,3);
c;
⇒ AMBIENT_DIM
⇒ 3
⇒ FACETS
⇒ 0,1,-1,

```

```

⇒ 0,1, 1
⇒ LINEAR_SPAN
⇒
⇒

```

### D.13.2.8 coneViaInequalities

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:**      `coneViaInequalities(IE); intmat IE`  
               `coneViaInequalities(IE,E); intmat IE, intmat E`  
               `coneViaInequalities(IE,E,flags); intmat IE, intmat E, int flags`

**Return:**     `cone`

**Purpose:**     cone consisting of all points  $x$ , such that  $IE \cdot x \geq 0$  in each component and (if stated)  $E \cdot x = 0$ ;  
               inequalities and (if stated) equations will be transformed, getting rid of redundancies;  
               flags may range between 0,...,3 defining an upper and lower bit ( $0=0 \cdot 2+0$ ,  $1=0 \cdot 2+1$ ,  $2=1 \cdot 2+0$ ,  $3=1 \cdot 2+1$ ),  
               if higher bit is 1, then program assumes each inequality yields a facet, if lower bit is 1, then program assumes the kernel of  $E$  is the span of the cone, if either bit is 0, then program computes the information itself.

**Example:**

```

LIB "gfan.lib";
// Let's define a cone in R^3 given by the following inequalities:
intmat IE[6][3]=
1,3,5,
1,5,3,
0,1,-1,
0,1,1,
1,0,0,
-1,0,0;
cone c=coneViaInequalities(IE);
c;
⇒ AMBIENT_DIM
⇒ 3
⇒ INEQUALITIES
⇒ 0,1,-1,
⇒ 0,1, 1,
⇒ 1,3, 5,
⇒ 1,5, 3
⇒ LINEAR_SPAN
⇒ -1,0,0
⇒
// Note that the last two inequalities yield  $x_1 = 0$ , hence also possible:
intmat IE[4][3]=
⇒ // ** redefining IE (intmat IE[4][3]=) ./examples/coneViaInequalities.sin\
g:13
0,1,-1,
0,1,1;
intmat E[1][3]=
1,0,0;

```

```

cone c=coneViaInequalities(IE,E);
⇒ // ** redefining c (cone c=coneViaInequalities(IE,E);) ./examples/coneVia\
  Inequalities.sing:18
c;
⇒ AMBIENT_DIM
⇒ 3
⇒ INEQUALITIES
⇒ 0,1,-1,
⇒ 0,1, 1
⇒ LINEAR_SPAN
⇒ 1,0,0
⇒
// each inequalities gives rise to a facet
intmat IE[2][3]=
⇒ // ** redefining IE (intmat IE[2][3]=) ./examples/coneViaInequalities.sin\
  g:21
0,1,-1,
0,1,1;
intmat E[1][3]=
⇒ // ** redefining E (intmat E[1][3]=) ./examples/coneViaInequalities.sing:\
  24
1,0,0;
cone c=coneViaInequalities(IE,E,1);
⇒ // ** redefining c (cone c=coneViaInequalities(IE,E,1);) ./examples/coneV\
  iaInequalities.sing:26
c;
⇒ AMBIENT_DIM
⇒ 3
⇒ INEQUALITIES
⇒ 0,1,-1,
⇒ 0,1, 1
⇒ LINEAR_SPAN
⇒ 1,0,0
⇒
// and the kernel of E is the span of the cone
intmat IE[2][3]=
⇒ // ** redefining IE (intmat IE[2][3]=) ./examples/coneViaInequalities.sin\
  g:29
0,1,-1,
0,1,1;
intmat E[1][3]=
⇒ // ** redefining E (intmat E[1][3]=) ./examples/coneViaInequalities.sing:\
  32
1,0,0;
cone c=coneViaInequalities(IE,E,3);
⇒ // ** redefining c (cone c=coneViaInequalities(IE,E,3);) ./examples/coneV\
  iaInequalities.sing:34
c;
⇒ AMBIENT_DIM
⇒ 3
⇒ FACETS
⇒ 0,1,-1,
⇒ 0,1, 1

```

```

↳ LINEAR_SPAN
↳ 1,0,0
↳

```

### D.13.2.9 coneLink

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:** `coneLink(c,w)`; `c` cone, `w` intvec/bigintmat

**Return:** cone, the link of `c` around `w`

**Example:**

```

LIB "gfan.lib";
intmat M[3][3]=
1,0,0,
0,1,0,
0,0,1;
cone c=coneViaPoints(M);
intvec v=1,0,0;
cone cv=coneLink(c,v);
rays(cv);
↳ 0,1,0,
↳ 0,0,1
generatorsOfLinealitySpace(cv);
↳ -1,0,0
intvec w=1,1,1;
cone cw=coneLink(c,w);
rays(cw);
↳
generatorsOfLinealitySpace(cw);
↳ -1, 0, 0,
↳ 0,-1, 0,
↳ 0, 0,-1

```

### D.13.2.10 containsAsFace

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:** `containsAsFace(c,d)`; `c` cone, `d` cone

**Return:** 1, if `d` is a face of `c`; 0 otherwise

**Example:**

```

LIB "gfan.lib";
intmat M[2][2]=
1,0,
0,1;
cone c=coneViaPoints(M);
intmat N1[1][2]=
1,1;
cone d1=coneViaPoints(N1);
containsInSupport(c,d1);
↳ 1
containsAsFace(c,d1);
↳ 0

```

```

intmat N2[1][2]=
0,1;
cone d2=coneViaPoints(N2);
containsInSupport(c,d2);
↳ 1
containsAsFace(c,d2);
↳ 1

```

### D.13.2.11 containsInSupport

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:**      `containsInSupport(c,d);` c cone, d cone  
               `containsInSupport(c,p);` c cone, p intvec/bigintmat

**Return:**     1, if d resp. p is contained in c; 0 otherwise

**Example:**

```

LIB "gfan.lib";
intmat M[2][2]=
1,0,
0,1;
cone c=coneViaPoints(M);
containsInSupport(c,c);
↳ 1
intmat N1[2][2]=
1,1,
0,1;
cone d1=coneViaPoints(N1);
containsInSupport(c,d1);
↳ 1
intmat N2[2][2]=
1,1,
1,-1;
cone d2=coneViaPoints(N2);
containsInSupport(c,d2);
↳ 0
intvec p1=0,1;
containsInSupport(c,p1);
↳ 1
intvec p2=1,-1;
containsInSupport(c,p2);
↳ 0

```

### D.13.2.12 containsPositiveVector

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:**      `containsPositiveVector(c);` c cone

**Return:**     1, if c contains a vector with only positive entries in its relative interior

**Example:**

```

LIB "gfan.lib";
intmat M1[2][2]=
1,1,

```



```

1,-1;
cone c1=coneViaPoints(M1);
containsPositiveVector(c1);
 $\mapsto$  1
intmat M2[2][2]=
0,1,
-1,0;
cone c2=coneViaPoints(M2);
containsPositiveVector(c2);
 $\mapsto$  0

```

### D.13.2.13 containsRelatively

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\], page 2050](#)).

**Usage:** `containsRelatively(c,p);` c cone, intvec p

**Return:** 1 iff the given cone contains the given point in its relative interior; 0 otherwise

**Example:**

```

LIB "gfan.lib";
intmat M[2][2]=
1,0,
0,1;
cone c=coneViaPoints(M);
intvec p1=1,1;
containsRelatively(c,p1);
 $\mapsto$  1
intvec p2=0,1;
containsRelatively(c,p2);
 $\mapsto$  0

```

### D.13.2.14 convexHull

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\], page 2050](#)).

**Usage:** `convexHull(c1,c2);` c1 cone, c2 cone  
`convexHull(c1,p1);` c1 cone, p1 polytope  
`convexHull(p1,c1);` p1 cone, c1 polytope  
`convexHull(p1,p2);` p1 polytope, p2 polytope

**Return:** cone resp polytope, the convex hull of its two input objects

**Example:**

```

LIB "gfan.lib";
intmat M1[2][2]=
1,0,
0,1;
cone c1=coneViaPoints(M1);
intmat M2[2][2]=
1,1,
1,-1;
cone c2=coneViaPoints(M2);
intmat M3[2][2]=
1,0,

```

```

0,-1;
cone c3=coneViaPoints(M3);
cone c12=convexHull(c1,c2);
c12;
↳ AMBIENT_DIM
↳ 2
↳ FACETS
↳ 1,0,
↳ 1,1
↳ LINEAR_SPAN
↳
↳
print(rays(c12));
↳ 1,-1,
↳ 0, 1
cone c23=convexHull(c2,c3);
c23;
↳ AMBIENT_DIM
↳ 2
↳ FACETS
↳ 1,-1,
↳ 1, 0
↳ LINEAR_SPAN
↳
↳
print(rays(c23));
↳ 0,-1,
↳ 1, 1
cone c13=convexHull(c1,c3);
c13;
↳ AMBIENT_DIM
↳ 2
↳ FACETS
↳ 1,0
↳ LINEAR_SPAN
↳
↳
print(rays(c13));
↳ 1,0

```

### D.13.2.15 convexIntersection

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:**      `convexIntersection(c1,c2);` c1 cone, c2 cone  
               `convexIntersection(c1,p1);` c1 cone, p1 polytope  
               `convexIntersection(p1,c1);` p1 cone, c1 polytope  
               `convexIntersection(p1,p2);` p1 polytope, p2 polytope

**Return:**     cone resp polytope, the convex hull of its two input objects

**Example:**

```

LIB "gfan.lib";
intmat M1[2][2]=

```

```

1,0,
0,1;
cone c1=coneViaPoints(M1);
intmat M2[2][2]=
1,1,
1,-1;
cone c2=coneViaPoints(M2);
intmat M3[2][2]=
1,0,
0,-1;
cone c3=coneViaPoints(M3);
cone c12=convexIntersection(c1,c2);
c12;
⇒ AMBIENT_DIM
⇒ 2
⇒ FACETS
⇒ 0, 1,
⇒ 1,-1
⇒ LINEAR_SPAN
⇒
⇒
print(rays(c12));
⇒ 1,0,
⇒ 1,1
cone c23=convexIntersection(c2,c3);
c23;
⇒ AMBIENT_DIM
⇒ 2
⇒ FACETS
⇒ 0,-1,
⇒ 1, 1
⇒ LINEAR_SPAN
⇒
⇒
print(rays(c23));
⇒ 1, 0,
⇒ 1,-1
cone c13=convexIntersection(c1,c3);
c13;
⇒ AMBIENT_DIM
⇒ 2
⇒ FACETS
⇒ 1,0
⇒ LINEAR_SPAN
⇒ 0,1
⇒
print(rays(c13));
⇒ 1,0

```

#### D.13.2.16 dimension

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:**      `dimension(c); c cone`  
               `dimension(f); f fan`  
               `dimension(p); p polytope`

**Return:**     `int`, the dimension of the input

**Example:**

```
LIB "gfan.lib";
intmat M1[1][2]=
1,0;
cone c1=coneViaPoints(M1);
dimension(c1);
↪ 1
intmat M2[2][2]=
1,0,
0,1;
cone c2=coneViaPoints(M2);
dimension(c2);
↪ 2
fan f = emptyFan(2);
dimension(f);
↪ -1
insertCone(f,c1);
dimension(f);
↪ 1
insertCone(f,c2);
dimension(f);
↪ 2
```

### D.13.2.17 dualCone

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:**      `dualCone(c); c cone`

**Return:**     `cone`, the dual of `c`

**Example:**

```
LIB "gfan.lib";
intmat M1[2][2]=
1,0,
0,1;
cone c1=coneViaPoints(M1);
cone d1=dualCone(c1);
d1;
↪ AMBIENT_DIM
↪ 2
↪ INEQUALITIES
↪ 0,1,
↪ 1,0
↪ LINEAR_SPAN
↪
↪
print(rays(d1));
↪ 1,0,
```

```

⇒ 0,1
intmat M2[2][2]=
1,1,
0,1;
cone c2=coneViaPoints(M2);
cone d2=dualCone(c2);
d2;
⇒ AMBIENT_DIM
⇒ 2
⇒ INEQUALITIES
⇒ 0,1,
⇒ 1,1
⇒ LINEAR_SPAN
⇒
⇒
print(rays(d2));
⇒ 1,0,
⇒ -1,1

```

### D.13.2.18 equations

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\], page 2050](#)).

**Usage:**      `equations(c)`; `c` cone  
               `equations(p)`; `p` polytope

**Return:**    `bigintmat`, defining equations of `c` resp `p`

**Note:**      neither unique nor complete, unless `c` resp `p` in canonical form

**Example:**

```

LIB "gfan.lib";
intmat M1[2][2]=
1,0,
0,1;
cone c1=coneViaPoints(M1);
bigintmat E1=equations(c1);
print(E1);
⇒
intmat M2[1][2]=
1,0;
cone c2=coneViaPoints(M2);
bigintmat E2=equations(c2);
print(E2);
⇒ 0,-1

```

### D.13.2.19 faceContaining

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\], page 2050](#)).

**Usage:**      `faceContaining(c,w)`; `c` cone, `w` intvec/`bigintmat`

**Assume:**    `containsInSupport(c,w)==1`

**Return:**    cone, the face of `c` containing `w` in its relative interior

**Example:**

```

LIB "gfan.lib";
intmat M[2][2]=
1,0,
0,1;
cone c=coneViaPoints(M);
faceContaining(c,intvec(1,0));
⇒ AMBIENT_DIM
⇒ 2
⇒ INEQUALITIES
⇒ 1,0
⇒ LINEAR_SPAN
⇒ 0,1
⇒
faceContaining(c,intvec(0,1));
⇒ AMBIENT_DIM
⇒ 2
⇒ INEQUALITIES
⇒ 0,1
⇒ LINEAR_SPAN
⇒ 1,0
⇒
faceContaining(c,intvec(1,1));
⇒ AMBIENT_DIM
⇒ 2
⇒ INEQUALITIES
⇒ 0,1,
⇒ 1,0
⇒ LINEAR_SPAN
⇒
⇒
faceContaining(c,intvec(0,0));
⇒ AMBIENT_DIM
⇒ 2
⇒ INEQUALITIES
⇒
⇒ LINEAR_SPAN
⇒ 1,0,
⇒ 0,1
⇒

```

### D.13.2.20 facets

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:**      `facets(c);` c cone  
               `facets(p);` p polytope

**Return:**     bigintmat, the facets of c resp p

**Example:**

```

LIB "gfan.lib";
intmat M1[2][2]=
1,0,
0,1;

```

```

cone c1=coneViaPoints(M1);
bigintmat F1=facets(c1);
print(F1);
↳ 0,1,
↳ 1,0
intmat M2[2][2]=
1,1,
0,-1;
cone c2=coneViaPoints(M2);
bigintmat F2=facets(c2);
print(F2);
↳ 1,-1,
↳ 1, 0

```

### D.13.2.21 generatorsOfLinealitySpace

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:** `generatorsOfLinealitySpace(c); c cone`

**Return:** `bigintmat`, generators of the lineality space of `c`

**Example:**

```

LIB "gfan.lib";
intmat M[5][3]=
1,0,0,
0,1,0,
0,0,1,
-1,0,0,
0,-1,0;
cone c=coneViaPoints(M);
bigintmat L=generatorsOfLinealitySpace(c);
print(L);
↳ -1, 0,0,
↳ 0,-1,0

```

### D.13.2.22 generatorsOfSpan

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:** `generatorsOfSpan(c); c cone`

**Return:** `bigintmat`, generators of the span of `c`

**Example:**

```

LIB "gfan.lib";
intmat M[3][5]=
1,0,0,0,0,
0,1,0,0,0,
0,0,1,0,0;
cone c=coneViaPoints(M);
bigintmat S=generatorsOfSpan(c);
print(S);
↳ -1, 0, 0,0,0,
↳ 0,-1, 0,0,0,
↳ 0, 0,-1,0,0

```

### D.13.2.23 getLinearForms

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\], page 2050](#)).

**Usage:**      `getLinearForms(c);` c cone  
               `getLinearForms(p);` p polytope

**Return:**     `bigintmat`, linear forms previously stored in c resp p

**Example:**

```
LIB "gfan.lib";
intmat M[2][3]=
-1,0,0,
0,-1,0;
cone c=coneViaPoints(M);
getLinearForms(c);
↪
intvec v=1,1,1;
setLinearForms(c,v);
getLinearForms(c);
↪ 1,1,1
```

### D.13.2.24 getMultiplicity

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\], page 2050](#)).

**Usage:**      `getMultiplicity(c);` c cone  
               `getMultiplicity(p);` p polytope

**Return:**     `bigint`, 1 or a multiplicity previously stored in c resp p

**Example:**

```
LIB "gfan.lib";
intmat M[2][3]=
-1,0,0,
0,-1,0;
cone c=coneViaPoints(M);
getMultiplicity(c);
↪ 1
setMultiplicity(c,3);
getMultiplicity(c);
↪ 3
```

### D.13.2.25 inequalities

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\], page 2050](#)).

**Usage:**      `inequalities(c);` c cone  
               `inequalities(p);` p polytope

**Return:**     `bigintmat`, the inequalities of c resp p

**Note:**       neither unique nor irredundant, unless c resp p in canonical form

**Example:**



```

LIB "gfan.lib";
intmat M1[2][2]=
1,0,
0,1;
cone c1=coneViaPoints(M1);
bigintmat I1=inequalities(c1);
print(I1);
↪ 0,1,
↪ 1,0
intmat M2[2][2]=
1,1,
0,-1;
cone c2=coneViaPoints(M2);
bigintmat I2=inequalities(c2);
print(I2);
↪ 1,-1,
↪ 1, 0

```

### D.13.2.26 isFullSpace

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\], page 2050](#)).

**Usage:** `isFullSpace(c)`;  $c$  cone

**Return:** 1, if  $c$  is the entire ambient space; 0 otherwise

**Example:**

```

LIB "gfan.lib";
cone c1;
isFullSpace(c1);
↪ 1
intmat M2[2][2]=
1,0,
0,1;
cone c2=coneViaPoints(M2);
isFullSpace(c2);
↪ 0
intmat M3[4][2]=
1,0,
0,1,
-1,0,
0,-1;
cone c3=coneViaPoints(M3);
isFullSpace(c3);
↪ 1

```

### D.13.2.27 isOrigin

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\], page 2050](#)).

**Usage:** `isOrigin(c)`;  $c$  cone

**Return:** 1, if  $c$  is the origin; 0 otherwise

**Example:**

```

LIB "gfan.lib";
cone c1;
isOrigin(c1);
 $\mapsto$  1
intmat M2[2][2]=
1,0,
0,1;
cone c2=coneViaPoints(M2);
isOrigin(c2);
 $\mapsto$  0
intmat M3[4][2]=
1,0,
0,1,
-1,0,
0,-1;
cone c3=coneViaPoints(M3);
isOrigin(c3);
 $\mapsto$  0

```

### D.13.2.28 isSimplicial

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:**     `isSimplicial(c)`; `c` cone  
               `isSimplicial(f)`; `f` fan

**Return:**    1, if `c` resp `f` is simplicial; 0 otherwise

**Example:**

```

LIB "gfan.lib";
intmat M1[3][3]=
1,0,0,
0,1,0,
0,0,1;
cone c1=coneViaPoints(M1);
isSimplicial(c1);
 $\mapsto$  1
intmat M2[4][3]=
1,0,0,
0,1,0,
0,0,1,
1,1,-1;
cone c2=coneViaPoints(M2);
isSimplicial(c2);
 $\mapsto$  0
/*****/
fan f=emptyFan(3);
isSimplicial(f);
 $\mapsto$  1
intmat N1[3][3]=
1,0,0,
0,1,0,
0,0,1;
cone d1=coneViaPoints(N1);

```

```

insertCone(f,d1);
isSimplicial(f);
 $\mapsto$  1
intmat N2[4][3]=
1,0,0,
0,1,0,
1,0,-1,
0,1,-1;
cone d2=coneViaPoints(N2);
insertCone(f,d2);
isSimplicial(f);
 $\mapsto$  0

```

### D.13.2.29 linealityDimension

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\], page 2050](#)).

**Usage:**      `linealityDimension(c); c cone`  
               `linealityDimension(f); f fan`

**Return:**    `int`, the dimension of the lineality space of `c` resp `f`

**Example:**

```

LIB "gfan.lib";
intmat M1[3][3]=
1,0,0,
0,1,0,
0,0,1;
cone c1=coneViaPoints(M1);
linealityDimension(c1);
 $\mapsto$  0
intmat M2[4][3]=
1,0,0,
0,1,0,
0,0,1,
-1,0,0;
cone c2=coneViaPoints(M2);
linealityDimension(c2);
 $\mapsto$  1

```

### D.13.2.30 linealitySpace

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\], page 2050](#)).

**Usage:**      `linealitySpace(c); c cone`

**Return:**    `cone`, the lineality space of `c`

**Example:**

```

LIB "gfan.lib";
intmat M1[3][3]=
1,0,0,
0,1,0,
0,0,1;
cone c1=coneViaPoints(M1);
cone l1=linealitySpace(c1);

```

```

l1;
↳ AMBIENT_DIM
↳ 3
↳ INEQUALITIES
↳
↳ LINEAR_SPAN
↳ 1,0,0,
↳ 0,1,0,
↳ 0,0,1
↳
intmat M2[4][3]=
1,0,0,
0,1,0,
0,0,1,
-1,0,0;
cone c2=coneViaPoints(M2);
cone l2=linealitySpace(c2);
l2;
↳ AMBIENT_DIM
↳ 3
↳ INEQUALITIES
↳
↳ LINEAR_SPAN
↳ 0,1,0,
↳ 0,0,1
↳

```

### D.13.2.31 negatedCone

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\], page 2050](#)).

**Usage:** `negatedCone(c); c cone`

**Return:** `cone`, the negative of `c`

**Example:**

```

LIB "gfan.lib";
intmat M[2][2]=
1,0,
0,1;
cone c=coneViaPoints(M);
cone cn=negatedCone(c);
print(rays(cn));
↳ -1, 0,
↳ 0, -1

```

### D.13.2.32 polytopeViaInequalities

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\], page 2050](#)).

**Usage:** `polytopeViaInequalities(EV [, E [, flags]]); intmat EV,E, int flags`

**Return:** `polytope` consisting of all points  $x$ , such that  $IE^*x \geq 0$  in each component and (if stated)  $E^*x = 0$ ;  
`flags` may range between 0,...,3 defining an upper and lower bit ( $0=0*2+0$ ,  $1=0*2+1$ ,

$2=1*2+0$ ,  $3=1*2+1$ ),  
 if higher bit is 1, then program assumes each inequality yields a facet, if lower bit is 1,  
 then program assumes the kernel of E is the span of the cone, if either bit is 0, then  
 program computes the information itself.

**Example:**

```
LIB "gfan.lib";
intmat IE[2][3]=
1,0,0,
0,1,0;
intmat E[1][3]=
0,0,1;
polytope p=polytopeViaInequalities(IE,E);
p;
↳ AMBIENT_DIM
↳ 2
↳ INEQUALITIES
↳ 0,1,0,
↳ 1,0,0
↳ EQUATIONS
↳ 0,0,1
↳
```

**D.13.2.33 polytopeViaPoints**

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:** `polytopeViaPoints(V [, flags]);` intmat V, int flags

**Return:** polytope which is the intersection of the cone generated by the row vectors of V with the hyperplane, in which the first coordinate equals 1; flags may be 0 or 1, if flags is 1, then program assumes that each row vector of M generates a ray in the cone, if flags is 0, then program computes that information itself

**Example:**

```
LIB "gfan.lib";
// This is a polytope in R^2 generated by (0,0), (1,0), (0,1), (0,0);
intmat V[4][3]=
1,0,0,
1,1,0,
1,0,1,
1,1,1;
polytope p1=polytopeViaPoints(V);
p1;
↳ AMBIENT_DIM
↳ 2
↳ INEQUALITIES
↳ 0, 0, 1,
↳ 0, 1, 0,
↳ 1,-1, 0,
↳ 1, 0,-1
↳ EQUATIONS
↳
↳
```

```

// This is a polytope in R^2 generated by (1/2,2/3), (3/4,4/5), (5/6,6/7):
intmat V[3][3]=
↳ // ** redefining V (intmat V[3][3]=) ./examples/polytopeViaPoints.sing:11
6,3,4,
20,15,16,
42,35,36;
polytope p2=polytopeViaPoints(V);
p2;
↳ AMBIENT_DIM
↳ 2
↳ INEQUALITIES
↳ -10,-24, 35,
↳ -6, -8, 15,
↳ 8, 12,-21
↳ EQUATIONS
↳
↳
// This polytope is the positive orthant in R^2:
// (0,1,0) and (0,0,1) imply that the polytope is unbounded in that direction
intmat V[3][3]=
↳ // ** redefining V (intmat V[3][3]=) ./examples/polytopeViaPoints.sing:19
1,0,0,
0,1,0,
0,0,1;
polytope p3=polytopeViaPoints(V);
p3;
↳ AMBIENT_DIM
↳ 2
↳ INEQUALITIES
↳ 0,0,1,
↳ 0,1,0,
↳ 1,0,0
↳ EQUATIONS
↳
↳

```

### D.13.2.34 quotientLatticeBasis

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:**      `quotientLatticeBasis(c);`  $c$  cone

**Return:**    `bigintmat`, a basis of  $\mathbb{Z}^n$  intersected with the span of  $c$  modulo  $\mathbb{Z}^n$  intersected with the lineality space of  $c$

**Example:**

```

LIB "gfan.lib";
intmat M[3][2]=
1,0,
0,1,
-1,0;
cone c=coneViaPoints(M);
bigintmat Q=quotientLatticeBasis(c);
print(Q);
↳ 0,1

```

### D.13.2.35 randomPoint

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\], page 2050](#)).

**Usage:**      `randomPoint(c); c cone`  
               `randomPoint(c,b); c cone, b int`

**Return:**     `bigintmat`, a random point in the relative interior of `c`

**Note:**        returns a weighted sum over all its rays  
               if `b` is given and `b>0`, only chooses weights between 1 and `b`

**Example:**

```
LIB "gfan.lib";
intmat M[2][2]=
1,0,
0,1;
cone c=coneViaPoints(M);
bigintmat Q=randomPoint(c);
print(Q);
↪ 1335380034,380636641
bigintmat P=randomPoint(c,5);
print(P);
↪ 4,4
```

### D.13.2.36 rays

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\], page 2050](#)).

**Usage:**      `rays(c); c cone`

**Return:**     `bigintmat`, generators of the rays of `c`, orthogonal to its lineality space

**Example:**

```
LIB "gfan.lib";
intmat M1[2][2]=
1,0,
0,1;
cone c1=coneViaPoints(M1);
bigintmat R1=rays(c1);
print(R1);
↪ 1,0,
↪ 0,1
intmat M2[3][2]=
1,0,
0,1,
-1,0;
cone c2=coneViaPoints(M2);
bigintmat R2=rays(c2);
print(R2);
↪ 0,1
```

### D.13.2.37 relativeInteriorPoint

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\], page 2050](#)).

**Usage:**      `relativeInteriorPoint(c); c cone`

**Return:** bigintmat, a point in the relative interior of  $c$

**Note:** not unique, unless  $c$  is in its canonical form

**Example:**

```
LIB "gfan.lib";
intmat M1[2][2]=
1,0,
0,1;
cone c1=coneViaPoints(M1);
relativeInteriorPoint(c1);
↦ 1,1
intmat M2[2][2]=
1,0,
1,1;
cone c2=coneViaPoints(M2);
relativeInteriorPoint(c2);
↦ 2,1
```

### D.13.2.38 semigroupGenerator

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:** `semigroupGenerator(c);`  $c$  cone

**Return:** bigintmat, the generator of  $Z^n$  intersected with  $c$  modulo  $Z^n$  intersected with the lineality space of  $c$

**Assume:** `dimension(c) == linealityDimension(c)+1`

**Example:**

```
LIB "gfan.lib";
intmat M[3][2]=
1,0,
0,1,
-1,0;
cone c=coneViaPoints(M);
semigroupGenerator(c);
↦ 0,1
```

### D.13.2.39 setLinearForms

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:** `setLinearForms(c);`  $c$  cone  
`setLinearForms(p);`  $p$  polytope

**Return:** none, stores linear forms in  $c$  resp  $p$

**Example:**

```
LIB "gfan.lib";
intmat M[2][3]=
-1,0,0,
0,-1,0;
cone c=coneViaPoints(M);
getLinearForms(c);
↦
```



```

intvec v=1,1,1;
setLinearForms(c,v);
getLinearForms(c);
↳ 1,1,1

```

#### D.13.2.40 setMultiplicity

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\], page 2050](#)).

**Usage:**     `setMultiplicity(c); c cone`  
               `setMultiplicity(p); p polytope`

**Return:**    none, stores a multiplicity in `c` resp `p`

**Example:**

```

LIB "gfan.lib";
intmat M[2][3]=
-1,0,0,
0,-1,0;
cone c=coneViaPoints(M);
getMultiplicity(c);
↳ 1
setMultiplicity(c,3);
getMultiplicity(c);
↳ 3

```

#### D.13.2.41 span

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\], page 2050](#)).

**Usage:**     `span(c); c cone`

**Return:**    `bigintmat`, unique irredundant equations of `c`

**Note:**     the name 'span' was chosen to be in line with `polymake`'s nomenclature

**Example:**

```

LIB "gfan.lib";
intmat M[3][5]=
1,0,0,0,0,
0,1,0,0,0,
0,0,1,0,0;
cone c=coneViaPoints(M);
bigintmat Eq=span(c);
print(Eq);
↳ 0,0,0,-1, 0,
↳ 0,0,0, 0,-1

```

#### D.13.2.42 uniquePoint

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\], page 2050](#)).

**Usage:**     `uniquePoint(c); c cone`

**Return:**    `bigintmat`, a unique point in `c` stable under reflections at coordinate hyperplanes

**Example:**

```

LIB "gfan.lib";
intmat M1[2][2]=
1,0,
0,1;
cone c1=coneViaPoints(M1);
uniquePoint(c1);
↪ 1,1
intmat M2[2][2]=
-1,0,
0,1;
cone c2=coneViaPoints(M2);
uniquePoint(c2);
↪ -1,1

```

### D.13.2.43 containsInCollection

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\], page 2050](#)).

**Usage:** `containsInCollection(f,c)`;  $f$  fan,  $c$  cone

**Return:** 1, if  $f$  contains  $c$ ; 0 otherwise

**Example:**

```

LIB "gfan.lib";
fan f=emptyFan(2);
intmat M[2][2]=
1,0,
0,1;
cone c=coneViaPoints(M);
containsInCollection(f,c);
↪ 0
insertCone(f,c);
containsInCollection(f,c);
↪ 1

```

### D.13.2.44 emptyFan

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\], page 2050](#)).

**Usage:** `emptyFan(n)`;  $n$  int

**Return:** fan, an empty fan in ambient dimension  $n$

**Example:**

```

LIB "gfan.lib";
fan f=emptyFan(2);
f;
↪ _application PolyhedralFan
↪ _version 2.2
↪ _type PolyhedralFan
↪
↪ AMBIENT_DIM
↪ 2
↪
↪ DIM
↪ -1

```

```

↳
↳ LINEALITY_DIM
↳ 2
↳
↳ RAYS
↳
↳ N_RAYS
↳ 0
↳
↳ LINEALITY_SPACE
↳ 1 0 # 0
↳ 0 1 # 1
↳
↳ ORTH_LINEALITY_SPACE
↳
↳ F_VECTOR
↳
↳
↳ SIMPLICIAL
↳ 1
↳
↳ PURE
↳ 1
↳
↳ CONES
↳
↳ MAXIMAL_CONES
↳

```

### D.13.2.45 fanViaCones

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:**      `fanViaCones(L)`; L list  
               `fanViaCones(c1[,...,ck])`; c1,...,ck cones

**Return:**     fan, creates a fan generated by the cones in L resp c1,...,ck

**Example:**

```

LIB "gfan.lib";
intmat M[2][2]=1,0,0,1;
cone c=coneViaPoints(M);
intmat N[2][2]=1,0,0,-1;
cone d=coneViaPoints(N);
fan f=fanViaCones(c,d);
f;
↳ _application PolyhedralFan
↳ _version 2.2
↳ _type PolyhedralFan
↳
↳ AMBIENT_DIM
↳ 2
↳
↳ DIM

```

```

⇒ 2
⇒
⇒ LINEALITY_DIM
⇒ 0
⇒
⇒ RAYS
⇒ 0 -1 # 0
⇒ 0 1 # 1
⇒ 1 0 # 2
⇒
⇒ N_RAYS
⇒ 3
⇒
⇒ LINEALITY_SPACE
⇒
⇒ ORTH_LINEALITY_SPACE
⇒ -1 0 # 0
⇒ 0 -1 # 1
⇒
⇒ F_VECTOR
⇒ 1 3 2
⇒
⇒ SIMPLICIAL
⇒ 1
⇒
⇒ PURE
⇒ 1
⇒
⇒ CONES
⇒ {} # Dimension 0
⇒ {0} # Dimension 1
⇒ {1}
⇒ {2}
⇒ {0 2} # Dimension 2
⇒ {1 2}
⇒
⇒ MAXIMAL_CONES
⇒ {0 2} # Dimension 2
⇒ {1 2}
⇒
list L=c,d;
fan g=fanViaCones(L);
g;
⇒ _application PolyhedralFan
⇒ _version 2.2
⇒ _type PolyhedralFan
⇒
⇒ AMBIENT_DIM
⇒ 2
⇒
⇒ DIM
⇒ 2
⇒

```

```

⇒ LINEALITY_DIM
⇒ 0
⇒
⇒ RAYS
⇒ 0 -1 # 0
⇒ 0 1 # 1
⇒ 1 0 # 2
⇒
⇒ N_RAYS
⇒ 3
⇒
⇒ LINEALITY_SPACE
⇒
⇒ ORTH_LINEALITY_SPACE
⇒ -1 0 # 0
⇒ 0 -1 # 1
⇒
⇒ F_VECTOR
⇒ 1 3 2
⇒
⇒ SIMPLICIAL
⇒ 1
⇒
⇒ PURE
⇒ 1
⇒
⇒ CONES
⇒ {} # Dimension 0
⇒ {0} # Dimension 1
⇒ {1}
⇒ {2}
⇒ {0 2} # Dimension 2
⇒ {1 2}
⇒
⇒ MAXIMAL_CONES
⇒ {0 2} # Dimension 2
⇒ {1 2}
⇒

```

### D.13.2.46 fullFan

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:**      `fullFan(n); n int`

**Return:**     `fan`, an full fan in ambient dimension `n`

**Example:**

```

LIB "gfan.lib";
fan f=fullFan(2);
f;
⇒ _application PolyhedralFan
⇒ _version 2.2
⇒ _type PolyhedralFan

```

```

↳
↳ AMBIENT_DIM
↳ 2
↳
↳ DIM
↳ 2
↳
↳ LINEALITY_DIM
↳ 2
↳
↳ RAYS
↳
↳ N_RAYS
↳ 0
↳
↳ LINEALITY_SPACE
↳ -1 0 # 0
↳ 0 -1 # 1
↳
↳ ORTH_LINEALITY_SPACE
↳
↳ F_VECTOR
↳ 1
↳
↳ SIMPLICIAL
↳ 1
↳
↳ PURE
↳ 1
↳
↳ CONES
↳ {} # Dimension 2
↳
↳ MAXIMAL_CONES
↳ {} # Dimension 2
↳

```

### D.13.2.47 fVector

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:** `fVector(f); f fan`

**Return:** `bigintmat`, the `f`-Vector of `f`

**Example:**

```

LIB "gfan.lib";
fan f=emptyFan(2);
fVector(f);
↳
intmat M[2][2]=1,0,0,1;
cone c=coneViaPoints(M);
insertCone(f,c);
fVector(f);
↳ 1,2,1

```

### D.13.2.48 `getCone`

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\], page 2050](#)).

**Usage:** `getCone(f,d,i[,m]);`  $f$  fan,  $d$  int,  $i$  int,  $m$  int

**Assume:**  $d$  is between 0 and `ambientDimension(f)`  
 $i$  is between 1 and `numberOfConesOfDimension(f,d,o,m)`

**Return:** cone, returns in the fan  $f$  of all cones in dimension  $d$  the  $i$ -th cone if  $m!=0$ , it will enumerate over maximal cones only

**Example:**

```
LIB "gfan.lib";
intmat M[3][3]=
1,0,0,
0,1,0,
0,0,1;
cone c=coneViaPoints(M);
fan f=emptyFan(3);
insertCone(f,c);
getCone(f,2,1,0);
⇒ AMBIENT_DIM
⇒ 3
⇒ FACETS
⇒ 0,0,1,
⇒ 0,1,0
⇒ LINEAR_SPAN
⇒ -1,0,0
⇒
getCone(f,2,2,0);
⇒ AMBIENT_DIM
⇒ 3
⇒ FACETS
⇒ 0,0,1,
⇒ 1,0,0
⇒ LINEAR_SPAN
⇒ 0,-1,0
⇒
```

### D.13.2.49 `insertCone`

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\], page 2050](#)).

**Usage:** `insertCone(f,c[,b]);`  $f$  fan,  $c$  cone,  $b$  int

**Assume:** `isCompatible(f,c)=1`

**Return:** none, inserts the cone  $c$  into  $f$   
if  $b=0$ , then skips check whether  $f$  and  $c$  are compatible

**Example:**

```
LIB "gfan.lib";
fan f=emptyFan(3);
f;
⇒ _application PolyhedralFan
⇒ _version 2.2
```

```

↳ _type PolyhedralFan
↳
↳ AMBIENT_DIM
↳ 3
↳
↳ DIM
↳ -1
↳
↳ LINEALITY_DIM
↳ 3
↳
↳ RAYS
↳
↳ N_RAYS
↳ 0
↳
↳ LINEALITY_SPACE
↳ 1 0 0 # 0
↳ 0 1 0 # 1
↳ 0 0 1 # 2
↳
↳ ORTH_LINEALITY_SPACE
↳
↳ F_VECTOR
↳
↳
↳ SIMPLICIAL
↳ 1
↳
↳ PURE
↳ 1
↳
↳ CONES
↳
↳ MAXIMAL_CONES
↳
intmat M[3][3]=
1,0,0,
0,1,0,
0,0,1;
cone c=coneViaPoints(M);
insertCone(f,c);
f;
↳ _application PolyhedralFan
↳ _version 2.2
↳ _type PolyhedralFan
↳
↳ AMBIENT_DIM
↳ 3
↳
↳ DIM
↳ 3
↳

```



```

⇒ LINEALITY_DIM
⇒ 0
⇒
⇒ RAYS
⇒ 0 0 1 # 0
⇒ 0 1 0 # 1
⇒ 1 0 0 # 2
⇒
⇒ N_RAYS
⇒ 3
⇒
⇒ LINEALITY_SPACE
⇒
⇒ ORTH_LINEALITY_SPACE
⇒ -1 0 0 # 0
⇒ 0 -1 0 # 1
⇒ 0 0 -1 # 2
⇒
⇒ F_VECTOR
⇒ 1 3 3 1
⇒
⇒ SIMPLICIAL
⇒ 1
⇒
⇒ PURE
⇒ 1
⇒
⇒ CONES
⇒ {} # Dimension 0
⇒ {0} # Dimension 1
⇒ {1}
⇒ {2}
⇒ {0 1} # Dimension 2
⇒ {0 2}
⇒ {1 2}
⇒ {0 1 2} # Dimension 3
⇒
⇒ MAXIMAL_CONES
⇒ {0 1 2} # Dimension 3
⇒

```

### D.13.2.50 isCompatible

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:**      `isCompatible(f,c);` `f` fan, `c` cone

**Return:**    1 if `f` and `c` live in the same ambient space and  
               if the intersection of `c` with any cone of `f` is a face of each; 0 otherwise

**Example:**

```

LIB "gfan.lib";
fan f=emptyFan(3);
intmat M1[3][3]=

```

```

1,0,0,
0,1,0,
0,0,1;
cone c1=coneViaPoints(M1);
isCompatible(f,c1);
⇒ 1
insertCone(f,c1);
intmat M2[3][3]=
1,1,1,
1,0,0,
0,1,0;
cone c2=coneViaPoints(M2);
isCompatible(f,c2);
⇒ 0
intmat M3[3][3]=
1,0,0,
0,1,0,
0,0,-1;
cone c3=coneViaPoints(M3);
isCompatible(f,c3);
⇒ 1

```

### D.13.2.51 isPure

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:**      `isPure(f); f fan`

**Return:**     1 if all maximal cones of `f` are of the same dimension 0 otherwise

**Example:**

```

LIB "gfan.lib";
fan f=fullFan(2);
isPure(f);
⇒ 1
fan g=emptyFan(2);
intmat M1[2][2]=
1,0,
0,1;
cone c1=coneViaPoints(M1);
insertCone(g,c1);
isPure(g);
⇒ 1
intmat M2[1][2]=
0,-1;
cone c2=coneViaPoints(M2);
insertCone(g,c2);
isPure(g,c2);
⇒ 0

```

### D.13.2.52 nmaxcones

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:**      `nmaxcones(f); f fan`

**Return:** int, the number of maximal cones in f

**Example:**

```
LIB "gfan.lib";
fan f=emptyFan(3);
nmaxcones(f);
↪ 0
intmat M1[3][3]=
1,0,0,
0,1,0,
0,0,1;
cone c1=coneViaPoints(M1);
insertCone(f,c1);
nmaxcones(f);
↪ 1
intmat M2[2][3]=
1,0,0,
0,-1,0;
cone c2=coneViaPoints(M2);
insertCone(f,c2);
nmaxcones(f);
↪ 2
```

### D.13.2.53 ncones

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:** ncones(f); f fan

**Return:** int, the number of cones in f

**Example:**

```
LIB "gfan.lib";
fan f=emptyFan(3);
ncones(f);
↪ 0
intmat M1[3][3]=
1,0,0,
0,1,0,
0,0,1;
cone c1=coneViaPoints(M1);
insertCone(f,c1);
ncones(f);
↪ 8
intmat M2[2][3]=
1,0,0,
0,-1,0;
cone c2=coneViaPoints(M2);
insertCone(f,c2);
ncones(f);
↪ 10
```

### D.13.2.54 numberOfConesOfDimension

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:**      `numberOfConesOfDimension(f,d[,m]);` f fan, d int, m int

**Assume:**    d is between 0 and `ambientDimension(f)`

**Return:**      cone, returns in the fan f the number of cones in dimension d if m!=0, it will only count maximal cones

**Example:**

```
LIB "gfan.lib";
fan f=emptyFan(3);
ncones(f);
↪ 0
intmat M[3][3]=
1,0,0,
0,1,0,
0,0,1;
cone c=coneViaPoints(M);
insertCone(f,c);
numberOfConesOfDimension(f,0,0);
↪ 1
numberOfConesOfDimension(f,0,1);
↪ 0
numberOfConesOfDimension(f,1,0);
↪ 3
numberOfConesOfDimension(f,0,1);
↪ 0
numberOfConesOfDimension(f,2,0);
↪ 3
numberOfConesOfDimension(f,2,1);
↪ 0
numberOfConesOfDimension(f,3,0);
↪ 1
numberOfConesOfDimension(f,3,1);
↪ 1
```

### D.13.2.55 removeCone

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:**      `removeCone(f,c[,b]);` f fan, c cone, b int

**Assume:**    `containsInCollection(f,c)=1`

**Return:**      none, removes the cone c from f  
if b=0, skips the check whether c is contained in f

**Example:**

```
LIB "gfan.lib";
intmat M[2][2]=1,0,0,1;
intmat N[2][2]=1,0,1,-1;
cone c=coneViaPoints(M);
cone d=coneViaPoints(N);
fan f=emptyFan(2);
insertCone(f,c);
insertCone(f,d);
f;
```

```

⇒ _application PolyhedralFan
⇒ _version 2.2
⇒ _type PolyhedralFan
⇒
⇒ AMBIENT_DIM
⇒ 2
⇒
⇒ DIM
⇒ 2
⇒
⇒ LINEALITY_DIM
⇒ 0
⇒
⇒ RAYS
⇒ 0 1 # 0
⇒ 1 -1 # 1
⇒ 1 0 # 2
⇒
⇒ N_RAYS
⇒ 3
⇒
⇒ LINEALITY_SPACE
⇒
⇒ ORTH_LINEALITY_SPACE
⇒ -1 0 # 0
⇒ 0 -1 # 1
⇒
⇒ F_VECTOR
⇒ 1 3 2
⇒
⇒ SIMPLICIAL
⇒ 1
⇒
⇒ PURE
⇒ 1
⇒
⇒ CONES
⇒ {} # Dimension 0
⇒ {0} # Dimension 1
⇒ {1}
⇒ {2}
⇒ {0 2} # Dimension 2
⇒ {1 2}
⇒
⇒ MAXIMAL_CONES
⇒ {0 2} # Dimension 2
⇒ {1 2}
⇒
removeCone(f,c);
f;
⇒ _application PolyhedralFan
⇒ _version 2.2
⇒ _type PolyhedralFan

```

```

⇒
⇒ AMBIENT_DIM
⇒ 2
⇒
⇒ DIM
⇒ 2
⇒
⇒ LINEALITY_DIM
⇒ 0
⇒
⇒ RAYS
⇒ 1 -1 # 0
⇒ 1 0 # 1
⇒
⇒ N_RAYS
⇒ 2
⇒
⇒ LINEALITY_SPACE
⇒
⇒ ORTH_LINEALITY_SPACE
⇒ -1 0 # 0
⇒ 0 -1 # 1
⇒
⇒ F_VECTOR
⇒ 1 2 1
⇒
⇒ SIMPLICIAL
⇒ 1
⇒
⇒ PURE
⇒ 1
⇒
⇒ CONES
⇒ {} # Dimension 0
⇒ {0} # Dimension 1
⇒ {1}
⇒ {0 1} # Dimension 2
⇒
⇒ MAXIMAL_CONES
⇒ {0 1} # Dimension 2
⇒

```

### D.13.2.56 dualPolytope

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:**      `dualPolytope(p)`; `p` polytope

**Return:**    polytope, the dual of `p`

**Example:**

```

LIB "gfan.lib";
intmat M[4][2]=
0,0,

```

```

1,0,
0,1,
1,1;
polytope p=polytopeViaPoints(M);
dualPolytope(p);
⇒ AMBIENT_DIM
⇒ 1
⇒ INEQUALITIES
⇒ 0,1,
⇒ 1,0
⇒ EQUATIONS
⇒
⇒

```

### D.13.2.57 newtonPolytope

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:**      `newtonPolytope(f); f poly`

**Return:**     `polytope`, the convex hull of all exponent vectors of `f`

**Example:**

```

LIB "gfan.lib";
ring r;
poly f=x+y+z;
polytope p=newtonPolytope(f);
p;
⇒ AMBIENT_DIM
⇒ 3
⇒ INEQUALITIES
⇒ 0,0,0,1,
⇒ 0,0,1,0,
⇒ 0,1,0,0
⇒ EQUATIONS
⇒ 1,-1,-1,-1
⇒

```

### D.13.2.58 vertices

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:**      `vertices(p); p polytope`

**Return:**     `bigintmat`, the vertices of `p` modulo its lineality space

**Example:**

```

LIB "gfan.lib";
intmat M[4][3]=
1,0,0,
1,2,0,
1,0,2,
1,2,2,
1,1,1;
polytope p=polytopeViaPoints(M);
vertices(p);

```

```

 $\mapsto 1,2,0,$ 
 $\mapsto 1,0,0,$ 
 $\mapsto 1,0,2,$ 
 $\mapsto 1,2,2$ 

```

### D.13.2.59 onesVector

Procedure from library `gfan.lib` (see [Section D.13.2 \[gfan.lib\]](#), page 2050).

**Usage:** `onesVector(n);`  $n$  int

**Return:** intvec, intvec of length  $n$  with all entries 1

**Example:**

```

LIB "gfan.lib";
intvec w = onesVector(3);
w;
 $\mapsto 1,1,1$ 

```

### D.13.3 gitfan\_lib

**Library:** `gitfan.lib`

**Purpose:** Compute GIT-fans.

**Authors:** Janko Boehm, boehm at mathematik.uni-kl.de  
 Simon Keicher, keicher at mail.mathematik.uni-tuebingen.de  
 Yue Ren, ren at mathematik.uni-kl.de

**Overview:** This library allows you to calculate GIT-fans, torus orbits and GKZ-fans.

It provides features to make use of symmetries of the torus action under consideration.

The main procedure is `GITfan` which can be directly applied to an ideal and a grading matrix encoding the torus action, and returns a fan, the associated GIT-fan. We also provide various procedures implementing substeps of the algorithm to deal with large computations.

The library uses the package 'gfanlib' by Anders N. Jensen.

For notation, background, and algorithms see [BKR16].

Functions produce debug output if `printlevel` is positive.

Elements of the symmetric group  $S_n$  of type permutation can be created by the function `permutationFromIntvec`.

The images of  $1, \dots, n$  can be obtained by `permutationToIntvec`. Composition of permutations can be done by the `*`-Operator, also powers can be computed in the usual way.

**References:**

[BKR16] J. Boehm, S. Keicher, Y. Ren: Computing GIT-Fans with Symmetry and the Mori Chamber Decomposition of  $M_{0,6}$ , <https://arxiv.org/abs/1603.09241>

**Types:** permutation; Permutation in map representation.

**Procedures:**



### D.13.3.1 isAface

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\], page 2090](#)).

**Usage:** `isAface(a,gam0)`; a: ideal, gam0:intvec

**Purpose:** Checks whether gam0 is an a-face w.r.t. the ideal a.

**Return:** int

**Example:**

```
LIB "gitfan.lib";
ring R = 0,(T(1..4)),dp;
ideal I = T(1)*T(2)-T(4);
intvec w = 1,4;
intvec v = 1,2,4;
isAface(I,w); // should be 0
↪ 0
isAface(I,v); // should be 1
↪ 1
```

### D.13.3.2 afaces

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\], page 2090](#)).

**Usage:** `afaces(a [,L])`; a: ideal, L: list of intvecs

**Purpose:** Returns a list of all a-faces (considered as intvecs of 0 and 1, where the i-th entry is 1 if the cone has the i-th unit basis vector as a generator), if L is specified only the faces of the simplex listed in L are considered (e.g. representatives with respect to a group action).

**Return:** a list of intvecs

**Example:**

```
LIB "gitfan.lib";
ring R = 0,T(1..3),dp;
ideal a = T(1)+T(2)+T(3);
list F = afaces(a);
print(F);
↪ [1]:
↪ 0
↪ [2]:
↪ 1,2
↪ [3]:
↪ 1,3
↪ [4]:
↪ 2,3
↪ [5]:
↪ 1,2,3
print(size(F));
↪ 5
// 2nd ex //
ring R2 = 0,T(1..3),dp;
ideal a2 = T(2)^2*T(3)^2+T(1)*T(3);
list F2 = afaces(a2);
print(F2);
```

```

↳ [1]:
↳ 0
↳ [2]:
↳ 1
↳ [3]:
↳ 2
↳ [4]:
↳ 1,2
↳ [5]:
↳ 3
↳ [6]:
↳ 1,2,3
print(size(F2));
↳ 6
// 3rd ex //
ring R3 = 0,T(1..3),dp;
ideal a3 = 0;
list F3 = afaces(a3);
print(F3);
↳ [1]:
↳ 0
↳ [2]:
↳ 1
↳ [3]:
↳ 2
↳ [4]:
↳ 1,2
↳ [5]:
↳ 3
↳ [6]:
↳ 1,3
↳ [7]:
↳ 2,3
↳ [8]:
↳ 1,2,3
print(size(F3));
↳ 8
// 4th ex //
ring R4 = 0,T(1..10),wp(1,1,1,1,1,1,1,1,1,1);
ideal J =
T(5)*T(10)-T(6)*T(9)+T(7)*T(8),
T(1)*T(9)-T(2)*T(7)+T(4)*T(5),
T(1)*T(8)-T(2)*T(6)+T(3)*T(5),
T(1)*T(10)-T(3)*T(7)+T(4)*T(6),
T(2)*T(10)-T(3)*T(9)+T(4)*T(8);
list F4 = afaces(J);
print(size(F4));
↳ 172

```

### D.13.3.3 fullDimImages

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\]](#), page 2090).

**Usage:**      `fullDimImages(afaces, Q);` `afaces`: list, `Q`: intmat

**Purpose:** Determines the a-faces (represented as intvecs) from the list afaces which have a full-dimensional projection with respect to  $Q$ .

**Return:** a list of intvecs

**Example:**

```
LIB "gitfan.lib";
ring R = 0,T(1..10),wp(1,1,1,1,1,1,1,1,1,1);
ideal J =
T(5)*T(10)-T(6)*T(9)+T(7)*T(8),
T(1)*T(9)-T(2)*T(7)+T(4)*T(5),
T(1)*T(8)-T(2)*T(6)+T(3)*T(5),
T(1)*T(10)-T(3)*T(7)+T(4)*T(6),
T(2)*T(10)-T(3)*T(9)+T(4)*T(8);
intmat Q[5][10] =
1, 1, 1, 1, 0, 0, 0, 0, 0, 0,
1, 0, 0, 0, 1, 1, 1, 0, 0, 0,
0, 1, 1, 0, 0, 0, -1, 1, 0, 0,
0, 1, 0, 1, 0, -1, 0, 0, 1, 0,
0, 0, 1, 1, -1, 0, 0, 0, 0, 1;
list AF= afaces(J,nrows(Q));
size(AF);
↳ 81
size(fullDimImages(AF,Q));
↳ 36
```

#### D.13.3.4 minimalAfaces

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\]](#), page 2090).

**Usage:** `minimalAfaces(listOfAfaces);` listOfAfaces: list

**Purpose:** Returns a list of all minimal a-faces. Note that listOfAfaces must only contain afaces which project to full dimension.

**Return:** a list of intvecs

**Example:**

```
LIB "gitfan.lib";
setcores(4);
↳ 4
ring R = 0,T(1..10),wp(1,1,1,1,1,1,1,1,1,1);
ideal J =
T(5)*T(10)-T(6)*T(9)+T(7)*T(8),
T(1)*T(9)-T(2)*T(7)+T(4)*T(5),
T(1)*T(8)-T(2)*T(6)+T(3)*T(5),
T(1)*T(10)-T(3)*T(7)+T(4)*T(6),
T(2)*T(10)-T(3)*T(9)+T(4)*T(8);
intmat Q[5][10] =
1, 1, 1, 1, 0, 0, 0, 0, 0, 0,
1, 0, 0, 0, 1, 1, 1, 0, 0, 0,
0, 1, 1, 0, 0, 0, -1, 1, 0, 0,
0, 1, 0, 1, 0, -1, 0, 0, 1, 0,
0, 0, 1, 1, -1, 0, 0, 0, 0, 1;
list AF= afaces(J,nrows(Q));
```

```

size(AF);
↳ 81
AF=fullDimImages(AF,Q);
size(AF);
↳ 36
AF=minimalAfaces(AF);
size(AF);
↳ 25

```

### D.13.3.5 orbitCones

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\], page 2090](#)).

**Usage:** `orbitCones(AF, Q[, d]);` AF: list of intvecs, Q: intmat, d: int

**Purpose:** Returns the list consisting of all cones  $Q(\text{gam0})$  where  $\text{gam0}$  in AF. If the optional argument  $d$  is given then the function returns only the orbit cones of dimension at least  $d$

**Return:** a list of cones

**Example:**

```

LIB "gitfan.lib";
ring R = 0,T(1..10),wp(1,1,1,1,1,1,1,1,1,1);
ideal J =
T(5)*T(10)-T(6)*T(9)+T(7)*T(8),
T(1)*T(9)-T(2)*T(7)+T(4)*T(5),
T(1)*T(8)-T(2)*T(6)+T(3)*T(5),
T(1)*T(10)-T(3)*T(7)+T(4)*T(6),
T(2)*T(10)-T(3)*T(9)+T(4)*T(8);
intmat Q[5][10] =
1, 1, 1, 1, 0, 0, 0, 0, 0, 0,
1, 0, 0, 0, 1, 1, 1, 0, 0, 0,
0, 1, 1, 0, 0, 0, -1, 1, 0, 0,
0, 1, 0, 1, 0, -1, 0, 0, 1, 0,
0, 0, 1, 1, -1, 0, 0, 0, 0, 1;
list AF= afaces(J);
print(size(AF));
↳ 172
list OC = orbitCones(AF,Q);
size(OC);
↳ 172

```

### D.13.3.6 GITcone

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\], page 2090](#)).

**Usage:** `GITcone(OCcones, w);` OCcones: list of orbit cones, w: bigintmat with one row

**Purpose:** Returns the intersection of all orbit cones containing  $w$ .

**Return:** cone,intvec with the GIT cone containing  $w$ , and the hash of this cone (the indices of the orbit cones contributing to the intersection)

**Example:**

```

LIB "gitfan.lib";
ring R = 0,T(1..10),wp(1,1,1,1,1,1,1,1,1,1);
ideal J =
T(5)*T(10)-T(6)*T(9)+T(7)*T(8),
T(1)*T(9)-T(2)*T(7)+T(4)*T(5),
T(1)*T(8)-T(2)*T(6)+T(3)*T(5),
T(1)*T(10)-T(3)*T(7)+T(4)*T(6),
T(2)*T(10)-T(3)*T(9)+T(4)*T(8);
intmat Q[5][10] =
1, 1, 1, 1, 0, 0, 0, 0, 0, 0,
1, 0, 0, 0, 1, 1, 1, 0, 0, 0,
0, 1, 1, 0, 0, 0, -1, 1, 0, 0,
0, 1, 0, 1, 0, -1, 0, 0, 1, 0,
0, 0, 1, 1, -1, 0, 0, 0, 0, 1;
list AF= afaces(J,nrows(Q));
AF=fullDimImages(AF,Q);
AF = minimalAfaces(AF);
list OC = orbitCones(AF,Q);
bigintmat w[1][nrows(Q)];
int j;
for(int i = 1; i <= nrows(Q); i++)
{
for(j=1;j<=nrows(Q);j++)
{
w[1,j]= w[1,j] + Q[1,i];
}
}
GITcone(OC,w);
⇒ AMBIENT_DIM
⇒ 5
⇒ FACETS
⇒ 0,0,0,0,1,
⇒ 0,1,0,0,0
⇒ LINEAR_SPAN
⇒ 1,0,0,0,-1,
⇒ 0,0,1,0,-1,
⇒ 0,0,0,1,-1
⇒ 3,5,6,8,10,11,12,14,16,17,18,19,20,21,22,23,24,25

```

### D.13.3.7 GITfan

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\]](#), page 2090).

**Usage:**     GITfan(J,Q [, G]); J:ideal, Q:intmat, G:list

**Purpose:**     Computes the GIT fan associated to J and Q. Optionally a symmetry group action on the column space of Q can be specified.

**Return:**    a fan, the GIT fan.

**Note:**       The procedure uses parallel computation for the construction of the GIT-cones. The a-faces are not computed in parallel. This can be done by calling the `aface` procedure specifying a list of simplex faces. If used with the optional argument G, the orbit decomposition of the simplex of columns of Q is computed. Refer to the Singular documentation on how to do this more efficiently using GAP.

**Example:**

```

LIB "gitfan.lib";
setcores(4);
 $\mapsto$  4
ring R = 0,T(1..10),wp(1,1,1,1,1,1,1,1,1,1);
ideal J =
T(5)*T(10)-T(6)*T(9)+T(7)*T(8),
T(1)*T(9)-T(2)*T(7)+T(4)*T(5),
T(1)*T(8)-T(2)*T(6)+T(3)*T(5),
T(1)*T(10)-T(3)*T(7)+T(4)*T(6),
T(2)*T(10)-T(3)*T(9)+T(4)*T(8);
intmat Q[5][10] =
1, 1, 1, 1, 0, 0, 0, 0, 0, 0,
1, 0, 0, 0, 1, 1, 1, 0, 0, 0,
0, 1, 1, 0, 0, 0, -1, 1, 0, 0,
0, 1, 0, 1, 0, -1, 0, 0, 1, 0,
0, 0, 1, 1, -1, 0, 0, 0, 0, 1;
fan GIT = GITfan(J,Q);
intmat Q[5][10] =
 $\mapsto$  // ** redefining Q (intmat Q[5][10] =) ./examples/GITfan.sing:17
1, 1, 1, 1, 0, 0, 0, 0, 0, 0,
1, 0, 0, 0, 1, 1, 1, 0, 0, 0,
0, 1, 1, 0, 0, 0, -1, 1, 0, 0,
0, 1, 0, 1, 0, -1, 0, 0, 1, 0,
0, 0, 1, 1, -1, 0, 0, 0, 0, 1;
list simplexSymmetryGroup = G25Action();
fan GIT2 = GITfan(J,Q,simplexSymmetryGroup);

```

**D.13.3.8 GITfanFromOrbitCones**

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\]](#), page 2090).

**Usage:** `GITfanFromOrbitCones(OC, Q, Qgamma [, file1, file2]);` OC:list, Q:bigintmat, Qgamma:cone, file1:string, file2:string

**Purpose:** Returns the common refinement of the cones given in the list OC which is supposed to contain the orbit cones intersected with Qgamma. The optional argument can be used to specify one or two strings with file names, where the first file will contain the hashes of the GIT-cones and the second argument the actual cones in their H-representation. To obtain the whole GIT-fan Qgamma has to be take the cone generated by the columns of Q.

**Return:** a list containing the bigint hashes of the GIT cones.

**Example:**

```

LIB "gitfan.lib";
ring R = 0,T(1..10),wp(1,1,1,1,1,1,1,1,1,1);
ideal J =
T(5)*T(10)-T(6)*T(9)+T(7)*T(8),
T(1)*T(9)-T(2)*T(7)+T(4)*T(5),
T(1)*T(8)-T(2)*T(6)+T(3)*T(5),
T(1)*T(10)-T(3)*T(7)+T(4)*T(6),
T(2)*T(10)-T(3)*T(9)+T(4)*T(8);
intmat Q[5][10] =

```

```

1, 1, 1, 1, 0, 0, 0, 0, 0, 0,
1, 0, 0, 0, 1, 1, 1, 0, 0, 0,
0, 1, 1, 0, 0, 0, -1, 1, 0, 0,
0, 1, 0, 1, 0, -1, 0, 0, 1, 0,
0, 0, 1, 1, -1, 0, 0, 0, 0, 1;
list AF= afaces(J,nrows(Q));
AF=fullDimImages(AF,Q);
AF = minimalAfaces(AF);
list OC = orbitCones(AF,Q);
cone Qgamma = coneViaPoints(transpose(Q));
list GIT = GITfanFromOrbitCones(OC,Q,Qgamma);
size(GIT);
↳ 76

```

### D.13.3.9 GITfanParallel

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\]](#), page 2090).

**Usage:** `GITfanParallel(OC, Q, Qgamma [, file1]);` OC:list, Q:intmat, Qgamma:cone, file1:string

**Purpose:** Returns the common refinement of the cones given in the list OC which is supposed to contain the orbit cones intersected with Qgamma. The optional argument can be used to specify a name for a file which will contain the hashes of the GIT-cones. To obtain the whole GIT-fan Qgamma has to be take the cone generated by the columns of Q.

**Return:** a list containing the bigint hashes of the GIT cones.

**Note:** The procedure uses parallel computation for the construction of the GIT-cones.

**Example:**

```

LIB "gitfan.lib";
setcores(4);
↳ 4
ring R = 0,T(1..10),wp(1,1,1,1,1,1,1,1,1,1);
ideal J =
T(5)*T(10)-T(6)*T(9)+T(7)*T(8),
T(1)*T(9)-T(2)*T(7)+T(4)*T(5),
T(1)*T(8)-T(2)*T(6)+T(3)*T(5),
T(1)*T(10)-T(3)*T(7)+T(4)*T(6),
T(2)*T(10)-T(3)*T(9)+T(4)*T(8);
intmat Q[5][10] =
1, 1, 1, 1, 0, 0, 0, 0, 0, 0,
1, 0, 0, 0, 1, 1, 1, 0, 0, 0,
0, 1, 1, 0, 0, 0, -1, 1, 0, 0,
0, 1, 0, 1, 0, -1, 0, 0, 1, 0,
0, 0, 1, 1, -1, 0, 0, 0, 0, 1;
list AF= afaces(J);
print(size(AF));
↳ 172
list OC = orbitCones(AF,Q);
cone Qgamma = coneViaPoints(transpose(Q));
list GIT = GITfanParallel(OC,Q,Qgamma);
size(GIT);
↳ 76

```

### D.13.3.10 GKZfan

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\]](#), page 2090).

### D.13.3.11 movingCone

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\]](#), page 2090).

**Usage:**      `movingCone(Q); Q: intmat`

**Purpose:**    Computes the moving cone from the grading matrix, with the degrees in the columns of `Q`.

**Return:**    a cone

**Example:**

```
LIB "gitfan.lib";
intmat Q[5][10] =
1, 1, 1, 1, 0, 0, 0, 0, 0, 0,
1, 0, 0, 0, 1, 1, 1, 0, 0, 0,
0, 1, 1, 0, 0, 0, -1, 1, 0, 0,
0, 1, 0, 1, 0, -1, 0, 0, 1, 0,
0, 0, 1, 1, -1, 0, 0, 0, 0, 1;
cone mov = movingCone(Q);
mov;
↳ AMBIENT_DIM
↳ 5
↳ FACETS
↳ -2,1, 1, 1, 1,
↳ -1,1, 0, 0, 1,
↳ -1,1, 0, 1, 0,
↳ -1,1, 1, 0, 0,
↳ 0,0, 0, 0, 1,
↳ 0,0, 0, 1, 0,
↳ 0,0, 1, 0, 0,
↳ 1,0,-1, 0, 0,
↳ 1,0, 0,-1, 0,
↳ 1,0, 0, 0,-1
↳ LINEAR_SPAN
↳
↳
rays(mov);
↳ 1,1,0,1,0,
↳ 1,1,1,1,0,
↳ 1,2,0,0,0,
↳ 2,1,1,1,1,
↳ 1,1,1,0,0,
↳ 1,1,1,0,1,
↳ 1,1,0,1,1,
↳ 1,1,0,0,1,
↳ 1,0,1,1,1,
↳ 0,1,0,0,0
// moving cone where the image of the positive orthant
// is the entire space
// (from bug reported by Donton-Bury and Grab)
```



```
intmat Q2[3][16] =
7, 7,7,7, 7,7,7, 7,7,7, 7,7,7, -7,0,0,
0, 0,2,1, 0,2,1, 3,2,1, 3,2,4, 0,-3,0,
0, 0,1,2, 0,1,2, 3,1,2, 3,4,2, 0,0,-3;
cone mov2 = movingCone(Q2);
⇒ // ** redefining Qdash (      intmat Qdash[nrows(Qt)-1][ncols(Qt)];) gitf\
   an.lib::movingCone:1012
⇒ // ** redefining Qdash (      intmat Qdash[nrows(Qt)-1][ncols(Qt)];) gitf\
   an.lib::movingCone:1012
⇒ // ** redefining Qdash (      intmat Qdash[nrows(Qt)-1][ncols(Qt)];) gitf\
   an.lib::movingCone:1012
⇒ // ** redefining Qdash (      intmat Qdash[nrows(Qt)-1][ncols(Qt)];) gitf\
   an.lib::movingCone:1012
⇒ // ** redefining Qdash (      intmat Qdash[nrows(Qt)-1][ncols(Qt)];) gitf\
   an.lib::movingCone:1012
⇒ // ** redefining Qdash (      intmat Qdash[nrows(Qt)-1][ncols(Qt)];) gitf\
   an.lib::movingCone:1012
⇒ // ** redefining Qdash (      intmat Qdash[nrows(Qt)-1][ncols(Qt)];) gitf\
   an.lib::movingCone:1012
⇒ // ** redefining Qdash (      intmat Qdash[nrows(Qt)-1][ncols(Qt)];) gitf\
   an.lib::movingCone:1012
⇒ // ** redefining Qdash (      intmat Qdash[nrows(Qt)-1][ncols(Qt)];) gitf\
   an.lib::movingCone:1012
⇒ // ** redefining Qdash (      intmat Qdash[nrows(Qt)-1][ncols(Qt)];) gitf\
   an.lib::movingCone:1012
⇒ // ** redefining Qdash (      intmat Qdash[nrows(Qt)-1][ncols(Qt)];) gitf\
   an.lib::movingCone:1012
⇒ // ** redefining Qdash (      intmat Qdash[nrows(Qt)-1][ncols(Qt)];) gitf\
   an.lib::movingCone:1012
⇒ // ** redefining Qdash (      intmat Qdash[nrows(Qt)-1][ncols(Qt)];) gitf\
   an.lib::movingCone:1012
⇒ // ** redefining Qdash (      intmat Qdash[nrows(Qt)-1][ncols(Qt)];) gitf\
   an.lib::movingCone:1012
mov2;
⇒ AMBIENT_DIM
⇒ 3
⇒ FACETS
⇒ 0,-1, 2,
⇒ 0, 2,-1,
⇒ 6,-7,-7
⇒ LINEAR_SPAN
⇒
⇒
rays(mov2);
⇒ 1,0,0,
⇒ 7,2,4,
⇒ 7,4,2
```

### D.13.3.12 computeAfaceOrbits

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\]](#), page 2090).

**Usage:**      `computeAfaceOrbits(AF,G);` AF: list of intvecs, G: list of permutations

**Purpose:**      Computes the orbits of the afaces in the list AF under the group action in G, where G is a list of permutations. We assume that the elements of G form a group and the first entry corresponds to the neutral element.

**Return:**      a list of lists of intvecs

**Example:**

```
LIB "gitfan.lib";
ring R = 0,T(1..10),wp(1,1,1,1,1,1,1,1,1,1);
ideal J =
T(5)*T(10)-T(6)*T(9)+T(7)*T(8),
T(1)*T(9)-T(2)*T(7)+T(4)*T(5),
T(1)*T(8)-T(2)*T(6)+T(3)*T(5),
T(1)*T(10)-T(3)*T(7)+T(4)*T(6),
T(2)*T(10)-T(3)*T(9)+T(4)*T(8);
intmat Q[5][10] =
1, 1, 1, 1, 0, 0, 0, 0, 0, 0,
1, 0, 0, 0, 1, 1, 1, 0, 0, 0,
0, 1, 1, 0, 0, 0, -1, 1, 0, 0,
0, 1, 0, 1, 0, -1, 0, 0, 1, 0,
0, 0, 1, 1, -1, 0, 0, 0, 0, 1;
list simplexSymmetryGroup = G25Action();
list simplexOrbitRepresentatives = intvec( 1, 2, 3, 4, 5 ),
intvec( 1, 2, 3, 5, 6 ),
intvec( 1, 2, 3, 5, 7 ),
intvec( 1, 2, 3, 5, 10 ),
intvec( 1, 2, 3, 7, 9 ),
intvec( 1, 2, 6, 9, 10 ),
intvec( 1, 2, 3, 4, 5, 6 ),
intvec( 1, 2, 3, 4, 5, 10 ),
intvec( 1, 2, 3, 5, 6, 8 ),
intvec( 1, 2, 3, 5, 6, 9 ),
intvec( 1, 2, 3, 5, 7, 10 ),
intvec( 1, 2, 3, 7, 9, 10 ),
intvec( 1, 2, 3, 4, 5, 6, 7 ),
intvec( 1, 2, 3, 4, 5, 6, 8 ),
intvec( 1, 2, 3, 4, 5, 6, 9 ),
intvec( 1, 2, 3, 5, 6, 9, 10 ),
intvec( 1, 2, 3, 4, 5, 6, 7, 8 ),
intvec( 1, 2, 3, 4, 5, 6, 9, 10 ),
intvec( 1, 2, 3, 4, 5, 6, 7, 8, 9 ),
intvec( 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 );
list afaceOrbitRepresentatives=afaces(J,simplexOrbitRepresentatives);
list fullDimAfaceOrbitRepresentatives=fullDimImages(afaceOrbitRepresentatives,Q);
list afaceOrbits=computeAfaceOrbits(fullDimAfaceOrbitRepresentatives,simplexSymmetryGroup);
apply(afaceOrbits,size);
↦ 10 15 10 1
```

### D.13.3.13 minimalAfaceOrbits

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\]](#), page 2090).

**Usage:**      `minimalAfaceOrbits(afaceOrbits);` afaceOrbits: list

**Purpose:** Returns a list of all minimal a-face orbits.

**Return:** a list of intvecs

**Example:**

```
LIB "gitfan.lib";
ring R = 0,T(1..10),wp(1,1,1,1,1,1,1,1,1,1);
ideal J =
T(5)*T(10)-T(6)*T(9)+T(7)*T(8),
T(1)*T(9)-T(2)*T(7)+T(4)*T(5),
T(1)*T(8)-T(2)*T(6)+T(3)*T(5),
T(1)*T(10)-T(3)*T(7)+T(4)*T(6),
T(2)*T(10)-T(3)*T(9)+T(4)*T(8);
intmat Q[5][10] =
1, 1, 1, 1, 0, 0, 0, 0, 0, 0,
1, 0, 0, 0, 1, 1, 1, 0, 0, 0,
0, 1, 1, 0, 0, 0, -1, 1, 0, 0,
0, 1, 0, 1, 0, -1, 0, 0, 1, 0,
0, 0, 1, 1, -1, 0, 0, 0, 0, 1;
list simplexSymmetryGroup = G25Action();
list simplexOrbitRepresentatives = intvec( 1, 2, 3, 4, 5 ),
intvec( 1, 2, 3, 5, 6 ),
intvec( 1, 2, 3, 5, 7 ),
intvec( 1, 2, 3, 5, 10 ),
intvec( 1, 2, 3, 7, 9 ),
intvec( 1, 2, 6, 9, 10 ),
intvec( 1, 2, 3, 4, 5, 6 ),
intvec( 1, 2, 3, 4, 5, 10 ),
intvec( 1, 2, 3, 5, 6, 8 ),
intvec( 1, 2, 3, 5, 6, 9 ),
intvec( 1, 2, 3, 5, 7, 10 ),
intvec( 1, 2, 3, 7, 9, 10 ),
intvec( 1, 2, 3, 4, 5, 6, 7 ),
intvec( 1, 2, 3, 4, 5, 6, 8 ),
intvec( 1, 2, 3, 4, 5, 6, 9 ),
intvec( 1, 2, 3, 5, 6, 9, 10 ),
intvec( 1, 2, 3, 4, 5, 6, 7, 8 ),
intvec( 1, 2, 3, 4, 5, 6, 9, 10 ),
intvec( 1, 2, 3, 4, 5, 6, 7, 8, 9 ),
intvec( 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 );
list afaceOrbitRepresentatives=afaces(J,simplexOrbitRepresentatives);
list fulldimAfaceOrbitRepresentatives=fullDimImages(afaceOrbitRepresentatives,Q);
list afaceOrbits=computeAfaceOrbits(fulldimAfaceOrbitRepresentatives,simplexSymmetryGroup);
apply(afaceOrbits,size);
↳ 10 15 10 1
list minAfaceOrbits = minimalAfaceOrbits(afaceOrbits);
apply(minAfaceOrbits,size);
↳ 10 15
```

### D.13.3.14 orbitConeOrbits

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\]](#), page 2090).

**Usage:** orbitConeOrbits(F, Q); F: list, Q: intmat

**Purpose:** Projects a list F of a-face orbits to the orbit cones with respect to Q. The function checks whether the projections are of full dimension and returns an error otherwise.

**Return:** a list of lists of cones

**Example:**

```
LIB "gitfan.lib";
// Note that simplexOrbitRepresentatives and simplexSymmetryGroup are subsets of the
ring R = 0,T(1..10),wp(1,1,1,1,1,1,1,1,1,1);
ideal J =
T(5)*T(10)-T(6)*T(9)+T(7)*T(8),
T(1)*T(9)-T(2)*T(7)+T(4)*T(5),
T(1)*T(8)-T(2)*T(6)+T(3)*T(5),
T(1)*T(10)-T(3)*T(7)+T(4)*T(6),
T(2)*T(10)-T(3)*T(9)+T(4)*T(8);
intmat Q[5][10] =
1, 1, 1, 1, 0, 0, 0, 0, 0, 0,
1, 0, 0, 0, 1, 1, 1, 0, 0, 0,
0, 1, 1, 0, 0, 0, -1, 1, 0, 0,
0, 1, 0, 1, 0, -1, 0, 0, 1, 0,
0, 0, 1, 1, -1, 0, 0, 0, 0, 1;
list simplexOrbitRepresentatives = intvec( 1, 2, 3, 4, 5 ),
intvec( 1, 2, 3, 5, 6 ),
intvec( 1, 2, 3, 5, 7 ),
intvec( 1, 2, 3, 5, 10 ),
intvec( 1, 2, 3, 7, 9 ),
intvec( 1, 2, 3, 4, 5, 6, 9, 10 ),
intvec( 1, 2, 3, 4, 5, 6, 7, 8, 9 ),
intvec( 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 );
list afaceOrbitRepresentatives=afaces(J,simplexOrbitRepresentatives);
list simplexSymmetryGroup = permutationFromIntvec(intvec( 1 .. 10 )),
permutationFromIntvec(intvec( 1, 2, 4, 3, 5, 7, 6, 9, 8, 10 )),
permutationFromIntvec(intvec( 1, 3, 2, 4, 6, 5, 7, 8, 10, 9 )),
permutationFromIntvec(intvec( 1, 3, 4, 2, 6, 7, 5, 10, 8, 9 )),
permutationFromIntvec(intvec( 1, 4, 2, 3, 7, 5, 6, 9, 10, 8 )),
permutationFromIntvec(intvec( 1, 4, 3, 2, 7, 6, 5, 10, 9, 8 ));
list fulldimAfaceOrbitRepresentatives=fullDimImages(afaceOrbitRepresentatives,Q);
list afaceOrbits=computeAfaceOrbits(fulldimAfaceOrbitRepresentatives,simplexSymmetryGroup);
apply(afaceOrbits,size);
↳ 3 3 1
list minAfaceOrbits = minimalAfaceOrbits(afaceOrbits);
apply(minAfaceOrbits,size);
↳ 3
list listOfOrbitConeOrbits = orbitConeOrbits(minAfaceOrbits,Q);
```

### D.13.3.15 minimalOrbitConeOrbits

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\]](#), page 2090).

**Usage:** `minimalOrbitConeOrbits(listOfOrbits);` listOfOrbits: list of lists of cones

**Purpose:** Minimizes a list of orbit cone orbits.

**Return:** a list of lists of cones

**Example:**

```

LIB "gitfan.lib";
ring R = 0,T(1..10),wp(1,1,1,1,1,1,1,1,1,1);
ideal J =
T(5)*T(10)-T(6)*T(9)+T(7)*T(8),
T(1)*T(9)-T(2)*T(7)+T(4)*T(5),
T(1)*T(8)-T(2)*T(6)+T(3)*T(5),
T(1)*T(10)-T(3)*T(7)+T(4)*T(6),
T(2)*T(10)-T(3)*T(9)+T(4)*T(8);
intmat Q[5][10] =
1, 1, 1, 1, 0, 0, 0, 0, 0, 0,
1, 0, 0, 0, 1, 1, 1, 0, 0, 0,
0, 1, 1, 0, 0, 0, -1, 1, 0, 0,
0, 1, 0, 1, 0, -1, 0, 0, 1, 0,
0, 0, 1, 1, -1, 0, 0, 0, 0, 1;
list simplexSymmetryGroup = G25Action();
list simplexOrbitRepresentatives = intvec( 1, 2, 3, 4, 5 ),
intvec( 1, 2, 3, 5, 6 ),
intvec( 1, 2, 3, 5, 7 ),
intvec( 1, 2, 3, 5, 10 ),
intvec( 1, 2, 3, 7, 9 ),
intvec( 1, 2, 6, 9, 10 ),
intvec( 1, 2, 3, 4, 5, 6 ),
intvec( 1, 2, 3, 4, 5, 10 ),
intvec( 1, 2, 3, 5, 6, 8 ),
intvec( 1, 2, 3, 5, 6, 9 ),
intvec( 1, 2, 3, 5, 7, 10 ),
intvec( 1, 2, 3, 7, 9, 10 ),
intvec( 1, 2, 3, 4, 5, 6, 7 ),
intvec( 1, 2, 3, 4, 5, 6, 8 ),
intvec( 1, 2, 3, 4, 5, 6, 9 ),
intvec( 1, 2, 3, 5, 6, 9, 10 ),
intvec( 1, 2, 3, 4, 5, 6, 7, 8 ),
intvec( 1, 2, 3, 4, 5, 6, 9, 10 ),
intvec( 1, 2, 3, 4, 5, 6, 7, 8, 9 ),
intvec( 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 );
list afaceOrbitRepresentatives=afaces(J,simplexOrbitRepresentatives);
list fulldimAfaceOrbitRepresentatives=fullDimImages(afaceOrbitRepresentatives,Q);
list afaceOrbits=computeAfaceOrbits(fulldimAfaceOrbitRepresentatives,simplexSymmetryGroup);
apply(afaceOrbits,size);
⇨ 10 15 10 1
list minAfaceOrbits = minimalAfaceOrbits(afaceOrbits);
apply(minAfaceOrbits,size);
⇨ 10 15
list listOfOrbitConeOrbits = orbitConeOrbits(minAfaceOrbits,Q);
apply(listOfOrbitConeOrbits,size);
⇨ 10 15
list listOfMinimalOrbitConeOrbits = minimalOrbitConeOrbits(listOfOrbitConeOrbits);
size(listOfMinimalOrbitConeOrbits);
⇨ 2

```

### D.13.3.16 intersectOrbitsWithMovingCone

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\]](#), page 2090).

**Usage:**      `intersectOrbitsWithMovingCone(OCmin, mov);` OCmin: list of lists of cones, mov: cone

**Purpose:**      Intersects all cones in the orbits in OCmin with mov discarding all orbits of cones which are not of full dimension. The resulting orbits are duplicate free.

**Return:**      a list of lists of cones

**Example:**

```
LIB "gitfan.lib";
ring R = 0,T(1..10),wp(1,1,1,1,1,1,1,1,1,1);
ideal J =
T(5)*T(10)-T(6)*T(9)+T(7)*T(8),
T(1)*T(9)-T(2)*T(7)+T(4)*T(5),
T(1)*T(8)-T(2)*T(6)+T(3)*T(5),
T(1)*T(10)-T(3)*T(7)+T(4)*T(6),
T(2)*T(10)-T(3)*T(9)+T(4)*T(8);
intmat Q[5][10] =
1, 1, 1, 1, 0, 0, 0, 0, 0, 0,
1, 0, 0, 0, 1, 1, 1, 0, 0, 0,
0, 1, 1, 0, 0, 0, -1, 1, 0, 0,
0, 1, 0, 1, 0, -1, 0, 0, 1, 0,
0, 0, 1, 1, -1, 0, 0, 0, 0, 1;
list simplexSymmetryGroup = G25Action();
list simplexOrbitRepresentatives = intvec( 1, 2, 3, 4, 5 ),
intvec( 1, 2, 3, 5, 6 ),
intvec( 1, 2, 3, 5, 7 ),
intvec( 1, 2, 3, 5, 10 ),
intvec( 1, 2, 3, 7, 9 ),
intvec( 1, 2, 6, 9, 10 ),
intvec( 1, 2, 3, 4, 5, 6 ),
intvec( 1, 2, 3, 4, 5, 10 ),
intvec( 1, 2, 3, 5, 6, 8 ),
intvec( 1, 2, 3, 5, 6, 9 ),
intvec( 1, 2, 3, 5, 7, 10 ),
intvec( 1, 2, 3, 7, 9, 10 ),
intvec( 1, 2, 3, 4, 5, 6, 7 ),
intvec( 1, 2, 3, 4, 5, 6, 8 ),
intvec( 1, 2, 3, 4, 5, 6, 9 ),
intvec( 1, 2, 3, 5, 6, 9, 10 ),
intvec( 1, 2, 3, 4, 5, 6, 7, 8 ),
intvec( 1, 2, 3, 4, 5, 6, 9, 10 ),
intvec( 1, 2, 3, 4, 5, 6, 7, 8, 9 ),
intvec( 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 );
list afaceOrbitRepresentatives=afaces(J,simplexOrbitRepresentatives);
list fullDimAfaceOrbitRepresentatives=fullDimImages(afaceOrbitRepresentatives,Q);
list afaceOrbits=computeAfaceOrbits(fullDimAfaceOrbitRepresentatives,simplexSymmetryGroup);
apply(afaceOrbits,size);
⇨ 10 15 10 1
list minAfaceOrbits = minimalAfaceOrbits(afaceOrbits);
apply(minAfaceOrbits,size);
⇨ 10 15
list listOfOrbitConeOrbits = orbitConeOrbits(minAfaceOrbits,Q);
cone mov = movingCone(Q);
```

```

intersectOrbitsWithMovingCone(listOfOrbitConeOrbits,mov);
↳ [1]:
↳      [1]:
↳      AMBIENT_DIM
↳      5
↳      FACETS
↳      -2,1, 1, 1, 1,
↳      -1,1, 0, 0, 1,
↳      -1,1, 0, 1, 0,
↳      -1,1, 1, 0, 0,
↳      0,0, 0, 0, 1,
↳      0,0, 0, 1, 0,
↳      0,0, 1, 0, 0,
↳      1,0,-1, 0, 0,
↳      1,0, 0,-1, 0,
↳      1,0, 0, 0,-1
↳      LINEAR_SPAN
↳
↳

```

### D.13.3.17 groupActionOnQImage

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\]](#), page 2090).

**Usage:** `groupActionOnQImage(G,Q)`; G: list of permutations, Q: intmat

**Purpose:** Given the group G of permutations acting on the simplex on `ncols(Q)` objects, computes the corresponding group action on the image of Q. We assume that the basering has characteristic 0.

**Return:** list of matrices

**Example:**

```

LIB "gitfan.lib";
ring R = 0,(x),dp;
intmat Q[5][10] =
1, 1, 1, 1, 0, 0, 0, 0, 0, 0,
1, 0, 0, 0, 1, 1, 1, 0, 0, 0,
0, 1, 1, 0, 0, 0, -1, 1, 0, 0,
0, 1, 0, 1, 0, -1, 0, 0, 1, 0,
0, 0, 1, 1, -1, 0, 0, 0, 0, 1;
list generatorsG = permutationFromIntvec(intvec( 1, 3, 2, 4, 6, 5, 7, 8, 10, 9 )),
permutationFromIntvec(intvec( 5, 7, 1, 6, 9, 2, 8, 4, 10, 3 ));
groupActionOnQImage(generatorsG,Q);
↳ [1]:
↳      1,0,0,0,0,
↳      0,1,0,0,0,
↳      0,0,1,0,0,
↳      0,0,0,0,1,
↳      0,0,0,1,0
↳ [2]:
↳      -1,1,1,0,1,
↳      1,0,0,0,0,
↳      -1,1,0,0,1,
↳      -1,1,1,0,0,

```

$\mapsto$  -2,1,1,1,1

### D.13.3.18 groupActionOnHashes

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\]](#), page 2090).

**Usage:** `groupActionOnHashes(Asigma,OCmov)`; Asigma: list, OCmov: list of list of cones

**Purpose:** From the list of orbits of orbitcones, and the symmetry group representation given by the matrices in Asigma, compute the corresponding permutation representation of the symmetry group on the orbit cones. The permutations are specified in a map representation of length the sum of the size of the orbits of OCmov.

**Return:** list of permutations

**Example:**

```
LIB "gitfan.lib";
ring R = 0,T(1..10),wp(1,1,1,1,1,1,1,1,1,1);
ideal J =
T(5)*T(10)-T(6)*T(9)+T(7)*T(8),
T(1)*T(9)-T(2)*T(7)+T(4)*T(5),
T(1)*T(8)-T(2)*T(6)+T(3)*T(5),
T(1)*T(10)-T(3)*T(7)+T(4)*T(6),
T(2)*T(10)-T(3)*T(9)+T(4)*T(8);
intmat Q[5][10] =
1, 1, 1, 1, 0, 0, 0, 0, 0, 0,
1, 0, 0, 0, 1, 1, 1, 0, 0, 0,
0, 1, 1, 0, 0, 0, -1, 1, 0, 0,
0, 1, 0, 1, 0, -1, 0, 0, 1, 0,
0, 0, 1, 1, -1, 0, 0, 0, 0, 1;
list AF= afaces(J);
list OC = orbitCones(AF,Q);
list generatorsG = permutationFromIntvec(intvec( 1, 3, 2, 4, 6, 5, 7, 8, 10, 9 )),
permutationFromIntvec(intvec( 5, 7, 1, 6, 9, 2, 8, 4, 10, 3 ));
list Asigmagens = groupActionOnQImage(generatorsG,Q);
groupActionOnHashes(Asigmagens,list(OC));
 $\mapsto$  [1]:
 $\mapsto$ 
| 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 \
18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 \
36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 5\
4 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72\
73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 \
91 92 93 94 95 96 97 98 99 100 101 102 103 104 105 106 107 108 1\
09 110 111 112 113 114 115 116 117 118 119 120 121 122 123 124 125 126 12\
7 128 129 130 131 132 133 134 135 136 137 138 139 140 141 142 143 144 145\
146 147 148 149 150 151 152 153 154 155 156 157 158 159 160 161 162 163 \
164 165 166 167 168 169 170 171 172|
 $\mapsto$  | 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 \
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 \
0 0 0 0 0 0 0 44 0 0 0 0 0 0 0 0 0 0 \
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 \
0 0 0 0 0 0 0 0 0 0 0 0 0 120 0 0 0 124 0\
126 127 128 129 0 0 0 0 0 0 0 0 0 0 0 0 0 0 \
0 0 0 0 0 0 0 0 0 0 86 0 0 0 90 0 92 93 \
94 95 0 0 0 0 0 0 0 0 137 0 0 0 141 0 143 144 145 14\
```



```

6  0  0  0  0  0  0  153 155 154 156  0 163 165 164 166  0 158 160\
159 161  0 168 170 169 171 172|
⇒
⇒ [2]:
⇒ |  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 \
   18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 \
  36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 5\
  4  55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72\
   73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 \
   91 92 93 94 95 96 97 98 99 100 101 102 103 104 105 106 107 108 1\
  09 110 111 112 113 114 115 116 117 118 119 120 121 122 123 124 125 126 12\
  7 128 129 130 131 132 133 134 135 136 137 138 139 140 141 142 143 144 145\
   146 147 148 149 150 151 152 153 154 155 156 157 158 159 160 161 162 163 \
  164 165 166 167 168 169 170 171 172|
⇒ |  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0\
   0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0\
   0  0  0  0  0  0  0  120  0  0  0  0  0  0  0  0  0  0  0 \
   0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0 \
  0  0  0  0  0  0  0  0  0  0  0  0  0  0  126  0  0  0  128  0\
   44 124 127 129  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0 \
   0  0  0  0  0  0  0  0  0  0  0  165  0  0  0  163  0 141  90 1\
  64 166  0  0  0  0  0  0  0  0  155  0  0  0  153  0 137  86 154 15\
  6  0  0  0  0  0  0 144  93 160 170  0 143  92 158 168  0 145  94\
  159 171  0 146  95 161 169 172|
⇒
list simplexSymmetryGroup = G25Action();
list orb = findOrbits(simplexSymmetryGroup,nrows(Q));
list simplexOrbitRepresentatives;
for (int i=1;i<=size(orb);i++){simplexOrbitRepresentatives[i]=orb[i][1];}
list afaceOrbitRepresentatives=afaces(J,simplexOrbitRepresentatives);
list fullDimAfaceOrbitRepresentatives=fullDimImages(afaceOrbitRepresentatives,Q);
list afaceOrbits=computeAfaceOrbits(fullDimAfaceOrbitRepresentatives,simplexSymmetryGroup);
list minAfaceOrbits = minimalAfaceOrbits(afaceOrbits);
list listOfOrbitConeOrbits = orbitConeOrbits(minAfaceOrbits,Q);
list listOfMinimalOrbitConeOrbits = minimalOrbitConeOrbits(listOfOrbitConeOrbits);
list Asigma = groupActionOnQImage(simplexSymmetryGroup,Q);
groupActionOnHashes(Asigma,listOfOrbitConeOrbits);
⇒ [1]:
⇒ |  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ |  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24\
   25|
⇒
⇒ [2]:
⇒ |  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ |  1  2  4  3  5  7  6  9  8 10 11 13 12 14 16 15 17 19 18 23 24 25 20 21\
   22|
⇒
⇒ [3]:
⇒ |  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ |  1  3  2  4  6  5  7  8 10  9 12 11 13 15 14 16 20 21 22 17 18 19 23 25\

```

```

24|
⇨
⇨ [4]:
⇨ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
24 25|
⇨ | 1 4 2 3 7 5 6 9 10 8 13 11 12 16 14 15 23 24 25 17 19 18 20 22\
21|
⇨
⇨ [5]:
⇨ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
24 25|
⇨ | 1 3 4 2 6 7 5 10 8 9 12 13 11 15 16 14 20 22 21 23 25 24 17 18\
19|
⇨
⇨ [6]:
⇨ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
24 25|
⇨ | 1 4 3 2 7 6 5 10 9 8 13 12 11 16 15 14 23 25 24 20 22 21 17 19\
18|
⇨
⇨ [7]:
⇨ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
24 25|
⇨ | 1 5 6 7 2 3 4 8 9 10 14 15 16 11 12 13 17 19 18 20 22 21 23 25\
24|
⇨
⇨ [8]:
⇨ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
24 25|
⇨ | 1 5 7 6 2 4 3 9 8 10 14 16 15 11 13 12 17 18 19 23 25 24 20 22\
21|
⇨
⇨ [9]:
⇨ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
24 25|
⇨ | 1 6 5 7 3 2 4 8 10 9 15 14 16 12 11 13 20 22 21 17 19 18 23 24\
25|
⇨
⇨ [10]:
⇨ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
24 25|
⇨ | 1 7 5 6 4 2 3 9 10 8 16 14 15 13 11 12 23 25 24 17 18 19 20 21\
22|
⇨
⇨ [11]:
⇨ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
24 25|
⇨ | 1 6 7 5 3 4 2 10 8 9 15 16 14 12 13 11 20 21 22 23 24 25 17 19\
18|
⇨
⇨ [12]:
⇨ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
24 25|

```

```

⇒ | 1 7 6 5 4 3 2 10 9 8 16 15 14 13 12 11 23 24 25 20 21 22 17 18\
   19|
⇒
⇒ [13]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ | 2 1 3 4 5 8 9 6 7 10 11 13 12 17 18 19 14 15 16 21 20 22 24 23\
   25|
⇒
⇒ [14]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ | 2 1 4 3 5 9 8 7 6 10 11 12 13 17 19 18 14 16 15 24 23 25 21 20\
   22|
⇒
⇒ [15]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ | 3 1 2 4 6 8 10 5 7 9 12 13 11 20 21 22 15 14 16 18 17 19 25 23\
   24|
⇒
⇒ [16]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ | 4 1 2 3 7 9 10 5 6 8 13 12 11 23 24 25 16 14 15 19 17 18 22 20\
   21|
⇒
⇒ [17]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ | 3 1 4 2 6 10 8 7 5 9 12 11 13 20 22 21 15 16 14 25 23 24 18 17\
   19|
⇒
⇒ [18]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ | 4 1 3 2 7 10 9 6 5 8 13 11 12 23 25 24 16 15 14 22 20 21 19 17\
   18|
⇒
⇒ [19]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ | 5 1 6 7 2 8 9 3 4 10 14 16 15 17 19 18 11 12 13 22 20 21 25 23\
   24|
⇒
⇒ [20]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ | 5 1 7 6 2 9 8 4 3 10 14 15 16 17 18 19 11 13 12 25 23 24 22 20\
   21|
⇒
⇒ [21]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\

```

```

      24 25|
⇒ | 6 1 5 7 3 8 10 2 4 9 15 16 14 20 22 21 12 11 13 19 17 18 24 23\
    25|
⇒
⇒ [22]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
    24 25|
⇒ | 7 1 5 6 4 9 10 2 3 8 16 15 14 23 25 24 13 11 12 18 17 19 21 20\
    22|
⇒
⇒ [23]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
    24 25|
⇒ | 6 1 7 5 3 10 8 4 2 9 15 14 16 20 21 22 12 13 11 24 23 25 19 17\
    18|
⇒
⇒ [24]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
    24 25|
⇒ | 7 1 6 5 4 10 9 3 2 8 16 14 15 23 24 25 13 12 11 21 20 22 18 17\
    19|
⇒
⇒ [25]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
    24 25|
⇒ | 2 3 1 4 8 5 9 6 10 7 13 11 12 18 17 19 21 20 22 14 15 16 24 25\
    23|
⇒
⇒ [26]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
    24 25|
⇒ | 2 4 1 3 9 5 8 7 10 6 12 11 13 19 17 18 24 23 25 14 16 15 21 22\
    20|
⇒
⇒ [27]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
    24 25|
⇒ | 3 2 1 4 8 6 10 5 9 7 13 12 11 21 20 22 18 17 19 15 14 16 25 24\
    23|
⇒
⇒ [28]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
    24 25|
⇒ | 4 2 1 3 9 7 10 5 8 6 12 13 11 24 23 25 19 17 18 16 14 15 22 21\
    20|
⇒
⇒ [29]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
    24 25|
⇒ | 3 4 1 2 10 6 8 7 9 5 11 12 13 22 20 21 25 23 24 15 16 14 18 19\
    17|
⇒
⇒ [30]:

```

```

⇒      | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
      24 25|
⇒      | 4 3 1 2 10 7 9 6 8 5 11 13 12 25 23 24 22 20 21 16 15 14 19 18\
      17|
⇒
⇒ [31]:
⇒      | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
      24 25|
⇒      | 5 6 1 7 8 2 9 3 10 4 16 14 15 19 17 18 22 20 21 11 12 13 25 24\
      23|
⇒
⇒ [32]:
⇒      | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
      24 25|
⇒      | 5 7 1 6 9 2 8 4 10 3 15 14 16 18 17 19 25 23 24 11 13 12 22 21\
      20|
⇒
⇒ [33]:
⇒      | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
      24 25|
⇒      | 6 5 1 7 8 3 10 2 9 4 16 15 14 22 20 21 19 17 18 12 11 13 24 25\
      23|
⇒
⇒ [34]:
⇒      | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
      24 25|
⇒      | 7 5 1 6 9 4 10 2 8 3 15 16 14 25 23 24 18 17 19 13 11 12 21 22\
      20|
⇒
⇒ [35]:
⇒      | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
      24 25|
⇒      | 6 7 1 5 10 3 8 4 9 2 14 15 16 21 20 22 24 23 25 12 13 11 19 18\
      17|
⇒
⇒ [36]:
⇒      | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
      24 25|
⇒      | 7 6 1 5 10 4 9 3 8 2 14 16 15 24 23 25 21 20 22 13 12 11 18 19\
      17|
⇒
⇒ [37]:
⇒      | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
      24 25|
⇒      | 2 3 4 1 8 9 5 10 6 7 13 12 11 18 19 17 21 22 20 24 25 23 14 15\
      16|
⇒
⇒ [38]:
⇒      | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
      24 25|
⇒      | 2 4 3 1 9 8 5 10 7 6 12 13 11 19 18 17 24 25 23 21 22 20 14 16\
      15|
⇒

```

```

⇒ [39]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
  24 25|
⇒ | 3 2 4 1 8 10 6 9 5 7 13 11 12 21 22 20 18 19 17 25 24 23 15 14\
  16|
⇒
⇒ [40]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
  24 25|
⇒ | 4 2 3 1 9 10 7 8 5 6 12 11 13 24 25 23 19 18 17 22 21 20 16 14\
  15|
⇒
⇒ [41]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
  24 25|
⇒ | 3 4 2 1 10 8 6 9 7 5 11 13 12 22 21 20 25 24 23 18 19 17 15 16\
  14|
⇒
⇒ [42]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
  24 25|
⇒ | 4 3 2 1 10 9 7 8 6 5 11 12 13 25 24 23 22 21 20 19 18 17 16 15\
  14|
⇒
⇒ [43]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
  24 25|
⇒ | 5 6 7 1 8 9 2 10 3 4 16 15 14 19 18 17 22 21 20 25 24 23 11 12\
  13|
⇒
⇒ [44]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
  24 25|
⇒ | 5 7 6 1 9 8 2 10 4 3 15 16 14 18 19 17 25 24 23 22 21 20 11 13\
  12|
⇒
⇒ [45]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
  24 25|
⇒ | 6 5 7 1 8 10 3 9 2 4 16 14 15 22 21 20 19 18 17 24 25 23 12 11\
  13|
⇒
⇒ [46]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
  24 25|
⇒ | 7 5 6 1 9 10 4 8 2 3 15 14 16 25 24 23 18 19 17 21 22 20 13 11\
  12|
⇒
⇒ [47]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
  24 25|
⇒ | 6 7 5 1 10 8 3 9 4 2 14 16 15 21 22 20 24 25 23 19 18 17 12 13\
  11|

```

```

⇒
⇒ [48]:
⇒   |  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ |  7  6  5  1 10  9  4  8  3  2 14 15 16 24 25 23 21 22 20 18 19 17 13 12\
   11|
⇒
⇒ [49]:
⇒   |  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ |  2  5  8  9  1  3  4  6  7 10 17 18 19 11 13 12 14 16 15 21 22 20 24 25\
   23|
⇒
⇒ [50]:
⇒   |  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ |  2  5  9  8  1  4  3  7  6 10 17 19 18 11 12 13 14 15 16 24 25 23 21 22\
   20|
⇒
⇒ [51]:
⇒   |  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ |  5  2  8  9  1  6  7  3  4 10 17 19 18 14 16 15 11 13 12 22 21 20 25 24\
   23|
⇒
⇒ [52]:
⇒   |  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ |  5  2  9  8  1  7  6  4  3 10 17 18 19 14 15 16 11 12 13 25 24 23 22 21\
   20|
⇒
⇒ [53]:
⇒   |  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ |  3  6  8 10  1  2  4  5  7  9 20 21 22 12 13 11 15 16 14 18 19 17 25 24\
   23|
⇒
⇒ [54]:
⇒   |  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ |  4  7  9 10  1  2  3  5  6  8 23 24 25 13 12 11 16 15 14 19 18 17 22 21\
   20|
⇒
⇒ [55]:
⇒   |  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ |  3  6 10  8  1  4  2  7  5  9 20 22 21 12 11 13 15 14 16 25 24 23 18 19\
   17|
⇒
⇒ [56]:
⇒   |  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ |  4  7 10  9  1  3  2  6  5  8 23 25 24 13 11 12 16 14 15 22 21 20 19 18\

```

```

17|
⇒
⇒ [57]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
24 25|
⇒ | 6 3 8 10 1 5 7 2 4 9 20 22 21 15 16 14 12 13 11 19 18 17 24 25\
23|
⇒
⇒ [58]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
24 25|
⇒ | 7 4 9 10 1 5 6 2 3 8 23 25 24 16 15 14 13 12 11 18 19 17 21 22\
20|
⇒
⇒ [59]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
24 25|
⇒ | 6 3 10 8 1 7 5 4 2 9 20 21 22 15 14 16 12 11 13 24 25 23 19 18\
17|
⇒
⇒ [60]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
24 25|
⇒ | 7 4 10 9 1 6 5 3 2 8 23 24 25 16 14 15 13 11 12 21 22 20 18 19\
17|
⇒
⇒ [61]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
24 25|
⇒ | 2 8 5 9 3 1 4 6 10 7 18 17 19 13 11 12 21 22 20 14 16 15 24 23\
25|
⇒
⇒ [62]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
24 25|
⇒ | 2 9 5 8 4 1 3 7 10 6 19 17 18 12 11 13 24 25 23 14 15 16 21 20\
22|
⇒
⇒ [63]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
24 25|
⇒ | 5 8 2 9 6 1 7 3 10 4 19 17 18 16 14 15 22 21 20 11 13 12 25 23\
24|
⇒
⇒ [64]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
24 25|
⇒ | 5 9 2 8 7 1 6 4 10 3 18 17 19 15 14 16 25 24 23 11 12 13 22 20\
21|
⇒
⇒ [65]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
24 25|

```



```

⇒ | 3 8 6 10 2 1 4 5 9 7 21 20 22 13 12 11 18 19 17 15 16 14 25 23\
   24|
⇒
⇒ [66]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ | 4 9 7 10 2 1 3 5 8 6 24 23 25 12 13 11 19 18 17 16 15 14 22 20\
   21|
⇒
⇒ [67]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ | 3 10 6 8 4 1 2 7 9 5 22 20 21 11 12 13 25 24 23 15 14 16 18 17\
   19|
⇒
⇒ [68]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ | 4 10 7 9 3 1 2 6 8 5 25 23 24 11 13 12 22 21 20 16 14 15 19 17\
   18|
⇒
⇒ [69]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ | 6 8 3 10 5 1 7 2 9 4 22 20 21 16 15 14 19 18 17 12 13 11 24 23\
   25|
⇒
⇒ [70]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ | 7 9 4 10 5 1 6 2 8 3 25 23 24 15 16 14 18 19 17 13 12 11 21 20\
   22|
⇒
⇒ [71]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ | 6 10 3 8 7 1 5 4 9 2 21 20 22 14 15 16 24 25 23 12 11 13 19 17\
   18|
⇒
⇒ [72]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ | 7 10 4 9 6 1 5 3 8 2 24 23 25 14 16 15 21 22 20 13 11 12 18 17\
   19|
⇒
⇒ [73]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ | 2 8 9 5 3 4 1 10 6 7 18 19 17 13 12 11 21 20 22 24 23 25 14 16\
   15|
⇒
⇒ [74]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\

```

```

      24 25|
⇒ | 2 9 8 5 4 3 1 10 7 6 19 18 17 12 13 11 24 23 25 21 20 22 14 15\
    16|
⇒
⇒ [75]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
    24 25|
⇒ | 5 8 9 2 6 7 1 10 3 4 19 18 17 16 15 14 22 20 21 25 23 24 11 13\
    12|
⇒
⇒ [76]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
    24 25|
⇒ | 5 9 8 2 7 6 1 10 4 3 18 19 17 15 16 14 25 23 24 22 20 21 11 12\
    13|
⇒
⇒ [77]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
    24 25|
⇒ | 3 8 10 6 2 4 1 9 5 7 21 22 20 13 11 12 18 17 19 25 23 24 15 16\
    14|
⇒
⇒ [78]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
    24 25|
⇒ | 4 9 10 7 2 3 1 8 5 6 24 25 23 12 11 13 19 17 18 22 20 21 16 15\
    14|
⇒
⇒ [79]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
    24 25|
⇒ | 3 10 8 6 4 2 1 9 7 5 22 21 20 11 13 12 25 23 24 18 17 19 15 14\
    16|
⇒
⇒ [80]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
    24 25|
⇒ | 4 10 9 7 3 2 1 8 6 5 25 24 23 11 12 13 22 20 21 19 17 18 16 14\
    15|
⇒
⇒ [81]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
    24 25|
⇒ | 6 8 10 3 5 7 1 9 2 4 22 21 20 16 14 15 19 17 18 24 23 25 12 13\
    11|
⇒
⇒ [82]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
    24 25|
⇒ | 7 9 10 4 5 6 1 8 2 3 25 24 23 15 14 16 18 17 19 21 20 22 13 12\
    11|
⇒
⇒ [83]:

```

```

⇒      |  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
      24 25|
⇒ |  6 10  8  3  7  5  1  9  4  2 21 22 20 14 16 15 24 23 25 19 17 18 12 11\
      13|
⇒
⇒ [84]:
⇒      |  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
      24 25|
⇒ |  7 10  9  4  6  5  1  8  3  2 24 25 23 14 15 16 21 20 22 18 17 19 13 11\
      12|
⇒
⇒ [85]:
⇒      |  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
      24 25|
⇒ |  8  2  5  9  3  6 10  1  4  7 18 19 17 21 22 20 13 11 12 16 14 15 23 24\
      25|
⇒
⇒ [86]:
⇒      |  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
      24 25|
⇒ |  9  2  5  8  4  7 10  1  3  6 19 18 17 24 25 23 12 11 13 15 14 16 20 21\
      22|
⇒
⇒ [87]:
⇒      |  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
      24 25|
⇒ |  8  5  2  9  6  3 10  1  7  4 19 18 17 22 21 20 16 14 15 13 11 12 23 25\
      24|
⇒
⇒ [88]:
⇒      |  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
      24 25|
⇒ |  9  5  2  8  7  4 10  1  6  3 18 19 17 25 24 23 15 14 16 12 11 13 20 22\
      21|
⇒
⇒ [89]:
⇒      |  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
      24 25|
⇒ |  8  3  6 10  2  5  9  1  4  7 21 22 20 18 19 17 13 12 11 16 15 14 23 25\
      24|
⇒
⇒ [90]:
⇒      |  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
      24 25|
⇒ |  9  4  7 10  2  5  8  1  3  6 24 25 23 19 18 17 12 13 11 15 16 14 20 22\
      21|
⇒
⇒ [91]:
⇒      |  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
      24 25|
⇒ |  8  6  3 10  5  2  9  1  7  4 22 21 20 19 18 17 16 15 14 13 12 11 23 24\
      25|
⇒

```

```

⇒ [92]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
  24 25|
⇒ | 9 7 4 10 5 2 8 1 6 3 25 24 23 18 19 17 15 16 14 12 13 11 20 21\
  22|
⇒
⇒ [93]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
  24 25|
⇒ | 10 3 6 8 4 7 9 1 2 5 22 21 20 25 24 23 11 12 13 14 15 16 17 18\
  19|
⇒
⇒ [94]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
  24 25|
⇒ | 10 4 7 9 3 6 8 1 2 5 25 24 23 22 21 20 11 13 12 14 16 15 17 19\
  18|
⇒
⇒ [95]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
  24 25|
⇒ | 10 6 3 8 7 4 9 1 5 2 21 22 20 24 25 23 14 15 16 11 12 13 17 19\
  18|
⇒
⇒ [96]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
  24 25|
⇒ | 10 7 4 9 6 3 8 1 5 2 24 25 23 21 22 20 14 16 15 11 13 12 17 18\
  19|
⇒
⇒ [97]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
  24 25|
⇒ | 8 2 9 5 3 10 6 4 1 7 18 17 19 21 20 22 13 12 11 23 24 25 16 14\
  15|
⇒
⇒ [98]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
  24 25|
⇒ | 9 2 8 5 4 10 7 3 1 6 19 17 18 24 23 25 12 13 11 20 21 22 15 14\
  16|
⇒
⇒ [99]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
  24 25|
⇒ | 8 5 9 2 6 10 3 7 1 4 19 17 18 22 20 21 16 15 14 23 25 24 13 11\
  12|
⇒
⇒ [100]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
  24 25|
⇒ | 9 5 8 2 7 10 4 6 1 3 18 17 19 25 23 24 15 16 14 20 22 21 12 11\
  13|

```

```

⇒
⇒ [101]:
⇒   | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ | 8 3 10 6 2 9 5 4 1 7 21 20 22 18 17 19 13 11 12 23 25 24 16 15\
   14|
⇒
⇒ [102]:
⇒   | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ | 9 4 10 7 2 8 5 3 1 6 24 23 25 19 17 18 12 11 13 20 22 21 15 16\
   14|
⇒
⇒ [103]:
⇒   | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ | 8 6 10 3 5 9 2 7 1 4 22 20 21 19 17 18 16 14 15 23 24 25 13 12\
   11|
⇒
⇒ [104]:
⇒   | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ | 9 7 10 4 5 8 2 6 1 3 25 23 24 18 17 19 15 14 16 20 21 22 12 13\
   11|
⇒
⇒ [105]:
⇒   | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ | 10 3 8 6 4 9 7 2 1 5 22 20 21 25 23 24 11 13 12 17 18 19 14 15\
   16|
⇒
⇒ [106]:
⇒   | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ | 10 4 9 7 3 8 6 2 1 5 25 23 24 22 20 21 11 12 13 17 19 18 14 16\
   15|
⇒
⇒ [107]:
⇒   | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ | 10 6 8 3 7 9 4 5 1 2 21 20 22 24 23 25 14 16 15 17 19 18 11 12\
   13|
⇒
⇒ [108]:
⇒   | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ | 10 7 9 4 6 8 3 5 1 2 24 23 25 21 20 22 14 15 16 17 18 19 11 13\
   12|
⇒
⇒ [109]:
⇒   | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ | 8 9 2 5 10 3 6 4 7 1 17 18 19 20 21 22 23 24 25 13 12 11 16 15\

```

```

14|
⇒
⇒ [110]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
24 25|
⇒ | 9 8 2 5 10 4 7 3 6 1 17 19 18 23 24 25 20 21 22 12 13 11 15 16\
14|
⇒
⇒ [111]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
24 25|
⇒ | 8 9 5 2 10 6 3 7 4 1 17 19 18 20 22 21 23 25 24 16 15 14 13 12\
11|
⇒
⇒ [112]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
24 25|
⇒ | 9 8 5 2 10 7 4 6 3 1 17 18 19 23 25 24 20 22 21 15 16 14 12 13\
11|
⇒
⇒ [113]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
24 25|
⇒ | 8 10 3 6 9 2 5 4 7 1 20 21 22 17 18 19 23 25 24 13 11 12 16 14\
15|
⇒
⇒ [114]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
24 25|
⇒ | 9 10 4 7 8 2 5 3 6 1 23 24 25 17 19 18 20 22 21 12 11 13 15 14\
16|
⇒
⇒ [115]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
24 25|
⇒ | 8 10 6 3 9 5 2 7 4 1 20 22 21 17 19 18 23 24 25 16 14 15 13 11\
12|
⇒
⇒ [116]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
24 25|
⇒ | 9 10 7 4 8 5 2 6 3 1 23 25 24 17 18 19 20 21 22 15 14 16 12 11\
13|
⇒
⇒ [117]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
24 25|
⇒ | 10 8 3 6 9 4 7 2 5 1 20 22 21 23 25 24 17 18 19 11 13 12 14 16\
15|
⇒
⇒ [118]:
⇒ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
24 25|

```

```

⇒ | 10  9  4  7  8  3  6  2  5  1 23 25 24 20 22 21 17 19 18 11 12 13 14 15\
   16|
⇒
⇒ [119]:
⇒ | 1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ | 10  8  6  3  9  7  4  5  2  1 20 21 22 23 24 25 17 19 18 14 16 15 11 13\
   12|
⇒
⇒ [120]:
⇒ | 1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23\
   24 25|
⇒ | 10  9  7  4  8  6  3  5  2  1 23 24 25 20 21 22 17 18 19 14 15 16 11 12\
   13|
⇒

```

### D.13.3.19 storeActionOnOrbitConeIndices

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\], page 2090](#)).

**Usage:** `storeActionOnOrbitConeIndices(generatorsGperm,st);` `generatorsGperm:` list, `fn:` string

**Purpose:** Write the action on the set of orbit cones to the file `fn` in Singular readable format.

**Return:** nothing

### D.13.3.20 permutationFromIntvec

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\], page 2090](#)).

**Usage:** `permutationFromIntvec(sigmaImage);` `sigmaImage:` intvec

**Purpose:** Create a permutation from an intvec of images.

**Return:** permutation

### D.13.3.21 permutationToIntvec

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\], page 2090](#)).

**Usage:** `permutationToIntvec(sigma);` `sigma:` permutation

**Purpose:** Convert a permutation to an intvec of images.

**Return:** intvec

**Example:**

```

LIB "gitfan.lib";
permutation sigma = permutationFromIntvec(intvec( 1, 2, 4, 3, 5, 7, 6, 9, 8, 10 ));
sigma;
⇒ | 1  2  3  4  5  6  7  8  9 10|
⇒ | 1  2  4  3  5  7  6  9  8 10|
⇒
permutationToIntvec(sigma);
⇒ 1,2,4,3,5,7,6,9,8,10

```

### D.13.3.22 evaluateProduct

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\]](#), page 2090).

**Usage:** `evaluateProduct(generatorsGperm,st);` generatorsGperm: list, st: string

**Purpose:** Evaluates a formal product of variables  $x_i$  in `st`, where  $x_i$  corresponds to the permutation `generatorsGperm[i]`.

**Return:** permutation

**Example:**

```
LIB "gitfan.lib";
ring R = 0,T(1..10),wp(1,1,1,1,1,1,1,1,1,1);
ideal J =
T(5)*T(10)-T(6)*T(9)+T(7)*T(8),
T(1)*T(9)-T(2)*T(7)+T(4)*T(5),
T(1)*T(8)-T(2)*T(6)+T(3)*T(5),
T(1)*T(10)-T(3)*T(7)+T(4)*T(6),
T(2)*T(10)-T(3)*T(9)+T(4)*T(8);
intmat Q[5][10] =
1, 1, 1, 1, 0, 0, 0, 0, 0, 0,
1, 0, 0, 0, 1, 1, 1, 0, 0, 0,
0, 1, 1, 0, 0, 0, -1, 1, 0, 0,
0, 1, 0, 1, 0, -1, 0, 0, 1, 0,
0, 0, 1, 1, -1, 0, 0, 0, 0, 1;
list AF= afaces(J);
list OC = orbitCones(AF,Q);
list generatorsG = permutationFromIntvec(intvec( 1, 3, 2, 4, 6, 5, 7, 8, 10, 9 )),
permutationFromIntvec(intvec( 5, 7, 1, 6, 9, 2, 8, 4, 10, 3 ));
list Asigmagens = groupActionOnQImage(generatorsG,Q);
//list actionOnOrbitconeIndicesForGenerators = groupActionOnHashes(Asigmagens,OC);
string elementInTermsOfGenerators =
"(x2^-1*x1^-1)^3*x1^-1";
//evaluateProduct(actionOnOrbitconeIndicesForGenerators, elementInTermsOfGenerators)
```

### D.13.3.23 GITfanSymmetric

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\]](#), page 2090).

**Usage:** `GITfanSymmetric(OC, Q, Qgamma, actiononorbitcones [, file1, file2]);` OC:list, Q:bigintmat, Qgamma:cone, actiononorbitcones: list of intvec, file1:string, file2:string

**Purpose:** Returns the common refinement of the cones given in the list `OC` which is supposed to contain the orbit cones intersected with `Qgamma`. The list `actiononorbitcones` is supposed to contain the symmetry group acting as permutations of on the list of orbit cones in `OC`. The optional argument can be used to specify one or two strings with file names, where the first file will contain the hashes of the GIT-cones and the second argument the actual cones in their H-representation. To obtain the whole GIT-fan `Qgamma` has to be take the cone generated by the columns of `Q`.

**Return:** a list containing the bigint hashes of the GIT cones.

**Example:**



```

LIB "gitfan.lib";
ring R = 0,T(1..10),wp(1,1,1,1,1,1,1,1,1,1);
ideal J =
T(5)*T(10)-T(6)*T(9)+T(7)*T(8),
T(1)*T(9)-T(2)*T(7)+T(4)*T(5),
T(1)*T(8)-T(2)*T(6)+T(3)*T(5),
T(1)*T(10)-T(3)*T(7)+T(4)*T(6),
T(2)*T(10)-T(3)*T(9)+T(4)*T(8);
intmat Q[5][10] =
1, 1, 1, 1, 0, 0, 0, 0, 0, 0,
1, 0, 0, 0, 1, 1, 1, 0, 0, 0,
0, 1, 1, 0, 0, 0, -1, 1, 0, 0,
0, 1, 0, 1, 0, -1, 0, 0, 1, 0,
0, 0, 1, 1, -1, 0, 0, 0, 0, 1;
list simplexSymmetryGroup = G25Action();
list simplexOrbitRepresentatives = intvec( 1, 2, 3, 4, 5 ),
intvec( 1, 2, 3, 5, 6 ),
intvec( 1, 2, 3, 5, 7 ),
intvec( 1, 2, 3, 5, 10 ),
intvec( 1, 2, 3, 7, 9 ),
intvec( 1, 2, 6, 9, 10 ),
intvec( 1, 2, 3, 4, 5, 6 ),
intvec( 1, 2, 3, 4, 5, 10 ),
intvec( 1, 2, 3, 5, 6, 8 ),
intvec( 1, 2, 3, 5, 6, 9 ),
intvec( 1, 2, 3, 5, 7, 10 ),
intvec( 1, 2, 3, 7, 9, 10 ),
intvec( 1, 2, 3, 4, 5, 6, 7 ),
intvec( 1, 2, 3, 4, 5, 6, 8 ),
intvec( 1, 2, 3, 4, 5, 6, 9 ),
intvec( 1, 2, 3, 5, 6, 9, 10 ),
intvec( 1, 2, 3, 4, 5, 6, 7, 8 ),
intvec( 1, 2, 3, 4, 5, 6, 9, 10 ),
intvec( 1, 2, 3, 4, 5, 6, 7, 8, 9 ),
intvec( 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 );
list afaceOrbitRepresentatives=afaces(J,simplexOrbitRepresentatives);
list fulldimAfaceOrbitRepresentatives=fullDimImages(afaceOrbitRepresentatives,Q);
list afaceOrbits=computeAfaceOrbits(fulldimAfaceOrbitRepresentatives,simplexSymmetryGroup);
apply(afaceOrbits,size);
⇨ 10 15 10 1
list minAfaceOrbits = minimalAfaceOrbits(afaceOrbits);
apply(minAfaceOrbits,size);
⇨ 10 15
list listOfOrbitConeOrbits = orbitConeOrbits(minAfaceOrbits,Q);
apply(listOfOrbitConeOrbits,size);
⇨ 10 15
list listOfMinimalOrbitConeOrbits = minimalOrbitConeOrbits(listOfOrbitConeOrbits);
size(listOfMinimalOrbitConeOrbits);
⇨ 2
list Asigma = groupActionOnQImage(simplexSymmetryGroup,Q);
list actionOnOrbitconeIndices = groupActionOnHashes(Asigma,listOfOrbitConeOrbits);
list OClist = listOfOrbitConeOrbits[1];
for (int i =2;i<=size(listOfOrbitConeOrbits);i++){

```

```

OClist = OClist + listOfOrbitConeOrbits[i];
}
cone mov = coneViaPoints(transpose(Q));
mov = canonicalizeCone(mov);
printlevel = 3;
list Sigma = GITfanSymmetric(OClist, Q, mov, actionOnOrbitconeIndices);
⇒ testing 215, 230, 113, 58, 85
⇒ computed cone of dimension 5
⇒ time for facets: 0
⇒ overall: 2   open: 1   time for loop: 0
⇒ time for facets: 0
⇒ overall: 3   open: 1   time for loop: 0
⇒ time for facets: 0
⇒ overall: 5   open: 2   time for loop: 0
⇒ time for facets: 0
⇒ overall: 6   open: 2   time for loop: 0
⇒ time for facets: 0
⇒ overall: 6   open: 1   time for loop: 0
⇒ time for facets: 0
⇒ overall: 6   open: 0   time for loop: 0
Sigma;
⇒ [1]:
⇒   7183
⇒ [2]:
⇒  224275
⇒ [3]:
⇒  605191
⇒ [4]:
⇒  4946947
⇒ [5]:
⇒  14416897
⇒ [6]:
⇒  33553408

```

### D.13.3.24 GITfanParallelSymmetric

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\]](#), page 2090).

**Usage:** `GITfanParallelSymmetric(OC, Q, Qgamma, actiononorbitcones [, file1]);` OC:list, Q:bigintmat, Qgamma:cone, actiononorbitcones: list of intvec, file1:string

**Purpose:** Returns the common refinement of the cones given in the list OC which is supposed to contain the orbit cones intersected with Qgamma. The list actiononorbitcones is supposed to contain the symmetry group acting as permutations of on the list of orbit cones in OC. The optional argument can be used to specify a name for a file which will contain the hashes of the GIT-cones. To obtain the whole GIT-fan Qgamma has to be take the cone generated by the columns of Q.

**Return:** a list containing the bigint hashes of the GIT cones.

**Note:** The procedure uses parallel computation for the construction of the GIT-cones.

**Example:**

```

LIB "gitfan.lib";
ring R = 0,T(1..10),wp(1,1,1,1,1,1,1,1,1,1);

```

```

ideal J =
T(5)*T(10)-T(6)*T(9)+T(7)*T(8),
T(1)*T(9)-T(2)*T(7)+T(4)*T(5),
T(1)*T(8)-T(2)*T(6)+T(3)*T(5),
T(1)*T(10)-T(3)*T(7)+T(4)*T(6),
T(2)*T(10)-T(3)*T(9)+T(4)*T(8);
intmat Q[5][10] =
1, 1, 1, 1, 0, 0, 0, 0, 0, 0,
1, 0, 0, 0, 1, 1, 1, 0, 0, 0,
0, 1, 1, 0, 0, 0, -1, 1, 0, 0,
0, 1, 0, 1, 0, -1, 0, 0, 1, 0,
0, 0, 1, 1, -1, 0, 0, 0, 0, 1;
list simplexSymmetryGroup = G25Action();
list simplexOrbitRepresentatives = intvec( 1, 2, 3, 4, 5 ),
intvec( 1, 2, 3, 5, 6 ),
intvec( 1, 2, 3, 5, 7 ),
intvec( 1, 2, 3, 5, 10 ),
intvec( 1, 2, 3, 7, 9 ),
intvec( 1, 2, 6, 9, 10 ),
intvec( 1, 2, 3, 4, 5, 6 ),
intvec( 1, 2, 3, 4, 5, 10 ),
intvec( 1, 2, 3, 5, 6, 8 ),
intvec( 1, 2, 3, 5, 6, 9 ),
intvec( 1, 2, 3, 5, 7, 10 ),
intvec( 1, 2, 3, 7, 9, 10 ),
intvec( 1, 2, 3, 4, 5, 6, 7 ),
intvec( 1, 2, 3, 4, 5, 6, 8 ),
intvec( 1, 2, 3, 4, 5, 6, 9 ),
intvec( 1, 2, 3, 5, 6, 9, 10 ),
intvec( 1, 2, 3, 4, 5, 6, 7, 8 ),
intvec( 1, 2, 3, 4, 5, 6, 9, 10 ),
intvec( 1, 2, 3, 4, 5, 6, 7, 8, 9 ),
intvec( 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 );
list afaceOrbitRepresentatives=afaces(J,simplexOrbitRepresentatives);
list fullDimAfaceOrbitRepresentatives=fullDimImages(afaceOrbitRepresentatives,Q);
list afaceOrbits=computeAfaceOrbits(fullDimAfaceOrbitRepresentatives,simplexSymmetryGroup);
apply(afaceOrbits,size);
⇨ 10 15 10 1
list minAfaceOrbits = minimalAfaceOrbits(afaceOrbits);
apply(minAfaceOrbits,size);
⇨ 10 15
list listOfOrbitConeOrbits = orbitConeOrbits(minAfaceOrbits,Q);
apply(listOfOrbitConeOrbits,size);
⇨ 10 15
list listOfMinimalOrbitConeOrbits = minimalOrbitConeOrbits(listOfOrbitConeOrbits);
size(listOfMinimalOrbitConeOrbits);
⇨ 2
list Asigma = groupActionOnQImage(simplexSymmetryGroup,Q);
list actionOnOrbitConeIndices = groupActionOnHashes(Asigma,listOfOrbitConeOrbits);
list OClist = listOfOrbitConeOrbits[1];
for (int i =2;i<=size(listOfOrbitConeOrbits);i++){
OClist = OClist + listOfOrbitConeOrbits[i];
}

```

```

cone mov = coneViaPoints(transpose(Q));
mov = canonicalizeCone(mov);
list Sigma = GITfanParallelSymmetric(OClist, Q, mov, actionOnOrbitconeIndices);
Sigma;
↳ [1]:
↳ 7183
↳ [2]:
↳ 224275
↳ [3]:
↳ 605191
↳ [4]:
↳ 4946947
↳ [5]:
↳ 14416897
↳ [6]:
↳ 33553408

```

### D.13.3.25 bigintToBinary

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\], page 2090](#)).

**Usage:** `bigintToBinary(n, r)`:  $n$  bigint,  $r$  int.

**Assume:**  $n$  is smaller than  $2^r$ .

**Return:** an intvec, with entries the positions of 1 in the binary representation of  $n$  with  $r$  bits.

**Example:**

```

LIB "gitfan.lib";
bigintToBinary(bigint(2)^90-1, 90);
↳ 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,2\
8,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,52\
,53,54,55,56,57,58,59,60,61,62,63,64,65,66,67,68,69,70,71,72,73,74,75,76,\
77,78,79,80,81,82,83,84,85,86,87,88,89,90

```

### D.13.3.26 binaryToBigint

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\], page 2090](#)).

### D.13.3.27 applyPermutationToIntvec

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\], page 2090](#)).

**Usage:** `applyPermutationToIntvec(v,g)`;  $v$ : intvec,  $g$ : permutation

**Purpose:** Apply  $g$  to the set of entries of  $v$ . The result is a sorted intvec. We assume that the entries of  $v$  are valid arguments of  $g$ . We do not require that the input is sorted.

**Return:** intvec.

**Example:**

```

LIB "gitfan.lib";
permutation g = permutationFromIntvec(intvec(10, 9, 7, 4, 8, 6, 3, 5, 2, 1));
applyPermutationToIntvec(intvec(1, 3, 4, 6, 8),g);
↳ 4,5,6,7,10

```

### D.13.3.28 hashToCone

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\]](#), page 2090).

**Usage:** `hashToCone(v, OC)`: `v` bigint, `OC` list of cones.

**Assume:** the elements of `OC` are the orbit cones used in the hash representation of the GIT cones.

**Return:** a cone, the intersection of the cones in `OC` according to the binary representation of the hash `v`.

**Example:**

```
LIB "gitfan.lib";
setcores(4);
↪ 4
ring R = 0,T(1..10),wp(1,1,1,1,1,1,1,1,1,1);
ideal J =
T(5)*T(10)-T(6)*T(9)+T(7)*T(8),
T(1)*T(9)-T(2)*T(7)+T(4)*T(5),
T(1)*T(8)-T(2)*T(6)+T(3)*T(5),
T(1)*T(10)-T(3)*T(7)+T(4)*T(6),
T(2)*T(10)-T(3)*T(9)+T(4)*T(8);
intmat Q[5][10] =
1, 1, 1, 1, 0, 0, 0, 0, 0, 0,
1, 0, 0, 0, 1, 1, 1, 0, 0, 0,
0, 1, 1, 0, 0, 0, -1, 1, 0, 0,
0, 1, 0, 1, 0, -1, 0, 0, 1, 0,
0, 0, 1, 1, -1, 0, 0, 0, 0, 1;
list AF= afaces(J,nrows(Q));
AF=fullDimImages(AF,Q);
AF = minimalAfaces(AF);
list OC = orbitCones(AF,Q);
bigint v = 21300544;
hashToCone(v, OC);
↪ AMBIENT_DIM
↪ 5
↪ FACETS
↪ -1, 0,0, 0,1,
↪ -1, 0,1, 0,0,
↪ 0, 0,0,-1,0,
↪ 0, 1,0, 1,0,
↪ 1,-1,0,-1,0
↪ LINEAR_SPAN
↪
↪
```

### D.13.3.29 hashesToFan

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\]](#), page 2090).

**Usage:** `hashesToFan(hashes, OC)`: `hashes` list of bigint, `OC` list of cones.

**Assume:** the elements of `OC` are the orbit cones used in the hash representation of the GIT cones.

**Return:** a fan, with maximal cones the intersections of the cones in OC according to the binary representation of the hashes.

**Example:**

```
LIB "gitfan.lib";
setcores(4);
↪ 4
ring R = 0,T(1..10),wp(1,1,1,1,1,1,1,1,1,1);
ideal J =
T(5)*T(10)-T(6)*T(9)+T(7)*T(8),
T(1)*T(9)-T(2)*T(7)+T(4)*T(5),
T(1)*T(8)-T(2)*T(6)+T(3)*T(5),
T(1)*T(10)-T(3)*T(7)+T(4)*T(6),
T(2)*T(10)-T(3)*T(9)+T(4)*T(8);
intmat Q[5][10] =
1, 1, 1, 1, 0, 0, 0, 0, 0, 0,
1, 0, 0, 0, 1, 1, 1, 0, 0, 0,
0, 1, 1, 0, 0, 0, 0, -1, 1, 0,
0, 1, 0, 1, 0, -1, 0, 0, 1, 0,
0, 0, 1, 1, -1, 0, 0, 0, 0, 1;
list AF= afaces(J,nrows(Q));
AF=fullDimImages(AF,Q);
AF = minimalAfaces(AF);
list OC = orbitCones(AF,Q);
cone Qgamma = coneViaPoints(transpose(Q));
list GIT = GITfanParallel(OC,Q,Qgamma);
fan Sigma = hashesToFan(GIT,OC);
```

### D.13.3.30 gitCone

Procedure from library `gitfan.lib` (see [Section D.13.3 \[gitfan.lib\]](#), page 2090).

**Usage:** `gitCone(a, Q, w)`; a: ideal, Q:bigintmat, w:bigintmat

**Purpose:** Returns the GIT-cone  $\lambda(w)$ , i.e. the intersection of all orbit cones containing the vector w.

**Note:** call this only if you are interested in a single GIT-cone.

**Return:** a cone.

**Example:**

```
LIB "gitfan.lib";
intmat Q[3][4] =
1,0,1,0,
0,1,0,1,
0,0,1,1;
ring R = 0,T(1..4),dp;
ideal a = 0;
bigintmat w[1][3] = 3,3,1;
cone lambda = gitCone(a, Q, w);
rays(lambda);
↪ 1,0,0,
↪ 0,1,0,
↪ 1,1,1
```

```

bigintmat w2[1][3] = 1,1,1;
cone lambda2 = gitCone(a, Q, w2);
rays(lambda2);
↳ 1,1,1

```

### D.13.4 polymake\_lib

**Library:** polymake.lib

**Purpose:** Computations with polytopes and fans, interface to TOPCOM

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Yue Ren, email: ren@mathematik.uni-kl.de

**Warning:** Most procedures will not work unless polymake or topcom is installed and if so, they will only work with the operating system LINUX! For more detailed information see IMPORTANT NOTE respectively consult the help string of the procedures.

The conventions used in this library for polytopes and fans, e.g. the length and labeling of their vertices resp. rays, differs from the conventions used in polymake and thus from the conventions used in the polymake extension polymake.so of Singular. We recommend to use the newer polymake.so whenever possible.

**Important note:**

Even though this is a Singular library for computing polytopes and fans such as the Newton polytope or the Groebner fan of a polynomial, most of the hard computations are NOT done by Singular but by the program

- TOPCOM by Joerg Rambau, Universitaet Bayreuth (see [http://www.uni-bayreuth.de/de/team/rambau\\_joerg/TOPCOM/](http://www.uni-bayreuth.de/de/team/rambau_joerg/TOPCOM/));

this library should rather be seen as an interface which allows to use a (very limited) number of options which topcom offers to compute with polytopes and fans and to make the results available in Singular for further computations; moreover, the user familiar with Singular does not have to learn the syntax of topcom, if the options offered here are sufficient for his purposes.

Note, though, that the procedures concerned with planar polygons are independent of topcom.

**Procedures using topcom:** Procedures concerned with planar polygons:

#### D.13.4.1 triangulations

Procedure from library `polymake.lib` (see [Section D.13.4 \[polymake\\_lib\]](#), page 2129).

**Usage:** `triangulations(polygon[,#]);` list polygon, list #

**Assume:** polygon is a list of integer vectors of the same size representing the affine coordinates of the lattice points

**Purpose:** the procedure considers the marked polytope given as the convex hull of the lattice points and with these lattice points as markings; it then computes all possible triangulations of this marked polytope

**Return:** list, each entry corresponds to one triangulation and the *i*th entry is itself a list of integer vectors of size three, where each integer vector defines one triangle in the triangulation by telling which points of the input are the vertices of the triangle

- Note:**
- the procedure calls for its computations the program `points2triangs` from the program `topcom` by Joerg Rambau, Universitaet Bayreuth; it therefore is necessary that this program is installed in order to use this procedure; see <http://www.rambau.wm.uni-bayreuth.de/TOPCOM/>;
  - if you only want to have the regular triangulations the procedure should be called with the string `'regular'` as optional argument
  - the procedure creates the files `/tmp/triangulationsinput` and `/tmp/triangulationsoutput`; the former is used as input for `points2triangs` and the latter is its output containing the triangulations of corresponding to points in the format of `points2triangs`; if you wish to use this for further computations with `topcom`, you have to call the procedure with the string `'keepfiles'` as optional argument
  - note that an integer `i` in an integer vector representing a triangle refers to the `i`th lattice point, i.e. `polygon[i]`; this convention is different from TOPCOM's convention, where `i` would refer to the `i-1`st lattice point

**Example:**

```
LIB "polymake.lib";
// the lattice points of the unit square in the plane
list polygon=intvec(0,0),intvec(0,1),intvec(1,0),intvec(1,1);
// the triangulations of this lattice point configuration are computed
list triang=triangulations(polygon);
⇒ Evaluating Commandline Options ...
⇒ ... done.
⇒ 0
⇒ 0
triang;
⇒ [1]:
⇒   [1]:
⇒     1,2,3
⇒   [2]:
⇒     2,3,4
⇒ [2]:
⇒   [1]:
⇒     1,3,4
⇒   [2]:
⇒     1,2,4
```

**D.13.4.2 secondaryPolytope**

Procedure from library `polymake.lib` (see [Section D.13.4 \[polymake.lib\]](#), page 2129).

**Usage:** `secondaryPolytope(polygon[,#]);` list polygon, list #

**Assume:**

- polygon is a list of integer vectors of the same size representing the affine coordinates of lattice points
- if the triangulations of the corresponding polygon have already been computed with the procedure `triangulations` then these can be given as a second (optional) argument in order to avoid doing this computation again

**Purpose:** the procedure considers the marked polytope given as the convex hull of the lattice points and with these lattice points as markings; it then computes the lattice points of the secondary polytope given by this marked polytope which correspond to the triangulations computed by the procedure `triangulations`



- Return:** list, say  $L$ , such that:  
 $L[1]$  = intmat, each row gives the affine coordinates of a lattice point in the secondary polytope given by the marked  
polytope corresponding to polygon  
 $L[2]$  = the list of corresponding triangulations
- Note:** if the triangulations are not handed over as optional argument the procedure calls for its computation of these triangulations the program `points2triangs` from the program `topcom` by Joerg Rambau, Universitaet Bayreuth; it therefore is necessary that this program is installed in order to use this procedure; see  
<http://www.rambau.wm.uni-bayreuth.de/TOPCOM/>;

**Example:**

```
LIB "polymake.lib";
// the lattice points of the unit square in the plane
list polygon=intvec(0,0),intvec(0,1),intvec(1,0),intvec(1,1);
// the secondary polytope of this lattice point configuration is computed
list secpoly=secondaryPolytope(polygon);
⇒ Evaluating Commandline Options ...
⇒ ... done.
⇒ 0
⇒ 0
// the points in the secondary polytope
print(secpoly[1]);
⇒      1      2      2      1
⇒      2      1      1      2
// the corresponding triangulations
secpoly[2];
⇒ [1]:
⇒      [1]:
⇒      1,2,3
⇒      [2]:
⇒      2,3,4
⇒ [2]:
⇒      [1]:
⇒      1,3,4
⇒      [2]:
⇒      1,2,4
```

**D.13.4.3 cycleLength**

Procedure from library `polymake.lib` (see [Section D.13.4 \[polymake.lib\]](#), page 2129).

- Usage:** `cycleLength(boundary,interior)`; list boundary, intvec interior
- Assume:** boundary is a list of integer vectors describing a cycle in some convex lattice polygon around the lattice point interior ordered clock wise
- Return:** string, the cycle length of the corresponding cycle in the dual tropical curve

**Example:**

```
LIB "polymake.lib";
// the integer vectors in boundary are lattice points on the boundary
// of a convex lattice polygon in the plane
list boundary=intvec(0,0),intvec(0,1),intvec(0,2),intvec(2,2),
```

```

intvec(2,1),intvec(2,0);
// interior is a lattice point in the interior of this lattice polygon
intvec interior=1,1;
// compute the general cycle length of a cycle of the corresponding cycle
// in the dual tropical curve, note that (0,1) and (2,1) do not contribute
cycleLength(boundary,interior);
↪ -4*u11+u00+u02+u22+u20

```

#### D.13.4.4 splitPolygon

Procedure from library `polymake.lib` (see [Section D.13.4 \[polymake.lib\]](#), page 2129).

**Usage:** `splitPolygon (markings);` markings list

**Assume:** markings is a list of integer vectors representing lattice points in the plane which we consider as the marked points of the convex lattice polytope spanned by them

**Purpose:** split the marked points in the vertices, the points on the facets which are not vertices, and the interior points

**Return:** list, L consisting of three lists  
`L[1]` : represents the vertices the polygon ordered clockwise  
`L[1][i][1]` = intvec, the coordinates of the *i*th vertex  
`L[1][i][2]` = int, the position of `L[1][i][1]` in markings  
`L[2][i]` : represents the lattice points on the facet of the polygon with endpoints `L[1][i]` and `L[1][i+1]`  
*(i considered modulo size(L[1]))*  
`L[2][i][j][1]` = intvec, the coordinates of the *j*th lattice point on that facet  
`L[2][i][j][2]` = int, the position of `L[2][i][j][1]` in markings  
`L[3]` : represents the interior lattice points of the polygon  
`L[3][i][1]` = intvec, coordinates of *i*th interior point  
`L[3][i][2]` = int, the position of `L[3][i][1]` in markings

**Example:**

```

LIB "polymake.lib";
// the lattice polygon spanned by the points (0,0), (3,0) and (0,3)
// with all integer points as markings
list polygon=intvec(1,1),intvec(3,0),intvec(2,0),intvec(1,0),
intvec(0,0),intvec(2,1),intvec(0,1),intvec(1,2),
intvec(0,2),intvec(0,3);
// split the polygon in its vertices, its facets and its interior points
list sp=splitPolygon(polygon);
// the vertices
sp[1];
↪ [1]:
↪   [1]:
↪     3,0
↪   [2]:
↪     2
↪ [2]:
↪   [1]:
↪     0,0
↪   [2]:
↪     5
↪ [3]:

```

```

⇒      [1]:
⇒      0,3
⇒      [2]:
⇒      10
// the points on facets which are not vertices
sp[2];
⇒ [1]:
⇒      [1]:
⇒      2,0
⇒      [2]:
⇒      3
⇒      [2]:
⇒      [1]:
⇒      1,0
⇒      [2]:
⇒      4
⇒ [2]:
⇒      [1]:
⇒      [1]:
⇒      0,1
⇒      [2]:
⇒      7
⇒      [2]:
⇒      [1]:
⇒      0,2
⇒      [2]:
⇒      9
⇒ [3]:
⇒      [1]:
⇒      [1]:
⇒      1,2
⇒      [2]:
⇒      8
⇒      [2]:
⇒      [1]:
⇒      2,1
⇒      [2]:
⇒      6
// the interior points
sp[3];
⇒ [1]:
⇒      [1]:
⇒      1,1
⇒      [2]:
⇒      1

```

#### D.13.4.5 eta

Procedure from library `polymake.lib` (see [Section D.13.4 \[polymake.lib\]](#), page 2129).

**Usage:**     `eta(triang,polygon);` triang, polygon list

- Assume:** polygon has the format of the output of `splitPolygon`, i.e. it is a list with three entries describing a convex lattice polygon in the following way:  
 polygon[1] : is a list of lists; for each  $i$  the entry `polygon[1][i][1]` is a lattice point which is a vertex of the lattice  
 polygon, and `polygon[1][i][2]` is an integer assigned to this lattice point as identifying index  
 polygon[2] : is a list of lists; for each vertex of the polygon, i.e. for each entry in `polygon[1]`, it contains a list `polygon[2][i]`, which contains the lattice points on the facet with endpoints `polygon[1][i]` and `polygon[1][i+1]` -  $i$  considered mod `size(polygon[1])`; each such lattice point contributes an entry `polygon[2][i][j][1]` which is an integer vector giving the coordinate of the lattice point and an entry `polygon[2][i][j][2]` which is the identifying index  
 polygon[3] : is a list of lists, where each entry corresponds to a lattice point in the interior of the polygon, with `polygon[3][j][1]` being the coordinates of the point and `polygon[3][j][2]` being the identifying index;  
 triang is a list of integer vectors all of size three describing a triangulation of the polygon described by `polygon`; if an entry of `triang` is the vector  $(i,j,k)$  then the triangle is built by the vertices with indices  $i, j$  and  $k$
- Return:** `intvec`, the integer vector `eta` describing that vertex of the Newton polytope discriminant of the polygon whose dual cone in the Groebner fan contains the cone of the secondary fan of the polygon corresponding to the given triangulation
- Note:** for a better description of `eta` see Gelfand, Kapranov, Zelevinski: Discriminants, Resultants and multidimensional Determinants. Chapter 10.

**Example:**

```
LIB "polymake.lib";
// the lattice polygon spanned by the points (0,0), (3,0) and (0,3)
// with all integer points as markings
list polygon=intvec(1,1),intvec(3,0),intvec(2,0),intvec(1,0),
intvec(0,0),intvec(2,1),intvec(0,1),intvec(1,2),
intvec(0,2),intvec(0,3);
// split the polygon in its vertices, its facets and its interior points
list sp=splitPolygon(polygon);
// define a triangulation by connecting the only interior point
// with the vertices
list triang=intvec(1,2,5),intvec(1,5,10),intvec(1,5,10);
// compute the eta-vector of this triangulation
eta(triang,sp);
⇒ 9,1,0,0,1,0,0,0,0,1
```

**D.13.4.6 findOrientedBoundary**

Procedure from library `polymake.lib` (see [Section D.13.4 \[polymake.lib\]](#), page 2129).

- Usage:** `findOrientedBoundary(polygon)`; `polygon` list
- Assume:** `polygon` is a list of integer vectors defining integer lattice points in the plane
- Return:** list `l` with the following interpretation  
`l[1]` = list of integer vectors such that the polygonal path defined by these is the

boundary of the convex hull of the lattice points in polygon  
 $l[2]$  = list, the redundant points in  $l[1]$  have been removed

**Example:**

```
LIB "polymake.lib";
// the following lattice points in the plane define a polygon
list polygon=intvec(0,0),intvec(3,1),intvec(1,0),intvec(2,0),
intvec(1,1),intvec(3,2),intvec(1,2),intvec(2,3),
intvec(2,4);
// we compute its boundary
list boundarypolygon=findOrientedBoundary(polygon);
// the points on the boundary ordered clockwise are boundarypolygon[1]
boundarypolygon[1];
⇒ [1]:
⇒ 0,0
⇒ [2]:
⇒ 1,2
⇒ [3]:
⇒ 2,4
⇒ [4]:
⇒ 3,2
⇒ [5]:
⇒ 3,1
⇒ [6]:
⇒ 2,0
⇒ [7]:
⇒ 1,0
// the vertices of the boundary are boundarypolygon[2]
boundarypolygon[2];
⇒ [1]:
⇒ 0,0
⇒ [2]:
⇒ 2,4
⇒ [3]:
⇒ 3,2
⇒ [4]:
⇒ 3,1
⇒ [5]:
⇒ 2,0
```

**D.13.4.7 cyclePoints**

Procedure from library `polymake.lib` (see [Section D.13.4 \[polymake.lib\]](#), page 2129).

**Usage:** `cyclePoints(triang,points,pt)` triang,points list, pt int

**Assume:**

- points is a list of integer vectors describing the lattice points of a marked polygon;
- triang is a list of integer vectors describing a triangulation of the marked polygon in the sense that an integer vector of the form  $(i,j,k)$  describes the triangle formed by `polygon[i]`, `polygon[j]` and `polygon[k]`;
- pt is an integer between 1 and `size(points)`, singling out a lattice point among the marked points

**Purpose:** consider the convex lattice polygon, say  $P$ , spanned by all lattice points in points which in the triangulation triang are connected to the point `points[pt]`; the procedure

computes all marked points in points which lie on the boundary of that polygon, ordered clockwise

**Return:** list, of integer vectors which are the coordinates of the lattice points on the boundary of the above mentioned polygon P, if this polygon is not the empty set (that would be the case if points[pt] is not a vertex of any triangle in the triangulation); otherwise return the empty list

**Example:**

```
LIB "polymake.lib";
// the lattice polygon spanned by the points (0,0), (3,0) and (0,3)
// with all integer points as markings
list points=intvec(1,1),intvec(3,0),intvec(2,0),intvec(1,0),
intvec(0,0),intvec(2,1),intvec(0,1),intvec(1,2),
intvec(0,2),intvec(0,3);
// define a triangulation
list triang=intvec(1,2,5),intvec(1,5,7),intvec(1,7,9),intvec(8,9,10),
intvec(1,8,9),intvec(1,2,8);
// compute the points connected to (1,1) in triang
cyclePoints(triang,points,1);
⇒ [1]:
⇒ 3,0
⇒ [2]:
⇒ 0,0
⇒ [3]:
⇒ 0,1
⇒ [4]:
⇒ 0,2
⇒ [5]:
⇒ 1,2
```

#### D.13.4.8 latticeArea

Procedure from library `polymake.lib` (see [Section D.13.4 \[polymake.lib\]](#), page 2129).

**Usage:** `latticeArea(polygon);` polygon list

**Assume:** polygon is a list of integer vectors in the plane

**Return:** int, the lattice area of the convex hull of the lattice points in polygon, i.e. twice the Euclidean area

**Example:**

```
LIB "polymake.lib";
// define a polygon with lattice area 5
list polygon=intvec(1,2),intvec(1,0),intvec(2,0),intvec(1,1),
intvec(2,1),intvec(0,0);
latticeArea(polygon);
⇒ 5
```

#### D.13.4.9 picksFormula

Procedure from library `polymake.lib` (see [Section D.13.4 \[polymake.lib\]](#), page 2129).

**Usage:** `picksFormula(polygon);` polygon list

**Assume:** polygon is a list of integer vectors in the plane and consider their convex hull C

**Return:** list, L of three integers  
 L[1] : the lattice area of C, i.e. twice the Euclidean area  
 L[2] : the number of lattice points on the boundary of C  
 L[3] : the number of interior lattice points of C

**Note:** the integers in L are related by Pick's formula, namely:  $L[1] = L[2] + 2 \cdot L[3] - 2$

**Example:**

```
LIB "polymake.lib";
// define a polygon with lattice area 5
list polygon=intvec(1,2),intvec(1,0),intvec(2,0),intvec(1,1),
intvec(2,1),intvec(0,0);
list pick=picksFormula(polygon);
// the lattice area of the polygon is:
pick[1];
⇒ 5
// the number of lattice points on the boundary is:
pick[2];
⇒ 5
// the number of interior lattice points is:
pick[3];
⇒ 1
// the number's are related by Pick's formula:
pick[1]-pick[2]-2*pick[3]+2;
⇒ 0
```

#### D.13.4.10 ellipticNF

Procedure from library `polymake.lib` (see [Section D.13.4 \[polymake.lib\]](#), page 2129).

**Usage:** `ellipticNF(polygon);` polygon list

**Assume:** polygon is a list of integer vectors in the plane such that their convex hull C has precisely one interior lattice point; i.e. C is the Newton polygon of an elliptic curve

**Purpose:** compute the normal form of the polygon with respect to the unimodular affine transformations  $T = A \cdot x + v$ ; there are sixteen different normal forms (see e.g. Bjorn Poonen, Fernando Rodriguez-Villegas: Lattice Polygons and the number 12. Amer. Math. Monthly 107 (2000), no. 3, 238–250.)

**Return:** list, L such that  
 L[1] : list whose entries are the vertices of the normal form of the polygon  
 L[2] : the matrix A of the unimodular transformation  
 L[3] : the translation vector v of the unimodular transformation  
 L[4] : list such that the ith entry is the image of `polygon[i]` under the unimodular transformation T

**Example:**

```
LIB "polymake.lib";
ring r=0,(x,y),dp;
// the Newton polygon of the following polynomial
// has precisely one interior point
poly f=x22y11+x19y10+x17y9+x16y9+x12y7+x9y6+x7y5+x2y3;
list polygon=list(intvec(22,11),intvec(19,10),intvec(17,9),
```

```

intvec(16,9), intvec(12,7),intvec(9,6),
intvec(7,5),intvec(2,3));
// its lattice points are
polygon;
⇒ [1]:
⇒ 22,11
⇒ [2]:
⇒ 19,10
⇒ [3]:
⇒ 17,9
⇒ [4]:
⇒ 16,9
⇒ [5]:
⇒ 12,7
⇒ [6]:
⇒ 9,6
⇒ [7]:
⇒ 7,5
⇒ [8]:
⇒ 2,3
// find its normal form
list nf=ellipticNF(polygon);
// the vertices of the normal form are
nf[1];
⇒ [1]:
⇒ 4,0
⇒ [2]:
⇒ 0,0
⇒ [3]:
⇒ 0,2
// it has been transformed by the unimodular affine transformation A*x+v
// with matrix A
nf[2];
⇒ 3,-7,
⇒ -2,5
// and translation vector v
nf[3];
⇒ 15,-11
// the 3rd lattice point ...
polygon[3];
⇒ 17,9
// ... has been transformed to
nf[4][3];
⇒ 3,0

```

### D.13.5 realizationMatroids\_lib

**Library:** realizationMatroids.lib

**Purpose:** Deciding Relative Realizability for Tropical Fan Curves in 2-Dimensional Matroidal Fans

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**Overview:** In tropical geometry, one question to ask is the following: given a one-dimensional balanced polyhedral fan  $C$  which is set theoretically contained in the tropicalization  $\text{trop}(Y)$  of an algebraic variety  $Y$ , does there exist a curve  $X$  in  $Y$  such that  $\text{trop}(X) = C$ ? This equality of  $C$  and  $\text{trop}(X)$  denotes an equality of both, the fans  $\text{trop}(X)$  and  $C$  and their weights on the maximal cones. The relative realization space of  $C$  with respect to  $Y$  is the space of all algebraic curves in  $Y$  which tropicalize to  $C$ .

This library provides procedures deciding relative realizability for tropical fan curves, i.e. one-dimensional weighted balanced polyhedral fans, contained in two-dimensional matroidal fans  $\text{trop}(Y)$  where  $Y$  is a projective plane.

**Notation:** If  $Y$  is a projective plane in  $(n-1)$ -dimensional projective space, we consider  $\text{trop}(Y)$  in  $\mathbb{R}^n/\langle 1 \rangle$ . Moreover, for the relative realization space of  $C$  with respect to  $Y$  we only consider algebraic curves of degree  $\deg(C)$  in  $Y$  which tropicalize to  $C$ .

**Procedures:**

### D.13.5.1 realizationDim

Procedure from library `realizationMatroids.lib` (see [Section D.13.5 \[realizationMatroids.lib\]](#), [page 2138](#)).

**Usage:** `realizationDim(I,C)`; where  $I$  is a homogeneous linear ideal defining the projective plane  $Y = V(I)$  and  $C$  is a list of intvectors such that each intvector represents a one-dimensional cone in the tropical fan curve whose relative realizability should be checked. This representation is done in the following way: the one-dimensional cone  $K$  is represented by a vector  $w$  whose equivalence class  $[w]$  in  $\mathbb{R}^n/\langle 1 \rangle$  can be written as  $[w] = m*[v]$  where  $[v]$  is the primitive generator of  $K$  and  $m$  is the weight of  $K$ .

**Returns:** the dimension of the relative realization space of the tropical curve  $C$  with respect to  $Y$ , and -1 if the relative realization space is empty.

**Example:**

```
LIB "realizationMatroids.lib";
ring r = 0,(x(1..4)),dp;
ideal I = x(1)+x(2)+x(3)+x(4);
list C = list(intvec(2,2,0,0),intvec(0,0,2,1),intvec(0,0,0,1));
//C represents the tropical fan curve which consists of the cones
//cone([(1,1,0,0)]) (with weight 2), cone([(0,0,2,1)]) (with weight 1)
//and cone([(0,0,0,1)]) (with weight 1)
realizationDim(I,C);
↦ -1
```

### D.13.5.2 irrRealizationDim

Procedure from library `realizationMatroids.lib` (see [Section D.13.5 \[realizationMatroids.lib\]](#), [page 2138](#)).

**Usage:** `irrRealizationDim(I,C)`; where  $I$  is a homogeneous linear ideal defining the projective plane  $Y = V(I)$  and  $C$  is a list of intvectors such that each intvector represents a one-dimensional cone in the tropical fan curve whose irreducible relative realizability should be checked. This representation is done in the following way: a one-dimensional cone  $K$  is represented by a vector  $w$  whose equivalence class  $[w]$  in  $\mathbb{R}^n/\langle 1 \rangle$  can be written as  $[w] = m*[v]$  where  $[v]$  is the primitive generator of  $K$  and  $m$  is the weight of  $K$ .

**Returns:** the dimension of the irreducible relative realization space of  $C$  with respect to  $Y$ , and -1 if the irreducible realization space is empty.

**Example:**

```
LIB "realizationMatroids.lib";
ring r = 0,(x(1..4)),dp;
ideal I = x(1)+x(2)+x(3)+x(4);
list C = list(intvec(2,2,0,0),intvec(0,0,2,2));
//C represents the tropical fan curve which consists of the cones
//cone([(1,1,0,0)]) and cone([(1,1,0,0)]), both with weight 2
realizationDim(I,C);
↦ 0
irrRealizationDim(I,C);
↦ -1
```

### D.13.5.3 realizationDimPoly

Procedure from library `realizationMatroids.lib` (see [Section D.13.5 \[realizationMatroids.lib\]](#), page 2138).

**Usage:** `realizationDimPoly(I,C)`; where  $I$  is a homogeneous linear ideal defining the projective plane  $Y = V(I)$  and  $C$  is a list of intvectors such that each intvector represents a one-dimensional cone in the tropical fan curve whose relative realizability should be checked. This representation is done in the following way: the one-dimensional cone  $K$  is represented by a vector  $w$  whose equivalence class  $[w]$  in  $R^n/\langle 1 \rangle$  can be written as  $[w] = m[v]$  where  $[v]$  is the primitive generator of  $K$  and  $m$  is the weight of  $K$ .

**Returns:** If the relative realization space of the tropical fan curve  $C$  is non-empty, this routine returns the tuple  $(r,f)$ , where  $r$  is the dimension of the relative realization space and  $f$  is an example of a homogeneous polynomial of degree  $\deg(C)$  cutting out a curve  $X$  in  $Y$  which tropicalizes to  $C$ . In case the relative realization space is empty, the output is set to -1.

**Example:**

```
LIB "realizationMatroids.lib";
ring r = 0,(x(1..4)),dp;
ideal I = x(1)+x(2)+x(3)+x(4);
list C = list(intvec(2,2,0,0),intvec(0,0,2,2));
//C represents the tropical fan curve which consists of the cones
//cone([(1,1,0,0)]) and cone([(1,1,0,0)]), both with weight 2
realizationDimPoly(I,C);
↦ 0 x(1)^2+2*x(1)*x(2)+x(2)^2
C = list(intvec(0,0,0,4),intvec(0,1,3,0),intvec(1,0,1,0),intvec(0,2,0,0),intvec(3,1,0,0));
//C represents the tropical fan curve which consists of the cones
//cone([(0,0,0,1)]) with weight 4,
//cone([(0,1,3,0)]), cone([(1,0,1,0)]) both with weight 1,
//cone([(0,1,0,0)]) with weight 2, and
//cone([(3,1,0,0)]) with weight 1
realizationDimPoly(I,C);
↦ 7 x(1)*x(2)^3+x(1)^3*x(3)+x(2)^3*x(3)+x(1)*x(3)^3
```

### D.13.6 tropical.lib

**Library:** `tropical.lib`

**Purpose:** Computations in Tropical Geometry

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**Warning:** - tropicalLifting will only work with LINUX and if in addition gfan is installed.  
 - drawTropicalCurve and drawTropicalNewtonSubdivision will only display the tropical curve with LINUX and if in addition latex and xdg-open are installed.  
 - For tropicalLifting in the definition of the basering the parameter  $t$  from the Puiseux series field  $C\{\{t\}\}$  must be defined as a variable, while for all other procedures it must be defined as a parameter.

**Theory:** Fix some base field  $K$  and a bunch of lattice points  $v_0, \dots, v_m$  in the integer lattice  $Z^n$ , then this defines a toric variety as the closure of  $(K^*)^n$  in the projective space  $P^m$ , where the torus is embedded via the map sending a point  $x$  in  $(K^*)^n$  to the point  $(x^{v_0}, \dots, x^{v_m})$ .

The generic hyperplane sections are just the images of the hypersurfaces in  $(K^*)^n$  defined by the polynomials  $f = a_0 x^{v_0} + \dots + a_m x^{v_m} = 0$ . Some properties of these hypersurfaces can be studied via tropicalisation.

For this we suppose that  $K = C\{\{t\}\}$  is the field of Puiseux series over the field of complex numbers (or any other field with a valuation into the real numbers). One associates to the hypersurface given by  $f = a_0 x^{v_0} + \dots + a_m x^{v_m}$  the tropical hypersurface defined by the tropicalisation  $\text{trop}(f) = \min\{\text{val}(a_0) + \langle v_0, x \rangle, \dots, \text{val}(a_m) + \langle v_m, x \rangle\}$ .

Here,  $\langle v, x \rangle$  denotes the standard scalar product of the integer vector  $v$  in  $Z^n$  with the vector  $x = (x_1, \dots, x_n)$  of variables, so that  $\text{trop}(f)$  is a piecewise linear function on  $R^n$ . The corner locus of this function (i.e. the points at which the minimum is attained at least twice) is the tropical hypersurface defined by  $\text{trop}(f)$ .

The theorem of Newton-Kapranov states that this tropical hypersurface is the same as if one computes pointwise the valuation of the hypersurface given by  $f$ . The analogue holds true if one replaces one equation  $f$  by an ideal  $I$ . A constructive proof of the theorem is given by an adapted version of the Newton-Puiseux algorithm. The hard part is to find a point in the variety over  $C\{\{t\}\}$  which corresponds to a given point in the tropical variety.

It is the purpose of this library to provide basic means to deal with tropical varieties. Of course we cannot represent the field of Puiseux series over  $C$  in its full strength, however, in order to compute interesting examples it will be sufficient to replace the complex numbers  $C$  by the rational numbers  $Q$  and to replace Puiseux series in  $t$  by rational functions in  $t$ , i.e. we replace  $C\{\{t\}\}$  by  $Q(t)$ , or sometimes even by  $Q[t]$ . Note, that this in particular forbids rational exponents for the  $t$ 's.

Moreover, in Singular no negative exponents of monomials are allowed, so that the integer vectors  $v_i$  will have to have non-negative entries. Shifting all exponents by a fixed integer vector does not change the tropicalisation nor does it change the toric variety. Thus this does not cause any restriction.

If, however, for some reason you prefer to work with general  $v_i$ , then you have to pass right away to the tropicalisation of the equations, wherever this is allowed – these are linear polynomials where the constant coefficient corresponds to the valuation of the original coefficient and where the non-constant coefficient correspond to the exponents of the monomials, thus they may be rational numbers respectively negative numbers: e.g. if  $f = t^{1/2} x^{-2} y^3 + 2t x^2 y + 4$  then  $\text{trop}(f) = \min\{1/2 - 2x + 3y, 1 + x + y, 0\}$ .

The main tools provided in this library are as follows:

- `tropicalLifting` implements the constructive proof of the Theorem of Newton-Kapranov and constructs a point in the variety over  $\mathbb{C}\{\{t\}\}$  corresponding to a given point in the corresponding tropical variety associated to an ideal  $I$ ; the generators of  $I$  have to be in the polynomial ring  $\mathbb{Q}[t, x_1, \dots, x_n]$  considered as a subring of  $\mathbb{C}\{\{t\}\}[x_1, \dots, x_n]$ ; a solution will be constructed up to given order; note that several field extensions of  $\mathbb{Q}$  might be necessary throughout the intermediate computations; the procedures use the external program `gfan`
- `puiseuxExpansion` computes a Newton-Puiseux expansion of a plane curve singularity
- `drawTropicalCurve` visualises a tropical plane curve either given by a polynomial in  $\mathbb{Q}(t)[x, y]$  or by a list of linear polynomials of the form  $ax+by+c$  with  $a, b$  in  $\mathbb{Z}$  and  $c$  in  $\mathbb{Q}$ ; latex must be installed on your computer
- `tropicalJInvariant` computes the tropical j-invariant of a tropical elliptic curve

**Procedures for tropical lifting: Procedures for drawing tropical curves: General procedures: Procedures for latex conversion: Auxiliary procedures: Procedures from binary library:**

### D.13.6.1 tropicalLifting

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\]](#), page 2140).

**Usage:** `tropicalLifting(i, w, ord[, opt])`;  $i$  ideal,  $w$  intvec,  $ord$  int,  $opt$  string

**Assume:**

- $i$  is an ideal in  $\mathbb{Q}[t, x_1, \dots, x_n]$ ,  $w = (w_0, w_1, \dots, w_n)$  and  $(w_1/w_0, \dots, w_n/w_0)$  is in the tropical variety of  $i$ , and  $ord$  is the order up to which a point in  $V(i)$  over  $\mathbb{Q}\{\{t\}\}$  lying over  $(w_1/w_0, \dots, w_n/w_0)$  shall be computed;
- $w_0$  may NOT be ZERO
- the basering should not have any parameters on its own and it should have a global monomial ordering, e.g. `ring r=0,(t,x(1..n)),dp;`
- the first variable of the basering will be treated as the parameter  $t$  in the Puiseux series field
- the optional parameter `opt` should be one or more strings among the following:
  - 'isZeroDimensional' : the dimension  $i$  is zero (not to be checked);
  - 'isPrime' : the ideal is prime over  $\mathbb{Q}(t)[x_1, \dots, x_n]$  (not to be checked);
  - 'isInTrop' :  $(w_1/w_0, \dots, w_n/w_0)$  is in the tropical variety (not to be checked);
  - 'oldGfan' : uses `gfan` version 0.2.1 or less
  - 'findAll' : find all solutions of a zero-dimensional ideal over  $(w_1/w_0, \dots, w_n/w_0)$
  - 'noAbs' : do NOT use absolute primary decomposition
  - 'puiseux' :  $n=1$  and  $i$  is generated by one equation
  - 'noResubst' : avoids the computation of the resubstitution

**Return:** IF THE OPTION 'findAll' WAS NOT SET THEN:  
list, containing one lifting of the given point  $(w_1/w_0, \dots, w_n/w_0)$  in the tropical variety of  $i$  to a point in  $V(i)$  over Puiseux series field up to the first  $ord$  terms; more precisely:  
IF THE OPTION 'noAbs' WAS NOT SET, THEN:

```
l[1] = ring Q[a]/m[[t]]
```

```
l[2] = int
```

```
l[3] = intvec
```

```
l[4] = list
```

```
IF THE OPTION 'noAbs' WAS SET, THEN:
```

```
l[1] = ring Q[X(1),...,X(k)]/m[[t]]
```

```
l[2] = int
```

```
l[3] = intvec
```

```
l[4] = list
```

```
l[5] = string
```

```
IF THE OPTION 'findAll' WAS SET, THEN:
```

list, containing ALL liftings of the given point  $((w_1/w_0, \dots, w_n/w_0))$  in the tropical variety of  $i$  to a point in  $V(i)$  over Puiseux series field up to the first ord terms, if the ideal is zero-dimensional over  $\mathbb{Q}\{\{t\}\}$ ;

more precisely, each entry of the list is a list  $l$  as computed if 'findAll' was NOT set

WE NOW DESCRIBE THE LIST ENTRIES IF 'findAll' WAS NOT SET:

- the ring  $l[1]$  contains an ideal LIFT, which contains a point in  $V(i)$  lying over  $w$  up to the first ord terms;

- and if the integer  $l[2]$  is  $N$  then  $t$  has to be replaced by  $t^{1/N}$  in the lift, or alternatively replace  $t$  by  $t^N$  in the defining ideal

- if the  $k+1$ st entry of  $l[3]$  is non-zero, then the  $k$ th component of LIFT has to be multiplied  $t^{(-l[3][k]/l[3][1])}$  AFTER substituting  $t$  by  $t^{1/N}$

- unless the option 'noResubst' was set, the  $k$ th entry of list  $l[4]$  is a string which represents the  $k$ th generator of

the ideal  $i$  where the coordinates have been replaced by the result of the lift;

the  $t$ -order of the  $k$ th entry should in principle be larger than the  $t$ -degree of LIFT

- if the option 'noAbs' was set, then the string in  $l[5]$  defines a maximal ideal in the field  $\mathbb{Q}[X(1), \dots, X(k)]$ , where  $X(1), \dots, X(k)$  are the parameters of the ring in  $l[1]$ ;

the basefield of the ring in  $l[1]$  should be considered modulo this ideal

**Remark:**

- it is best to use the procedure `displayTropicalLifting` to display the result

- the option 'findAll' cannot be used if 'noAbs' is set

- if the parameter 'findAll' is set AND the ideal  $i$  is zero-dimensional in  $\mathbb{Q}\{\{t\}\}[x_1, \dots, x_n]$  then ALL points in  $V(i)$  lying over  $w$  are computed up to order ord; if the ideal is not-zero dimensional, then only the points in the ideal after cutting down to dimension zero will be computed

- the procedure requires that the program GFAN is installed on your computer; if you have GFAN version less than 0.3.0 then you must use the optional parameter 'oldGfan'

- the procedure requires the Singular procedure `absPrimdecGTZ` to be present in the package `primdec.lib`, unless the option 'noAbs' is set; but even if `absPrimdecGTZ` is present it might be necessary to set the option 'noAbs' in order to avoid the costly absolute primary decomposition; the side effect is that the field extension which is computed throughout the recursion might need more than one parameter to be described

- since  $\mathbb{Q}$  is infinite, the procedure finishes with probability one

- you can call the procedure with  $\mathbb{Z}/p\mathbb{Z}$  as base field instead of  $\mathbb{Q}$ , but there are some problems you should be aware of:

- + the Puiseux series field over the algebraic closure of  $\mathbb{Z}/p\mathbb{Z}$  is NOT algebraically closed, and thus there may not exist a point in  $V(i)$  over the Puiseux series field with the desired valuation; so there is no chance that the procedure produced a sensible

output - e.g. if  $i = tx^p - tx - 1$   
 + if the dimension of  $i$  over  $\mathbb{Z}/p\mathbb{Z}(t)$  is not zero the process of reduction to zero might not work if the characteristic is small and you are unlucky  
 + the option 'noAbs' has to be used since absolute primary decomposition in Singular only works in characteristic zero  
 - the basefield should either be  $\mathbb{Q}$  or  $\mathbb{Z}/p\mathbb{Z}$  for some prime  $p$ ; field extensions will be computed if necessary; if you need parameters or field extensions from the beginning they should rather be simulated as variables possibly adding their relations to the ideal; the weights for the additional variables should be zero

**Example:**

```
LIB "tropical.lib";
int oldprintlevel=printlevel;
printlevel=1;
////////////////////////////////////
// 1st EXAMPLE:
////////////////////////////////////
ring r=0,(t,x),dp;
ideal i=(1+t2)*x2+t5x+t2;
intvec w=1,-1;
list LIST=tropicalLifting(i,w,4);
⇒ // ** name conflict var(1) and var(3): 'x(1)', rename to '@x(1)' in >> \
    ring EXTENSIONRING = ring(RL);<<
⇒ in tropical.lib::findzerosAndBasictransform:6334
⇒ LP algorithm being used: "cddgmp".
⇒ Groebner basis Engine being used: "gfan".
⇒ LP algorithm being used: "cddgmp".
⇒ Groebner basis Engine being used: "gfan".
⇒ LP algorithm being used: "cddgmp".
⇒ Groebner basis Engine being used: "gfan".
⇒ 0
⇒ 0
⇒ 0
⇒ 0
⇒ The procedure has created a list of lists. The jth entry of this list
⇒ contains a ring, an integer and an intvec.
⇒ In this ring lives an ideal representing the wanted lifting,
⇒ if the integer is N then in the parametrisation t has to be replaced by t\
  ^1/N,
⇒ and if the ith component of the intvec is w[i] then the ith component in \
  LIFT
⇒ should be multiplied by t^-w[i]/N in order to get the parametrisation.
⇒
⇒ Suppose your list has the name L, then you can access the 1st ring via:
⇒
⇒ def LIFTRing=L[1]; setring LIFTRing; LIFT;
⇒
def LIFTRing=LIST[1];
setring LIFTRing;
// LIFT contains the first 4 terms of a point in the variety of i
// over the Puiseux series field C{{t}} whose order is -w[1]/w[0]=1
LIFT;
⇒ LIFT[1]=(a)*t+(-a)/2*t3+(3a-4)/8*t5+(-5a+8)/16*t7
```

```

// Since the computations were done over Q a field extension was necessary,
// and the parameter "a" satisfies the equation given by minpoly
minpoly;
⇒ 0
////////////////////////////////////
// 2nd EXAMPLE
////////////////////////////////////
setring r;
LIST=tropicalLifting(x12-t11,intvec(12,-11),2,"isPrime","isInTrop");
⇒ 0
⇒ The procedure has created a list of lists. The jth entry of this list
⇒ contains a ring, an integer and an intvec.
⇒ In this ring lives an ideal representing the wanted lifting,
⇒ if the integer is N then in the parametrisation t has to be replaced by t\
  ^1/N,
⇒ and if the ith component of the intvec is w[i] then the ith component in \
  LIFT
⇒ should be multiplied by t^-w[i]/N in order to get the parametrisation.
⇒
⇒ Suppose your list has the name L, then you can access the 1st ring via:
⇒
⇒ def LIFTRing=L[1]; setring LIFTRing; LIFT;
⇒
def LIFTRing2=LIST[1];
setring LIFTRing2;
// This time, LIFT contains the lifting of the point -w[1]/w[0]=11/12
// only after we replace in LIFT the variable t by t^1/N with N=LIST[3]
LIFT;
⇒ LIFT[1]=t11
LIST[3];
⇒ 12,0
////////////////////////////////////
// 3rd EXAMPLE
////////////////////////////////////
ring R=0,(t,x,y,z),dp;
ideal i=-y2t4+x2,yt3+xz+y;
w=1,-2,0,2;
LIST=tropicalLifting(i,w,3);
⇒ LP algorithm being used: "cddgmp".
⇒ Groebner basis Engine being used: "gfan".
⇒ 0
⇒ 0
⇒ The procedure has created a list of lists. The jth entry of this list
⇒ contains a ring, an integer and an intvec.
⇒ In this ring lives an ideal representing the wanted lifting,
⇒ if the integer is N then in the parametrisation t has to be replaced by t\
  ^1/N,
⇒ and if the ith component of the intvec is w[i] then the ith component in \
  LIFT
⇒ should be multiplied by t^-w[i]/N in order to get the parametrisation.
⇒
⇒ Suppose your list has the name L, then you can access the 1st ring via:
⇒

```



```

⇒ def LIFTRing=L[1]; setring LIFTRing; LIFT;
⇒
// This time, LIFT contains the lifting of the point v=(-2,0,2)
// only after we multiply LIFT[3] by t^k with k=-LIST[4][3];
// NOTE: since the last component of v is positive, the lifting
//       must start with a negative power of t, which in Singular
//       is not allowed for a variable.
def LIFTRing3=LIST[1];
setring LIFTRing3;
LIFT;
⇒ LIFT[1]=t2
⇒ LIFT[2]=1
⇒ LIFT[3]=-1-t3
LIST[4];
⇒ [1]:
⇒ 0
⇒ [2]:
⇒ 0
// An easier way to display this is via displayTropicalLifting.
setring R;
displayTropicalLifting(LIST,"subst");
⇒ The lifting of the point in the tropical variety lives in the ring
⇒ Q[[t]]
⇒
⇒ The lifting has the form:
⇒ x=(1)*t^2
⇒ y=(1)
⇒ z=(-1)*1/t^2 + (-1)*t
⇒
⇒ Substituting the solution into the ideal gives:
⇒ i[1]=0
⇒ i[2]=0
printlevel=oldprintlevel;

```

### D.13.6.2 displayTropicalLifting

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\]](#), page 2140).

**Usage:** `displaytropcallifting(troplift[,#]);` troplift list, # list

**Assume:** troplift is the output of `tropicalLifting`; the optional parameter # can be the string 'subst'

**Return:** none

**Note:** - the procedure displays the output of the procedure `tropicalLifting`  
 - if the optional parameter 'subst' is given, then the lifting is substituted into the ideal and the result is displayed

**Example:**

```

LIB "tropical.lib";
ring r=0,(t,x,y,z),dp;
ideal i=-y2t4+x2,yt3+xz+y;
intvec w=2,-4,0,4;
displayTropicalLifting(tropicalLifting(i,w,3),"subst");

```



```

⇒ LP algorithm being used: "cddgmp".
⇒ Groebner basis Engine being used: "gfan".
⇒ 0
⇒ 0
⇒ The procedure has created a list of lists. The jth entry of this list
⇒ contains a ring, an integer and an intvec.
⇒ In this ring lives an ideal representing the wanted lifting,
⇒ if the integer is N then in the parametrisation t has to be replaced by t\
  ^1/N,
⇒ and if the ith component of the intvec is w[i] then the ith component in \
  LIFT
⇒ should be multiplied by t^-w[i]/N in order to get the parametrisation.
⇒
⇒ Suppose your list has the name L, then you can access the 1st ring via:
⇒
⇒ def LIFTRing=L[1]; setring LIFTRing; LIFT;
⇒
⇒ The lifting of the point in the tropical variety lives in the ring
⇒ Q[[t^(1/2)]]
⇒
⇒ The lifting has the form:
⇒ x=(1)*t^(4/2)
⇒ y=(1)
⇒ z=(-1)*1/t^(4/2) + (-1)*t^(2/2)
⇒
⇒ Substituting the solution into the ideal gives:
⇒ i[1]=0
⇒ i[2]=0

```

### D.13.6.3 `puiseuxExpansion`

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\]](#), page 2140).

**Usage:** `puiseuxExpansion(f,n,#)`; f poly, n int, # list

**Assume:** f is a non-constant polynomial in two variables which is not divisible by the first variable and which is squarefree as a power series over the complex numbers;  
the base field is either the field of rational numbers or a finite extension thereof; monomial ordering is assumed to be local;  
the optional parameter # can be the string 'subst'

**Return:** list, where each entry of the list l describes the Newton-Puiseux parametrisations of one branch of the plane curve singularity at the origin defined by f; only the terms up to order n of each parametrisation are computed  
 $l[i][1]$  = is a ring  
 $l[i][2]$  = int  
 $l[i][3]$  = string

WE NOW DESCRIBE THE LIST ENTRIES  $l[i]$  IN MORE DETAIL:

- the ring  $l[i][1]$  contains an ideal LIFT and the Newton-Puiseux parametrisation of the branch is given by  $x=t^N$  and  $y=LIFT[1]$ , where  $N=l[i][2]$
- if the base field had a parameter and a minimal polynomial, then the new base field will have a parameter and a new minimal polynomial, and  $LIFT[2]$  describes how the

old parameter can be computed from the new one

- if a field extension with minimal polynomial of degree  $k$  was necessary, then to the one extension produced actually  $k$  extensions correspond by replacing the parameter  $a$  successively by all zeros of the minimal polynomial
- if the option `subst` was set `l[i][3]` contains the polynomial where  $y$  has been substituted by  $y(t^{1/N})$  as a string

**Remark:**

- it is best to use the procedure `displayPuisseuxExpansion` to display the result
- the procedure requires the Singular procedure `absPrimdecGTZ` to be present in the package `primdec.lib`
- if  $f$  is not squarefree it will be replaced by its squarefree part

**Example:**

```
LIB "tropical.lib";
printlevel=1;
ring r=0,(x,y),ds;
poly f=x2-y4+x5y7;
puisseuxExpansion(f,3,"subst");
⇒ // ** name conflict var(1) and var(3): 'x(1)', rename to '@x(1)' in >> \
    ring EXTENSIONRING = ring(RL);<<
⇒ in tropical.lib::findzerosAndBasictransform:6334
⇒ The procedure has created a list of lists. The jth entry of this list
⇒ contains a ring, an integer and an intvec.
⇒ In this ring lives an ideal representing the wanted lifting,
⇒ if the integer is N then in the parametrisation t has to be replaced by t\
  ^1/N,
⇒ and if the ith component of the intvec is w[i] then the ith component in \
  LIFT
⇒ should be multiplied by t^-w[i]/N in order to get the parametrisation.
⇒
⇒ Suppose your list has the name L, then you can access the 1st ring via:
⇒
⇒ def LIFTRing=L[1][1]; setring LIFTRing; LIFT;
⇒
⇒ !!!! WARNING: The number of terms computed in the Puiseux expansion were
⇒ !!!!          not enough to find all branches of the curve singularity!
⇒ [1]:
⇒   [1]:
⇒       // coefficients: QQ
⇒ // number of vars : 1
⇒ //      block   1 : ordering ls
⇒ //              : names      t
⇒ //      block   2 : ordering C
⇒   [2]:
⇒       2
⇒   [3]:
⇒       (11/8)*t^(30/2) + (5/4)*t^(43/2) + (139/256)*t^(56/2) + (35/256)*t^\
        (69/2) + (21/1024)*t^(82/2) + (7/4096)*t^(95/2) + (1/16384)*t^(108/2)
⇒ [2]:
⇒   [1]:
⇒       // coefficients: QQ
⇒ // number of vars : 1
⇒ //      block   1 : ordering ls
```

```

⇒ //          : names      t
⇒ //          block      2 : ordering C
⇒ [2]:
⇒      2
⇒ [3]:
⇒      (11/8)*t^(30/2) + (-5/4)*t^(43/2) + (139/256)*t^(56/2) + (-35/256)*\
      t^(69/2) + (21/1024)*t^(82/2) + (-7/4096)*t^(95/2) + (1/16384)*t^(108/2)
⇒ [3]:
⇒ [1]:
⇒      // coefficients: QQ(a)
⇒ // number of vars : 1
⇒ //          block      1 : ordering ls
⇒ //          : names      t
⇒ //          block      2 : ordering C
⇒ [2]:
⇒      2
⇒ [3]:
⇒      (-a4+1)*t^(4/2) + (a7-a3)*t^(17/2) + (14a6-3a2)/8*t^(30/2) + (21a5-\
      a)/16*t^(43/2) + (140a4-1)/256*t^(56/2) + (35a3)/256*t^(69/2) + (21a2)/10\
      24*t^(82/2) + (7a)/4096*t^(95/2) + (1/16384)*t^(108/2)
displayPuisseuxExpansion(puisseuxExpansion(f,3));
⇒ // ** name conflict var(1) and var(3): 'x(1)', rename to '@x(1)' in >> \
      ring EXTENSIONRING = ring(RL);<<
⇒ in tropical.lib::findzerosAndBasictransform:6334
⇒ The procedure has created a list of lists. The jth entry of this list
⇒ contains a ring, an integer and an intvec.
⇒ In this ring lives an ideal representing the wanted lifting,
⇒ if the integer is N then in the parametrisation t has to be replaced by t\
      ^1/N,
⇒ and if the ith component of the intvec is w[i] then the ith component in \
      LIFT
⇒ should be multiplied by t^-w[i]/N in order to get the parametrisation.
⇒
⇒ Suppose your list has the name L, then you can access the 1st ring via:
⇒
⇒ def LIFTRing=L[1][1]; setring LIFTRing; LIFT;
⇒
⇒ !!!! WARNING: The number of terms computed in the Puiseux expansion were
⇒ !!!!          not enough to find all branches of the curve singularity!
⇒ =====
⇒ 1. Expansion:
⇒
⇒ The Puiseux expansion lives in the ring
⇒ Q[[t^(1/2)]]
⇒
⇒ The expansion has the form:
⇒ y=(1)*t^(1/2) + (1/4)*t^(14/2)
⇒
⇒ =====
⇒ 2. Expansion:
⇒
⇒ The Puiseux expansion lives in the ring
⇒ Q[[t^(1/2)]]

```

```

↳
↳ The expansion has the form:
↳  $y = (-1) \cdot t^{(1/2)} + (1/4) \cdot t^{(14/2)}$ 
↳
↳ =====
↳ 3. Expansion:
↳
↳ The Puiseux expansion lives in the ring
↳  $\mathbb{Q}[a]/\mathbb{Q}[[t^{(1/2)}]]$ 
↳
↳ The expansion has the form:
↳  $y = (a) \cdot t^{(1/2)} + (1/4) \cdot t^{(14/2)}$ 
↳

```

#### D.13.6.4 displayPuisseuxExpansion

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\]](#), page 2140).

**Usage:** `displayPuisseuxExpansion(puisseux[,#]);` `puisseux` list, `#` list

**Assume:** `puisseux` is the output of `puisseuxExpansion`; the optional parameter `#` can be the string `'subst'`

**Return:** none

**Note:**

- the procedure displays the output of the procedure `puisseuxExpansion`
- if the optional parameter `'subst'` is given, then the expansion is substituted into the polynomial and the result is displayed
- if the base field had a parameter and a minimal polynomial, then the new base field will have a parameter and a minimal polynomial; `var(2)` is the old parameter and it is displayed how the old parameter can be computed from the new one

**Example:**

```

LIB "tropical.lib";
ring r=0,(x,y),ds;
poly f=x2-y4+x5y7;
displayPuisseuxExpansion(puisseuxExpansion(f,3));
↳ // ** name conflict var(1) and var(3): 'x(1)', rename to '@x(1)' in >> \
    ring EXTENSIONRING = ring(RL);<<
↳ in tropical.lib::findzerosAndBasictransform:6334
↳ !!!! WARNING: The number of terms computed in the Puiseux expansion were
↳ !!!! not enough to find all branches of the curve singularity!
↳ =====
↳ 1. Expansion:
↳
↳ The Puiseux expansion lives in the ring
↳  $\mathbb{Q}[[t^{(1/2)}]]$ 
↳
↳ The expansion has the form:
↳  $y = (1) \cdot t^{(1/2)} + (1/4) \cdot t^{(14/2)}$ 
↳
↳ =====
↳ 2. Expansion:
↳
↳ The Puiseux expansion lives in the ring

```

```

⇒ Q[[t^(1/2)]]
⇒
⇒ The expansion has the form:
⇒ y=(-1)*t^(1/2) + (1/4)*t^(14/2)
⇒
⇒ =====
⇒ 3. Expansion:
⇒
⇒ The Puiseux expansion lives in the ring
⇒ Q[a]/O[[t^(1/2)]]
⇒
⇒ The expansion has the form:
⇒ y=(a)*t^(1/2) + (1/4)*t^(14/2)
⇒

```

### D.13.6.5 tropicalCurve

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\]](#), page 2140).

**Usage:** `tropicalCurve(tp[,#]);` tp list, # optional list

**Assume:** tp is list of linear polynomials of the form  $ax+by+c$  with integers a, b and a rational number c representing a tropical Laurent polynomial defining a tropical plane curve; alternatively tp can be a polynomial in  $\mathbb{Q}(t)[x,y]$  defining a tropical plane curve via the valuation map; the basering must have a global monomial ordering, two variables and up to one parameter!

**Return:** list, each entry  $i=1,\dots,\text{size}(l)-1$  corresponds to a vertex in the tropical plane curve defined by tp  
 $l[i][1]$  = x-coordinate of the ith vertex  
 $l[i][2]$  = y-coordinate of the ith vertex  
 $l[i][3]$  = intmat, if j is an entry in the first row of intmat then the ith vertex of the tropical curve is connected to the jth vertex with multiplicity given by the corresponding entry in the second row  
 $l[i][4]$  = list of lists, the first entry of a list is a primitive integer vector defining the direction of an unbounded edge emerging from the ith vertex of the graph, the corresponding second entry in the list is the multiplicity of the unbounded edge  
 $l[i][5]$  = a polynomial whose monomials mark the vertices in the Newton polygon corresponding to the entries in tp which take the common minimum at the ith vertex – if some coefficient a or b of the linear polynomials in the input was negative, then each monomial has to be shifted by the values in  $l[\text{size}(l)][3]$   
 $l[\text{size}(l)][1]$  = list, the entries describe the boundary points of the Newton subdivision  
 $l[\text{size}(l)][2]$  = list, the entries are pairs of integer vectors defining an interior edge of the Newton subdivision  
 $l[\text{size}(l)][3]$  = intvec, the monomials occurring in  $l[i][5]$  have to be shifted by this vector

in order to represent marked  
vertices in the Newton polygon

**Note:** here the tropical polynomial is supposed to be the MINIMUM of the linear forms in  $tp$ , unless the optional input  $\# [1]$  is the string 'max'

**Example:**

```
LIB "tropical.lib";
ring r=(0,t),(x,y),dp;
poly f=t*(x7+y7+1)+1/t*(x4+y4+x2+y2+x3y+xy3)+1/t7*x2y2;
list graph=tropicalCurve(f);
// the tropical curve has size(graph)-1 vertices
size(graph)-1;
⇒ 7
// the coordinates of the first vertex are graph[1][1],graph[1][2];
graph[1][1],graph[1][2];
⇒ -8/3 -8/3
// the first vertex is connected to the vertices
// graph[1][3][1,1..ncols(graph[1][3])]
intmat M=graph[1][3];
M[1,1..ncols(graph[1][3])];
⇒ 2 3
// the weights of the edges to these vertices are
// graph[1][3][2,1..ncols(graph[1][3])]
M[2,1..ncols(graph[1][3])];
⇒ 1 1
// from the first vertex emerge size(graph[1][4]) unbounded edges
size(graph[1][4]);
⇒ 1
// the primitive integral direction vector of the first unbounded edge
// of the first vertex
graph[1][4][1][1];
⇒ -1,-1
// the weight of the first unbounded edge of the first vertex
graph[1][4][1][2];
⇒ 7
// the monomials which are part of the Newton subdivision of the first vertex
graph[1][5];
⇒ x7+y7+x2y2
// connecting the points in graph[size(graph)][1] we get
// the boundary of the Newton polytope
graph[size(graph)][1];
⇒ [1]:
⇒ 0,7
⇒ [2]:
⇒ 7,0
⇒ [3]:
⇒ 0,0
// an entry in graph[size(graph)][2] is a pair of points
// in the Newton polytope bounding an inner edge
graph[size(graph)][2][1];
⇒ [1]:
⇒ 7,0
⇒ [2]:
```

$\mapsto$  2,2

### D.13.6.6 drawTropicalCurve

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\]](#), page 2140).

**Usage:** `drawTropicalCurve(f[,#]);` `f` poly or list, `#` optional list

**Assume:** `f` is list of linear polynomials of the form  $ax+by+c$  with integers `a`, `b` and a rational number `c` representing a tropical Laurent polynomial defining a tropical plane curve; alternatively `f` can be a polynomial in  $\mathbb{Q}(t)[x,y]$  defining a tropical plane curve via the valuation map;  
the basering must have a global monomial ordering, two variables and up to one parameter!

**Return:** NONE

**Note:**

- the procedure creates the files `/tmp/tropicalcurveNUMBER.tex` and `/tmp/tropicalcurveNUMBER.ps`, where `NUMBER` is a random four digit integer; moreover it displays the tropical curve via `xdg-open`;
- if you wish to remove all these files from `/tmp`, call the procedure `cleanTmp`
- edges with multiplicity greater than one carry this multiplicity
- if `#` is empty, then the tropical curve is computed w.r.t. minimum, if `#[1]` is the string 'max', then it is computed w.r.t. maximum
- if the last optional argument is 'onlytexfile' then only the latex file is produced; this option should be used if `xdg-utils` is not installed on your system
- note that lattice points in the Newton subdivision which are black correspond to markings of the marked subdivision, while lattice points in grey are not marked

**Example:**

```
LIB "tropical.lib";
ring r=(0,t),(x,y),dp;
poly f=t*(x3+y3+1)+1/t*(x2+y2+x+y+x2y+xy2)+1/t2*xy;
// the command drawTropicalCurve(f) computes the graph of the tropical curve
// given by f and displays a post script image, provided you have xdg-open
drawTropicalCurve(f);
 $\mapsto$  This is pdfTeX, Version 3.14159265-2.6-1.40.18 (TeX Live 2017) (preloaded\
format=latex)
 $\mapsto$  restricted \write18 enabled.
 $\mapsto$  entering extended mode
 $\mapsto$  (/tmp/tropicalcurve6034.tex
 $\mapsto$  LaTeX2e <2017/01/01> patch level 3
 $\mapsto$  Babel <3.10> and hyphenation patterns for 3 language(s) loaded.
 $\mapsto$  (/usr/share/texmf-dist/tex/latex/amscls/amsart.cls
 $\mapsto$  Document Class: amsart 2015/03/04 v2.20.2
 $\mapsto$  (/usr/share/texmf-dist/tex/latex/amsmath/amsmath.sty
 $\mapsto$  For additional information on amsmath, use the '?' option.
 $\mapsto$  (/usr/share/texmf-dist/tex/latex/amsmath/amstext.sty
 $\mapsto$  (/usr/share/texmf-dist/tex/latex/amsmath/amsgen.sty))
 $\mapsto$  (/usr/share/texmf-dist/tex/latex/amsmath/amsbsy.sty)
 $\mapsto$  (/usr/share/texmf-dist/tex/latex/amsmath/amsopn.sty))
 $\mapsto$  (/usr/share/texmf-dist/tex/latex/amsfonts/umsa.fd)
 $\mapsto$  (/usr/share/texmf-dist/tex/latex/amsfonts/amsfonts.sty))
```

```

⇒
⇒ ! LaTeX Error: File 'texdraw.sty' not found.
⇒
⇒ Type X to quit or <RETURN> to proceed,
⇒ or enter new name. (Default extension: sty)
⇒
⇒ Enter file name: warning: kpathsea: $.sty: Unrecognized variable construc\
  t '$.'.
⇒
⇒ ! LaTeX Error: File '$.sty' not found.
⇒
⇒ Type X to quit or <RETURN> to proceed,
⇒ or enter new name. (Default extension: sty)
⇒
⇒ Enter file name:
⇒ ! Emergency stop.
⇒ <read *>
⇒
⇒ 1.3 \setlength
  {\topmargin}{30mm}^^M
⇒ No pages of output.
⇒ Transcript written on tropicalcurve6034.log.
⇒ This is dvips(k) 5.997 Copyright 2017 Radical Eye Software (www.radicley\
  e.com)
⇒ dvips: DVI file can't be opened: /tmp/tropicalcurve6034.dvi: No such file\
  or directory
⇒ rm: das Entfernen von 'tropicalcurve6034.aux' ist nicht möglich: Datei o\
  der Verzeichnis nicht gefunden
⇒ rm: das Entfernen von 'tropicalcurve6034.ps?' ist nicht möglich: Datei o\
  der Verzeichnis nicht gefunden
⇒ rm: das Entfernen von 'tropicalcurve6034.dvi' ist nicht möglich: Datei o\
  der Verzeichnis nicht gefunden
⇒ sh: xdg-open: Kommando nicht gefunden.
⇒ 0
// we can instead apply the procedure to a tropical polynomial and use "maximum"
poly g=1/t3*(x7+y7+1)+t3*(x4+y4+x2+y2+x3y+xy3)+t21*x2y2;
list tropical_g=tropicalise(g);
tropical_g;
⇒ [1]:
  7*x-3
⇒ [2]:
  7*y-3
⇒ [3]:
  4*x+3
⇒ [4]:
  3*x+y+3
⇒ [5]:
  2*x+2*y+21
⇒ [6]:
  x+3*y+3
⇒ [7]:
  4*y+3
⇒ [8]:

```



```

⇒      2*x+3
⇒ [9]:
⇒      2*y+3
⇒ [10]:
⇒      -3
drawTropicalCurve(tropical_g,"max");
⇒ This is pdfTeX, Version 3.14159265-2.6-1.40.18 (TeX Live 2017) (preloaded\
format=latex)
⇒ restricted \write18 enabled.
⇒ entering extended mode
⇒ (/tmp/tropicalcurve9641.tex
⇒ LaTeX2e <2017/01/01> patch level 3
⇒ Babel <3.10> and hyphenation patterns for 3 language(s) loaded.
⇒ (/usr/share/texmf-dist/tex/latex/amscls/amsart.cls
⇒ Document Class: amsart 2015/03/04 v2.20.2
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsmath.sty
⇒ For additional information on amsmath, use the '?' option.
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amstext.sty
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsgen.sty))
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsbsy.sty)
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsopn.sty))
⇒ (/usr/share/texmf-dist/tex/latex/amsfonts/umsa.fd)
⇒ (/usr/share/texmf-dist/tex/latex/amsfonts/amsfonts.sty))
⇒
⇒ ! LaTeX Error: File 'texdraw.sty' not found.
⇒
⇒ Type X to quit or <RETURN> to proceed,
⇒ or enter new name. (Default extension: sty)
⇒
⇒ Enter file name:
⇒ ! Emergency stop.
⇒ <read *>
⇒
⇒ 1.3 \setlength
⇒      {\topmargin}{30mm}^^M
⇒ No pages of output.
⇒ Transcript written on tropicalcurve9641.log.
⇒ This is dvips(k) 5.997 Copyright 2017 Radical Eye Software (www.radicley\
e.com)
⇒ dvips: DVI file can't be opened: /tmp/tropicalcurve9641.dvi: No such file\
or directory
⇒ rm: das Entfernen von 'tropicalcurve9641.aux' ist nicht mglich: Datei o\
der Verzeichnis nicht gefunden
⇒ rm: das Entfernen von 'tropicalcurve9641.ps?' ist nicht mglich: Datei o\
der Verzeichnis nicht gefunden
⇒ rm: das Entfernen von 'tropicalcurve9641.dvi' ist nicht mglich: Datei o\
der Verzeichnis nicht gefunden
⇒ 0
⇒ sh: xdg-open: Kommando nicht gefunden.

```

### D.13.6.7 drawNewtonSubdivision

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\]](#), page 2140).

- Usage:** `drawTropicalCurve(f[,#]);` `f` poly, `#` optional list
- Assume:** `f` is list of linear polynomials of the form  $ax+by+c$  with integers  $a$ ,  $b$  and a rational number  $c$  representing a tropical Laurent polynomial defining a tropical plane curve; alternatively `f` can be a polynomial in  $\mathbb{Q}(t)[x,y]$  defining a tropical plane curve via the valuation map;  
the basering must have a global monomial ordering, two variables and up to one parameter!
- Return:** NONE
- Note:**
- the procedure creates the files `/tmp/newtonsubdivisionNUMBER.tex`, and `/tmp/newtonsubdivisionNUMBER.ps`, where `NUMBER` is a random four digit integer;
  - moreover it displays the tropical curve defined by `f` via `xdg-open`; if you wish to remove all these files from `/tmp`, call the procedure `cleanTmp`;
  - if `#` is empty, then the tropical curve is computed w.r.t. minimum, if `#[1]` is the string 'max', then it is computed w.r.t. maximum
  - note that lattice points in the Newton subdivision which are black correspond to markings of the marked subdivision, while lattice points in grey are not marked

**Example:**

```
LIB "tropical.lib";
ring r=(0,t),(x,y),dp;
poly f=t*(x3+y3+1)+1/t*(x2+y2+x+y+x2y+xy2)+1/t2*xy;
// the command drawTropicalCurve(f) computes the graph of the tropical curve
// given by f and displays a post script image, provided you have xdg-open
drawNewtonSubdivision(f);
⇒ This is pdfTeX, Version 3.14159265-2.6-1.40.18 (TeX Live 2017) (preloaded\
  format=latex)
⇒ restricted \write18 enabled.
⇒ entering extended mode
⇒ (/tmp/newtonsubdivision6034.tex
⇒ LaTeX2e <2017/01/01> patch level 3
⇒ Babel <3.10> and hyphenation patterns for 3 language(s) loaded.
⇒ (/usr/share/texmf-dist/tex/latex/amscls/amsart.cls
⇒ Document Class: amsart 2015/03/04 v2.20.2
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsmath.sty
⇒ For additional information on amsmath, use the '?' option.
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amstext.sty
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsgen.sty))
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsbsy.sty)
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsopn.sty))
⇒ (/usr/share/texmf-dist/tex/latex/amsfonts/umsa.fd)
⇒ (/usr/share/texmf-dist/tex/latex/amsfonts/amsfonts.sty))
⇒
⇒ ! LaTeX Error: File 'texdraw.sty' not found.
⇒
⇒ Type X to quit or <RETURN> to proceed,
⇒ or enter new name. (Default extension: sty)
⇒
⇒ Enter file name: warning: kpathsea: $.sty: Unrecognized variable construc\
  t '$.'.
⇒
```

```

⇒ ! LaTeX Error: File '$.sty' not found.
⇒
⇒ Type X to quit or <RETURN> to proceed,
⇒ or enter new name. (Default extension: sty)
⇒
⇒ Enter file name:
⇒ ! Emergency stop.
⇒ <read *>
⇒
⇒ 1.3 \begin
⇒          {document}^^M
⇒ No pages of output.
⇒ Transcript written on newtonsubdivision6034.log.
⇒ This is dvips(k) 5.997 Copyright 2017 Radical Eye Software (www.radical-eye.com)
⇒ dvips: DVI file can't be opened: /tmp/newtonsubdivision6034.dvi: No such \
  file or directory
⇒ rm: das Entfernen von 'newtonsubdivision6034.aux' ist nicht möglich: Dat\
  ei oder Verzeichnis nicht gefunden
⇒ rm: das Entfernen von 'newtonsubdivision6034.ps?' ist nicht möglich: Dat\
  ei oder Verzeichnis nicht gefunden
⇒ rm: das Entfernen von 'newtonsubdivision6034.dvi' ist nicht möglich: Dat\
  ei oder Verzeichnis nicht gefunden
⇒ 0
// we can instead apply the procedure to a tropical polynomial
⇒ sh: xdg-open: Kommando nicht gefunden.
poly g=x+y+x2y+xy2+1/t*xy;
list tropical_g=tropicalise(g);
tropical_g;
⇒ [1]:
⇒      2*x+y
⇒ [2]:
⇒      x+2*y
⇒ [3]:
⇒      x+y-1
⇒ [4]:
⇒      x
⇒ [5]:
⇒      y
drawNewtonSubdivision(tropical_g);
⇒ This is pdfTeX, Version 3.14159265-2.6-1.40.18 (TeX Live 2017) (preloaded\
  format=latex)
⇒ restricted \write18 enabled.
⇒ entering extended mode
⇒ (/tmp/newtonsubdivision9641.tex
⇒ LaTeX2e <2017/01/01> patch level 3
⇒ Babel <3.10> and hyphenation patterns for 3 language(s) loaded.
⇒ (/usr/share/texmf-dist/tex/latex/amscs/amsart.cls
⇒ Document Class: amsart 2015/03/04 v2.20.2
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsmath.sty
⇒ For additional information on amsmath, use the '?' option.
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amstext.sty
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsgen.sty))

```

```

⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsbsy.sty)
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsopn.sty))
⇒ (/usr/share/texmf-dist/tex/latex/amsfonts/umsa.fd)
⇒ (/usr/share/texmf-dist/tex/latex/amsfonts/amsfonts.sty))
⇒
⇒ ! LaTeX Error: File 'texdraw.sty' not found.
⇒
⇒ Type X to quit or <RETURN> to proceed,
⇒ or enter new name. (Default extension: sty)
⇒
⇒ Enter file name:
⇒ ! Emergency stop.
⇒ <read *>
⇒
⇒ 1.3 \begin
⇒           {document}^^M
⇒ No pages of output.
⇒ Transcript written on newtonsubdivision9641.log.
⇒ This is dvips(k) 5.997 Copyright 2017 Radical Eye Software (www.radical-eye.com)
⇒ dvips: DVI file can't be opened: /tmp/newtonsubdivision9641.dvi: No such \
  file or directory
⇒ rm: das Entfernen von 'newtonsubdivision9641.aux' ist nicht möglich: Dat\
  ei oder Verzeichnis nicht gefunden
⇒ rm: das Entfernen von 'newtonsubdivision9641.ps?' ist nicht möglich: Dat\
  ei oder Verzeichnis nicht gefunden
⇒ rm: das Entfernen von 'newtonsubdivision9641.dvi' ist nicht möglich: Dat\
  ei oder Verzeichnis nicht gefunden
⇒ sh: xdg-open: Kommando nicht gefunden.
⇒ 0

```

### D.13.6.8 tropicalJInvariant

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\]](#), page 2140).

**Usage:** `tropicalJInvariant(f[,#]);` `f` poly or list, `#` optional list

**Assume:** `f` is list of linear polynomials of the form  $ax+by+c$  with integers  $a$ ,  $b$  and a rational number  $c$  representing a tropical Laurent polynomial defining a tropical plane curve; alternatively `f` can be a polynomial in  $\mathbb{Q}(t)[x,y]$  defining a tropical plane curve via the valuation map; the basering must have a global monomial ordering, two variables and up to one parameter!

**Return:** number, if the graph underlying the tropical curve has precisely one loop then its weighted lattice length is returned, otherwise the result will be -1

**Note:**

- if the tropical curve is elliptic and its embedded graph has precisely one loop, then the weighted lattice length of the loop is its tropical j-invariant
- the procedure checks if the embedded graph of the tropical curve has genus one, but it does NOT check if the loop can be resolved, so that the curve is not a proper tropical elliptic curve
- if the embedded graph of a tropical elliptic curve has more than one loop, then all but one can be resolved, but this is not observed by this procedure, so it will not compute

the j-invariant

- if # is empty, then the tropical curve is computed w.r.t. minimum, if #[1] is the string 'max', then it is computed w.r.t. maximum
- the tropicalJInvariant of a plane tropical cubic is the 'cycle length' of the cubic as introduced in the paper: Eric Katz, Hannah Markwig, Thomas Markwig: The j-invariant of a cubic tropical plane curve.

**Example:**

```
LIB "tropical.lib";
ring r=(0,t),(x,y),dp;
// tropicalJInvariant computes the tropical j-invariant of an elliptic curve
tropicalJInvariant(t*(x3+y3+1)+1/t*(x2+y2+x+y+x2y+xy2)+1/t2*xy);
⇒ 6
// the Newton polygon need not be the standard simplex
tropicalJInvariant(x+y+x2y+xy2+1/t*xy);
⇒ 8
// the curve can have arbitrary degree
tropicalJInvariant(t*(x7+y7+1)+1/t*(x4+y4+x2+y2+x3y+xy3)+1/t7*x2y2);
⇒ 14/3
// the procedure does not realise, if the embedded graph of the tropical
// curve has a loop that can be resolved
tropicalJInvariant(1+x+y+xy+tx2y+txy2);
⇒ 4
// but it does realise, if the curve has no loop at all ...
tropicalJInvariant(x+y+1);
⇒ The embedded graph of the curve has not genus one.
⇒ -1
// or if the embedded graph has more than one loop - even if only one
// cannot be resolved
tropicalJInvariant(1+x+y+xy+tx2y+txy2+t3x5+t3y5+tx2y2+t2xy4+t2yx4);
⇒ The embedded graph of the curve has not genus one.
⇒ -1
```

### D.13.6.9 weierstrassForm

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\]](#), page 2140).

### D.13.6.10 conicWithTangents

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\]](#), page 2140).

**Usage:** `conicWithTangents(points[,#]);` points list, # optional list

**Assume:** points is a list of five points in the plane over  $K(t)$

**Return:** list,  $l[1]$  = the list points of the five given points  
 $l[2]$  = the conic  $f$  passing through the five points  
 $l[3]$  = list of equations of tangents to  $f$  in the given points  
 $l[4]$  = ideal, tropicalisation of  $f$  (i.e. list of linear forms)  
 $l[5]$  = a list of the tropicalisation of the tangents  
 $l[6]$  = a list containing the vertices of the tropical conic  $f$   
 $l[7]$  = a list containing lists with vertices of the tangents  
 $l[8]$  = a string which contains the latex-code to draw the tropical conic and its tropicalised tangents

$l[9]$  = if  $\#$  is non-empty, this is the same data for the dual conic and the points dual to the computed tangents

**Note:** the points must be generic, i.e. no three on a line

**Example:**

```
LIB "tropical.lib";
ring r=(0,t),(x,y),dp;
// the input consists of a list of five points in the plane over Q(t)
list points=list(1/t^2,t),list(1/t,t^2),list(1,1),list(t,1/t^2),list(t^2,1/t);
list conic=conicWithTangents(points);
// conic[1] is the list of the given five points
conic[1];
⇒ [1]:
⇒ [1]:
⇒ 1/(t^2)
⇒ [2]:
⇒ (t)
⇒ [2]:
⇒ [1]:
⇒ 1/(t)
⇒ [2]:
⇒ (t^2)
⇒ [3]:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 1
⇒ [4]:
⇒ [1]:
⇒ (t)
⇒ [2]:
⇒ 1/(t^2)
⇒ [5]:
⇒ [1]:
⇒ (t^2)
⇒ [2]:
⇒ 1/(t)
// conic[2] is the equation of the conic f passing through the five points
conic[2];
⇒ (2t^3)*x^2+(-t^6+t^5+2t^4+t^3+2t^2+t-1)*xy+(2t^3)*y^2+(-t^5-2t^4-3t^3-2t^2-t)*x+(-t^5-2t^4-3t^3-2t^2-t)*y+(t^6+t^5+2t^4+t^3+2t^2+t+1)
// conic[3] is a list containing the equations of the tangents
// through the five points
conic[3];
⇒ [1]:
⇒ (-t^7+t^6+t^5-t^4-t^3-t^2+2t)*x+(-t^7+t^6-2t^5+2t^2+t-1)/(t^2)*y+(t^7+t^5-t^4+t^3-t^2-1)/(t)
⇒ [2]:
⇒ (-t^8+t^7+2t^6-2t^3+t^2-t)*x+(2t^6-t^5-t^4-t^3+t^2+t-1)/(t)*y+(-t^7-t^5+t^4-t^3+t^2+1)
⇒ [3]:
⇒ (-t^6+2t^3-1)*x+(-t^6+2t^3-1)*y+(2t^6-4t^3+2)
⇒ [4]:
```

```

⇒      (-t7+t6-2t5+2t2+t-1)/(t2)*x+(-t7+t6+t5-t4-t3-t2+2t)*y+(t7+t5-t4+t3-t2-\
      1)/(t)
⇒ [5]:
⇒      (2t6-t5-t4-t3+t2+t-1)/(t)*x+(-t8+t7+2t6-2t3+t2-t)*y+(-t7-t5+t4-t3+t2+1\
      )
// conic[4] is an ideal representing the tropicalisation of the conic f
conic[4];
⇒ [1]:
⇒      2*x+3
⇒ [2]:
⇒      x+y
⇒ [3]:
⇒      2*y+3
⇒ [4]:
⇒      x+1
⇒ [5]:
⇒      y+1
⇒ [6]:
⇒      0
// conic[5] is a list containing the tropicalisation
//      of the five tangents in conic[3]
conic[5];
⇒ [1]:
⇒      [1]:
⇒      x+1
⇒      [2]:
⇒      y-2
⇒      [3]:
⇒      -1
⇒ [2]:
⇒      [1]:
⇒      x+1
⇒      [2]:
⇒      y-1
⇒      [3]:
⇒      0
⇒ [3]:
⇒      [1]:
⇒      x
⇒      [2]:
⇒      y
⇒      [3]:
⇒      0
⇒ [4]:
⇒      [1]:
⇒      x-2
⇒      [2]:
⇒      y+1
⇒      [3]:
⇒      -1
⇒ [5]:
⇒      [1]:
⇒      x-1

```

```

    ↪      [2]:
    ↪      y+1
    ↪      [3]:
    ↪      0
    ↪      // conic[6] is a list containing the vertices of the tropical conic
    ↪      conic[6];
    ↪      [1]:
    ↪      [1]:
    ↪      -2
    ↪      [2]:
    ↪      1
    ↪      [2]:
    ↪      [1]:
    ↪      1
    ↪      [2]:
    ↪      -2
    ↪      [3]:
    ↪      [1]:
    ↪      -1
    ↪      [2]:
    ↪      1
    ↪      [4]:
    ↪      [1]:
    ↪      1
    ↪      [2]:
    ↪      -1
    ↪      // conic[7] is a list containing the vertices of the five tangents
    ↪      conic[7];
    ↪      [1]:
    ↪      [1]:
    ↪      [1]:
    ↪      -2
    ↪      [2]:
    ↪      1
    ↪      [2]:
    ↪      [1]:
    ↪      [1]:
    ↪      -1
    ↪      [2]:
    ↪      1
    ↪      [3]:
    ↪      [1]:
    ↪      [1]:
    ↪      0
    ↪      [2]:
    ↪      0
    ↪      [4]:
    ↪      [1]:
    ↪      [1]:
    ↪      1
    ↪      [2]:
    ↪      -2
    ↪      [5]:

```



```

⇒      [1]:
⇒      [1]:
⇒      1
⇒      [2]:
⇒      -1
// conic[8] contains the latex code to draw the tropical conic and
//      its tropicalised tangents; it can be written in a file, processed and
//      displayed via xdg-open
write(":w /tmp/conic.tex",conic[8]);
system("sh","cd /tmp; latex /tmp/conic.tex; dvips /tmp/conic.dvi -o;
xdg-open conic.ps &");
⇒ This is pdfTeX, Version 3.14159265-2.6-1.40.18 (TeX Live 2017) (preloaded\
format=latex)
⇒ restricted \write18 enabled.
⇒ entering extended mode
⇒ (/tmp/conic.tex
⇒ LaTeX2e <2017/01/01> patch level 3
⇒ Babel <3.10> and hyphenation patterns for 3 language(s) loaded.
⇒ (/usr/share/texmf-dist/tex/latex/amscls/amsart.cls
⇒ Document Class: amsart 2015/03/04 v2.20.2
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsmath.sty
⇒ For additional information on amsmath, use the '?' option.
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amstext.sty
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsgen.sty))
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsbsy.sty)
⇒ (/usr/share/texmf-dist/tex/latex/amsmath/amsopn.sty))
⇒ (/usr/share/texmf-dist/tex/latex/amsfonts/umsa.fd)
⇒ (/usr/share/texmf-dist/tex/latex/amsfonts/amsfonts.sty))
⇒
⇒ ! LaTeX Error: File 'texdraw.sty' not found.
⇒
⇒ Type X to quit or <RETURN> to proceed,
⇒ or enter new name. (Default extension: sty)
⇒
⇒ Enter file name: warning: kpathsea: $.sty: Unrecognized variable construc\
t '$.'.
⇒
⇒ ! LaTeX Error: File '$.sty' not found.
⇒
⇒ Type X to quit or <RETURN> to proceed,
⇒ or enter new name. (Default extension: sty)
⇒
⇒ Enter file name:
⇒ ! Emergency stop.
⇒ <read *>
⇒
⇒ l.3 \begin
⇒      {document}~^M
⇒ No pages of output.
⇒ Transcript written on conic.log.
⇒ This is dvips(k) 5.997 Copyright 2017 Radical Eye Software (www.radicley\
e.com)
⇒ dvips: DVI file can't be opened: /tmp/conic.dvi: No such file or director\

```

```

      y
  ↪ 0
  // with an optional argument the same information for the dual conic is computed
  //      and saved in conic[9]
  conic=conicWithTangents(points,1);
  ↪ sh: Zeile 1: xdg-open: Kommando nicht gefunden.
  conic[9][2]; // the equation of the dual conic
  ↪ (t6-6t5+11t4-6t3+t2)*x2+(-4t8+8t7-2t6-4t5+6t4-4t3-2t2+8t-4)*xy+(t6-6t5+11\
      t4-6t3+t2)*y2+(-2t7+2t6+4t5-6t4+4t3+2t2-2t)*x+(-2t7+2t6+4t5-6t4+4t3+2t2-2\
      t)*y+(t8-4t7+2t6+8t5-13t4+8t3+2t2-4t+1)

```

### D.13.6.11 tropicalise

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\]](#), page 2140).

**Usage:** `tropicalise(f[#]);`  $f$  polynomial,  $\#$  optional list

**Assume:**  $f$  is a polynomial in  $\mathbb{Q}(t)[x_1, \dots, x_n]$

**Return:** list, the linear forms of the tropicalisation of  $f$

**Note:** if  $\#$  is empty, then the valuation of  $t$  will be 1,  
 if  $\#$  is the string 'max' it will be -1;  
 the latter supposes that we consider the maximum of the computed linear forms, the  
 former that we consider their minimum

**Example:**

```

LIB "tropical.lib";
ring r=(0,t),(x,y),dp;
tropicalise(2t3x2-1/t*xy+2t3y2+(3t3-t)*x+ty+(t6+1));
↪ [1]:
↪ 2*x+3
↪ [2]:
↪ x+y-1
↪ [3]:
↪ 2*y+3
↪ [4]:
↪ x+1
↪ [5]:
↪ y+1
↪ [6]:
↪ 0

```

### D.13.6.12 tropicaliseSet

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\]](#), page 2140).

**Usage:** `tropicaliseSet(i);`  $i$  ideal

**Assume:**  $i$  is an ideal in  $\mathbb{Q}(t)[x_1, \dots, x_n]$

**Return:** list, the  $j$ th entry is the tropicalisation of the  $j$ th generator of  $i$

**Example:**

```

LIB "tropical.lib";
ring r=(0,t),(x,y),dp;
ideal i=txy-y2+1,2t3x2+1/t*y-t6;

```

```

tropicaliseSet(i);
↪ [1]:
↪   [1]:
↪     x+y+1
↪   [2]:
↪     2*y
↪   [3]:
↪     0
↪ [2]:
↪   [1]:
↪     2*x+3
↪   [2]:
↪     y-1
↪   [3]:
↪     6

```

### D.13.6.13 tInitialForm

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\], page 2140](#)).

**Usage:** `tInitialForm(f,w)`;  $f$  a polynomial,  $w$  an integer vector

**Assume:**  $f$  is a polynomial in  $\mathbb{Q}[t, x_1, \dots, x_n]$  and  $w=(w_0, w_1, \dots, w_n)$

**Return:** `poly`, the  $t$ -initialform of  $f(t, x)$  w.r.t.  $w$  evaluated at  $t=1$

**Note:** the  $t$ -initialform is the sum of the terms with MAXIMAL weighted order w.r.t.  $w$

**Example:**

```

LIB "tropical.lib";
ring r=0,(t,x,y),dp;
poly f=t4x2+y2-t2xy+t4x-t9;
intvec w=-1,-2,-3;
tInitialForm(f,w);
↪ y2+x

```

### D.13.6.14 tInitialIdeal

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\], page 2140](#)).

**Usage:** `tInitialIdeal(i,w)`;  $i$  ideal,  $w$  intvec

**Assume:**  $i$  is an ideal in  $\mathbb{Q}[t, x_1, \dots, x_n]$  and  $w=(w_0, \dots, w_n)$

**Return:** `ideal ini`, the  $t$ -initial ideal of  $i$  with respect to  $w$

**Example:**

```

LIB "tropical.lib";
ring r=0,(t,x,y),dp;
ideal i=t2x-y+t3,t2x-y-2t3x;
intvec w=-1,2,0;
// the t-initial forms of the generators are
tInitialForm(i[1],w),tInitialForm(i[2],w);
↪ x-y x-y
// and they do not generate the t-initial ideal of i
tInitialIdeal(i,w);

```

```

 $\mapsto$   $\_ [1] = -x + y$ 
 $\mapsto$   $\_ [2] = 2xy$ 
 $\mapsto$   $\_ [3] = 2x$ 
 $\mapsto$   $\_ [4] = 2xy^2$ 

```

### D.13.6.15 initialForm

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\]](#), page 2140).

**Usage:** `initialForm(f,w)`;  $f$  a polynomial,  $w$  an integer vector

**Assume:**  $f$  is a polynomial in  $\mathbb{Q}[x_1, \dots, x_n]$  and  $w = (w_1, \dots, w_n)$

**Return:** `poly`, the initial form of  $f(x)$  w.r.t.  $w$

**Note:** the `initialForm` consists of the terms with MAXIMAL weighted order w.r.t.  $w$

**Example:**

```

LIB "tropical.lib";
ring r=0,(x,y),dp;
poly f=x3+y2-xy+x-1;
intvec w=2,3;
initialForm(f,w);
 $\mapsto$   $x^3 + y^2$ 

```

### D.13.6.16 initialIdeal

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\]](#), page 2140).

**Usage:** `initialIdeal(i,w)`;  $i$  ideal,  $w$  intvec

**Assume:**  $i$  is an ideal in  $\mathbb{Q}[x_1, \dots, x_n]$  and  $w = (w_1, \dots, w_n)$

**Return:** `ideal`, the `initialIdeal` of  $i$  w.r.t.  $w$

**Note:** the `initialIdeal` consists of the terms with MAXIMAL weighted order w.r.t.  $w$

**Example:**

```

LIB "tropical.lib";
ring r=0,(x,y),dp;
poly f=x3+y2-xy+x-1;
intvec w=2,3;
initialIdeal(f,w);
 $\mapsto$   $\_ [1] = x^3 + y^2$ 

```

### D.13.6.17 texNumber

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\]](#), page 2140).

**Usage:** `texNumber(f)`;  $f$  poly

**Return:** string, tex command representing leading coefficient of  $f$  using `\frac`

**Example:**

```

LIB "tropical.lib";
ring r=(0,t),x,dp;
texNumber((3t2-1)/t3);
 $\mapsto$  \tfrac{-1+3\cdot t^2}{t^3}

```

### D.13.6.18 texPolynomial

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\], page 2140](#)).

**Usage:** `texPolynomial(f); f poly`

**Return:** string, the tex command representing `f`

**Example:**

```
LIB "tropical.lib";
ring r=(0,t),x,dp;
texPolynomial(1/t*x^2-t^2x+1/t);
⇒ \tfrac{1}{t}\cdot x^2-t^2\cdot x+\tfrac{1}{t}
```

### D.13.6.19 texMatrix

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\], page 2140](#)).

**Usage:** `texMatrix(M); M matrix`

**Return:** string, the tex command representing `M`

**Example:**

```
LIB "tropical.lib";
ring r=(0,t),x,dp;
matrix M[2][2]=3/2,1/t*x^2-t^2x+1/t,5,-2x;
texMatrix(M);
⇒ \left(\begin{array}{cc}
⇒ \tfrac{3}{2} & \tfrac{1}{t}\cdot x^2-t^2\cdot x+\tfrac{1}{t} \\
⇒ 5 & -2\cdot x \\
⇒ \end{array}\right)
```

### D.13.6.20 texDrawBasic

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\], page 2140](#)).

**Usage:** `texDrawBasic(texdraw); list texdraw`

**Assume:** `texdraw` is a list of strings representing `texdraw` commands (as produced by `texDrawTropical`) which should be embedded into a `texdraw` environment

**Return:** string, a `texdraw` environment enclosing the input

**Note:** is called from `conicWithTangents`

**Example:**

```
LIB "tropical.lib";
ring r=(0,t),(x,y),dp;
poly f=x+y+1;
string texf=texDrawTropical(tropicalCurve(f),list("",1));
texDrawBasic(texf);
⇒
⇒ \begin{texdraw}
⇒ \drawdim cm \relunitscale 0.7 \arrowheadtype t:V
⇒ %\linewidth 0.05 \lpatt (0.1 0.4)
⇒ %\move (-4 0) \avec (9 0) \move (0 -4) \avec (0 9)
⇒ \linewidth 0.1 \lpatt (1 0)
⇒
```

```

⇒
⇒      \setgray 0.6
⇒
⇒
⇒
⇒      \relunitscale 12
⇒      \move (0 0) \fcir f:0 r:0.016
⇒      \move (0 0) \rlvec (0 0.25)
⇒      \move (0 0) \rlvec (0.25 0)
⇒      \move (0 0) \rlvec (-0.25 -0.25)
⇒
⇒      %% HERE STARTS THE CODE FOR THE LATTICE
⇒      \move (0 0) \fcir f:0.8 r:0.008
⇒      %% HERE ENDS THE CODE FOR THE LATTICE
⇒
⇒      \end{texdraw}

```

### D.13.6.21 texDrawTropical

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\]](#), page 2140).

**Usage:** `texDrawTropical(graph[,#]);` graph list, # optional list

**Assume:** graph is the output of `tropicalCurve`

**Return:** string, the texdraw code of the tropical plane curve encoded by graph

**Note:**

- if the list # is non-empty, the first entry should be a string; if this string is 'max', then the tropical curve is considered with respect to the maximum
- the procedure computes a scalefactor for the texdraw command which should help to display the curve in the right way; this may, however, be a bad idea if several `texDrawTropical` outputs are put together to form one image; the scalefactor can be prescribed by the further optional entry of type poly
- one can add a string as last optional argument to the list #; it can be used to insert further texdraw commands (e.g. to have a lighter image as when called from inside `conicWithTangents`);
- the list # is optional and may as well be empty

**Example:**

```

LIB "tropical.lib";
ring r=(0,t),(x,y),dp;
poly f=x+y+x2y+xy2+1/t*xy;
list graph=tropicalCurve(f);
// compute the texdraw code of the tropical curve defined by f
texDrawTropical(graph);
⇒
⇒
⇒      \setgray 0.6
⇒
⇒
⇒
⇒      \relunitscale 6
⇒      \move (-1 -1) \fcir f:0 r:0.03
⇒      \move (-1 -1) \lvec (-1 1)
⇒      \move (-1 -1) \lvec (1 -1)
⇒      \move (-1 -1) \rlvec (-0.5 -0.5)
⇒      \move (-1 1) \fcir f:0 r:0.03

```

```

⇒      \move (-1 1) \lvec (1 1)
⇒      \move (-1 1) \rlvec (-0.5 0.5)
⇒      \move (1 -1) \fcir f:0 r:0.03
⇒      \move (1 -1) \lvec (1 1)
⇒      \move (1 -1) \rlvec (0.5 -0.5)
⇒      \move (1 1) \fcir f:0 r:0.03
⇒      \move (1 1) \rlvec (0.5 0.5)
⇒
⇒      %% HERE STARTS THE CODE FOR THE LATTICE
⇒      \move (-1 -1) \fcir f:0.8 r:0.01
⇒      \move (-1 0) \fcir f:0.8 r:0.01
⇒      \move (-1 1) \fcir f:0.8 r:0.01
⇒      \move (0 -1) \fcir f:0.8 r:0.01
⇒      \move (0 0) \fcir f:0.8 r:0.01
⇒      \move (0 1) \fcir f:0.8 r:0.01
⇒      \move (1 -1) \fcir f:0.8 r:0.01
⇒      \move (1 0) \fcir f:0.8 r:0.01
⇒      \move (1 1) \fcir f:0.8 r:0.01
⇒      %% HERE ENDS THE CODE FOR THE LATTICE
⇒
// compute the texdraw code again, but set the scalefactor to 1
texDrawTropical(graph,"",1);
⇒
⇒
⇒      \setgray 0.6
⇒
⇒
⇒      \relunitscale 6
⇒      \move (-1 -1) \fcir f:0 r:0.03
⇒      \move (-1 -1) \lvec (-1 1)
⇒      \move (-1 -1) \lvec (1 -1)
⇒      \move (-1 -1) \rlvec (-0.5 -0.5)
⇒      \move (-1 1) \fcir f:0 r:0.03
⇒      \move (-1 1) \lvec (1 1)
⇒      \move (-1 1) \rlvec (-0.5 0.5)
⇒      \move (1 -1) \fcir f:0 r:0.03
⇒      \move (1 -1) \lvec (1 1)
⇒      \move (1 -1) \rlvec (0.5 -0.5)
⇒      \move (1 1) \fcir f:0 r:0.03
⇒      \move (1 1) \rlvec (0.5 0.5)
⇒
⇒      %% HERE STARTS THE CODE FOR THE LATTICE
⇒      \move (-1 -1) \fcir f:0.8 r:0.01
⇒      \move (-1 0) \fcir f:0.8 r:0.01
⇒      \move (-1 1) \fcir f:0.8 r:0.01
⇒      \move (0 -1) \fcir f:0.8 r:0.01
⇒      \move (0 0) \fcir f:0.8 r:0.01
⇒      \move (0 1) \fcir f:0.8 r:0.01
⇒      \move (1 -1) \fcir f:0.8 r:0.01
⇒      \move (1 0) \fcir f:0.8 r:0.01
⇒      \move (1 1) \fcir f:0.8 r:0.01
⇒      %% HERE ENDS THE CODE FOR THE LATTICE
⇒

```

### D.13.6.22 texDrawNewtonSubdivision

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\]](#), page 2140).

**Usage:** `texDrawNewtonSubdivision(graph[,#]);` graph list, # optional list

**Assume:** graph is the output of `tropicalCurve`

**Return:** string, the texdraw code of the Newton subdivision of the tropical plane curve encoded by graph

**Note:**

- the list # may contain optional arguments, of which only one will be considered, namely the first entry of type 'poly'; this entry should be a rational number which specifies the scaling factor to be used; if it is missing, the factor will be computed; the list # may as well be empty
- note that lattice points in the Newton subdivision which are black correspond to markings of the marked subdivision, while lattice points in grey are not marked

**Example:**

```
LIB "tropical.lib";
ring r=(0,t),(x,y),dp;
poly f=x+y+x2y+xy2+1/t*xy;
list graph=tropicalCurve(f);
// compute the texdraw code of the Newton subdivision of the tropical curve
texDrawNewtonSubdivision(graph);
↳
↳      \begin{texdraw}
↳          \drawdim cm \relunitscale 1
↳          \linewidth 0.05
↳          \move (1 2)
↳          \lvec (2 1)
↳          \move (2 1)
↳          \lvec (1 0)
↳          \move (1 0)
↳          \lvec (0 1)
↳          \move (0 1)
↳          \lvec (1 2)
↳
↳
↳          \move (2 1)
↳          \lvec (1 1)
↳          \move (1 1)
↳          \lvec (1 2)
↳          \move (1 0)
↳          \lvec (1 1)
↳          \move (1 1)
↳          \lvec (0 1)
↳          \move (0 0) \fcir f:0.6 r:0.03
↳          \move (0 1) \fcir f:0.6 r:0.03
↳          \move (0 2) \fcir f:0.6 r:0.03
↳          \move (1 0) \fcir f:0.6 r:0.03
↳          \move (1 1) \fcir f:0.6 r:0.03
↳          \move (1 2) \fcir f:0.6 r:0.03
↳          \move (2 0) \fcir f:0.6 r:0.03
↳          \move (2 1) \fcir f:0.6 r:0.03
```



```

⇒      \move (2 2) \fcir f:0.6 r:0.03
⇒      \move (2 1)
⇒      \fcir f:0 r:0.04
⇒      \move (1 2)
⇒      \fcir f:0 r:0.04
⇒      \move (1 1)
⇒      \fcir f:0 r:0.04
⇒      \move (1 0)
⇒      \fcir f:0 r:0.04
⇒      \move (0 1)
⇒      \fcir f:0 r:0.04
⇒      \end{texdraw}
⇒

```

### D.13.6.23 texDrawTriangulation

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\]](#), page 2140).

**Usage:** `texDrawTriangulation(triang,polygon);` triang,polygon list

**Assume:** polygon is a list of integer vectors describing the lattice points of a marked polygon;  
 triang is a list of integer vectors describing a triangulation of the marked polygon  
 in the sense that an integer vector of the form (i,j,k) describes the triangle formed by polygon[i], polygon[j] and polygon[k]

**Return:** string, a texdraw code for the triangulation described by triang without the texdraw environment

**Example:**

```

LIB "tropical.lib";
// the lattice polygon spanned by the points (0,0), (3,0) and (0,3)
// with all integer points as markings
list polygon=intvec(1,1),intvec(3,0),intvec(2,0),intvec(1,0),intvec(0,0),
intvec(2,1),intvec(0,1),intvec(1,2),intvec(0,2),intvec(0,3);
// define a triangulation by connecting the only interior point
//      with the vertices
list triang=intvec(1,2,5),intvec(1,5,10),intvec(1,2,10);
// produce the texdraw output of the triangulation triang
texDrawTriangulation(triang,polygon);
⇒
⇒      \drawdim cm \relunitscale 1.2 \arrowheadtype t:V
⇒
⇒      \setgray 0
⇒      \move (1 1)
⇒      \fcir f:0 r:0.08
⇒      \move (3 0)
⇒      \fcir f:0 r:0.08
⇒      \move (0 0)
⇒      \fcir f:0 r:0.08
⇒      \move (0 3)
⇒      \fcir f:0 r:0.08
⇒      \move (1 1)
⇒      \lvec (3 0)

```

```

↳      \move (3 0)
↳      \lvec (0 0)
↳      \move (0 0)
↳      \lvec (1 1)
↳      \move (0 0)
↳      \lvec (0 3)
↳      \move (0 3)
↳      \lvec (1 1)
↳      \move (3 0)
↳      \lvec (0 3)
↳      \move (1 1)
↳      \fcir f:0.7 r:0.04
↳      \move (3 0)
↳      \fcir f:0.7 r:0.04
↳      \move (2 0)
↳      \fcir f:0.7 r:0.04
↳      \move (1 0)
↳      \fcir f:0.7 r:0.04
↳      \move (0 0)
↳      \fcir f:0.7 r:0.04
↳      \move (2 1)
↳      \fcir f:0.7 r:0.04
↳      \move (0 1)
↳      \fcir f:0.7 r:0.04
↳      \move (1 2)
↳      \fcir f:0.7 r:0.04
↳      \move (0 2)
↳      \fcir f:0.7 r:0.04
↳      \move (0 3)
↳      \fcir f:0.7 r:0.04

```

#### D.13.6.24 radicalMemberShip

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\], page 2140](#)).

**Usage:**      `radicalMemberShip (f,i);`  $f$  poly,  $i$  ideal

**Return:**    `int`, 1 if  $f$  is in the radical of  $i$ , 0 else

**Example:**

```

LIB "tropical.lib";
ring r=0,(x,y),dp;
ideal i=(x+1)*y2;
// y is NOT in the radical of i
radicalMemberShip(y,i);
↳ 0
ring rr=0,(x,y),ds;
ideal i=(x+1)*y2;
// since this time the ordering is local, y is in the radical of i
radicalMemberShip(y,i);
↳ 1

```

#### D.13.6.25 tInitialFormPar

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\], page 2140](#)).

**Usage:** `tInitialFormPar(f,w)`;  $f$  a polynomial,  $w$  an integer vector  
**Assume:**  $f$  is a polynomial in  $\mathbb{Q}(t)[x_1, \dots, x_n]$  and  $w=(w_1, \dots, w_n)$   
**Return:** poly, the  $t$ -initialform of  $f(t,x)$  w.r.t.  $(1,w)$  evaluated at  $t=1$   
**Note:** the  $t$ -initialform are the terms with MINIMAL weighted order w.r.t.  $(1,w)$

**Example:**

```
LIB "tropical.lib";
ring r=(0,t),(x,y),dp;
poly f=t4x2+y2-t2xy+t4x-t9;
intvec w=2,3;
tInitialFormPar(f,w);
↦ y2+x
```

### D.13.6.26 tInitialFormParMax

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\], page 2140](#)).

**Usage:** `tInitialFormParMax(f,w)`;  $f$  a polynomial,  $w$  an integer vector  
**Assume:**  $f$  is a polynomial in  $\mathbb{Q}(t)[x_1, \dots, x_n]$  and  $w=(w_1, \dots, w_n)$   
**Return:** poly, the  $t$ -initialform of  $f(t,x)$  w.r.t.  $(-1,w)$  evaluated at  $t=1$   
**Note:** the  $t$ -initialform are the terms with MAXIMAL weighted order w.r.t.  $(1,w)$

**Example:**

```
LIB "tropical.lib";
ring r=(0,t),(x,y),dp;
poly f=t4x2+y2-t2xy+t4x-1/t6;
intvec w=2,3;
tInitialFormParMax(f,w);
↦ y2-1
```

### D.13.6.27 solveTInitialFormPar

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\], page 2140](#)).

**Usage:** `solveTInitialFormPar(i)`;  $i$  ideal  
**Assume:**  $i$  is a zero-dimensional ideal in  $\mathbb{Q}(t)[x_1, \dots, x_n]$  generated by the  $(1,w)$ -homogeneous elements for some integer vector  $w$  - i.e. by the  $(1,w)$ -initialforms of polynomials  
**Return:** none  
**Note:** the procedure just displays complex approximations of the solution set of  $i$

**Example:**

```
LIB "tropical.lib";
ring r=(0,t),(x,y),dp;
ideal i=t2x2+y2,x-t2;
solveTInitialFormPar(i);
↦ [1]:
↦ [1]:
↦ 1
↦ [2]:
```

```

↳ -i
↳ [2]:
↳ [1]:
↳ 1
↳ [2]:
↳ i
↳ [1]:
↳ // coefficients: real[i](complex:8 digits, additional 8 digits)/(i^2+1\
)
↳ // number of vars : 2
↳ //      block 1 : ordering lp
↳ //      : names x y
↳ //      block 2 : ordering C

```

### D.13.6.28 detropicalise

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\], page 2140](#)).

**Usage:** `detropicalise(f)`; `f` poly or `f` list

**Assume:** if `f` is of type `poly` then `t` is a linear polynomial with an arbitrary constant term and positive integer coefficients as further coefficients;  
if `f` is of type `list` then `f` is a list of polynomials of the type just described in before

**Return:** `poly`, the detropicalisation of `f` ignoring the constant parts

**Note:** the output will be a monomial and the constant coefficient has been ignored

**Example:**

```

LIB "tropical.lib";
ring r=(0,t),(x,y),dp;
detropicalise(3x+4y-1);
↳ x3y4

```

### D.13.6.29 tDetropicalise

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\], page 2140](#)).

**Usage:** `tDetropicalise(f)`; `f` poly or `f` list

**Assume:** if `f` is of type `poly` then `f` is a linear polynomial with an integer constant term and positive integer coefficients as further coefficients;  
if `f` is of type `list` then it is a list of polynomials of the type just described in before

**Return:** `poly`, the detropicalisation of `f` over the field  $\mathbb{Q}(t)$

**Note:** the output will be a term where the coefficient is a Laurent monomial in the variable `t`

**Example:**

```

LIB "tropical.lib";
ring r=(0,t),(x,y),dp;
tDetropicalise(3x+4y-1);
↳ 1/(t)*x3y4

```

### D.13.6.30 dualConic

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\], page 2140](#)).

**Usage:** `dualConic(f); f poly`

**Assume:** `f` is an affine conic in two variables `x` and `y`

**Return:** `poly`, the equation of the dual conic

**Example:**

```
LIB "tropical.lib";
ring r=0,(x,y),dp;
poly conic=2x2+1/2y2-1;
dualConic(conic);
↦ 1/2x2+2y2-1
```

### D.13.6.31 parameterSubstitute

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\], page 2140](#)).

**Usage:** `parameterSubstitute(f,N); f poly, N int`

**Assume:** `f` is a polynomial in  $\mathbb{Q}(t)[x_1, \dots, x_n]$  describing a plane curve over  $\mathbb{Q}(t)$

**Return:** `poly f` with `t` replaced by  $t^N$

**Example:**

```
LIB "tropical.lib";
ring r=(0,t),(x,y),dp;
poly f=t2xy+1/t*y+t3;
parameterSubstitute(f,3);
↦ (t6)*xy+1/(t3)*y+(t9)
parameterSubstitute(f,-1);
↦ 1/(t2)*xy+(t)*y+1/(t3)
```

### D.13.6.32 tropicalSubst

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\], page 2140](#)).

**Usage:** `tropicalSubst(f,N,L); f poly, N int, L list`

**Assume:** `f` is a polynomial in  $\mathbb{Q}(t)[x_1, \dots, x_k]$   
and `L` is a list of the form `var(i.1),poly_1,...,v(i.k),poly_k`

**Return:** `list`, the list is the tropical polynomial which we get from `f` by replacing the `i`-th variable be the `i`-th polynomial but in the `i`-th polynomial the parameter `t` is replaced by  $t^{1/N}$

**Example:**

```
LIB "tropical.lib";
ring r=(0,t),(x,y),dp;
poly f=t2x+1/t*y-1;
tropicalSubst(f,2,x,x+t,y,tx+y+t2);
↦ [1]:
↦ x-1/2
↦ [2]:
↦ y-1
```

```

⇒ [3]:
⇒ 5/2
// The procedure can be used to study the effect of a transformation of
// the form  $x \rightarrow x+t^b$ , with  $b$  a rational number, on the tropicalisation and
// the  $j$ -invariant of a cubic over the Puiseux series.
f=t7*y3+t3*y2+t*(x3+xy2+y+1)+xy;
// -  $b=3/2$ , then the cycle length of the tropical cubic equals  $-\text{val}(j\text{-inv})$ 
list g32=tropicalSubst(f,2,x,x+t3,y,y);
tropicalJInvariant(g32);
⇒ 5
// -  $b=1$ , then it is still true, but only just ...
list g1=tropicalSubst(f,1,x,x+t,y,y);
tropicalJInvariant(g1);
⇒ 5
// -  $b=2/3$ , as soon as  $b<1$ , the cycle length is strictly less than  $-\text{val}(j\text{-inv})$ 
list g23=tropicalSubst(f,3,x,x+t2,y,y);
tropicalJInvariant(g23);
⇒ 14/3

```

### D.13.6.33 randomPolyInT

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\]](#), page 2140).

**Usage:** `randomPolyInT(d,ug,og[#]);`  $d$ ,  $ug$ ,  $og$  int,  $\#$  list

**Assume:** the basering has a parameter  $t$

**Return:** `poly`, a polynomial of degree  $d$  where the coefficients are of the form  $t^j$  with  $j$  a random integer between  $ug$  and  $og$

**Note:** if an optional argument  $\#$  is given, then the coefficients are instead either of the form  $t^j$  as above or they are zero, and this is chosen randomly

**Example:**

```

LIB "tropical.lib";
ring r=(0,t),(x,y),dp;
randomPolyInT(3,-2,5);
⇒ (t5)*x3+1/(t2)*x2y+(t3)*xy2+(t4)*y3+1/(t)*x2+(t5)*xy+y2+1/(t2)*x+1/(t)*y+\
1
randomPolyInT(3,-2,5,1);
⇒ (t4)*y3+1/(t)*x2+(t2)*xy+(t2)*y2+(t3)*x

```

### D.13.6.34 cleanTmp

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\]](#), page 2140).

**Usage:** `cleanTmp()`

**Purpose:** some procedures create latex and ps-files in the directory `/tmp`; in order to remove them simply call `cleanTmp()`;

**Return:** none

### D.13.6.35 groebnerCone

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\]](#), page 2140).

**Usage:**      `groebnerCone(I,w)`; I ideal or poly, w intvec or bigintmat

**Assume:**    I a reduced standard basis and w contained in the maximal Groebner cone

**Return:**     cone, the Groebner cone of I with respect to w

**Example:**

```
LIB "tropical.lib";
ring r = 0,(x,y,z),dp;
ideal I = cyclic(3);
option(redSB);
ideal stdI = std(I);
// w lies in the interior of a maximal Groebner cone
intvec w = 3,2,1;
cone CwI = groebnerCone(stdI,w);
print(rays(CwI));
⇒ 1,1,0,
⇒ 1,0,0,
⇒ 1,1,1
// v lies on a facet of a maximal Groebner cone
intvec v = 2,1,0;
cone CvI = groebnerCone(stdI,v);
print(rays(CvI));
⇒ 1,0,0,
⇒ 1,1,0
// v lies on a ray of a maximal Groebner cone
intvec u = 1,1,1;
cone CuI = groebnerCone(stdI,u);
print(rays(CuI));
⇒ 1,1,1
```

### D.13.6.36 maximalGroebnerCone

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\]](#), page 2140).

**Usage:**      `maximalGroebnerCone(I)`; I ideal or poly

**Assume:**    I a reduced standard basis

**Return:**     cone, the maximal Groebner cone of I with respect to the current ordering

**Example:**

```
LIB "tropical.lib";
ring r = 0,(x,y,z),dp;
ideal I = cyclic(3);
option(redSB);
ideal stdI = std(I);
cone CI = maximalGroebnerCone(stdI);
print(rays(CI));
⇒ 1,1,0,
⇒ 1,0,0,
⇒ 1,1,1
ring s = 0,(x,y,z,u),dp;
ideal Ih = homog(cyclic(3),u);
ideal stdI = std(Ih);
cone CIh = maximalGroebnerCone(stdI);
```

```

print(rays(CIh));
↪ 1, 1,-1,-1,
↪ 3,-1,-1,-1,
↪ 1, 1, 1,-3
print(generatorsOfLinealitySpace(CIh));
↪ -1,-1,-1,-1
ring rw = 0,(x,y,z),(a(1,0,1),lp);
ideal I = cyclic(3);
ideal stdI = std(I);
CI = maximalGroebnerCone(stdI);
print(rays(CI));
↪ 1,0,0,
↪ 1,0,1,
↪ 1,1,1

```

### D.13.6.37 homogeneitySpace

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\], page 2140](#)).

**Usage:** `homogeneitySpace(I)`; I ideal or poly

**Assume:** I a reduced standard basis

**Return:** cone, the set of all weight vectors with respect to whom I is weighted homogeneous

**Example:**

```

LIB "tropical.lib";
ring r = 0,(x,y,z),dp;
ideal I = cyclic(3);
option(redSB);
ideal stdI = std(I);
cone COI = homogeneitySpace(stdI);
print(generatorsOfLinealitySpace(COI));
↪
ring s = 0,(x,y,z,u),dp;
ideal Ih = homog(cyclic(3),u);
ideal stdI = std(Ih);
cone COIh = homogeneitySpace(stdI);
print(generatorsOfLinealitySpace(COIh));
↪ -1,-1,-1,-1

```

### D.13.6.38 initial

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\], page 2140](#)).

**Usage:** `initial(f,w)`; f poly, w intvec or bigintmat  
`initial(I,w)`; I ideal, w intvec or bigintmat

**Assume:** I reduced Groebner basis,  
w in the maximal Groebner cone of I with respect to the current ordering

**Return:** poly or ideal, the initial form of f or the initial ideal of I with respect to w

**Example:**

```

LIB "tropical.lib";
ring r = 0,(x,y,z),dp;

```



```

ideal I = cyclic(3);
intvec w = 1,1,1;
option(redSB);
ideal stdI = std(I);
stdI;
↳ stdI[1]=x+y+z
↳ stdI[2]=y2+yz+z2
↳ stdI[3]=z3-1
ideal inI = initial(stdI,w);
inI;
↳ inI[1]=x+y+z
↳ inI[2]=y2+yz+z2
↳ inI[3]=z3

```

### D.13.6.39 tropicalVariety

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\]](#), page 2140).

**Usage:**      `tropicalVariety(f[p]);` f poly, p optional number  
               `tropicalVariety(I[p]);` I ideal, p optional number

**Assume:**    I homogeneous, p prime number

**Return:**     fan, the tropical variety of f resp. I with respect to the trivial valuation or the p-adic valuation

**Note:**       set `printlevel=1` for output during traversal

**Example:**

```

LIB "tropical.lib";
ring r = 0,(x,y,z,w),dp;
ideal I = x-2y+3z,3y-4z+5w;
tropicalVariety(I);
↳ _application PolyhedralFan
↳ _version 2.2
↳ _type PolyhedralFan
↳
↳ AMBIENT_DIM
↳ 4
↳
↳ DIM
↳ 2
↳
↳ LINEALITY_DIM
↳ 1
↳
↳ RAYS
↳ -3 1 1 1 # 0
↳ 1 -3 1 1 # 1
↳ 1 1 -3 1 # 2
↳ 1 1 1 -3 # 3
↳
↳ N_RAYS
↳ 4
↳

```

```

⇒ LINEALITY_SPACE
⇒ -1 -1 -1 -1 # 0
⇒
⇒ ORTH_LINEALITY_SPACE
⇒ 1 -1 0 0 # 0
⇒ 1 0 -1 0 # 1
⇒ 1 0 0 -1 # 2
⇒
⇒ F_VECTOR
⇒ 1 4
⇒
⇒ SIMPLICIAL
⇒ 1
⇒
⇒ PURE
⇒ 1
⇒
⇒ CONES
⇒ {} # Dimension 1
⇒ {0} # Dimension 2
⇒ {1}
⇒ {2}
⇒ {3}
⇒
⇒ MAXIMAL_CONES
⇒ {0} # Dimension 2
⇒ {1}
⇒ {2}
⇒ {3}
⇒
tropicalVariety(I,number(2));
⇒ _application PolyhedralFan
⇒ _version 2.2
⇒ _type PolyhedralFan
⇒
⇒ AMBIENT_DIM
⇒ 5
⇒
⇒ DIM
⇒ 3
⇒
⇒ LINEALITY_DIM
⇒ 1
⇒
⇒ RAYS
⇒ -2 -1 1 -1 1 # 0
⇒ -1 1 -1 1 -1 # 1
⇒ 0 -3 1 1 1 # 2
⇒ 0 1 -3 1 1 # 3
⇒ 0 1 1 -3 1 # 4
⇒ 0 1 1 1 -3 # 5
⇒
⇒ N_RAYS

```

```

⇒ 6
⇒
⇒ LINEALITY_SPACE
⇒ 0 -1 -1 -1 -1 # 0
⇒
⇒ ORTH_LINEALITY_SPACE
⇒ -1 0 0 0 0 # 0
⇒ 0 1 -1 0 0 # 1
⇒ 0 1 0 -1 0 # 2
⇒ 0 1 0 0 -1 # 3
⇒
⇒ F_VECTOR
⇒ 1 6 5
⇒
⇒ SIMPLICIAL
⇒ 1
⇒
⇒ PURE
⇒ 1
⇒
⇒ CONES
⇒ {} # Dimension 1
⇒ {0} # Dimension 2
⇒ {1}
⇒ {2}
⇒ {3}
⇒ {4}
⇒ {5}
⇒ {0 1} # Dimension 3
⇒ {0 2}
⇒ {0 4}
⇒ {1 3}
⇒ {1 5}
⇒
⇒ MAXIMAL_CONES
⇒ {0 1} # Dimension 3
⇒ {0 2}
⇒ {0 4}
⇒ {1 3}
⇒ {1 5}
⇒
tropicalVariety(I,number(3));
⇒ _application PolyhedralFan
⇒ _version 2.2
⇒ _type PolyhedralFan
⇒
⇒ AMBIENT_DIM
⇒ 5
⇒
⇒ DIM
⇒ 3
⇒
⇒ LINEALITY_DIM

```

```

⇒ 1
⇒
⇒ RAYS
⇒ -2 -1 -1 1 1 # 0
⇒ -2 1 1 -1 -1 # 1
⇒ 0 -3 1 1 1 # 2
⇒ 0 1 -3 1 1 # 3
⇒ 0 1 1 -3 1 # 4
⇒ 0 1 1 1 -3 # 5
⇒
⇒ N_RAYS
⇒ 6
⇒
⇒ LINEALITY_SPACE
⇒ 0 -1 -1 -1 -1 # 0
⇒
⇒ ORTH_LINEALITY_SPACE
⇒ -1 0 0 0 0 # 0
⇒ 0 1 -1 0 0 # 1
⇒ 0 1 0 -1 0 # 2
⇒ 0 1 0 0 -1 # 3
⇒
⇒ F_VECTOR
⇒ 1 6 5
⇒
⇒ SIMPLICIAL
⇒ 1
⇒
⇒ PURE
⇒ 1
⇒
⇒ CONES
⇒ {} # Dimension 1
⇒ {0} # Dimension 2
⇒ {1}
⇒ {2}
⇒ {3}
⇒ {4}
⇒ {5}
⇒ {0 1} # Dimension 3
⇒ {0 2}
⇒ {0 3}
⇒ {1 4}
⇒ {1 5}
⇒
⇒ MAXIMAL_CONES
⇒ {0 1} # Dimension 3
⇒ {0 2}
⇒ {0 3}
⇒ {1 4}
⇒ {1 5}
⇒
tropicalVariety(I,number(5));

```

```

⇒ _application PolyhedralFan
⇒ _version 2.2
⇒ _type PolyhedralFan
⇒
⇒ AMBIENT_DIM
⇒ 5
⇒
⇒ DIM
⇒ 3
⇒
⇒ LINEALITY_DIM
⇒ 1
⇒
⇒ RAYS
⇒ -4 -1 -1 -1 3 # 0
⇒ 0 -3 1 1 1 # 1
⇒ 0 1 -3 1 1 # 2
⇒ 0 1 1 -3 1 # 3
⇒ 0 1 1 1 -3 # 4
⇒
⇒ N_RAYS
⇒ 5
⇒
⇒ LINEALITY_SPACE
⇒ 0 -1 -1 -1 -1 # 0
⇒
⇒ ORTH_LINEALITY_SPACE
⇒ -1 0 0 0 0 # 0
⇒ 0 1 -1 0 0 # 1
⇒ 0 1 0 -1 0 # 2
⇒ 0 1 0 0 -1 # 3
⇒
⇒ F_VECTOR
⇒ 1 5 4
⇒
⇒ SIMPLICIAL
⇒ 1
⇒
⇒ PURE
⇒ 1
⇒
⇒ CONES
⇒ {} # Dimension 1
⇒ {0} # Dimension 2
⇒ {1}
⇒ {2}
⇒ {3}
⇒ {4}
⇒ {0 1} # Dimension 3
⇒ {0 2}
⇒ {0 3}
⇒ {0 4}
⇒

```

```

⇒ MAXIMAL_CONES
⇒ {0 1} # Dimension 3
⇒ {0 2}
⇒ {0 3}
⇒ {0 4}
⇒
tropicalVariety(I,number(7));
⇒ _application PolyhedralFan
⇒ _version 2.2
⇒ _type PolyhedralFan
⇒
⇒ AMBIENT_DIM
⇒ 5
⇒
⇒ DIM
⇒ 3
⇒
⇒ LINEALITY_DIM
⇒ 1
⇒
⇒ RAYS
⇒ -1 0 0 0 0 # 0
⇒ 0 -3 1 1 1 # 1
⇒ 0 1 -3 1 1 # 2
⇒ 0 1 1 -3 1 # 3
⇒ 0 1 1 1 -3 # 4
⇒
⇒ N_RAYS
⇒ 5
⇒
⇒ LINEALITY_SPACE
⇒ 0 -1 -1 -1 -1 # 0
⇒
⇒ ORTH_LINEALITY_SPACE
⇒ -1 0 0 0 0 # 0
⇒ 0 1 -1 0 0 # 1
⇒ 0 1 0 -1 0 # 2
⇒ 0 1 0 0 -1 # 3
⇒
⇒ F_VECTOR
⇒ 1 5 4
⇒
⇒ SIMPLICIAL
⇒ 1
⇒
⇒ PURE
⇒ 1
⇒
⇒ CONES
⇒ {} # Dimension 1
⇒ {0} # Dimension 2
⇒ {1}
⇒ {2}

```

```

⇒ {3}
⇒ {4}
⇒ {0 1} # Dimension 3
⇒ {0 2}
⇒ {0 3}
⇒ {0 4}
⇒
⇒ MAXIMAL_CONES
⇒ {0 1} # Dimension 3
⇒ {0 2}
⇒ {0 3}
⇒ {0 4}
⇒

```

#### D.13.6.40 groebnerFan

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\]](#), page 2140).

**Usage:**      `groebnerFan(f); f poly`  
               `groebnerFan(I); I ideal`

**Assume:**    `I` homogeneous

**Return:**     `fan`, the Groebner fan of `f` resp. `I`

**Note:**        set `printlevel=1` for output during traversal

**Example:**

```

LIB "tropical.lib";
ring r = 0,(x,y,z,w),dp;
ideal I = x-2y+3z,3y-4z+5w;
groebnerFan(I);
⇒ _application PolyhedralFan
⇒ _version 2.2
⇒ _type PolyhedralFan
⇒
⇒ AMBIENT_DIM
⇒ 4
⇒
⇒ DIM
⇒ 4
⇒
⇒ LINEALITY_DIM
⇒ 1
⇒
⇒ RAYS
⇒ -3 1 1 1 # 0
⇒ -1 -1 -1 3 # 1
⇒ -1 -1 3 -1 # 2
⇒ -1 3 -1 -1 # 3
⇒ 1 -3 1 1 # 4
⇒ 1 1 -3 1 # 5
⇒ 1 1 1 -3 # 6
⇒ 3 -1 -1 -1 # 7
⇒

```

```

⇒ N_RAYS
⇒ 8
⇒
⇒ LINEALITY_SPACE
⇒ -1 -1 -1 -1 # 0
⇒
⇒ ORTH_LINEALITY_SPACE
⇒ 1 -1 0 0 # 0
⇒ 1 0 -1 0 # 1
⇒ 1 0 0 -1 # 2
⇒
⇒ F_VECTOR
⇒ 1 8 12 6
⇒
⇒ SIMPLICIAL
⇒ 0
⇒
⇒ PURE
⇒ 1
⇒
⇒ CONES
⇒ {} # Dimension 1
⇒ {0} # Dimension 2
⇒ {1}
⇒ {2}
⇒ {3}
⇒ {4}
⇒ {5}
⇒ {6}
⇒ {7}
⇒ {0 1} # Dimension 3
⇒ {0 2}
⇒ {0 3}
⇒ {1 4}
⇒ {2 4}
⇒ {1 5}
⇒ {2 6}
⇒ {3 5}
⇒ {3 6}
⇒ {4 7}
⇒ {5 7}
⇒ {6 7}
⇒ {0 1 2 4} # Dimension 4
⇒ {0 1 3 5}
⇒ {0 2 3 6}
⇒ {1 4 5 7}
⇒ {2 4 6 7}
⇒ {3 5 6 7}
⇒
⇒ MAXIMAL_CONES
⇒ {0 1 2 4} # Dimension 4
⇒ {0 1 3 5}
⇒ {0 2 3 6}

```



```

⇒ {1 4 5 7}
⇒ {2 4 6 7}
⇒ {3 5 6 7}
⇒

```

### D.13.6.41 groebnerComplex

Procedure from library `tropical.lib` (see [Section D.13.6 \[tropical.lib\]](#), page 2140).

**Usage:**      `groebnerComplex(f,p)`; `f` poly, `p` number  
               `groebnerComplex(I,p)`; `I` ideal, `p` number

**Assume:**    `I` homogeneous, `p` prime number

**Return:**     `fan`, the Groebner complex of `f` resp. `I` with respect to the  $p$ -adic valuation

**Note:**        set `printlevel=1` for output during traversal

**Example:**

```

LIB "tropical.lib";
ring r = 0,(x,y,z,w),dp;
ideal I = x-2y+3z,3y-4z+5w;
groebnerComplex(I,number(2));
⇒ _application PolyhedralFan
⇒ _version 2.2
⇒ _type PolyhedralFan
⇒
⇒ AMBIENT_DIM
⇒ 5
⇒
⇒ DIM
⇒ 5
⇒
⇒ LINEALITY_DIM
⇒ 1
⇒
⇒ RAYS
⇒ -2 -1 1 -1 1 # 0
⇒ -1 1 -1 1 -1 # 1
⇒ 0 -3 1 1 1 # 2
⇒ 0 -1 -1 -1 3 # 3
⇒ 0 -1 -1 3 -1 # 4
⇒ 0 -1 3 -1 -1 # 5
⇒ 0 1 -3 1 1 # 6
⇒ 0 1 1 -3 1 # 7
⇒ 0 1 1 1 -3 # 8
⇒ 0 3 -1 -1 -1 # 9
⇒
⇒ N_RAYS
⇒ 10
⇒
⇒ LINEALITY_SPACE
⇒ 0 -1 -1 -1 -1 # 0
⇒
⇒ ORTH_LINEALITY_SPACE

```

```

⇒ -1 0 0 0 0 # 0
⇒ 0 1 -1 0 0 # 1
⇒ 0 1 0 -1 0 # 2
⇒ 0 1 0 0 -1 # 3
⇒
⇒ F_VECTOR
⇒ 1 10 23 22 8
⇒
⇒ SIMPLICIAL
⇒ 0
⇒
⇒ PURE
⇒ 1
⇒
⇒ CONES
⇒ {} # Dimension 1
⇒ {0} # Dimension 2
⇒ {1}
⇒ {2}
⇒ {3}
⇒ {4}
⇒ {5}
⇒ {6}
⇒ {7}
⇒ {8}
⇒ {9}
⇒ {0 1} # Dimension 3
⇒ {0 2}
⇒ {0 3}
⇒ {0 5}
⇒ {0 7}
⇒ {1 4}
⇒ {1 6}
⇒ {1 8}
⇒ {1 9}
⇒ {2 3}
⇒ {2 4}
⇒ {2 5}
⇒ {2 7}
⇒ {3 6}
⇒ {4 6}
⇒ {3 7}
⇒ {4 8}
⇒ {5 7}
⇒ {5 8}
⇒ {6 8}
⇒ {6 9}
⇒ {7 9}
⇒ {8 9}
⇒ {0 1 2 4} # Dimension 4
⇒ {0 1 3 6}
⇒ {0 1 5 8}
⇒ {0 1 7 9}

```

```

⇒ {0 2 3}
⇒ {0 2 5}
⇒ {0 2 7}
⇒ {0 3 7}
⇒ {0 5 7}
⇒ {1 4 6}
⇒ {1 4 8}
⇒ {1 6 8}
⇒ {1 6 9}
⇒ {1 8 9}
⇒ {2 3 4 6}
⇒ {2 4 5 8}
⇒ {2 3 7}
⇒ {2 5 7}
⇒ {4 6 8}
⇒ {3 6 7 9}
⇒ {5 7 8 9}
⇒ {6 8 9}
⇒ {0 1 2 3 4 6} # Dimension 5
⇒ {0 1 2 4 5 8}
⇒ {0 1 3 6 7 9}
⇒ {0 1 5 7 8 9}
⇒ {0 2 3 7}
⇒ {0 2 5 7}
⇒ {1 4 6 8}
⇒ {1 6 8 9}
⇒
⇒ MAXIMAL_CONES
⇒ {0 1 2 3 4 6} # Dimension 5
⇒ {0 1 2 4 5 8}
⇒ {0 1 3 6 7 9}
⇒ {0 1 5 7 8 9}
⇒ {0 2 3 7}
⇒ {0 2 5 7}
⇒ {1 4 6 8}
⇒ {1 6 8 9}
⇒
groebnerComplex(I,number(3));
⇒ _application PolyhedralFan
⇒ _version 2.2
⇒ _type PolyhedralFan
⇒
⇒ AMBIENT_DIM
⇒ 5
⇒
⇒ DIM
⇒ 5
⇒
⇒ LINEALITY_DIM
⇒ 1
⇒
⇒ RAYS
⇒ -2 -1 -1 1 1 # 0

```

```

⇒ -2 1 1 -1 -1 # 1
⇒ 0 -3 1 1 1 # 2
⇒ 0 -1 -1 -1 3 # 3
⇒ 0 -1 -1 3 -1 # 4
⇒ 0 -1 3 -1 -1 # 5
⇒ 0 1 -3 1 1 # 6
⇒ 0 1 1 -3 1 # 7
⇒ 0 1 1 1 -3 # 8
⇒ 0 3 -1 -1 -1 # 9
⇒
⇒ N_RAYS
⇒ 10
⇒
⇒ LINEALITY_SPACE
⇒ 0 -1 -1 -1 -1 # 0
⇒
⇒ ORTH_LINEALITY_SPACE
⇒ -1 0 0 0 0 # 0
⇒ 0 1 -1 0 0 # 1
⇒ 0 1 0 -1 0 # 2
⇒ 0 1 0 0 -1 # 3
⇒
⇒ F_VECTOR
⇒ 1 10 23 22 8
⇒
⇒ SIMPLICIAL
⇒ 0
⇒
⇒ PURE
⇒ 1
⇒
⇒ CONES
⇒ {} # Dimension 1
⇒ {0} # Dimension 2
⇒ {1}
⇒ {2}
⇒ {3}
⇒ {4}
⇒ {5}
⇒ {6}
⇒ {7}
⇒ {8}
⇒ {9}
⇒ {0 1} # Dimension 3
⇒ {0 2}
⇒ {0 3}
⇒ {0 4}
⇒ {0 6}
⇒ {1 5}
⇒ {1 7}
⇒ {1 8}
⇒ {1 9}
⇒ {2 3}

```

```

⇒ {2 4}
⇒ {2 5}
⇒ {2 6}
⇒ {3 6}
⇒ {4 6}
⇒ {3 7}
⇒ {4 8}
⇒ {5 7}
⇒ {5 8}
⇒ {7 8}
⇒ {6 9}
⇒ {7 9}
⇒ {8 9}
⇒ {0 1 2 5} # Dimension 4
⇒ {0 1 3 7}
⇒ {0 1 4 8}
⇒ {0 1 6 9}
⇒ {0 2 3}
⇒ {0 2 4}
⇒ {0 2 6}
⇒ {0 3 6}
⇒ {0 4 6}
⇒ {1 5 7}
⇒ {1 5 8}
⇒ {1 7 8}
⇒ {1 7 9}
⇒ {1 8 9}
⇒ {2 3 5 7}
⇒ {2 4 5 8}
⇒ {2 3 6}
⇒ {2 4 6}
⇒ {5 7 8}
⇒ {3 6 7 9}
⇒ {4 6 8 9}
⇒ {7 8 9}
⇒ {0 1 2 3 5 7} # Dimension 5
⇒ {0 1 2 4 5 8}
⇒ {0 1 3 6 7 9}
⇒ {0 1 4 6 8 9}
⇒ {0 2 3 6}
⇒ {0 2 4 6}
⇒ {1 5 7 8}
⇒ {1 7 8 9}
⇒
⇒ MAXIMAL_CONES
⇒ {0 1 2 3 5 7} # Dimension 5
⇒ {0 1 2 4 5 8}
⇒ {0 1 3 6 7 9}
⇒ {0 1 4 6 8 9}
⇒ {0 2 3 6}
⇒ {0 2 4 6}
⇒ {1 5 7 8}
⇒ {1 7 8 9}

```

```

↳
groebnerComplex(I,number(5));
↳ _application PolyhedralFan
↳ _version 2.2
↳ _type PolyhedralFan
↳
↳ AMBIENT_DIM
↳ 5
↳
↳ DIM
↳ 5
↳
↳ LINEALITY_DIM
↳ 1
↳
↳ RAYS
↳ -4 -1 -1 -1 3 # 0
↳ 0 -3 1 1 1 # 1
↳ 0 -1 -1 -1 3 # 2
↳ 0 -1 -1 3 -1 # 3
↳ 0 -1 3 -1 -1 # 4
↳ 0 1 -3 1 1 # 5
↳ 0 1 1 -3 1 # 6
↳ 0 1 1 1 -3 # 7
↳ 0 3 -1 -1 -1 # 8
↳
↳ N_RAYS
↳ 9
↳
↳ LINEALITY_SPACE
↳ 0 -1 -1 -1 -1 # 0
↳
↳ ORTH_LINEALITY_SPACE
↳ -1 0 0 0 0 # 0
↳ 0 1 -1 0 0 # 1
↳ 0 1 0 -1 0 # 2
↳ 0 1 0 0 -1 # 3
↳
↳ F_VECTOR
↳ 1 9 20 18 6
↳
↳ SIMPLICIAL
↳ 0
↳
↳ PURE
↳ 1
↳
↳ CONES
↳ {} # Dimension 1
↳ {0} # Dimension 2
↳ {1}
↳ {2}
↳ {3}

```

```

⇒ {4}
⇒ {5}
⇒ {6}
⇒ {7}
⇒ {8}
⇒ {0 1} # Dimension 3
⇒ {0 2}
⇒ {0 3}
⇒ {0 4}
⇒ {0 5}
⇒ {0 6}
⇒ {0 7}
⇒ {0 8}
⇒ {1 2}
⇒ {1 3}
⇒ {1 4}
⇒ {2 5}
⇒ {3 5}
⇒ {2 6}
⇒ {3 7}
⇒ {4 6}
⇒ {4 7}
⇒ {5 8}
⇒ {6 8}
⇒ {7 8}
⇒ {0 1 2} # Dimension 4
⇒ {0 1 3}
⇒ {0 1 4}
⇒ {0 2 5}
⇒ {0 3 5}
⇒ {0 2 6}
⇒ {0 3 7}
⇒ {0 4 6}
⇒ {0 4 7}
⇒ {0 5 8}
⇒ {0 6 8}
⇒ {0 7 8}
⇒ {1 2 3 5}
⇒ {1 2 4 6}
⇒ {1 3 4 7}
⇒ {2 5 6 8}
⇒ {3 5 7 8}
⇒ {4 6 7 8}
⇒ {0 1 2 3 5} # Dimension 5
⇒ {0 1 2 4 6}
⇒ {0 1 3 4 7}
⇒ {0 2 5 6 8}
⇒ {0 3 5 7 8}
⇒ {0 4 6 7 8}
⇒
⇒ MAXIMAL_CONES
⇒ {0 1 2 3 5} # Dimension 5
⇒ {0 1 2 4 6}

```

```

⇒ {0 1 3 4 7}
⇒ {0 2 5 6 8}
⇒ {0 3 5 7 8}
⇒ {0 4 6 7 8}
⇒
groebnerComplex(I,number(7));
⇒ _application PolyhedralFan
⇒ _version 2.2
⇒ _type PolyhedralFan
⇒
⇒ AMBIENT_DIM
⇒ 5
⇒
⇒ DIM
⇒ 5
⇒
⇒ LINEALITY_DIM
⇒ 1
⇒
⇒ RAYS
⇒ -1 0 0 0 0 # 0
⇒ 0 -3 1 1 1 # 1
⇒ 0 -1 -1 -1 3 # 2
⇒ 0 -1 -1 3 -1 # 3
⇒ 0 -1 3 -1 -1 # 4
⇒ 0 1 -3 1 1 # 5
⇒ 0 1 1 -3 1 # 6
⇒ 0 1 1 1 -3 # 7
⇒ 0 3 -1 -1 -1 # 8
⇒
⇒ N_RAYS
⇒ 9
⇒
⇒ LINEALITY_SPACE
⇒ 0 -1 -1 -1 -1 # 0
⇒
⇒ ORTH_LINEALITY_SPACE
⇒ -1 0 0 0 0 # 0
⇒ 0 1 -1 0 0 # 1
⇒ 0 1 0 -1 0 # 2
⇒ 0 1 0 0 -1 # 3
⇒
⇒ F_VECTOR
⇒ 1 9 20 18 6
⇒
⇒ SIMPLICIAL
⇒ 0
⇒
⇒ PURE
⇒ 1
⇒
⇒ CONES
⇒ {} # Dimension 1

```



```

⇒ {0} # Dimension 2
⇒ {1}
⇒ {2}
⇒ {3}
⇒ {4}
⇒ {5}
⇒ {6}
⇒ {7}
⇒ {8}
⇒ {0 1} # Dimension 3
⇒ {0 2}
⇒ {0 3}
⇒ {0 4}
⇒ {0 5}
⇒ {0 6}
⇒ {0 7}
⇒ {0 8}
⇒ {1 2}
⇒ {1 3}
⇒ {1 4}
⇒ {2 5}
⇒ {3 5}
⇒ {2 6}
⇒ {3 7}
⇒ {4 6}
⇒ {4 7}
⇒ {5 8}
⇒ {6 8}
⇒ {7 8}
⇒ {0 1 2} # Dimension 4
⇒ {0 1 3}
⇒ {0 1 4}
⇒ {0 2 5}
⇒ {0 3 5}
⇒ {0 2 6}
⇒ {0 3 7}
⇒ {0 4 6}
⇒ {0 4 7}
⇒ {0 5 8}
⇒ {0 6 8}
⇒ {0 7 8}
⇒ {1 2 3 5}
⇒ {1 2 4 6}
⇒ {1 3 4 7}
⇒ {2 5 6 8}
⇒ {3 5 7 8}
⇒ {4 6 7 8}
⇒ {0 1 2 3 5} # Dimension 5
⇒ {0 1 2 4 6}
⇒ {0 1 3 4 7}
⇒ {0 2 5 6 8}
⇒ {0 3 5 7 8}
⇒ {0 4 6 7 8}

```

```

↳
↳ MAXIMAL_CONES
↳ {0 1 2 3 5} # Dimension 5
↳ {0 1 2 4 6}
↳ {0 1 3 4 7}
↳ {0 2 5 6 8}
↳ {0 3 5 7 8}
↳ {0 4 6 7 8}
↳

```

### D.13.7 tropicalNewton\_lib

**Library:** tropicalNewton.lib

**Purpose:** Computations in Tropical Geometry using Newton Polygon methods

**Authors:** Tommy Hofman, email: thofmann@mathematik.uni.kl.de Yue Ren, email: reny@cs.bgu.ac.il

**Overview:** This libraries contains algorithms for computing

- non-trivial points on tropical varieties,
- zero-dimensional tropical varieties,
- one-codimensional links of tropical varieties

based on Newton polygon methods.

**References:**

Hofmann, Ren: Computing tropical points and tropical links, arXiv:1611.02878 (WARNING: this library follows the max convention instead and triangular sets follow the definition of the Singular book)

**Procedures:** See also: [Section D.13.6.39 \[tropicalVariety\]](#), page 2179; [Section D.13.6 \[tropical\\_lib\]](#), page 2140.

#### D.13.7.1 setUniformizingParameter

Procedure from library tropicalNewton.lib (see [Section D.13.7 \[tropicalNewton\\_lib\]](#), page 2196).

**Usage:** setUniformizingParameter(p); p number

**Return:** none, sets the uniformizing parameter as p

**Assume:** char(K)==0 and p prime number or  
trdeg(K)>0 and p transcendental variable or  
p==0

**Example:**

```

LIB "tropicalNewton.lib";
// poor man's polynomials over Puiseux series:
ring r = (0,t),x,dp;
setUniformizingParameter(t);
val(t2+t3);
↳ 2
val(t^-2+t^-3);
↳ -3
// poor man's polynomials over p-adic numbers:
ring s = 0,x,dp;

```

```

setUniformizingParameter(2);
val(12);
 $\mapsto$  2
val(1/12);
 $\mapsto$  -2

```

### D.13.7.2 val

Procedure from library `tropicalNewton.lib` (see [Section D.13.7 \[tropicalNewton.lib\]](#), page 2196).

**Usage:** `val(c)`;  $c$  number

**Return:** int, the valuation of a element in the ground field

**Assume:** `uniformizingParameter` is set and  $c \neq 0$

**Example:**

```

LIB "tropicalNewton.lib";
// poor man's polynomials over Puiseux series:
ring r = (0,t),x,dp;
setUniformizingParameter(t);
val(t2+t3);
 $\mapsto$  2
val(t^-2+t^-3);
 $\mapsto$  -3
// poor man's polynomials over p-adic numbers:
ring s = 0,x,dp;
setUniformizingParameter(2);
val(12);
 $\mapsto$  2
val(1/12);
 $\mapsto$  -2

```

### D.13.7.3 newtonPolygonNegSlopes

Procedure from library `tropicalNewton.lib` (see [Section D.13.7 \[tropicalNewton.lib\]](#), page 2196).

**Usage:** `newtonPolygonNegSlopes(g,b)`;  $g$  poly,  $b$  int

**Return:** list, the negative slopes of the Newton Polygon of  $g$   
if  $b=1$ , computes root (type poly) instead if (easily) possible

**Assume:** `uniformizingParameter` is set and  $g$  univariate

**Example:**

```

LIB "tropicalNewton.lib";
ring r = (0,t),x,dp;
setUniformizingParameter(t);
poly g = tx2+x+1;
newtonPolygonNegSlopes(g);
 $\mapsto$  [1]:
 $\mapsto$  0
 $\mapsto$  [2]:
 $\mapsto$  -1
// poor man's polynomials over p-adic numbers:
ring s = 0,x,dp;
setUniformizingParameter(3);

```

```

poly g = x2+9x+1;
newtonPolygonNegSlopes(g);
↳ [1]:
↳ 0

```

#### D.13.7.4 cccMatrixToPositiveIntvec

Procedure from library `tropicalNewton.lib` (see [Section D.13.7 \[tropicalNewton.lib\]](#), page 2196).

**Usage:** `cccMatrixToPositiveIntvec(M)`; M matrix

**Return:** intvec, strictly positive equivalent as weight vector to row vector in M

**Assume:** constant coefficient case only,  
will scale weight vector and add vectors of ones to it

**Example:**

```

LIB "tropicalNewton.lib";
ring r = (0,t),(p01,p02,p12,p03,p13,p23,p04,p14,p24,p34),dp;
number uniformizingParameter = t;
export(uniformizingParameter);
ideal I =
p23*p14-p13*p24+p12*p34,
p23*p04-p03*p24+p02*p34,
p13*p04-p03*p14+p01*p34,
p12*p04-p02*p14+p01*p24,
p12*p03-p02*p13+p01*p23;
system("--random",1337);
matrix p = tropicalPointNewton(I);
print(p);
↳ -59,-14,-87,57,-16,-63,41,-32,-90,-8
intvec w = cccMatrixToPositiveIntvec(p);
print(w);
↳ 32,
↳ 77,
↳ 4,
↳ 148,
↳ 75,
↳ 28,
↳ 132,
↳ 59,
↳ 1,
↳ 83
def s = switchRingsAndComputeInitialIdeal(I,w);
kill uniformizingParameter;

```

#### D.13.7.5 tropicalPointNewton

Procedure from library `tropicalNewton.lib` (see [Section D.13.7 \[tropicalNewton.lib\]](#), page 2196).

**Usage:** `tropicalPointsLasVegas(I)`; I ideal

**Return:** matrix, a matrix containing a tropical point as row vector

**Assume:** uniformizingParameter is set and I monomial free

**Note:** if printlevel sufficiently high will print intermediate data and timings returns error if randomly chosen hyperplanes are not generic

**Example:**

```
LIB "tropicalNewton.lib";
ring r = (0,t),(p01,p02,p12,p03,p13,p23,p04,p14,p24,p34),dp;
number uniformizingParameter = t;
export(uniformizingParameter);
ideal I =
p23*p14-p13*p24+p12*p34,
p23*p04-p03*p24+p02*p34,
p13*p04-p03*p14+p01*p34,
p12*p04-p02*p14+p01*p24,
p12*p03-p02*p13+p01*p23;
system("--random",1337);
printlevel = 3;
matrix p = tropicalPointNewton(I);
⇒ maximal independent set: 1,1,1,0,1,1,0,0,1,1
⇒ substituting p01 with number of valuation 59
⇒ substituting p02 with number of valuation 14
⇒ substituting p12 with number of valuation 87
⇒ substituting p13 with number of valuation 16
⇒ substituting p23 with number of valuation 63
⇒ substituting p24 with number of valuation 90
⇒ substituting p34 with number of valuation 8
⇒ computing triangular decomposition (picking first factor)
⇒ starting analysis of Newton polygons
⇒ possible valuations for p14 (picking first): (968*t^104+1870*t^95+4136*t^89+7990*t^80+1738*t^74+3960*t^73+7650*t^64+7426*t^59-5658*t^58-3936*t^56-3198*t^55+7110*t^43-2760*t^42-1920*t^40-1560*t^39-2760*t^35-1920*t^33-1560*t^32)/(29*t^22+71)
⇒ possible valuations for p04 (picking first): (-9889*t^175-14674*t^174-422\53*t^160-62698*t^159-13079*t^156-24211*t^153-35926*t^152-40455*t^144-6003\0*t^143-55883*t^141-103447*t^138-153502*t^137-32021*t^134-53505*t^125-990\45*t^122-146970*t^121-136817*t^119-130995*t^103+36300*t^102+70125*t^93+15\5100*t^87+299625*t^78+7744*t^75+68563*t^72+148500*t^71+14960*t^66+6545*t^63+286875*t^62+33088*t^60+292951*t^57-212175*t^56-147600*t^54-119925*t^53\+63920*t^51+27965*t^48+13904*t^45+31680*t^44+6083*t^42+280485*t^41-103500\*t^40-72000*t^38-58500*t^37+61200*t^35-103500*t^33+26775*t^32-72000*t^31+\908*t^30-45264*t^29-5497*t^27-45387*t^26-13776*t^24-11193*t^23+56880*t^14\ -22080*t^13+9525*t^11-22140*t^10-6720*t^8-5460*t^7-22080*t^6-15360*t^4-22\140*t^3-6720*t-5460)/(1189*t^86+580*t^70+2911*t^64+580*t^63+1420*t^48+142\0*t^41)
⇒ possible valuations for p03 (picking first): (-899*t^133-1334*t^132-1189*\t^114-2201*t^111-3266*t^110-2911*t^92+3300*t^60+6375*t^51+704*t^33+6233*t\^30+1360*t^24+595*t^21+1264*t^3+553)/(82*t^80+40*t^64+40*t^57)
⇒ time used total: 0
⇒ computing independent set: 0
⇒ computing triangular decomposition: 0
⇒ analyzing newton polygons: 0
print(p);
⇒ -59,-14,-87,57,-16,-63,41,-32,-90,-8
intvec w = cccMatrixToPositiveIntvec(p);
```

```

print(w);
↳ 32,
↳ 77,
↳ 4,
↳ 148,
↳ 75,
↳ 28,
↳ 132,
↳ 59,
↳ 1,
↳ 83
def s = switchRingsAndComputeInitialIdeal(I,w);
↳ time used computing initial ideal: 0
kill uniformizingParameter;

```

### D.13.7.6 switchRingsAndComputeInitialIdeal

Procedure from library `tropicalNewton.lib` (see [Section D.13.7 \[tropicalNewton.lib\]](#), page 2196).

**Usage:** `switchRingsAndComputeInitialIdeal(I,w);` I ideal, w intvec

**Return:** ring, a ring containing the initial ideal with respect to w

**Assume:** constant coefficient case and w strictly positive integer

**Note:** if printlevel sufficiently high will print timing

**Example:**

```

LIB "tropicalNewton.lib";
ring r = (0,t),(p01,p02,p12,p03,p13,p23,p04,p14,p24,p34),dp;
number uniformizingParameter = t;
export(uniformizingParameter);
ideal I =
p23*p14-p13*p24+p12*p34,
p23*p04-p03*p24+p02*p34,
p13*p04-p03*p14+p01*p34,
p12*p04-p02*p14+p01*p24,
p12*p03-p02*p13+p01*p23;
system("--random",1337);
printlevel = 3;
matrix p = tropicalPointNewton(I);
↳ maximal independent set: 1,1,1,0,1,1,0,0,1,1
↳ substituting p01 with number of valuation 59
↳ substituting p02 with number of valuation 14
↳ substituting p12 with number of valuation 87
↳ substituting p13 with number of valuation 16
↳ substituting p23 with number of valuation 63
↳ substituting p24 with number of valuation 90
↳ substituting p34 with number of valuation 8
↳ computing triangular decomposition (picking first factor)
↳ starting analysis of Newton polygons
↳ possible valuations for p14 (picking first): (968*t^104+1870*t^95+4136*t^89+7990*t^80+1738*t^74+3960*t^73+7650*t^64+7426*t^59-5658*t^58-3936*t^56-3198*t^55+7110*t^43-2760*t^42-1920*t^40-1560*t^39-2760*t^35-1920*t^33-1560*t^32)/(29*t^22+71)

```

```

⇒ possible valuations for p04 (picking first): (-9889*t^175-14674*t^174-422\
53*t^160-62698*t^159-13079*t^156-24211*t^153-35926*t^152-40455*t^144-6003\
0*t^143-55883*t^141-103447*t^138-153502*t^137-32021*t^134-53505*t^125-990\
45*t^122-146970*t^121-136817*t^119-130995*t^103+36300*t^102+70125*t^93+15\
5100*t^87+299625*t^78+7744*t^75+68563*t^72+148500*t^71+14960*t^66+6545*t^5\
63+286875*t^62+33088*t^60+292951*t^57-212175*t^56-147600*t^54-119925*t^53\
+63920*t^51+27965*t^48+13904*t^45+31680*t^44+6083*t^42+280485*t^41-103500\
*t^40-72000*t^38-58500*t^37+61200*t^35-103500*t^33+26775*t^32-72000*t^31+\
908*t^30-45264*t^29-5497*t^27-45387*t^26-13776*t^24-11193*t^23+56880*t^14\
-22080*t^13+9525*t^11-22140*t^10-6720*t^8-5460*t^7-22080*t^6-15360*t^4-22\
140*t^3-6720*t-5460)/(1189*t^86+580*t^70+2911*t^64+580*t^63+1420*t^48+142\
0*t^41)
⇒ possible valuations for p03 (picking first): (-899*t^133-1334*t^132-1189*\
t^114-2201*t^111-3266*t^110-2911*t^92+3300*t^60+6375*t^51+704*t^33+6233*t\
^30+1360*t^24+595*t^21+1264*t^3+553)/(82*t^80+40*t^64+40*t^57)
⇒ time used total: 0
⇒ computing independent set: 0
⇒ computing triangular decomposition: 0
⇒ analyzing newton polygons: 0
print(p);
⇒ -59,-14,-87,57,-16,-63,41,-32,-90,-8
intvec w = cccMatrixToPositiveIntvec(p);
print(w);
⇒ 32,
⇒ 77,
⇒ 4,
⇒ 148,
⇒ 75,
⇒ 28,
⇒ 132,
⇒ 59,
⇒ 1,
⇒ 83
def s = switchRingsAndComputeInitialIdeal(I,w);
⇒ time used computing initial ideal: 0
kill uniformizingParameter;

```

### D.13.7.7 tropicalVarietyNewton

Procedure from library `tropicalNewton.lib` (see [Section D.13.7 \[tropicalNewton.lib\]](#), page 2196).

**Usage:** `tropicalVarietyNewton(I);` I ideal

**Return:** matrix, a matrix containing all elements of the tropical variety

**Assume:** `uniformizingParameter` is set, I monomial free and zero-dimensional

**Example:**

```

LIB "tropicalNewton.lib";
ring r = (0,t),(z,y,x),lp;
number uniformizingParameter = t;
export(uniformizingParameter);
ideal I = tx2+x+1,txy2+xy+1,xyz+1;
list TI = tropicalVarietyNewton(I);
for (int i=1; i<=size(TI); i++)

```

```

{ print(TI[i]); }
↦ 0,0,0
↦ 0,-1,1
↦ -1,1,0
↦ -2,1,1
kill uniformizingParameter;

```

### D.13.7.8 tropicalLinkNewton

Procedure from library `tropicalNewton.lib` (see [Section D.13.7 \[tropicalNewton.lib\]](#), page 2196).

**Usage:** `tropicalLinkNewton(inI)`; `inI` ideal

**Return:** matrix, a matrix containing generators of all rays of the tropical variety

**Assume:** constant coefficient case, `inI` is monomial free,  
its tropical variety has codimension one lineality space and is a polyhedral fan

**Note:** if `printlevel` sufficiently high will print intermediate results

**Example:**

```

LIB "tropicalNewton.lib";
// a 10 valent facet in tropical Grass(3,7)
ring r = (0,t),
(p012,p013,p023,p123,p014,p024,p124,p034,p134,p234,
p015,p025,p125,p035,p135,p235,p045,p145,p245,p345,
p016,p026,p126,p036,p136,p236,p046,p146,p246,p346,
p056,p156,p256,p356,p456),
wp(4,7,5,7,4,4,4,7,5,7,2,1,2,4,4,4,2,1,2,4,7,5,7,7,
5,7,7,5,7,4,4,4,4,4,4);
number uniformizingParameter = t;
export(uniformizingParameter);
ideal inI =
p345*p136+p134*p356, p125*p045+p015*p245, p124*p015-p014*p125,
p135*p245-p125*p345, p135*p045+p015*p345, p124*p045+p014*p245,
p024*p125-p012*p245, p145*p236-p124*p356, p124*p135-p123*p145,
p024*p015+p012*p045, p134*p026+p023*p146-p024*p136,
p145*p036+p014*p356, p014*p135-p013*p145, p234*p145+p124*p345,
p034*p145-p014*p345, p024*p135-p012*p345, p125*p035+p015*p235,
p235*p045-p035*p245, p234*p136-p134*p236, p134*p036-p034*p136,
p146*p356-p136*p456, p135*p146-p134*p156,
p135*p026+p023*p156+p012*p356, p124*p035+p014*p235,
p123*p025+p012*p235, p013*p025-p012*p035, p345*p146+p134*p456,
p125*p036+p015*p236, p345*p026-p023*p456+p024*p356,
p123*p015-p013*p125, p234*p025-p024*p235, p034*p025-p024*p035,
p234*p125+p123*p245, p245*p036-p045*p236, p123*p045+p013*p245,
p034*p125-p013*p245, p234*p015+p013*p245, p245*p156+p125*p456,
p034*p015+p013*p045, p045*p156-p015*p456, p135*p236-p123*p356,
p235*p146-p134*p256, p135*p036+p013*p356, p124*p036+p014*p236,
p123*p014-p013*p124, p035*p146-p134*p056, p145*p126+p124*p156,
p234*p045-p034*p245, p235*p026+p023*p256-p025*p236,
p145*p016+p014*p156, p035*p026+p023*p056-p025*p036,
p345*p236+p234*p356, p234*p135+p123*p345, p345*p036+p034*p356,
p034*p135-p013*p345, p345*p156+p135*p456, p124*p034+p014*p234,
p145*p246-p124*p456, p123*p024+p012*p234, p145*p046+p014*p456,

```



$p_{013}p_{024}p_{012}p_{034}$ ,  $p_{024}p_{156}p_{012}p_{456}$ ,  $p_{125}p_{056}p_{015}p_{256}$ ,  
 $p_{245}p_{056}p_{045}p_{256}$ ,  $p_{236}p_{146}p_{136}p_{246}$ ,  $p_{134}p_{126}p_{123}p_{146}$ ,  
 $p_{136}p_{046}p_{036}p_{146}$ ,  $p_{235}p_{036}p_{035}p_{236}$ ,  $p_{134}p_{016}p_{013}p_{146}$ ,  
 $p_{123}p_{035}p_{013}p_{235}$ ,  $p_{235}p_{156}p_{135}p_{256}$ ,  
 $p_{123}p_{026}p_{023}p_{126}p_{012}p_{236}$ ,  $p_{135}p_{056}p_{035}p_{156}$ ,  
 $p_{023}p_{016}p_{013}p_{026}p_{012}p_{036}$ ,  $p_{124}p_{056}p_{014}p_{256}$ ,  
 $p_{234}p_{146}p_{134}p_{246}$ ,  $p_{025}p_{126}p_{012}p_{256}$ ,  $p_{134}p_{046}p_{034}p_{146}$ ,  
 $p_{025}p_{016}p_{012}p_{056}$ ,  $p_{234}p_{035}p_{034}p_{235}$ ,  $p_{345}p_{256}p_{235}p_{456}$ ,  
 $p_{234}p_{026}p_{023}p_{246}p_{024}p_{236}$ ,  $p_{345}p_{056}p_{035}p_{456}$ ,  
 $p_{034}p_{026}p_{023}p_{046}p_{024}p_{036}$ ,  $p_{125}p_{016}p_{015}p_{126}$ ,  
 $p_{025}p_{246}p_{024}p_{256}$ ,  $p_{025}p_{046}p_{024}p_{056}$ ,  $p_{245}p_{126}p_{125}p_{246}$ ,  
 $p_{125}p_{046}p_{015}p_{246}$ ,  $p_{045}p_{126}p_{015}p_{246}$ ,  $p_{245}p_{016}p_{015}p_{246}$ ,  
 $p_{045}p_{016}p_{015}p_{046}$ ,  $p_{123}p_{036}p_{013}p_{236}$ ,  $p_{236}p_{156}p_{126}p_{356}$ ,  
 $p_{135}p_{126}p_{123}p_{156}$ ,  $p_{036}p_{156}p_{016}p_{356}$ ,  $p_{135}p_{016}p_{013}p_{156}$ ,  
 $p_{124}p_{016}p_{014}p_{126}$ ,  $p_{235}p_{056}p_{035}p_{256}$ ,  $p_{245}p_{046}p_{045}p_{246}$ ,  
 $p_{234}p_{036}p_{034}p_{236}$ ,  $p_{123}p_{034}p_{013}p_{234}$ ,  $p_{246}p_{356}p_{236}p_{456}$ ,  
 $p_{234}p_{156}p_{123}p_{456}$ ,  $p_{135}p_{246}p_{123}p_{456}$ ,  $p_{345}p_{126}p_{123}p_{456}$ ,  
 $p_{046}p_{356}p_{036}p_{456}$ ,  $p_{034}p_{156}p_{013}p_{456}$ ,  $p_{135}p_{046}p_{013}p_{456}$ ,  
 $p_{345}p_{016}p_{013}p_{456}$ ,  $p_{124}p_{046}p_{014}p_{246}$ ,  $p_{024}p_{126}p_{012}p_{246}$ ,  
 $p_{024}p_{016}p_{012}p_{046}$ ,  $p_{345}p_{246}p_{234}p_{456}$ ,  $p_{345}p_{046}p_{034}p_{456}$ ,  
 $p_{235}p_{126}p_{123}p_{256}$ ,  $p_{236}p_{056}p_{036}p_{256}$ ,  $p_{123}p_{056}p_{013}p_{256}$ ,  
 $p_{035}p_{126}p_{013}p_{256}$ ,  $p_{235}p_{016}p_{013}p_{256}$ ,  $p_{035}p_{016}p_{013}p_{056}$ ,  
 $p_{235}p_{246}p_{234}p_{256}$ ,  $p_{234}p_{056}p_{034}p_{256}$ ,  $p_{035}p_{246}p_{034}p_{256}$ ,  
 $p_{235}p_{046}p_{034}p_{256}$ ,  $p_{035}p_{046}p_{034}p_{056}$ ,  $p_{126}p_{036}p_{016}p_{236}$ ,  
 $p_{123}p_{016}p_{013}p_{126}$ ,  $p_{234}p_{126}p_{123}p_{246}$ ,  $p_{236}p_{046}p_{036}p_{246}$ ,  
 $p_{123}p_{046}p_{013}p_{246}$ ,  $p_{034}p_{126}p_{013}p_{246}$ ,  $p_{234}p_{016}p_{013}p_{246}$ ,  
 $p_{246}p_{156}p_{126}p_{456}$ ,  $p_{034}p_{016}p_{013}p_{046}$ ,  $p_{046}p_{156}p_{016}p_{456}$ ,  
 $p_{234}p_{046}p_{034}p_{246}$ ,  $p_{126}p_{056}p_{016}p_{256}$ ,  $p_{246}p_{056}p_{046}p_{256}$ ,  
 $p_{126}p_{046}p_{016}p_{246}$ ,  $p_{024}p_{235}p_{145}p_{124}p_{025}p_{345}$ ,  
 $p_{024}p_{035}p_{145}p_{014}p_{025}p_{345}$ ,  $p_{123}p_{145}p_{245}p_{124}p_{125}p_{345}$ ,  
 $p_{013}p_{145}p_{245}p_{014}p_{125}p_{345}$ ,  $p_{013}p_{045}p_{145}p_{014}p_{015}p_{345}$ ,  
 $p_{024}p_{235}p_{136}p_{134}p_{025}p_{236}$ ,  $p_{123}p_{245}p_{136}p_{134}p_{125}p_{236}$ ,  
 $p_{013}p_{245}p_{136}p_{134}p_{015}p_{236}$ ,  $p_{034}p_{245}p_{136}p_{134}p_{045}p_{236}$ ,  
 $p_{134}p_{156}p_{356}p_{135}p_{136}p_{456}$ ,  $p_{123}p_{145}p_{146}p_{124}p_{134}p_{156}$ ,  
 $p_{013}p_{145}p_{146}p_{014}p_{134}p_{156}$ ,  $p_{013}p_{145}p_{026}p_{023}p_{014}p_{156}p_{012}p_{014}p_{356}$ ,  
 $p_{124}p_{025}p_{156}p_{012}p_{145}p_{256}$ ,  $p_{012}p_{145}p_{056}p_{014}p_{025}p_{156}$ ,  
 $p_{024}p_{145}p_{256}p_{124}p_{025}p_{456}$ ,  $p_{024}p_{145}p_{056}p_{014}p_{025}p_{456}$ ,  
 $p_{034}p_{235}p_{136}p_{134}p_{035}p_{236}$ ,  $p_{134}p_{256}p_{356}p_{235}p_{136}p_{456}$ ,  
 $p_{134}p_{056}p_{356}p_{035}p_{136}p_{456}$ ,  $p_{025}p_{036}p_{146}p_{024}p_{136}p_{056}$ ,  
 $p_{013}p_{125}p_{026}p_{023}p_{015}p_{126}p_{012}p_{015}p_{236}$ ,  
 $p_{123}p_{245}p_{146}p_{134}p_{125}p_{246}$ ,  $p_{013}p_{245}p_{146}p_{134}p_{015}p_{246}$ ,  
 $p_{013}p_{245}p_{026}p_{023}p_{015}p_{246}p_{012}p_{045}p_{236}$ ,  
 $p_{013}p_{045}p_{026}p_{023}p_{015}p_{046}p_{012}p_{045}p_{036}$ ,  
 $p_{034}p_{245}p_{146}p_{134}p_{045}p_{246}$ ,  $p_{013}p_{124}p_{026}p_{023}p_{014}p_{126}p_{012}p_{014}p_{236}$ ,  
 $p_{013}p_{145}p_{056}p_{014}p_{035}p_{156}$ ,  $p_{024}p_{256}p_{356}p_{025}p_{236}p_{456}$ ,  
 $p_{024}p_{056}p_{356}p_{025}p_{036}p_{456}$ ,  $p_{234}p_{256}p_{356}p_{235}p_{236}p_{456}$ ,  
 $p_{034}p_{256}p_{356}p_{035}p_{236}p_{456}$ ,  $p_{034}p_{056}p_{356}p_{035}p_{036}p_{456}$ ,  
 $p_{012}p_{235}p_{145}p_{245}p_{124}p_{025}p_{125}p_{345}$ ,  
 $p_{012}p_{035}p_{145}p_{245}p_{014}p_{025}p_{125}p_{345}$ ,  
 $p_{012}p_{035}p_{045}p_{145}p_{014}p_{015}p_{025}p_{345}$ ,  
 $p_{012}p_{235}p_{245}p_{136}p_{134}p_{025}p_{125}p_{236}$ ,  
 $p_{012}p_{035}p_{245}p_{136}p_{134}p_{015}p_{025}p_{236}$ ,  
 $p_{024}p_{035}p_{245}p_{136}p_{134}p_{025}p_{045}p_{236}$ ,

```

p014*p025*p125*p156+p012*p015*p145*p256,
p012*p145*p245*p256-p124*p025*p125*p456,
p012*p045*p145*p256+p014*p025*p125*p456,
p012*p245*p256*p356-p025*p125*p236*p456,
p012*p045*p256*p356+p015*p025*p236*p456,
p012*p045*p056*p356+p015*p025*p036*p456,
p123*p245*p256*p356+p125*p235*p236*p456,
p013*p245*p256*p356+p015*p235*p236*p456,
p013*p045*p256*p356+p015*p035*p236*p456,
p013*p045*p056*p356+p015*p035*p036*p456;
system("--random",1337);
printlevel = 3;
list TinI = tropicalLinkNewton(inI);
⇒ reducing to one-dimensional fan
⇒ intersecting with pairs of affine hyperplanes
⇒ 1: empty
⇒ 2: empty
⇒ 3: empty
⇒ 4: empty
⇒ 5: empty
⇒ 6: empty
⇒ 7: empty
⇒ 8: empty
⇒ 9: empty
⇒ 10: empty
⇒ 11: non-empty, computing tropical variety
⇒ total number of rays: 2
⇒ 12: empty
⇒ 13: non-empty, computing tropical variety
⇒ total number of rays: 4
⇒ 14: non-empty, computing tropical variety
⇒ total number of rays: 6
⇒ 15: non-empty, computing tropical variety
⇒ total number of rays: 8
⇒ 16: empty
⇒ 17: empty
⇒ 18: empty
⇒ 19: empty
⇒ 20: empty
⇒ 21: empty
⇒ 22: non-empty, computing tropical variety
⇒ total number of rays: 10
⇒ 23: empty
for (int i=1; i<=size(TinI); i++)
{ print(TinI[i]); }
⇒ 0,0,0,0,0,0,-1,-1,0,1,0,0,-1,-1,0,1,1,-1,-1,-1,0,1,-1,(-t+1),(-t+1),(t-1)\
,1,1,
⇒ -1,0,1,1,-1,(t-1),-1
⇒ 0,0,0,0,0,0,-1,-1,0,1,0,0,-1,-1,0,1,1,-1,-1,-1,0,-1,-1,_[1,24],_[1,25],_[\
1,26],
⇒ 1,1,-1,0,1,1,-1,_[1,34],-1
⇒ 0,0,0,0,0,0,-1,-1,0,1,0,0,-1,-1,0,1,1,-1,-1,-1,0,(-t-1),-1,1,(-t),(t),1,1\
,-1,0,

```

```

⇒ 1,1,-1,(t),-1
⇒ 0,0,0,0,0,0,-1,-1,0,1,0,0,-1,-1,0,1,1,-1,-1,-1,0,_[1,22],-1,-1,-1/(t),1/(\
t),1,1,
⇒ -1,0,1,1,-1,1/(t),-1
⇒ 0,0,0,0,0,0,-1,-1,0,1,0,0,-1,-1,0,1,1,-1,-1,-1,0,(-t-1),-1,(-t),1,(t),1,1,\
-1,0,
⇒ 1,1,-1,(t),-1
⇒ 0,0,0,0,0,0,-1,-1,0,1,0,0,-1,-1,0,1,1,-1,-1,-1,0,_[1,22],-1,-1/(t),-1,1/(\
t),1,1,
⇒ -1,0,1,1,-1,1/(t),-1
⇒ 0,0,0,0,0,0,-1,-1,0,1,0,0,-1,-1,0,1,1,-1,-1,-1,0,(t-1),-1,(t),(t),1,1,1,-\
1,0,1,
⇒ 1,-1,(-t),-1
⇒ 0,0,0,0,0,0,-1,-1,0,1,0,0,-1,-1,0,1,1,-1,-1,-1,0,_[1,22],-1,1/(t),1/(t),-\
1,1,1,
⇒ -1,0,1,1,-1,-1/(t),-1
⇒ 0,0,0,0,0,0,-1,-1,0,1,0,0,-1,-1,0,1,1,-1,-1,-1,0,(t-1),-1,(t),(t),(-t),1,\
1,-1,0,
⇒ 1,1,-1,1,-1
⇒ 0,0,0,0,0,0,-1,-1,0,1,0,0,-1,-1,0,1,1,-1,-1,-1,0,_[1,22],-1,1/(t),1/(t),-\
1/(t),
⇒ 1,1,-1,0,1,1,-1,-1,-1

```

## D.14 Miscellaneous libraries

### D.14.1 arr\_lib

**Library:** arr.lib

**Purpose:** a library of algorithms for arrangements of hyperplanes

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**Overloads:**

```

// OPERATORS
= arrAdd assignment
+ arrAdd union of two arrs
[ arrGet access to a single/multiple hyperplane(s) - arrMinus deletes given hyperplanes
from the arr <= arrLEQ comparison
>= arrGEQ comparison
== arrEQ comparison
!= arrNEQ comparison
< arrLNEQ comparison
> arrGNEQ comparison
// TYPECASTING
matrix arr2mat coeff matrix
poly arr2poly defining polynomial
// OTHER
variables arrVariables ideal generated by the variables the arr depends on nvars arrN-
vars number of variables the arr depends on delete arrDelete deletes hyperplanes by
indices print arrPrint prints the arr on the screen

```

```
// IDEAL INHERITED FUNCTIONS
homog arrHomog checks if arrangement is homogeneous simplify arrSimplify simplifies
arrangement size arrSize number of planes
subst arrSubst substitute variables
// MULTI-ARRANGEMENTS
= multarrAdd assignment of multarr
+ multarrAdd union of multarr
poly multarr2poly defining polynomial
size multarrSize number of hyperplanes with mult. print multarrPrint displays multiarr
delete multarrDelete deletes hyperplane
```

**Procedures:****D.14.1.1 arrSet**

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\], page 2205](#)).

**Usage:** `arrSet(A, k, p);` arr A, int k, poly p;

**Return:** [arr] Arrangement where the k-th hyperplane is replaced by p.

**Note:** p must be linear

**Example:**

```
LIB "arr.lib";
ring R = 0, (x,y,z), dp;
arr A = ideal(x,y,z);
arrSet(A,1,x+1);
↪ _[1]=x+1
↪ _[2]=y
↪ _[3]=z
↪
```

**D.14.1.2 type2arr**

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\], page 2205](#)).

**Usage:** `type2arr(#);` # def

**Return:** [arr] Arrangement defined by the input

**Note:** The procedure tries to cast the input to [arr] using `arrAdd`

**Example:**

```
LIB "arr.lib";
ring R = 0, (x,y,z), dp;
ideal I = x,y,z;
typeof(type2arr(I));
↪ arr
```

**D.14.1.3 mat2arr**

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\], page 2205](#)).

**Usage:** `mat2arr(M);` matrix (M|b)

**Return:** [arr] interprets the rows of the matrix as the defining polynomial equations of the arrangement where the last column will be considered as the constant terms, i.e. if  $M$  is an  $m \times (n+1)$  matrix we have  $H_i = \text{Ker}(M_{i1}x_1 + \dots + M_{in}x_n + M_{i(n+1)})$  for  $i=1\dots m$  and  $A = \{H_1, \dots, H_m\}$  the resulting arrangement.

**Example:**

```
LIB "arr.lib";
ring R = 0,(x,y,z),dp;
matrix M[4][4] = 1,0,1,1,1,1,0,2,0,1,1,3,2,1,1,4;
print(M);
  ↳ 1,0,1,1,
  ↳ 1,1,0,2,
  ↳ 0,1,1,3,
  ↳ 2,1,1,4
mat2arr(M);
  ↳ _[1]=x+z+1
  ↳ _[2]=x+y+2
  ↳ _[3]=y+z+3
  ↳ _[4]=2x+y+z+4
  ↳
```

See also: [Section D.14.1.4 \[mat2carr\]](#), page 2207.

#### D.14.1.4 mat2carr

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:** `mat2carr(M);` matrix  $M$

**Return:** [arr] interprets the rows of the matrix as the defining polynomial equations of the arrangement. I.e. if  $M$  is an  $m \times n$  matrix we have  $H_i = \text{Ker}(M_{i1}x_1 + \dots + M_{in}x_n)$  for  $i=1\dots m$  and  $A = \{H_1, \dots, H_m\}$  the resulting arrangement.

**Example:**

```
LIB "arr.lib";
ring R = 0,(x,y,z),dp;
matrix M[4][3] = 1,0,1,1,1,0,0,1,1,2,1,1;
print(M);
  ↳ 1,0,1,
  ↳ 1,1,0,
  ↳ 0,1,1,
  ↳ 2,1,1
mat2carr(M);
  ↳ _[1]=x+z
  ↳ _[2]=x+y
  ↳ _[3]=y+z
  ↳ _[4]=2x+y+z
  ↳
```

See also: [Section D.14.1.3 \[mat2arr\]](#), page 2206.

#### D.14.1.5 arrPrintMatrix

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:**     `arPrintMatrix(arr A)`

**Return:**    `[]` prints arr in matrix form

**Note:**     differs `print(matrix(arr A))` since variables included

**Example:**

```
LIB "arr.lib";
ring R = 0,(x,y,z),dp;
arr A = mat2arr(random(20,5,4));
A;
⇒ _[1]=10x+y-2z-12
⇒ _[2]=2x-16z-14
⇒ _[3]=11y+16z-5
⇒ _[4]=-16x-18y-5z+11
⇒ _[5]=3x+8y+6
⇒
arrPrintMatrix(A);
⇒ 10x, y, -2z, -12,
⇒ 2x, 0, -16z, -14,
⇒ 0, 11y, 16z, -5,
⇒ -16x, -18y, -5z, 11,
⇒ 3x, 8y, 0, 6
```

#### D.14.1.6 varMat

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:**     `varMat(v); v intvec`

**Return:**    `[matrix]` M containing the corresponding ring-variables

**Example:**

```
LIB "arr.lib";
ring R = 0,(x(1..6)),dp;
intvec v = 2,4,6;
varMat(v);
⇒ _[1,1]=x(2)
⇒ _[2,1]=x(4)
⇒ _[3,1]=x(6)
```

See also: [Section D.14.1.9 \[arrLastVar\]](#), page 2209; [Section D.14.1.8 \[arrSwapVar\]](#), page 2209; [Section D.14.1.6 \[varMat\]](#), page 2208; [Section D.14.1.7 \[varNum\]](#), page 2208.

#### D.14.1.7 varNum

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:**     `varnum(string s);`  
               `varnum(ring-variable);`

**Return:**    `[int]` number of given ring variable, or 0 if it does not appear

**Note:**     This procedure has the same functionality as `varNum` from the `dmod.lib` package, but also accepts polys as input.

**Example:**

```

LIB "arr.lib";
ring R = 0,(x,y,z),dp;
varNum(y);
↳ 2
ring S = 0,(x(1..5),y(1..5)),dp;
varNum("y(3)");
↳ 8

```

See also: [Section D.14.1.9 \[arrLastVar\]](#), page 2209; [Section D.14.1.8 \[arrSwapVar\]](#), page 2209; [Section D.14.1.6 \[varMat\]](#), page 2208; [Section D.14.1.7 \[varNum\]](#), page 2208.

### D.14.1.8 arrSwapVar

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:** `arrSwapVar(A, i, j);` arr A, ring\_variables/integers i,j

**Return:** [arr] A where variables i and j are swapped.

**Note:** if i and/or j are integers the algorithm considers the variables `variables(A)[i]` and/or `variables(A)[j]`

**Example:**

```

LIB "arr.lib";
ring R = 0,(x,y,z),lp;
arr A = ideal(x+1,x+y,z);
arrSwapVar(A,x,z);
↳ _[1]=z+1
↳ _[2]=y+z
↳ _[3]=x
↳

```

See also: [Section D.14.1.9 \[arrLastVar\]](#), page 2209; [Section D.14.1.8 \[arrSwapVar\]](#), page 2209; [Section D.14.1.6 \[varMat\]](#), page 2208; [Section D.14.1.7 \[varNum\]](#), page 2208.

### D.14.1.9 arrLastVar

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:** `arrLastVar(A);` arr A

**Return:** [int] number of the last variable A uses

**Note:** useful if you want a list containing all variables `x_1 ... x_k` used in A, but you do not want to skip any like `variables(A)` does.

**Example:**

```

LIB "arr.lib";
ring R = 0,x(1..10),dp;
arr A = ideal(x(1), x(2), x(3), x(6));
int n = arrLastVar(A);
varMat(1..n);
↳ _[1,1]=x(1)
↳ _[2,1]=x(2)
↳ _[3,1]=x(3)
↳ _[4,1]=x(4)
↳ _[5,1]=x(5)
↳ _[6,1]=x(6)

```

```

variables(A);
↪ _[1]=x(1)
↪ _[2]=x(2)
↪ _[3]=x(3)
↪ _[4]=x(6)

```

See also: [Section D.14.1.9 \[arrLastVar\]](#), page 2209; [Section D.14.1.8 \[arrSwapVar\]](#), page 2209; [Section D.14.1.6 \[varMat\]](#), page 2208; [Section D.14.1.7 \[varNum\]](#), page 2208.

### D.14.1.10 arrCenter

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:** `arrCenter(A); arr A`

**Return:** [list] L entry 0 if A not centered or entries 1, x, H, where x is any particular point of the center and H is a matrix consisting of vectors which spanning linear intersection space.  
If there is exactly one solution, then  $H = 0$ .

**Note:** The intersection of all hyperplanes can be expressed in forms of a linear system  $Ax=b$ , where  $(A|b)$  is the coeff. matrix of the arrangement, which is then solved using L-U decomposition

**Example:**

```

LIB "arr.lib";
ring R = 0,(x,y,z),dp;
arr A= ideal(x,y,x-y+1);    // centerless
arrCenter(A);
↪ [1]:
↪    0
arr B= ideal(x,y,z);        // center is a single point
arrCenter(B);
↪ [1]:
↪    1
↪ [2]:
↪    _[1,1]=0
↪    _[2,1]=0
↪    _[3,1]=0
↪ [3]:
↪    _[1,1]=0
arr C= ideal(x,z,x+z);      // center is a line
// here we get a wrong result because the matrix is simplified since A doesn't
// contain any "y" the matrix (A|b) will be 3x3 only.
arrCenter(C);
↪ [1]:
↪    1
↪ [2]:
↪    _[1,1]=0
↪    _[2,1]=0
↪    _[3,1]=0
↪ [3]:
↪    _[1,1]=0
↪    _[2,1]=-1
↪    _[3,1]=0

```



See also: [Section D.14.1.10 \[arrCenter\]](#), page 2210; [Section D.14.1.12 \[arrCentered\]](#), page 2211; [Section D.14.1.11 \[arrCentral\]](#), page 2211; [Section D.14.1.13 \[arrCentralize\]](#), page 2212.

### D.14.1.11 arrCentral

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:** `arrCentral(A);` arr A

**Return:** `[0,1]` true if arr is central (i.e. all planes intersect in 0)

**Note:** This is the same as `homog(A)`

**Example:**

```
LIB "arr.lib";
ring R = 0,(x,y,z),dp;
// centered and central
arr A = ideal(x,y,z);
arrCentered(A);
⇒ 1
arrCentral(A);
⇒ 1
// centered but not central (center: (-1,-1/2, 1))
arr B = ideal(x+1,2y+1,-z+1);
arrCentered(B);
⇒ 1
arrCentral(B);
⇒ 0
```

See also: [Section D.14.1.10 \[arrCenter\]](#), page 2210; [Section D.14.1.12 \[arrCentered\]](#), page 2211; [Section D.14.1.11 \[arrCentral\]](#), page 2211; [Section D.14.1.13 \[arrCentralize\]](#), page 2212.

### D.14.1.12 arrCentered

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:** `arrCentered(A);` arr A

**Return:** `[0,1]` true if A is centered (i.e. intersection of all planes not empty)

**Note:** The algorithm uses the rank of matrix:  $Ax=b$  has a solution iff  $\text{rank}(A) = \text{rank}(A|b)$

**Example:**

```
LIB "arr.lib";
ring R = 0,(x,y,z),dp;
arr A= ideal(x,y,x-y+1);           // centerless
arrCentral(A);
⇒ 0
arr B= ideal(x,y,z);               // central with center being the origin
arrCentral(B);
⇒ 1
arr C= ideal(x+1,2y+1,-z+1);       // central with center (-1,-1/2, 1)
arrCentral(C);
⇒ 0
```

See also: [Section D.14.1.10 \[arrCenter\]](#), page 2210; [Section D.14.1.12 \[arrCentered\]](#), page 2211; [Section D.14.1.11 \[arrCentral\]](#), page 2211; [Section D.14.1.13 \[arrCentralize\]](#), page 2212.

### D.14.1.13 arrCentralize

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:** `arrCentralize(A); arr A`

**Return:** `[arr]` A after centralization via coordinate change

**Note:** The coordinate change only does translation, vector of translation is the second output of `arrCenter`

**Example:**

```
LIB "arr.lib";
ring R = 0,(x,y,z),dp;
arr A = ideal(x-1,y,x-z-1,x-z-1);
arrCentralize(A);
↳ _[1]=x-2
↳ _[2]=y
↳ _[3]=x-z-2
↳
```

See also: [Section D.14.1.10 \[arrCenter\]](#), page 2210; [Section D.14.1.12 \[arrCentered\]](#), page 2211; [Section D.14.1.11 \[arrCentral\]](#), page 2211; [Section D.14.1.13 \[arrCentralize\]](#), page 2212.

### D.14.1.14 arrCoordChange

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:** `arrCoordChange(A, T); arr A, (m*n mat) n*n or n*n+1 matrix T arrCoordChange(A, T, c); arr A, n*n matrix T, n*1 matrix/vector`

**Return:** `[arr]`: Arrangement A `[A|b]` after a coordinate change  $f: x \rightarrow Tx + c$  with T invertible i.e.  $[A|b] \Rightarrow [AT^{-1}|b+AT^{-1}c]$  since we have  $f(H) = f(\ker(a_1x_1 + \dots + a_nx_n - b)) = \{f(x) : a'x - b = 0\} = \{y : a'f^{-1}(y) - b = 0\} = \{y : a'(T^{-1}(y-c)) - b = 0\} = \{y : a'T^{-1}y - (b + a'T^{-1}c) = 0\}$

**Note:** There are 3 options how you can give the input (in each case  $n \leq \text{nvars}(\text{basering})$ ) 1. Just a  $n \times n$  matrix with  $\Rightarrow$  Will automatically complete T by a unit matrix and perform  $x \rightarrow Tx$  2. A  $n \times n$  matrix T and a  $n \times 1$  vector/matrix c with  $\Rightarrow$  Will automatically complete T and c and perform  $x \rightarrow Tx + c$  3. A  $n \times n+1$  matrix T with  $\Rightarrow$  will use last column as translation vector c

**Example:**

```
LIB "arr.lib";
ring r = 0,(x,y,z),lp;
arr A = x,y,z;
arrCoordChange(A,1,[0,0,1]); //lifts z-hyperplane by 1 unit
↳ _[1]=x
↳ _[2]=y
↳ _[3]=z-1
↳
matrix T[2][2] = [0,1,1,0]; // swaps x and y
arrCoordChange(A,T);
↳ _[1]=y
```

```

  ↪ _[2]=x
  ↪ _[3]=z
  ↪
  matrix c[2][1] = [1,0];
  T = concat(T,c);           // now swap x and y and add 1 to x afterwards
  arrCoordChange(A,T);
  ↪ _[1]=y
  ↪ _[2]=x-1
  ↪ _[3]=z
  ↪
  // Note how T doesn't even need to be a full 3x3 base change matrix.

```

See also: [Section D.14.1.14 \[arrCoordChange\], page 2212](#); [Section D.14.1.15 \[arrCoordNormalize\], page 2213](#).

### D.14.1.15 arrCoordNormalize

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\], page 2205](#)).

**Usage:** `arrCoordChange(A, v);`

**Return:** `[arr]`: Arrangement after a coordinate change that transforms the arrangement such that after a transformation  $x \rightarrow Tx + c$  we have the arrangement has the matrix representation  $[AT^{-1}|b+AT^{-1}c]$  such that  $[AT^{-1}]_v = I$  and  $[b+AT^{-1}c]_v = 0$ ;

**Note:** algorithm performs a base change if  $H_k$  is homogeneous (i.e. has no) constant term and an affine transformation otherwise  
 $Ax+b = 0$ , Transformation  $x = Ty+c$ :  $AT^{-1}y + AT^{-1}c + b = 0$  Now we want to have  $(AT^{-1})_v = I$  and  $(AT^{-1}c + b)_v = AT^{-1}_v * c + b_v = 0$

**Example:**

```

LIB "arr.lib";
ring r = 0,(x,y,z),lp;
arr A = ideal(x,y,z,x+z+4);
intvec v = 1,2,4;
arrCoordNormalize(A,v);
↪ _[1]=x
↪ _[2]=y
↪ _[3]=-x+z-4
↪ _[4]=z
↪

```

See also: [Section D.14.1.14 \[arrCoordChange\], page 2212](#); [Section D.14.1.15 \[arrCoordNormalize\], page 2213](#).

### D.14.1.16 arrCone

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\], page 2205](#)).

**Usage:** `arrCone(A);`  
`arrCone(A, ring_variable);` `arr` `A` arrangement in variables `x_1...x_n`;

**Return:** `arr`, the coned hyperplane Arrangement `cA` with respect to the given `ring_variable`, or the last `ring_variable` if none was given.

**Note:** The hyperplanes are homogenized w.r.t. `v` and a new hyperplane  $H = \ker(x_{n+1})$  is added.

**Example:**

```

LIB "arr.lib";
ring R = 0,(x,y,z),dp;
arr A = ideal(x+1, x,x-2,x-1);
arrCone(A, y);
  ↳ _[1]=x+y
  ↳ _[2]=x
  ↳ _[3]=x-2y
  ↳ _[4]=x-y
  ↳ _[5]=y
  ↳
arr B= ideal(x,y,x+y-1);
arrCone(B);
  ↳ _[1]=x
  ↳ _[2]=y
  ↳ _[3]=x+y-z
  ↳ _[4]=z
  ↳

```

See also: [Section D.14.1.16 \[arrCone\]](#), page 2213; [Section D.14.1.17 \[arrDecone\]](#), page 2214; [Section D.14.1.21 \[arrEssentialize\]](#), page 2216; [Section D.14.1.20 \[arrIsEssential\]](#), page 2216; [Section D.14.1.19 \[arrRestrict\]](#), page 2215.

**D.14.1.17 arrDecone**

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:** `arrDecone(A, k)`; arrangement A, integer k;

**Return:** `arr`: the deconed hyperplane Arrangement dA

**Note:** A has to be non-empty and central. `arrDecone` is an inverse operation to `arrCone` since  $A == \text{arrDecone}(\text{arrCone}(A), \text{size}(A)+1)$  for any A. One can also decone a central arrangement with respect to any hyper- plane k, but than a coordinate change is necessary to make  $H_k = \ker(x_k)$ . Since such a coordinate change is not unique, use `arrCoordchange` to do so.

**Example:**

```

LIB "arr.lib";
ring R = 0,(x,y,z),dp;
arr A= ideal(x,y,z,x+y-z);
arrDecone(A,3);
  ↳ _[1]=x
  ↳ _[2]=y
  ↳ _[3]=x+y-1
  ↳

```

See also: [Section D.14.1.16 \[arrCone\]](#), page 2213; [Section D.14.1.17 \[arrDecone\]](#), page 2214; [Section D.14.1.21 \[arrEssentialize\]](#), page 2216; [Section D.14.1.20 \[arrIsEssential\]](#), page 2216; [Section D.14.1.19 \[arrRestrict\]](#), page 2215.

**D.14.1.18 arrLocalize**

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:** `arrLocalize(A, v)`; arrangement A, intvec v;

**Return:** arr: the localized arrangement  $A_X$ , i.e.  $A_X$  only contains the hyperplanes which contain the flat  $X$ , which is defined by the equations  $A[v]$

**Example:**

```
LIB "arr.lib";
ring R = 0,(x,y,z),dp;
arr A=arrTypeB(3);
intvec v=5,8;
arr B=arrLocalize(A,v);
B;
↳ _[1]=x-y
↳ _[2]=x+y
↳ _[3]=x
↳ _[4]=y
↳
```

See also: [Section D.14.1.16 \[arrCone\]](#), page 2213; [Section D.14.1.17 \[arrDecone\]](#), page 2214; [Section D.14.1.21 \[arrEssentialize\]](#), page 2216; [Section D.14.1.20 \[arrIsEssential\]](#), page 2216; [Section D.14.1.19 \[arrRestrict\]](#), page 2215.

### D.14.1.19 arrRestrict

Procedure from library `arr.lib` (see [Section D.14.1 \[arr\\_lib\]](#), page 2205).

**Return:** arr: the restricted hyperplane Arrangement ( $A^X$ )

**Note:** A has to be non-empty.

**Remarks:** We restrict  $A$  to the flat  $X$ , defined by the equations in  $A[v]$ . The restriction will only be performed, if the ideal defining the flat  $X$  is monomial (i.e.  $X$  is an intersection of coordinate planes). If the optional argument `CC` is given, the arrangement is transformed in such a way that  $X$  has the above form.

**Example:**

```
LIB "arr.lib";
ring S = 0,(x,y,z),dp;
arr A = arrTypeB(3);
A;
↳ _[1]=x-y
↳ _[2]=x+y
↳ _[3]=x-z
↳ _[4]=x+z
↳ _[5]=x
↳ _[6]=y-z
↳ _[7]=y+z
↳ _[8]=y
↳ _[9]=z
↳
arrRestrict(A,9);
↳ _[1]=x-y
↳ _[2]=x+y
↳ _[3]=x
↳ _[4]=y
↳
arrRestrict(A,4,"CC");
```

```

↳ _[1]=1/2y-z
↳ _[2]=1/2y+z
↳ _[3]=y
↳ _[4]=z
↳
intvec v=5,8;
arrRestrict(A,v);
↳ _[1]=-z
↳

```

See also: [Section D.14.1.16 \[arrCone\]](#), page 2213; [Section D.14.1.17 \[arrDecone\]](#), page 2214; [Section D.14.1.21 \[arrEssentialize\]](#), page 2216; [Section D.14.1.20 \[arrIsEssential\]](#), page 2216; [Section D.14.1.19 \[arrRestrict\]](#), page 2215.

### D.14.1.20 arrIsEssential

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:** `arrIsEssential(A)`; arrangement A;

**Return:** boolean: 1 if arr is essential, i.e. rank of maximal element of poset is dimension

**Note:** A has to be non-empty.

**Example:**

```

LIB "arr.lib";
ring S = 0,(x,y,z),lp;
arr A = ideal(x,y,z);
arr B = ideal(x+y+z,x,y+z);
arrIsEssential(A);
↳ 1
arrIsEssential(B);
↳ 0

```

See also: [Section D.14.1.16 \[arrCone\]](#), page 2213; [Section D.14.1.17 \[arrDecone\]](#), page 2214; [Section D.14.1.21 \[arrEssentialize\]](#), page 2216; [Section D.14.1.20 \[arrIsEssential\]](#), page 2216; [Section D.14.1.19 \[arrRestrict\]](#), page 2215.

### D.14.1.21 arrEssentialize

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:** `arrEssentialize(A)`; arrangement A;

**Return:** essential arrangement by transformation

**Note:** A has to be non-empty.

**Example:**

```

LIB "arr.lib";
ring S = 0,(x,y,z),dp;
arr A=arrBraid(3);
arrEssentialize(A);
↳ _[1]=x-y
↳ _[2]=x
↳ _[3]=y
↳
arr B = ideal(x+y+z,x,y+z);

```

```

arrEssentialize(B);
⇨ _[1]=x+y
⇨ _[2]=x
⇨ _[3]=y
⇨

```

See also: [Section D.14.1.16 \[arrCone\]](#), page 2213; [Section D.14.1.17 \[arrDecone\]](#), page 2214; [Section D.14.1.21 \[arrEssentialize\]](#), page 2216; [Section D.14.1.20 \[arrIsEssential\]](#), page 2216; [Section D.14.1.19 \[arrRestrict\]](#), page 2215.

### D.14.1.22 arrBoolean

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:**      `arrBoolean(v); int v`

**Return:**     `arr`, which uses the first `v` variables of ring for boolean arrangement

**Example:**

```

LIB "arr.lib";
ring R = 0,x(1..10),dp;
arrBoolean(7);
⇨ _[1]=x(1)
⇨ _[2]=x(2)
⇨ _[3]=x(3)
⇨ _[4]=x(4)
⇨ _[5]=x(5)
⇨ _[6]=x(6)
⇨ _[7]=x(7)
⇨

```

See also: [Section D.14.1.22 \[arrBoolean\]](#), page 2217; [Section D.14.1.23 \[arrBraid\]](#), page 2217; [Section D.14.1.28 \[arrEdelmanReiner\]](#), page 2220; [Section D.14.1.26 \[arrRandom\]](#), page 2219; [Section D.14.1.24 \[arrTypeB\]](#), page 2218; [Section D.14.1.25 \[arrTypeD\]](#), page 2219.

### D.14.1.23 arrBraid

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:**      `arrBraid(v); int v`

**Return:**     Type A (braid) arrangement of dimension `v`

**Example:**

```

LIB "arr.lib";
ring R = 0,x(1..10),dp;
arrBraid(7);
⇨ _[1]=x(1)-x(2)
⇨ _[2]=x(1)-x(3)
⇨ _[3]=x(1)-x(4)
⇨ _[4]=x(1)-x(5)
⇨ _[5]=x(1)-x(6)
⇨ _[6]=x(1)-x(7)
⇨ _[7]=x(2)-x(3)
⇨ _[8]=x(2)-x(4)
⇨ _[9]=x(2)-x(5)
⇨ _[10]=x(2)-x(6)

```

```

↳ _[11]=x(2)-x(7)
↳ _[12]=x(3)-x(4)
↳ _[13]=x(3)-x(5)
↳ _[14]=x(3)-x(6)
↳ _[15]=x(3)-x(7)
↳ _[16]=x(4)-x(5)
↳ _[17]=x(4)-x(6)
↳ _[18]=x(4)-x(7)
↳ _[19]=x(5)-x(6)
↳ _[20]=x(5)-x(7)
↳ _[21]=x(6)-x(7)
↳

```

See also: [Section D.14.1.22 \[arrBoolean\]](#), page 2217; [Section D.14.1.23 \[arrBraid\]](#), page 2217; [Section D.14.1.28 \[arrEdelmanReiner\]](#), page 2220; [Section D.14.1.26 \[arrRandom\]](#), page 2219; [Section D.14.1.24 \[arrTypeB\]](#), page 2218; [Section D.14.1.25 \[arrTypeD\]](#), page 2219.

### D.14.1.24 arrTypeB

Procedure from library `arr.lib` (see [Section D.14.1 \[arr\\_lib\]](#), page 2205).

**Usage:**      `arrTypeB(v); int v`

**Return:**     arrangement, which uses first `v` variables of ring for reflection arrangement of type B

**Example:**

```

LIB "arr.lib";
ring R = 0,x(1..10),dp;
arrTypeB(5);
↳ _[1]=x(1)-x(2)
↳ _[2]=x(1)+x(2)
↳ _[3]=x(1)-x(3)
↳ _[4]=x(1)+x(3)
↳ _[5]=x(1)-x(4)
↳ _[6]=x(1)+x(4)
↳ _[7]=x(1)-x(5)
↳ _[8]=x(1)+x(5)
↳ _[9]=x(1)
↳ _[10]=x(2)-x(3)
↳ _[11]=x(2)+x(3)
↳ _[12]=x(2)-x(4)
↳ _[13]=x(2)+x(4)
↳ _[14]=x(2)-x(5)
↳ _[15]=x(2)+x(5)
↳ _[16]=x(2)
↳ _[17]=x(3)-x(4)
↳ _[18]=x(3)+x(4)
↳ _[19]=x(3)-x(5)
↳ _[20]=x(3)+x(5)
↳ _[21]=x(3)
↳ _[22]=x(4)-x(5)
↳ _[23]=x(4)+x(5)
↳ _[24]=x(4)
↳ _[25]=x(5)
↳

```



See also: [Section D.14.1.22 \[arrBoolean\]](#), page 2217; [Section D.14.1.23 \[arrBraid\]](#), page 2217; [Section D.14.1.28 \[arrEdelmanReiner\]](#), page 2220; [Section D.14.1.26 \[arrRandom\]](#), page 2219; [Section D.14.1.24 \[arrTypeB\]](#), page 2218; [Section D.14.1.25 \[arrTypeD\]](#), page 2219.

### D.14.1.25 arrTypeD

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:** `arrTypeD(v); int v`

**Return:** arrangement, which uses first  $v$  variables of ring for reflection arrangement of type D

**Example:**

```
LIB "arr.lib";
ring R = 0,x(1..10),dp;
arrTypeD(5);
⇨ _[1]=x(1)-x(2)
⇨ _[2]=x(1)+x(2)
⇨ _[3]=x(1)-x(3)
⇨ _[4]=x(1)+x(3)
⇨ _[5]=x(1)-x(4)
⇨ _[6]=x(1)+x(4)
⇨ _[7]=x(1)-x(5)
⇨ _[8]=x(1)+x(5)
⇨ _[9]=x(2)-x(3)
⇨ _[10]=x(2)+x(3)
⇨ _[11]=x(2)-x(4)
⇨ _[12]=x(2)+x(4)
⇨ _[13]=x(2)-x(5)
⇨ _[14]=x(2)+x(5)
⇨ _[15]=x(3)-x(4)
⇨ _[16]=x(3)+x(4)
⇨ _[17]=x(3)-x(5)
⇨ _[18]=x(3)+x(5)
⇨ _[19]=x(4)-x(5)
⇨ _[20]=x(4)+x(5)
⇨
```

See also: [Section D.14.1.22 \[arrBoolean\]](#), page 2217; [Section D.14.1.23 \[arrBraid\]](#), page 2217; [Section D.14.1.28 \[arrEdelmanReiner\]](#), page 2220; [Section D.14.1.26 \[arrRandom\]](#), page 2219; [Section D.14.1.24 \[arrTypeB\]](#), page 2218; [Section D.14.1.25 \[arrTypeD\]](#), page 2219.

### D.14.1.26 arrRandom

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:** `arrRandom(n,v,N); int n,v,N`

**Return:** Random arrangement, where  $m$  is the number of hyperplanes,  $n$  the dimension,  $d$  the upper bound for absolute value of coefficients.

**Note:** You can also write `arr = random(d,m,n)` to create random arrangements

**Example:**

```
LIB "arr.lib";
ring R = 0,x(1..20),dp;
arrRandom(7,3,15);
```

```

↳ _[1]=2*x(1)-6*x(2)-3*x(3)+4*x(5)+x(6)-4*x(7)-3*x(8)-x(9)+x(10)-3*x(11)-6*\
x(12)-x(13)+2*x(14)-6*x(15)-4
↳ _[2]=x(1)-3*x(2)-4*x(3)-7*x(5)-3*x(6)+x(8)-3*x(9)-4*x(10)-x(11)-7*x(12)-2*\
*x(13)-3*x(14)-1
↳ _[3]=4*x(1)+2*x(2)+5*x(3)-x(4)+6*x(5)+5*x(6)-6*x(7)+5*x(8)+x(10)+3*x(11)-\
5*x(12)+2*x(13)+6*x(14)+x(15)+3
↳

```

See also: [Section D.14.1.22 \[arrBoolean\]](#), page 2217; [Section D.14.1.23 \[arrBraid\]](#), page 2217; [Section D.14.1.28 \[arrEdelmanReiner\]](#), page 2220; [Section D.14.1.26 \[arrRandom\]](#), page 2219; [Section D.14.1.24 \[arrTypeB\]](#), page 2218; [Section D.14.1.25 \[arrTypeD\]](#), page 2219.

### D.14.1.27 arrRandomCentral

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:** `arrRandomCentral(d,m,n); int d,m,n`

**Return:** Random central arrangement, where m is the number of hyperplanes, n the dimension, d the upper bound for absolute value of coefficients.

**Example:**

```

LIB "arr.lib";
ring R = 0,x(1..20),dp;
arrRandomCentral(7,3,15);
↳ _[1]=2*x(1)-6*x(2)-3*x(3)+4*x(5)+x(6)-4*x(7)-3*x(8)-x(9)+x(10)-3*x(11)-6*\
x(12)-x(13)+2*x(14)-6*x(15)
↳ _[2]=-4*x(1)+x(2)-3*x(3)-4*x(4)-7*x(6)-3*x(7)+x(9)-3*x(10)-4*x(11)-x(12)-\
7*x(13)-2*x(14)-3*x(15)
↳ _[3]=-x(2)+4*x(3)+2*x(4)+5*x(5)-x(6)+6*x(7)+5*x(8)-6*x(9)+5*x(10)+x(12)+3*\
*x(13)-5*x(14)+2*x(15)
↳

```

See also: [Section D.14.1.22 \[arrBoolean\]](#), page 2217; [Section D.14.1.23 \[arrBraid\]](#), page 2217; [Section D.14.1.28 \[arrEdelmanReiner\]](#), page 2220; [Section D.14.1.26 \[arrRandom\]](#), page 2219; [Section D.14.1.24 \[arrTypeB\]](#), page 2218; [Section D.14.1.25 \[arrTypeD\]](#), page 2219.

### D.14.1.28 arrEdelmanReiner

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:** `arrEdelmanReiner();`

**Return:** the Edelman-Reiner arrangement, which is a free arrangement but the restriction to the 6-th hyperplane is nonfree.  
(i.e. counterexample for Orlik-Conjecture)

**Note:** the active ring must have at least five variables

**Example:**

```

LIB "arr.lib";
ring r=0,x(1..5),dp;
arrEdelmanReiner();
↳ _[1]=x(1)
↳ _[2]=x(2)
↳ _[3]=x(3)
↳ _[4]=x(4)

```

```

⇒ _[5]=x(5)
⇒ _[6]=x(1)-x(2)-x(3)-x(4)-x(5)
⇒ _[7]=x(1)-x(2)-x(3)-x(4)+x(5)
⇒ _[8]=x(1)-x(2)-x(3)+x(4)-x(5)
⇒ _[9]=x(1)-x(2)-x(3)+x(4)+x(5)
⇒ _[10]=x(1)-x(2)+x(3)-x(4)-x(5)
⇒ _[11]=x(1)-x(2)+x(3)-x(4)+x(5)
⇒ _[12]=x(1)-x(2)+x(3)+x(4)-x(5)
⇒ _[13]=x(1)-x(2)+x(3)+x(4)+x(5)
⇒ _[14]=x(1)+x(2)-x(3)-x(4)-x(5)
⇒ _[15]=x(1)+x(2)-x(3)-x(4)+x(5)
⇒ _[16]=x(1)+x(2)-x(3)+x(4)-x(5)
⇒ _[17]=x(1)+x(2)-x(3)+x(4)+x(5)
⇒ _[18]=x(1)+x(2)+x(3)-x(4)-x(5)
⇒ _[19]=x(1)+x(2)+x(3)-x(4)+x(5)
⇒ _[20]=x(1)+x(2)+x(3)+x(4)-x(5)
⇒ _[21]=x(1)+x(2)+x(3)+x(4)+x(5)
⇒

```

See also: [Section D.14.1.22 \[arrBoolean\]](#), page 2217; [Section D.14.1.23 \[arrBraid\]](#), page 2217; [Section D.14.1.28 \[arrEdelmanReiner\]](#), page 2220; [Section D.14.1.26 \[arrRandom\]](#), page 2219; [Section D.14.1.24 \[arrTypeB\]](#), page 2218; [Section D.14.1.25 \[arrTypeD\]](#), page 2219.

### D.14.1.29 arrOrlikSolomon

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:**      `arrOrlikSolomon(A); arr A`

**Return:**    `[ring]` exterior Algebra E as ring with Orlik-Solomon ideal as attribute I. The Orlik-Solomon ideal is generated by the differentials of dependent tuples of hyperplanes. For a complex arrangement the quotient  $E/I$  is isomorphic to the cohomology ring of the complement of the arrangement.

**Note:**      In order to access this ideal I activate this exterior algebra with `setring`.

**Example:**

```

LIB "arr.lib";
ring R = 0,(x,y,z),dp;
arr A = arrTypeB(3);
def E = arrOrlikSolomon(A);
setring E;
//The generators of the Orlik-Solomon-Ideal are:
I;
⇒ I[1]=e(7)*e(8)-e(7)*e(9)+e(8)*e(9)
⇒ I[2]=e(6)*e(8)-e(6)*e(9)+e(8)*e(9)
⇒ I[3]=e(6)*e(7)-e(6)*e(9)+e(7)*e(9)
⇒ I[4]=e(4)*e(5)-e(4)*e(9)+e(5)*e(9)
⇒ I[5]=e(3)*e(5)-e(3)*e(9)+e(5)*e(9)
⇒ I[6]=e(3)*e(4)-e(3)*e(9)+e(4)*e(9)
⇒ I[7]=e(6)*e(7)-e(6)*e(8)+e(7)*e(8)
⇒ I[8]=e(2)*e(5)-e(2)*e(8)+e(5)*e(8)
⇒ I[9]=e(1)*e(5)-e(1)*e(8)+e(5)*e(8)
⇒ I[10]=e(1)*e(2)-e(1)*e(8)+e(2)*e(8)
⇒ I[11]=e(1)*e(4)-e(1)*e(7)+e(4)*e(7)

```

```

⇒ I[12]=e(2)*e(3)-e(2)*e(7)+e(3)*e(7)
⇒ I[13]=e(2)*e(4)-e(2)*e(6)+e(4)*e(6)
⇒ I[14]=e(1)*e(3)-e(1)*e(6)+e(3)*e(6)
⇒ I[15]=e(3)*e(4)-e(3)*e(5)+e(4)*e(5)
⇒ I[16]=e(1)*e(2)-e(1)*e(5)+e(2)*e(5)
⇒ I[17]=-e(2)*e(4)*e(8)+e(2)*e(4)*e(9)-e(2)*e(8)*e(9)+e(4)*e(8)*e(9)
⇒ I[18]=-e(1)*e(4)*e(8)+e(1)*e(4)*e(9)-e(1)*e(8)*e(9)+e(4)*e(8)*e(9)
⇒ I[19]=-e(2)*e(3)*e(8)+e(2)*e(3)*e(9)-e(2)*e(8)*e(9)+e(3)*e(8)*e(9)
⇒ I[20]=-e(1)*e(3)*e(8)+e(1)*e(3)*e(9)-e(1)*e(8)*e(9)+e(3)*e(8)*e(9)
⇒ I[21]=-e(2)*e(5)*e(7)+e(2)*e(5)*e(9)-e(2)*e(7)*e(9)+e(5)*e(7)*e(9)
⇒ I[22]=-e(1)*e(5)*e(7)+e(1)*e(5)*e(9)-e(1)*e(7)*e(9)+e(5)*e(7)*e(9)
⇒ I[23]=-e(2)*e(4)*e(7)+e(2)*e(4)*e(9)-e(2)*e(7)*e(9)+e(4)*e(7)*e(9)
⇒ I[24]=-e(1)*e(3)*e(7)+e(1)*e(3)*e(9)-e(1)*e(7)*e(9)+e(3)*e(7)*e(9)
⇒ I[25]=-e(1)*e(2)*e(7)+e(1)*e(2)*e(9)-e(1)*e(7)*e(9)+e(2)*e(7)*e(9)
⇒ I[26]=-e(2)*e(5)*e(6)+e(2)*e(5)*e(9)-e(2)*e(6)*e(9)+e(5)*e(6)*e(9)
⇒ I[27]=-e(1)*e(5)*e(6)+e(1)*e(5)*e(9)-e(1)*e(6)*e(9)+e(5)*e(6)*e(9)
⇒ I[28]=-e(1)*e(4)*e(6)+e(1)*e(4)*e(9)-e(1)*e(6)*e(9)+e(4)*e(6)*e(9)
⇒ I[29]=-e(2)*e(3)*e(6)+e(2)*e(3)*e(9)-e(2)*e(6)*e(9)+e(3)*e(6)*e(9)
⇒ I[30]=-e(1)*e(2)*e(6)+e(1)*e(2)*e(9)-e(1)*e(6)*e(9)+e(2)*e(6)*e(9)
⇒ I[31]=-e(1)*e(2)*e(4)+e(1)*e(2)*e(9)-e(1)*e(4)*e(9)+e(2)*e(4)*e(9)
⇒ I[32]=-e(1)*e(2)*e(3)+e(1)*e(2)*e(9)-e(1)*e(3)*e(9)+e(2)*e(3)*e(9)
⇒ I[33]=-e(4)*e(5)*e(7)+e(4)*e(5)*e(8)-e(4)*e(7)*e(8)+e(5)*e(7)*e(8)
⇒ I[34]=-e(3)*e(5)*e(7)+e(3)*e(5)*e(8)-e(3)*e(7)*e(8)+e(5)*e(7)*e(8)
⇒ I[35]=-e(3)*e(4)*e(7)+e(3)*e(4)*e(8)-e(3)*e(7)*e(8)+e(4)*e(7)*e(8)
⇒ I[36]=-e(2)*e(4)*e(7)+e(2)*e(4)*e(8)-e(2)*e(7)*e(8)+e(4)*e(7)*e(8)
⇒ I[37]=-e(1)*e(3)*e(7)+e(1)*e(3)*e(8)-e(1)*e(7)*e(8)+e(3)*e(7)*e(8)
⇒ I[38]=-e(4)*e(5)*e(6)+e(4)*e(5)*e(8)-e(4)*e(6)*e(8)+e(5)*e(6)*e(8)
⇒ I[39]=-e(3)*e(5)*e(6)+e(3)*e(5)*e(8)-e(3)*e(6)*e(8)+e(5)*e(6)*e(8)
⇒ I[40]=-e(3)*e(4)*e(6)+e(3)*e(4)*e(8)-e(3)*e(6)*e(8)+e(4)*e(6)*e(8)
⇒ I[41]=-e(1)*e(4)*e(6)+e(1)*e(4)*e(8)-e(1)*e(6)*e(8)+e(4)*e(6)*e(8)
⇒ I[42]=-e(2)*e(3)*e(6)+e(2)*e(3)*e(8)-e(2)*e(6)*e(8)+e(3)*e(6)*e(8)
⇒ I[43]=-e(2)*e(3)*e(4)+e(2)*e(3)*e(8)-e(2)*e(4)*e(8)+e(3)*e(4)*e(8)
⇒ I[44]=-e(1)*e(3)*e(4)+e(1)*e(3)*e(8)-e(1)*e(4)*e(8)+e(3)*e(4)*e(8)
⇒ I[45]=-e(4)*e(5)*e(6)+e(4)*e(5)*e(7)-e(4)*e(6)*e(7)+e(5)*e(6)*e(7)
⇒ I[46]=-e(3)*e(5)*e(6)+e(3)*e(5)*e(7)-e(3)*e(6)*e(7)+e(5)*e(6)*e(7)
⇒ I[47]=-e(2)*e(5)*e(6)+e(2)*e(5)*e(7)-e(2)*e(6)*e(7)+e(5)*e(6)*e(7)
⇒ I[48]=-e(1)*e(5)*e(6)+e(1)*e(5)*e(7)-e(1)*e(6)*e(7)+e(5)*e(6)*e(7)
⇒ I[49]=-e(3)*e(4)*e(6)+e(3)*e(4)*e(7)-e(3)*e(6)*e(7)+e(4)*e(6)*e(7)
⇒ I[50]=-e(1)*e(2)*e(6)+e(1)*e(2)*e(7)-e(1)*e(6)*e(7)+e(2)*e(6)*e(7)
⇒ I[51]=-e(2)*e(4)*e(5)+e(2)*e(4)*e(7)-e(2)*e(5)*e(7)+e(4)*e(5)*e(7)
⇒ I[52]=-e(1)*e(3)*e(5)+e(1)*e(3)*e(7)-e(1)*e(5)*e(7)+e(3)*e(5)*e(7)
⇒ I[53]=-e(1)*e(4)*e(5)+e(1)*e(4)*e(6)-e(1)*e(5)*e(6)+e(4)*e(5)*e(6)
⇒ I[54]=-e(2)*e(3)*e(5)+e(2)*e(3)*e(6)-e(2)*e(5)*e(6)+e(3)*e(5)*e(6)
⇒ I[55]=-e(1)*e(2)*e(3)+e(1)*e(2)*e(4)-e(1)*e(3)*e(4)+e(2)*e(3)*e(4)

```

See also: [Section D.14.1.29 \[arrOrlikSolomon\]](#), page 2221.

### D.14.1.30 arrDer

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:** `arrDer(A); arr A , multarr A`

**Return:** [module] The module `Der(A)` of derivations of the (multi-)arrangement `A`, i.e. the derivations tangent to each hyperplane of `A` (resp. with multiplicities)

**Note:** This is only defined for central (multi-)arrangements

**Example:**

```
LIB "arr.lib";
ring R = 0,(x,y,z),dp;
arr A3 = arrBoolean(3);
arr B3 = arrTypeB(3);
arr G = ideal(x,y,z,x+y+z);
//The derivation module of the Boolean 3-arrangement:
arrDer(A3);
⇒ _[1]=z*gen(3)
⇒ _[2]=y*gen(2)
⇒ _[3]=x*gen(1)
//The derivation module of the Braid 3-arrangement:
arrDer(B3);
⇒ _[1]=x*gen(1)+y*gen(2)+z*gen(3)
⇒ _[2]=x2y*gen(2)-y3*gen(2)+x2z*gen(3)-z3*gen(3)
⇒ _[3]=x2y2z*gen(3)-x2z3*gen(3)-y2z3*gen(3)+z5*gen(3)
//The derivation module of the generic arrangement:
arrDer(G);
⇒ _[1]=x*gen(1)+y*gen(2)+z*gen(3)
⇒ _[2]=yz*gen(3)-yz*gen(2)
⇒ _[3]=xz*gen(3)+yz*gen(2)+z2*gen(3)
⇒ _[4]=xy*gen(2)+y2*gen(2)+yz*gen(2)
```

See also: [Section D.14.1.30 \[arrDer\]](#), page 2222; [Section D.14.1.32 \[arrExponents\]](#), page 2224; [Section D.14.1.31 \[arrIsFree\]](#), page 2223.

### D.14.1.31 arrIsFree

Procedure from library `arr.lib` (see [Section D.14.1 \[arr\\_lib\]](#), page 2205).

**Usage:** `arrIsFree(A)`; `arr A`, `multarr A`

**Return:** `[0,1]` 1 if the (multi-)arrangement is free, i.e. `Der(A)` is a free module

**Note:** only defined for central arrangements

**Example:**

```
LIB "arr.lib";
ring R = 0,(x,y,z),dp;
arr A3 = arrBoolean(3);
arr B3 = arrTypeB(3);
arr G = ideal(x,y,z,x+y+z);
arrIsFree(A3);
⇒ 1
arrIsFree(B3);
⇒ 1
arrIsFree(G);
⇒ 0
```

See also: [Section D.14.1.30 \[arrDer\]](#), page 2222; [Section D.14.1.32 \[arrExponents\]](#), page 2224; [Section D.14.1.31 \[arrIsFree\]](#), page 2223.

### D.14.1.32 arrExponents

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:** `arrExponents(A);` arr A, multarr A

**Return:** [intvec] The exponents of a free (multi-) arrangement, i.e. the degrees of a basis of  $D(A)$  the derivation module.

**Note:** only defined for central arrangements

**Example:**

```
LIB "arr.lib";
ring R = 0,(x,y,z),dp;
arr A3 = arrBoolean(3);
arr B3 = arrTypeB(3);
arr G = ideal(x,y,z,x+y+z);
arrExponents(A3);
  ↪ 1,1,1
arrExponents(B3);
  ↪ 1,3,5
```

See also: [Section D.14.1.30 \[arrDer\]](#), page 2222; [Section D.14.1.32 \[arrExponents\]](#), page 2224; [Section D.14.1.31 \[arrIsFree\]](#), page 2223.

### D.14.1.33 arr2multarr

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:** `multArrFromIntvec(arr A, intvec v);`

**Return:** [multarr] multiarrangement MA, which is the arrangement A with multiplicities v

**Note:** the size of v must match the number of hyperplanes of the arrangement A

**Example:**

```
LIB "arr.lib";
ring R = 0,(x,y,z),dp;
arr A = arrTypeB(3);           A;
  ↪ _[1]=x-y
  ↪ _[2]=x+y
  ↪ _[3]=x-z
  ↪ _[4]=x+z
  ↪ _[5]=x
  ↪ _[6]=y-z
  ↪ _[7]=y+z
  ↪ _[8]=y
  ↪ _[9]=z
  ↪
intvec v=2:9;                  v;
  ↪ 2,2,2,2,2,2,2,2,2
multarr MA=arr2multarr(A,v);
MA;
  ↪ _[1]=(x-y)^2
  ↪ _[2]=(x+y)^2
  ↪ _[3]=(x-z)^2
  ↪ _[4]=(x+z)^2
```

```

↳ _[5]=(x)^2
↳ _[6]=(y-z)^2
↳ _[7]=(y+z)^2
↳ _[8]=(y)^2
↳ _[9]=(z)^2
↳

```

See also: [Section D.14.1.33 \[arr2multarr\]](#), page 2224; [Section D.14.1.34 \[multarr2arr\]](#), page 2225.

### D.14.1.34 multarr2arr

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:** `multarr2arr(multarr A, intvec v);`

**Return:** `[arr]` arrangement A, with all multiplicities removed

**Example:**

```

LIB "arr.lib";
ring r = 0,(x,y,z),dp;
multarr A=x2y3z5; A;
↳ _[1]=(x)^2
↳ _[2]=(y)^3
↳ _[3]=(z)^5
↳
arr AS = multarr2arr(A); AS;
↳ _[1]=x
↳ _[2]=y
↳ _[3]=z
↳

```

See also: [Section D.14.1.33 \[arr2multarr\]](#), page 2224; [Section D.14.1.34 \[multarr2arr\]](#), page 2225.

### D.14.1.35 multarrRestrict

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Return:** `[multarr]` the restricted hyperplane Multi-Arrangement ( $A^X$ ) with multiplicities i.e. counting how often one element of the restricted arrangement occurs as intersection of hyperplane of the first arrangement. This definition is due to Guenter M. Ziegler.

**Note:** A has to be non-empty.

**Remarks:** We restrict A to the flat X, defined by the equations in  $A[v]$ . The restriction will only be performed, if the ideal defining the flat X is monomial (i.e. X is an intersection of coordinate planes). If the optional argument CC is given, the arrangement is transformed in such a way that X has the above form.

**Example:**

```

LIB "arr.lib";
ring R = 0,x(1..5),dp;
arr A = arrEdelmanReiner(); A;
↳ _[1]=x(1)
↳ _[2]=x(2)
↳ _[3]=x(3)
↳ _[4]=x(4)
↳ _[5]=x(5)

```

```

↳ _[6]=x(1)-x(2)-x(3)-x(4)-x(5)
↳ _[7]=x(1)-x(2)-x(3)-x(4)+x(5)
↳ _[8]=x(1)-x(2)-x(3)+x(4)-x(5)
↳ _[9]=x(1)-x(2)-x(3)+x(4)+x(5)
↳ _[10]=x(1)-x(2)+x(3)-x(4)-x(5)
↳ _[11]=x(1)-x(2)+x(3)-x(4)+x(5)
↳ _[12]=x(1)-x(2)+x(3)+x(4)-x(5)
↳ _[13]=x(1)-x(2)+x(3)+x(4)+x(5)
↳ _[14]=x(1)+x(2)-x(3)-x(4)-x(5)
↳ _[15]=x(1)+x(2)-x(3)-x(4)+x(5)
↳ _[16]=x(1)+x(2)-x(3)+x(4)-x(5)
↳ _[17]=x(1)+x(2)-x(3)+x(4)+x(5)
↳ _[18]=x(1)+x(2)+x(3)-x(4)-x(5)
↳ _[19]=x(1)+x(2)+x(3)-x(4)+x(5)
↳ _[20]=x(1)+x(2)+x(3)+x(4)-x(5)
↳ _[21]=x(1)+x(2)+x(3)+x(4)+x(5)
↳
multarr AR = multarrRestrict(A,6,"CC");  AR;
↳ _[1]=(x(2)+1/4*x(3)+1/4*x(4)+1/4*x(5))^2
↳ _[2]=(x(2)+4*x(3)-x(4)-x(5))^2
↳ _[3]=(x(2)-x(3)+4*x(4)-x(5))^2
↳ _[4]=(x(2)-x(3)-x(4)+4*x(5))^2
↳ _[5]=(x(2)-x(3)-x(4)-x(5))^2
↳ _[6]=(x(2)-x(3)-x(4)+3/2*x(5))^1
↳ _[7]=(x(2)-x(3)+3/2*x(4)-x(5))^1
↳ _[8]=(x(2)-x(3)+3/2*x(4)+3/2*x(5))^1
↳ _[9]=(x(2)-x(3)+2/3*x(4)+2/3*x(5))^1
↳ _[10]=(x(2)+3/2*x(3)-x(4)-x(5))^1
↳ _[11]=(x(2)+3/2*x(3)-x(4)+3/2*x(5))^1
↳ _[12]=(x(2)+2/3*x(3)-x(4)+2/3*x(5))^1
↳ _[13]=(x(2)+3/2*x(3)+3/2*x(4)-x(5))^1
↳ _[14]=(x(2)+2/3*x(3)+2/3*x(4)-x(5))^1
↳ _[15]=(x(2)+2/3*x(3)+2/3*x(4)+2/3*x(5))^1
↳

```

See also: [Section D.14.1.19 \[arrRestrict\]](#), page 2215; [Section D.14.1.36 \[multarrMultRestrict\]](#), page 2226; [Section D.14.1.35 \[multarrRestrict\]](#), page 2225.

### D.14.1.36 multarrMultRestrict

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:** `multarrMultRestrict(A, k);` multiarrangement A, integer k;

**Return:** `[multarr]` the restricted hyperplane Multi-Arrangement  $(A^H_k)$  with multiplicities, i.e. counting with multiplicities how often one element of the restricted arrangement occurs as intersection of hyperplane of the first multiarrangement. This definition is due to Guenter M. Ziegler.

**Note:** A has to be non-empty.

**Remarks:** The restriction will only be performed, if  $H_k = \ker(x_i)$  for some i. One can also restrict an arrangement with respect to any hyperplane k, but then a coordinate change is necessary first to make  $H_k = \ker(x_k)$ . Since such a coordinate change is not unique, please use `arrCoordchange` to do so.



**Example:**

```

LIB "arr.lib";
ring R = 0,(x,y,z),dp;
multarr A =ideal(x2,y2,z2,(x-y)^3,(x-z)^2,(y-z));    A;
↳ _[1]=(x)^2
↳ _[2]=(y)^2
↳ _[3]=(z)^2
↳ _[4]=(x-y)^3
↳ _[5]=(x-z)^2
↳ _[6]=(y-z)^1
↳
//The restriction of the multiarrangement is:
multarr AR = multarrMultRestrict(A,1);  AR;
↳ _[1]=(y)^5
↳ _[2]=(z)^4
↳ _[3]=(y-z)^1
↳

```

See also: [Section D.14.1.19 \[arrRestrict\]](#), page 2215; [Section D.14.1.36 \[multarrMultRestrict\]](#), page 2226; [Section D.14.1.35 \[multarrRestrict\]](#), page 2225.

**D.14.1.37 arrFlats**

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:**      `size(A); A arr`

**Return:**    `[arrposet]` Intersection lattice

**Example:**

```

LIB "arr.lib";
ring R = 0,(x(1..5)),dp;
arrFlats(arrBraid(5));
↳
↳
↳ === Computing poset ===
↳
↳
↳ rank: 2, expected OPS: 45, executed OPS: 60
↳ Cleaned up: 35 hyperplanes
↳
↳
↳ rank: 3, expected OPS: 300, executed OPS: 75
↳ Cleaned up: 60 hyperplanes
↳
↳
↳ rank: 4, expected OPS: 105, executed OPS: 15
↳ Cleaned up: 14 hyperplanes
↳
↳
↳
↳ Matrix tests: 1484
↳ Given Arrangement:
↳ _[1]=x(1)-x(2)

```

```

⇒ _[2]=x(1)-x(3)
⇒ _[3]=x(1)-x(4)
⇒ _[4]=x(1)-x(5)
⇒ _[5]=x(2)-x(3)
⇒ _[6]=x(2)-x(4)
⇒ _[7]=x(2)-x(5)
⇒ _[8]=x(3)-x(4)
⇒ _[9]=x(3)-x(5)
⇒ _[10]=x(4)-x(5)
⇒
⇒ Corresponding poset:
⇒ ===== rank 1: 10 flats =====
⇒ (1), (2), (3), (4), (5), (6), (7), (8), (9), (10),
⇒ ===== rank 2: 25 flats =====
⇒ (1,2,5), (1,3,6), (1,4,7), (1,8), (1,9), (1,10), (2,3,8), (2,4,9)\
, (2,6), (2,7), (2,10), (3,4,10), (3,5), (3,7), (3,9), (4,5), (4\
,6), (4,8), (5,6,8), (5,7,9), (5,10), (6,7,10), (6,9), (7,8), (8,\
9,10),
⇒ ===== rank 3: 15 flats =====
⇒ (1,2,3,5,6,8), (1,2,4,5,7,9), (1,2,5,10), (1,3,4,6,7,10), (1,3,6,9)\
(1,4,7,8), (1,8,9,10), (2,3,4,8,9,10), (2,3,7,8), (2,4,6,9), (2,6,\
7,10), (3,4,5,10), (3,5,7,9), (4,5,6,8), (5,6,7,8,9,10),
⇒ ===== rank 4: 1 flats =====
⇒ (1,2,3,4,5,6,7,8,9,10),
⇒ ===== rank 5: 0 flats =====
⇒
⇒

```

See also: [Section D.14.1.37 \[arrFlats\]](#), page 2227.

### D.14.1.38 arrLattice

Procedure from library `arr.lib` (see [Section D.14.1 \[arr\\_lib\]](#), page 2205).

**Usage:** `arrLattice(arr ARR)`

**Return:** `[arrposet]` intersection poset of the arrangement

**Note:** The algorithm works by a bottom up approach, i.e. it calculates the

**Example:**

```

LIB "arr.lib";
ring r;
arrLattice(arrTypeB(3));
⇒
⇒
⇒ === Computing poset ===
⇒
⇒
⇒ rank 2: found 13 flats in 0s
⇒ rank 3: found 1 flats in 0s
⇒
⇒
⇒ Matrix tests: 86
⇒ Given Arrangement:
⇒ _[1]=x-y

```

```

↳ _[2]=x+y
↳ _[3]=x-z
↳ _[4]=x+z
↳ _[5]=x
↳ _[6]=y-z
↳ _[7]=y+z
↳ _[8]=y
↳ _[9]=z
↳
↳ Corresponding poset:
↳ ===== rank 1: 9 flats =====
↳ (1), (2), (3), (4), (5), (6), (7), (8), (9),
↳ ===== rank 2: 13 flats =====
↳ (1,2,5,8), (1,3,6), (1,4,7), (1,9), (2,3,7), (2,4,6), (2,9), (3,4\
,5,9), (3,8), (4,8), (5,6), (5,7), (6,7,8,9),
↳ ===== rank 3: 1 flats =====
↳ (1,2,3,4,5,6,7,8,9),
↳

```

See also: [Section D.14.1.37 \[arrFlats\], page 2227](#); [Section D.14.1.38 \[arrLattice\], page 2228](#).

### D.14.1.39 moebius

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\], page 2205](#)).

**Usage:**        `moebius(arrposet P)`

**Return:**      `[arrposet]` fills in the moebius values of the flats in the poset

**Example:**

```

LIB "arr.lib";
ring R = 0,(x,y,z,t),dp;
arr A = arrBraid(4);
arrposet P = arrLattice(A);
↳
↳
↳ === Computing poset ===
↳
↳
↳ rank 2: found 7 flats in 0s
↳ rank 3: found 1 flats in 0s
↳
↳
↳ Matrix tests: 38
P;
↳ Given Arrangement:
↳ _[1]=x-y
↳ _[2]=x-z
↳ _[3]=x-t
↳ _[4]=y-z
↳ _[5]=y-t
↳ _[6]=z-t
↳
↳ Corresponding poset:
↳ ===== rank 1: 6 flats =====

```

```

↳ (1), (2), (3), (4), (5), (6),
↳ ===== rank 2: 7 flats =====
↳ (1,2,4), (1,3,5), (1,6), (2,3,6), (2,5), (3,4), (4,5,6),
↳ ===== rank 3: 1 flats =====
↳ (1,2,3,4,5,6),
↳ ===== rank 4: 0 flats =====
↳
↳
//As you can see the values are not calculated yet:
printMoebius(P);
↳ Moebius values:
↳ ===== rank 1: 6 flats =====
↳ (-1), (-1), (-1), (-1), (-1), (-1),
↳ ===== rank 2: 7 flats =====
↳ (0), (0), (0), (0), (0), (0), (0),
↳ ===== rank 3: 1 flats =====
↳ (0),
↳ ===== rank 4: 0 flats =====
↳
P = moebius(P);
//Now all entries are initialized:
printMoebius(P);
↳ Moebius values:
↳ ===== rank 1: 6 flats =====
↳ (-1), (-1), (-1), (-1), (-1), (-1),
↳ ===== rank 2: 7 flats =====
↳ (2), (2), (1), (2), (1), (1), (2),
↳ ===== rank 3: 1 flats =====
↳ (-6),
↳ ===== rank 4: 0 flats =====
↳

```

See also: [Section D.14.1.39 \[moebius\]](#), page 2229.

#### D.14.1.40 arrCharPoly

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:** `arrCharPoly(arr A)`

**Return:** `[intvec]` coefficients of the characteristic polynomial of `A` in increasing order

**Remarks:** The algorithm only returns the coefficients of the characteristic polynomial since they are whole numbers but the basering could be something different.

**Example:**

```

LIB "arr.lib";
ring R = 0,(x,y,z,u,v),dp;
arr A = arrBraid(5);
intvec v = arrCharPoly(A);
x*(x-1)*(x-2)*(x-3)*(x-4);
↳ x5-10x4+35x3-50x2+24x
v;
↳ 24,-50,35,-10,1

```

See also: [Section D.14.1.40 \[arrCharPoly\]](#), page 2230; [Section D.14.1.41 \[arrPoincare\]](#), page 2231.

### D.14.1.41 arrPoincare

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:** `arrPoincare(A); arr A`

**Return:** `[intvec]` The Poincare polynomial as integer vector of the arrangement, which is equal to the second kind Poincare-Series of the Orlik-Solomon Algebra.

**Example:**

```
LIB "arr.lib";
ring R = 0,(x,y,z,u,v),dp;
arr A = arrBraid(5);
intvec v = arrPoincare(A);
(1+x)*(1+2x)*(1+3x)*(1+4x);
↦ 24x4+50x3+35x2+10x+1
v;
↦ 1,10,35,50,24
```

See also: [Section D.14.1.43 \[arrBoundedChambers\]](#), page 2231; [Section D.14.1.42 \[arrChambers\]](#), page 2231; [Section D.14.1.41 \[arrPoincare\]](#), page 2231.

### D.14.1.42 arrChambers

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:** `arrChambers(A); arr A`

**Return:** `[int]` The number of chambers of an arrangement, which is equal to the evaluation of the Poincare polynomial at 1.

**Example:**

```
LIB "arr.lib";
ring R = 0,(x,y),dp;
arr A = ideal(x,y,x+y-1);
arrChambers(A);
↦ 7
```

See also: [Section D.14.1.43 \[arrBoundedChambers\]](#), page 2231; [Section D.14.1.42 \[arrChambers\]](#), page 2231; [Section D.14.1.41 \[arrPoincare\]](#), page 2231.

### D.14.1.43 arrBoundedChambers

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\]](#), page 2205).

**Usage:** `arrBoundedChambers(A); arr A`

**Return:** `[int]` The number of bounded chambers of an arrangement, which is equal to the evaluation of the Poincare polynomial at -1.

**Example:**

```
LIB "arr.lib";
ring R = 0,(x,y),dp;
arr A = ideal(x,y,x+y-1);
arrBoundedChambers(A);
↦ 1
```

See also: [Section D.14.1.43 \[arrBoundedChambers\]](#), page 2231; [Section D.14.1.42 \[arrChambers\]](#), page 2231; [Section D.14.1.41 \[arrPoincare\]](#), page 2231.

### D.14.1.44 printMoebius

Procedure from library `arr.lib` (see [Section D.14.1 \[arr.lib\], page 2205](#)).

**Usage:**     `printMoebius(A); arr A`

**Return:**    `[]` displays the moebius values of all the flats in the poset

**Remarks:**  Mainly used for debugging.

**Example:**

```
LIB "arr.lib";
ring R = 0,(x,y,z,t),dp;
arr A = arrBraid(4);
arrposet P = arrLattice(A);
↳
↳
↳ === Computing poset ===
↳
↳
↳ rank 2: found 7 flats in 0s
↳ rank 3: found 1 flats in 0s
↳
↳
↳ Matrix tests: 38
P;
↳ Given Arrangement:
↳ _[1]=x-y
↳ _[2]=x-z
↳ _[3]=x-t
↳ _[4]=y-z
↳ _[5]=y-t
↳ _[6]=z-t
↳
↳ Corresponding poset:
↳ ===== rank 1: 6 flats =====
↳ (1), (2), (3), (4), (5), (6),
↳ ===== rank 2: 7 flats =====
↳ (1,2,4), (1,3,5), (1,6), (2,3,6), (2,5), (3,4), (4,5,6),
↳ ===== rank 3: 1 flats =====
↳ (1,2,3,4,5,6),
↳ ===== rank 4: 0 flats =====
↳
↳
//As you can see the values are not calculated yet:
printMoebius(P);
↳ Moebius values:
↳ ===== rank 1: 6 flats =====
↳ (-1), (-1), (-1), (-1), (-1), (-1),
↳ ===== rank 2: 7 flats =====
↳ (0), (0), (0), (0), (0), (0), (0),
↳ ===== rank 3: 1 flats =====
↳ (0),
↳ ===== rank 4: 0 flats =====
↳
```

```

P = moebius(P);
//Now all entries are initialized:
printMoebius(P);
↳ Moebius values:
↳ ===== rank 1: 6 flats =====
↳ (-1), (-1), (-1), (-1), (-1), (-1),
↳ ===== rank 2: 7 flats =====
↳ (2), (2), (1), (2), (1), (1), (2),
↳ ===== rank 3: 1 flats =====
↳ (-6),
↳ ===== rank 4: 0 flats =====
↳

```

## D.14.2 combinat\_lib

**Library:** combinat.lib

**Purpose:** Some useful functions

**Authors:** J. Boehm, boehm @ mathematik.uni-kl.de

**Overview:** Some useful basic functions from combinatorics.

**Procedures:**

### D.14.2.1 intersectLists

Procedure from library `combinat.lib` (see [Section D.14.2 \[combinat\\_lib\]](#), page 2233).

**Usage:** `intersectLists(L,M)`; L list, M list

**Return:** list, containing all elements of L which are elements of M

**Theory:** we require the elements of the lists to have a compare operator `==`.

**Example:**

```

LIB "combinat.lib";
intersectLists(list(1,1,2,2,3),list(2,3,3,4));
↳ [1]:
↳ 2
↳ [2]:
↳ 3
intersectLists(list(1,1,2,2,3),list(1,1,2,2,3));
↳ [1]:
↳ 1
↳ [2]:
↳ 2
↳ [3]:
↳ 3

```

### D.14.2.2 sublists

Procedure from library `combinat.lib` (see [Section D.14.2 \[combinat\\_lib\]](#), page 2233).

**Usage:** `sublists(L)`; L list

**Assume:** L is a list

**Return:** list of all sublists of L.

**Example:**

```
LIB "combinat.lib";
list L = 1,2,3,4,5;
sublists(L);
↳ [1]:
↳   empty list
↳ [2]:
↳   [1]:
↳     1
↳ [3]:
↳   [1]:
↳     2
↳ [4]:
↳   [1]:
↳     1
↳   [2]:
↳     2
↳ [5]:
↳   [1]:
↳     3
↳ [6]:
↳   [1]:
↳     1
↳   [2]:
↳     3
↳ [7]:
↳   [1]:
↳     2
↳   [2]:
↳     3
↳ [8]:
↳   [1]:
↳     1
↳   [2]:
↳     2
↳   [3]:
↳     3
↳ [9]:
↳   [1]:
↳     4
↳ [10]:
↳   [1]:
↳     1
↳   [2]:
↳     4
↳ [11]:
↳   [1]:
↳     2
↳   [2]:
```



```

↳      4
↳ [12]:
↳   [1]:
↳      1
↳   [2]:
↳      2
↳   [3]:
↳      4
↳ [13]:
↳   [1]:
↳      3
↳   [2]:
↳      4
↳ [14]:
↳   [1]:
↳      1
↳   [2]:
↳      3
↳   [3]:
↳      4
↳ [15]:
↳   [1]:
↳      2
↳   [2]:
↳      3
↳   [3]:
↳      4
↳ [16]:
↳   [1]:
↳      1
↳   [2]:
↳      2
↳   [3]:
↳      3
↳   [4]:
↳      4
↳ [17]:
↳   [1]:
↳      5
↳ [18]:
↳   [1]:
↳      1
↳   [2]:
↳      5
↳ [19]:
↳   [1]:
↳      2
↳   [2]:
↳      5
↳ [20]:
↳   [1]:
↳      1
↳   [2]:

```

```

↳      2
↳    [3]:
↳      5
↳  [21]:
↳    [1]:
↳      3
↳    [2]:
↳      5
↳  [22]:
↳    [1]:
↳      1
↳    [2]:
↳      3
↳    [3]:
↳      5
↳  [23]:
↳    [1]:
↳      2
↳    [2]:
↳      3
↳    [3]:
↳      5
↳  [24]:
↳    [1]:
↳      1
↳    [2]:
↳      2
↳    [3]:
↳      3
↳    [4]:
↳      5
↳  [25]:
↳    [1]:
↳      4
↳    [2]:
↳      5
↳  [26]:
↳    [1]:
↳      1
↳    [2]:
↳      4
↳    [3]:
↳      5
↳  [27]:
↳    [1]:
↳      2
↳    [2]:
↳      4
↳    [3]:
↳      5
↳  [28]:
↳    [1]:
↳      1

```

```

⇒      [2]:
⇒      2
⇒      [3]:
⇒      4
⇒      [4]:
⇒      5
⇒ [29]:
⇒      [1]:
⇒      3
⇒      [2]:
⇒      4
⇒      [3]:
⇒      5
⇒ [30]:
⇒      [1]:
⇒      1
⇒      [2]:
⇒      3
⇒      [3]:
⇒      4
⇒      [4]:
⇒      5
⇒ [31]:
⇒      [1]:
⇒      2
⇒      [2]:
⇒      3
⇒      [3]:
⇒      4
⇒      [4]:
⇒      5
⇒ [32]:
⇒      [1]:
⇒      1
⇒      [2]:
⇒      2
⇒      [3]:
⇒      3
⇒      [4]:
⇒      4
⇒      [5]:
⇒      5

```

### D.14.2.3 member

Procedure from library `combinat.lib` (see [Section D.14.2 \[combinat.lib\]](#), page 2233).

**Usage:**      `member(e,L)`; e def, L list

**Return:**     1 if e is an element of L, 0 otherwise

**Theory:**     we require the elements involved to have a compare operator `==`.

**Example:**

```
LIB "combinat.lib";
member(1,list(1,4,5));
⇨ 1
```

### D.14.3 customstd.lib

**Library:** customstd.lib

**Purpose:** Load customstd.so

**Authors:** Hans Schoenemann, hannes at mathematik.uni-kl.de  
Yue Ren, ren at mathematik.uni-kl.de

**Overview:** This library offers customly modified standard bases algorithms in order to increase the performance of other algorithms. If you require a customly modified standard bases algorithm, please contact the authors.

**Procedures:**

#### D.14.3.1 monomialabortstd

Procedure from library `customstd.lib` (see [Section D.14.3 \[customstd.lib\]](#), page 2238).

**Usage:** monomialabortstd(I); I ideal

**Purpose:** computes a standard basis and aborts if a monomial generator is found. Returns all standard basis elements that have been computed.

**Note:** Due to sequencing of the standard basis computation:  
- If aborted, there is no guarantee that the monomial is the final standard bases element  
- If ideal has a monomial generator, there is no guarantee that the computation aborts because of it

**Example:**

```
LIB "customstd.lib";
LIB "polylib.lib";
ring r = 0,(x,y,z,u,v),dp;
// yields normal standard basis since no monomial is found
ideal I = homog(cyclic(4),v);
monomialabortstd(I);
⇨ _[1]=x+y+z+u
⇨ _[2]=y2+2yu+u2
⇨ _[3]=yz2+z2u-yu2-u3
⇨ _[4]=yzu2+z2u2-yu3+zu3-u4-v4
⇨ _[5]=yu4+u5-yv4-uv4
⇨ _[6]=z3u2+z2u3-zv4-uv4
⇨ _[7]=z2u4+yzv4-yuv4+zuv4-2u2v4
// will not start standard basis computation in the first place,
// as one of the generators is a monomial
I = v,homog(cyclic(4),v);
monomialabortstd(I);
⇨ _[1]=v
// aborts standard basis computation when it encounters monomial u
// note that u is not the final element in the standard basis!!!
I = x+y+z,homog(cyclic(4),v);
monomialabortstd(I);
```

```

↳ _[1]=u
↳ _[2]=x+y+z+u
// aborts standard basis computation when it encounters monomial z2
// note that it is not the monomial generator x which lead to the abortion!!!
I = x,homog(cyclic(4),v);
monomialabortstd(I);
↳ _[1]=y+z+u
↳ _[2]=x+y+z+u
↳ _[3]=z2

```

### D.14.3.2 satstd

Procedure from library `customstd.lib` (see [Section D.14.3 \[customstd.lib\]](#), page 2238).

**Usage:** `satstd(I[,J]);` I ideal, J optional ideal

**Assume:** J generated by variables

**Purpose:** computes a standard basis of I and, if possible, divides each polynomial during the computation by the variables in J. By default, J is assumed to be the ideal generated by all variables.

**Note:** Even if I contains a monomial generated by the variables in J, there is no guarantee that it is found during the computation. If it is found, however, 1 is immediately returned.

**Note:** The result is a standard basis of a partially saturated ideal wrt. the the variables in J. If the I is homogeneous and the ordering dp, the result is completely saturated wrt. to the last variable (wrt. to the first for Dp).

**Example:**

```

LIB "customstd.lib";
ring r = 0,(x,y,z,u,v),dp;
ideal I = x2+x,y2+y,z2+z;
// returns normal standard basis, no changes during (trivial) computation
satstd(I,ideal(u,v));
↳ _[1]=z2+z
↳ _[2]=y2+y
↳ _[3]=x2+x
// returns x+1 instead of x2+x
satstd(I,ideal(x));
↳ _[1]=x+1
↳ _[2]=z2+z
↳ _[3]=y2+y
// returns standard basis with elements of degree up to 8 (instead of 16)
deg(satstd(I^8));
↳ 8

```

### D.14.4 methods\_lib

**Library:** `methods.lib`

**Purpose:** installing methods in Singular

**Authors:** J. Boehm, boehm @ mathematik.uni-kl.de

**Overview:** Methods select the function to execute by the types of the input tuple. The central function is `installMethod`, which takes a hashtable associating a tuple of input types to function names and creates a corresponding procedure.

HashTables are lists with arbitrary index sets. They can be created by the command `hashTable`. Their size can be determined by the command `size`. Values can be extracted by `selectKey` or the `*` operator. HashTables can also be added using `addHashTables` or the `+` operator.

Methods can be added with the `+` operator.

**Types:** Method the class of all methods  
 HashTable the class of all hash tables

**Procedures:**

### D.14.5 nets\_lib

**Library:** `net.lib`

**Purpose:** Net structures for pretty printing

**Authors:** J. Boehm, boehm@mathematik.uni-kl.de  
 M. Mueller, mkmuelle@mathematik.uni-kl.de  
 H. Rombach, rombach@mathematik.uni-kl.de  
 M. Stein, maxstein77@web.de

**Overview:** Nets are arrays of characters, which are printed in a matrix format. They can be concatenated horizontally and vertically. When concatenating horizontally, empty rows are filled with spaces. All Singular types can be converted to a Net by applying the command `net`.

**Types:** Net The class of all nets

**Procedures:**

#### D.14.5.1 catNets

Procedure from library `nets.lib` (see [Section D.14.5 \[nets\\_lib\], page 2240](#)).

**Usage:** `catNets(N,M);` N,M Nets

**Return:** horizontal concatenation of N and M

**Theory:** The nets are concated horizontally with alignment at the top row. Can also be called with `+`.

**Example:**

```
LIB "nets.lib";
Net A = net("aaa");
Net B = net("b");
catNets(A,B);
↪ aaab
↪
A+B;
↪ aaab
↪
```

### D.14.5.2 net

Procedure from library `nets.lib` (see [Section D.14.5 \[nets.lib\]](#), page 2240).

### D.14.5.3 netBigIntMat

Procedure from library `nets.lib` (see [Section D.14.5 \[nets.lib\]](#), page 2240).

**Usage:**      `netBigIntMat(M)`; M bigint Matrix

**Assume:**    M is a bigintmatrix

**Return:**     visual presentation of M

**Theory:**     A Singular object is converted into a character array (a Net) for on screen printing.

**Example:**

```
LIB "nets.lib";
ring r1=101,(x,y,z),lp;
int a=111111;
int b=22222;
int c=3333;
int d=444;
bigintmat M[2][2]=a,b,c,d;
netBigIntMat(M);
↪ | 111111 22222 |
↪ | 3333   444   |
↪
```

### D.14.5.4 netBigIntMatShort

Procedure from library `nets.lib` (see [Section D.14.5 \[nets.lib\]](#), page 2240).

**Usage:**      `netBigIntMatShort(M)`; M bigint matrix

**Assume:**    M is a bigintmatrix

**Return:**     visual presentation of M, only the first digits of each entry

**Theory:**     A Singular object is converted into a character array (a Net) for on screen printing.

**Example:**

```
LIB "nets.lib";
ring r1=101,(x,y,z),lp;
bigint a=1111111111;
bigint b=2222222222;
bigint c=3333333333;
bigint d=4444444444;
bigintmat M[2][2]=a,b,c,d;
netBigIntMatShort(M);
↪ | 1111111111 2222222222 |
↪ | 3333333333 4444444444 |
↪
```

### D.14.5.5 netCoefficientRing

Procedure from library `nets.lib` (see [Section D.14.5 \[nets.lib\], page 2240](#)).

**Usage:** `netCoefficientRing(R);` R ring

**Assume:** R is a ring

**Return:** visual presentation of the coefficient ring of R

**Theory:** A Singular object is converted into a character array (a Net) for on screen printing.

**Example:**

```
LIB "nets.lib";
ring r=0,x,lp;
netCoefficientRing(r);
↦ QQ
↦
```

### D.14.5.6 netIdeal

Procedure from library `nets.lib` (see [Section D.14.5 \[nets.lib\], page 2240](#)).

**Usage:** `netIdeal(P);` P ideal

**Assume:** P is a poly

**Return:** visual presentation of I

**Theory:** A Singular object is converted into a character array (a Net) for on screen printing.

**Example:**

```
LIB "nets.lib";
ring r;
ideal I=2x3y4,2x3y4z+x3y4z5,5x6y7z10-2x3y+4y5z8;
netIdeal(I);
↦ <2x3y4, x3y4z5+2x3y4z, 5x6y7z10+4y5z8-2x3y>
↦
```

### D.14.5.7 netInt

Procedure from library `nets.lib` (see [Section D.14.5 \[nets.lib\], page 2240](#)).

**Usage:** `netInt(M);` M integer

**Assume:** M is an integer

**Return:** visual presentation of M

**Theory:** A Singular object is converted into a character array (a Net) for on screen printing.

**Example:**

```
LIB "nets.lib";
ring r;
int M=5;
netInt(M);
↦ 5
↦
```



### D.14.5.8 netBigInt

Procedure from library `nets.lib` (see [Section D.14.5 \[nets.lib\], page 2240](#)).

**Usage:**     `netBigInt(M)`; M integer

**Assume:**   M is a bigint

**Return:**    visual presentation of M

**Theory:**    A Singular object is converted into a character array (a Net) for on screen printing.

**Example:**

```
LIB "nets.lib";
ring r;
bigint M=5;
netBigInt(M);
⇒ 5
⇒
```

### D.14.5.9 netIntMat

Procedure from library `nets.lib` (see [Section D.14.5 \[nets.lib\], page 2240](#)).

**Usage:**     `netIntMat(M)`; M int matrix

**Assume:**    M is a int matrix

**Return:**    visual presentation of M

**Theory:**    A Singular object is converted into a character array (a Net) for on screen printing.

**Example:**

```
LIB "nets.lib";
ring r1=101,(x,y,z),lp;
int a=111111;
int b=222222;
int c=333333;
int d=444444;
intmat M[2][2]=a,b,c,d;
netIntMat(M);
⇒ | 111111 222222 |
⇒ | 333333 444444 |
⇒
```

### D.14.5.10 netIntMatShort

Procedure from library `nets.lib` (see [Section D.14.5 \[nets.lib\], page 2240](#)).

**Usage:**     `netIntMatShort(M)`; M int matrix

**Assume:**    M is a int matrix

**Return:**    visual presentation of M, only the first digits of each entry

**Theory:**    A Singular object is converted into a character array (a Net) for on screen printing.

**Example:**

```

LIB "nets.lib";
ring r1=101,(x,y,z),lp;
int a=111111;
int b=222222;
int c=333333;
int d=444444;
intmat M[2][2]=a,b,c,d;
print(M);
↳ 111111 222222
↳ 333333 444444
netIntMatShort(M);
↳ | 111111 222222 |
↳ | 333333 444444 |
↳
//
print(M);
↳ 111111 222222
↳ 333333 444444
netIntMatShort(M,2);
↳ | 11... 22... |
↳ | 33... 44... |
↳
//

```

**D.14.5.11 netIntVector**

Procedure from library `nets.lib` (see [Section D.14.5 \[nets.lib\]](#), page 2240).

**Usage:**     `netIntVector(V)`; V int vector

**Assume:**   V is a int vector

**Return:**    visual presentation of V

**Theory:**    A Singular object is converted into a character array (a Net) for on screen printing.

**Example:**

```

LIB "nets.lib";
ring r1=101,(x,y,z),lp;
int a=111111;
int b=222222;
int c=333333;
int d=444444;
intvec V=a,b,c,d;
netIntVector(V);
↳ | 111111 |
↳ | 222222 |
↳ | 333333 |
↳ | 444444 |
↳

```

**D.14.5.12 netIntVectorShort**

Procedure from library `nets.lib` (see [Section D.14.5 \[nets.lib\]](#), page 2240).

**Usage:**      `netIntVectorShort(V); V int vector`

**Assume:**    `V` is a int vector

**Return:**     visual presentation of `V`, only the first digits of each entry

**Theory:**     A Singular object is converted into a character array (a Net) for on screen printing.

**Example:**

```
LIB "nets.lib";
ring r1=101,(x,y,z),lp;
int a=111111;
int b=222222;
int c=333333;
int d=444444;
intvec V=a,b,c,d;
netIntVectorShort(V,4);
↪ | 1111... |
↪ | 2222... |
↪ | 3333... |
↪ | 4444... |
↪
```

#### D.14.5.13 netNumber

Procedure from library `nets.lib` (see [Section D.14.5 \[nets.lib\], page 2240](#)).

**Usage:**      `netNumber(R); a number`

**Assume:**    `a` is a number

**Return:**     visual presentation of `a`

**Theory:**     A Singular object is converted into a character array (a Net) for on screen printing.

**Example:**

```
LIB "nets.lib";
ring r1=101,(x,y,z),lp;
number a = 5;
netNumber(a);
↪
↪ 5
↪
```

#### D.14.5.14 netList

Procedure from library `nets.lib` (see [Section D.14.5 \[nets.lib\], page 2240](#)).

**Usage:**      `netList(L); L list`

**Assume:**    `L` is a list of elements

**Return:**     visual presentation of `L`

**Theory:**     A Singular object is converted into a character array (a Net) for on screen printing.

**Example:**

```

LIB "nets.lib";
ring r1=101,(x,y,z),lp;
list L=111, x+y, x3y-z, y+3z4;
print("list L=111, x+y, x3y-z, y+3z4");
↪ list L=111, x+y, x3y-z, y+3z4
netList(L);
↪ [111, x+y, x3y-z, y+3z4]
↪

```

**D.14.5.15 netMap**

Procedure from library `nets.lib` (see [Section D.14.5 \[nets.lib\]](#), page 2240).

**Usage:**      `netMap(f); f map`

**Assume:**    `f` is a map from a ring to the basering

**Return:**     visual presentation of the map `f`

**Theory:**     A Singular object is converted into a character array (a Net) for on screen printing.

**Example:**

```

LIB "nets.lib";
ring r1=101,(x,y,z),lp;
ring r2=103,(a,b,c),lp;
map f=r1,ab,ac,bc;
netMap(f);
↪ Map: r1 --> r2 , x -> ab
↪           , y -> ac
↪           , z -> bc
↪

```

**D.14.5.16 netMap2**

Procedure from library `nets.lib` (see [Section D.14.5 \[nets.lib\]](#), page 2240).

**Usage:**      `netMap2(f); f map`

**Assume:**    `f` is a map from a ring to the basering

**Return:**     visual presentation of the map `f`, alternative version

**Theory:**     A Singular object is converted into a character array (a Net) for on screen printing.

**Example:**

```

LIB "nets.lib";
ring r1=101,(x,y,z),lp;
ring r2=103,(a,b,c),lp;
map f=r1,ab,ac,bc;
netMap2(f);
↪ f: r1 --> r2 , x -> ab , y -> ac , z -> bc
↪

```

### D.14.5.17 netmatrix

Procedure from library `nets.lib` (see [Section D.14.5 \[nets.lib\], page 2240](#)).

**Usage:**      `netmatrix(V)`; M matrix

**Assume:**    M is a matrix

**Return:**     visual presentation of M

**Theory:**     A Singular object is converted into a character array (a Net) for on screen printing.

**Example:**

```
LIB "nets.lib";
ring r1=101,(x,y,z),lp;
poly a=2x3y4+300xy-234z23;
poly b=2x3y4z;
poly c=x3y4z5;
poly d=5x6y7z10;
poly e=2x3y;
poly f=4y5z8;
matrix M[2][3]=a,b,c,d,e,f;
print(M);
  ↳ 2x3y4-3xy-32z23, 2x3y4z, x3y4z5,
  ↳ 5x6y7z10,      2x3y,  4y5z8
netmatrix(M);
  ↳ | 2x3y4-3xy-32z23 2x3y4z x3y4z5 |
  ↳ | 5x6y7z10      2x3y  4y5z8 |
  ↳
```

### D.14.5.18 netmatrixShort

Procedure from library `nets.lib` (see [Section D.14.5 \[nets.lib\], page 2240](#)).

**Usage:**      `netmatrixShort(M)`; M matrix

**Assume:**    M is a matrix

**Return:**     visual presentation of M

**Theory:**     A Singular object is converted into a character array (a Net) for on screen printing.

**Example:**

```
LIB "nets.lib";
ring r1=101,(x,y,z),lp;
poly a=2x3y4+300xy-234z23;
poly b=2x3y4z;
poly c=x3y4z5;
poly d=5x6y7z10;
poly e=2x3y-2x3y4+300xy-234z23;
poly f=4y5z8;
matrix M[2][3]=a,b,c,d,e,f;
netmatrixShort(M, 10);
  ↳ | 2x3y4-3xy-32z23 2x3y4z          x3y4z5 |
  ↳ | 5x6y7z10      -2x3y4+2x3y-3xy-32z23 4y5z8 |
  ↳
```

### D.14.5.19 netPoly

Procedure from library `nets.lib` (see [Section D.14.5 \[nets.lib\]](#), page 2240).

**Usage:**      `netPoly(P);` P poly

**Assume:**    P is a poly

**Return:**     visual presentation of P over two rows

**Theory:**     A Singular object is converted into a character array (a Net) for on screen printing.

**Example:**

```
LIB "nets.lib";
// from 3.3.1 Examples of ring declarations
//
ring R1 = 32003,(x,y,z),dp;
poly q6=1;
print(q6);
↳ 1
netPoly(q6);
↳
↳ 1
↳
poly q7=-1;
print(q7);
↳ -1
netPoly(q7);
↳
↳ -1
↳
poly q8=2;
print(q8);
↳ 2
netPoly(q8);
↳
↳ 2
↳
poly q9=-2;
print(q9);
↳ -2
netPoly(q9);
↳
↳ -2
↳
poly q1=x+y+z;
print(q1);
↳ x+y+z
netPoly(q1);
↳
↳ x+y+z
↳
poly q2=xy+xz+yz;
print(q2);
```

```

↳ xy+xz+yz
netPoly(q2);
↳
↳ xy+xz+yz
↳
poly q3=2x3y3z4-3x4y5z6;
print(q3);
↳ -3x4y5z6+2x3y3z4
netPoly(q3);
↳ 4 5 6 3 3 4
↳ -3x y z +2x y z
↳
poly q4=x3y3z4-x4y5z6;
print(q4);
↳ -x4y5z6+x3y3z4
netPoly(q4);
↳ 4 5 6 3 3 4
↳ -x y z +x y z
↳
poly q5=-x3y3z4+x4y5z6;
print(q5);
↳ x4y5z6-x3y3z4
netPoly(q5);
↳ 4 5 6 3 3 4
↳ x y z -x y z
↳
ring R2 = 32003,(x(1..10)),dp;
poly w6=1;
print(w6);
↳ 1
netPoly(w6);
↳
↳ 1
↳
poly w7=-1;
print(w7);
↳ -1
netPoly(w7);
↳
↳ -1
↳
poly w2=-x(1)-(2)-x(3);
print(w2);
↳ -x(1)-x(3)-2
netPoly(w2);
↳
↳ -x(1)-x(3)-2
↳
poly w3=x(1)*x(2)+x(1)*x(2)+x(2)*x(3);
print(w3);
↳ 2*x(1)*x(2)+x(2)*x(3)
netPoly(w3);
↳

```

```

↳ 2x(1)x(2)+x(2)x(3)
↳
poly w4=x(1)*x(2)-x(1)*x(2)-x(2)*x(3);
print(w4);
↳ -x(2)*x(3)
netPoly(w4);
↳
↳ -x(2)x(3)
↳
poly w5=x(1)^2*x(2)^3*x(3)^4;
print(w5);
↳ x(1)^2*x(2)^3*x(3)^4
netPoly(w5);
↳      2      3      4
↳ x(1) x(2) x(3)
↳
poly w8=x(1)+x(2)+x(3);
print(w8);
↳ x(1)+x(2)+x(3)
netPoly(w8);
↳
↳ x(1)+x(2)+x(3)
↳
poly w9=x(1)+x(2)+x(3);
print(w9);
↳ x(1)+x(2)+x(3)
netPoly(w9);
↳
↳ x(1)+x(2)+x(3)
↳
ring R3 = 32003,(x(1..5)(1..8)),dp;
poly e1=x(1)(1)+x(2)(2)+x(3)(3);
print(e1);
↳ x(1)(1)+x(2)(2)+x(3)(3)
netPoly(e1);
↳
↳ x(1)(1)+x(2)(2)+x(3)(3)
↳
poly e2=x(1)(1)*x(2)(2)*x(3)(3);
print(e2);
↳ x(1)(1)*x(2)(2)*x(3)(3)
netPoly(e2);
↳
↳ x(1)(1)x(2)(2)x(3)(3)
↳
poly e3=x(1)(1)^2*x(2)(2)^3*x(3)(3)^4;
print(e3);
↳ x(1)(1)^2*x(2)(2)^3*x(3)(3)^4
netPoly(e3);
↳      2      3      4
↳ x(1)(1) x(2)(2) x(3)(3)
↳
poly e4=-x(1)(1)^2*x(2)(2)^3*x(3)(3)^4-x(1)(1)^3*x(2)(2)^3*x(3)(3)^4;

```



```

print(e4);
 $\mapsto -x(1)(1)^3x(2)(2)^3x(3)(3)^4-x(1)(1)^2x(2)(2)^3x(3)(3)^4$ 
netPoly(e4);
 $\mapsto -x(1)(1)^3x(2)(2)^3x(3)(3)^4-x(1)(1)^2x(2)(2)^3x(3)(3)^4$ 
 $\mapsto -x(1)(1)^3x(2)(2)^3x(3)(3)^4-x(1)(1)^2x(2)(2)^3x(3)(3)^4$ 
 $\mapsto -x(1)(1)^3x(2)(2)^3x(3)(3)^4-x(1)(1)^2x(2)(2)^3x(3)(3)^4$ 
ring r=32003,(x,y,z),lp;
poly p=x4+4y4+4z4-x3-3y3-3z3+1x2+2y2+z2-x-1y-z1;
p;
 $\mapsto x^4-x^3+x^2-x+4y^4-3y^3+2y^2-y+4z^4-3z^3+z^2-z$ 
netPoly(p);
 $\mapsto x^4-x^3+x^2-x+4y^4-3y^3+2y^2-y+4z^4-3z^3+z^2-z$ 
 $\mapsto x^4-x^3+x^2-x+4y^4-3y^3+2y^2-y+4z^4-3z^3+z^2-z$ 
 $\mapsto x^4-x^3+x^2-x+4y^4-3y^3+2y^2-y+4z^4-3z^3+z^2-z$ 
poly p2=x3yz+xy3z+xyz3-2x2yz-2xy2z-2xyz2+1xyz+x1yzxy1z;
p2;
 $\mapsto x^3yz+x^2y^2z^2-2x^2yz+xy^3z-2xy^2z+xyz^3-2xyz^2+xyz$ 
netPoly(p2);
 $\mapsto x^3yz+x^2y^2z^2-2x^2yz+xy^3z-2xy^2z+xyz^3-2xyz^2+xyz$ 
 $\mapsto x^3yz+x^2y^2z^2-2x^2yz+xy^3z-2xy^2z+xyz^3-2xyz^2+xyz$ 
 $\mapsto x^3yz+x^2y^2z^2-2x^2yz+xy^3z-2xy^2z+xyz^3-2xyz^2+xyz$ 
poly p3=x+y+z-x2-3y-4z4+xy+xz+2xy-x2y-xz2-y2z2;
p3;
 $\mapsto -x^2y-x^2+3xy-xz^2+xz+x-y^2z^2-2y-4z^4+z$ 
netPoly(p3);
 $\mapsto -x^2y-x^2+3xy-xz^2+xz+x-y^2z^2-2y-4z^4+z$ 
 $\mapsto -x^2y-x^2+3xy-xz^2+xz+x-y^2z^2-2y-4z^4+z$ 
 $\mapsto -x^2y-x^2+3xy-xz^2+xz+x-y^2z^2-2y-4z^4+z$ 
ring r2=32003,(x(1..10)),lp;
poly p=x(1)*x(2)*x(3)+2*x(1)^2+2*x(1)*x(2);
p;
 $\mapsto 2*x(1)^2+x(1)*x(2)*x(3)+2*x(1)*x(2)$ 
netPoly(p);
 $\mapsto 2*x(1)^2+x(1)*x(2)*x(3)+2*x(1)*x(2)$ 
 $\mapsto 2x(1)^2+x(1)x(2)x(3)+2x(1)x(2)$ 
 $\mapsto 2x(1)^2+x(1)x(2)x(3)+2x(1)x(2)$ 
poly p2=x(1)^2*x(2)^3*x(3)^4-2*x(1)^1*x(2)^2+2*x(1)*x(2)*x(10);
p2;
 $\mapsto x(1)^2*x(2)^3*x(3)^4-2*x(1)*x(2)^2+2*x(1)*x(2)*x(10)$ 
netPoly(p2);
 $\mapsto x(1)^2*x(2)^3*x(3)^4-2*x(1)*x(2)^2+2*x(1)*x(2)*x(10)$ 
 $\mapsto x(1)^2*x(2)^3*x(3)^4-2*x(1)*x(2)^2+2*x(1)*x(2)*x(10)$ 
 $\mapsto x(1)^2*x(2)^3*x(3)^4-2*x(1)*x(2)^2+2*x(1)*x(2)*x(10)$ 
ring r3=7,(x,y,z),lp;
poly p=17x2+24y;
p;
 $\mapsto 3x^2+3y$ 
netPoly(p);
 $\mapsto 3x^2+3y$ 
 $\mapsto 3x^2+3y$ 
 $\mapsto 3x^2+3y$ 
ring r4=(7,a,b,c),(x,y,z),Dp;
poly p=2ax2+by-cz3;

```

```

p;
↪ (-c)*z3+(2a)*x2+(b)*y
netPoly(p);
↪      3      2
↪ (-c)z +(2a)x +(b)y
↪
ring r5=(7,a),(x,y,z),dp;
minpoly = a^2+a+3;
poly p=2ax2+y-az3;
p;
↪ (-a)*z3+(2a)*x2+y
netPoly(p);
↪      3      2
↪ (-a)z +(2a)x +y
↪
ring r6 = (complex,30,j),(x,y,z),dp;
poly p=2x2+y-z3+20*j;
p;
↪ -z3+2*x2+y+(j*20)
netPoly(p);
↪      3      2
↪ -z +2x +y+(j*20)
↪

```

#### D.14.5.20 netPrimePower

Procedure from library `nets.lib` (see [Section D.14.5 \[nets.lib\]](#), page 2240).

**Usage:** `netPrimePower(n,m);` n,m int

**Return:** visual presentation of the prime power  $n^m$

**Theory:** A Singular object is converted into a character array (a Net) for on screen printing.

**Example:**

```

LIB "nets.lib";
ring r=0,x,lp;
int n=2;
int m=5;
netPrimePower(n,m);
↪ 2^5
↪

```

#### D.14.5.21 netRing

Procedure from library `nets.lib` (see [Section D.14.5 \[nets.lib\]](#), page 2240).

**Usage:** `netRing(f);` f ring

**Assume:** R is a ring

**Return:** visual presentation of R

**Theory:** A Singular object is converted into a character array (a Net) for on screen printing.

**Example:**

```

LIB "nets.lib";
// from 3.3.1 Examples of ring declarations
ring r1 = 32003,(x,y,z),dp;
netRing(r1);
⇨ FF_32003[x,y,z]
⇨
//
ring r2 = 32003,(x(1..10)),dp;
netRing(r2);
⇨ FF_32003[x(1),x(2),x(3),x(4),x(5),x(6),x(7),x(8),x(9),x(10)]
⇨
//
ring r3 = 32003,(x(1..5)(1..8)),dp;
netRing(r3);
⇨ FF_32003[x(1)(1),x(1)(2),x(1)(3),x(1)(4),x(1)(5),x(1)(6),x(1)(7),x(1)(8),\
  x(2)(1),x(2)(2),x(2)(3),x(2)(4),x(2)(5),x(2)(6),x(2)(7),x(2)(8),x(3)(1),x(\
  3)(2),x(3)(3),x(3)(4),x(3)(5),x(3)(6),x(3)(7),x(3)(8),x(4)(1),x(4)(2),x(\
  4)(3),x(4)(4),x(4)(5),x(4)(6),x(4)(7),x(4)(8),x(5)(1),x(5)(2),x(5)(3),x(5)\
  )(4),x(5)(5),x(5)(6),x(5)(7),x(5)(8)]
⇨
//
ring r4 = 0,(a,b,c,d),lp;
netRing(r4);
⇨ QQ[a,b,c,d]
⇨
//
ring r5 = 7,(x,y,z),ds;
netRing(r5);
⇨ FF_7[x,y,z]
⇨
//
ring r6 = 10,(x,y,z),ds;
⇨ // ** 10 is invalid as characteristic of the ground field. 32003 is used.
netRing(r6);
⇨ FF_32003[x,y,z]
⇨
//
ring r7 = 7,(x(1..6)),(lp(3),dp);
netRing(r7);
⇨ FF_7[x(1),x(2),x(3),x(4),x(5),x(6)]
⇨
//
ring r8 = 0,(x,y,z,a,b,c),(ds(3), dp(3));
netRing(r8);
⇨ QQ[x,y,z,a,b,c]
⇨
//
ring r9 = 0,(x,y,z),(c,wp(2,1,3));
netRing(r9);
⇨ QQ[x,y,z]
⇨
//

```

```

ring r10 = (7,a,b,c),(x,y,z),Dp;
netRing(r10);
⇒ FF_7(a,b,c)[x,y,z]
⇒
//
ring r11 = (7,a),(x,y,z),dp;
minpoly = a^2+a+3;
netRing(r11);
⇒ FF_7[a]/(a^2+a+3)[x,y,z]
⇒
//
ring r12 = (7^2,a),(x,y,z),dp;
netRing(r12);
⇒ FF_7^2[x,y,z]
⇒
//
ring r13 = real,(x,y,z),dp;
netRing(r13);
⇒ QQ(6,6)[x,y,z]
⇒
//
ring r14 = (real,50),(x,y,z),dp;
netRing(r14);
⇒ QQ(50,50)[x,y,z]
⇒
//
ring r15 = (real,10,50),(x,y,z),dp;
netRing(r15);
⇒ QQ(10,50)[x,y,z]
⇒
//
ring r16 = (complex,30,j),(x,y,z),dp;
netRing(r16);
⇒ QQ(30,30)[x,y,z]
⇒
//
ring r17 = complex,(x,y,z),dp;
netRing(r17);
⇒ QQ(6,6)[x,y,z]
⇒
//
ring R = 7,(x,y,z), dp;
qring r18 = std(maxideal(2));
netRing(r18);
⇒ FF_7[x,y,z] / <z^2, yz, xz, y^2, xy, x^2>
⇒

```

#### D.14.5.22 netString

Procedure from library `nets.lib` (see [Section D.14.5 \[nets.lib\]](#), page 2240).

**Usage:**      `netString(M)`; M string

**Assume:** M is a string

**Return:** visual presentation of M

**Theory:** A Singular object is converted into a character array (a Net) for on screen printing.

**Example:**

```
LIB "nets.lib";
string M="Hallo";
netString(M);
↦ Hallo
↦
```

### D.14.5.23 netvector

Procedure from library `nets.lib` (see [Section D.14.5 \[nets.lib\]](#), page 2240).

**Usage:** netvector(V); V vector

**Assume:** V is a vector

**Return:** visual presentation of V

**Theory:** A Singular object is converted into a character array (a Net) for on screen printing.

**Example:**

```
LIB "nets.lib";
ring r1=101,(x,y,z),lp;
poly a=2x3y4;
poly b=2x3y4z;
poly c=x3y4z5;
poly d=5x6y7z10;
poly e=2x3y;
poly f=4y5z8;
vector V=[a,b,c,d,e,f];
netvector(V);
↦ | 2x3y4      |
↦ | 2x3y4z     |
↦ | x3y4z5     |
↦ | 5x6y7z10   |
↦ | 2x3y       |
↦ | 4y5z8      |
↦
```

### D.14.5.24 netvectorShort

Procedure from library `nets.lib` (see [Section D.14.5 \[nets.lib\]](#), page 2240).

**Usage:** netvectorShort(V); V vector

**Assume:** V is a vector

**Return:** visual presentation of V, short version

**Theory:** A Singular object is converted into a character array (a Net) for on screen printing.

**Example:**

```

LIB "nets.lib";
ring r1=101,(x,y,z),lp;
poly a=2x3y4;
poly b=2x3y4z-5x6y7z10;
poly c=x3y4z5;
poly d=5x6y7z10;
poly e=2x3y;
poly f=4y5z8+5x6y7z10;
vector V=[a,b,c,d,e,f];
netvectorShort(V);
⇒ | 2x3y4          |
⇒ | -5x6y7z10+2x3y4z |
⇒ | x3y4z5         |
⇒ | 5x6y7z10       |
⇒ | 2x3y           |
⇒ | 5x6y7z10+4y5z8  |
⇒

```

#### D.14.5.25 stackNets

Procedure from library `nets.lib` (see [Section D.14.5 \[nets.lib\]](#), page 2240).

**Usage:**      `stackNets(N,M)`; N,M Nets

**Return:**     vertical concatenation of N and M

**Theory:**     The nets are concated vertically with alignment at the left column.

**Example:**

```

LIB "nets.lib";
Net A = net("aaa");
Net B = net("b");
stackNets(A,B);
⇒ aaa
⇒ b
⇒
A+B;
⇒ aaab
⇒

```

#### D.14.6 phindex.lib

**Library :**    `phindex.lib`

**Purpose:**     Procedures to compute the index of real analytic vector fields

**Author:**    Victor Castellanos

**Note:**       To compute the Poincare-Hopf index of a real analytic vector field with an algebraically isolated singularity at 0 (w. an a. i. s), we use the algebraic formula for the degree of the real analytic map germ found by Eisenbud-Levine in 1997. This result was also proved by Khimshiashvili. If the isolated singularity is non algebraically isolated and the vector field has similar reduced complex zeroes of codimension 1, we use a formula as the Eisenbud-Levine found by Victor Castellanos, in both cases is necessary to use a local order (ds,...). To compute the signature of a quadratic form (or symmetric matrix) we use the method of Lagrange.

**Procedures:****D.14.6.1 signatureL**

Procedure from library `phindex.lib` (see [Section D.14.6 \[phindex.lib\]](#), page 2256).

**Usage:** `signatureL(M[,r]);`  $M$  symmetric matrix,  $r$  int (optional).

**Return:** the signature of  $M$  of type int or if  $r$  is given and  $!=0$  then intvec with (signature, nr. of +, nr. of -) is returned.

**Theory:** Given the matrix  $M$ , we construct the quadratic form associated. Afterwards we use the method of Lagrange to compute the signature. The law of inertia for a real quadratic form  $A(x,x)$  says that in a representation of  $A(x,x)$  as a sum of independent squares  $A(x,x)=\sum_{i=1}^r a_i X_i^2$ .

The number of positive and the number of negative squares are independent of the choice of representation. The signature  $-s-$  of  $A(x,x)$  is the difference between the number  $-pi-$  of positive squares and the number  $-nu-$  of negative squares in the representation of  $A(x,x)$ . The rank  $-r-$  of  $M$  (or  $A(x,x)$ ) and the signature  $-s-$  determine the numbers  $-pi-$  and  $-nu-$  uniquely, since

$$r=pi+nu, \quad s=pi-nu.$$

The method of Lagrange is a procedure to reduce any real quadratic form to a sum of squares.

Ref. Gantmacher, The theory of matrices, Vol. I, Chelsea Publishing Company, NY 1960, page 299.

**Example:**

```
LIB "phindex.lib";
ring r=0,(x),ds;
matrix M[5][5]=0,0,0,1,0,0,1,0,0,-1,0,0,1,0,0,1,0,0,3,0,0,-1,0,0,1;
signatureL(M,1); //The rank of M is 3+1=4
  ↳ 2,3,1
matrix H[5][5]=0,-7,0,1,0,-7,1,0,0,-1,0,0,1,0,0,1,0,0,-3,5,0,-1,0,5,1;
signatureL(H);
  ↳ 1
```

**D.14.6.2 signatureLqf**

Procedure from library `phindex.lib` (see [Section D.14.6 \[phindex.lib\]](#), page 2256).

**Usage:** `signatureLqf(h);`  $h$  quadratic form (poly type).

**Return:** the signature of  $h$  of type int or if  $r$  is given and  $!=0$  then intvec with (signature, nr. of +, nr. of -) is returned.

**Theory:** To compute the signature we use the method of Lagrange. The law of inertia for a real quadratic form  $h(x,x)$  says that in a representation of  $h(x,x)$  as a sum of independent squares  $h(x,x)=\sum_{i=1}^r a_i X_i^2$  the number of positive and the number of negative squares are independent of the choice of representation. The signature  $-s-$  of  $h(x,x)$  is the difference between the number  $-pi-$  of positive squares and the number  $-nu-$  of negative squares in the representation of  $h(x,x)$ . The rank  $-r-$  of  $h(x,x)$  and the signature  $-s-$  determine the numbers  $-pi-$  and  $-nu-$  uniquely, since

$$r=pi+nu, \quad s=pi-nu.$$

The method of Lagrange is a procedure to reduce any real quadratic form to a sum of squares.

Ref. Gantmacher, The theory of matrices, Vol. I, Chelsea Publishing Company, NY 1960, page 299.

**Example:**

```
LIB "phindex.lib";
ring r=0,(x(1..4)),ds;
poly Ax=4*x(1)^2+x(2)^2+x(3)^2+x(4)^2-4*x(1)*x(2)-4*x(1)*x(3)+4*x(1)*x(4)+4*x(2)*x(3);
signatureLqf(Ax,1); //The rank of Ax is 3+1=4
↦ 2,3,1
poly Bx=2*x(1)*x(4)+x(2)^2+x(3)^2;
signatureLqf(Bx);
↦ 2
```

### D.14.6.3 PH\_ais

Procedure from library `phindex.lib` (see [Section D.14.6 \[phindex.lib\]](#), page 2256).

**Usage:** PH\_ais(I); I ideal of coordinates of the vector field.

**Return:** the Poincare-Hopf index of type int.

**Note:** the isolated singularity must be algebraically isolated.

**Theory:** The Poincare-Hopf index of a real vector field  $X$  at the isolated singularity  $0$  is the degree of the map  $(X/|X|) : S_{\epsilon} \rightarrow S$ , where  $S$  is the unit sphere, and the spheres are oriented as  $(n-1)$ -spheres in  $\mathbb{R}^n$ . The degree depends only on the germ,  $X$ , of  $X$  at  $0$ . If the vector field  $X$  is real analytic, then an invariant of the germ is its local ring

$$Q_X = R[[x_1, \dots, x_n]] / I_X$$

where  $R[[x_1, \dots, x_n]]$  is the ring of germs at  $0$  of real-valued analytic functions on  $\mathbb{R}^n$ , and  $I_X$  is the ideal generated by the components of  $X$ . The isolated singularity of  $X$  is algebraically isolated if the algebra  $Q_X$  is finite dimensional as real vector space, geometrically this means that  $0$  is also an isolated singularity for the complexified vector field. In this case the Poincare-Hopf index is the signature of the non degenerate bilinear form  $\langle, \rangle$  obtained by composition of the product in the algebra  $Q_X$  with a linear functional map

$$\langle, \rangle : (Q_X) \times (Q_X) \rightarrow Q_X \xrightarrow{L} \mathbb{R}$$

with  $L(J_0) > 0$ , where  $J_0$  is the residue class of the Jacobian determinant in  $Q_X$ . Here, we use a natural linear functional defined as follows. Suppose that  $E = \{E_1, \dots, E_r\}$  is a basis of  $Q_X$ , then  $J_0$  can be written as

$$J_0 = a_1 E_{j_1} + \dots + a_k E_{j_k}, \quad j_s \in \{1, \dots, r\}, \quad s = 1, \dots, k, \quad k \leq r, \quad \text{where } a_s \text{ are constant.}$$

The linear functional  $L: Q_X \rightarrow \mathbb{R}$  is defined as  $L(E_{j_1}) = (a_1) / |a_1| = \text{sign of } a_1$ ,

the other elements of the base are sent to 0.

Refs. -Eisenbud & Levine, An algebraic formula for the degree of a  $C^\infty$  map germ, Ann. Math., 106, (1977), 19-38.

-Khimshiashvili, On a local degree of a smooth map, Trudi Tbilisi Math. Inst., (1980), 105-124.

**Example:**

```
LIB "phindex.lib";
ring r=0,(x,y,z),ds;
ideal I=x^3-3xy^2,-y^3+3yx^2,z^3;
PH_ais(I);
↦ 3
```



#### D.14.6.4 PH\_nais

Procedure from library `phindex.lib` (see [Section D.14.6 \[phindex.lib\]](#), page 2256).

**Usage:** `PH_nais(I)`;  $I$  ideal of coordinates of the vector field.

**Return:** the Poincare-Hopf index of type `int`.

**Note:** the vector field must be a non algebraically isolated singularity at 0, with reduced complex zeros of codimension 1.

**Theory:** Suppose that 0 is an algebraically isolated singularity of the real analytic vector field  $X$ , geometrically this corresponds to the fact that the complexified vector field has positive dimension singular locus, algebraically this mean that the local ring  $Qx = R[[x_1..x_n]]/Ix$  where  $R[[x_1..x_n]]$  is the ring of germs at 0 of real-valued analytic functions on  $R^n$ , and  $Ix$  is the ideal generated by the components of  $X$  is infinite dimensional as real vector space. In the case that  $X$  has a reduced hypersurface as complex zeros we have the next. There exist a real analytic function  $f: R^n \rightarrow R$ , and a real analytic vector field  $Y$  s. t.  $X = fY$ . The function  $f$  does not change of sign out of 0 and

$$Mx = R[[x_1..x_n]]/(Ix : \text{radical}(Ix))$$

is a finite dimensional sub-algebra of  $Qx$ . The Poincare-Hopf index of  $X$  at 0 is the sign of  $f$  times the signature of the non degenerate bilinear form  $\langle, \rangle$  obtained by composition of the product in the algebra  $Mx$  with a linear functional map

$$\langle, \rangle : (Mx) \times (Mx) \rightarrow Mx \xrightarrow{(L)} R$$

with  $L(Jp) > 0$ , where  $Jp$  is the residue class of the Jacobian determinant of  $X$ ,  $JX$ , over  $f^n$ ,  $JX/(f^n)$  in  $Mx$ . Here, we use a natural linear functional defined as follows. Suppose that  $E = \{E_1, \dots, E_r\}$  is a basis of  $Mx$ , then  $Jp$  is writing as  $Jp = a_1 E_{j_1} + \dots + a_k E_{j_k}$ ,  $j_s \in \{1 \dots r\}$ ,  $s = 1..k$ ,  $k \leq r$ , where  $a_s$  are constant. The linear functional  $L: M \rightarrow R$  is defined as  $L(E_{j_1}) = (a_1)/|a_1| = \text{sign of } a_1$ , the other elements of the base are sent to 0.

Refs. -Castellanos-Vargas, V., Una formula algebraica del indice de Poincare-Hopf para campos vectoriales reales con una variedad de ceros complejos, Ph. D. thesis CIMAT (2000), chapter 1, Guanajuato Mexico.

-Castellanos -Vargas, V. The index of non algebraically isolated singularity, Bol. Soc. Mat. Mexicana, (3)

Vol. 8, 2002, 141-147.

**Example:**

```
LIB "phindex.lib";
ring r=0,(x,y,z),ds;
ideal I=x5-2x3y2-3xy4+x3z2-3xy2z2,-3x4y-2x2y3+y5-3x2yz2+y3z2,x2z3+y2z3+z5;
PH_nais(I);
--> -3
```

#### D.14.7 polybori.lib

**Library:** `polybori.lib`

**Purpose:** A Singular Library Interface for PolyBoRi

**Authors:** Maximilian Kammermeier: Max0791@gmx.de  
Susanne Scherer: sscherer90@yahoo.de

**Overview:** A library for using PolyBoRi in the SINGULAR interface, with procedures that convert structures (polynomials, rings, ideals) in both directions. Therefore, it is possible to

compute boolean groebner basis via [Section D.14.7.1 \[boolean\\_std\]](#), page 2260. Polynomials can be converted to zero-suppressed decision diagrams (zdd) and vice versa.

For usability it defines the PolyBoRi types `bideal`, `bpoly`, and `bring` which are equivalent to Singular's `ideal`, `poly`, and `ring`, as well as `bset` which corresponds to the type `zdd` introduced here. In addition `bvar(i)` constructs the Boolean variable corresponding to `var(i)` from current `ring`;

For convenience, the corresponding types can be converted explicitly or implicitly while assigning. Also several SINGULAR operators were overloaded: `bring` comes with `nvars`, `bpoly` implements `lead`, `leadmonom` and `leadcoef`. Objects of this type may be added and multiplied, too. Finally, `bideal` yields `std` and `size` as well as addition and element access.

Hence, by using these types PolyBoRi functionality can be carried out seamlessly in SINGULAR:

```
> LIB "polybori.lib";
> ring r0=2,x(1..4),lp;
> def x=bvar; // enforce Boolean variables

> bpoly f1=x(1)+x(4);
> bpoly f2=x(1)+x(3)*x(1);
> bideal bI=list(f1,f2);

> std(bI);
_[1] = x(1) + x(4)
_[2] = x(3)*x(4) + x(4)
```

**Note:** For using this library SINGULAR's `python` interface must be available on your system. Please `./configure --with-python` when building SINGULAR for this purpose.

There are prebuilt binary packages for PolyBoRi available from <http://polybori.sf.net/>.

For building your own PolyBoRi please ensure that you have `scons` and a development version of the boost libraries installed on you system. Then you may execute the following commands in a `bash`-style shell to build PolyBoRi available to `python`:

```
PBDIR=/path/to/custom/polybori
wget http://downloads.sf.net/project/polybori/polybori/\
0.8.2/polybori-0.8.2.tar.gz
tar -xvzf polybori-0.8.2.tar.gz
cd polybori-0.8.2
scons install PREFIX=$PBDIR PYINSTALLPREFIX=$PBDIR/python
export PYTHONPATH=$PBDIR/python:$PYTHONPATH
```

#### References:

See <http://polybori.sf.net> for details about PolyBoRi.

**Procedures:** See also: [Section 3.8 \[Libraries\]](#), page 55; [Section 4.24 \[User defined types\]](#), page 136; [Section 4.28 \[pyobject\]](#), page 140.

### D.14.7.1 boolean\_std

Procedure from library `polybori.lib` (see [Section D.14.7 \[polybori\\_lib\]](#), page 2259).

**Usage:**     `boolean_std(Is);` Is ideal

**Return:**    Singular ideal of the boolean groebner basis of Is

**Example:**

```
LIB "polybori.lib";
ring r0=2,x(1..4),lp;
poly f1=x(1)^2+2*x(2)*(x(3))-x(4)^3;
poly f2=x(1)^2-x(3)*x(1);
poly f3=x(2)+5-2*x(1);
poly f4=x(1)*x(2)-x(3);
ideal I=f1,f2,f3,f4;
boolean_std(I);           // implicitly add x(i)^2-x(i)
bideal bI=I;              // alternative syntax
bideal re = std(bI);      // Continue PolyBoRi computations
std(re[1..2]);
ring r1=0,x,Dp;
poly f1=x3+2*x+1;
poly f2=x10-x5+2x;
poly f3=19;
ideal I=f1,f2,f3;
boolean_std(I);
ring r2=32003,(x,y,z),Dp;
poly f1=xz+y+20*x^2*y;
poly f2=32002*xy+xz2+y;
ideal I=f1,f2;
boolean_std(I);
ring r2=32003,(x,y,z),Dp;
poly f1=xyz+20*x^2*y-3*xz+15;
poly f2=32002*xy+z2;
poly f3=19*x5y;
ideal I=f1,f2,f3;
boolean_std(I);
```

See also: [Section D.14.7.7 \[boolean\\_ideal\]](#), page 2263; [Section D.14.7.4 \[boolean\\_poly\]](#), page 2262; [Section D.14.7.2 \[boolean\\_poly\\_ring\]](#), page 2261; [Section D.14.7.13 \[from\\_boolean\\_ideal\]](#), page 2266; [Section D.14.7.10 \[from\\_boolean\\_poly\]](#), page 2264.

### D.14.7.2 boolean\_poly\_ring

Procedure from library `polybori.lib` (see [Section D.14.7 \[polybori.lib\]](#), page 2259).

### D.14.7.3 boolean\_constant

Procedure from library `polybori.lib` (see [Section D.14.7 \[polybori.lib\]](#), page 2259).

**Usage:**     `boolean_constant(const[, rb]);` const constant and rb boolean ring

**Return:**    default:     constant   const   in the representation of the boolean ring  
               rb==boolean\_poly\_ring(basering); optional input: rb=boolean ring rb

**Example:**

```
LIB "polybori.lib";
ring r=7,(x,y),Dp;
pyobject rb=boolean_poly_ring(r);
boolean_constant(int(3));
```

```

typeof(boolean_constant(int(3)));
boolean_constant(int(0));
typeof(boolean_constant(int(0)));

```

See also: [Section D.14.7.7 \[boolean\\_ideal\]](#), page 2263; [Section D.14.7.1 \[boolean\\_std\]](#), page 2260.

#### D.14.7.4 boolean\_poly

Procedure from library `polybori.lib` (see [Section D.14.7 \[polybori.lib\]](#), page 2259).

**Usage:** `boolean_poly(ps[, dir, rb]);` ps polynomial, dir integer zero or one, rb boolean ring

**Return:** default: polynomial ps in the representation of the boolean ring  
`rb==boolean_poly_ring(basering);` optional input: boolean ring rb

**Note:** via the optional input dir, one can choose the computation method (either direct[`dir==0`] or recursive[`dir==1`]). default: recursive

**Example:**

```

LIB "polybori.lib";
ring r=0,x(1..5),Dp;
poly f=x(2)*(x(3)-x(1))+x(4)*x(5);
bring rb=r;
boolean_poly(f);
boolean_poly(f,0);
boolean_poly(f,0,boolean_poly_ring(r));
boolean_poly(f,0,rb);
poly g=0;
boolean_poly(g);
poly g=1;
boolean_poly(g);

```

See also: [Section D.14.7.7 \[boolean\\_ideal\]](#), page 2263; [Section D.14.7.1 \[boolean\\_std\]](#), page 2260.

#### D.14.7.5 direct\_boolean\_poly

Procedure from library `polybori.lib` (see [Section D.14.7 \[polybori.lib\]](#), page 2259).

**Usage:** `boolean_poly(ps[, rb]);` ps polynomial, rb boolean ring

**Return:** default: polynomial ps in the representation of the boolean ring  
`rb==boolean_poly_ring(basering);` optional input: boolean ring rb

**Example:**

```

LIB "polybori.lib";
ring r0=2,x(1..4),lp;
bring rb=r0;
poly f=x(1)^2+2*x(2)*(x(3))-x(4)^3;
direct_boolean_poly(f);
direct_boolean_poly(f,rb);
ring r1=0,x,Dp;
poly f=x3+2x+1;
direct_boolean_poly(f);
ring r2=32003,(x,y,z),Dp;
poly f=xyz+20*x^2*y-3*xz+15;
direct_boolean_poly(f);

```

See also: [Section D.14.7.7 \[boolean\\_ideal\]](#), page 2263; [Section D.14.7.1 \[boolean\\_std\]](#), page 2260.

### D.14.7.6 recursive\_boolean\_poly

Procedure from library `polybori.lib` (see [Section D.14.7 \[polybori.lib\]](#), page 2259).

**Usage:** `boolean_poly(ps[, rb]);` ps polynomial, rb boolean ring

**Return:** default: polynomial ps in the representation of the boolean ring  
`rb==boolean_poly_ring(basering);` optional input: rb boolean ring

**Example:**

```
LIB "polybori.lib";
ring r0=2,x(1..4),lp;
poly f=x(1)^2+2*x(2)*(x(3))-x(4)^3;
recursive_boolean_poly(f);
ring r1=0,x,Dp;
poly f=x3+2x+1;
recursive_boolean_poly(f);
ring r2=32003,(x,y,z),Dp;
def br2=boolean_poly_ring(r2);
bring bbr2=r2;
poly f=xyz+20*x^2*y-3*xz+15;
recursive_boolean_poly(f,br2);
recursive_boolean_poly(f,bbr2);
```

See also: [Section D.14.7.7 \[boolean\\_ideal\]](#), page 2263; [Section D.14.7.1 \[boolean\\_std\]](#), page 2260.

### D.14.7.7 boolean\_ideal

Procedure from library `polybori.lib` (see [Section D.14.7 \[polybori.lib\]](#), page 2259).

**Usage:** `boolean_ideal(Is[, rb]);` Is Ideal, rb boolean ring

**Return:** default: ideal Is in the representation of the boolean ring  
`rb==boolean_poly_ring(basering);` optional input: rb boolean ring

**Example:**

```
LIB "polybori.lib";
ring r0=2,x(1..4),lp;
poly f1=x(1)^2+2*x(2)*(x(3))-x(4)^3;
poly f2=x(1)^2-x(3)*x(1);
poly f3=x(2)+5-2*x(1);
poly f4=x(1)*x(2)-x(3);
ideal I=f1,f2,f3,f4;
boolean_ideal(I);
ring r1=0,x,Dp;
poly f1=x3+2*x+1;
poly f2=x10-x5+2x;
poly f3=19;
ideal I=f1,f2,f3;
boolean_ideal(I);
ring r2=32003,(x,y,z),Dp;
bring bbr2=r2;
poly f1=xyz+20*x^2*y-3*xz+15;
poly f2=32002*xy+z2;
poly f3=19;
ideal I=f1,f2,f3;
```

```
boolean_ideal(I);
boolean_ideal(I,bbr2);
```

See also: [Section D.14.7.1 \[boolean\\_std\]](#), page 2260.

### D.14.7.8 boolean\_set

Procedure from library `polybori.lib` (see [Section D.14.7 \[polybori\\_lib\]](#), page 2259).

**Usage:** `boolean_set(ss[, rb]);` ss zdd, rb boolean ring

**Return:** default: boolean set ss in the representation of a Polybori boolean set in the ring  
rb==boolean\_poly\_ring(basing); optional input: boolean ring rb

**Example:**

```
LIB "polybori.lib";
ring rs=0,(x,y,z),Dp;
poly ps=(x+1)*(y+1)*(z+1);
zdd fz=ps;
boolean_set(fz);
poly g=x*y*z+1;
zdd gz=g;
boolean_set(gz);
ring R=0,(x(1..4)),Dp;
def Rb=boolean_poly_ring(R);
poly h=(x(1)+1)*(x(2)+1)*(x(3)+1)*(x(4)+1);
zdd hz=h;
boolean_set(hz);
```

See also: [Section D.14.7.7 \[boolean\\_ideal\]](#), page 2263; [Section D.14.7.1 \[boolean\\_std\]](#), page 2260.

### D.14.7.9 from\_boolean\_constant

Procedure from library `polybori.lib` (see [Section D.14.7 \[polybori\\_lib\]](#), page 2259).

**Usage:** `from_boolean_constant(pb);` pb pyobject

**Return:** constant polynomial

**Example:**

```
LIB "polybori.lib";
ring rs=0,(x,y,z),Dp;
def rsb=boolean_poly_ring(rs);
poly f=(x+y)*x+z;
bpoly pp=f;
from_boolean_constant(0);
from_boolean_constant(1);
from_boolean_constant(pp);
```

See also: [Section D.14.7.7 \[boolean\\_ideal\]](#), page 2263; [Section D.14.7.1 \[boolean\\_std\]](#), page 2260.

### D.14.7.10 from\_boolean\_poly

Procedure from library `polybori.lib` (see [Section D.14.7 \[polybori\\_lib\]](#), page 2259).

**Usage:** `from_boolean_poly(ps[, dir]);` ps polynomial, dir integer zero or one

**Return:** default: polynomial ps in the representation of the boolean ring

**Note:** via the optional input `dir`, one can choose the computation method (either `direct[dir==0]` or `recursive[dir==1]`). default: `direct`

**Example:**

```
LIB "polybori.lib";
ring r=0,(x,y,z),Dp;
def rb=boolean_poly_ring(r);
poly f=x^2+2*y+5*z^4;
bpoly pp=f;
from_boolean_poly(pp);
from_boolean_poly(pp,1);
ring r2=5,(x,y,z),Dp;
def rb2=boolean_poly_ring(r2);
poly f2=x+y+z;
bpoly pp2=f2;
from_boolean_poly(pp);
from_boolean_poly(pp,1);
```

See also: [Section D.14.7.7 \[boolean\\_ideal\]](#), page 2263; [Section D.14.7.1 \[boolean\\_std\]](#), page 2260.

### D.14.7.11 `direct_from_boolean_poly`

Procedure from library `polybori.lib` (see [Section D.14.7 \[polybori.lib\]](#), page 2259).

**Usage:** `from_boolean_poly(pb)`; `pb` boolean polynomial

**Return:** polynomial in Singular

**Example:**

```
LIB "polybori.lib";
ring r=0,(x,y,z),Dp;
pyobject rb=boolean_poly_ring(r);
poly f=x^2+2*y^3+3*z^3;
bpoly pp=f;
direct_from_boolean_poly(pp);
poly g=0;
bpoly pp=g;
direct_from_boolean_poly(pp);
```

See also: [Section D.14.7.7 \[boolean\\_ideal\]](#), page 2263; [Section D.14.7.4 \[boolean\\_poly\]](#), page 2262; [Section D.14.7.2 \[boolean\\_poly\\_ring\]](#), page 2261; [Section D.14.7.1 \[boolean\\_std\]](#), page 2260; [Section D.14.7.13 \[from\\_boolean\\_ideal\]](#), page 2266.

### D.14.7.12 `recursive_from_boolean_poly`

Procedure from library `polybori.lib` (see [Section D.14.7 \[polybori.lib\]](#), page 2259).

**Usage:** `recursive_from_boolean_poly(pb)`; `pb` boolean polynomial

**Return:** polynomial in Singular

**Example:**

```
LIB "polybori.lib";
ring rs=0,(x,y,z),Dp;
def rsb=boolean_poly_ring(rs);
poly f=(x+y)*x+z;
bpoly pp=f;
```

```

recursive_from_boolean_poly(pp);
ring rs2=2,(x,y,z),Dp;
def rsb2=boolean_poly_ring(rs2);
poly f2=(x+y)*x+x;
bpoly pp2=f2;
recursive_from_boolean_poly(pp);

```

See also: [Section D.14.7.7 \[boolean\\_ideal\]](#), page 2263; [Section D.14.7.4 \[boolean\\_poly\]](#), page 2262; [Section D.14.7.2 \[boolean\\_poly\\_ring\]](#), page 2261; [Section D.14.7.13 \[from\\_boolean\\_ideal\]](#), page 2266; [Section D.14.7.10 \[from\\_boolean\\_poly\]](#), page 2264.

### D.14.7.13 from\_boolean\_ideal

Procedure from library `polybori.lib` (see [Section D.14.7 \[polybori.lib\]](#), page 2259).

**Usage:** `from_boolean_ideal(I);` I boolean ideal

**Return:** ideal in Singular

**Example:**

```

LIB "polybori.lib";
ring rs=0,(x,y,z),Dp;
def rb3=boolean_poly_ring(rs);
poly f1=x+y;
poly f2=x+z;
bpoly pp =f1;
bpoly p = f2;
bideal Ib;
list K=(p,pp);
Ib=K;
from_boolean_ideal(Ib);
ring rs2=5,(x,y,z),Dp;
def rb4=boolean_poly_ring(rs2);
poly p1=x+y;
poly p2=x+z;
poly p3=y+z;
bpoly p = p1;
bpoly pp = p2;
bpoly ppp = p3;
bideal Ib;
list K=(p,pp,ppp);
Ib=K;
from_boolean_ideal(Ib);

```

See also: [Section D.14.7.7 \[boolean\\_ideal\]](#), page 2263; [Section D.14.7.4 \[boolean\\_poly\]](#), page 2262; [Section D.14.7.2 \[boolean\\_poly\\_ring\]](#), page 2261; [Section D.14.7.1 \[boolean\\_std\]](#), page 2260; [Section D.14.7.10 \[from\\_boolean\\_poly\]](#), page 2264.

### D.14.7.14 from\_boolean\_set

Procedure from library `polybori.lib` (see [Section D.14.7 \[polybori.lib\]](#), page 2259).

**Usage:** `from_boolean_set(sb);` sb boolean set

**Return:** Boolean set sb in the representation of a zdd

**Example:**



```

LIB "polybori.lib";
ring r=0,(x,y,z),Dp;
poly f=(x+1)*(y+1)*(z+1);
bpoly fb=f;
bset fs=fb;
from_boolean_set(fs);
poly g=x*y*z+1;
bpoly gb=g;
bset gs=gb;
from_boolean_set(gs);
ring R=0,(x(1..4)),Dp;
poly h=(x(1)+1)*(x(2)+1)*(x(3)+1)*(x(4)+1);
pyobject hb=boolean_poly(h);
def hs=hb.set();
from_boolean_set(hs);

```

See also: [Section D.14.7.7 \[boolean\\_ideal\]](#), page 2263; [Section D.14.7.1 \[boolean\\_std\]](#), page 2260.

### D.14.7.15 bvar

Procedure from library `polybori.lib` (see [Section D.14.7 \[polybori.lib\]](#), page 2259).

**Usage:**      `bvar(i); int i`

**Return:**    `i`-th variable of Boolean ring corresponding to current basering

**Example:**

```

LIB "polybori.lib";
ring r = 2,(x,y,z),Dp;
bvar(1); // -> x

```

See also: [Section D.14.7.2 \[boolean\\_poly\\_ring\]](#), page 2261; [Section 5.1.165 \[var\]](#), page 284.

### D.14.7.16 poly2zdd

Procedure from library `polybori.lib` (see [Section D.14.7 \[polybori.lib\]](#), page 2259).

**Usage:**      `poly2zdd(poly ps); polynomial ps`

**Return:**    polynomial `ps` in zdd representation

**Example:**

```

LIB "polybori.lib";
ring r=0,x(1..5),Dp;
poly f=(x(1)+1)*(x(2)+1)*(x(3)+1)*x(4)*x(5);
poly2zdd(f);
poly g=x(3);
poly2zdd(g);

```

See also: [Section D.14.7.4 \[boolean\\_poly\]](#), page 2262; [Section D.14.7.14 \[from\\_boolean\\_set\]](#), page 2266.

### D.14.7.17 zdd2poly

Procedure from library `polybori.lib` (see [Section D.14.7 \[polybori.lib\]](#), page 2259).

**Usage:**      `zdd2poly(ss); zero-supressed decision diagram ss`

**Return:**    zdd `ss` in polynomial representation

**Example:**

```

LIB "polybori.lib";
ring r=0,x(1..5),Dp;
poly f=(x(1)+1)*(x(2)+1)*(x(3)+1)*x(4)*x(5);
zdd2poly(poly2zdd(f));
poly g=x(3);
zdd2poly(poly2zdd(g));
poly g=0;
zdd2poly(poly2zdd(0));
poly g=1;
zdd2poly(poly2zdd(01));

```

See also: [Section D.14.7.8 \[boolean\\_set\]](#), page 2264; [Section D.14.7.10 \[from\\_boolean\\_poly\]](#), page 2264.

**D.14.7.18 disp\_zdd**

Procedure from library `polybori.lib` (see [Section D.14.7 \[polybori.lib\]](#), page 2259).

**Usage:** `disp_zdd(ss)`; zero-suppressed decision diagram `ss`

**Return:** string containing visualization of `ss`

**Note:** the resulting string is the visualization of the polynomial that corresponds to `ss`, but with a additional structure that comes from the `zdd`. Every reached else- Branch induces a new line in the string.

**Example:**

```

LIB "polybori.lib";
ring r1=0,(x,y,z),Dp;
poly f1=xyz+xy+xz+yz+y+z+x+1;
zdd s1=f1;
disp_zdd(s1);
ring r2=0,x(1..6),Dp;
poly f2=x(1)+x(2)+x(3)+x(5)^2+x(6);
zdd s2=f2;
disp_zdd(s2);
ring r4=0,x(1..6),Dp;
poly f2=x(1)+1;
zdd s2=f2;
disp_zdd(s2);
ring r2=0,x(1..6),Dp;
poly f2=x(1)*x(2)*(x(3)-x(5)^2*x(6))+3*x(4)*x(5)-3;
zdd s2=f2;
disp_zdd(s2);
poly f4=0;
zdd s4=f4;
disp_zdd(s4);
poly f5=1;
zdd s5=f5;
disp_zdd(s5);

```

See also: [Section D.14.7.16 \[poly2zdd\]](#), page 2267; [Section D.14.7.17 \[zdd2poly\]](#), page 2267.

### D.14.8 sets\_lib

**Library:** sets.lib

**Purpose:** Sets in Singular

**Authors:** J. Boehm, boehm @ mathematik.uni-kl.de  
 D. Wienholz, wienholz @ mathematik.uni-kl.de  
 S. Zillien, zillien @ rhrk.uni-kl.de

**Overview:** We implement the new class set and all basic methods needed to work with sets. A set is generated from a list. After the generating of a set, the adding of an element or the union of two sets, automatically every double element is removed to secure that no element occurs in a set more than once.

There is a comparison operator, we access the operator via the function isEqual. This function isEqual can be used to compare two elements of the same type (Set, list, int, bigint, string, intmat, bigintmat, intvec, ring, map, poly, matrix, ideal, module, vector, resolution) and also works for comparing of int, bigint and number with each other, similarly for matrix, bigintmat and intmat.

The function size can be used to determine the number of elements.

The + operator is used for the union, the \* operator for the intersection.

The operators < and > can be used for inclusion tests.

The print function can be used for printing sets.

Note that the implementation of the underlying data structure and algorithms is very trivial and will at some point be replaced with something more efficient.

**Types:** Set The class of all sets

**Procedures:**

#### D.14.8.1 set

Procedure from library `sets.lib` (see [Section D.14.8 \[sets.lib\]](#), page 2269).

**Usage:** set(l) or \*=l (short form of \* = set(l)); l list

**Return:** Set, the set created from the entered list

**Example:**

```
LIB "sets.lib";
//example for set
Set S0a = list(list(1,2,3),list(list(1,2)),list(10,11));
Set S0b = list(list(10,11),list(list(1,2)));
S0b<S0a;
⇒ 1
S0a<S0b;
⇒ 0
S0a==S0a;
⇒ 1
S0a==S0b;
⇒ 0
list L = 1,1,2,3;
Set S1 = L;
S1;
```

```

⇒ {3; 2; 1}
⇒ Set with 3 elements
⇒
ring R1;
ring R2 = 0,(x,y),dp;
Set S2 = list(R1,R1,R2);
S2;
⇒ {(QQ),(x,y),(dp(2),C); (ZZ/32003),(x,y,z),(dp(3),C)}
⇒ Set with 2 elements
⇒
ideal I1 = x+y;
ideal I2 = y^2;
ideal I3 = x+y, (x+y)^3;
Set S3 = list(I1,I2,I3);
S3;
⇒ {y2; x+y}
⇒ Set with 2 elements
⇒
isEqual(I1,I3);
⇒ 1
isEqual(I1,I2);
⇒ 0
module M1 = x*gen(1), y*gen(1);
module M2 = y^2*gen(2);
module M3 = (x+y)*gen(1), (x-y)*gen(1);
Set S4 = list(M1,M2,M3);
S4;
⇒ {y2*gen(2); x*gen(1),y*gen(1)}
⇒ Set with 2 elements
⇒
intmat m1[2][3]= 1,2,3,4,5,6;
intmat m2[2][3]= 1,2,3,4,5,7;
Set S5 = list(m1,m2,m1);
S5;
⇒ {1,2,3,4,5,7 ; 1,2,3,4,5,6 }
⇒ Set with 2 elements
⇒
bigintmat b1[2][3]= 1,2,3,4,5,6;
bigintmat b2[2][3]= 1,2,3,4,5,7;
Set S6 = list(b1,b2,b1);
S6;
⇒ {1, 2, 3, 4, 5, 7; 1, 2, 3, 4, 5, 6}
⇒ Set with 2 elements
⇒
resolution r1 = res(maxideal(3),0);
resolution r2 = res(maxideal(4),0);
Set S7 = list(r1,r1,r2);
size(S7);
⇒ 2

```

### D.14.8.2 union

Procedure from library `sets.lib` (see [Section D.14.8 \[sets.lib\]](#), page 2269).

**Usage:**      `union(N,M)` or `N+M`;  $N, M$  sets

**Return:**    Set, the union of the sets  $N$  and  $M$

**Example:**

```
LIB "sets.lib";
list l =1,2,3;
list j =2,3,4;
Set N=l;
Set M=j;
N;
⇒ {3; 2; 1}
⇒ Set with 3 elements
⇒
M;
⇒ {4; 3; 2}
⇒ Set with 3 elements
⇒
N+M;
⇒ {1; 4; 3; 2}
⇒ Set with 4 elements
⇒
```

#### D.14.8.3 `intersectionSet`

Procedure from library `sets.lib` (see [Section D.14.8 \[sets.lib\]](#), page 2269).

**Usage:**      `intersectionSet(N,M)` or `N*M`;  $N, M$  sets

**Return:**    Set, the intersection of the sets  $N$  and  $M$

**Example:**

```
LIB "sets.lib";
list l =1,2,3;
list j =2,3,4;
Set N=l;
Set M=j;
N;
⇒ {3; 2; 1}
⇒ Set with 3 elements
⇒
M;
⇒ {4; 3; 2}
⇒ Set with 3 elements
⇒
N*M;
⇒ {3; 2}
⇒ Set with 2 elements
⇒
```

#### D.14.8.4 `complement`

Procedure from library `sets.lib` (see [Section D.14.8 \[sets.lib\]](#), page 2269).

**Usage:**      `complement(N,M)`;  $N, M$  sets

**Return:** Set, the complement of the set N in M

**Example:**

```
LIB "sets.lib";
list l =1,2;
list j =1,2,3,4;
Set N=l;
Set M=j;
N;
⇒ {2; 1}
⇒ Set with 2 elements
⇒
M;
⇒ {4; 3; 2; 1}
⇒ Set with 4 elements
⇒
complement(N,M);
⇒ {3; 4}
⇒ Set with 2 elements
⇒
```

#### D.14.8.5 isElement

Procedure from library `sets.lib` (see [Section D.14.8 \[sets.lib\]](#), page 2269).

**Usage:** `isElement(a,M)`; M set a def

**Return:** bool, 1 if a is an element of M, 0 if not

**Example:**

```
LIB "sets.lib";
int i=1;
int j=5;
list k =1,2,3,4;
Set M=k;
i;
⇒ 1
j;
⇒ 5
M;
⇒ {4; 3; 2; 1}
⇒ Set with 4 elements
⇒
isElement(i,M);
⇒ 1
isElement(j,M);
⇒ 0
```

#### D.14.8.6 isSubset

Procedure from library `sets.lib` (see [Section D.14.8 \[sets.lib\]](#), page 2269).

**Usage:** `isSubset(N,M)` or `N<M` ; N,M sets

**Return:** bool, 1 if N is a Subset of M or 0 if not

**Example:**

```

LIB "sets.lib";
list l =1,2;
list j =1,2,3,4;
Set N=l;
Set M=j;
N;
⇒ {2; 1}
⇒ Set with 2 elements
⇒
M;
⇒ {4; 3; 2; 1}
⇒ Set with 4 elements
⇒
N<M;
⇒ 1
M<N;
⇒ 0

```

**D.14.8.7 isSuperset**

Procedure from library `sets.lib` (see [Section D.14.8 \[sets.lib\]](#), page 2269).

**Return:** bool, 1 if N is a Superset of M or 0 if not

**Example:**

```

LIB "sets.lib";
list l =1,2;
list j =1,2,3,4;
Set N=l;
Set M=j;
N;
⇒ {2; 1}
⇒ Set with 2 elements
⇒
M;
⇒ {4; 3; 2; 1}
⇒ Set with 4 elements
⇒
N>M;
⇒ 0
M>N;
⇒ 1

```

**D.14.8.8 addElement**

Procedure from library `sets.lib` (see [Section D.14.8 \[sets.lib\]](#), page 2269).

**Usage:** addElement(M,a) ; M Set, a freely chosen element

**Return:** adds the Element a to the Set M

**Example:**

```

LIB "sets.lib";
//example for addElement

```

```

int a=4;
list L = 1,2,3;
Set S = L;
S;
↳ {3; 2; 1}
↳ Set with 3 elements
↳
a;
↳ 4
addElement(S,a);
↳ {1; 2; 3; 4}
↳ Set with 4 elements
↳

```

## D.15 Experimental libraries

This sections collect libraries in the beta test phase. Everything in these libraries may change.

For the minimal requirements and guidelines see [Section 3.8 \[Libraries\]](#), page 55.

Comments should be send to the author of the library directly.

### D.15.1 autgradalg\_lib

**Library:** autgradalg.lib

**Purpose:** Compute automorphism groups of pointedly graded algebras and of Mori dream spaces.

**Authors:** Simon Keicher

**Overview:** This library provides a framework for computing automorphisms of integral, finitely generated algebras that are graded by a finitely generated abelian group. This library also contains functions to compute automorphism groups of Mori dream spaces. The results are ideals  $I$  such that the respective automorphism group is isomorphic to the subgroup  $V(I)$  in some  $GL(n)$ .

**Assumptions:**

\* the algebra  $R$  is given as factor algebra  $S/I$  with a graded polynomial ring  $S = KK[T_1, \dots, T_r]$ . We will always assume that the basering is  $S$  and it is given over the rationals  $QQ$  or a number field  $QQ(a)$ . \*  $R$  must be minimally presented, i.e.,  $I$  is contained in  $\langle T_1, \dots, T_r \rangle^2$ . \*  $S$  (and hence  $R$ ) are graded via 'setBaseMultigrading(Q)' from 'multigrading.lib'. The last rows of the matrix  $Q$  are interpreted over  $\mathbb{Z}/a_i\mathbb{Z}$  if the respective entry of the list  $TOR$  is  $a_i$  and has been provided as parameter to the respective function. (See the functions for more details.) \* For all  $1 \leq i \leq r$ :  $L_{w_i} = 0$  where  $w_i := \deg(T_i)$ . \* the grading is pointed, i.e., no generator has degree 0 and the cone over all generator degrees is pointed. \* For Mori dream spaces  $X$ , we assume them to be given as  $X = X(R, w)$  with the Cox ring  $R$  of  $X$  (given as the algebra  $R$  before) and an ample class  $w$  in the grading group  $K$  with the torsion entries removed.

**Note:** we require the user to execute 'LIB'gfanlib.so" before using this library.

**Procedures:**

**Note:** the following functions were taken from 'quotsingcox.lib' by M.Donten-Bury and S.Keicher: 'hilbertBas'.

**Note:** This library comes without any warranty whatsoever. Use it at your own risk.



### D.15.1.1 autKS

Procedure from library `autgradalg.lib` (see [Section D.15.1 \[autgradalg.lib\]](#), page 2274).

**Usage:** `autKS(TOR)`; TOR: optional list of elementary divisors in case of torsion.

**Assume:** the basering is multigraded having used the command `setBaseMultigrading(Q)` from `'multigrading.lib'`.

**Purpose:** Compute the subgroup  $\text{Aut}_K(S)$  of  $\text{GL}(n)$  of graded automorphisms of the polynomial ring  $S$  (the basering).

**Return:** returns a ring  $S$  and exports an ideal `Iexported` in the coordinate ring  $S = K[Y_{ij}]$  of  $\text{GL}(n)$  such that  $\text{Aut}_K(S) = V(I)$ .

**Example:**

```
LIB "autgradalg.lib";
//////////
// example: fano 15:
intmat Q[1][5] = 3,3,2,2,1;
ring R = 0,T(1..5),dp;
// attach degree matrix Q to R:
setBaseMultigrading(Q);
//ideal I = T(1)*T(2) + T(3)^2*T(4) + T(5)^6;
def S = autKS();
⇒ // coefficients: QQ
⇒ // number of vars : 50
⇒ //          block 1 : ordering dp
⇒ //          : names  T(1) T(2) T(3) T(4) T(5) Y(1) Y(2) Y(3) Y(\
4) Y(5) Y(6) Y(7) Y(8) Y(9) Y(10) Y(11) Y(12) Y(13) Y(14) Y(15) Y(16) Y(1\
7) Y(18) Y(19) Y(20) Y(21) Y(22) Y(23) Y(24) Y(25) Y(26) Y(27) Y(28) Y(29\
) Y(30) Y(31) Y(32) Y(33) Y(34) Y(35) Y(36) Y(37) Y(38) Y(39) Y(40) Y(41)\
Y(42) Y(43) Y(44) Y(45)
⇒ //          block 2 : ordering C
setring S;
dim(std(Iexported));
⇒ 17
basing;
⇒ // coefficients: QQ
⇒ // number of vars : 46
⇒ //          block 1 : ordering dp
⇒ //          : names  Y(1) Y(2) Y(3) Y(4) Y(5) Y(6) Y(7) Y(8) Y(\
9) Y(10) Y(11) Y(12) Y(13) Y(14) Y(15) Y(16) Y(17) Y(18) Y(19) Y(20) Y(21\
) Y(22) Y(23) Y(24) Y(25) Y(26) Y(27) Y(28) Y(29) Y(30) Y(31) Y(32) Y(33)\
Y(34) Y(35) Y(36) Y(37) Y(38) Y(39) Y(40) Y(41) Y(42) Y(43) Y(44) Y(45)
⇒ //          block 2 : ordering dp
⇒ //          : names  Z
⇒ //          block 3 : ordering C
autKSexported;
⇒ [1]:
⇒ [1]:
⇒ [1,1]=Y(1)
⇒ [1,2]=Y(2)
⇒ [1,3]=0
⇒ [1,4]=0
```

```

↳      _[1,5]=0
↳      _[1,6]=Y(6)
↳      _[1,7]=Y(7)
↳      _[1,8]=0
↳      _[1,9]=Y(9)
↳      _[2,1]=Y(10)
↳      _[2,2]=Y(11)
↳      _[2,3]=0
↳      _[2,4]=0
↳      _[2,5]=0
↳      _[2,6]=Y(15)
↳      _[2,7]=Y(16)
↳      _[2,8]=0
↳      _[2,9]=Y(18)
↳      _[3,1]=0
↳      _[3,2]=0
↳      _[3,3]=Y(21)
↳      _[3,4]=Y(22)
↳      _[3,5]=0
↳      _[3,6]=0
↳      _[3,7]=0
↳      _[3,8]=Y(26)
↳      _[3,9]=0
↳      _[4,1]=0
↳      _[4,2]=0
↳      _[4,3]=Y(30)
↳      _[4,4]=Y(31)
↳      _[4,5]=0
↳      _[4,6]=0
↳      _[4,7]=0
↳      _[4,8]=Y(35)
↳      _[4,9]=0
↳      _[5,1]=0
↳      _[5,2]=0
↳      _[5,3]=0
↳      _[5,4]=0
↳      _[5,5]=Y(41)
↳      _[5,6]=0
↳      _[5,7]=0
↳      _[5,8]=0
↳      _[5,9]=0
↳      _[6,1]=0
↳      _[6,2]=0
↳      _[6,3]=0
↳      _[6,4]=0
↳      _[6,5]=0
↳      _[6,6]=Y(21)*Y(41)
↳      _[6,7]=Y(22)*Y(41)
↳      _[6,8]=0
↳      _[6,9]=Y(26)*Y(41)
↳      _[7,1]=0
↳      _[7,2]=0
↳      _[7,3]=0

```

```

↳      _[7,4]=0
↳      _[7,5]=0
↳      _[7,6]=Y(30)*Y(41)
↳      _[7,7]=Y(31)*Y(41)
↳      _[7,8]=0
↳      _[7,9]=Y(35)*Y(41)
↳      _[8,1]=0
↳      _[8,2]=0
↳      _[8,3]=0
↳      _[8,4]=0
↳      _[8,5]=0
↳      _[8,6]=0
↳      _[8,7]=0
↳      _[8,8]=Y(41)^2
↳      _[8,9]=0
↳      _[9,1]=0
↳      _[9,2]=0
↳      _[9,3]=0
↳      _[9,4]=0
↳      _[9,5]=0
↳      _[9,6]=0
↳      _[9,7]=0
↳      _[9,8]=0
↳      _[9,9]=Y(41)^3
↳      [2]:
↳      1
↳      [3]:
↳      _[1]=Y(3)
↳      _[2]=Y(4)
↳      _[3]=Y(5)
↳      _[4]=Y(8)
↳      _[5]=Y(12)
↳      _[6]=Y(13)
↳      _[7]=Y(14)
↳      _[8]=Y(17)
↳      _[9]=Y(19)
↳      _[10]=Y(20)
↳      _[11]=Y(23)
↳      _[12]=Y(24)
↳      _[13]=Y(25)
↳      _[14]=Y(27)
↳      _[15]=Y(28)
↳      _[16]=Y(29)
↳      _[17]=Y(32)
↳      _[18]=Y(33)
↳      _[19]=Y(34)
↳      _[20]=Y(36)
↳      _[21]=Y(37)
↳      _[22]=Y(38)
↳      _[23]=Y(39)
↳      _[24]=Y(40)
↳      _[25]=Y(42)
↳      _[26]=Y(43)

```

```

⇒      _[27]=Y(44)
⇒      _[28]=Y(45)
⇒      _[29]=-Y(2)*Y(10)*Y(22)^2*Y(30)^2*Y(41)^8*Z+Y(1)*Y(11)*Y(22)^2*Y(30\
)~2*Y(41)^8*Z+2*Y(2)*Y(10)*Y(21)*Y(22)*Y(30)*Y(31)*Y(41)^8*Z-2*Y(1)*Y(11)\
*Y(21)*Y(22)*Y(30)*Y(31)*Y(41)^8*Z-Y(2)*Y(10)*Y(21)^2*Y(31)^2*Y(41)^8*Z+Y\
(1)*Y(11)*Y(21)^2*Y(31)^2*Y(41)^8*Z-1
⇒      [4]:
⇒      T(1),T(2),T(3),T(4),T(5),T(3)*T(5),T(4)*T(5),T(5)^2,T(5)^3
getVariableWeights();
⇒ 3,3,2,2,1,3,3,2,3,3,3,2,2,1,3,3,2,3,3,3,2,2,1,3,3,2,3,3,3,2,2,1,3,3,2,3,3\
,3,2,2,1,3,3,2,3,-22
kill S, Q, R;
//////////
// example 3.14 from the paper
intmat Q[3][5] =
1,1,1,1,1,
1,-1,0,0,1,
1,1,1,0,0;
list TOR = 2;
ring R = 0,T(1..5),dp;
// attach degree matrix Q to R:
setBaseMultigrading(Q);
//ideal I = T(1)*T(2) + T(3)^2 + T(4)^2;
def S = autKS();
⇒ // coefficients: QQ
⇒ // number of vars : 30
⇒ //      block 1 : ordering dp
⇒ //      : names      T(1) T(2) T(3) T(4) T(5) Y(1) Y(2) Y(3) Y(\
4) Y(5) Y(6) Y(7) Y(8) Y(9) Y(10) Y(11) Y(12) Y(13) Y(14) Y(15) Y(16) Y(1\
7) Y(18) Y(19) Y(20) Y(21) Y(22) Y(23) Y(24) Y(25)
⇒ //      block 2 : ordering C
⇒ // ** redefining adMons ( list adMons;) autgradalg.lib::autKS:2325
⇒ // ** redefining MONexported ( export(MONexported);)
setring S;
Iexported;
⇒ Iexported[1]=Y(23)
⇒ Iexported[2]=Y(22)
⇒ Iexported[3]=Y(21)
⇒ Iexported[4]=Y(18)
⇒ Iexported[5]=Y(17)
⇒ Iexported[6]=Y(16)
⇒ Iexported[7]=Y(15)
⇒ Iexported[8]=Y(14)
⇒ Iexported[9]=Y(12)
⇒ Iexported[10]=Y(11)
⇒ Iexported[11]=Y(10)
⇒ Iexported[12]=Y(9)
⇒ Iexported[13]=Y(8)
⇒ Iexported[14]=Y(5)
⇒ Iexported[15]=Y(4)
⇒ Iexported[16]=Y(3)
⇒ Iexported[17]=Y(24)*Y(25)
⇒ Iexported[18]=Y(20)*Y(25)

```

```

⇒ Iexported[19]=Y(6)*Y(25)
⇒ Iexported[20]=Y(2)*Y(25)
⇒ Iexported[21]=Y(19)*Y(24)
⇒ Iexported[22]=Y(7)*Y(24)
⇒ Iexported[23]=Y(1)*Y(24)
⇒ Iexported[24]=Y(19)*Y(20)
⇒ Iexported[25]=Y(7)*Y(20)
⇒ Iexported[26]=Y(1)*Y(20)
⇒ Iexported[27]=Y(6)*Y(19)
⇒ Iexported[28]=Y(2)*Y(19)
⇒ Iexported[29]=Y(6)*Y(7)
⇒ Iexported[30]=Y(2)*Y(7)
⇒ Iexported[31]=Y(1)*Y(6)
⇒ Iexported[32]=Y(1)*Y(2)
⇒ Iexported[33]=Y(2)*Y(6)*Y(13)*Y(20)*Y(24)*Z+Y(1)*Y(7)*Y(13)*Y(19)*Y(25)*Z\
-1
⇒ Iexported[34]=Y(1)*Y(7)*Y(13)*Y(19)*Y(25)^2*Z-Y(25)
⇒ Iexported[35]=Y(1)*Y(7)*Y(13)*Y(19)^2*Y(25)*Z-Y(19)
⇒ Iexported[36]=Y(1)*Y(7)^2*Y(13)*Y(19)*Y(25)*Z-Y(7)
⇒ Iexported[37]=Y(1)^2*Y(7)*Y(13)*Y(19)*Y(25)*Z-Y(1)
print(getVariableWeights());
⇒      1      1      1      1      1      1      1      1      1      1      1      1 \
      1      1      1      1      1      1      1      1      1      1      1      1 \
      1      -5
⇒      1      -1      0      0      1      1      -1      0      0      1      1      -1 \
      0      0      1      1      -1      0      0      1      1      -1      0      0 \
      1      -1
⇒      1      1      1      0      0      1      1      1      0      0      1      1 \
      1      0      0      1      1      1      0      0      1      1      1      0 \
      0      -3
kill S, R, Q;

```

### D.15.1.2 autGradAlg

Procedure from library `autgradalg.lib` (see [Section D.15.1 \[autgradalg.lib\]](#), page 2274).

**Usage:** `autGradAlg(I, TOR)`;  $I$  is an ideal,  $TOR$  is an optional list of integers representing the torsion part of the grading group.

**Assume:** minimally presented, degrees of the generators of  $I$  are the minimal degrees, basering multigraded pointedly,  $I_w = 0$  for all  $w = \deg(\text{var}(i))$

**Return:** a ring. Also exports an ideal `Jexported` and a list `stabExported`.

**Example:**

```

LIB "autgradalg.lib";
intmat Q[1][3] =
1,1,1;
ring R = 0,T(1..3), dp;
setBaseMultigrading(Q);
ideal I = 0; //T(1)*T(2) + T(3)*T(4);
def RR = autGradAlg(I);
⇒ // coefficients: QQ
⇒ // number of vars : 12
⇒ //      block 1 : ordering dp

```

```

⇒ //          : names      T(1) T(2) T(3) Y(1) Y(2) Y(3) Y(4) Y(5) Y(\
    6) Y(7) Y(8) Y(9)
⇒ //          block      2 : ordering C
⇒ // ** redefining adMons ( list adMons;) autgradalg.lib::autKS:2324
setring RR;
"resulting ideal:";
⇒ resulting ideal:
Jexported;
⇒ Jexported[1]=-Y(3)*Y(5)*Y(7)*Z+Y(2)*Y(6)*Y(7)*Z+Y(3)*Y(4)*Y(8)*Z-Y(1)*Y(6)\
    )*Y(8)*Z-Y(2)*Y(4)*Y(9)*Z+Y(1)*Y(5)*Y(9)*Z-1
"dimension:";
⇒ dimension:
dim(std(Jexported));
⇒ 9
"as a detailed list:";
⇒ as a detailed list:
stabExported;
⇒ [1]:
⇒      [1]:
⇒      _[1,1]=Y(1)
⇒      _[1,2]=Y(2)
⇒      _[1,3]=Y(3)
⇒      _[2,1]=Y(4)
⇒      _[2,2]=Y(5)
⇒      _[2,3]=Y(6)
⇒      _[3,1]=Y(7)
⇒      _[3,2]=Y(8)
⇒      _[3,3]=Y(9)
⇒      [2]:
⇒      1
⇒      [3]:
⇒      _[1]=-Y(3)*Y(5)*Y(7)*Z+Y(2)*Y(6)*Y(7)*Z+Y(3)*Y(4)*Y(8)*Z-Y(1)*Y(6)*\
    Y(8)*Z-Y(2)*Y(4)*Y(9)*Z+Y(1)*Y(5)*Y(9)*Z-1

```

### D.15.1.3 autGenWeights

Procedure from library `autgradalg.lib` (see [Section D.15.1 \[autgradalg.lib\]](#), page 2274).

**Usage:** `autGenWeights(Q)`:  $Q$  is an intmat (columns must contain a lattice basis).

**Assume:** the cone over  $Q$  must be pointed and the columns of  $Q$  contain a lattice basis; there must be no 0-columns in  $Q$ . We assume that, in the torsion case, the torsion rows of  $Q$  are reduced (for example, a row of  $Q$  standing for entries in  $\mathbb{Z}\mathbb{Z}/5\mathbb{Z}\mathbb{Z}$  must not contain elements  $> 5$  or  $< 0$ ).

**Purpose:** computes generators for the subgroup  $\text{aut}(\Omega_S)$  of  $\text{GL}(n, \mathbb{Z}\mathbb{Z})$  that consists of all invertible integer  $k \times k$  matrices which fix the set  $\Omega_S$  of degrees of the variables of the basering  $S$ . The set of columns of  $Q$  equals  $\Omega_S$ .

**Reference:**  
Remark 3.1.

**Return:** a list of integral matrices  $A$  with  $|\det A| = 1$  such that  $A \cdot \{\text{columns of } Q\} = \{\text{columns of } Q\}$ .

**Example:**

```
LIB "autgradalg.lib";
// torsion example
// ZZ + ZZ/5ZZ:
intmat Q[2][5] =
1,1,1,1,1,
2,3,1,4,0;
list TOR = 5;
autGenWeights(Q, TOR);
⇨ [1]:
⇨ 1,0,
⇨ 0,1
⇨ [2]:
⇨ 1,0,
⇨ 0,2
⇨ [3]:
⇨ 1,0,
⇨ 0,3
⇨ [4]:
⇨ 1,0,
⇨ 0,4
⇨ [5]:
⇨ 1,0,
⇨ 1,1
⇨ [6]:
⇨ 1,0,
⇨ 1,2
⇨ [7]:
⇨ 1,0,
⇨ 1,3
⇨ [8]:
⇨ 1,0,
⇨ 1,4
⇨ [9]:
⇨ 1,0,
⇨ 2,1
⇨ [10]:
⇨ 1,0,
⇨ 2,2
⇨ [11]:
⇨ 1,0,
⇨ 2,3
⇨ [12]:
⇨ 1,0,
⇨ 2,4
⇨ [13]:
⇨ 1,0,
⇨ 3,1
⇨ [14]:
⇨ 1,0,
⇨ 3,2
⇨ [15]:
⇨ 1,0,
⇨ 3,3
```

```

⇒ [16]:
⇒ 1,0,
⇒ 3,4
⇒ [17]:
⇒ 1,0,
⇒ 4,1
⇒ [18]:
⇒ 1,0,
⇒ 4,2
⇒ [19]:
⇒ 1,0,
⇒ 4,3
⇒ [20]:
⇒ 1,0,
⇒ 4,4
kill Q, TOR;
// another free example
intmat Q[2][6] =
-2,2,-1,1,-1,1,
1,1,1,1,1,1;
autGenWeights(Q);
⇒ [1]:
⇒ 1,0,
⇒ 0,1
⇒ [2]:
⇒ -1,0,
⇒ 0,1
kill Q;
//-----
// 2nd free example
intmat Q[2][4] =
1,0,1,1,
0,1,1,1;
autGenWeights(Q);
⇒ [1]:
⇒ 1,0,
⇒ 0,1
⇒ [2]:
⇒ 0,1,
⇒ 1,0
kill Q;

```

#### D.15.1.4 stabilizer

Procedure from library `autgradalg.lib` (see [Section D.15.1 \[autgradalg.lib\]](#), page 2274).

**Usage:** `stabilizer(RL, A, BB, AMON, n0)`: RL is an ideal, A a matrix (standing for a subgroup of  $GL(n)$ ), BB is an intmat (standing for an automorphism of the grading group), AMON a list of monomials corresponding to the rows/columns of A, n0 an integer such that the first n0 variables of the basering are the  $T(i)$ , optional: a list of elementary divisors if there is torsion.

**Assume:** the basering must be graded (see `setBaseMultigrading()`) and the cone over the degrees of the variables must be pointed; there mustn't be 0-degrees. The vector w must be an



element of the cone over the degrees of the variables. Moreover,  $B$  must be such that it permutes the degrees of the variables and the degrees of the generators of  $RL$ .

**Purpose:** returns relations such that  $A \cdot I = I$ .

**Return:** a ring. Also exports an ideal `Jexported` and a list `stabExported`.

**Example:**

```
LIB "autgradalg.lib";
//////////
// example: fano 15:
intmat Q[1][5] = 3,3,2,2,1;
ring R = 0,T(1..5),dp;
// attach degree matrix Q to R:
setBaseMultigrading(Q);
ideal I = T(1)*T(2) + T(3)^2*T(4) + T(5)^6;
def RR = stabilizer(I);
⇒ // coefficients: QQ
⇒ // number of vars : 50
⇒ //          block 1 : ordering dp
⇒ //          : names  T(1) T(2) T(3) T(4) T(5) Y(1) Y(2) Y(3) Y(\
4) Y(5) Y(6) Y(7) Y(8) Y(9) Y(10) Y(11) Y(12) Y(13) Y(14) Y(15) Y(16) Y(1\
7) Y(18) Y(19) Y(20) Y(21) Y(22) Y(23) Y(24) Y(25) Y(26) Y(27) Y(28) Y(29\
) Y(30) Y(31) Y(32) Y(33) Y(34) Y(35) Y(36) Y(37) Y(38) Y(39) Y(40) Y(41)\
Y(42) Y(43) Y(44) Y(45)
⇒ //          block 2 : ordering C
setring RR;
RR;
⇒ // coefficients: QQ
⇒ // number of vars : 46
⇒ //          block 1 : ordering dp
⇒ //          : names  Y(1) Y(2) Y(3) Y(4) Y(5) Y(6) Y(7) Y(8) Y(\
9) Y(10) Y(11) Y(12) Y(13) Y(14) Y(15) Y(16) Y(17) Y(18) Y(19) Y(20) Y(21\
) Y(22) Y(23) Y(24) Y(25) Y(26) Y(27) Y(28) Y(29) Y(30) Y(31) Y(32) Y(33)\
Y(34) Y(35) Y(36) Y(37) Y(38) Y(39) Y(40) Y(41) Y(42) Y(43) Y(44) Y(45)
⇒ //          block 2 : ordering dp
⇒ //          : names  Z
⇒ //          block 3 : ordering C
Jexported;
⇒ Jexported[1]=Y(3)
⇒ Jexported[2]=Y(4)
⇒ Jexported[3]=Y(5)
⇒ Jexported[4]=Y(8)
⇒ Jexported[5]=Y(12)
⇒ Jexported[6]=Y(13)
⇒ Jexported[7]=Y(14)
⇒ Jexported[8]=Y(17)
⇒ Jexported[9]=Y(19)
⇒ Jexported[10]=Y(20)
⇒ Jexported[11]=Y(23)
⇒ Jexported[12]=Y(24)
⇒ Jexported[13]=Y(25)
⇒ Jexported[14]=Y(27)
⇒ Jexported[15]=Y(28)
```

```

⇒ Jexported[16]=Y(29)
⇒ Jexported[17]=Y(32)
⇒ Jexported[18]=Y(33)
⇒ Jexported[19]=Y(34)
⇒ Jexported[20]=Y(36)
⇒ Jexported[21]=Y(37)
⇒ Jexported[22]=Y(38)
⇒ Jexported[23]=Y(39)
⇒ Jexported[24]=Y(40)
⇒ Jexported[25]=Y(42)
⇒ Jexported[26]=Y(43)
⇒ Jexported[27]=Y(44)
⇒ Jexported[28]=Y(45)
⇒ Jexported[29]=-Y(2)*Y(10)*Y(22)^2*Y(30)^2*Y(41)^8*Z+Y(1)*Y(11)*Y(22)^2*Y(\
30)^2*Y(41)^8*Z+2*Y(2)*Y(10)*Y(21)*Y(22)*Y(30)*Y(31)*Y(41)^8*Z-2*Y(1)*Y(1\
1)*Y(21)*Y(22)*Y(30)*Y(31)*Y(41)^8*Z-Y(2)*Y(10)*Y(21)^2*Y(31)^2*Y(41)^8*Z\
+Y(1)*Y(11)*Y(21)^2*Y(31)^2*Y(41)^8*Z-1
⇒ Jexported[30]=Y(26)^2*Y(31)+2*Y(22)*Y(26)*Y(35)+Y(9)*Y(16)+Y(7)*Y(18)
⇒ Jexported[31]=2*Y(22)*Y(26)*Y(31)+Y(22)^2*Y(35)+Y(7)*Y(16)
⇒ Jexported[32]=Y(22)^2*Y(31)
⇒ Jexported[33]=Y(26)^2*Y(30)+2*Y(21)*Y(26)*Y(35)+Y(9)*Y(15)+Y(6)*Y(18)
⇒ Jexported[34]=2*Y(22)*Y(26)*Y(30)+2*Y(21)*Y(26)*Y(31)+2*Y(21)*Y(22)*Y(35)\
+Y(7)*Y(15)+Y(6)*Y(16)
⇒ Jexported[35]=Y(22)^2*Y(30)+2*Y(21)*Y(22)*Y(31)
⇒ Jexported[36]=2*Y(21)*Y(26)*Y(30)+Y(21)^2*Y(35)+Y(6)*Y(15)
⇒ Jexported[37]=-Y(41)^6+2*Y(21)*Y(22)*Y(30)+Y(21)^2*Y(31)-Y(26)^2*Y(35)-Y(\
9)*Y(18)
⇒ Jexported[38]=Y(21)^2*Y(30)
⇒ Jexported[39]=Y(9)*Y(11)+Y(2)*Y(18)
⇒ Jexported[40]=Y(7)*Y(11)+Y(2)*Y(16)
⇒ Jexported[41]=Y(6)*Y(11)+Y(2)*Y(15)
⇒ Jexported[42]=Y(2)*Y(11)
⇒ Jexported[43]=Y(9)*Y(10)+Y(1)*Y(18)
⇒ Jexported[44]=Y(7)*Y(10)+Y(1)*Y(16)
⇒ Jexported[45]=Y(6)*Y(10)+Y(1)*Y(15)
⇒ Jexported[46]=-2*Y(21)*Y(22)*Y(30)-Y(21)^2*Y(31)+Y(2)*Y(10)+Y(1)*Y(11)
⇒ Jexported[47]=Y(1)*Y(10)
dim(std(Jexported));
⇒ 3
getVariableWeights();
⇒ 3,3,2,2,1,3,3,2,3,3,2,2,1,3,3,2,3,3,2,2,1,3,3,2,3,3,3,2,2,1,3,3,2,3,3\
,3,2,2,1,3,3,2,3,-22
kill RR, Q, R;
//////////
// example 3.14 from the paper
intmat Q[3][5] =
1,1,1,1,1,
1,-1,0,0,1,
1,1,1,0,0;
list TOR = 2;
ring R = 0,T(1..5),dp;
// attach degree matrix Q to R:
setBaseMultigrading(Q);

```

```

ideal I = T(1)*T(2) + T(3)^2 + T(4)^2;
list TOR = 2;
⇒ // ** redefining TOR (list TOR = 2;) ./examples/stabilizer.sing:27
def RR = stabilizer(I, TOR);
⇒ // coefficients: QQ
⇒ // number of vars : 30
⇒ //          block 1 : ordering dp
⇒ //          : names  T(1) T(2) T(3) T(4) T(5) Y(1) Y(2) Y(3) Y(\
4) Y(5) Y(6) Y(7) Y(8) Y(9) Y(10) Y(11) Y(12) Y(13) Y(14) Y(15) Y(16) Y(1\
7) Y(18) Y(19) Y(20) Y(21) Y(22) Y(23) Y(24) Y(25)
⇒ //          block 2 : ordering C
⇒ // ** redefining adMons ( list adMons;) autgradalg.lib::autKS:2324
⇒ // ** redefining MONexported ( export(MONexported);)
⇒ // ** redefining k (int k = 1;) autgradalg.lib::stabilizer:1975
setring RR;
RR;
⇒ // coefficients: QQ
⇒ // number of vars : 26
⇒ //          block 1 : ordering dp
⇒ //          : names  Y(1) Y(2) Y(3) Y(4) Y(5) Y(6) Y(7) Y(8) Y(\
9) Y(10) Y(11) Y(12) Y(13) Y(14) Y(15) Y(16) Y(17) Y(18) Y(19) Y(20) Y(21\
) Y(22) Y(23) Y(24) Y(25)
⇒ //          block 2 : ordering dp
⇒ //          : names  Z
⇒ //          block 3 : ordering C
Jexported;
⇒ Jexported[1]=Y(24)
⇒ Jexported[2]=Y(23)
⇒ Jexported[3]=Y(22)
⇒ Jexported[4]=Y(21)
⇒ Jexported[5]=Y(20)
⇒ Jexported[6]=Y(17)
⇒ Jexported[7]=Y(16)
⇒ Jexported[8]=Y(15)
⇒ Jexported[9]=Y(12)
⇒ Jexported[10]=Y(11)
⇒ Jexported[11]=Y(10)
⇒ Jexported[12]=Y(9)
⇒ Jexported[13]=Y(8)
⇒ Jexported[14]=Y(6)
⇒ Jexported[15]=Y(5)
⇒ Jexported[16]=Y(4)
⇒ Jexported[17]=Y(3)
⇒ Jexported[18]=Y(2)
⇒ Jexported[19]=Y(18)*Y(19)
⇒ Jexported[20]=Y(14)*Y(19)
⇒ Jexported[21]=Y(13)*Y(18)
⇒ Jexported[22]=Y(14)^2-Y(18)^2
⇒ Jexported[23]=Y(13)*Y(14)
⇒ Jexported[24]=Y(13)^2-Y(19)^2
⇒ Jexported[25]=Y(1)*Y(7)-Y(18)^2-Y(19)^2
⇒ Jexported[26]=Y(14)*Y(18)^3*Y(25)*Z-Y(13)*Y(19)^3*Y(25)*Z+1
⇒ Jexported[27]=Y(19)^5*Y(25)*Z-Y(13)

```

```

⇒ Jexported[28]=Y(13)*Y(19)^4*Y(25)*Z-Y(19)
⇒ Jexported[29]=Y(18)^5*Y(25)*Z+Y(14)
dim(std(Jexported));
⇒ 3
stabExported;
⇒ [1]:
⇒ [1]:
⇒ _[1,1]=Y(1)
⇒ _[1,2]=0
⇒ _[1,3]=0
⇒ _[1,4]=0
⇒ _[1,5]=0
⇒ _[2,1]=0
⇒ _[2,2]=Y(7)
⇒ _[2,3]=0
⇒ _[2,4]=0
⇒ _[2,5]=0
⇒ _[3,1]=0
⇒ _[3,2]=0
⇒ _[3,3]=Y(13)
⇒ _[3,4]=0
⇒ _[3,5]=0
⇒ _[4,1]=0
⇒ _[4,2]=0
⇒ _[4,3]=0
⇒ _[4,4]=Y(19)
⇒ _[4,5]=0
⇒ _[5,1]=0
⇒ _[5,2]=0
⇒ _[5,3]=0
⇒ _[5,4]=0
⇒ _[5,5]=Y(25)
⇒ [2]:
⇒ 1,0,0,
⇒ 0,1,0,
⇒ 0,0,1
⇒ [3]:
⇒ _[1]=Y(2)
⇒ _[2]=Y(3)
⇒ _[3]=Y(4)
⇒ _[4]=Y(5)
⇒ _[5]=Y(6)
⇒ _[6]=Y(8)
⇒ _[7]=Y(9)
⇒ _[8]=Y(10)
⇒ _[9]=Y(11)
⇒ _[10]=Y(12)
⇒ _[11]=Y(14)
⇒ _[12]=Y(15)
⇒ _[13]=Y(16)
⇒ _[14]=Y(17)
⇒ _[15]=Y(18)
⇒ _[16]=Y(20)

```

```

↳      _[17]=Y(21)
↳      _[18]=Y(22)
↳      _[19]=Y(23)
↳      _[20]=Y(24)
↳      _[21]=Y(1)*Y(7)*Y(13)*Y(19)*Y(25)*Z-1
↳      _[22]=Y(13)^2-Y(19)^2
↳      _[23]=Y(1)*Y(7)-Y(13)^2
↳ [2] :
↳   [1] :
↳      _[1,1]=Y(1)
↳      _[1,2]=0
↳      _[1,3]=0
↳      _[1,4]=0
↳      _[1,5]=0
↳      _[2,1]=0
↳      _[2,2]=Y(7)
↳      _[2,3]=0
↳      _[2,4]=0
↳      _[2,5]=0
↳      _[3,1]=0
↳      _[3,2]=0
↳      _[3,3]=0
↳      _[3,4]=Y(14)
↳      _[3,5]=0
↳      _[4,1]=0
↳      _[4,2]=0
↳      _[4,3]=Y(18)
↳      _[4,4]=0
↳      _[4,5]=0
↳      _[5,1]=0
↳      _[5,2]=0
↳      _[5,3]=0
↳      _[5,4]=0
↳      _[5,5]=Y(25)
↳ [2] :
↳      1,0,0,
↳      0,1,0,
↳      1,1,1
↳ [3] :
↳      _[1]=Y(2)
↳      _[2]=Y(3)
↳      _[3]=Y(4)
↳      _[4]=Y(5)
↳      _[5]=Y(6)
↳      _[6]=Y(8)
↳      _[7]=Y(9)
↳      _[8]=Y(10)
↳      _[9]=Y(11)
↳      _[10]=Y(12)
↳      _[11]=Y(13)
↳      _[12]=Y(15)
↳      _[13]=Y(16)
↳      _[14]=Y(17)

```

```

⇒      _[15]=Y(19)
⇒      _[16]=Y(20)
⇒      _[17]=Y(21)
⇒      _[18]=Y(22)
⇒      _[19]=Y(23)
⇒      _[20]=Y(24)
⇒      _[21]=-Y(1)*Y(7)*Y(14)*Y(18)*Y(25)*Z-1
⇒      _[22]=-Y(14)^2+Y(18)^2
⇒      _[23]=Y(1)*Y(7)-Y(18)^2
kill RR, Q, R;

```

### D.15.1.5 autXhat

Procedure from library `autgradalg.lib` (see [Section D.15.1 \[autgradalg.lib\]](#), page 2274).

**Usage:** `autXhat(RL, w0, TOR)`: RL is an ideal, w an intvec, TOR a list of integers

**Assume:** the basering is multigraded, the elements of TOR stand for the torsion rows of the matrix `getVariableWeights()`, w is an ample class or the free part of such a class.

**Purpose:** compute an ideal J such that  $V(J)$  in some  $GL(n)$  is isomorphic to the H-equivariant automorphisms  $\widehat{X} \rightarrow \widehat{X}$ .

**Example:**

```

LIB "autgradalg.lib";
intmat Q[3][5] =
1,1,1,1,1,
1,-1,0,0,1,
1,1,1,0,0;
list TOR = 2;
ring R = 0,T(1..5),dp;
setBaseMultigrading(Q);
ideal I = T(1)*T(2) + T(3)^2 + T(4)^2;
intvec w0 = 2,1,0;
def RR = autXhat(I, w0, TOR);
⇒ // coefficients: QQ
⇒ // number of vars : 30
⇒ //          block 1 : ordering dp
⇒ //          : names  T(1) T(2) T(3) T(4) T(5) Y(1) Y(2) Y(3) Y(\
4) Y(5) Y(6) Y(7) Y(8) Y(9) Y(10) Y(11) Y(12) Y(13) Y(14) Y(15) Y(16) Y(1\
7) Y(18) Y(19) Y(20) Y(21) Y(22) Y(23) Y(24) Y(25)
⇒ //          block 2 : ordering C
⇒ // ** redefining adMons ( list adMons;) autgradalg.lib::autKS:2324
⇒ // ** redefining k (int k = 1;) autgradalg.lib::stabilizer:1975
setring RR;
RES;
⇒ [1]:
⇒      [1]:
⇒      _[1,1]=Y(1)
⇒      _[1,2]=0
⇒      _[1,3]=0
⇒      _[1,4]=0
⇒      _[1,5]=0
⇒      _[2,1]=0
⇒      _[2,2]=Y(7)

```

```

↳      _[2,3]=0
↳      _[2,4]=0
↳      _[2,5]=0
↳      _[3,1]=0
↳      _[3,2]=0
↳      _[3,3]=Y(13)
↳      _[3,4]=0
↳      _[3,5]=0
↳      _[4,1]=0
↳      _[4,2]=0
↳      _[4,3]=0
↳      _[4,4]=Y(19)
↳      _[4,5]=0
↳      _[5,1]=0
↳      _[5,2]=0
↳      _[5,3]=0
↳      _[5,4]=0
↳      _[5,5]=Y(25)
↳      [2]:
↳      1,0,0,
↳      0,1,0,
↳      0,0,1
↳      [3]:
↳      _[1]=Y(2)
↳      _[2]=Y(3)
↳      _[3]=Y(4)
↳      _[4]=Y(5)
↳      _[5]=Y(6)
↳      _[6]=Y(8)
↳      _[7]=Y(9)
↳      _[8]=Y(10)
↳      _[9]=Y(11)
↳      _[10]=Y(12)
↳      _[11]=Y(14)
↳      _[12]=Y(15)
↳      _[13]=Y(16)
↳      _[14]=Y(17)
↳      _[15]=Y(18)
↳      _[16]=Y(20)
↳      _[17]=Y(21)
↳      _[18]=Y(22)
↳      _[19]=Y(23)
↳      _[20]=Y(24)
↳      _[21]=Y(1)*Y(7)*Y(13)*Y(19)*Y(25)*Z-1
↳      _[22]=Y(13)^2-Y(19)^2
↳      _[23]=Y(1)*Y(7)-Y(13)^2
↳      [2]:
↳      [1]:
↳      _[1,1]=Y(1)
↳      _[1,2]=0
↳      _[1,3]=0
↳      _[1,4]=0
↳      _[1,5]=0

```

```

↳      _[2,1]=0
↳      _[2,2]=Y(7)
↳      _[2,3]=0
↳      _[2,4]=0
↳      _[2,5]=0
↳      _[3,1]=0
↳      _[3,2]=0
↳      _[3,3]=0
↳      _[3,4]=Y(14)
↳      _[3,5]=0
↳      _[4,1]=0
↳      _[4,2]=0
↳      _[4,3]=Y(18)
↳      _[4,4]=0
↳      _[4,5]=0
↳      _[5,1]=0
↳      _[5,2]=0
↳      _[5,3]=0
↳      _[5,4]=0
↳      _[5,5]=Y(25)
↳      [2]:
↳      1,0,0,
↳      0,1,0,
↳      1,1,1
↳      [3]:
↳      _[1]=Y(2)
↳      _[2]=Y(3)
↳      _[3]=Y(4)
↳      _[4]=Y(5)
↳      _[5]=Y(6)
↳      _[6]=Y(8)
↳      _[7]=Y(9)
↳      _[8]=Y(10)
↳      _[9]=Y(11)
↳      _[10]=Y(12)
↳      _[11]=Y(13)
↳      _[12]=Y(15)
↳      _[13]=Y(16)
↳      _[14]=Y(17)
↳      _[15]=Y(19)
↳      _[16]=Y(20)
↳      _[17]=Y(21)
↳      _[18]=Y(22)
↳      _[19]=Y(23)
↳      _[20]=Y(24)
↳      _[21]=-Y(1)*Y(7)*Y(14)*Y(18)*Y(25)*Z-1
↳      _[22]=-Y(14)^2+Y(18)^2
↳      _[23]=Y(1)*Y(7)-Y(18)^2
kill RR, Q, R;

```

### D.15.1.6 autX

Procedure from library `autgradalg.lib` (see [Section D.15.1 \[autgradalg.lib\]](#), page 2274).



**Usage:** autX(RL, w, TOR); RL: ideal, w: intvec, TOR: optional list of integers.

**Purpose:** compute generators for the hopf algebra  $O(\text{Aut}(X))$  of the Mori dream space  $X$  given by  $\text{Cox}(X) := \text{basing}/\text{RL}$  and the ample class  $w$ .

**Assume:** there is no torsion.

**Return:** a ring. Also exports an ideal Iexported.

**Example:**

```
LIB "autgradalg.lib";
//////////
//// CAREFUL: the following examples seems to be unfeasible at the moment, see remark
//echo=2;
//////////
//// PP2
//intmat Q[1][4] =
// 1,1,1,1;
//ring R = 0,T(1..ncols(Q)),dp;
//// attach degree matrix Q to R:
//setBaseMultigrading(Q);
//ideal I = 0;
//intvec w0 = 1;
//def RR = autX(I, w0);
//setring RR;
//Iexported;
//basing;
//dim(std(Iexported));
//kill RR, Q, R;
//////////
//// example 3.14 from the paper
//intmat Q[3][5] =
// 1,1,1,1,1,
// 1,-1,0,0,1,
// 1,1,1,0,0;
//list TOR = 2;
//ring R = 0,T(1..5),dp;
//// attach degree matrix Q to R:
//setBaseMultigrading(Q);
//ideal I = T(1)*T(2) + T(3)^2 + T(4)^2;
//list TOR = 2;
//intvec w0 = 2,1,0;
//def RR = autX(I, w0, TOR);
//setring RR;
//kill RR, Q, R;
```

### D.15.2 difform\_lib

**Library:** difform.lib

**Purpose:** Procedures for differential forms

**Author:** Peter Chini, chini@rhrk.uni-kl.de

**Overview:** A library for computing with elements of the differential algebra over a (quotient) ring. To compute in this algebra, a non-commutative ring with additional variables

$dx_1, \dots, dx_n$  and 'exterior' relations between this variables is used. In the case of a quotient ring, the defining ideal and its image under the universal derivation are added as relations. The differential forms themselves are defined via an additional type 'diffform'. Objects of this type carry as an attribute a polynomial in the differential algebra and make it available over the basering.

Additionally, the universal derivation is available as a procedure and the differentials between the graded parts of the differential algebra can be applied to differential forms. The library also supports derivations: maps from the first graded part of the differential algebra to the basering. These are defined via the type 'derivation' and there are procedures for basic arithmetic operations, evaluation and Lie-derivative.

#### Procedures:

##### D.15.2.1 diffAlgebra

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

#### Side effects:

If  $R$  is the basering, the differential algebra is constructed with name `Omega_R` and the differential forms  $dx_1, \dots, dx_n$  are available. The name of the differential algebra is stored in the attribute `attrib(R, "diffAlgebra")`.

**Note:**

- computations with differential forms need the structure of the differential algebra, so this procedure should be executed first.
- the variable names 'd' or 'D' should be avoided.
- the procedure also works for quotient rings

#### Example:

```
LIB "diffform.lib";
/////////////////////////////////////////////////////////////////
// Example for a differential algebra over a polynomial ring //
/////////////////////////////////////////////////////////////////
ring R = 0,(a,b,c),ds;
diffAlgebra();
↳ // The differential algebra Omega_R was constructed and the differential \
    forms dDa, dDb, dDc, da, db, dc are available.
setring Omega_R;
// The differential algebra is given by:
basing;
↳ // coefficients: QQ
↳ // number of vars : 6
↳ //          block  1 : ordering dp
↳ //                      : names    Da Db Dc
↳ //          block  2 : ordering ds
↳ //                      : names    a b c
↳ //          block  3 : ordering C
↳ // noncommutative relations:
↳ //      DbDa=-Da*Db
↳ //      DcDa=-Da*Dc
↳ //      DcDb=-Db*Dc
↳ // quotient ring from ideal
↳ _[1]=Da^2
↳ _[2]=Db^2
↳ _[3]=Dc^2
```

```

kill R, Omega_R, da, db, dc;
/////////////////////////////////////////////////////////////////
// Example for a differential algebra over a quotient ring //
/////////////////////////////////////////////////////////////////
ring R = 0, (x, y, z), lp;
ideal I = x+y+z, xyz;
qring S = std(I);
diffAlgebra();
⇒ // The differential algebra Omega_R was constructed and the differential \
    forms dDx, dDy, dDz, dx, dy, dz are available.
setring Omega_R;
// The differential algebra is given by:
basing;
⇒ // coefficients: QQ
⇒ // number of vars : 6
⇒ //      block  1 : ordering dp
⇒ //      : names  Dx Dy Dz
⇒ //      block  2 : ordering lp
⇒ //      : names  x y z
⇒ //      block  3 : ordering C
⇒ // noncommutative relations:
⇒ //      DyDx=-Dx*Dy
⇒ //      DzDx=-Dx*Dz
⇒ //      DzDy=-Dy*Dz
⇒ // quotient ring from ideal
⇒ _[1]=y^2*z+y*z^2
⇒ _[2]=x+y+z
⇒ _[3]=Dz*y^4+3*Dz*y^3*z+2*Dz*y^2*z^2
⇒ _[4]=Dy*z^3+2*Dz*y^3+5*Dz*y^2*z+2*Dz*y*z^2
⇒ _[5]=2*Dy*y*z+Dy*z^2+Dz*y^2+2*Dz*y*z
⇒ _[6]=Dx+Dy+Dz
⇒ _[7]=Dy*Dz*y^2+2*Dy*Dz*y*z
⇒ _[8]=Dz^2
⇒ _[9]=Dy^2
⇒ _[10]=Dx^2
kill Omega_R, dx, dy, dz;

```

See also: [Section D.15.2.3 \[diffAlgebraGens\]](#), page 2294; [Section D.15.2.2 \[diffAlgebraStructure\]](#), page 2293; [Section D.15.2.4 \[diffAlgebraUnivDerIdeal\]](#), page 2294.

### D.15.2.2 diffAlgebraStructure

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `diffAlgebraStructure();`

**Return:** the structure of the differential algebra

**Remarks:** The differential algebra is constructed as non-commutative ring with additional variables  $Dx_1, \dots, Dx_n$  and 'exterior' relations between them. In the case, that the basering is a quotient ring, the defining ideal and its image under the universal derivation are added as relations.

**Note:** the monomial ordering of the basering is preserved in the differential algebra

See also: [Section D.15.2.1 \[diffAlgebra\]](#), page 2292; [Section D.15.2.3 \[diffAlgebraGens\]](#), page 2294; [Section D.15.2.4 \[diffAlgebraUnivDerIdeal\]](#), page 2294.

### D.15.2.3 diffAlgebraGens

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `diffAlgebraGens();`

**Side effects:**

The differential forms  $dx_1, \dots, dx_n$  are constructed.

See also: [Section D.15.2.1 \[diffAlgebra\]](#), page 2292; [Section D.15.2.2 \[diffAlgebraStructure\]](#), page 2293; [Section D.15.2.4 \[diffAlgebraUnivDerIdeal\]](#), page 2294.

### D.15.2.4 diffAlgebraUnivDerIdeal

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `diffAlgebraUnivDerIdeal(I); I ideal`

**Assume:** current basering is the differential algebra of a polynomial ring and  $I$  is lifted from the polynomial ring

**Return:** the image of  $I$  under the universal derivation

**Remark:** The procedure computes the universal derivation of every generator of the ideal.

**Note:** for differential forms use the procedure `diffformUnivDer` or `diffformDiff`

See also: [Section D.15.2.1 \[diffAlgebra\]](#), page 2292; [Section D.15.2.3 \[diffAlgebraGens\]](#), page 2294; [Section D.15.2.2 \[diffAlgebraStructure\]](#), page 2293.

### D.15.2.5 diffAlgebraChangeOrd

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `diffAlgebraChangeOrd(#); # list`

**Assume:** the current basering is the differential algebra

**Return:** the differential algebra with changed monomial ordering

**Note:**

- an ordering is defined via the following pattern:
- `#[i] = 'gen'` defines the ordering on the generators  $dx_i$  - `#[i+1]` must then be a valid monomial ordering as string - `#[i+2]` an optional weight vector
- `#[i] = 'ringvar'` defines the ordering on the ringvariables - `#[i+1]` must then be a valid monomial ordering as string - `#[i+2]` an optional weight vector
- only use for interior computations
- differential forms are polynomials in the differential algebra - not in the returned ring - do not define differential forms as polynomials in the returned ring since this is another data-ring - an error occurs if:
  - no valid monomial ordering is given - no weight vector is given but a weighted monomial ordering - a given weight vector has wrong dimension
- weight vectors are ignored if the given ordering is not weighted

See also: [Section D.15.2.21 \[diffformIsBigger\]](#), page 2306; [Section D.15.2.22 \[diffformIsSmaller\]](#), page 2306; [Section D.15.2.27 \[diffformListSort\]](#), page 2310.

### D.15.2.6 diffAlgebraListGen

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `diffAlgebraListGen(#); # list`

**Return:** - a list of generators of the differential algebra as module over the basering - a list of generators of a graded part of the differential algebra

**Remarks:** In order to find all generators, they are counted 'binary': The generators are in 1:1 - correspondence to the dual number representations of 1 up to  $(2^n-1)$

**Note:**

- if all generators of the differential algebra are needed, apply the procedure without input
- if the generator(s) of a graded part are needed, apply the procedure with an integer which specifies the wanted degree
- the list of generators is sorted with respect to the monomial ordering on the differential algebra

**Example:**

```
LIB "diffform.lib";
ring R = 11,(x,y,z),dp;
diffAlgebra();
⇒ // The differential algebra Omega_R was constructed and the differential \
    forms dDx, dDy, dDz, dx, dy, dz are available.
////////////////////////////////////
// Generators of the differential algebra //
////////////////////////////////////
diffAlgebraListGen();
⇒ [1]:
⇒ 1
⇒
⇒ [2]:
⇒ dz
⇒
⇒ [3]:
⇒ dy
⇒
⇒ [4]:
⇒ dx
⇒
⇒ [5]:
⇒ dy*dz
⇒
⇒ [6]:
⇒ dx*dz
⇒
⇒ [7]:
⇒ dx*dy
⇒
⇒ [8]:
⇒ dx*dy*dz
⇒
////////////////////////////////////
// Generators of the second graded part //
```

```

////////////////////////////////////
diffAlgebraListGen(2);
↳ [1]:
↳   dy*dz
↳
↳ [2]:
↳   dx*dz
↳
↳ [3]:
↳   dx*dy
↳
kill Omega_R,dx,dy,dz;

```

### D.15.2.7 diffFromPoly

Procedure from library `diffFrom.lib` (see [Section D.15.2 \[diffFrom.lib\], page 2291](#)).

**Usage:** `diffFrom df = f; f poly`

**Return:** the differential form of degree 0 defined by `f`

**Remark:** The given polynomial gets lifted to the differential algebra and the differential form is defined there.

**Example:**

```

LIB "diffFrom.lib";
ring R = 0,(x,y,z),ds;
diffAlgebra();
↳ // The differential algebra Omega_R was constructed and the differential \
   forms dDx, dDy, dDz, dx, dy, dz are available.
////////////////////////////////////
// Construction of differential forms //
////////////////////////////////////
poly f = 3x3z*(y4-y5) + 2;
diffFrom df = f;
df;
↳ 2+3x3y4z-3x3y5z
↳
// For the construction of more general differential forms,
// the constructor diffFromPoly is used implicitly:
diffFrom dg = 3*x*dx - y*dy + dx*dy*dz + 1;
dg;
↳ 1+3x*dx+(-y)*dy+dx*dy*dz
↳
kill Omega_R,df,dg,dx,dy,dz;

```

### D.15.2.8 diffCoef

Procedure from library `diffFrom.lib` (see [Section D.15.2 \[diffFrom.lib\], page 2291](#)).

**Usage:** `diffCoef(df); df diffFrom`

**Return:** list of lists of differential forms and polynomials:  
- the first entry is a generator of the differential algebra which appears in `df` - the second entry is the corresponding coefficient

**Remarks:** Via the procedure `coef`, the coefficients are found - therefore the ring has to be changed to the differential algebra. After that, the coefficients have to be mapped back to the original ring.

**Note:** the returned list can be sorted with the procedure `diffFormListSort` and the optional string 'Llist'

**Example:**

```
LIB "diffForm.lib";
ring R = 0,(x,y,z),lp;
diffAlgebra();
⇨ // The differential algebra Omega_R was constructed and the differential \
    forms dDx, dDy, dDz, dx, dy, dz are available.
diffForm df = 3*x25*dx - y*dx*dy + 12*dx*dy*dz - dz*dy + 3 + 12*x*dx + 24*(y4-y5) + d
////////////////////
// Unsorted Coefficient List //
////////////////////
list L_1 = diffFormCoef(df);
L_1;
⇨ [1]:
⇨   [1]:
⇨     dx*dy*dz
⇨
⇨   [2]:
⇨     x3+12
⇨ [2]:
⇨   [1]:
⇨     dx*dy
⇨
⇨   [2]:
⇨     -y+1
⇨ [3]:
⇨   [1]:
⇨     dy*dz
⇨
⇨   [2]:
⇨     1
⇨ [4]:
⇨   [1]:
⇨     dx
⇨
⇨   [2]:
⇨     3x25+12x
⇨ [5]:
⇨   [1]:
⇨     dy
⇨
⇨   [2]:
⇨     yz5
⇨ [6]:
⇨   [1]:
⇨     dz
⇨
⇨   [2]:
```

```

↳      x2+1
↳ [7]:
↳      [1]:
↳      1
↳
↳      [2]:
↳      -24y5+24y4+3
////////////////////////////////////
// Sorted Coefficient List //
////////////////////////////////////
L_1 = diffformListSort(L_1,"Llist","gen","ds");
L_1;
↳ [1]:
↳      [1]:
↳      dx*dy*dz
↳
↳      [2]:
↳      x3+12
↳ [2]:
↳      [1]:
↳      dy*dz
↳
↳      [2]:
↳      1
↳ [3]:
↳      [1]:
↳      dx*dy
↳
↳      [2]:
↳      -y+1
↳ [4]:
↳      [1]:
↳      dz
↳
↳      [2]:
↳      x2+1
↳ [5]:
↳      [1]:
↳      dy
↳
↳      [2]:
↳      yz5
↳ [6]:
↳      [1]:
↳      dx
↳
↳      [2]:
↳      3x25+12x
↳ [7]:
↳      [1]:
↳      1
↳
↳      [2]:

```



```

    ↪      -24y5+24y4+3
    kill Omega_R,df,dx,dy,dz,L_1;

```

### D.15.2.9 diffformGenToString

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `diffformGenToString(df);` `df` diffform

**Return:** the differential form `df` as a string - but with unsorted coefficients

**Note:** - this is only used to print generators  
 - the procedure replaces the 'D' from the variables of the differential algebra by an 'd'

See also: [Section D.15.2.11 \[diffformToString\]](#), page 2300.

### D.15.2.10 diffformHomogDecomp

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `diffformHomogDecomp(df);` `df` diffform

**Return:** list of differential forms: homogeneous decomposition

**Note:** the output list always has as length the maximal possible degree plus one and the degree-0 part is the last element in the list

**Example:**

```

LIB "diffform.lib";
ring R = 0,(x,y,z),ds;
diffAlgebra();
↪ // The differential algebra Omega_R was constructed and the differential \
    forms dDx, dDy, dDz, dx, dy, dz are available.
diffform df = 3*dx*dz - x8*dx*dy + 12 + dy*dz + dz*dx - (y4-y5)*x12*dx*dy*dz - dx - dy
////////////////////////////////////
// Homogeneous decomposition //
////////////////////////////////////
list L = diffformHomogDecomp(df);
L;
↪ [1]:
↪      (-1)*dx+(-1)*dy+dz
↪
↪ [2]:
↪      (x2-x8)*dx*dy+2*dx*dz+dy*dz
↪
↪ [3]:
↪      (-x12y4+x12y5)*dx*dy*dz
↪
↪ [4]:
↪      12
↪
kill Omega_R,df,L,dx,dy,dz;

```

See also: [Section D.15.2.8 \[diffformCoef\]](#), page 2296; [Section D.15.2.23 \[diffformDeg\]](#), page 2307.

### D.15.2.11 diffformToString

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `diffformToString(df,#);` df diffform, # list

**Return:** df as a string, sorted by a given ordering on the generators (standard: the ordering chosen for the differential algebra)

**Remarks:** The differential form is decomposed via `diffformCoef`, the coefficient list is sorted and then the string is built as concatenation of coefficients and generators

**Note:** to get a string, respecting a certain monomial ordering on the generators, use: - `#[1]` = 'gen'  
 - `#[2]`: a monomial ordering as string  
 - `#[3]`: an optional weight vector

**Example:**

```
LIB "diffform.lib";
ring R = 0,(x,y,z,a,b),ds;
diffAlgebra();
⇨ // The differential algebra Omega_R was constructed and the differential \
    forms dDx, dDy, dDz, dDa, dDb, dx, dy, dz, da, db are available.
diffform df = 3*x*dx -2*db + 24*a*dy - y*dx*dy*db + 12*dx*dy*dz - dz*dy*da*db + 3 + 12
////////////////////////////////////
// String sorted with respect to the monomial order on the differential algebra //
////////////////////////////////////
string df_str = diffformToString(df);
print(df_str);
⇨ dy*dz*da*db+12*dx*dy*dz+(-y)*dx*dy*db+1/77*dx*da+15x*dx+24a*dy+(-2)*db+3
////////////////////////////////////
// String sorted with respect to the weighted order wp //
////////////////////////////////////
df_str = diffformToString(df,"gen","wp",intvec(-1,-1,-1,1,1));
print(df_str);
⇨ (-2)*db+dy*dz*da*db+1/77*dx*da+3+(-y)*dx*dy*db+15x*dx+24a*dy+12*dx*dy*dz
kill Omega_R,df,df_str,dx,dy,dz,da,db;
```

See also: [Section D.15.2.9 \[diffformGenToString\]](#), page 2299; [Section D.15.2.12 \[diffformPrint\]](#), page 2300.

### D.15.2.12 diffformPrint

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `df;` df diffform

**Side effects:**  
 prints the differential form

**Remarks:** Uses the procedure `diffformToString` with a ds-ordering on the generators

**Example:**

```
LIB "diffform.lib";
ring R = 0,(x,y,z),ds;
diffAlgebra();
⇨ // The differential algebra Omega_R was constructed and the differential \
```

```

    forms dDx, dDy, dDz, dx, dy, dz are available.
    //////////////////////////////////
    // Application of Print //
    //////////////////////////////////
    diffform df = 3*x*dx - y*dx*dy + 12*dx*dy*dz - dz*dy + 3 + 12*x*dx;
    df;
    ↪ 3+15x*dx+(-y)*dx*dy+dy*dz+12*dx*dy*dz
    ↪
    kill Omega_R,df,dx,dy,dz;

```

See also: [Section D.15.2.9 \[diffformGenToString\], page 2299](#); [Section D.15.2.11 \[diffformToString\], page 2300](#).

### D.15.2.13 diffformIsGen

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\], page 2291](#)).

**Usage:** `diffformIsGen(df); df diffform`

**Return:** 1, if `df` is a generator of the differential algebra - 0, otherwise

**Remarks:** Uses the procedure `diffformCoef` and tests for a single coefficient which is one

**Example:**

```

LIB "diffform.lib";
ring R = 0,(x,y,z,a,b,c),lp;
diffAlgebra();
↪ // The differential algebra Omega_R was constructed and the differential \
    forms dDx, dDy, dDz, dDa, dDb, dDc, dx, dy, dz, da, db, dc are available.
    //////////////////////////////////
    // Generators //
    //////////////////////////////////
    diffformIsGen(1);
    ↪ 1
    diffformIsGen(dx);
    ↪ 1
    diffformIsGen(da*dc);
    ↪ 1
    diffformIsGen(dy*da*db);
    ↪ 1
    diffformIsGen(-da*dz);
    ↪ 1
    //////////////////////////////////
    // No generators //
    //////////////////////////////////
    diffformIsGen(-1);
    ↪ 0
    diffformIsGen(-dx);
    ↪ 0
    diffformIsGen(dc*da);
    ↪ 0
    diffformIsGen(dy*db*da);
    ↪ 0
    diffformIsGen(dx*dz*dy);
    ↪ 0
    kill Omega_R,dx,dy,dz,da,db,dc;

```

See also: [Section D.15.2.8 \[diffformCoef\]](#), page 2296.

### D.15.2.14 diffformAdd

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `df+dg; df,dg diffform`

**Return:** the sum of the differential forms as differential form

**Example:**

```
LIB "diffform.lib";
ring R = 0,(x,y,z),ds;
diffAlgebra();
⇒ // The differential algebra Omega_R was constructed and the differential \
   forms dDx, dDy, dDz, dx, dy, dz are available.
////////////////////////////////////
// Addition of differential forms //
////////////////////////////////////
diffform df = dx + 4*dy - dz*dx + 4 + 3*dx*dy + 4*dz;
diffform dg = dx + dy + 27*dz*dy;
df+dg;
⇒ 4+2*dx+5*dy+4*dz+3*dx*dy+dx*dz+(-27)*dy*dz
⇒
////////////////////////////////////
// Addition of polynomials and differential forms //
////////////////////////////////////
df + x2y2z2;
⇒ 4+x2y2z2+dx+4*dy+4*dz+3*dx*dy+dx*dz
⇒
12 + dg;
⇒ 12+dx+dy+(-27)*dy*dz
⇒
kill Omega_R,df,dg,dx,dy,dz;
```

See also: [Section D.15.2.15 \[diffformSub\]](#), page 2302.

### D.15.2.15 diffformSub

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `df-dg; df,dg diffform`

**Return:** the difference of the differential forms as differential form

**Example:**

```
LIB "diffform.lib";
ring R = 0,(x,y,z),ds;
diffAlgebra();
⇒ // The differential algebra Omega_R was constructed and the differential \
   forms dDx, dDy, dDz, dx, dy, dz are available.
////////////////////////////////////
// Subtraction of differential forms //
////////////////////////////////////
diffform df = 5*dx*x2 - 7*dy*z - 2x2*dz - 3;
diffform dg = dx - 8x*dz*dy;
```

```

df-dg;
 $\mapsto -3+(-1+5x^2)*dx+(-7z)*dy+(-2x^2)*dz+(-8x)*dy*dz$ 
 $\mapsto$ 
////////////////////////////////////
// Subtraction of polynomials and differential forms //
////////////////////////////////////
df - 2x3;
 $\mapsto -3-2x^3+5x^2*dx+(-7z)*dy+(-2x^2)*dz$ 
 $\mapsto$ 
1 - dg;
 $\mapsto 1+(-1)*dx+(-8x)*dy*dz$ 
 $\mapsto$ 
kill Omega_R,df,dg,dx,dy,dz;

```

See also: [Section D.15.2.14 \[diffformAdd\]](#), page 2302; [Section D.15.2.16 \[diffformNeg\]](#), page 2303.

### D.15.2.16 diffformNeg

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `-df; df diffform`

**Return:** the negation of the differential form

**Example:**

```

LIB "diffform.lib";
ring R = 0,(x,y,z),ds;
diffAlgebra();
 $\mapsto$  // The differential algebra Omega_R was constructed and the differential \
      forms dDx, dDy, dDz, dx, dy, dz are available.
////////////////////////////////////
// Negation of a differential form //
////////////////////////////////////
diffform df = 13*dx*dy + 2*dy*dz - 6*dx*dy*dz - 3;
-df;
 $\mapsto 3+(-13)*dx*dy+(-2)*dy*dz+6*dx*dy*dz$ 
 $\mapsto$ 
kill Omega_R,df,dx,dy,dz;

```

See also: [Section D.15.2.15 \[diffformSub\]](#), page 2302.

### D.15.2.17 diffformMul

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `df*dg; df,dg diffform`

**Return:** the product of the differential forms as differential form

**Example:**

```

LIB "diffform.lib";
ring R = 0,(x,y,z),ds;
diffAlgebra();
 $\mapsto$  // The differential algebra Omega_R was constructed and the differential \
      forms dDx, dDy, dDz, dx, dy, dz are available.
////////////////////////////////////
// Multiplication of differential forms //

```

```

////////////////////////////////////
diffform df = 13*dx*dy - 7*dy*dz - 6*dx*dy*dz;
diffform dg = dx - 8x;
df*dg;
↳ (-104x)*dx*dy+56x*dy*dz+(-7+48x)*dx*dy*dz
↳
////////////////////////////////////
// Multiplication of polynomials and differential forms //
////////////////////////////////////
df*(y2-x);
↳ (-13x+13y2)*dx*dy+(7x-7y2)*dy*dz+(6x-6y2)*dx*dy*dz
↳
12*dg;
↳ -96x+12*dx
↳
kill Omega_R,df,dg,dx,dy,dz;

```

See also: [Section D.15.2.18 \[diffformDiv\]](#), page 2304.

### D.15.2.18 diffformDiv

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `df/dg; df,dg diffform`

**Return:** the quotient `df/dg` as differential form

**Example:**

```

LIB "diffform.lib";
ring R = 0,(x,y,z),lp;
diffAlgebra();
↳ // The differential algebra Omega_R was constructed and the differential \
    forms dDx, dDy, dDz, dx, dy, dz are available.
////////////////////////////////////
// Divisions without remainder //
////////////////////////////////////
dx / dx;
↳ 1
↳
dx*dy*dz / dz;
↳ dx*dy
↳
(dx*x2 - yx2) / x2;
↳ -y+dx
↳
////////////////////////////////////
// Divisions with reaminder //
////////////////////////////////////
(dx + dx*dy + 1) / dx;
↳ 1+(-1)*dy
↳
(x2*dx - x*dy) / (dx-dy);
↳ x2
↳
kill Omega_R,dx,dy,dz;

```

See also: [Section D.15.2.17 \[diffformMul\]](#), page 2303.

### D.15.2.19 diffformEqu

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform\\_lib\]](#), page 2291).

**Usage:** `df == dg; df, dg diffform`

**Return:** 1, if df and dg are euqal - 0, otherwise

**Example:**

```
LIB "diffform.lib";
ring R = 0,(x,y,z),ds;
diffAlgebra();
⇨ // The differential algebra Omega_R was constructed and the differential \
    forms dDx, dDy, dDz, dx, dy, dz are available.
////////////////////////////////////
// Applications of comparison operator //
////////////////////////////////////
diffform df = 3*dx - x8*dx*dy;
diffform dg = 3 + x8*dy;
df == dg;
⇨ 0
dg = dg*dx;
df == dg;
⇨ 1
kill Omega_R,df,dg,dx,dy,dz;
```

See also: [Section D.15.2.20 \[diffformNeq\]](#), page 2305.

### D.15.2.20 diffformNeq

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform\\_lib\]](#), page 2291).

**Usage:** `df != dg; df, dg diffform`

**Return:** 0, if df and dg are euqal - 1, otherwise

**Example:**

```
LIB "diffform.lib";
ring R = 0,(x,y,z),ds;
diffAlgebra();
⇨ // The differential algebra Omega_R was constructed and the differential \
    forms dDx, dDy, dDz, dx, dy, dz are available.
////////////////////////////////////
// Applications of negated comparison operator //
////////////////////////////////////
diffform df = 3*dx - x8*dx*dy;
diffform dg = 3 + x8*dy;
df != dg;
⇨ 1
dg = dg*dx;
df != dg;
⇨ 0
kill Omega_R,df,dg,dx,dy,dz;
```

See also: [Section D.15.2.19 \[diffformEqu\]](#), page 2305.

### D.15.2.21 diffformIsBigger

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `df > dg` OR `diffformIsBigger(df,dg,#)`; `df,dg` diffform, `#` list

**Return:** - 1, if `df` is bigger than `dg` with respect to the monomial ordering in the differential algebra - 0, otherwise - 1, if `df` is bigger than `dg` with respect to a given monomial ordering on the generators/ringvariables - 0, otherwise

**Note:** the procedure uses `diffAlgebraChangeOrd` to change the order on the differential algebra, therefore an ordering is defined via the following pattern:  
 - `#[i] = 'gen'` defines the ordering on the generators `dx.i` - `#[i+1]` must then be a valid monomial ordering as string - `#[i+2]` an optional weight vector  
 - `#[i] = 'ringvar'` defines the ordering on the ringvariables - `#[i+1]` must then be a valid monomial ordering as string - `#[i+2]` an optional weight vector

**Example:**

```
LIB "diffform.lib";
ring R = 0,(x,y,z),dp;
diffAlgebra();
⇒ // The differential algebra Omega_R was constructed and the differential \
   forms dDx, dDy, dDz, dx, dy, dz are available.
////////////////////////////////////
// With standard ordering //
////////////////////////////////////
dx > dy;
⇒ 1
x37*dy > dz;
⇒ 1
x*dz > y*x*dy;
⇒ 0
x3*dx - y*dx*dz > dx*dy*dz;
⇒ 0
////////////////////////////////////
// With changed ordering //
////////////////////////////////////
diffformIsBigger(dx,dy,"gen","ls");
⇒ 0
diffformIsBigger(x37*dy,dz,"gen","wp",intvec(1,-1,1));
⇒ 0
diffformIsBigger(x*dz,y*x*dy,"gen","wp",intvec(1,-1,1),"ringvar","wp",intvec(1,-1,1));
⇒ 1
diffformIsBigger(x3*dx - y*dx*dz,dx*dy*dz,"gen","wp",intvec(-1,-1,1));
⇒ 1
kill Omega_R,dx,dy,dz;
```

See also: [Section D.15.2.5 \[diffAlgebraChangeOrd\]](#), page 2294; [Section D.15.2.22 \[diffformIsSmaller\]](#), page 2306.

### D.15.2.22 diffformIsSmaller

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `df < dg` OR `diffformIsSmaller(df,dg,#)`; `df,dg` diffform, `#` list



**Return:** - 1, if df is smaller than dg with respect to the monomial ordering in the differential algebra - 0, otherwise - 1, if df is smaller than dg with respect to a given monomial ordering on the generators/ringvariables - 0, otherwise

**Note:** the procedure uses `diffAlgebraChangeOrd` to change the order on the differential algebra, therefore an ordering is defined via the following pattern:  
 - `#[i] = 'gen'` defines the ordering on the generators `dx_i - #[i+1]` must then be a valid monomial ordering as string - `#[i+2]` an optional weight vector  
 - `#[i] = 'ringvar'` defines the ordering on the ringvariables - `#[i+1]` must then be a valid monomial ordering as string - `#[i+2]` an optional weight vector

**Example:**

```
LIB "diffform.lib";
ring R = 0,(x,y,z),lp;
diffAlgebra();
⇒ // The differential algebra Omega_R was constructed and the differential \
    forms dDx, dDy, dDz, dx, dy, dz are available.
////////////////////////////////////
// With standard ordering //
////////////////////////////////////
dz < dy;
⇒ 1
x*dz < y*dz;
⇒ 0
y2*z2*dy < x;
⇒ 0
dx*dz < dy;
⇒ 0
////////////////////////////////////
// With changed ordering //
////////////////////////////////////
diffformIsSmaller(dz,dy,"gen","ls");
⇒ 0
diffformIsSmaller(x*dz,y*dz,"ringvar","ls");
⇒ 1
diffformIsSmaller(y2*z2*dy,x,"gen","wp",intvec(1,-1,1));
⇒ 1
diffformIsSmaller(dx*dz,dy,"gen","ws",intvec(1,-1,-1));
⇒ 1
kill Omega_R,dx,dy,dz;
```

See also: [Section D.15.2.5 \[diffAlgebraChangeOrd\]](#), page 2294; [Section D.15.2.21 \[diffformIsBigger\]](#), page 2306.

### D.15.2.23 diffformDeg

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `deg(df); df diffform`

**Return:** degree of df - degree of the highest generator, -1 if df = 0

**Note:** - the procedure does not check if df is homogeneous  
 - be careful: `diffformDeg` does not cast polynomials to differential forms. So before applying to a polynomial, a type cast should be done

```
LIB "diffform.lib";
ring R = 0,(x,y,z),ds;
diffAlgebra();
--> // The differential algebra Omega_R was constructed and the differential \
      forms dDx, dDy, dDz, dx, dy, dz are available.
//////////
// Homogeneous //
//////////
homog(3*dx*dz - x8*dx*dy);
--> 0
homog(12x*dx + dy - (y4-y5)*dz);
--> 0
//////////
// Not homogeneous //
//////////
homog(3 + x8*dy);
--> 0
```

```

homog(x*dx+dy*dx);
↳ 0
// When applying homog to a polynomial which is considered
// as a differential form, a type cast has to be done first
homog(3x-y2);
↳ 0
diffform df = 3x-y2;
homog(df);
↳ 1
kill Omega_R,dx,dy,dz,df;

```

See also: [Section D.15.2.23 \[diffformDeg\]](#), page 2307; [Section D.15.2.25 \[diffformIsHomogDeg\]](#), page 2309.

### D.15.2.25 diffformIsHomogDeg

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `diffformIsHomogDeg(df,p);` df diffform, p int

**Return:** 1, if df is homogeneous of degree p - 0, otherwise

**Note:** - 0 is homogeneous of degree -1

**Example:**

```

LIB "diffform.lib";
ring R = 0,(x,y,z),ds;
diffAlgebra();
↳ // The differential algebra Omega_R was constructed and the differential \
    forms dDx, dDy, dDz, dx, dy, dz are available.
diffform df = 3*dx*dz - x8*dx*dy;
diffform dg = 3 + x8*dy;
diffform dh = 2;
diffform dt = 0;
////////////////////////////////////
// Homogeneous of given degree //
////////////////////////////////////
diffformIsHomogDeg(df,2);
↳ 0
diffformIsHomogDeg(dh,0);
↳ 1
diffformIsHomogDeg(dt,-1);
↳ 1
////////////////////////////////////
// Not homogeneous of given degree //
////////////////////////////////////
diffformIsHomogDeg(df,1);
↳ 0
diffformIsHomogDeg(dg,1);
↳ 0
diffformIsHomogDeg(dh,1);
↳ 0
kill Omega_R,df,dg,dh,dt,dx,dy,dz;

```

See also: [Section D.15.2.23 \[diffformDeg\]](#), page 2307; [Section D.15.2.24 \[diffformIsHomog\]](#), page 2308.

### D.15.2.26 diffformListCont

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `diffformListCont(L,df)`; L list, df diffform

**Return:** 1, if df is in the list L - 0, otherwise

**Note:** lists with arbitrary input are allowed

**Example:**

```
LIB "diffform.lib";
ring R = 17,(a,b,c),lp;
diffAlgebra();
⇒ // The differential algebra Omega_R was constructed and the differential \
   forms dDa, dDb, dDc, da, db, dc are available.
list L = "",3,12*a,da-db,16 + dc, 23*da - 4*db*dc*da, db - 4, "entry", dc - db*da, a
//////////
// Elements in the list //
//////////
diffformListCont(L,da - db);
⇒ 1
diffformListCont(L,16 + dc);
⇒ 1
diffformListCont(L,dc - db*da);
⇒ 1
//////////
// Elements not in the list //
//////////
diffformListCont(L,22*da);
⇒ 0
diffformListCont(L,1);
⇒ 0
diffformListCont(L,a);
⇒ 0
kill Omega_R,L,da,db,dc;
```

See also: [Section D.15.2.19 \[diffformEqu\]](#), page 2305.

### D.15.2.27 diffformListSort

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Return:** the sorted list L in ascending order, depending on the optional monomial ordering given

**Remarks:** Classical insertion sort is used to sort the list

**Note:**

- the procedure uses `diffformIsBigger` to compare list elements, therefore an optional ordering is defined via the pattern in `diffformIsBigger` - the standard ordering is the ordering on the differential algebra - the procedure can also handle special lists of lists by using the optional input "Dlist" or "Llist":
  - "Dlist" is used for lists with structure:
    - L[1] is a list of differential forms
    - L[2] is a list of polynomials of same size,
 The list gets sorted by the elements in L[1]. This is mainly used for the structure of derivations.
  - "Llist" allows the structure: L[i] is a list with two entries: L[i][1] is a differential form

$L[i][2]$  is a polynomial

The list gets sorted by the elements  $L[i][1]$ . This is used to sort coefficient lists.

**Example:**

```
LIB "diffform.lib";
ring R = 0,(x,y,z,t),dp;
diffAlgebra();
⇨ // The differential algebra Omega_R was constructed and the differential \
    forms dDx, dDy, dDz, dDt, dx, dy, dz, dt are available.
////////////////////////////////////
// Sortation of a list //
////////////////////////////////////
list L = dx*x, x2 - y*t, 12, dt*dy*dx*dz;
// Sort list with standard ordering
diffformListSort(L);
⇨ [1]:
⇨    12
⇨
⇨ [2]:
⇨    x2-yt
⇨
⇨ [3]:
⇨    x*dx
⇨
⇨ [4]:
⇨    dx*dy*dz*dt
⇨
// Sort list with changed ordering
diffformListSort(L,"gen","ls","ringvar","wp",intvec(-1,1,1,1));
⇨ [1]:
⇨    dx*dy*dz*dt
⇨
⇨ [2]:
⇨    x*dx
⇨
⇨ [3]:
⇨    12
⇨
⇨ [4]:
⇨    x2-yt
⇨
////////////////////////////////////
// Sortation of list with structure "Dlist" //
////////////////////////////////////
list DL;
DL[1] = list(dx,x,t,dt); DL[2] = list(y,t*z,4,x);
// This list has the structure described by "Dlist"
diffformListSort(DL,"Dlist","ringvar","ls");
⇨ [1]:
⇨    [1]:
⇨        x
⇨
⇨    [2]:
```

```

↳      t
↳
↳      [3]:
↳      dt
↳
↳      [4]:
↳      dx
↳
↳ [2]:
↳ [1]:
↳      zt
↳ [2]:
↳      4
↳ [3]:
↳      x
↳ [4]:
↳      y
////////////////////////////////////
// Sortation of list with structure "Llist" //
////////////////////////////////////
list LL;
LL[1] = list(dx,x); LL[2] = list(t*dt,y); LL[3] = list(x,2);
// This list has the structure described by "Llist"
diffformListSort(LL,"Llist");
↳ [1]:
↳ [1]:
↳      x
↳
↳ [2]:
↳      2
↳ [2]:
↳ [1]:
↳      t*dt
↳
↳ [2]:
↳      y
↳ [3]:
↳ [1]:
↳      dx
↳
↳ [2]:
↳      x
kill Omega_R,dx,dy,dz,dt,L,DL,LL;

```

See also: [Section D.15.2.21 \[diffformIsBigger\]](#), page 2306; [Section D.15.2.22 \[diffformIsSmaller\]](#), page 2306.

### D.15.2.28 diffformUnivDer

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `diffformUnivDer(f); f poly`

**Return:** a differential form: the image of the universal derivation applied to `f`

**Example:**

```

LIB "diffform.lib";
ring R = 0,(x,y,z),lp;
diffAlgebra();
⇨ // The differential algebra Omega_R was constructed and the differential \
    forms dDx, dDy, dDz, dx, dy, dz are available.
////////////////////////////////////
// Universal derivation applied to polynomials //
////////////////////////////////////
diffformUnivDer(3x);
⇨ 3*dx
⇨
diffformUnivDer(xyz+x2y2z2);
⇨ (2xy2z2+yz)*dx+(2x2yz2+xz)*dy+(2x2y2z+xy)*dz
⇨
diffformUnivDer(x+y+z);
⇨ dx+dy+dz
⇨
kill Omega_R,dx,dy,dz;

```

See also: [Section D.15.2.29 \[diffformDiff\]](#), page 2313.

### D.15.2.29 diffformDiff

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `diffformDiff(df); df diffform`

**Return:** the image of `df` under the differential

**Remark:** To compute the image, the universal derivation is applied to each coefficient and multiplied with the corresponding generator

**Note:** - the differential is a map  $\Omega_R^p \rightarrow \Omega_R^{p+1}$  and this procedure applies the differential to all homogeneous parts of `df`  
 - this procedure can also be applied to polynomials - in this case it is just the universal derivation

**Example:**

```

LIB "diffform.lib";
ring R = 0,(x,y,z,a,b,c),lp;
diffAlgebra();
⇨ // The differential algebra Omega_R was constructed and the differential \
    forms dDx, dDy, dDz, dDa, dDb, dDc, dx, dy, dz, da, db, dc are available.
////////////////////////////////////
// Construction of differential forms //
////////////////////////////////////
diffform df_1 = y*dx + z*dy + x*dz + a*db + b*dc + c*da;
diffform df_2 = -5*c4*dc*dz*dy + 3*dx*dz - 13*a4*da*db + 12*a4*da*db + x8*dx*dy + 12 -
poly f = 3x2y2 - z3*c;
////////////////////////////////////
// Differential applied to the forms //
////////////////////////////////////
diffformDiff(df_1);
⇨ (-1)*dx*dy+dx*dz+(-1)*dy*dz+da*db+(-1)*da*dc+db*dc
⇨
diffformDiff(df_2);

```

```

    ↪ (-2x)*dx*dy*db+(5x12y4-4x12y3)*dx*dy*dz*db
    ↪
    diffformDiff(f);
    ↪ 6xy2*dx+6x2y*dy+(-3z2c)*dz+(-z3)*dc
    ↪
    // The composition of differentials is the zero-map:
    diffformDiff(diffformDiff(df_1));
    ↪ 0
    ↪
    kill Omega_R,df_1,df_2,f,dx,dy,dz,da,db,dc;

```

See also: [Section D.15.2.28 \[diffformUnivDer\]](#), page 2312.

### D.15.2.30 derivationFromList

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:**      `derivation phi = derivationFromList(L);` L list

**Return:**    the derivation defined by the list L

**Remarks:** The structure of L is checked and L is sorted,  
then it is set as structure list of phi

**Note:**      the structure of L must follow the rules:  
- L[1] is a list of all degree-1 generators: all  $dx_i$  must occur once and no other differential forms are allowed. The order of the list is not important - L[2] is the list of images of the  $dx_i$ : these must be polynomials Since the map is linear, it is enough to store the images of the  $dx_i$

**Example:**

```

LIB "diffform.lib";
ring R = 11,(u,v,w,x),lp;
diffAlgebra();
↪ // The differential algebra Omega_R was constructed and the differential \
    forms dDu, dDv, dDw, dDx, du, dv, dw, dx are available.
////////////////////////////////////
// Construction of structure lists //
////////////////////////////////////
list L_1;
L_1[1] = list(du,dv,dw,dx);
L_1[2] = list(u,v,w,x);
list L_2;
L_2[1] = list(dx,dw,du,dv);
L_2[2] = list(x2,w2,u2,v-wu);
////////////////////////////////////
// Construction of derivations //
////////////////////////////////////
derivation phi = derivationFromList(L_1); phi;
↪ Omega_R^1 --> R
↪      du |--> u
↪      dv |--> v
↪      dw |--> w
↪      dx |--> x
↪
↪

```



```

derivation psi = derivationFromList(L_2); psi;
↳ Omega_R^1 --> R
↳ du |--> u2
↳ dv |--> -uw+v
↳ dw |--> w2
↳ dx |--> x2
↳
↳
kill Omega_R,du,dv,dw,dx,phi,psi,L_1,L_2;

```

See also: [Section D.15.2.31 \[derivationCheckList\]](#), page 2315; [Section D.15.2.33 \[derivationConstructor\]](#), page 2316; [Section D.15.2.32 \[derivationFromPoly\]](#), page 2315.

### D.15.2.31 derivationCheckList

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `derivationCheckList(L)`; L list

**Remarks:** The procedure checks if a given list has the right form for a derivation and throws an error if this is not the case. In particular: - Only degree-1 generators are allowed in L[1] - this is checked via `diffformIsGen` - Any degree-1 generator must occur once - this is checked via `diffformListCont`

**Note:** like in `derivationFromList`, the structure of L must follow the rules: - L[1] is a list of all degree-1 generators: all `dx.i` must occur once and no other differential forms are allowed. The order of the list is not important - L[2] is the list of images of the `dx.i`: these must be polynomials

See also: [Section D.15.2.30 \[derivationFromList\]](#), page 2314; [Section D.15.2.13 \[diffformIsGen\]](#), page 2301; [Section D.15.2.26 \[diffformListCont\]](#), page 2310.

### D.15.2.32 derivationFromPoly

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `derivation phi = derivationFromPoly(f)`; f poly

**Return:** a derivation which maps any degree-1 generator to f

**Remarks:** The degree-1 generators are returned by `diffAlgebraListGen`

**Note:** the procedure allows to interpret polynomials as derivations

**Example:**

```

LIB "diffform.lib";
ring R = 0,(x,y,z),lp;
diffAlgebra();
↳ // The differential algebra Omega_R was constructed and the differential \
   forms dDx, dDy, dDz, dx, dy, dz are available.
////////////////////////////////////
// Construction of derivations from polynomials //
////////////////////////////////////
derivation phi = derivationFromPoly(3x*y - 12*y4-z2); phi;
↳ Omega_R^1 --> R
↳ dx |--> 3xy-12y4-z2
↳ dy |--> 3xy-12y4-z2

```

```

    ↪          dz |--> 3xy-12y4-z2
    ↪
    ↪
    ↪          derivation psi = derivationFromPoly(0); psi;
    ↪      Omega_R^1 --> R
    ↪          dx |--> 0
    ↪          dy |--> 0
    ↪          dz |--> 0
    ↪
    ↪
    ↪      kill Omega_R,dx,dy,dz,phi,psi;

```

See also: [Section D.15.2.33 \[derivationConstructor\]](#), page 2316; [Section D.15.2.30 \[derivationFromList\]](#), page 2314.

### D.15.2.33 derivationConstructor

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:**      `derivation phi = inp;` `inp` of any type

**Return:**    the derivation defined by `inp`:

**Remarks:** the output depends on the type of `inp`:  
 - if `inp` is of type list, the constructor `derivationFromList` is used - if `inp` is of type poly,  
 number, int or bigint, `derivationFromPoly` is used

**Note:**      for other than the mentioned types, there is no output

**Example:**

```

LIB "diffform.lib";
ring R = 31,(x,y,z),dp;
diffAlgebra();
    ↪ // The differential algebra Omega_R was constructed and the differential \
    ↪      forms dDx, dDy, dDz, dx, dy, dz are available.
    ↪ //////////////////////////////////////
    ↪ // Construction of derivations from lists and polynomials //
    ↪ //////////////////////////////////////
    ↪ list L; L[1] = list(dx,dz,dy); L[2] = list(x2,y-x,z);
    ↪ derivation phi = L; phi;
    ↪      Omega_R^1 --> R
    ↪          dx |--> x2
    ↪          dy |--> z
    ↪          dz |--> -x+y
    ↪
    ↪
    ↪      derivation psi = 3x2-12z; psi;
    ↪      Omega_R^1 --> R
    ↪          dx |--> 3x2-12z
    ↪          dy |--> 3x2-12z
    ↪          dz |--> 3x2-12z
    ↪
    ↪
    ↪      kill Omega_R,dx,dy,dz,phi,psi;

```

See also: [Section D.15.2.30 \[derivationFromList\]](#), page 2314; [Section D.15.2.32 \[derivationFromPoly\]](#), page 2315.

### D.15.2.34 derivationToString

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `derivationToString(phi,#);` phi derivation, # list

**Return:** the derivation as a string, describing the image of the degree-1 generators, optionally ordered by a given monomial ordering on the generators

**Remarks:** To sort the images of the generators in the output string, `diffformListSort` is used.

**Note:** to define an ordering for the generators, one can use:

- `#[1] = 'gen'`
  - `#[2]`: a monomial ordering as string
  - `#[3]`: an optional weight vector
- the standard ordering is the ordering of the differential algebra

**Example:**

```
LIB "diffform.lib";
ring R = 0,(x,y,z),dp;
diffAlgebra();
⇨ // The differential algebra Omega_R was constructed and the differential \
    forms dDx, dDy, dDz, dx, dy, dz are available.
list L; L[1] = list(dx,dy,dz); L[2] = list(x2,y-23xz,xz4);
derivation phi = L;
////////////////////////////////////
// String with standard ordering //
////////////////////////////////////
print(derivationToString(phi));
⇨ Omega_R^1 --> R
⇨      dz |--> xz4
⇨      dy |--> -23xz+y
⇨      dx |--> x2
⇨
////////////////////////////////////
// String with changed ordering //
////////////////////////////////////
print(derivationToString(phi,"gen","wp",intvec(-1,-1,1)));
⇨ Omega_R^1 --> R
⇨      dy |--> -23xz+y
⇨      dx |--> x2
⇨      dz |--> xz4
⇨
kill Omega_R,dx,dy,dz,L,phi;
```

See also: [Section D.15.2.35 \[derivationPrint\]](#), page 2317; [Section D.15.2.9 \[diffformGenToString\]](#), page 2299; [Section D.15.2.27 \[diffformListSort\]](#), page 2310.

### D.15.2.35 derivationPrint

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `phi;` phi derivation

**Side effects:**  
prints the given derivation

**Remarks:** Prints the string returned by `derivationToString` with a ls-ordering on the generators

**Example:**

```
LIB "diffform.lib";
ring R = 0,(a,b,c,x,y,z),lp;
diffAlgebra();
⇨ // The differential algebra Omega_R was constructed and the differential \
    forms dDa, dDb, dDc, dDx, dDy, dDz, da, db, dc, dx, dy, dz are available.
//////////
// Derivations //
//////////
list L; L[1] = list(dx,dy,dz,da,db,dc); L[2] = list(1,12x-y,z4aby, 2*b5x,0,xyz-abc);
derivation phi = L;
derivation phi_poly = 3ab - c2*x + z;
//////////
// Applications of Print //
//////////
phi;
⇨ Omega_R^1 --> R
⇨      da |--> 2b5x
⇨      db |--> 0
⇨      dc |--> -abc+xyz
⇨      dx |--> 1
⇨      dy |--> 12x-y
⇨      dz |--> aby4
⇨
⇨
phi_poly;
⇨ Omega_R^1 --> R
⇨      da |--> 3ab-c2x+z
⇨      db |--> 3ab-c2x+z
⇨      dc |--> 3ab-c2x+z
⇨      dx |--> 3ab-c2x+z
⇨      dy |--> 3ab-c2x+z
⇨      dz |--> 3ab-c2x+z
⇨
⇨
kill Omega_R,da,db,dc,dx,dy,dz,L,phi,phi_poly;
```

See also: [Section D.15.2.34 \[derivationToString\]](#), page 2317.

### D.15.2.36 derivationAdd

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `phi+psi; phi,psi derivation`

**Return:** the sum of the given derivations

**Remark:** The sum is computed componentwise - this works since the structure lists of derivations are sorted the same way.

**Note:** once can also add polynomials and derivations

**Example:**

```

LIB "diffform.lib";
ring R = 0,(x,y,z),ds;
diffAlgebra();
⇨ // The differential algebra Omega_R was constructed and the differential \
    forms dDx, dDy, dDz, dx, dy, dz are available.
list L_1; L_1[1] = list(dx,dy,dz); L_1[2] = list(2x,2y,2z);
list L_2; L_2[1] = list(dx,dy,dz); L_2[2] = list(y2-x,z4+x+y,y2);
//////////
// Derivations //
//////////
derivation phi_1 = L_1; phi_1;
⇨ Omega_R^1 --> R
⇨      dx |--> 2x
⇨      dy |--> 2y
⇨      dz |--> 2z
⇨
⇨
derivation phi_2 = L_2; phi_2;
⇨ Omega_R^1 --> R
⇨      dx |--> -x+y2
⇨      dy |--> x+y+z4
⇨      dz |--> y2
⇨
⇨
//////////
// Sum of derivations //
//////////
phi_1 + phi_2;
⇨ Omega_R^1 --> R
⇨      dx |--> x+y2
⇨      dy |--> x+3y+z4
⇨      dz |--> 2z+y2
⇨
⇨
phi_1 + phi_2 + phi_2;
⇨ Omega_R^1 --> R
⇨      dx |--> 2y2
⇨      dy |--> 2x+4y+2z4
⇨      dz |--> 2z+2y2
⇨
⇨
phi_1 + phi_2 + 3x2;
⇨ Omega_R^1 --> R
⇨      dx |--> x+3x2+y2
⇨      dy |--> x+3y+3x2+z4
⇨      dz |--> 2z+3x2+y2
⇨
⇨
kill Omega_R,dx,dy,dz,L_1,L_2,phi_1,phi_2;

```

See also: [Section D.15.2.37 \[derivationSub\]](#), page 2320.

**D.15.2.37 derivationSub**

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:**      `phi-psi; phi,psi derivation`

**Return:**    the difference of the given derivations

**Remarks:** The difference is computed componentwise - this works since the structure lists of derivations are sorted the same way.

**Note:**      one can also subtract polynomials from derivations

**Example:**

```
LIB "diffform.lib";
ring R = 0,(x,y),lp;
diffAlgebra();
⇨ // The differential algebra Omega_R was constructed and the differential \
    forms dDx, dDy, dx, dy are available.
list L_1; L_1[1] = list(dx,dy); L_1[2] = list(x+y,1);
list L_2; L_2[1] = list(dy,dx); L_2[2] = list(x,y2);
//////////
// Derivations //
//////////
derivation phi_1 = L_1; phi_1;
⇨ Omega_R^1 --> R
⇨      dx |--> x+y
⇨      dy |--> 1
⇨
⇨
derivation phi_2 = L_2; phi_2;
⇨ Omega_R^1 --> R
⇨      dx |--> y2
⇨      dy |--> x
⇨
⇨
//////////
// Difference of derivations //
//////////
phi_1-phi_2;
⇨ Omega_R^1 --> R
⇨      dx |--> x-y2+y
⇨      dy |--> -x+1
⇨
⇨
phi_1-phi_2-phi_1;
⇨ Omega_R^1 --> R
⇨      dx |--> -y2
⇨      dy |--> -x
⇨
⇨
phi_1 - (x+y);
⇨ Omega_R^1 --> R
⇨      dx |--> 0
⇨      dy |--> -x-y+1
```

```

↳
↳
kill Omega_R,dx,dy,L_1,L_2,phi_1,phi_2;

```

See also: [Section D.15.2.36 \[derivationAdd\]](#), page 2318; [Section D.15.2.38 \[derivationNeg\]](#), page 2321.

### D.15.2.38 derivationNeg

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:**      `-phi; phi derivation`

**Return:**     the negation of a given derivation

**Example:**

```

LIB "diffform.lib";
ring R = 0,(x,y,z,t),dp;
diffAlgebra();
↳ // The differential algebra Omega_R was constructed and the differential \
   forms dDx, dDy, dDz, dDt, dx, dy, dz, dt are available.
list L_1; L_1[1] = list(dy,dx,dt,dz); L_1[2] = list(x2-y,23y+t,tz4,z-y);
////////////////////////////////
// Derivations //
////////////////////////////////
derivation phi_1 = L_1; phi_1;
↳ Omega_R^1 --> R
↳      dx |--> 23y+t
↳      dy |--> x2-y
↳      dz |--> -y+z
↳      dt |--> z4t
↳
↳
derivation phi_poly = 3xyz; phi_poly;
↳ Omega_R^1 --> R
↳      dx |--> 3xyz
↳      dy |--> 3xyz
↳      dz |--> 3xyz
↳      dt |--> 3xyz
↳
↳
////////////////////////////////
// Negation of derivations //
////////////////////////////////
-phi_1;
↳ Omega_R^1 --> R
↳      dx |--> -23y-t
↳      dy |--> -x2+y
↳      dz |--> y-z
↳      dt |--> -z4t
↳
↳
-(-phi_1);
↳ Omega_R^1 --> R
↳      dx |--> 23y+t

```

```

    ↪      dy |--> x2-y
    ↪      dz |--> -y+z
    ↪      dt |--> z4t
    ↪
    ↪
    ↪ -(phi_poly);
    ↪ Omega_R^1 --> R
    ↪      dx |--> -3xyz
    ↪      dy |--> -3xyz
    ↪      dz |--> -3xyz
    ↪      dt |--> -3xyz
    ↪
    ↪
    kill Omega_R,dx,dy,dz,dt,L_1,phi_1,phi_poly;

```

See also: [Section D.15.2.37 \[derivationSub\]](#), page 2320.

### D.15.2.39 derivationMul

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:**     `phi*psi; phi,psi derivation`

**Return:**    the componentwise product of `phi` and `psi`

**Remarks:** The product is computed componentwise - this works since the structure lists of derivations are sorted the same way.

**Note:**       one can also multiply polynomials and derivations

**Example:**

```

LIB "diffform.lib";
ring R = 0,(a,b,t),ls;
diffAlgebra();
↪ // The differential algebra Omega_R was constructed and the differential \
   forms dDa, dDb, dDt, da, db, dt are available.
list L_1; L_1[1] = list(da,dt,db); L_1[2] = list(2a,2t-b,2t);
list L_2; L_2[1] = list(dt,db,da); L_2[2] = list(-a,-b,-t);
//////////
// Derivations //
//////////
derivation phi_1 = L_1; phi_1;
↪ Omega_R^1 --> R
↪      da |--> 2a
↪      db |--> 2t
↪      dt |--> 2t-b
↪
↪
derivation phi_2 = L_2; phi_2;
↪ Omega_R^1 --> R
↪      da |--> -t
↪      db |--> -b
↪      dt |--> -a
↪
↪
//////////

```



```

// Multiplication of derivations //
////////////////////////////////////
phi_1*phi_2;
  ↳ Omega_R^1 --> R
  ↳      da |--> -2at
  ↳      db |--> -2bt
  ↳      dt |--> -2at+ab
  ↳
  ↳
phi_1*phi_2*phi_2;
  ↳ Omega_R^1 --> R
  ↳      da |--> 2at2
  ↳      db |--> 2b2t
  ↳      dt |--> 2a2t-a2b
  ↳
  ↳
phi_2*(3a2-bt);
  ↳ Omega_R^1 --> R
  ↳      da |--> bt2-3a2t
  ↳      db |--> b2t-3a2b
  ↳      dt |--> abt-3a3
  ↳
  ↳
kill Omega_R,da,db,dt,L_1,L_2,phi_1,phi_2;

```

#### D.15.2.40 derivationEqu

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:**     `phi == psi; phi,psi derivation`

**Return:**    1, if phi and psi are equal - 0, otherwise

**Remarks:** The images of the generators are compared componentwise - this works since the structure lists of derivations are sorted the same way.

**Note:**       derivations can also be compared to polynomials

**Example:**

```

LIB "diffform.lib";
ring R = 0,(u,v),lp;
diffAlgebra();
  ↳ // The differential algebra Omega_R was constructed and the differential \
      forms dDu, dDv, du, dv are available.
list L_1; L_1[1] = list(dv,du); L_1[2] = list(u,-v);
////////////////////////////////////
// Derivations //
////////////////////////////////////
derivation phi_1 = L_1; phi_1;
  ↳ Omega_R^1 --> R
  ↳      du |--> -v
  ↳      dv |--> u
  ↳
  ↳
derivation phi_poly = u*v; phi_poly;

```

```

↳ Omega_R^1 --> R
↳      du |--> uv
↳      dv |--> uv
↳
↳
////////////////////////////////
// Comparison of derivations //
////////////////////////////////
phi_1 == phi_1;
↳ 1
phi_1 == phi_poly;
↳ 0
phi_poly == u*v;
↳ 1
kill Omega_R,du,dv,phi_1,phi_poly;

```

See also: [Section D.15.2.41 \[derivationNeq\]](#), page 2324.

### D.15.2.41 derivationNeq

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:**     `phi != psi; phi,psi derivation`

**Return:**    0, if phi and psi are equal - 1, otherwise

**Remarks:** The comparison is done by `diffformEqu`

**Note:**      derivations can also be compared to polynomials

**Example:**

```

LIB "diffform.lib";
ring R = 0,(u,v),lp;
diffAlgebra();
↳ // The differential algebra Omega_R was constructed and the differential \
   forms dDu, dDv, du, dv are available.
list L_1; L_1[1] = list(dv,du); L_1[2] = list(u,-v);
////////////////////////////////
// Derivations //
////////////////////////////////
derivation phi_1 = L_1; phi_1;
↳ Omega_R^1 --> R
↳      du |--> -v
↳      dv |--> u
↳
↳
derivation phi_poly = u*v; phi_poly;
↳ Omega_R^1 --> R
↳      du |--> uv
↳      dv |--> uv
↳
↳
////////////////////////////////
// Comparison of derivations //
////////////////////////////////
phi_1 != phi_1;

```

```

    ↪ 0
    phi_1 != phi_poly;
    ↪ 1
    phi_poly != u*v;
    ↪ 0
    kill Omega_R,du,dv,phi_1,phi_poly;

```

See also: [Section D.15.2.40 \[derivationEqu\]](#), page 2323.

### D.15.2.42 derivationEval

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:**     `phi(df);` phi derivation, df diffform

**Return:**    the polynomial `phi(df)`, the derivation phi evaluated at df

**Remarks:** - By linearity it is enough to compute the sum of all differential forms: (coefficient of `dx_i`)\*(image of `dx_i`)  
 - The coefficient list of df is computed via `diffformCoef` - To avoid searching generators in lists, the coefficient list of df and the structure list of phi are sorted the same way

**Note:**     - the differential form 0 is allowed as input  
 - an error will occur if the given differential form is not of degree 1 or -1

**Example:**

```

LIB "diffform.lib";
ring R = 13,(x,y,z,t),dp;
diffAlgebra();
↪ // The differential algebra Omega_R was constructed and the differential \
    forms dDx, dDy, dDz, dDt, dx, dy, dz, dt are available.
////////////////////////////////////
// Construction of derivations //
////////////////////////////////////
list L_1; L_1[1] = list(dx,dt,dz,dy); L_1[2] = list(x,y,z,t);
derivation phi_1 = L_1; phi_1;
↪ Omega_R^1 --> R
↪      dx |--> x
↪      dy |--> t
↪      dz |--> z
↪      dt |--> y
↪
↪
list L_2; L_2[1] = list(dx,dy,dz,dt); L_2[2] = list(y^2x-zt,zt + y, t^3-x, y^4-y^5);
derivation phi_2 = L_2; phi_2;
↪ Omega_R^1 --> R
↪      dx |--> xy^2-zt
↪      dy |--> zt+y
↪      dz |--> t^3-x
↪      dt |--> -y^5+y^4
↪
↪
list L_3; L_3[1] = list(dx,dy,dz,dt); L_3[2] = list(0,0,0,0);
derivation phi_3 = L_3; phi_3;
↪ Omega_R^1 --> R
↪      dx |--> 0

```

```

↳      dy |--> 0
↳      dz |--> 0
↳      dt |--> 0
↳
↳
////////////////////////////////
// Evaluation of derivations //
////////////////////////////////
phi_1(0);
↳ 0
phi_1(dx+dy+dz+dt);
↳      ? Cannot apply derivation to non-homogeneous element!
↳      ? leaving diffform.lib::derivationEval (0)
phi_1(3*dx - dt);
↳      ? Cannot apply derivation to non-homogeneous element!
↳      ? leaving diffform.lib::derivationEval (0)
phi_2(dt);
↳ -y5+y4
phi_2(dx+dt);
↳      ? Cannot apply derivation to non-homogeneous element!
↳      ? leaving diffform.lib::derivationEval (0)
phi_2(dx - dy + (x3-y2)*dz + 12*dt);
↳      ? Cannot apply derivation to non-homogeneous element!
↳      ? leaving diffform.lib::derivationEval (0)
phi_3(dx);
↳ 0
phi_3(dy);
↳ 0
phi_3(dx - 24*(dx + dz) - x4*dy);
↳ 0
kill Omega_R,dx,dy,dz,dt,L_1,L_2,L_3,phi_1,phi_2,phi_3;

```

See also: [Section D.15.2.8 \[diffformCoef\]](#), page 2296; [Section D.15.2.27 \[diffformListSort\]](#), page 2310.

### D.15.2.43 derivationContractionGen

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `derivationContractionGen(phi,d_gen);` phi derivation, d\_gen diffform

**Assume:** d\_gen is a generator of the differential algebra

**Return:** the image of d\_gen under the contraction map `i_phi`

**Remarks:** The formula for the contraction map applied to a generator of degree `l` is given by:  
 $i\_phi^l(dx\_k*...*dx\_j) = \sum(i=1,..l)(-1)^{i+1} * phi(dx\_i) * (dx\_k*...*dx\_j / dx\_i)$

**Note:** this procedure should only be applied to generators

See also: [Section D.15.2.44 \[derivationContraction\]](#), page 2326.

### D.15.2.44 derivationContraction

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `derivationContraction(phi,df);` phi derivation, df diffform

**Return:** the image of the contraction map `i_phi` applied to df

**Remarks:** Since the contraction map is linear, it is only applied to the generators: So the image of  $df$  under  $i_\phi$  is a sum, where the coefficients are multiplied by the image of the generators.

**Note:** over the basering, the contraction map is the 0-map

**Example:**

```
LIB "diffform.lib";
ring R = 0,(x,y,z),lp;
diffAlgebra();
⇒ // The differential algebra Omega_R was constructed and the differential \
    forms dDx, dDy, dDz, dx, dy, dz are available.
////////////////////////////////////
// Construction of derivations //
////////////////////////////////////
list L_1; L_1[1] = list(dx,dy,dz); L_1[2] = list(x,y,z);
derivation phi_1 = L_1; phi_1;
⇒ Omega_R^1 --> R
⇒      dx |--> x
⇒      dy |--> y
⇒      dz |--> z
⇒
⇒
list L_2; L_2[1] = list(dx,dy,dz); L_2[2] = list(y-x,z-y,x-z);
derivation phi_2 = L_2; phi_2;
⇒ Omega_R^1 --> R
⇒      dx |--> -x+y
⇒      dy |--> -y+z
⇒      dz |--> x-z
⇒
⇒
////////////////////////////////////
// Contractions of derivations //
////////////////////////////////////
derivationContraction(phi_1,dx+dy+dz);
⇒ x+y+z
⇒
derivationContraction(phi_1,x2*y4-z);
⇒ 0
⇒
derivationContraction(phi_1,x2*dx*dy + dx*dy*dz);
⇒ (-x2y)*dx+(-x3)*dy+z*dx*dy+y*dx*dz+x*dy*dz
⇒
derivationContraction(phi_2,dx+dy+dz);
⇒ 0
⇒
derivationContraction(phi_2,dx*dy*dz - dx*dy + dx*dz);
⇒ (-x-y+2z)*dx+(-x+y)*dy+(x-y)*dz+(x-z)*dx*dy+(-y+z)*dx*dz+(-x+y)*dy*dz
⇒
kill Omega_R,dx,dy,dz,L_1,L_2,phi_1,phi_2;
```

See also: [Section D.15.2.44 \[derivationContraction\]](#), page 2326; [Section D.15.2.45 \[derivationLie\]](#), page 2328.

### D.15.2.45 derivationLie

Procedure from library `diffform.lib` (see [Section D.15.2 \[diffform.lib\]](#), page 2291).

**Usage:** `diff(phi,df);` phi derivation, df diffform

**Return:** the image of df under the Lie-derivative  $L_{\text{phi}}$

**Remarks:** The map  $L_{\text{phi}}$  is the anticommutator of the contraction map  $i_{\text{phi}}$  and the differential  $d$ :  

$$(i_{\text{phi}} \circ d) + (d \circ i_{\text{phi}})$$

**Example:**

```
LIB "diffform.lib";
ring R = 0,(x,y,z),lp;
diffAlgebra();
⇨ // The differential algebra Omega_R was constructed and the differential \
    forms dDx, dDy, dDz, dx, dy, dz are available.
////////////////////////////////////
// Construction of derivations //
////////////////////////////////////
list L; L[1] = list(dx,dy,dz); L[2] = list(x2,y2,z2);
derivation phi = L; phi;
⇨ Omega_R^1 --> R
⇨      dx |--> x2
⇨      dy |--> y2
⇨      dz |--> z2
⇨
⇨
derivation phi_poly = x-y;
////////////////////////////////////
// Lie-derivative of derivations //
////////////////////////////////////
diff(phi,dx);
⇨ 2x*dx
⇨
diff(phi,dx*dy);
⇨ (-2x+2y)*dx*dy
⇨
diff(phi,dx*dy*dz);
⇨ (2x-2y+2z)*dx*dy*dz
⇨
diff(phi,dx*dy + dy*dx);
⇨ 0
⇨
diff(phi,dx*dy - dy*dx);
⇨ (-4x+4y)*dx*dy
⇨
diff(phi_poly,dx);
⇨ dx+(-1)*dy
⇨
diff(phi_poly,dx-dy);
⇨ 0
⇨
diff(phi_poly,dx+dy);
```

```

    ↪ 2*dx+(-2)*dy
    ↪
    diff(phi_poly,dx*(x2-y4) + 1);
    ↪ (3x2-4xy3-2xy+3y4)*dx+(-x2-8xy3+9y4)*dy
    ↪
    kill Omega_R,dx,dy,dz,L,phi,phi_poly;

```

See also: [Section D.15.2.44 \[derivationContraction\]](#), page 2326; [Section D.15.2.29 \[diffformDiff\]](#), page 2313.

### D.15.3 enumpoints.lib

**Library:** enumpoints.lib

**Purpose:** enumerating rational points

**Author:** Jieao Song (8d1h)

**Procedures:**

#### D.15.3.1 points

Procedure from library `enumpoints.lib` (see [Section D.15.3 \[enumpoints.lib\]](#), page 2329).

**Usage:** points(I); I homogeneous ideal

**Return:** list of coordinates of points on the zero set of I

**Assumes:**  $\dim(I)=0$

#### D.15.3.2 projPoints

Procedure from library `enumpoints.lib` (see [Section D.15.3 \[enumpoints.lib\]](#), page 2329).

**Return:** list of coordinates of points on the zero set of I

**Assumes:**  $\dim(I)=0$

**Example:**

```

LIB "enumpoints.lib";
// The set of nodes is a 0-dimensional variety over a number field.
ring R2 = (0,q),(x,y,z,w),dp;
minpoly = q16-q12+q8-q4+1;
poly s = 2q13-q9-q7+q5-q3-q;
ideal Togliatti = 64*(x-w)*(x^4 -4*x^3*w -10*x^2*y^2 -4*x^2*w^2 +16*x*w^3 -20*x*y^2*
matrix Jac = jacob(Togliatti);
ideal I2 = Togliatti+Jac;
list L=projPoints(std(I2));
L[1];
↪ [1]:
↪ 1
↪ [2]:
↪ 0
↪ [3]:
↪ (1/2q13-1/4q9-1/4q7+1/4q5-1/4q3-1/4q)
↪ [4]:
↪ (-q12+q8+3/2)
size(L);
↪ 31

```

### D.15.4 finitediff.lib

**Issues:**

- installation of qepcadfilter.pl needs to be solved
- tests for (nearly) all procedures are missing
- global variables needs to be cleaned
- temporary files needs to be cleaned
- temporary file names need to be unique (think about multiple instances)
- pollution of global Top namespace must be solved
- u is not a good name for a procedure

**Library:** finitediff.lib

**Purpose:** procedures to compute finite difference schemes for linear differential equations

**Author:** Christian Dingler

**Overview:** Using qepcad/qepcadsystem from this library requires the program qepcad to be installed. You can download qepcad from <http://www.usna.edu/CS/qepcadweb/B/QEPCAD.html>

**Procedures:**

#### D.15.4.1 visualize

Procedure from library `finitediff.lib` (see [Section D.15.4 \[finitediff.lib\]](#), page 2330).

**Usage:** visualize(f); f of type poly.

**Return:** type string; translates the polynomial form of a finite difference scheme into an indexed one as often seen in literature

**Example:**

```
LIB "finitediff.lib";
list D="Ux","Ut","U";
list P="a";
list V="t","x";
setinitials(V,D,P);
scheme(u(Ut)+a*u(Ux),trapezoid(Ux,U,x),backward(Ut,U,t));
 $\mapsto (dx)*x+(dx)$ 
visualize(_);
 $\mapsto (dx)*U(0,1)+(dx)*U(0,0)$ 
```

#### D.15.4.2 u

Procedure from library `finitediff.lib` (see [Section D.15.4 \[finitediff.lib\]](#), page 2330).

**Usage:** u(D[,#]); D a string that occurs in the list of @derivatives, # an optional list of integers.

**Return:** type vector; gives the vector, that corresponds with  $\text{gen}(n)^*m$ , where m is the monomial defined by #

**Example:**



```

LIB "finitediff.lib";
list D="Ux","Uy","Ut","U";
list P="a","b";
list V="t","x","y";
setinitials(V,D,P);
u(Ux);
 $\mapsto [1]$ 
u(Ux,2,3,7);
 $\mapsto [t^2x^3y^7]$ 
u(Uy)+u(Ut)-u(Ux);
 $\mapsto [-1,1,1]$ 
u(U)*234-dx*dt*dy*3*u(Uy);
 $\mapsto [0,(-3*dt*dx*dy),0,234]$ 

```

### D.15.4.3 scheme

Procedure from library `finitediff.lib` (see [Section D.15.4 \[finitediff.lib\]](#), page 2330).

**Usage:** `scheme([v1,...,vn]);`  $v_1, \dots, v_n$  of type vector

**Return:** poly

**Purpose:** performs substitutions by the means of Groebner basis computation of the submodule, generated by the input vectors, then intersects the intermediate result with the suitable component in order to get a finite difference scheme

**Note:** works only for a single PDE, for the case of a system use `matrixsystem`

**Example:**

```

LIB "finitediff.lib";
list D="Ux","Ut","U";
list P="a";
list V="t","x";
setinitials(V,D,P);
def s1=scheme(u(Ut)+a*u(Ux),backward(Ux,U,x),forward(Ut,U,t));
s1;
 $\mapsto (-a*dt+dx)/(dx)*x+(a*dt)/(dx)$ 

```

### D.15.4.4 laxfrT

Procedure from library `finitediff.lib` (see [Section D.15.4 \[finitediff.lib\]](#), page 2330).

**Usage:** `laxfrT(U1,U2,var);`  $U_1, U_2$  are the names of occurring derivatives, `var` is a variable in the basering;

**Return:** type vector; gives a predefined approximation of the Lax-Friedrich-approximation for the derivation in the timevariable as often used in literature;

**Note:** see also `laxfrX`, `setinitials`, `scheme`; Warning: `laxfrT` is not to be interchanged with `laxfrX`

**Example:**

```

LIB "finitediff.lib";
list D="Ux","Ut","U";
list P="a";
list V="t","x";
setinitials(V,D,P);

```

```
laxfrT(Ux,U,x);
↦ [(dt)*x,0,-t*x+1/2*x^2+1/2]
```

#### D.15.4.5 laxfrX

Procedure from library `finitediff.lib` (see [Section D.15.4 \[finitediff.lib\], page 2330](#)).

**Usage:** `laxfrX(U1,U2,var)`; U1, U2 are the names of occuring derivatives, var is a variable in the basering;

**Return:** type vector; gives a predefined approximation of the Lax-Friedrich-approximation for the derivation in one of the spatial variables as often used in literature;

**Note:** see also `laxfrT`, `setinitials`, `scheme`; Warning: `laxfrX` is not to be interchanged with `laxfrT`

**Example:**

```
LIB "finitediff.lib";
list D="Ux","Ut","U";
list P="a";
list V="t","x";
setinitials(V,D,P);
laxfrX(Ux,U,x);
↦ [(2*dx)*x,0,-x^2+1]
```

#### D.15.4.6 forward

Procedure from library `finitediff.lib` (see [Section D.15.4 \[finitediff.lib\], page 2330](#)).

**Usage:** `forward(U1,U2,var)`; U1, U2 are the names of occuring derivatives, var is a variable in the basering;

**Return:** type vector; gives a predefined approximation of the forward approximation as often used in literature;

**Note:** see also `laxfrT`, `setinitials`, `scheme`;

**Example:**

```
LIB "finitediff.lib";
list D="Ut","Ux","Uy","U";
list V="t","x","y";
list P="a","b";
setinitials(V,D,P);
forward(Ux,U,x);
↦ [0,(dx),0,-x+1]
forward(Uy,U,y);
↦ [0,0,(dy),-y+1]
forward(Ut,U,t);
↦ [(dt),0,0,-t+1]
```

#### D.15.4.7 backward

Procedure from library `finitediff.lib` (see [Section D.15.4 \[finitediff.lib\], page 2330](#)).

**Usage:** `backward(U1,U2,var)`; U1, U2 are the names of occuring derivatives, var is a variable in the basering;

**Return:** type vector; gives a predefined approximation of the backward approximation as often used in literature;

**Note:** see also `forward`, `laxfrT`, `setinitials`, `scheme`;

**Example:**

```
LIB "finitediff.lib";
list D="Ut","Ux","Uy","U";
list V="t","x","y";
list P="a","b";
setinitials(V,D,P);
backward(Ux,U,x);
 $\mapsto [0, (dx)*x, 0, -x+1]$ 
backward(Uy,U,y);
 $\mapsto [0, 0, (dy)*y, -y+1]$ 
backward(Ut,U,t);
 $\mapsto [(dt)*t, 0, 0, -t+1]$ 
```

#### D.15.4.8 central1st

Procedure from library `finitediff.lib` (see [Section D.15.4 \[finitediff.lib\]](#), page 2330).

**Usage:** `central1st(U1,U2,var)`; U1, U2 are the names of occuring derivatives, var is a variable in the basering;

**Return:** type vector; gives a predefined approximation of the first-order-central-approximation as often used in literature;

**Note:** see also `forward`, `laxfrT`, `setinitials`, `scheme`;

**Example:**

```
LIB "finitediff.lib";
list D="Ut","Ux","Uy","U";
list V="t","x","y";
list P="a","b";
setinitials(V,D,P);
central1st(Ux,U,x);
 $\mapsto [0, (2*dx)*x, 0, -x^2+1]$ 
central1st(Uy,U,y);
 $\mapsto [0, 0, (2*dy)*y, -y^2+1]$ 
```

#### D.15.4.9 central2nd

Procedure from library `finitediff.lib` (see [Section D.15.4 \[finitediff.lib\]](#), page 2330).

**Usage:** `central2nd(U1,U2,var)`; U1, U2 are the names of occuring derivatives, var is a variable in the basering;

**Return:** type vector; gives a predefined approximation of the second-order-central-approximation as often used in literature;

**Note:** see also `forward`, `laxfrT`, `setinitials`, `scheme`;

**Example:**

```
LIB "finitediff.lib";
list D="Uxx","Ux","Utt","Ut","U";
list P="lambda";
```

```

list V="t","x";
setinitials(V,D,P);
central2nd(Uxx,U,x);
 $\mapsto [(dx^2)*x,0,0,0,-x^2+2*x-1]$ 
central2nd(Utt,U,t);
 $\mapsto [0,0,(dt^2)*t,0,-t^2+2*t-1]$ 

```

#### D.15.4.10 trapezoid

Procedure from library `finitediff.lib` (see [Section D.15.4 \[finitediff.lib\]](#), page 2330).

**Usage:** `trapezoid(U1,U2,var)`; U1, U2 are the names of occurring derivatives, var is a variable in the basering;

**Return:** type vector; gives a predefined approximation of the trapezoid-approximation as often used in literature;

**Note:** see also `forward`, `laxfrT`, `setinitials`, `scheme`;

**Example:**

```

LIB "finitediff.lib";
list D="Uxx","Ux","Utt","Ut","U";
list P="lambda";
list V="t","x";
setinitials(V,D,P);
trapezoid(Uxx,Ux,x);
 $\mapsto [(dx)/2*x+(dx)/2,-x+1]$ 
trapezoid(Ux,U,x);
 $\mapsto [0,(dx)/2*x+(dx)/2,0,0,-x+1]$ 

```

#### D.15.4.11 midpoint

Procedure from library `finitediff.lib` (see [Section D.15.4 \[finitediff.lib\]](#), page 2330).

**Usage:** `midpoint(U1,U2,var)`; U1, U2 are the names of occurring derivatives, var is a variable in the basering;

**Return:** type vector; gives a predefined approximation of the midpoint-approximation as often used in literature;

**Note:** see also `forward`, `laxfrT`, `setinitials`, `scheme`;

**Example:**

```

LIB "finitediff.lib";
list D="Uxx","Ux","Utt","Ut","U";
list P="lambda";
list V="t","x";
setinitials(V,D,P);
midpoint(Ux,U,x);
 $\mapsto [0,(2*dx)*x,0,0,-x^2+1]$ 

```

#### D.15.4.12 pyramid

Procedure from library `finitediff.lib` (see [Section D.15.4 \[finitediff.lib\]](#), page 2330).

**Usage:** `pyramid(U1,U2,var)`; U1, U2 are the names of occurring derivatives, var is a variable in the basering;

**Return:** type vector; gives a predefined approximation of the pyramid-approximation as often used in literature;

**Note:** see also forward,laxfrT,setinitials,scheme;

**Example:**

```
LIB "finitediff.lib";
list D="Uxx","Ux","Utt","Ut","U";
list P="lambda";
list V="t","x";
setinitials(V,D,P);
pyramid(Ux,U,x);
⇒ [0,(dx)/3*x^2+(dx)/3*x+(dx)/3,0,0,-x^3+x]
```

### D.15.4.13 setinitials

Procedure from library `finitediff.lib` (see [Section D.15.4 \[finitediff\\_lib\]](#), page 2330).

**Usage:** `setinitials(V,D[,P]);` V,D,P are lists with strings as elements

**Return:** no return value: sets the dependence order of the occurring derivatives, constructs the suitable ring to compute in containing user chosen parameters, sets new basering

**Note:** P is optional, used to introduce some additional parameters into the ring. The Sine and Cosine values needed for the fourier transformation are symbolically introduced under the names `string(c)+nameof(variable)`, i.e. if x is any spatial variable then `cx:=cosine(dx*ksi)`, when regarding the fourier transform after ksi (for sine respectively). Artificial parameters I,T,Px,Py are introduced for the later eigenvalue analysis. Variables can be transformed into parameters of similar name

**Example:**

```
LIB "finitediff.lib";
list D="Ut","Ux","Uy","U";
list V="t","x","y";
list P="alpha","beta","gamma";
setinitials(V,D,P);////does not show the ring, since there is no output
basing;////does show the ring
⇒ // coefficients: QQ(I, T, Px, Py, Cx, Cy, Sx, Sy, alpha, beta, gamma, dt,\
    dx, dy)
⇒ // number of vars : 8
⇒ //      block  1 : ordering c
⇒ //      block  2 : ordering lp
⇒ //      : names    i t x y cx cy sx sy
⇒ // quotient ring from ideal
⇒ _[1]=cy^2+sy^2-1
⇒ _[2]=cx^2+sx^2-1
⇒ _[3]=i^2+1
```

### D.15.4.14 errormap

Procedure from library `finitediff.lib` (see [Section D.15.4 \[finitediff\\_lib\]](#), page 2330).

**Usage:** `errormap(f);` f of type poly

**Return:** type poly; performs the fouriertransformation of a single polynomial

**Example:**

```

LIB "finitediff.lib";
list D="Ux","Ut","U";
list P="a";
list V="t","x";
setinitials(V,D,P);
scheme(u(Ut)+a*u(Ux),central1st(Ux,U,x),backward(Ut,U,t));
↪ (2*dx)*x
errormap(_);
↪ (2*dx)*i*sx+(2*dx)*cx

```

### D.15.4.15 matrixsystem

Procedure from library `finitediff.lib` (see [Section D.15.4 \[finitediff.lib\], page 2330](#)).

**Usage:** `matrixsystem(M,A)`; where the  $M=M_t, M_1, \dots, M_n$  is a list with square matrices of the same dimension as entries, and  $A=A_t, A_1, \dots, A_n$  gives the corresponding approximations for the several variables  $(t, x_1, \dots, x_n)$  as vector. Intended to solve  $M_t \cdot U_t + M_1 \cdot U_{x_1} + \dots + M_n \cdot U_{x_n} = 0$  as a linear system of partial differential equations numerically by a finite difference scheme;

**Return:** type matrix; gives back the matrices  $B_1, B_2$  that represent the finite difference scheme, partitioned into different time levels in the form:  $B_1 \cdot U(t=N) = B_2 \cdot U(t < N)$ , where  $N$  is the maximal occurring degree (timelevel) of  $t$ .

**Example:**

```

LIB "finitediff.lib";
list D="Ut","Ux","Uy","U";
list V="t","x","y";
list P="a","b";
setinitials(V,D,P);
list Mat=unitmat(2),unitmat(2);
list Appr=forward(Ut,U,t),forward(Ux,U,x);
matrixsystem(Mat,Appr);
↪ _[1,1]=-1/(dt)*t-1/(dx)*x+(dt+dx)/(dt*dx)
↪ _[1,2]=0
↪ _[2,1]=0
↪ _[2,2]=-1/(dt)*t-1/(dx)*x+(dt+dx)/(dt*dx)

```

### D.15.4.16 timestep

Procedure from library `finitediff.lib` (see [Section D.15.4 \[finitediff.lib\], page 2330](#)).

**Usage:** `timestep(M)`;  $M$  a square matrix with polynomials over the basering as entries;

**Return:** type list; gives two matrices  $M_1, M_2$  that are the splitting of  $M$  with respect to the degree of the variable  $t$  in the entries, where the first list-entry  $M_1$  consists of the polynomials of the highest timelevel and  $M_2$  of the lower levels in the form:  $M=0 \Rightarrow M_1=M_2$ , i.e.  $M_1-M_2=M$

**Note:** intended to be used for the finite-difference-scheme-construction and partition into the several time steps

**Example:**

```

LIB "finitediff.lib";
list D="Ut","Ux","Uy","U";
list V="t","x","y";

```

```

list P="a","b";
setinitials(V,D,P);
list Mat=unitmat(2),unitmat(2);
list Apr=forward(Ut,U,t),forward(Ux,U,x);
matrixsystem(Mat,Apr);
⇒ _[1,1]=-1/(dt)*t-1/(dx)*x+(dt+dx)/(dt*dx)
⇒ _[1,2]=0
⇒ _[2,1]=0
⇒ _[2,2]=-1/(dt)*t-1/(dx)*x+(dt+dx)/(dt*dx)
timestep(_);
⇒ [1]:
⇒ _[1,1]=t
⇒ _[1,2]=0
⇒ _[2,1]=0
⇒ _[2,2]=t
⇒ [2]:
⇒ _[1,1]=(-dt)/(dx)*x+(dt+dx)/(dx)
⇒ _[1,2]=0
⇒ _[2,1]=0
⇒ _[2,2]=(-dt)/(dx)*x+(dt+dx)/(dx)

```

#### D.15.4.17 fouriersystem

Procedure from library `finitediff.lib` (see [Section D.15.4 \[finitediff.lib\]](#), page 2330).

**Usage:** `fouriersystem(M,A)`; M a list of matrices, A a list of approximations;

**Return:** type list; each entry is some matrix obtained by performing the substitution of the single approximations into the system of pde's, partitioning the equation into the several timesteps and fouriertransforming these parts

**Example:**

```

LIB "finitediff.lib";
list D="Ut","Ux","Uy","U";
list V="t","x","y";
list P="a","b";
setinitials(V,D,P);
matrix M[2][2]=0,-a,-a,0;
list Mat=unitmat(2),M;
list Apr=forward(Ut,U,t),trapezoid(Ux,U,x);
def s=fouriersystem(Mat,Apr);s;
⇒ [1]:
⇒ [1]:
⇒ _[1,1]=(I*T*Sx+T*Cx)
⇒ _[1,2]=0
⇒ _[2,1]=0
⇒ _[2,2]=(I*T*Sx+T*Cx)
⇒ [2]:
⇒ _[1,1]=(I*Sx+Cx)
⇒ _[1,2]=(2*I*Sx*a*dt+2*Cx*a*dt-2*a*dt)/(dx)
⇒ _[2,1]=(2*I*Sx*a*dt+2*Cx*a*dt-2*a*dt)/(dx)
⇒ _[2,2]=(I*Sx+Cx)
⇒ [2]:

```

```

⇒      [1]:
⇒      _[1]=(I*T*Sx+T*Cx)
⇒      [2]:
⇒      _[1]=(-2*I*Sx*a*dt+I*Sx*dx-2*Cx*a*dt+Cx*dx+2*a*dt)/(dx)
⇒      _[2]=(2*I*Sx*a*dt+I*Sx*dx+2*Cx*a*dt+Cx*dx-2*a*dt)/(dx)
⇒ [2]:
⇒      [1]:
⇒      [1]:
⇒      (T^2)
⇒      [2]:
⇒      [1]:
⇒      (-8*Cx*a^2*dt^2+4*Cx*a*dt*dx+8*a^2*dt^2-4*a*dt*dx+dx^2)/(dx^2)
⇒      [2]:
⇒      (-8*Cx*a^2*dt^2-4*Cx*a*dt*dx+8*a^2*dt^2+4*a*dt*dx+dx^2)/(dx^2)

```

#### D.15.4.18 PartitionVar

Procedure from library `finitediff.lib` (see [Section D.15.4 \[finitediff.lib\]](#), page 2330).

**Usage:** PartitionVar(f); f a poly in the basering;

**Return:** type poly; gives back a list L=f1,f2 obtained by the partition of f into two parts f1,f2 with  $\deg\_var\_n(f1) > 0$   $\deg\_var\_n(f2)=0$

**Example:**

```

LIB "finitediff.lib";
list D="Ut","Ux","Uy","U";
list V="t","x","y";
list P="a","b";
setinitials(V,D,P);////does not show the ring, since there is no output
basing;////does show the ring
⇒ // coefficients: QQ(I, T, Px, Py, Cx, Cy, Sx, Sy, a, b, dt, dx, dy)
⇒ // number of vars : 8
⇒ //      block 1 : ordering c
⇒ //      block 2 : ordering lp
⇒ //      : names      i t x y cx cy sx sy
⇒ // quotient ring from ideal
⇒ _[1]=cy^2+sy^2-1
⇒ _[2]=cx^2+sx^2-1
⇒ _[3]=i^2+1
poly f=t**3*cx**2-cy**2*dt+i**3*sx;
PartitionVar(f,1); ////i is the first variable
⇒ [1]:
⇒      i^3*sx
⇒ [2]:
⇒      t^3*cx^2+(-dt)*cy^2

```

#### D.15.4.19 ComplexValue

Procedure from library `finitediff.lib` (see [Section D.15.4 \[finitediff.lib\]](#), page 2330).

**Usage:** ComplexValue(f); f a poly in the basering;

**Return:** type poly; gives back the formal complex-value of f, where `var(1)` is redarded as the imaginary unit. Does only make sense, if the proc `<setinitials>` is executed before -> `nvars <= npars`



**Example:**

```

LIB "finitediff.lib";
list D="Ut","Ux","Uy","U";
list V="t","x","y";
list P="a","b";
setinitials(V,D,P);////does not show the ring, as there is no output
basing;////does show the ring
⇒ // coefficients: QQ(I, T, Px, Py, Cx, Cy, Sx, Sy, a, b, dt, dx, dy)
⇒ // number of vars : 8
⇒ //          block 1 : ordering c
⇒ //          block 2 : ordering lp
⇒ //          : names      i t x y cx cy sx sy
⇒ // quotient ring from ideal
⇒ _[1]=cy^2+sy^2-1
⇒ _[2]=cx^2+sx^2-1
⇒ _[3]=i^2+1
poly f=t**3*cx**2-cy**2*dt+i**3*sx;
f;
⇒ i^3*sx+t^3*cx^2+(-dt)*cy^2
ComplexValue(f);
⇒ t^6*sx^4-2*t^6*sx^2+t^6+(-2*dt)*t^3*sx^2*sy^2+(2*dt)*t^3*sx^2+(2*dt)*t^3*
sy^2+(-2*dt)*t^3*sx^2+(dt^2)*sy^4+(-2*dt^2)*sy^2+(dt^2)

```

**D.15.4.20 VarToPar**

Procedure from library `finitediff.lib` (see [Section D.15.4 \[finitediff.lib\]](#), page 2330).

**Usage:**      `VarToPar(f)`; `f` a poly in the basering;

**Return:**    type poly; gives back the poly obtained by substituting `var(i)` by `par(i)`, for all variables.  
Does only make sense, if the proc `<setinitials>` is executed before `-> nvars <= npars`;

**Example:**

```

LIB "finitediff.lib";
list D="Ut","Ux","Uy","U";
list V="t","x","y";
list P="a","b";
setinitials(V,D,P);////does not show the ring, as there is no output
basing;////does show the ring
⇒ // coefficients: QQ(I, T, Px, Py, Cx, Cy, Sx, Sy, a, b, dt, dx, dy)
⇒ // number of vars : 8
⇒ //          block 1 : ordering c
⇒ //          block 2 : ordering lp
⇒ //          : names      i t x y cx cy sx sy
⇒ // quotient ring from ideal
⇒ _[1]=cy^2+sy^2-1
⇒ _[2]=cx^2+sx^2-1
⇒ _[3]=i^2+1
poly f=t**3*cx**2-cy**2*dt+i**3*sx;
f;
⇒ i^3*sx+t^3*cx^2+(-dt)*cy^2
VarToPar(f);
⇒ (I^3*Sx+T^3*Cx^2-Cy^2*dt)

```

### D.15.4.21 ParToVar

Procedure from library `finitediff.lib` (see [Section D.15.4 \[finitediff.lib\], page 2330](#)).

**Usage:** `ParToVar(f)`;  $f$  a poly in the basering;

**Return:** type poly; gives back the poly obtained by substituting `par(i)` by `var(i)`, for the first `nvars(basing parameters)`. Does only make sense, if `setinitials` is executed before  $\rightarrow$  `nvars <= npars`. Is the opposite action to `VarToPar`, see example `ParToVar`;

**Example:**

```
LIB "finitediff.lib";
list D="Ut","Ux","Uy","U";
list V="t","x","y";
list P="a","b";
setinitials(V,D,P);////does not show the ring, as there is no output
basing;////does show the ring
 $\mapsto$  // coefficients: QQ(I, T, Px, Py, Cx, Cy, Sx, Sy, a, b, dt, dx, dy)
 $\mapsto$  // number of vars : 8
 $\mapsto$  //          block 1 : ordering c
 $\mapsto$  //          block 2 : ordering lp
 $\mapsto$  //          : names    i t x y cx cy sx sy
 $\mapsto$  // quotient ring from ideal
 $\mapsto$  _[1]=cy^2+sy^2-1
 $\mapsto$  _[2]=cx^2+sx^2-1
 $\mapsto$  _[3]=i^2+1
poly f=t**3*cx**2-cy**2*dt+i**3*sx/dt*dx;
f;
 $\mapsto$  (dx)/(dt)*i^3*sx+t^3*cx^2+(-dt)*cy^2
def g=VarToPar(f);
g;
 $\mapsto$  (I^3*Sx*dx+T^3*Cx^2*dt-Cy^2*dt^2)/(dt)
def h=ParToVar(g);
h==f;
 $\mapsto$  1
```

### D.15.4.22 qepcad

Procedure from library `finitediff.lib` (see [Section D.15.4 \[finitediff.lib\], page 2330](#)).

**Usage:** `qepcad(f)`;  $f$  a poly in the basering;

**Return:** type list; gives back some constraints that are equivalent to  $f < 1$  (computed by QEP-CAD);

**Example:**

```
LIB "finitediff.lib";
list D="Ux","Ut","U";
list P="a";
list V="t","x";
setinitials(V,D,P);
def s1=scheme(u(Ut)+a*u(Ux),laxfrX(Ux,U,x),laxfrT(Ut,U,x));
s1;
 $\mapsto$  (-a*dt+dx)/(2*dx)*x^2+(a*dt+dx)/(2*dx)
def s2=errormap(s1);
s2;
```

```

⇒ (-a*dt+dx)/(dx)*i*cx*sx+(a*dt-dx)/(dx)*sx^2+1
def s3=ComplexValue(s2);s3;
⇒ (a^2*dt^2-dx^2)/(dx^2)*sx^2+1
qepcad(s3);
⇒ 0
⇒ 0
⇒ 0
⇒ sh: qepcad: Kommando nicht gefunden.
⇒ sh: qepcadfilter.pl: Kommando nicht gefunden.
⇒ 32512
⇒
⇒ Try manually

```

#### D.15.4.23 qepcadsystem

Procedure from library `finitediff.lib` (see [Section D.15.4 \[finitediff.lib\]](#), page 2330).

**Usage:** `qepcadsystem(f); l` a list;

**Return:** list

**Purpose:** gives back some constraints that are equivalent to the eigenvalues of the matrices in the list `l` being  $< 1$  (computed by QEPCAD)

**Example:**

```

LIB "finitediff.lib";
list D="Ut","Ux","Uy","U";
list V="t","x","y";
list P="a","b";
setinitials(V,D,P);
matrix M[2][2]=0,-a,-a,0;
list Mat=unitmat(2),M;
list Appr=forward(Ut,U,t),forward(Ux,U,x);
//matrixsystem(Mat,Appr);
//timestep(_);
fourierysystem(Mat,Appr);
⇒ [1]:
⇒   [1]:
⇒     _[1,1]=(T)
⇒     _[1,2]=0
⇒     _[2,1]=0
⇒     _[2,2]=(T)
⇒   [2]:
⇒     _[1,1]=1
⇒     _[1,2]=(I*Sx*a*dt+Cx*a*dt-a*dt)/(dx)
⇒     _[2,1]=(I*Sx*a*dt+Cx*a*dt-a*dt)/(dx)
⇒     _[2,2]=1
⇒ [2]:
⇒   [1]:
⇒     _[1]=(T)
⇒   [2]:
⇒     _[1]=(-I*Sx*a*dt-Cx*a*dt+a*dt+dx)/(dx)
⇒     _[2]=(I*Sx*a*dt+Cx*a*dt-a*dt+dx)/(dx)
⇒ [2]:

```

```

⇒      [1]:
⇒      [1]:
⇒      (T^2)
⇒      [2]:
⇒      [1]:
⇒      (-2*Cx*a^2*dt^2-2*Cx*a*dt*dx+2*a^2*dt^2+2*a*dt*dx+dx^2)/(dx^2)
⇒      [2]:
⇒      (-2*Cx*a^2*dt^2+2*Cx*a*dt*dx+2*a^2*dt^2-2*a*dt*dx+dx^2)/(dx^2)
qepcadsystem(_[2]);
⇒ 0
⇒ 0
⇒ 0
⇒ sh: qepcad: Kommando nicht gefunden.
⇒ sh: qepcadfilter.pl: Kommando nicht gefunden.
⇒ 32512
⇒
⇒      ? Try manually
⇒      ? leaving finitediff.lib::qepcadsystem (0)

```

### D.15.5 GND.lib

**Library :** GND.lib

**Author :** Adrian Popescu, popescu@mathematik.uni-kl.de

**Overview :**

A method to compute the General Neron Desingularization in the frame of one dimensional local domains

**References:**

[1] A. Popescu, D. Popescu, "A method to compute the General Neron Desingularization in the frame of one dimensional local domains", [arxiv.org/abs/1508.05511](http://arxiv.org/abs/1508.05511)

**Procedures:**

#### D.15.5.1 desingularization

Procedure from library `GND.lib` (see [Section D.15.5 \[GND.lib\]](#), page 2342).

**Usage :** Returns as output a General Neron Desingularization as in the Paper <http://arxiv.org/abs/1508.05511>

**Example:**

```

LIB "GND.lib";
//Example 1
ring All = 0, (a1,a2,a3,x1,x2,x3,Y1,Y2,Y3), dp;
int nra = 3;
int nrx = 3;
int nry = 3;
ideal xid = x2^3-x3^2,x1^3-x3^2;
ideal yid = Y1^3-Y2^3;
ideal aid = a3^2-a1*a2;
poly y;
int i;

```

```

for(i=0;i<=30;i++)
{
y = y + a1*x3^i/factorial(i);
}
for(i=31;i<=50;i++)
{
y = y + a2*x3^i/factorial(i);
}
ideal f = a3*x1,a3*x2,y;
desingularization(All, nra,nrx,nry,xid,yid,aid,f);
⇒ h =
⇒ h[1]=Y1+(x1^3*x3^6)*T3+(-a3*x1)
⇒ h[2]=Y2+(-x1^2*x3^6)*T1+(x1^2*x2*x3^6)*T3+(-a3*x2)
⇒ h[3]=Y3+(x1^3*x3^6)*T2+(-a1*x3^7-7*a1*x3^6-42*a1*x3^5-210*a1*x3^4-840*a1*\
x3^3-2520*a1*x3^2-5040*a1*x3-5040*a1)/5040
⇒ h[4]=Y4+(-x1^2*x3^4)*T4+1
// With debug output
desingularization(All, nra,nrx,nry,xid,yid,aid,f,"debug");
⇒ Computing the kernel:
⇒ ker[1]=x2*Y1-x1*Y2
⇒ ker[2]=Y1^3-Y2^3
⇒ ker[3]=x1*Y1^2-x2*Y2^2
⇒ ker[4]=x1^2*Y1-x2^2*Y2
⇒ ker[5]=x2^3-x3^2
⇒ ker[6]=x1^3-x3^2
⇒ ker[7]=x1*x2^2*Y2-x3^2*Y1
⇒ ker[8]=x1^2*x2*Y2^2-x3^2*Y1^2
⇒ This is Plist:
⇒ [1]:
⇒ -x1*x3^2
⇒ [2]:
⇒ x3^2
⇒ [3]:
⇒ -x1
⇒ [4]:
⇒ _[1]=x2*Y1-x1*Y2
⇒ [5]:
⇒ [1]:
⇒ 1
⇒ The minor comes from these vars:
⇒ [1]:
⇒ 1
⇒ [2]:
⇒ 2
⇒ P' = -x1*x3^2
⇒ v(P'):
⇒ -x1*x3^2
⇒ d' = x1*x3^2
⇒ z = -1
⇒ P is constant (no Y), so d = d' = P = P'
⇒ P = P' = -x1*x3^2
⇒ d = -x1*x3^2
⇒ vidjet:

```

```

⇒ vidjet[1]=x1
⇒ vidjet[2]=x2
⇒ vidjet[3]=x3
⇒ vidjet[4]=(a3)*x1
⇒ vidjet[5]=(a3)*x2
⇒ vidjet[6]=(a1)/5040*x3^7+(a1)/720*x3^6+(a1)/120*x3^5+(a1)/24*x3^4+(a1)/6*\
  x3^3+(a1)/2*x3^2+(a1)*x3+(a1)
⇒ vidjet[7]=-1
⇒ Py = -x1*x3^2
⇒ This is C:
⇒ // coefficients: QQ
⇒ // number of vars : 5
⇒ //          block 1 : ordering dp
⇒ //          : names  a1 a3 x1 x2 x3
⇒ //          block 2 : ordering C
⇒ // quotient ring from ideal
⇒ _[1]=x2^3-x3^2
⇒ _[2]=x1^3-x3^2
⇒ _[3]=x3^8
⇒ This is D:
⇒ // coefficients: QQ
⇒ // number of vars : 5
⇒ //          block 1 : ordering dp
⇒ //          : names  a1 a3 x1 x2 x3
⇒ //          block 2 : ordering C
⇒ // quotient ring from ideal
⇒ _[1]=x2^3-x3^2
⇒ _[2]=x1^3-x3^2
⇒ This is the minor bordered matrix (H)
⇒ (x2),(-x1),0,0,
⇒ 0, 0, 1,0,
⇒ 1, 0, 0,0,
⇒ 0, 0, 0,(-x1*x3^2)
⇒ This is G:
⇒ 0, 0, (x1^2*x3^4)*Y4^2, 0,
⇒ (-x1*x3^4)*Y4^2,0, (x1*x2*x3^4)*Y4^2,0,
⇒ 0, (x1^2*x3^4)*Y4^2,0, 0,
⇒ 0, 0, 0, (-x1*x3^2)*Y4^2
⇒ G[1,1]=0
⇒ G[1,2]=0
⇒ G[1,3]=(x1^2*x3^4)*Y4^2
⇒ G[1,4]=0
⇒ G[2,1]=(-x1*x3^4)*Y4^2
⇒ G[2,2]=0
⇒ G[2,3]=(x1*x2*x3^4)*Y4^2
⇒ G[2,4]=0
⇒ G[3,1]=0
⇒ G[3,2]=(x1^2*x3^4)*Y4^2
⇒ G[3,3]=0
⇒ G[3,4]=0
⇒ G[4,1]=0
⇒ G[4,2]=0
⇒ G[4,3]=0

```

```

⇒ G[4,4]=(-x1*x3^2)*Y4^2
⇒ s = 1
⇒ This is cc
⇒ cc[1]=0
⇒ cc[2]=0
⇒ h =
⇒ h[1]=Y1+(x1^3*x3^6)*T3+(-a3*x1)
⇒ h[2]=Y2+(-x1^2*x3^6)*T1+(x1^2*x2*x3^6)*T3+(-a3*x2)
⇒ h[3]=Y3+(x1^3*x3^6)*T2+(-a1*x3^7-7*a1*x3^6-42*a1*x3^5-210*a1*x3^4-840*a1*
    x3^3-2520*a1*x3^2-5040*a1*x3-5040*a1)/5040
⇒ h[4]=Y4+(-x1^2*x3^4)*T4+1
⇒ m = 1
⇒ s^m = 1
⇒ QT =
⇒ QT[1]=0
⇒ QT[2]=0
⇒ f =
⇒ f[1]=(x2)*Y1+(-x1)*Y2
⇒ f[2]=(-x1*x3^2)*Y4+(x1*x3^2)
⇒ g =
⇒ g[1]=T1
⇒ g[2]=T2
kill All,nra,nrx,nry,i;
//Example 4
ring All = 0,(a1,a2,a3,a4,x,Y1,Y2,Y3),dp;
int nra = 4;
int nrx = 1;
int nry = 3;
ideal xid = 0;
ideal yid = Y1^3-Y2^3;
ideal aid = a3^2-a1*a2,a4^2+a4+1;
poly y;
int i;
for(i=0;i<=30;i++)
{
y = y + a1*x3^i/factorial(i);
}
for(i=31;i<=50;i++)
{
y = y + a2*x3^i/factorial(i);
}
ideal f = a3*x,a3*a4*x,y;
desingularization(All, nra,nrx,nry,xid,yid,aid,f);
⇒ h =
⇒ _[1]=Y1+(-4*a3^2*a4^2*x^6-4*a3^2*a4*x^6-a3^2*x^6-8*a3*a4*x^6-4*a3*x^6-4*x\
    ^6)/(4*a4^2+4*a4+1)*T3+(-a3*x)
⇒ _[2]=Y2+(-2*a3*a4*x^5-a3*x^5-2*x^5)/(2*a4+1)*T1+(8*a3^2*a4^2*x^6+8*a3^2*a\
    4*x^6+2*a3^2*x^6+10*a3*a4*x^6+5*a3*x^6+2*x^6)/(4*a4^2+4*a4+1)*T3+(-x)/(2*\
    a4+1)
⇒ _[3]=Y3+(-4*a3^2*a4^2*x^6-4*a3^2*a4*x^6-a3^2*x^6-8*a3*a4*x^6-4*a3*x^6-4*x\
    ^6)/(4*a4^2+4*a4+1)*T2+(-a1*x^3-a1)
⇒ _[4]=Y4+(2*x^5)*T1+(-3*a3*x^6)*T3+(-2*a3*a4*x^5-a3*x^5-2*x^5)/(2*a4+1)*T4\
    +(-x)

```

### D.15.6 gradedModules.lib

**Library:** gradedModules.lib

**Purpose:** Operations with graded modules/matrices/resolutions

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**Overview:** The library contains several procedures for constructing and manipulating graded modules/matrices/resolutions. Basics about graded objects can be found in [DL]. Throughout this library graded objects are graded maps, that is, matrices with polynomials, together with grading weights for source and destination. Graded modules are implicitly given as coker of a graded map. Note that in special cases we may also consider submodules in  $S^r$  generated by columns of a graded polynomial matrix (or a graded map).

**Note:** set assumeLevel to positive integer value in order to auto-check all assumptions. We denote the current basering by  $S$ .

**References:**

[DL] Decker, W., Lossen, Ch.: Computing in Algebraic Geometry, Springer, 2006

**Procedures:**

#### D.15.6.1 grobj

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\], page 2346](#)).

**Usage:** `grobj(M, w[, d])`, matrix/ideal/module  $M$ , intvec  $w$ ,  $d$

**Return:** graded object with matrix presentation  $M$ , row weighting  $w$  [and total graded degrees  $d$  of columns]

**Purpose:** create a valid graded object with a given matrix presentation, weighting [and total graded degrees (in case of zero columns)]

**Example:**

```
LIB "gradedModules.lib";
ring r=32003,(x,y,z),dp;
def A = grobj( module([x+y, x, 0, 0], [0, x+y, y, 0]), intvec(0,0,0,1) );
grview(A);
↳ Graded homomorphism:  $r^3 + r(-1) \leftarrow r(-1)^2$ , given by a matrix, with degr\
ees:
↳      ..1 ..2 ....
↳      --- --- +...
↳  0 :  1   - |..1
↳  0 :  1   1 |..2
↳  0 :  -   1 |..3
↳  1 :  -   - |..4
↳      === ===
↳      1   1
def F = grobj( module([x,y,0]), intvec(1,1,5) );
grview(F);
↳ Graded homomorphism:  $r(-1)^2 + r(-5) \leftarrow r(-2)$ , given by a matrix, with de\
grees:
↳      ..1 ....
```



```

↳      --- +...
↳      1 :  1 |..1
↳      1 :  1 |..2
↳      5 :  - |..3
↳      ===
↳      2
int d = 666; // zero can have any degree...
def Z = grobj( module([x,0], [0,0,0], [0, y]), intvec(1,2,3), intvec(2, d, 3) );
grview(Z);
↳ Graded homomorphism: r(-1) + r(-2) + r(-3) <- r(-2) + r(-666) + r(-3), gi\
  ven by a square matrix, with degrees:
↳      ...1 ...2 ...3 .....
↳      ---- ---- ---- +....
↳      1 :    1    -    - |...1
↳      2 :    -    -    1 |...2
↳      3 :    -    -    - |...3
↳      =====
↳      2    666    3
print(Z);
↳ x,0,0,
↳ 0,0,y,
↳ 0,0,0
attrib(Z);
↳ attr:degHomog, type intvec
↳ attr:isHomog, type intvec
grrange(Z); // module weights
↳ 1,2,3
attrib(Z, "degHomog"); // total degrees
↳ 2,666,3
// Zero object:
matrix z[3][0]; grview( grobj( z, intvec(1,2,3) ) );
↳ Graded homomorphism: r(-1) + r(-2) + r(-3) <- 0, given by zero (3 x 0) ma\
  trix.
grview( grobj( freemodule(0), intvec(1,2,3) ) );
↳ Graded homomorphism: r(-1) + r(-2) + r(-3) <- 0, given by zero (3 x 0) ma\
  trix.
matrix z1[0][3]; grview( grobj( z1, 0:0, intvec(1,2,3) ) );
↳ Graded homomorphism: 0 <- r(-1) + r(-2) + r(-3), given by zero (0 x 3) ma\
  trix.
grview( grobj( freemodule(0), 0:0, intvec(1,2,3) ) );
↳ Graded homomorphism: 0 <- r(-1) + r(-2) + r(-3), given by zero (0 x 3) ma\
  trix.
matrix z0[0][0]; grview( grobj( z0, 0:0 ) );
↳ Graded homomorphism: 0 <- 0, given by zero (0^2) matrix.
grview( grobj( freemodule(0), 0:0 ) );
↳ Graded homomorphism: 0 <- 0, given by zero (0^2) matrix.

```

### D.15.6.2 grtest

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\]](#), page 2346).

**Usage:** `grtest(M[,b])`, anything M, optionally int b

**Return:** 1 if M is a valid graded object, 0 otherwise

**Purpose:** validate a graded object. Print an invalid object message if b is not given

**Note:** M should be an ideal or module or matrix, with weighting attribute 'isHomog' and optionally total graded degrees attribute 'degHomog'. Attributes should be compatible with the presentation matrix.

**Example:**

```
LIB "gradedModules.lib";
ring r=32003,(x,y,z),dp;
// the following calls will fail due to tests in grtest:
grobj( module([x+y, x, 0, 0], [0, x+y, y, 0]), intvec(0,0,0,0) ); // enough row weights
⇒ _[1]=x*gen(2)+x*gen(1)+y*gen(1)
⇒ _[2]=x*gen(2)+y*gen(3)+y*gen(2)
// grobj( module([x+y, x, 0, 0], [0, x+y, y, 0]), intvec(0,0) ); // not enough row weights
// grobj( module([x,0], [0,0,0], [0, y]), intvec(1,2,3) ); // zero column needs other weights
grobj( module([x,0], [0,0,0], [0, y]), intvec(1,2,3), intvec(2, 10, 3) ); // compatible weights
⇒ _[1]=x*gen(1)
⇒ _[2]=0
⇒ _[3]=y*gen(2)
// grobj( module([x,0], [0,0,0], [0, y]), intvec(1,2,3), intvec(2-1, 10, 3+1) ); // not compatible weights
```

### D.15.6.3 grdeg

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\]](#), page 2346).

**Usage:** `grdeg(M)`, graded object M

**Return:** `intvec` of degrees

**Purpose:** graded degrees of columns (generators) of M, describing the source of M

**Assume:** M must be a graded object (matrix/module/ideal/mapping)

**Note:** if M has zero cols it should have `attrib(M,'degHomog')` set.

**Example:**

```
LIB "gradedModules.lib";
ring r=32003,(x,y,z),dp;
module A = grobj( module([x+y, x, 0, 0], [0, x+y, y, 0]), intvec(0,0,0,1) );
grview(A);
⇒ Graded homomorphism:  $r^3 + r(-1) \leftarrow r(-1)^2$ , given by a matrix, with degrees:
⇒      ..1 ..2 ....
⇒      --- --- +...
⇒  0 :  1  -  |..1
⇒  0 :  1   1 |..2
⇒  0 :  -   1 |..3
⇒  1 :  -   - |..4
⇒      === ===
⇒      1   1
module B = grobj( module([0,x,y]), intvec(15,1,1) );
grview(B);
⇒ Graded homomorphism:  $r(-15) + r(-1)^2 \leftarrow r(-2)$ , given by a matrix, with degrees:
⇒      ..1 ....
⇒      --- +...
```

```

⇒ 15 : - |..1
⇒ 1 : 1 |..2
⇒ 1 : 1 |..3
⇒      ===
⇒      2
module D = grsum(
  grsum(grpower(A,2), grtwist(1,1)),
  grsum(grtwist(1,2), grpower(B,2))
);
grview(D);
⇒ Graded homomorphism:
⇒  $r^3 + r(-1) + r^3 + r(-1) + r(1) + r(2) + r(-15) + r(-1)^2 + r(-15) + r(-1)^2 <-$ 
⇒  $r(-1)^4 + r(-2)^2$ , given by a matrix, with degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 .....
⇒      --- --- --- --- --- --- +....
⇒ 0 : 1 - - - - - |..1
⇒ 0 : 1 1 - - - - |..2
⇒ 0 : - 1 - - - - |..3
⇒ 1 : - - - - - |..4
⇒ 0 : - - 1 - - - |..5
⇒ 0 : - - 1 1 - - |..6
⇒ 0 : - - - 1 - - |..7
⇒ 1 : - - - - - |..8
⇒ -1 : - - - - - |..9
⇒ -2 : - - - - - |..10
⇒ 15 : - - - - - |..11
⇒ 1 : - - - - 1 - |..12
⇒ 1 : - - - - 1 - |..13
⇒ 15 : - - - - - |..14
⇒ 1 : - - - - 1 |..15
⇒ 1 : - - - - 1 |..16
⇒      === === === === === ===
⇒      1 1 1 1 2 2
grdeg(D);
⇒ 1,1,1,1,2,2
def D10 = grshift(D, 10);
grview(D10);
⇒ Graded homomorphism:
⇒  $r(10)^3 + r(9) + r(10)^3 + r(9) + r(11) + r(12) + r(-5) + r(9)^2 + r(-5) \setminus$ 
⇒  $+ r(9)^2 <-$ 
⇒  $r(9)^4 + r(8)^2$ , given by a matrix, with degrees:
⇒      ...1 ...2 ...3 ...4 ...5 ...6 .....
⇒      --- --- --- --- --- --- +....
⇒ -10 : 1 - - - - - |...1
⇒ -10 : 1 1 - - - - |...2
⇒ -10 : - 1 - - - - |...3
⇒ -9 : - - - - - |...4
⇒ -10 : - - 1 - - - |...5
⇒ -10 : - - 1 1 - - |...6
⇒ -10 : - - - 1 - - |...7
⇒ -9 : - - - - - |...8
⇒ -11 : - - - - - |...9

```

```

⇒ -12 :  -  -  -  -  -  -  - |..10
⇒  5 :  -  -  -  -  -  -  - |..11
⇒ -9 :  -  -  -  -  1  -  - |..12
⇒ -9 :  -  -  -  -  1  -  - |..13
⇒  5 :  -  -  -  -  -  -  - |..14
⇒ -9 :  -  -  -  -  -  -  1 |..15
⇒ -9 :  -  -  -  -  -  -  1 |..16
⇒      ====  ====  ====  ====  ====  ====
⇒      -9  -9  -9  -9  -8  -8
grdeg(D10);
⇒ -9,-9,-9,-9,-8,-8

```

#### D.15.6.4 grview

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\]](#), page 2346).

**Usage:** `grview(M)`, graded object `M`

**Return:** nothing

**Purpose:** print the degree/grading data about the GRADED matrix/module/ideal/mapping object `M`

**Assume:** `M` must be graded

**Example:**

```

LIB "gradedModules.lib";
ring r=32003,(x,y,z),dp;
module A = grobj( module([x+y, x, 0, 0], [0, x+y, y, 0]), intvec(0,0,0,1) );
grview(A);
⇒ Graded homomorphism: r^3 + r(-1) <- r(-1)^2, given by a matrix, with degr\
  ees:
⇒      ..1 ..2 ....
⇒      --- --- +...
⇒  0 :  1  -  |..1
⇒  0 :  1  1  |..2
⇒  0 :  -  1  |..3
⇒  1 :  -  -  |..4
⇒      === ===
⇒      1  1
module B = grobj( module([0,x,y]), intvec(15,1,1) );
grview(B);
⇒ Graded homomorphism: r(-15) + r(-1)^2 <- r(-2), given by a matrix, with d\
  eegrees:
⇒      ..1 ....
⇒      --- +...
⇒  15 :  -  |..1
⇒   1 :  1  |..2
⇒   1 :  1  |..3
⇒      ===
⇒      2
module D = grsum( grsum(grpower(A,2), grtwist(1,1)), grsum(grtwist(1,2), grpower(B,2) );
grview(D);
⇒ Graded homomorphism:
⇒ r^3 + r(-1) + r^3 + r(-1) + r(1) + r(2) + r(-15) + r(-1)^2 + r(-15) + r(-\

```

```

1)^2 <-
⇒ r(-1)^4 + r(-2)^2, given by a matrix, with degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ....
⇒      --- --- --- --- --- --- +...
⇒      0 :  1  -  -  -  -  -  |..1
⇒      0 :  1  1  -  -  -  -  |..2
⇒      0 :  -  1  -  -  -  -  |..3
⇒      1 :  -  -  -  -  -  -  |..4
⇒      0 :  -  -  1  -  -  -  |..5
⇒      0 :  -  -  1  1  -  -  |..6
⇒      0 :  -  -  -  1  -  -  |..7
⇒      1 :  -  -  -  -  -  -  |..8
⇒     -1 :  -  -  -  -  -  -  |..9
⇒     -2 :  -  -  -  -  -  -  |.10
⇒     15 :  -  -  -  -  -  -  |.11
⇒      1 :  -  -  -  -  1  -  |.12
⇒      1 :  -  -  -  -  1  -  |.13
⇒     15 :  -  -  -  -  -  -  |.14
⇒      1 :  -  -  -  -  -  1  |.15
⇒      1 :  -  -  -  -  -  1  |.16
⇒      === === === === === ===
⇒      1  1  1  1  2  2
ring R = 0,(w,x,y,z), dp; def I = grobj( ideal(y2-xz, xy-wz, x2z-wyz), intvec(0) );
list res1 = grres(I, 0); // non-minimal
grview(res1);
⇒ Graded resolution:
⇒ R <-- d_1 --
⇒ R(-2)^2 + R(-3) <-- d_2 --
⇒ R(-3) + R(-4) <-- d_3 --
⇒ 0, given by maps:
⇒ d_1 :
⇒ Graded homomorphism: R <- R(-2)^2 + R(-3), given by a matrix, with degree\
s:
⇒      .1 .2 .3 ...
⇒      -- -- -- +..
⇒     0 : 2  2  3 |.1
⇒      == == ==
⇒      2  2  3
⇒ d_2 :
⇒ Graded homomorphism: R(-2)^2 + R(-3) <- R(-3) + R(-4), given by a matrix,\
with degrees:
⇒      .1 .2 ...
⇒      -- -- +..
⇒     2 : 1  2 |.1
⇒     2 : 1  2 |.2
⇒     3 : 0  1 |.3
⇒      == ==
⇒      3  4
⇒ d_3 :
⇒ Graded homomorphism: R(-3) + R(-4) <- 0, given by zero (2 x 0) matrix.
print(betti(res1,0), "beti");
⇒      0      1      2
⇒ -----

```

```

⇒      0:      1      -      -
⇒      1:      -      2      1
⇒      2:      -      1      1
⇒ -----
⇒ total:      1      3      2
⇒
list res2 = grres(grshift(I, -10), 0, 1); // minimal!
grview(res2);
⇒ Graded resolution:
⇒ R(-10) <-- d_1 --
⇒ R(-12)^2 <-- d_2 --
⇒ R(-14) <-- d_3 --
⇒ 0, given by maps:
⇒ d_1 :
⇒ Graded homomorphism: R(-10) <- R(-12)^2, given by a matrix, with degrees:
⇒      ..1 ..2 ....
⇒      --- --- +...
⇒ 10 :  2   2 |..1
⇒      === ===
⇒      12  12
⇒ d_2 :
⇒ Graded homomorphism: R(-12)^2 <- R(-14), given by a matrix, with degrees:
⇒      ..1 ....
⇒      --- +...
⇒ 12 :  2 |..1
⇒ 12 :  2 |..2
⇒      ===
⇒      14
⇒ d_3 :
⇒ Graded homomorphism: R(-14) <- 0, given by zero (1 x 0) matrix.
print(betti(res2,0), "betti");
⇒      0      1      2
⇒ -----
⇒ 10:      1      -      -
⇒ 11:      -      2      -
⇒ 12:      -      -      1
⇒ -----
⇒ total:      1      2      1
⇒

```

### D.15.6.5 grshift

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\]](#), page 2346).

**Usage:** `grshift(A, d)`, graded objects A, int d

**Return:** shifted graded object

**Purpose:** shift the grading on A by d:  $A(i) \rightarrow A(i+d)$

**Example:**

```

LIB "gradedModules.lib";
ring r=32003,(x,y,z),dp;
module A = grobj( module([x+y, x, 0, 0], [0, x+y, y, 0]), intvec(0,0,0,1) );
grview(A);

```

```

⇒ Graded homomorphism:  $r^3 + r(-1) \leftarrow r(-1)^2$ , given by a matrix, with degr\
ees:
⇒      ..1 ..2 ....
⇒      --- --- +...
⇒      0 :  1  -  |..1
⇒      0 :  1   1 |..2
⇒      0 :  -   1 |..3
⇒      1 :  -   - |..4
⇒      === ===
⇒      1   1
module S = grshift( A, 6);
grview(S);
⇒ Graded homomorphism:  $r(6)^3 + r(5) \leftarrow r(5)^2$ , given by a matrix, with deg\
rees:
⇒      ..1 ..2 ....
⇒      --- --- +...
⇒     -6 :  1  -  |..1
⇒     -6 :  1   1 |..2
⇒     -6 :  -   1 |..3
⇒     -5 :  -   - |..4
⇒      === ===
⇒     -5  -5
grview( grshift( grzero(), 100 ) ); // does nothing...
⇒ !! Warning: shifting '0 <- 0' leaves it as it unchanged!
⇒ Graded homomorphism:  $0 \leftarrow 0$ , given by zero ( $0^2$ ) matrix.

```

### D.15.6.6 grzero

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\]](#), page 2346).

**Usage:** `grzero()`

**Return:** graded object representing  $S(0)^1$

**Purpose:** compute presentation of  $S(0)^1$

**Example:**

```

LIB "gradedModules.lib";
ring r=32003,(x,y,z),dp;
grview( grobj(freemodule(0), intvec(0:0), intvec(0:0)) );
⇒ Graded homomorphism:  $0 \leftarrow 0$ , given by zero ( $0^2$ ) matrix.
grview( grobj(freemodule(0), intvec(0:0)) );
⇒ Graded homomorphism:  $0 \leftarrow 0$ , given by zero ( $0^2$ ) matrix.
grview( grzero() );
⇒ Graded homomorphism:  $0 \leftarrow 0$ , given by zero ( $0^2$ ) matrix.
// def M = grpower( grshift( grzero(), 3), 2 ); grview(M);

```

### D.15.6.7 grtwist

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\]](#), page 2346).

**Usage:** `grtwist(a,d)`, int a, d

**Return:** graded object representing  $S(d)^a$

**Purpose:** compute presentation of  $S(d)^a$

**Example:**

```
LIB "gradedModules.lib";
ring r=32003,(x,y,z),dp;
// grview(grpower( grshift(grzero(), 10), 5 ) );
grview( grtwist (5, 10) );
⇒ Graded homomorphism:  $r(10)^5 \leftarrow 0$ , given by zero (5 x 0) matrix.
```

**D.15.6.8 grtwists**

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\], page 2346](#)).

**Usage:** `grtwists(v)`, `intvec v`

**Return:** graded object representing  $S(v[1]) + \dots + S(v[\text{size}(v)])$

**Purpose:** compute presentation of  $S(v[1]) + \dots + S(v[\text{size}(v)])$

**Example:**

```
LIB "gradedModules.lib";
ring r=32003,(x,y,z),dp;
grview( grtwists ( intvec(-4, 1, 6) ) );
⇒ Graded homomorphism:  $r(-4) + r(1) + r(6) \leftarrow 0$ , given by zero (3 x 0) matrix.
grview( grtwists ( intvec(0:0) ) );
⇒ Graded homomorphism:  $0 \leftarrow 0$ , given by zero (0^2) matrix.
```

**D.15.6.9 grsum**

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\], page 2346](#)).

**Usage:** `grsum(A, B)`, graded objects A and B

**Return:** graded direct sum of input objects

**Purpose:** compute the graded direct sum of A and B

**Example:**

```
LIB "gradedModules.lib";
// if( defined(assumeLevel) ){ int assumeLevel0 = assumeLevel; } else { int assumeLevel0 = 0; }
ring r=32003,(x,y,z),dp;
module A = grobj( module([x+y, x, 0, 0], [0, x+y, y, 0]), intvec(0,0,0,1) );
grview(A);
⇒ Graded homomorphism:  $r^3 + r(-1) \leftarrow r(-1)^2$ , given by a matrix, with degrees:
⇒      ..1 ..2 ....
⇒      --- --- +...
⇒      0 : 1   - |..1
⇒      0 : 1   1 |..2
⇒      0 : -   1 |..3
⇒      1 : -   - |..4
⇒      === ===
⇒      1   1
module B = grobj( module([0,x,y]), intvec(15,1,1) );
grview(B);
⇒ Graded homomorphism:  $r(-15) + r(-1)^2 \leftarrow r(-2)$ , given by a matrix, with degrees:
```



```

↳      ..1 ....
↳      --- +...
↳  15 :  - |..1
↳   1 :  1 |..2
↳   1 :  1 |..3
↳      ===
↳      2
module C = grsum(A,B);
print(C);
↳ x+y,0,  0,
↳ x,  x+y,0,
↳ 0,  y,  0,
↳ 0,  0,  0,
↳ 0,  0,  0,
↳ 0,  0,  x,
↳ 0,  0,  y
homog(C);
↳ 1
grview(C);
↳ Graded homomorphism: r^3 + r(-1) + r(-15) + r(-1)^2 <- r(-1)^2 + r(-2), g\
  given by a matrix, with degrees:
↳      ..1 ..2 ..3 ....
↳      --- --- --- +...
↳   0 :  1  -  - |..1
↳   0 :  1  1  - |..2
↳   0 :  -  1  - |..3
↳   1 :  -  -  - |..4
↳  15 :  -  -  - |..5
↳   1 :  -  -  1 |..6
↳   1 :  -  -  1 |..7
↳      === === ===
↳      1  1  2
module D = grsum(
  grsum(grpower(A,2), grtwist(1,1)),
  grsum(grtwist(1,2), grpower(B,2))
);
print(D);
↳ x+y,0,  0,  0,  0,0,
↳ x,  x+y,0,  0,  0,0,
↳ 0,  y,  0,  0,  0,0,
↳ 0,  0,  0,  0,  0,0,
↳ 0,  0,  x+y,0,  0,0,
↳ 0,  0,  x,  x+y,0,0,
↳ 0,  0,  0,  y,  0,0,
↳ 0,  0,  0,  0,  0,0,
↳ 0,  0,  0,  0,  0,0,
↳ 0,  0,  0,  0,  0,0,
↳ 0,  0,  0,  0,  0,0,
↳ 0,  0,  0,  0,  x,0,
↳ 0,  0,  0,  0,  y,0,
↳ 0,  0,  0,  0,  0,0,
↳ 0,  0,  0,  0,  0,x,
↳ 0,  0,  0,  0,  0,y

```

```

homog(D);
↳ 1
grview(D);
↳ Graded homomorphism:
↳  $r^3 + r(-1) + r^3 + r(-1) + r(1) + r(2) + r(-15) + r(-1)^2 + r(-15) + r(-1)^2 <-$ 
↳  $r(-1)^4 + r(-2)^2$ , given by a matrix, with degrees:
↳
  ..1 ..2 ..3 ..4 ..5 ..6 ....
  --- --- --- --- --- --- +...
↳ 0 : 1 - - - - - |..1
↳ 0 : 1 1 - - - - |..2
↳ 0 : - 1 - - - - |..3
↳ 1 : - - - - - - |..4
↳ 0 : - - 1 - - - |..5
↳ 0 : - - 1 1 - - |..6
↳ 0 : - - - 1 - - |..7
↳ 1 : - - - - - - |..8
↳ -1 : - - - - - - |..9
↳ -2 : - - - - - - |..10
↳ 15 : - - - - - - |..11
↳ 1 : - - - - 1 - |..12
↳ 1 : - - - - 1 - |..13
↳ 15 : - - - - - - |..14
↳ 1 : - - - - - 1 |..15
↳ 1 : - - - - - 1 |..16
↳
  === === === === === ===
↳
  1 1 1 1 2 2
module F = grobj( module([x,y,0]), intvec(1,1,5) );
grview(F);
↳ Graded homomorphism:  $r(-1)^2 + r(-5) <- r(-2)$ , given by a matrix, with de\
grees:
↳
  ..1 ....
  --- +...
↳ 1 : 1 |..1
↳ 1 : 1 |..2
↳ 5 : - |..3
↳
  ===
↳
  2
module T = grsum( F, grsum( grtwist(1, 10), B ) );
grview(T);
↳ Graded homomorphism:  $r(-1)^2 + r(-5) + r(10) + r(-15) + r(-1)^2 <- r(-2)^2$ 
↳ , given by a matrix, with degrees:
↳
  ...1 ...2 .....
  ---- ---- +....
↳ 1 : 1 - |...1
↳ 1 : 1 - |...2
↳ 5 : - - |...3
↳ -10 : - - |...4
↳ 15 : - - |...5
↳ 1 : - 1 |...6
↳ 1 : - 1 |...7
↳
  =====

```

```

    ↪          2      2
    // if( defined(assumeLevel0) ){ assumeLevel = assumeLevel0; } else { kill assumeLevel0; }

```

### D.15.6.10 grpowers

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\]](#), page 2346).

**Usage:** `grpowers(A, p)`, graded object A, int  $p > 0$

**Return:** graded direct power  $A^p$

**Purpose:** compute the graded direct power  $A^p$

**Note:** the power  $p$  must be positive

**Example:**

```

LIB "gradedModules.lib";
ring r=32003,(x,y,z),dp;
module A = grobj( module([x+y, x, 0], [0, x+y, y]), intvec(1,1,1) );
grview(A);
↪ Graded homomorphism: r(-1)^3 <- r(-2)^2, given by a matrix, with degrees:
↪      ..1 ..2 ....
↪      --- --- +...
↪      1 : 1   - |..1
↪      1 : 1   1 |..2
↪      1 : -   1 |..3
↪      === ===
↪      2      2
module B = grobj( module([x,y]), intvec(2,2) );
grview(B);
↪ Graded homomorphism: r(-2)^2 <- r(-3), given by a matrix, with degrees:
↪      .1 ...
↪      -- +..
↪      2 : 1 |.1
↪      2 : 1 |.2
↪      ==
↪      3
module D = grsum( grpowers(A,2), grpowers(B,2) );
print(D);
↪ x+y,0, 0, 0, 0,0,
↪ x,  x+y,0, 0, 0,0,
↪ 0,  y, 0, 0, 0,0,
↪ 0,  0, x+y,0, 0,0,
↪ 0,  0, x,  x+y,0,0,
↪ 0,  0, 0,  y, 0,0,
↪ 0,  0, 0,  0, x,0,
↪ 0,  0, 0,  0, y,0,
↪ 0,  0, 0,  0, 0,x,
↪ 0,  0, 0,  0, 0,y
homog(D);
↪ 1
grview(D);
↪ Graded homomorphism: r(-1)^6 + r(-2)^4 <- r(-2)^4 + r(-3)^2, given by a m\
    atrix, with degrees:
↪      ..1 ..2 ..3 ..4 ..5 ..6 ....

```

```

⇒      --- --- --- --- --- --- +...
⇒  1 :  1  -  -  -  -  - |..1
⇒  1 :  1  1  -  -  -  - |..2
⇒  1 :  -  1  -  -  -  - |..3
⇒  1 :  -  -  1  -  -  - |..4
⇒  1 :  -  -  1  1  -  - |..5
⇒  1 :  -  -  -  1  -  - |..6
⇒  2 :  -  -  -  -  1  - |..7
⇒  2 :  -  -  -  -  1  - |..8
⇒  2 :  -  -  -  -  -  1 |..9
⇒  2 :  -  -  -  -  -  1 |.10
⇒      === === === === === ===
⇒      2  2  2  2  3  3

```

### D.15.6.11 grtranspose

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\]](#), page 2346).

**Usage:** `grtranspose(M)`, graded object `M`

**Return:** graded object

**Purpose:** graded transpose of `M`

**Note:** no reordering is performed by this procedure

**Example:**

```

LIB "gradedModules.lib";
ring r=32003,(x,y,z),dp;
module M = grtwists( intvec(-2, 0, 4, 4) ); grview(M);
⇒ Graded homomorphism: r(-2) + r + r(4)^2 <- 0, given by zero (4 x 0) matrix.
module N = grsyz( grtranspose( M ) ); grview(N);
⇒ Graded homomorphism: r(2) + r + r(-4)^2 <- r(2) + r + r(-4)^2, given by a\
  diagonal matrix, with degrees:
⇒      ..1 ..2 ..3 ..4 ....
⇒      --- --- --- --- +...
⇒ -2 :  0  -  -  -  - |..1
⇒  0 :  -  0  -  -  - |..2
⇒  4 :  -  -  0  -  - |..3
⇒  4 :  -  -  -  0  - |..4
⇒      === === === ===
⇒      -2  0  4  4
module L = grtranspose(N); grview( L );
⇒ Graded homomorphism: r(-2) + r + r(4)^2 <- r(-2) + r + r(4)^2, given by a\
  diagonal matrix, with degrees:
⇒      ..1 ..2 ..3 ..4 ....
⇒      --- --- --- --- +...
⇒  2 :  0  -  -  -  - |..1
⇒  0 :  -  0  -  -  - |..2
⇒ -4 :  -  -  0  -  - |..3
⇒ -4 :  -  -  -  0  - |..4
⇒      === === === ===
⇒      2  0 -4 -4
module K = grsyz( L ); grview(K);

```

```

⇒ Graded homomorphism:  $r(-2) + r + r(4)^2 \leftarrow 0$ , given by zero (4 x 0) matrix.
x.
// Corner cases: 0 ← 0!
module Z = grzero(); grview(Z);
⇒ Graded homomorphism:  $0 \leftarrow 0$ , given by zero (0^2) matrix.
grview( grtranspose( Z ) );
⇒ Graded homomorphism:  $0 \leftarrow 0$ , given by zero (0^2) matrix.
// Corner cases: * ← 0
matrix M1[3][0];
module Z1 = grobj( M1, intvec(-1, 0, 1) ); grview(Z1);
⇒ Graded homomorphism:  $r(1) + r + r(-1) \leftarrow 0$ , given by zero (3 x 0) matrix.
grview( grtranspose( Z1 ) );
⇒ Graded homomorphism:  $0 \leftarrow r(-1) + r + r(1)$ , given by zero (0 x 3) matrix.
// Corner cases: 0 ← *
matrix M2[0][3];
module Z2 = grobj( M2, 0:0, intvec(-1, 0, 1) ); grview(Z2);
⇒ Graded homomorphism:  $0 \leftarrow r(1) + r + r(-1)$ , given by zero (0 x 3) matrix.
grview( grtranspose( Z2 ) );
⇒ Graded homomorphism:  $r(-1) + r + r(1) \leftarrow 0$ , given by zero (3 x 0) matrix.

```

### D.15.6.12 grgens

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\]](#), page 2346).

**Usage:** `grgens(M)`, graded object `M` (map)

**Return:** graded object

**Purpose:** try compute graded generators of `coker(M)` and return them as columns of a graded map.

**Note:** presentation of resulting generated submodule may be different to `M`!

**Example:**

```

LIB "gradedModules.lib";
ring r=32003,(x,y,z),dp;
module M = grtwists( intvec(-2, 0, 4, 4) ); grview(M);
⇒ Graded homomorphism:  $r(-2) + r + r(4)^2 \leftarrow 0$ , given by zero (4 x 0) matrix.
x.
module N = grgens(M);
grview( N ); print(N); // fine == M
⇒ Graded homomorphism:  $r(-2) + r + r(4)^2 \leftarrow r(-2) + r + r(4)^2$ , given by a\
diagonal matrix, with degrees:
⇒      ..1 ..2 ..3 ..4 ....
⇒      --- --- --- --- +...
⇒      2 :  0   -   -   - |..1
⇒      0 :  -   0   -   - |..2
⇒     -4 :  -   -   0   - |..3
⇒     -4 :  -   -   -   0 |..4
⇒      === === === ===
⇒           2    0  -4  -4
⇒ 1,0,0,0,
⇒ 0,1,0,0,
⇒ 0,0,1,0,
⇒ 0,0,0,1

```

```

module A = grobj( module([x+y, x, 0, 3], [0, x+y, y, 2], [y, y, z, 1]), intvec(0,0,0)
A = grgroebner(A); grview(A);
⇒ Graded homomorphism:  $r^3 + r(-1) \leftarrow r(-1)^3 + r(-2) + r(-3)$ , given by a matrix, with degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ....
⇒      --- --- --- --- --- +...
⇒      0 : 1  1  1  2  - |..1
⇒      0 : 1  - 1  -  - |..2
⇒      0 : 1  1  1  2  3 |..3
⇒      1 : 0  0  0  1  2 |..4
⇒      === === === === ===
⇒      1  1  1  2  3
module B = grgens(A);
grview( B ); print(B); // Ups :( != A
⇒ Graded homomorphism:  $r(2) \leftarrow r^3 + r(-1)$ , given by a matrix, with degrees\
:
⇒      ..1 ..2 ..3 ..4 ....
⇒      --- --- --- --- +...
⇒      -2 : 2  2  2  3 |..1
⇒      === === === ===
⇒      0  0  0  1
⇒ xy-3y2+xz+3yz,-xy+2y2+2xz+2yz,x2-xy-4y2,y3-x2z-2xyz-y2z
grview( grgens( grzero() ) );
⇒ Graded homomorphism:  $0 \leftarrow 0$ , given by zero ( $0^2$ ) matrix.

```

### D.15.6.13 grpres

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\]](#), page 2346).

**Usage:** `grpres(M)`, graded object `M` (submodule gens)

**Return:** graded module (via coker)

**Purpose:** compute graded presentation matrix of submodule generated by columns of `M`

**Example:**

```

LIB "gradedModules.lib";
ring r=32003,(x,y,z),dp;
def A = grgroebner( grobj( module([x+y, x, 0, 3], [0, x+y, y, 2], [y, y, z, 1]), intvec(0,0,0)
grview(A);
⇒ Graded homomorphism:  $r^3 + r(-1) \leftarrow r(-1)^3 + r(-2) + r(-3)$ , given by a matrix, with degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ....
⇒      --- --- --- --- --- +...
⇒      0 : 1  1  1  2  - |..1
⇒      0 : 1  - 1  -  - |..2
⇒      0 : 1  1  1  2  3 |..3
⇒      1 : 0  0  0  1  2 |..4
⇒      === === === === ===
⇒      1  1  1  2  3
"graded transpose: "; def B = grtranspose(A); grview( B ); print(B);
⇒ graded transpose:
⇒ Graded homomorphism:  $r(1)^3 + r(2) + r(3) \leftarrow r^3 + r(1)$ , given by a matrix, with degrees:
⇒      ..1 ..2 ..3 ..4 ....

```

```

↳      --- --- --- --- +...
↳ -1 :  1  1  1  0 |..1
↳ -1 :  1  -  1  0 |..2
↳ -1 :  1  1  1  0 |..3
↳ -2 :  2  -  2  1 |..4
↳ -3 :  -  -  3  2 |..5
↳      === === === ===
↳      0  0  0 -1
↳ y,   y,z,           1,
↳ x+2y,0,-y+z,        2,
↳ -y,   x,y-z,        1,
↳ y2,   0,-xz,        -x+3y,
↳ 0,    0,y3-x2z-2xyz-y2z,-x2+xy+4y2
"... syzygy: "; def C = grsyz(B); grview(C);
↳ ... syzygy:
↳ Graded homomorphism: r^3 + r(1) <- r(-2), given by a matrix, with degrees\
:
↳      ..1 ....
↳      --- +...
↳  0 :  2 |..1
↳  0 :  2 |..2
↳  0 :  2 |..3
↳ -1 :  3 |..4
↳      ===
↳      2
"... transposed: "; def D = grtranspose(C); grview( D ); print (D);
↳ ... transposed:
↳ Graded homomorphism: r(2) <- r^3 + r(-1), given by a matrix, with degrees\
:
↳      ..1 ..2 ..3 ..4 ....
↳      --- --- --- --- +...
↳ -2 :  2  2  2  3 |..1
↳      === === === ===
↳      0  0  0  1
↳ xy-3y2+xz+3yz,-xy+2y2+2xz+2yz,x2-xy-4y2,y3-x2z-2xyz-y2z
"... and back to presentation: "; def E = grsyz( D ); grview(E); print(E);
↳ ... and back to presentation:
↳ Graded homomorphism: r^3 + r(-1) <- r(-1)^3, given by a matrix, with degr\
ees:
↳      ..1 ..2 ..3 ....
↳      --- --- --- +...
↳  0 :  1  1  1 |..1
↳  0 :  1  1  1 |..2
↳  0 :  1  1  1 |..3
↳  1 :  0  -  - |..4
↳      === === ===
↳      1  1  1
↳ y,x,   x-2y,
↳ y,-2y, x-3y,
↳ z,-y-z,-3z,
↳ 1,0,   0
def F = grgens( E ); grview(F); print(F);
↳ Graded homomorphism: r(2) <- r^3 + r(-1), given by a matrix, with degrees\

```

```

:
⇒      ..1 ..2 ..3 ..4 ....
⇒      --- --- --- --- +...
⇒  -2 :  2   2   2   3 |..1
⇒      === === === ===
⇒      0   0   0   1
⇒ xy-3y2+xz+3yz,-xy+2y2+2xz+2yz,x2-xy-4y2,y3-x2z-2xyz-y2z
def G = grpres( F ); grview(G); print(G);
⇒ Graded homomorphism:  $r^3 + r(-1) \leftarrow r(-1)^3$ , given by a matrix, with degr\
  ees:
⇒      ..1 ..2 ..3 ....
⇒      --- --- --- --- +...
⇒   0 :  1   1   1 |..1
⇒   0 :  1   1   1 |..2
⇒   0 :  1   1   1 |..3
⇒   1 :  0   -   - |..4
⇒      === === ===
⇒      1   1   1
⇒ y,x,   x-2y,
⇒ y,-2y, x-3y,
⇒ z,-y-z,-3z,
⇒ 1,0,   0
def M = grtwists( intvec(-2, 0, 4, 4) ); grview(M);
⇒ Graded homomorphism:  $r(-2) + r + r(4)^2 \leftarrow 0$ , given by zero (4 x 0) matri\
  x.
def N = grgens(M); grview( N ); print(N);
⇒ Graded homomorphism:  $r(-2) + r + r(4)^2 \leftarrow r(-2) + r + r(4)^2$ , given by a\
  diagonal matrix, with degrees:
⇒      ..1 ..2 ..3 ..4 ....
⇒      --- --- --- --- +...
⇒   2 :  0   -   -   - |..1
⇒   0 :  -   0   -   - |..2
⇒  -4 :  -   -   0   - |..3
⇒  -4 :  -   -   -   0 |..4
⇒      === === === ===
⇒      2   0  -4  -4
⇒ 1,0,0,0,
⇒ 0,1,0,0,
⇒ 0,0,1,0,
⇒ 0,0,0,1
def L = grpres( N ); grview( L ); print(L);
⇒ Graded homomorphism:  $r(-2) + r + r(4)^2 \leftarrow 0$ , given by zero (4 x 0) matri\
  x.
⇒ 4 x 0 zero matrix

```

#### D.15.6.14 grorder

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\]](#), page 2346).

**Usage:** `grorder(M)`, graded object or list M

**Return:** same as input

**Purpose:** reorder/transform graded object or chain complex M into block form



**Assume:** M must be a graded object or a list of graded objects

**Example:**

```
LIB "gradedModules.lib";
"Surface Name: 'rat.d10.g9.quart2' in P^4";
↳ Surface Name: 'rat.d10.g9.quart2' in P^4
int @p=31991; ring R = (@p),(x,y,z,u,v), dp;
ideal J = x3yu2-48/11x2y2u2-8356xy3u2+35/121y4u2+31/66x3zu2-54/83x2yzu2-61/18xy2zu2+...
def I = grobj( groebner(J), intvec(0) ); // ASSUME: no zero entries in J!
ASSUME(0, grtest(I));
"Input degrees: "; grview(I);
↳ Input degrees:
↳ Graded homomorphism: R <- R(-4)^2 + R(-5)^6 + R(-6), given by a matrix, w\
ith degrees:
↳      .1 .2 .3 .4 .5 .6 .7 .8 .9 ...
↳      -- -- -- -- -- -- -- -- -- +..
↳ 0 : 4  4  5  5  5  5  5  5  6 |.1
↳      == == == == == == == == ==
↳      4  4  5  5  5  5  5  5  6
def RR = grres(I, 0, 1);
list L = RR;
" = Non-minimal betti numbers: "; print(betti(L, 0), "betti");
↳ = Non-minimal betti numbers:
↳      0      1      2      3      4
↳ -----
↳ 0:      1      -      -      -      -
↳ 1:      -      -      -      -      -
↳ 2:      -      -      -      -      -
↳ 3:      -      2      -      -      -
↳ 4:      -      5      9      3      -
↳ 5:      -      1      3      3      1
↳ -----
↳ total:      1      8      12      6      1
↳
"Graded reordered structure of 'res(Input,0)': "; grview(grorder(L));
↳ Graded reordered structure of 'res(Input,0)':
↳ Graded resolution:
↳ R <-- d_1 --
↳ R(-4)^2 + R(-5)^5 + R(-6) <-- d_2 --
↳ R(-6)^9 + R(-7)^3 <-- d_3 --
↳ R(-7)^3 + R(-8)^3 <-- d_4 --
↳ R(-9), given by maps:
↳ d_1 :
↳ Graded homomorphism: R <- R(-4)^2 + R(-5)^5 + R(-6), given by a matrix, w\
ith degrees:
↳      .1 .2 .3 .4 .5 .6 .7 .8 ...
↳      -- -- -- -- -- -- -- -- -- +..
↳ 0 : 4  4  5  5  5  5  5  6 |.1
↳      == == == == == == == == ==
↳      4  4  5  5  5  5  5  6
↳ d_2 :
↳ Graded homomorphism: R(-4)^2 + R(-5)^5 + R(-6) <- R(-6)^9 + R(-7)^3, give\
n by a matrix, with degrees:
```

```

⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 .11 .12 ....
⇒      --- --- --- --- --- --- --- --- --- --- --- --- +...
⇒      4 :  2   2   2   2   2   2   2   2   2   -   3   3 |..1
⇒      4 :  2   2   2   2   2   2   2   2   2   -   3   3 |..2
⇒      5 :  1   1   1   1   1   1   1   1   1   2   2   2 |..3
⇒      5 :  1   1   1   1   1   1   1   1   1   2   2   2 |..4
⇒      5 :  1   1   1   1   1   1   1   1   1   2   2   2 |..5
⇒      5 :  1   1   1   1   1   1   1   1   1   -   2   2 |..6
⇒      5 :  1   1   1   1   1   1   1   1   1   -   2   2 |..7
⇒      6 :  -   -   -   -   -   -   -   -   -   1   1   1 |..8
⇒      === === === === === === === === === === === ===
⇒      6   6   6   6   6   6   6   6   6   7   7   7
⇒ d_3 :
⇒ Graded homomorphism: R(-6)^9 + R(-7)^3 <- R(-7)^3 + R(-8)^3, given by a m\
  arix, with degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ....
⇒      --- --- --- --- --- --- +...
⇒      6 :  1   1   1   2   2   2 |..1
⇒      6 :  1   1   1   2   2   2 |..2
⇒      6 :  1   1   1   2   2   2 |..3
⇒      6 :  1   1   1   2   2   2 |..4
⇒      6 :  1   1   1   2   2   2 |..5
⇒      6 :  1   1   1   2   2   2 |..6
⇒      6 :  1   1   1   2   2   2 |..7
⇒      6 :  1   1   1   2   2   2 |..8
⇒      6 :  1   1   1   -   -   2 |..9
⇒      7 :  -   -   -   1   1   1 |.10
⇒      7 :  -   -   -   1   1   1 |.11
⇒      7 :  -   -   -   1   1   1 |.12
⇒      === === === === === ===
⇒      7   7   7   8   8   8
⇒ d_4 :
⇒ Graded homomorphism: R(-7)^3 + R(-8)^3 <- R(-9), given by a matrix, with \
  degrees:
⇒      .1 ...
⇒      -- +..
⇒      7 : 2 |.1
⇒      7 : 2 |.2
⇒      7 : 2 |.3
⇒      8 : 1 |.4
⇒      8 : 1 |.5
⇒      8 : 1 |.6
⇒      ==
⇒      9

```

### D.15.6.15 grtranspose1

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\]](#), page 2346).

**Usage:** `grtranspose1(M)`, graded object or list M

**Return:** same as input

**Purpose:** graded transpose of graded object or chain complex M

**Assume:** M must be a graded object or a list of graded objects

**Example:**

```
LIB "gradedModules.lib";
"Surface Name: 'k3.d10.g9.quart2' in P^4";
↳ Surface Name: 'k3.d10.g9.quart2' in P^4
int @p=31991; ring R = (@p),(x,y,z,u,v), dp;
ideal J = x3yz2+31/15x2y2z2-7231xy3z2+99/37y4z2+28/95x3z3+97/32x2yz3+13247xy2z3+1271
def I = grobj( groebner(J), intvec(0) ); // ASSUME: no zero entries in J!
ASSUME(0, grtest(I));
"Input degrees: "; grview(I);
↳ Input degrees:
↳ Graded homomorphism: R <- R(-4)^2 + R(-5)^5 + R(-6)^3, given by a matrix,\
  with degrees:
↳      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 ....
↳      --- --- --- --- --- --- --- --- --- --- +...
↳      0 : 4   4   5   5   5   5   5   6   6   6 |..1
↳      === === === === === === === === === ===
↳      4   4   5   5   5   5   5   6   6   6
def RR = grres(I, 0, 1); list L = RR;
" = Non-minimal betti numbers: "; print(betti(L, 0), "betti");
↳ = Non-minimal betti numbers:
↳      0      1      2      3      4
↳ -----
↳      0:      1      -      -      -      -
↳      1:      -      -      -      -      -
↳      2:      -      -      -      -      -
↳      3:      -      2      -      -      -
↳      4:      -      4      7      2      -
↳      5:      -      3      8      7      2
↳ -----
↳ total:      1      9     15      9      2
↳
"Graded (original) structure of 'res(Input,0)': "; grview(L);
↳ Graded (original) structure of 'res(Input,0)':
↳ Graded resolution:
↳ R <-- d_1 --
↳ R(-4)^2 + R(-5)^4 + R(-6)^3 <-- d_2 --
↳ R(-7)^2 + R(-6)^7 + R(-7)^6 <-- d_3 --
↳ R(-7)^2 + R(-8)^7 <-- d_4 --
↳ R(-9)^2 <-- d_5 --
↳ 0, given by maps:
↳ d_1 :
↳ Graded homomorphism: R <- R(-4)^2 + R(-5)^4 + R(-6)^3, given by a matrix,\
  with degrees:
↳      .1 .2 .3 .4 .5 .6 .7 .8 .9 ...
↳      -- -- -- -- -- -- -- -- -- +..
↳      0 : 4   4   5   5   5   5   6   6   6 |.1
↳      == == == == == == == == ==
↳      4   4   5   5   5   5   6   6   6
↳ d_2 :
↳ Graded homomorphism: R(-4)^2 + R(-5)^4 + R(-6)^3 <- R(-7)^2 + R(-6)^7 + R\
  (-7)^6
```

```

⇒ , given by a matrix, with degrees:
⇒   ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 .11 .12 .13 .14 .15 ....
⇒   --- --- --- --- --- --- --- --- --- --- --- --- --- --- --- +...
⇒   4 : - - 2 2 2 2 2 2 2 3 3 3 3 3 3 |..1
⇒   4 : - - 2 2 2 2 2 2 2 2 3 3 3 3 3 3 |..2
⇒   5 : 2 2 1 1 1 1 1 1 1 2 2 2 2 2 2 |..3
⇒   5 : 2 2 1 1 1 1 1 1 1 2 2 2 2 2 2 |..4
⇒   5 : - - 1 1 1 1 1 1 1 2 2 2 2 2 2 |..5
⇒   5 : - - 1 1 1 1 1 1 1 2 2 2 2 2 2 |..6
⇒   6 : 1 1 - - - - - - - 1 1 1 1 1 1 |..7
⇒   6 : 1 1 - - - - - - - 1 1 1 1 1 1 |..8
⇒   6 : 1 1 - - - - - - - 1 1 1 1 1 1 |..9
⇒   === === === === === === === === === === === === ===
⇒   7 7 6 6 6 6 6 6 6 7 7 7 7 7 7
⇒ d_3 :
⇒ Graded homomorphism: R(-7)^2 + R(-6)^7 + R(-7)^6 <- R(-7)^2 + R(-8)^7, gi\
  ven by a matrix, with degrees:
⇒   ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 ....
⇒   --- --- --- --- --- --- --- --- --- +...
⇒   7 : - - 1 1 1 1 1 1 1 |..1
⇒   7 : - - 1 1 1 1 1 1 1 |..2
⇒   6 : 1 1 2 2 2 2 2 2 2 |..3
⇒   6 : 1 1 2 2 2 2 2 2 2 |..4
⇒   6 : 1 1 2 2 2 2 2 2 2 |..5
⇒   6 : 1 1 2 2 2 2 2 2 2 |..6
⇒   6 : 1 1 2 2 2 2 2 2 2 |..7
⇒   6 : 1 1 2 2 2 2 2 2 2 |..8
⇒   6 : 1 1 2 2 2 2 2 2 2 |..9
⇒   7 : - - 1 1 1 1 1 1 1 |..10
⇒   7 : - - 1 1 1 1 1 1 1 |..11
⇒   7 : - - 1 1 1 1 1 1 1 |..12
⇒   7 : - - 1 1 1 1 1 1 1 |..13
⇒   7 : - - 1 1 1 1 1 1 1 |..14
⇒   7 : - - 1 1 1 1 1 1 1 |..15
⇒   === === === === === === === === ===
⇒   7 7 8 8 8 8 8 8 8
⇒ d_4 :
⇒ Graded homomorphism: R(-7)^2 + R(-8)^7 <- R(-9)^2, given by a matrix, wit\
  h degrees:
⇒   .1 .2 ...
⇒   -- -- +..
⇒   7 : 2 2 |.1
⇒   7 : 2 2 |.2
⇒   8 : 1 1 |.3
⇒   8 : 1 1 |.4
⇒   8 : 1 1 |.5
⇒   8 : 1 1 |.6
⇒   8 : 1 1 |.7
⇒   8 : 1 1 |.8
⇒   8 : 1 1 |.9
⇒   == ==
⇒   9 9
⇒ d_5 :

```

```

⇒ Graded homomorphism:  $R(-9)^2 \leftarrow 0$ , given by zero (2 x 0) matrix.
"Graded transpose of the previous resolution "; list LLL = grtranspose1( L ); grview
⇒ Graded transpose of the previous resolution
⇒ Graded resolution:
⇒  $R(9)^2 \leftarrow d_1$  --
⇒  $R(8)^7 + R(7)^2 \leftarrow d_2$  --
⇒  $R(7)^8 + R(6)^7 \leftarrow d_3$  --
⇒  $R(6)^3 + R(5)^4 + R(4)^2 \leftarrow d_4$  --
⇒ R, given by maps:
⇒  $d_1$  :
⇒ Graded homomorphism:  $R(9)^2 \leftarrow R(8)^7 + R(7)^2$ , given by a matrix, with d\
degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 ....
⇒      --- --- --- --- --- --- --- --- --- +...
⇒ -9 :  1   1   1   1   1   1   1   2   2 |..1
⇒ -9 :  1   1   1   1   1   1   1   2   2 |..2
⇒      === === === === === === === === ===
⇒      -8  -8  -8  -8  -8  -8  -8  -7  -7
⇒  $d_2$  :
⇒ Graded homomorphism:  $R(8)^7 + R(7)^2 \leftarrow R(7)^8 + R(6)^7$ , given by a matri\
x, with degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 .11 .12 .13 .14 .15 ....
⇒      --- --- --- --- --- --- --- --- --- --- --- --- --- --- --- +...
⇒ -8 :  1   1   1   1   1   1   1   1   2   2   2   2   2   2   2 |..1
⇒ -8 :  1   1   1   1   1   1   1   1   2   2   2   2   2   2   2 |..2
⇒ -8 :  1   1   1   1   1   1   1   1   2   2   2   2   2   2   2 |..3
⇒ -8 :  1   1   1   1   1   1   1   1   2   2   2   2   2   2   2 |..4
⇒ -8 :  1   1   1   1   1   1   1   1   2   2   2   2   2   2   2 |..5
⇒ -8 :  1   1   1   1   1   1   1   1   2   2   2   2   2   2   2 |..6
⇒ -8 :  1   1   1   1   1   1   1   1   2   2   2   2   2   2   2 |..7
⇒ -7 :  -   -   -   -   -   -   -   -   1   1   1   1   1   1   1 |..8
⇒ -7 :  -   -   -   -   -   -   -   -   1   1   1   1   1   1   1 |..9
⇒      === === === === === === === === ===
⇒      -7  -7  -7  -7  -7  -7  -7  -7  -6  -6  -6  -6  -6  -6  -6
⇒  $d_3$  :
⇒ Graded homomorphism:  $R(7)^8 + R(6)^7 \leftarrow R(6)^3 + R(5)^4 + R(4)^2$ , given b\
y a matrix, with degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 ....
⇒      --- --- --- --- --- --- --- --- --- +...
⇒ -7 :  1   1   1   2   2   -   -   -   - |..1
⇒ -7 :  1   1   1   2   2   -   -   -   - |..2
⇒ -7 :  1   1   1   2   2   2   2   3   3 |..3
⇒ -7 :  1   1   1   2   2   2   2   3   3 |..4
⇒ -7 :  1   1   1   2   2   2   2   3   3 |..5
⇒ -7 :  1   1   1   2   2   2   2   3   3 |..6
⇒ -7 :  1   1   1   2   2   2   2   3   3 |..7
⇒ -7 :  1   1   1   2   2   2   2   3   3 |..8
⇒ -6 :  -   -   -   1   1   1   1   2   2 |..9
⇒ -6 :  -   -   -   1   1   1   1   2   2 |.10
⇒ -6 :  -   -   -   1   1   1   1   2   2 |.11
⇒ -6 :  -   -   -   1   1   1   1   2   2 |.12
⇒ -6 :  -   -   -   1   1   1   1   2   2 |.13
⇒ -6 :  -   -   -   1   1   1   1   2   2 |.14

```

```

⇒ -6 : - - - 1 1 1 1 2 2 |.15
⇒      === === === === === === === ===
⇒      -6 -6 -6 -5 -5 -5 -5 -4 -4
⇒ d_4 :
⇒ Graded homomorphism: R(6)^3 + R(5)^4 + R(4)^2 <- R, given by a matrix, wi\
th degrees:
⇒      ..1 ....
⇒      --- +...
⇒ -6 : 6 |..1
⇒ -6 : 6 |..2
⇒ -6 : 6 |..3
⇒ -5 : 5 |..4
⇒ -5 : 5 |..5
⇒ -5 : 5 |..6
⇒ -5 : 5 |..7
⇒ -4 : 4 |..8
⇒ -4 : 4 |..9
⇒      ===
⇒      0
"Its non-minimal betti numbers: "; print(betti(LL, 0), "betti");
⇒ Its non-minimal betti numbers:
⇒      0      1      2      3      4
⇒ -----
⇒ -9:      2      7      8      3      -
⇒ -8:      -      2      7      4      -
⇒ -7:      -      -      -      2      -
⇒ -6:      -      -      -      -      -
⇒ -5:      -      -      -      -      -
⇒ -4:      -      -      -      -      1
⇒ -----
⇒ total:      2      9     15      9      1
⇒

```

### D.15.6.16 TestGRRes

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\]](#), page 2346).

**Usage:** `TestGRRes(name, I)`, string name, ideal I

**Return:** nothing

**Purpose:** compute/test/output/order/transpose a graded resolution of I

**Example:**

```

LIB "gradedModules.lib";
// if( defined(assumeLevel) ){ int assumeLevel0 = assumeLevel; } else { int assumeLevel0 = 0; }
// note: data from random generation 2
string Name = "castelnuovo"; int @p=31991; ring R = (@p),(x,y,z,u,v), dp; ideal I = 5x^2y^2z^2u^2v^2 -
⇒ =====
⇒
⇒ === Example: [ castelnuovo ]
⇒ = Ring: (ZZ/31991),(x,y,z,u,v),(dp(5),C)
⇒ ! Resolution via 'grres':
⇒ = Non-minimal betti numbers:
⇒      0      1      2

```

```

↳ -----
↳      0:      1      -      -
↳      1:      -      1      -
↳      2:      -      2      2
↳ -----
↳ total:      1      3      2
↳
↳ = Degrees of (ordered) maps:
↳ Graded resolution:
↳ R <-- d_1 --
↳ R(-2) + R(-3)^2 <-- d_2 --
↳ R(-4)^2, given by maps:
↳ d_1 :
↳ Graded homomorphism: R <- R(-2) + R(-3)^2, given by a matrix, with degree\
  s:
↳      .1 .2 .3 ...
↳      -- -- -- +..
↳ 0 : 2  3  3 |.1
↳      == == ==
↳      2  3  3
↳ d_2 :
↳ Graded homomorphism: R(-2) + R(-3)^2 <- R(-4)^2, given by a matrix, with \
  degrees:
↳      .1 .2 ...
↳      -- -- +..
↳ 2 : 2  2 |.1
↳ 3 : 1  1 |.2
↳ 3 : 1  1 |.3
↳      == ==
↳      4  4
↳ = TRANSPOSE'd complex: %%%%%%%%%%
↳      0      1      2
↳ -----
↳     -4:      2      2      -
↳     -3:      -      1      -
↳     -2:      -      -      1
↳ -----
↳ total:      2      3      1
↳
↳ Graded resolution:
↳ R(4)^2 <-- d_1 --
↳ R(3)^2 + R(2) <-- d_2 --
↳ R, given by maps:
↳ d_1 :
↳ Graded homomorphism: R(4)^2 <- R(3)^2 + R(2), given by a matrix, with deg\
  rees:
↳      ..1 ..2 ..3 ....
↳      --- --- --- +...
↳ -4 :  1   1   2 |..1
↳ -4 :  1   1   2 |..2
↳      === === ===
↳      -3  -3  -2
↳ d_2 :

```

```

⇒ Graded homomorphism: R(3)^2 + R(2) <- R, given by a matrix, with degrees:
⇒      ..1 ....
⇒      --- +...
⇒    -3 :  3 |..1
⇒    -3 :  3 |..2
⇒    -2 :  2 |..3
⇒      ===
⇒      0
⇒
⇒
string Name = "ell.d8.g7"; int @p=31991; ring R = (@p),(x,y,z,u,v), dp;ideal I = x2y
⇒ =====
⇒
⇒ === Example: [ ell.d8.g7 ]
⇒ = Ring: (ZZ/31991),(x,y,z,u,v),(dp(5),C)
⇒ ! Resolution via 'grres':
⇒ = Non-minimal betti numbers:
⇒      0      1      2
⇒ -----
⇒    0:      1      -      -
⇒    1:      -      -      -
⇒    2:      -      2      -
⇒    3:      -      1      2
⇒ -----
⇒ total:      1      3      2
⇒
⇒ = Degrees of (ordered) maps:
⇒ Graded resolution:
⇒ R <-- d_1 --
⇒ R(-3)^2 + R(-4) <-- d_2 --
⇒ R(-5)^2, given by maps:
⇒ d_1 :
⇒ Graded homomorphism: R <- R(-3)^2 + R(-4), given by a matrix, with degree\
  s:
⇒      .1 .2 .3 ...
⇒      -- -- -- +..
⇒    0 : 3  3  4 |.1
⇒      == == ==
⇒      3  3  4
⇒ d_2 :
⇒ Graded homomorphism: R(-3)^2 + R(-4) <- R(-5)^2, given by a matrix, with \
  degrees:
⇒      .1 .2 ...
⇒      -- -- -- +..
⇒    3 : 2  2 |.1
⇒    3 : 2  2 |.2
⇒    4 : 1  1 |.3
⇒      == ==
⇒      5  5
⇒ = TRANSPOSE'd complex: %%%%%%%%%%
⇒      0      1      2
⇒ -----
⇒    -5:      2      1      -

```



```

⇒      -4:      -      2      -
⇒      -3:      -      -      -
⇒      -2:      -      -      1
⇒      -----
⇒ total:      2      3      1
⇒
⇒ Graded resolution:
⇒ R(5)^2 <-- d_1 --
⇒ R(4) + R(3)^2 <-- d_2 --
⇒ R, given by maps:
⇒ d_1 :
⇒ Graded homomorphism: R(5)^2 <- R(4) + R(3)^2, given by a matrix, with deg\
rees:
⇒      ..1 ..2 ..3 ....
⇒      --- --- --- +...
⇒ -5 :  1   2   2 |..1
⇒ -5 :  1   2   2 |..2
⇒      === === ===
⇒      -4  -3  -3
⇒ d_2 :
⇒ Graded homomorphism: R(4) + R(3)^2 <- R, given by a matrix, with degrees:
⇒      ..1 ....
⇒      --- +...
⇒ -4 :  4 |..1
⇒ -3 :  3 |..2
⇒ -3 :  3 |..3
⇒      ===
⇒      0
⇒
⇒
string Name = "ell.d7.g6"; int @p=31991; ring R = (@p),(x,y,z,u,v), dp;ideal I = 497;
⇒ =====
⇒
⇒ === Example: [ ell.d7.g6 ]
⇒ = Ring:  (ZZ/31991),(x,y,z,u,v),(dp(5),C)
⇒ ! Resolution via 'grres':
⇒ = Non-minimal betti numbers:
⇒      0      1      2
⇒      -----
⇒      0:      1      -      -
⇒      1:      -      1      -
⇒      2:      -      -      -
⇒      3:      -      2      2
⇒      -----
⇒ total:      1      3      2
⇒
⇒ = Degrees of (ordered) maps:
⇒ Graded resolution:
⇒ R <-- d_1 --
⇒ R(-2) + R(-4)^2 <-- d_2 --
⇒ R(-5)^2, given by maps:
⇒ d_1 :
⇒ Graded homomorphism: R <- R(-2) + R(-4)^2, given by a matrix, with degree\

```

```

s:
⇒      .1 .2 .3 ...
⇒      -- -- -- +..
⇒  0 : 2  4  4 |.1
⇒      == == ==
⇒      2  4  4
⇒ d_2 :
⇒ Graded homomorphism: R(-2) + R(-4)^2 <- R(-5)^2, given by a matrix, with \
degrees:
⇒      .1 .2 ...
⇒      -- -- +..
⇒  2 : 3  3 |.1
⇒  4 : 1  1 |.2
⇒  4 : 1  1 |.3
⇒      == ==
⇒      5  5
⇒ = TRANSPOSE'd complex: %%%%%%%%%%%
⇒      0      1      2
⇒ -----
⇒  -5:      2      2      -
⇒  -4:      -      -      -
⇒  -3:      -      1      -
⇒  -2:      -      -      1
⇒ -----
⇒ total:      2      3      1
⇒
⇒ Graded resolution:
⇒ R(5)^2 <-- d_1 --
⇒ R(4)^2 + R(2) <-- d_2 --
⇒ R, given by maps:
⇒ d_1 :
⇒ Graded homomorphism: R(5)^2 <- R(4)^2 + R(2), given by a matrix, with deg\
rees:
⇒      ..1 ..2 ..3 ....
⇒      --- --- --- +...
⇒  -5 :  1   1   3 |..1
⇒  -5 :  1   1   3 |..2
⇒      === === ===
⇒      -4  -4  -2
⇒ d_2 :
⇒ Graded homomorphism: R(4)^2 + R(2) <- R, given by a matrix, with degrees:
⇒      ..1 ....
⇒      --- +...
⇒  -4 :  4 |..1
⇒  -4 :  4 |..2
⇒  -2 :  2 |..3
⇒      ===
⇒      0
⇒
⇒
string Name = "k3.d7.g5"; int @p=31991; ring R = (@p),(x,y,z,u,v), dp;ideal I = -97/
⇒ =====
⇒

```

```

⇒ === Example: [ k3.d7.g5 ]
⇒ = Ring: (ZZ/31991),(x,y,z,u,v),(dp(5),C)
⇒ ! Resolution via 'grres':
⇒ = Non-minimal betti numbers:
⇒      0      1      2
⇒ -----
⇒      0:      1      -      -
⇒      1:      -      -      -
⇒      2:      -      3      1
⇒      3:      -      -      1
⇒ -----
⇒ total:      1      3      2
⇒
⇒ = Degrees of (ordered) maps:
⇒ Graded resolution:
⇒ R <-- d_1 --
⇒ R(-3)^3 <-- d_2 --
⇒ R(-4) + R(-5), given by maps:
⇒ d_1 :
⇒ Graded homomorphism: R <- R(-3)^3, given by a matrix, with degrees:
⇒      .1 .2 .3 ...
⇒      -- -- -- +..
⇒      0 : 3  3  3 |.1
⇒      == == ==
⇒      3  3  3
⇒ d_2 :
⇒ Graded homomorphism: R(-3)^3 <- R(-4) + R(-5), given by a matrix, with de\
grees:
⇒      .1 .2 ...
⇒      -- -- +..
⇒      3 : 1  2 |.1
⇒      3 : 1  2 |.2
⇒      3 : 1  2 |.3
⇒      == ==
⇒      4  5
⇒ = TRANSPOSE'd complex: %%%%%%%%%%
⇒      0      1      2
⇒ -----
⇒      -5:      1      -      -
⇒      -4:      1      3      -
⇒      -3:      -      -      -
⇒      -2:      -      -      1
⇒ -----
⇒ total:      2      3      1
⇒
⇒ Graded resolution:
⇒ R(5) + R(4) <-- d_1 --
⇒ R(3)^3 <-- d_2 --
⇒ R, given by maps:
⇒ d_1 :
⇒ Graded homomorphism: R(5) + R(4) <- R(3)^3, given by a matrix, with dege\
es:
⇒      ..1 ..2 ..3 ....

```

```

↳      --- --- --- +...
↳    -5 :  2   2   2 |..1
↳    -4 :  1   1   1 |..2
↳      === === ===
↳      -3  -3  -3
↳ d_2 :
↳ Graded homomorphism: R(3)^3 <- R, given by a matrix, with degrees:
↳      ..1 ....
↳      --- +...
↳    -3 :  3 |..1
↳    -3 :  3 |..2
↳    -3 :  3 |..3
↳      ===
↳      0
↳
↳
↳ string Name = "rat.d8.g6"; int @p=31991; ring R = (@p),(x,y,z,u,v), dp;ideal I = -19,
↳ =====
↳
↳ === Example: [ rat.d8.g6 ]
↳ = Ring: (ZZ/31991),(x,y,z,u,v),(dp(5),C)
↳ ! Resolution via 'grres':
↳ = Non-minimal betti numbers:
↳      0      1      2      3
↳ -----
↳    0:      1      -      -      -
↳    1:      -      -      -      -
↳    2:      -      1      -      -
↳    3:      -      4      5      1
↳ -----
↳ total:      1      5      5      1
↳
↳ = Degrees of (ordered) maps:
↳ Graded resolution:
↳ R <-- d_1 --
↳ R(-3) + R(-4)^4 <-- d_2 --
↳ R(-5)^5 <-- d_3 --
↳ R(-6), given by maps:
↳ d_1 :
↳ Graded homomorphism: R <- R(-3) + R(-4)^4, given by a matrix, with degree\
↳ s:
↳      .1 .2 .3 .4 .5 ...
↳      -- -- -- -- -- +..
↳    0 : 3  4  4  4  4 |.1
↳      == == == == ==
↳      3  4  4  4  4
↳ d_2 :
↳ Graded homomorphism: R(-3) + R(-4)^4 <- R(-5)^5, given by a square matrix\
↳ , with degrees:
↳      ..1 ..2 ..3 ..4 ..5 ....
↳      --- --- --- --- --- +...
↳    3 :  -   2   2   2   2 |..1
↳    4 :  1   1   1   1   1 |..2

```

```

⇒ 4 : 1 1 1 1 1 |..3
⇒ 4 : 1 1 1 1 1 |..4
⇒ 4 : 1 1 1 1 1 |..5
⇒      === === === === ===
⇒      5 5 5 5 5
⇒ d_3 :
⇒ Graded homomorphism: R(-5)^5 <- R(-6), given by a matrix, with degrees:
⇒      .1 ...
⇒      -- +..
⇒ 5 : 1 |.1
⇒ 5 : 1 |.2
⇒ 5 : 1 |.3
⇒ 5 : 1 |.4
⇒ 5 : 1 |.5
⇒      ==
⇒      6
⇒ = TRANSPOSE'd complex: %%%%%%%%%%
⇒      0      1      2      3
⇒ -----
⇒ -6:      1      5      4      -
⇒ -5:      -      -      1      -
⇒ -4:      -      -      -      -
⇒ -3:      -      -      -      1
⇒ -----
⇒ total:      1      5      5      1
⇒
⇒ Graded resolution:
⇒ R(6) <-- d_1 --
⇒ R(5)^5 <-- d_2 --
⇒ R(4)^4 + R(3) <-- d_3 --
⇒ R, given by maps:
⇒ d_1 :
⇒ Graded homomorphism: R(6) <- R(5)^5, given by a matrix, with degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ....
⇒      --- --- --- --- --- +...
⇒ -6 : 1 1 1 1 1 |..1
⇒      === === === === ===
⇒      -5 -5 -5 -5 -5
⇒ d_2 :
⇒ Graded homomorphism: R(5)^5 <- R(4)^4 + R(3), given by a square matrix, w\
ith degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ....
⇒      --- --- --- --- --- +...
⇒ -5 : 1 1 1 1 - |..1
⇒ -5 : 1 1 1 1 2 |..2
⇒ -5 : 1 1 1 1 2 |..3
⇒ -5 : 1 1 1 1 2 |..4
⇒ -5 : 1 1 1 1 2 |..5
⇒      === === === === ===
⇒      -4 -4 -4 -4 -3
⇒ d_3 :
⇒ Graded homomorphism: R(4)^4 + R(3) <- R, given by a matrix, with degrees:
⇒      ..1 ....

```

```

↳      --- +...
↳    -4 :  4 |..1
↳    -4 :  4 |..2
↳    -4 :  4 |..3
↳    -4 :  4 |..4
↳    -3 :  3 |..5
↳      ===
↳      0
↳
↳
↳
string Name = "k3.d14.g19"; int @p=31991; ring R = (@p),(x,y,z,u,v), dp;ideal I = x4
↳ =====
↳
↳ === Example: [ k3.d14.g19 ]
↳ = Ring:  (ZZ/31991),(x,y,z,u,v),(dp(5),C)
↳ ! Resolution via 'grres':
↳ = Non-minimal betti numbers:
↳
↳      0      1      2      3
↳ -----
↳    0:      1      -      -      -
↳    1:      -      -      -      -
↳    2:      -      -      -      -
↳    3:      -      -      -      -
↳    4:      -      4      2      -
↳    5:      -      4      8      3
↳ -----
↳ total:      1      8     10      3
↳
↳ = Degrees of (ordered) maps:
↳ Graded resolution:
↳ R <-- d_1 --
↳ R(-5)^4 + R(-6)^4 <-- d_2 --
↳ R(-6)^2 + R(-7)^8 <-- d_3 --
↳ R(-8)^3, given by maps:
↳ d_1 :
↳ Graded homomorphism: R <- R(-5)^4 + R(-6)^4, given by a matrix, with degr\
  ees:
↳      .1 .2 .3 .4 .5 .6 .7 .8 ...
↳      -- -- -- -- -- -- -- -- +..
↳    0 : 5  5  5  5  6  6  6  6 |.1
↳      == == == == == == == ==
↳      5  5  5  5  6  6  6  6
↳ d_2 :
↳ Graded homomorphism: R(-5)^4 + R(-6)^4 <- R(-6)^2 + R(-7)^8, given by a m\
  atrix, with degrees:
↳      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 ....
↳      --- --- --- --- --- --- --- --- --- +...
↳    5 :  1  1  2  2  2  2  2  2  2  2 |..1
↳    5 :  1  1  2  2  2  2  2  2  2  2 |..2
↳    5 :  1  1  2  2  2  2  2  2  2  2 |..3
↳    5 :  1  1  2  2  2  2  2  2  2  2 |..4
↳    6 :  -  -  1  1  1  1  1  1  1  1 |..5
↳    6 :  -  -  1  1  1  1  1  1  1  1 |..6

```

```

⇒ 6 : - - 1 1 1 1 1 1 1 1 |..7
⇒ 6 : - - 1 1 1 1 1 1 1 1 |..8
⇒      === === === === === === === === ===
⇒      6 6 7 7 7 7 7 7 7 7
⇒ d_3 :
⇒ Graded homomorphism: R(-6)^2 + R(-7)^8 <- R(-8)^3, given by a matrix, with
degrees:
⇒      ..1 ..2 ..3 ....
⇒      --- --- --- +...
⇒ 6 : 2 2 2 |..1
⇒ 6 : 2 2 2 |..2
⇒ 7 : 1 1 1 |..3
⇒ 7 : 1 1 1 |..4
⇒ 7 : 1 1 1 |..5
⇒ 7 : 1 1 1 |..6
⇒ 7 : 1 1 1 |..7
⇒ 7 : 1 1 1 |..8
⇒ 7 : 1 1 1 |..9
⇒ 7 : 1 1 1 |..10
⇒      === === ===
⇒      8 8 8
⇒ = TRANSPOSE'd complex: %%%%%%%%%%
⇒      0 1 2 3
⇒ -----
⇒ -8: 3 8 4 -
⇒ -7: - 2 4 -
⇒ -6: - - - -
⇒ -5: - - - -
⇒ -4: - - - -
⇒ -3: - - - 1
⇒ -----
⇒ total: 3 10 8 1
⇒
⇒ Graded resolution:
⇒ R(8)^3 <-- d_1 --
⇒ R(7)^8 + R(6)^2 <-- d_2 --
⇒ R(6)^4 + R(5)^4 <-- d_3 --
⇒ R, given by maps:
⇒ d_1 :
⇒ Graded homomorphism: R(8)^3 <- R(7)^8 + R(6)^2, given by a matrix, with d\
degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 ....
⇒      --- --- --- --- --- --- --- --- --- +...
⇒ -8 : 1 1 1 1 1 1 1 1 2 2 |..1
⇒ -8 : 1 1 1 1 1 1 1 1 2 2 |..2
⇒ -8 : 1 1 1 1 1 1 1 1 2 2 |..3
⇒      === === === === === === === === ===
⇒      -7 -7 -7 -7 -7 -7 -7 -7 -6 -6
⇒ d_2 :
⇒ Graded homomorphism: R(7)^8 + R(6)^2 <- R(6)^4 + R(5)^4, given by a matrix\
x, with degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ....
⇒      --- --- --- --- --- --- --- --- +...

```

```

⇒ -7 : 1 1 1 1 2 2 2 2 |..1
⇒ -7 : 1 1 1 1 2 2 2 2 |..2
⇒ -7 : 1 1 1 1 2 2 2 2 |..3
⇒ -7 : 1 1 1 1 2 2 2 2 |..4
⇒ -7 : 1 1 1 1 2 2 2 2 |..5
⇒ -7 : 1 1 1 1 2 2 2 2 |..6
⇒ -7 : 1 1 1 1 2 2 2 2 |..7
⇒ -7 : 1 1 1 1 2 2 2 2 |..8
⇒ -6 : - - - - 1 1 1 1 |..9
⇒ -6 : - - - - 1 1 1 1 |.10
⇒      === === === === === === ===
⇒      -6 -6 -6 -6 -5 -5 -5 -5
⇒ d_3 :
⇒ Graded homomorphism: R(6)^4 + R(5)^4 <- R, given by a matrix, with degree\
s:
⇒      ..1 ....
⇒      --- +...
⇒ -6 : 6 |..1
⇒ -6 : 6 |..2
⇒ -6 : 6 |..3
⇒ -6 : 6 |..4
⇒ -5 : 5 |..5
⇒ -5 : 5 |..6
⇒ -5 : 5 |..7
⇒ -5 : 5 |..8
⇒      ===
⇒      0
⇒
⇒
string Name = "k3.d11.g11.ss0"; int @p=31991; ring R = (@p),(x,y,z,u,v), dp;ideal I =
⇒ =====
⇒
⇒ === Example: [ k3.d11.g11.ss0 ]
⇒ = Ring: (ZZ/31991),(x,y,z,u,v),(dp(5),C)
⇒ ! Resolution via 'grres':
⇒ = Non-minimal betti numbers:
⇒      0 1 2 3 4
⇒ -----
⇒ 0: 1 - - - -
⇒ 1: - - - - -
⇒ 2: - - - - -
⇒ 3: - - - - -
⇒ 4: - 9 8 - -
⇒ 5: - - 5 7 2
⇒ -----
⇒ total: 1 9 13 7 2
⇒
⇒ = Degrees of (ordered) maps:
⇒ Graded resolution:
⇒ R <-- d_1 --
⇒ R(-5)^9 <-- d_2 --
⇒ R(-6)^8 + R(-7)^5 <-- d_3 --
⇒ R(-8)^7 <-- d_4 --

```



```

⇒ R(-9)^2, given by maps:
⇒ d_1 :
⇒ Graded homomorphism: R <- R(-5)^9, given by a matrix, with degrees:
⇒      .1 .2 .3 .4 .5 .6 .7 .8 .9 ...
⇒      -- -- -- -- -- -- -- -- -- +..
⇒ 0 : 5  5  5  5  5  5  5  5  5 |.1
⇒      == == == == == == == == ==
⇒      5  5  5  5  5  5  5  5  5
⇒ d_2 :
⇒ Graded homomorphism: R(-5)^9 <- R(-6)^8 + R(-7)^5, given by a matrix, wit\
h degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 .11 .12 .13 ....
⇒      --- --- --- --- --- --- --- --- --- --- --- --- --- +...
⇒ 5 :  1  1  1  1  1  1  1  1  1  2  2  2  2  2 |..1
⇒ 5 :  1  1  1  1  1  1  1  1  1  2  2  2  2  2 |..2
⇒ 5 :  1  1  1  1  1  1  1  1  1  2  2  2  2  2 |..3
⇒ 5 :  1  1  1  1  1  1  1  1  1  2  2  2  2  2 |..4
⇒ 5 :  1  1  1  1  1  1  1  1  1  2  2  2  2  2 |..5
⇒ 5 :  1  1  1  1  1  1  1  1  1  2  2  2  2  2 |..6
⇒ 5 :  1  1  1  1  1  1  1  1  1  2  2  2  2  2 |..7
⇒ 5 :  1  1  1  1  1  1  1  1  1  2  2  2  2  2 |..8
⇒ 5 :  1  1  1  1  1  1  1  1  1  -  -  -  -  - |..9
⇒      === === === === === === === === ===
⇒      6  6  6  6  6  6  6  6  6  7  7  7  7  7
⇒ d_3 :
⇒ Graded homomorphism: R(-6)^8 + R(-7)^5 <- R(-8)^7, given by a matrix, wit\
h degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ....
⇒      --- --- --- --- --- --- --- +...
⇒ 6 :  -  -  2  2  2  2  2 |..1
⇒ 6 :  -  -  2  2  2  2  2 |..2
⇒ 6 :  -  -  2  2  2  2  2 |..3
⇒ 6 :  -  -  2  2  2  2  2 |..4
⇒ 6 :  -  -  2  2  2  2  2 |..5
⇒ 6 :  -  -  2  2  2  2  2 |..6
⇒ 6 :  -  -  2  2  2  2  2 |..7
⇒ 6 :  -  -  2  2  2  2  2 |..8
⇒ 7 :  1  1  1  1  1  1  1 |..9
⇒ 7 :  1  1  1  1  1  1  1 |..10
⇒ 7 :  1  1  1  1  1  1  1 |..11
⇒ 7 :  1  1  1  1  1  1  1 |..12
⇒ 7 :  1  1  1  1  1  1  1 |..13
⇒      === === === === === === ===
⇒      8  8  8  8  8  8  8
⇒ d_4 :
⇒ Graded homomorphism: R(-8)^7 <- R(-9)^2, given by a matrix, with degrees:
⇒      .1 .2 ...
⇒      -- -- +..
⇒ 8 :  1  1 |.1
⇒ 8 :  1  1 |.2
⇒ 8 :  1  1 |.3
⇒ 8 :  1  1 |.4
⇒ 8 :  1  1 |.5

```

```

⇒ 8 : 1 1 |.6
⇒ 8 : 1 1 |.7
⇒ == ==
⇒ 9 9
⇒ = TRANSPOSE'd complex: %%%%%%%%%%
⇒      0      1      2      3      4
⇒ -----
⇒ -9:      2      7      5      -      -
⇒ -8:      -      -      8      9      -
⇒ -7:      -      -      -      -      -
⇒ -6:      -      -      -      -      -
⇒ -5:      -      -      -      -      -
⇒ -4:      -      -      -      -      1
⇒ -----
⇒ total:      2      7      13      9      1
⇒
⇒ Graded resolution:
⇒ R(9)^2 <-- d_1 --
⇒ R(8)^7 <-- d_2 --
⇒ R(7)^5 + R(6)^8 <-- d_3 --
⇒ R(5)^9 <-- d_4 --
⇒ R, given by maps:
⇒ d_1 :
⇒ Graded homomorphism: R(9)^2 <- R(8)^7, given by a matrix, with degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ....
⇒      --- --- --- --- --- --- --- +...
⇒ -9 : 1 1 1 1 1 1 1 |..1
⇒ -9 : 1 1 1 1 1 1 1 |..2
⇒      === === === === === === ===
⇒      -8 -8 -8 -8 -8 -8 -8
⇒ d_2 :
⇒ Graded homomorphism: R(8)^7 <- R(7)^5 + R(6)^8, given by a matrix, with d\
egrees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 .11 .12 .13 ....
⇒      --- --- --- --- --- --- --- --- --- --- --- --- --- +...
⇒ -8 : 1 1 1 1 1 - - - - - - - - |..1
⇒ -8 : 1 1 1 1 1 - - - - - - - - |..2
⇒ -8 : 1 1 1 1 1 2 2 2 2 2 2 2 2 |..3
⇒ -8 : 1 1 1 1 1 2 2 2 2 2 2 2 2 |..4
⇒ -8 : 1 1 1 1 1 2 2 2 2 2 2 2 2 |..5
⇒ -8 : 1 1 1 1 1 2 2 2 2 2 2 2 2 |..6
⇒ -8 : 1 1 1 1 1 2 2 2 2 2 2 2 2 |..7
⇒      === === === === === === === === === === ===
⇒      -7 -7 -7 -7 -7 -6 -6 -6 -6 -6 -6 -6 -6
⇒ d_3 :
⇒ Graded homomorphism: R(7)^5 + R(6)^8 <- R(5)^9, given by a matrix, with d\
egrees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 ....
⇒      --- --- --- --- --- --- --- --- --- +...
⇒ -7 : 2 2 2 2 2 2 2 2 - |..1
⇒ -7 : 2 2 2 2 2 2 2 2 - |..2
⇒ -7 : 2 2 2 2 2 2 2 2 - |..3
⇒ -7 : 2 2 2 2 2 2 2 2 - |..4

```

```

↳ -7 : 2 2 2 2 2 2 2 2 - |..5
↳ -6 : 1 1 1 1 1 1 1 1 1 |..6
↳ -6 : 1 1 1 1 1 1 1 1 1 |..7
↳ -6 : 1 1 1 1 1 1 1 1 1 |..8
↳ -6 : 1 1 1 1 1 1 1 1 1 |..9
↳ -6 : 1 1 1 1 1 1 1 1 1 |.10
↳ -6 : 1 1 1 1 1 1 1 1 1 |.11
↳ -6 : 1 1 1 1 1 1 1 1 1 |.12
↳ -6 : 1 1 1 1 1 1 1 1 1 |.13
↳      === === === === === === === ===
↳      -5 -5 -5 -5 -5 -5 -5 -5 -5
↳ d_4 :
↳ Graded homomorphism: R(5)^9 <- R, given by a matrix, with degrees:
↳      ..1 ....
↳      --- +...
↳ -5 : 5 |..1
↳ -5 : 5 |..2
↳ -5 : 5 |..3
↳ -5 : 5 |..4
↳ -5 : 5 |..5
↳ -5 : 5 |..6
↳ -5 : 5 |..7
↳ -5 : 5 |..8
↳ -5 : 5 |..9
↳      ===
↳      0
↳
↳
↳ string Name = "ell.d10.g9"; int @p=31991; ring R = (@p),(x,y,z,u,v), dp;ideal I = -4
↳ =====
↳
↳ === Example: [ ell.d10.g9 ]
↳ = Ring: (ZZ/31991),(x,y,z,u,v),(dp(5),C)
↳ ! Resolution via 'grres':
↳ = Non-minimal betti numbers:
↳      0      1      2      3      4
↳ -----
↳ 0:      1      -      -      -      -
↳ 1:      -      -      -      -      -
↳ 2:      -      -      -      -      -
↳ 3:      -      1      -      -      -
↳ 4:      -      9     14      5      -
↳ 5:      -      -      1      2      1
↳ -----
↳ total:      1     10     15      7      1
↳
↳ = Degrees of (ordered) maps:
↳ Graded resolution:
↳ R <-- d_1 --
↳ R(-4) + R(-5)^9 <-- d_2 --
↳ R(-6)^14 + R(-7) <-- d_3 --
↳ R(-7)^5 + R(-8)^2 <-- d_4 --
↳ R(-9), given by maps:

```

```

⇒ d_1 :
⇒ Graded homomorphism: R <- R(-4) + R(-5)^9, given by a matrix, with degree\
s:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 ....
⇒      --- --- --- --- --- --- --- --- --- --- +...
⇒ 0 : 4   5   5   5   5   5   5   5   5   5 |..1
⇒      === === === === === === === === === ===
⇒      4   5   5   5   5   5   5   5   5   5
⇒ d_2 :
⇒ Graded homomorphism: R(-4) + R(-5)^9 <- R(-6)^14 + R(-7), given by a matr\
ix, with degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 .11 .12 .13 .14 .15 ....
⇒      --- --- --- --- --- --- --- --- --- --- --- --- --- --- +...
⇒ 4 : -   -   -   -   -   2   2   2   2   2   2   2   2   2   - |..1
⇒ 5 : 1   1   1   1   1   1   1   1   1   1   1   1   1   1   2 |..2
⇒ 5 : 1   1   1   1   1   1   1   1   1   1   1   1   1   1   2 |..3
⇒ 5 : 1   1   1   1   1   1   1   1   1   1   1   1   1   1   2 |..4
⇒ 5 : 1   1   1   1   1   1   1   1   1   1   1   1   1   1   2 |..5
⇒ 5 : 1   1   1   1   1   1   1   1   1   1   1   1   1   1   2 |..6
⇒ 5 : 1   1   1   1   1   1   1   1   1   1   1   1   1   1   - |..7
⇒ 5 : 1   1   1   1   1   1   1   1   1   1   1   1   1   1   - |..8
⇒ 5 : 1   1   1   1   1   1   1   1   1   1   1   1   1   1   - |..9
⇒ 5 : 1   1   1   1   1   1   1   1   1   1   1   1   1   1   - |..10
⇒      === === === === === === === === === ===
⇒      6   6   6   6   6   6   6   6   6   6   6   6   6   6   7
⇒ d_3 :
⇒ Graded homomorphism: R(-6)^14 + R(-7) <- R(-7)^5 + R(-8)^2, given by a ma\
trix, with degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ....
⇒      --- --- --- --- --- --- --- +...
⇒ 6 : 1   1   1   1   1   2   - |..1
⇒ 6 : 1   1   1   1   1   2   - |..2
⇒ 6 : 1   1   1   1   1   2   - |..3
⇒ 6 : 1   1   1   1   1   2   - |..4
⇒ 6 : 1   1   1   1   1   2   - |..5
⇒ 6 : 1   1   1   1   1   -   2 |..6
⇒ 6 : 1   1   1   1   1   -   2 |..7
⇒ 6 : 1   1   1   1   1   -   2 |..8
⇒ 6 : 1   1   1   1   1   -   2 |..9
⇒ 6 : 1   1   1   1   1   -   2 |..10
⇒ 6 : 1   1   1   1   1   -   2 |..11
⇒ 6 : 1   1   1   1   1   -   2 |..12
⇒ 6 : 1   1   1   1   1   -   2 |..13
⇒ 6 : 1   1   1   1   1   -   - |..14
⇒ 7 : -   -   -   -   -   1   1 |..15
⇒      === === === === === ===
⇒      7   7   7   7   7   8   8
⇒ d_4 :
⇒ Graded homomorphism: R(-7)^5 + R(-8)^2 <- R(-9), given by a matrix, with \
degrees:
⇒      .1 ...
⇒      -- +..
⇒ 7 : 2 |.1

```

```

⇒ 7 : 2 |.2
⇒ 7 : 2 |.3
⇒ 7 : 2 |.4
⇒ 7 : 2 |.5
⇒ 8 : 1 |.6
⇒ 8 : 1 |.7
⇒ ==
⇒ 9
⇒ = TRANSPOSE'd complex: %%%%%%%%%%
⇒      0      1      2      3      4
⇒ -----
⇒ -9:      1      2      1      -      -
⇒ -8:      -      5     14      9      -
⇒ -7:      -      -      -      1      -
⇒ -6:      -      -      -      -      -
⇒ -5:      -      -      -      -      -
⇒ -4:      -      -      -      -      1
⇒ -----
⇒ total:      1      7     15     10      1
⇒
⇒ Graded resolution:
⇒ R(9) <-- d_1 --
⇒ R(8)^2 + R(7)^5 <-- d_2 --
⇒ R(7) + R(6)^14 <-- d_3 --
⇒ R(5)^9 + R(4) <-- d_4 --
⇒ R, given by maps:
⇒ d_1 :
⇒ Graded homomorphism: R(9) <- R(8)^2 + R(7)^5, given by a matrix, with deg\
rees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ....
⇒      --- --- --- --- --- --- --- +...
⇒ -9 :  1   1   2   2   2   2   2 |..1
⇒      === === === === === === ===
⇒      -8  -8  -7  -7  -7  -7  -7
⇒ d_2 :
⇒ Graded homomorphism: R(8)^2 + R(7)^5 <- R(7) + R(6)^14, given by a matrix\
, with degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 .11 .12 .13 .14 .15 ....
⇒      --- --- --- --- --- --- --- --- --- --- --- --- --- --- --- +...
⇒ -8 :  1   2   2   2   2   2   -   -   -   -   -   -   -   -   - |..1
⇒ -8 :  1   -   -   -   -   -   2   2   2   2   2   2   2   2   - |..2
⇒ -7 :  -   1   1   1   1   1   1   1   1   1   1   1   1   1   1 |..3
⇒ -7 :  -   1   1   1   1   1   1   1   1   1   1   1   1   1   1 |..4
⇒ -7 :  -   1   1   1   1   1   1   1   1   1   1   1   1   1   1 |..5
⇒ -7 :  -   1   1   1   1   1   1   1   1   1   1   1   1   1   1 |..6
⇒ -7 :  -   1   1   1   1   1   1   1   1   1   1   1   1   1   1 |..7
⇒      === === === === === === === === === === === === === === ===
⇒      -7  -6  -6  -6  -6  -6  -6  -6  -6  -6  -6  -6  -6  -6  -6
⇒ d_3 :
⇒ Graded homomorphism: R(7) + R(6)^14 <- R(5)^9 + R(4), given by a matrix, \
with degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 ....
⇒      --- --- --- --- --- --- --- --- --- --- +...

```

```

⇒ -7 : 2 2 2 2 2 - - - - - |..1
⇒ -6 : 1 1 1 1 1 1 1 1 1 - |..2
⇒ -6 : 1 1 1 1 1 1 1 1 1 - |..3
⇒ -6 : 1 1 1 1 1 1 1 1 1 - |..4
⇒ -6 : 1 1 1 1 1 1 1 1 1 - |..5
⇒ -6 : 1 1 1 1 1 1 1 1 1 - |..6
⇒ -6 : 1 1 1 1 1 1 1 1 1 2 |..7
⇒ -6 : 1 1 1 1 1 1 1 1 1 2 |..8
⇒ -6 : 1 1 1 1 1 1 1 1 1 2 |..9
⇒ -6 : 1 1 1 1 1 1 1 1 1 2 |.10
⇒ -6 : 1 1 1 1 1 1 1 1 1 2 |.11
⇒ -6 : 1 1 1 1 1 1 1 1 1 2 |.12
⇒ -6 : 1 1 1 1 1 1 1 1 1 2 |.13
⇒ -6 : 1 1 1 1 1 1 1 1 1 2 |.14
⇒ -6 : 1 1 1 1 1 1 1 1 1 2 |.15
⇒      === === === === === === === === ===
⇒      -5 -5 -5 -5 -5 -5 -5 -5 -5 -4
⇒ d_4 :
⇒ Graded homomorphism: R(5)^9 + R(4) <- R, given by a matrix, with degrees:
⇒      ..1 ....
⇒      --- +...
⇒ -5 : 5 |..1
⇒ -5 : 5 |..2
⇒ -5 : 5 |..3
⇒ -5 : 5 |..4
⇒ -5 : 5 |..5
⇒ -5 : 5 |..6
⇒ -5 : 5 |..7
⇒ -5 : 5 |..8
⇒ -5 : 5 |..9
⇒ -4 : 4 |.10
⇒      ===
⇒      0
⇒
⇒
string Name = "k3.d10.g9.quart2"; int @p=31991; ring R = (@p),(x,y,z,u,v), dp;ideal I
⇒ =====
⇒
⇒ === Example: [ k3.d10.g9.quart2 ]
⇒ = Ring: (ZZ/31991),(x,y,z,u,v),(dp(5),C)
⇒ ! Resolution via 'grres':
⇒ = Non-minimal betti numbers:
⇒      0      1      2      3      4
⇒ -----
⇒ 0:      1      -      -      -      -
⇒ 1:      -      -      -      -      -
⇒ 2:      -      -      -      -      -
⇒ 3:      -      2      -      -      -
⇒ 4:      -      4      7      2      -
⇒ 5:      -      3      8      7      2
⇒ -----
⇒ total:      1      9      15      9      2
⇒

```

[illegible]

```

⇒ d_4 :
⇒ Graded homomorphism: R(-7)^2 + R(-8)^7 <- R(-9)^2, given by a matrix, with
  degrees:
⇒      .1 .2 ...
⇒      -- -- +..
⇒      7 : 2  2 |.1
⇒      7 : 2  2 |.2
⇒      8 : 1  1 |.3
⇒      8 : 1  1 |.4
⇒      8 : 1  1 |.5
⇒      8 : 1  1 |.6
⇒      8 : 1  1 |.7
⇒      8 : 1  1 |.8
⇒      8 : 1  1 |.9
⇒      == ==
⇒      9  9
⇒ = TRANSPOSE'd complex: %%%%%%%%%%
⇒      0      1      2      3      4
⇒ -----
⇒      -9:      2      7      8      3      -
⇒      -8:      -      2      7      4      -
⇒      -7:      -      -      -      2      -
⇒      -6:      -      -      -      -      -
⇒      -5:      -      -      -      -      -
⇒      -4:      -      -      -      -      1
⇒ -----
⇒ total:      2      9     15      9      1
⇒
⇒ Graded resolution:
⇒ R(9)^2 <-- d_1 --
⇒ R(8)^7 + R(7)^2 <-- d_2 --
⇒ R(7)^8 + R(6)^7 <-- d_3 --
⇒ R(6)^3 + R(5)^4 + R(4)^2 <-- d_4 --
⇒ R, given by maps:
⇒ d_1 :
⇒ Graded homomorphism: R(9)^2 <- R(8)^7 + R(7)^2, given by a matrix, with d\
  egresses:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 ....
⇒      --- --- --- --- --- --- --- --- --- +...
⇒      -9 :  1   1   1   1   1   1   1   2   2 |..1
⇒      -9 :  1   1   1   1   1   1   1   2   2 |..2
⇒      === === === === === === === === ===
⇒      -8  -8  -8  -8  -8  -8  -8  -7  -7
⇒ d_2 :
⇒ Graded homomorphism: R(8)^7 + R(7)^2 <- R(7)^8 + R(6)^7, given by a matrix\
  x, with degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 .11 .12 .13 .14 .15 ....
⇒      --- --- --- --- --- --- --- --- --- --- --- --- --- --- +...
⇒      -8 :  1   1   1   1   1   1   1   1   2   2   2   2   2   2   2 |..1
⇒      -8 :  1   1   1   1   1   1   1   1   2   2   2   2   2   2   2 |..2
⇒      -8 :  1   1   1   1   1   1   1   1   2   2   2   2   2   2   2 |..3
⇒      -8 :  1   1   1   1   1   1   1   1   2   2   2   2   2   2   2 |..4
⇒      -8 :  1   1   1   1   1   1   1   1   2   2   2   2   2   2   2 |..5

```



```

⇒ -8 : 1 1 1 1 1 1 1 1 2 2 2 2 2 2 2 |..6
⇒ -8 : 1 1 1 1 1 1 1 1 2 2 2 2 2 2 2 |..7
⇒ -7 : - - - - - - - - 1 1 1 1 1 1 1 |..8
⇒ -7 : - - - - - - - - 1 1 1 1 1 1 1 |..9
⇒      === === === === === === === === === ===
⇒      -7 -7 -7 -7 -7 -7 -7 -7 -6 -6 -6 -6 -6 -6 -6
⇒ d_3 :
⇒ Graded homomorphism: R(7)^8 + R(6)^7 <- R(6)^3 + R(5)^4 + R(4)^2, given b\
y a matrix, with degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 ....
⇒      --- --- --- --- --- --- --- --- --- +...
⇒ -7 : 1 1 1 2 2 - - - - |..1
⇒ -7 : 1 1 1 2 2 - - - - |..2
⇒ -7 : 1 1 1 2 2 2 2 3 3 |..3
⇒ -7 : 1 1 1 2 2 2 2 3 3 |..4
⇒ -7 : 1 1 1 2 2 2 2 3 3 |..5
⇒ -7 : 1 1 1 2 2 2 2 3 3 |..6
⇒ -7 : 1 1 1 2 2 2 2 3 3 |..7
⇒ -7 : 1 1 1 2 2 2 2 3 3 |..8
⇒ -6 : - - - 1 1 1 1 2 2 |..9
⇒ -6 : - - - 1 1 1 1 2 2 |.10
⇒ -6 : - - - 1 1 1 1 2 2 |.11
⇒ -6 : - - - 1 1 1 1 2 2 |.12
⇒ -6 : - - - 1 1 1 1 2 2 |.13
⇒ -6 : - - - 1 1 1 1 2 2 |.14
⇒ -6 : - - - 1 1 1 1 2 2 |.15
⇒      === === === === === === === === ===
⇒      -6 -6 -6 -5 -5 -5 -5 -4 -4
⇒ d_4 :
⇒ Graded homomorphism: R(6)^3 + R(5)^4 + R(4)^2 <- R, given by a matrix, wi\
th degrees:
⇒      ..1 ....
⇒      --- +...
⇒ -6 : 6 |..1
⇒ -6 : 6 |..2
⇒ -6 : 6 |..3
⇒ -5 : 5 |..4
⇒ -5 : 5 |..5
⇒ -5 : 5 |..6
⇒ -5 : 5 |..7
⇒ -4 : 4 |..8
⇒ -4 : 4 |..9
⇒      ===
⇒      0
⇒
⇒
string Name = "rat.d10.g9.quart2"; int @p=31991; ring R = (@p),(x,y,z,u,v), dp;ideal
⇒ =====
⇒
⇒ === Example: [ rat.d10.g9.quart2 ]
⇒ = Ring: (ZZ/31991),(x,y,z,u,v),(dp(5),C)
⇒ ! Resolution via 'grres':
⇒ = Non-minimal betti numbers:

```

```

⇒      0      1      2      3      4
⇒ -----
⇒    0:      1      -      -      -      -
⇒    1:      -      -      -      -      -
⇒    2:      -      -      -      -      -
⇒    3:      -      2      -      -      -
⇒    4:      -      5      9      3      -
⇒    5:      -      1      3      3      1
⇒ -----
⇒ total:      1      8     12      6      1
⇒
⇒ = Degrees of (ordered) maps:
⇒ Graded resolution:
⇒ R <-- d_1 --
⇒ R(-4)^2 + R(-5)^5 + R(-6) <-- d_2 --
⇒ R(-6)^9 + R(-7)^3 <-- d_3 --
⇒ R(-7)^3 + R(-8)^3 <-- d_4 --
⇒ R(-9), given by maps:
⇒ d_1 :
⇒ Graded homomorphism: R <- R(-4)^2 + R(-5)^5 + R(-6), given by a matrix, w\
ith degrees:
⇒      .1 .2 .3 .4 .5 .6 .7 .8 ...
⇒      -- -- -- -- -- -- -- -- +..
⇒    0 : 4  4  5  5  5  5  5  6 |.1
⇒      == == == == == == == ==
⇒      4  4  5  5  5  5  5  6
⇒ d_2 :
⇒ Graded homomorphism: R(-4)^2 + R(-5)^5 + R(-6) <- R(-6)^9 + R(-7)^3, give\
n by a matrix, with degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 .11 .12 ....
⇒      --- --- --- --- --- --- --- --- --- --- --- --- +...
⇒    4 :  2   2   2   2   2   2   2   2   2   -   3   3 |..1
⇒    4 :  2   2   2   2   2   2   2   2   2   -   3   3 |..2
⇒    5 :  1   1   1   1   1   1   1   1   1   1   2   2 |..3
⇒    5 :  1   1   1   1   1   1   1   1   1   1   2   2 |..4
⇒    5 :  1   1   1   1   1   1   1   1   1   1   2   2 |..5
⇒    5 :  1   1   1   1   1   1   1   1   1   1   -   2   2 |..6
⇒    5 :  1   1   1   1   1   1   1   1   1   1   -   2   2 |..7
⇒    6 :  -   -   -   -   -   -   -   -   -   -   1   1   1 |..8
⇒      === === === === === === === === === === === ===
⇒      6   6   6   6   6   6   6   6   6   6   7   7   7
⇒ d_3 :
⇒ Graded homomorphism: R(-6)^9 + R(-7)^3 <- R(-7)^3 + R(-8)^3, given by a m\
atrix, with degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ....
⇒      --- --- --- --- --- --- --- +...
⇒    6 :  1   1   1   2   2   2 |..1
⇒    6 :  1   1   1   2   2   2 |..2
⇒    6 :  1   1   1   2   2   2 |..3
⇒    6 :  1   1   1   2   2   2 |..4
⇒    6 :  1   1   1   2   2   2 |..5
⇒    6 :  1   1   1   2   2   2 |..6
⇒    6 :  1   1   1   2   2   2 |..7

```

```

⇒ 6 : 1 1 1 2 2 2 |..8
⇒ 6 : 1 1 1 - - 2 |..9
⇒ 7 : - - - 1 1 1 |.10
⇒ 7 : - - - 1 1 1 |.11
⇒ 7 : - - - 1 1 1 |.12
⇒      === === === === === ===
⇒      7 7 7 8 8 8
⇒ d_4 :
⇒ Graded homomorphism: R(-7)^3 + R(-8)^3 <- R(-9), given by a matrix, with \
degrees:
⇒      .1 ...
⇒      -- +..
⇒ 7 : 2 |.1
⇒ 7 : 2 |.2
⇒ 7 : 2 |.3
⇒ 8 : 1 |.4
⇒ 8 : 1 |.5
⇒ 8 : 1 |.6
⇒      ==
⇒      9
⇒ = TRANSPOSE'd complex: %%%%%%%%%%
⇒      0 1 2 3 4
⇒ -----
⇒ -9: 1 3 3 1 -
⇒ -8: - 3 9 5 -
⇒ -7: - - - 2 -
⇒ -6: - - - - -
⇒ -5: - - - - -
⇒ -4: - - - - 1
⇒ -----
⇒ total: 1 6 12 8 1
⇒
⇒ Graded resolution:
⇒ R(9) <-- d_1 --
⇒ R(8)^3 + R(7)^3 <-- d_2 --
⇒ R(7)^3 + R(6)^9 <-- d_3 --
⇒ R(6) + R(5)^5 + R(4)^2 <-- d_4 --
⇒ R, given by maps:
⇒ d_1 :
⇒ Graded homomorphism: R(9) <- R(8)^3 + R(7)^3, given by a matrix, with deg\
rees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ....
⇒      --- --- --- --- --- --- +...
⇒ -9 : 1 1 1 2 2 2 |..1
⇒      === === === === === ===
⇒      -8 -8 -8 -7 -7 -7
⇒ d_2 :
⇒ Graded homomorphism: R(8)^3 + R(7)^3 <- R(7)^3 + R(6)^9, given by a matri\
x, with degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 .11 .12 ....
⇒      --- --- --- --- --- --- --- --- --- --- --- --- +...
⇒ -8 : 1 1 1 2 2 2 2 2 2 2 2 - |..1
⇒ -8 : 1 1 1 2 2 2 2 2 2 2 2 - |..2

```

```

⇒ -8 : 1 1 1 2 2 2 2 2 2 2 2 2 |..3
⇒ -7 : - - - 1 1 1 1 1 1 1 1 1 |..4
⇒ -7 : - - - 1 1 1 1 1 1 1 1 1 |..5
⇒ -7 : - - - 1 1 1 1 1 1 1 1 1 |..6
⇒      === === === === === === === === === ===
⇒      -7 -7 -7 -6 -6 -6 -6 -6 -6 -6 -6 -6
⇒ d_3 :
⇒ Graded homomorphism: R(7)^3 + R(6)^9 <- R(6) + R(5)^5 + R(4)^2, given by \
  a matrix, with degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ....
⇒      --- --- --- --- --- --- --- --- +...
⇒ -7 : 1 2 2 2 - - - - |..1
⇒ -7 : 1 2 2 2 2 2 3 3 |..2
⇒ -7 : 1 2 2 2 2 2 3 3 |..3
⇒ -6 : - 1 1 1 1 1 2 2 |..4
⇒ -6 : - 1 1 1 1 1 2 2 |..5
⇒ -6 : - 1 1 1 1 1 2 2 |..6
⇒ -6 : - 1 1 1 1 1 2 2 |..7
⇒ -6 : - 1 1 1 1 1 2 2 |..8
⇒ -6 : - 1 1 1 1 1 2 2 |..9
⇒ -6 : - 1 1 1 1 1 2 2 |.10
⇒ -6 : - 1 1 1 1 1 2 2 |.11
⇒ -6 : - 1 1 1 1 1 2 2 |.12
⇒      === === === === === === === ===
⇒      -6 -5 -5 -5 -5 -5 -4 -4
⇒ d_4 :
⇒ Graded homomorphism: R(6) + R(5)^5 + R(4)^2 <- R, given by a matrix, with\
  degrees:
⇒      ..1 ....
⇒      --- +...
⇒ -6 : 6 |..1
⇒ -5 : 5 |..2
⇒ -5 : 5 |..3
⇒ -5 : 5 |..4
⇒ -5 : 5 |..5
⇒ -5 : 5 |..6
⇒ -4 : 4 |..7
⇒ -4 : 4 |..8
⇒      ===
⇒      0
⇒
⇒
// if( defined(assumeLevel0) ){ assumeLevel = assumeLevel0; } else { kill assumeLevel0; }

```

### D.15.6.17 KeneshlouMatrixPresentation

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\]](#), page 2346).

**Usage:** `KeneshlouMatrixPresentation(intvec a), intvec a.`

**Return:** graded object

**Purpose:** matrix presentation for direct sum of  $\omega^a[i](i)$  in form of a graded object

**Example:**

```

LIB "gradedModules.lib";
ring r = 32003,(x(0..4)),dp;
def N1 = KeneshlouMatrixPresentation(intvec(2,0,0,0,0));
grview(N1);
⇒ Graded homomorphism:  $r^2 \leftarrow 0$ , given by zero (2 x 0) matrix.
def N2 = KeneshlouMatrixPresentation(intvec(0,0,0,0,3));
grview(N2);
⇒ Graded homomorphism:  $r(-1)^3 \leftarrow 0$ , given by zero (3 x 0) matrix.
def N = KeneshlouMatrixPresentation(intvec(2,0,0,0,3));
grview(N);
⇒ Graded homomorphism:  $r^2 + r(-1)^3 \leftarrow 0$ , given by zero (5 x 0) matrix.
def M1 = KeneshlouMatrixPresentation(intvec(0,1,0,0,0));
grview(M1);
⇒ Graded homomorphism:  $r(-1)^{10} \leftarrow r(-2)^{10}$ , given by a square matrix, with\
degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 ....
⇒      --- --- --- --- --- --- --- --- --- --- +...
⇒  1 :  1  1  -  -  1  -  -  -  -  -  |..1
⇒  1 :  1  -  1  -  -  1  -  -  -  -  |..2
⇒  1 :  1  -  -  1  -  -  1  -  -  -  -  |..3
⇒  1 :  -  1  1  -  -  -  -  1  -  -  -  |..4
⇒  1 :  -  1  -  1  -  -  -  -  1  -  -  |..5
⇒  1 :  -  -  1  1  -  -  -  -  -  1  |..6
⇒  1 :  -  -  -  -  1  1  -  1  -  -  |..7
⇒  1 :  -  -  -  -  1  -  1  -  1  -  |..8
⇒  1 :  -  -  -  -  -  1  1  -  -  1  |..9
⇒  1 :  -  -  -  -  -  -  -  1  1  1  |..10
⇒      === === === === === === === === === ===
⇒      2  2  2  2  2  2  2  2  2  2
def M2 = KeneshlouMatrixPresentation(intvec(0,1,1,0,0));
grview(M2);
⇒ Graded homomorphism:  $r(-1)^{20} \leftarrow r(-2)^{15}$ , given by a matrix, with degree\
s:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 .11 .12 .13 .14 .15 ....
⇒      --- --- --- --- --- --- --- --- --- --- --- --- --- --- +...
⇒  1 :  1  1  -  -  1  -  -  -  -  -  -  -  -  -  -  |..1
⇒  1 :  1  -  1  -  -  1  -  -  -  -  -  -  -  -  -  |..2
⇒  1 :  1  -  -  1  -  -  -  -  1  -  -  -  -  -  -  |..3
⇒  1 :  -  1  1  -  -  -  -  1  -  -  -  -  -  -  |..4
⇒  1 :  -  1  -  1  -  -  -  -  -  1  -  -  -  -  -  |..5
⇒  1 :  -  -  1  1  -  -  -  -  -  1  -  -  -  -  -  |..6
⇒  1 :  -  -  -  -  1  1  -  1  -  -  -  -  -  -  |..7
⇒  1 :  -  -  -  -  1  -  1  -  1  -  -  -  -  -  -  |..8
⇒  1 :  -  -  -  -  -  1  1  -  -  1  -  -  -  -  -  |..9
⇒  1 :  -  -  -  -  -  -  -  1  1  1  -  -  -  -  -  |..10
⇒  1 :  -  -  -  -  -  -  -  -  -  -  1  1  -  -  -  |..11
⇒  1 :  -  -  -  -  -  -  -  -  -  -  -  1  -  1  -  |..12
⇒  1 :  -  -  -  -  -  -  -  -  -  -  -  -  1  -  1  |..13
⇒  1 :  -  -  -  -  -  -  -  -  -  -  -  -  -  1  1  |..14
⇒  1 :  -  -  -  -  -  -  -  -  -  -  -  -  -  -  1  |..15
⇒  1 :  -  -  -  -  -  -  -  -  -  -  -  -  -  -  -  |..16
⇒  1 :  -  -  -  -  -  -  -  -  -  -  -  -  -  -  1  |..17
⇒  1 :  -  -  -  -  -  -  -  -  -  -  -  -  1  1  -  |..18

```

```

⇒ 1 : - - - - - - - - - - - - - 1 - 1 |.19
⇒ 1 : - - - - - - - - - - - - - - 1 1 |.20
⇒      === === === === === === === === === === === === ===
⇒      2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2
def M3 = KeneshlouMatrixPresentation(intvec(0,0,0,1,0));
grview(M3);
⇒ Graded homomorphism: r(-1)^5 <- r(-2), given by a matrix, with degrees:
⇒      .1 ...
⇒      -- +..
⇒ 1 : 1 |.1
⇒ 1 : 1 |.2
⇒ 1 : 1 |.3
⇒ 1 : 1 |.4
⇒ 1 : 1 |.5
⇒      ==
⇒      2
def M = KeneshlouMatrixPresentation(intvec(1,1,1,0,0));
grview(M);
⇒ Graded homomorphism: r + r(-1)^20 <- r(-2)^15, given by a matrix, with de\
grees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 .11 .12 .13 .14 .15 ....
⇒      --- --- --- --- --- --- --- --- --- --- --- --- --- --- --- +...
⇒ 0 : - - - - - - - - - - - - - - - - - |..1
⇒ 1 : 1 1 - - 1 - - - - - - - - - - - |..2
⇒ 1 : 1 - 1 - - 1 - - - - - - - - - - - |..3
⇒ 1 : 1 - - 1 - - - 1 - - - - - - - - - |..4
⇒ 1 : - 1 1 - - - - 1 - - - - - - - - - |..5
⇒ 1 : - 1 - 1 - - - - 1 - - - - - - - - - |..6
⇒ 1 : - - 1 1 - - - - - 1 - - - - - - - - |..7
⇒ 1 : - - - - 1 1 - 1 - - - - - - - - - |..8
⇒ 1 : - - - - 1 - 1 - 1 - - - - - - - - - |..9
⇒ 1 : - - - - - 1 1 - - 1 - - - - - - - - |..10
⇒ 1 : - - - - - - - 1 1 1 - - - - - - - - |..11
⇒ 1 : - - - - - - - - - 1 1 - - - - - - - |..12
⇒ 1 : - - - - - - - - - - 1 - 1 - - - - - |..13
⇒ 1 : - - - - - - - - - - 1 - - 1 - - - - |..14
⇒ 1 : - - - - - - - - - - - 1 - - - 1 |..15
⇒ 1 : - - - - - - - - - - - - 1 1 - - - |..16
⇒ 1 : - - - - - - - - - - - - 1 - 1 - - |..17
⇒ 1 : - - - - - - - - - - - - - 1 - 1 |..18
⇒ 1 : - - - - - - - - - - - - - - 1 1 |..19
⇒ 1 : - - - - - - - - - - - - - - - 1 |..20
⇒ 1 : - - - - - - - - - - - - - - - 1 1 |..21
⇒      === === === === === === === === === === === === ===
⇒      2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2

```

### D.15.6.18 grsyz

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\]](#), page 2346).

**Usage:** `grsyz(M)`, graded object M

**Return:** graded object

**Purpose:** compute graded syzygy of M

**Example:**

```

LIB "gradedModules.lib";
ring r=32003,(x,y,z),dp;
module A = grgroebner( grobj( module([x+y, x, 0, 3], [0, x+y, y, 2], [y, y, z, 1]),
grview(A);
⇒ Graded homomorphism:  $r^3 + r(-1) \leftarrow r(-1)^3 + r(-2) + r(-3)$ , given by a m\
atrix, with degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ....
⇒      --- --- --- --- --- +...
⇒      0 :  1   1   1   2   - |..1
⇒      0 :  1   -   1   -   - |..2
⇒      0 :  1   1   1   2   3 |..3
⇒      1 :  0   0   0   1   2 |..4
⇒      === === === === ===
⇒      1   1   1   2   3
grview(grsyz(A));
⇒ Graded homomorphism:  $r(-1)^3 + r(-2) + r(-3) \leftarrow r(-2) + r(-3)$ , given by a\
matrix, with degrees:
⇒      ..1 ..2 ....
⇒      --- --- +...
⇒      1 :  1   - |..1
⇒      1 :  1   2 |..2
⇒      1 :  1   - |..3
⇒      2 :  0   1 |..4
⇒      3 :  -   0 |..5
⇒      === ===
⇒      2   3
module X = grgroebner( grobj( module([x]), intvec(2) ) );
grview(X);
⇒ Graded homomorphism:  $r(-2) \leftarrow r(-3)$ , given by a diagonal matrix, with deg\
rees:
⇒      .1 ...
⇒      -- +..
⇒      2 : 1 |.1
⇒      ==
⇒      3
// syzygy module should be zero!
grview(grsyz(X));
⇒ Graded homomorphism:  $r(-3) \leftarrow 0$ , given by zero (1 x 0) matrix.

```

**D.15.6.19 grres**

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\]](#), page 2346).

**Usage:** `grres(M, l[, b])`, graded object M, int l, int b

**Return:** graded resolution = list of graded objects

**Purpose:** compute graded resolution of M (of length l) and minimise it if b was given

**Example:**

```

LIB "gradedModules.lib";
ring r=32003,(x,y,z),dp;
module A = grobj( module([x+y, x, 0, 3], [0, x+y, y, 2], [y, y, z, 1]), intvec(0,0,0)
grview(A);

```

```

⇒ Graded homomorphism:  $r^3 + r(-1) \leftarrow r(-1)^3$ , given by a matrix, with degrees:
    ..1 ..2 ..3 ....
    --- --- --- +...
    0 : 1  -  1 |..1
    0 : 1  1  1 |..2
    0 : -  1  1 |..3
    1 : 0  0  0 |..4
    === === ===
    1  1  1
module B = grgroebner(A);
grview(B);
⇒ Graded homomorphism:  $r^3 + r(-1) \leftarrow r(-1)^3 + r(-2) + r(-3)$ , given by a matrix, with degrees:
    ..1 ..2 ..3 ..4 ..5 ....
    --- --- --- --- --- +...
    0 : 1  1  1  2  - |..1
    0 : 1  -  1  -  - |..2
    0 : 1  1  1  2  3 |..3
    1 : 0  0  0  1  2 |..4
    === === === === ===
    1  1  1  2  3
"graded resolution of B: "; def C = grres(B, 0); grview(C);
⇒ graded resolution of B:
⇒ Graded resolution:
⇒  $r^3 + r(-1) \leftarrow d_1$  --
⇒  $r(-1) + r(-2) + r(-1)^2 + r(-3) \leftarrow d_2$  --
⇒  $r(-3) + r(-2) \leftarrow d_3$  --
⇒ 0, given by maps:
⇒  $d_1$  :
⇒ Graded homomorphism:  $r^3 + r(-1) \leftarrow r(-1) + r(-2) + r(-1)^2 + r(-3)$ , given by a matrix, with degrees:
    ..1 ..2 ..3 ..4 ..5 ....
    --- --- --- --- --- +...
    0 : 1  2  1  1  - |..1
    0 : -  -  1  1  - |..2
    0 : 1  2  1  1  3 |..3
    1 : 0  1  0  0  2 |..4
    === === === === ===
    1  2  1  1  3
⇒  $d_2$  :
⇒ Graded homomorphism:  $r(-1) + r(-2) + r(-1)^2 + r(-3) \leftarrow r(-3) + r(-2)$ , given by a matrix, with degrees:
    ..1 ..2 ....
    --- --- +...
    1 : 2  1 |..1
    2 : 1  0 |..2
    1 : -  1 |..3
    1 : -  1 |..4
    3 : 0  - |..5
    === ===
    3  2
⇒  $d_3$  :

```



```

⇒ Graded homomorphism:  $r(-3) + r(-2) \leftarrow 0$ , given by zero (2 x 0) matrix.
int i; int l = size(C);
"D^2 == 0: "; for (i = 1; i < l; i++) { i; grview( grprod(C[i], C[i+1]) ); }
⇒ D^2 == 0:
⇒ 1
⇒ Graded homomorphism:  $r^3 + r(-1) \leftarrow r(-3) + r(-2)$ , given by zero (4 x 2) \
matrix.
⇒ 2
⇒ Graded homomorphism:  $r(-1) + r(-2) + r(-1)^2 + r(-3) \leftarrow 0$ , given by zero \
(5 x 0) matrix.

```

### D.15.6.20 grlift

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\]](#), page 2346).

**Usage:** `grlift(M, N)`, graded objects M and N

**Return:** transformation matrix (graded object???)

**Purpose:** compute graded matrix which the generators of submodule  $\text{Im}(N)$  in terms of  $\text{Im}(M)$ .

**Example:**

```

LIB "gradedModules.lib";
ring r=32003,(x,y,z),dp;
module P=grobj(module([xy,0,xz]),intvec(0,1,0));
grview(P);
⇒ Graded homomorphism:  $r + r(-1) + r \leftarrow r(-2)$ , given by a matrix, with degr\
ees:
⇒      ..1 ....
⇒      --- +...
⇒  0 :  2 |..1
⇒  1 :  - |..2
⇒  0 :  2 |..3
⇒      ===
⇒      2
module D=grobj(module([y,0,z],[x2+y2,z,0]),intvec(0,1,0));
grview(D);
⇒ Graded homomorphism:  $r + r(-1) + r \leftarrow r(-1) + r(-2)$ , given by a matrix, w\
ith degrees:
⇒      ..1 ..2 ....
⇒      --- --- +...
⇒  0 :  1  2 |..1
⇒  1 :  -  1 |..2
⇒  0 :  1  - |..3
⇒      === ===
⇒      1  2
def G=grlift(D,P);
grview(G);
⇒ Graded homomorphism:  $r(-1) + r(-2) \leftarrow r(-2)$ , given by a matrix, with degr\
ees:
⇒      ..1 ....
⇒      --- +...
⇒  1 :  1 |..1
⇒  2 :  - |..2
⇒      ===

```

```

    ↪      2
    ASSUME(0, grisequal( grprod(D, G), P) );

```

### D.15.6.21 grprod

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\], page 2346](#)).

**Usage:** `grprod(M, N)`, graded objects M and N

**Return:** graded object

**Purpose:** compute graded product  $M * N$  (as composition of maps)

**Example:**

```

LIB "gradedModules.lib";
ring r=32003,(x,y,z),dp;
module A = grobj( module([x+y, x, 0, 3], [0, x+y, y, 2], [y, y, z, 1]), intvec(0,0,0) );
grview(A);
↪ Graded homomorphism:  $r^3 + r(-1) \leftarrow r(-1)^3$ , given by a matrix, with degrees:
    ees:
    ↪      ..1 ..2 ..3 ....
    ↪      --- --- --- +...
    ↪      0 :  1  -   1 |..1
    ↪      0 :  1   1   1 |..2
    ↪      0 :  -   1   1 |..3
    ↪      1 :  0   0   0 |..4
    ↪      === === ===
    ↪      1   1   1
A = grgroebner(A);
grview(A);
↪ Graded homomorphism:  $r^3 + r(-1) \leftarrow r(-1)^3 + r(-2) + r(-3)$ , given by a matrix, with degrees:
    ↪      ..1 ..2 ..3 ..4 ..5 ....
    ↪      --- --- --- --- --- +...
    ↪      0 :  1   1   1   2   - |..1
    ↪      0 :  1   -   1   -   - |..2
    ↪      0 :  1   1   1   2   3 |..3
    ↪      1 :  0   0   0   1   2 |..4
    ↪      === === === === ===
    ↪      1   1   1   2   3
module B = grsyz(A);
grview(B);
↪ Graded homomorphism:  $r(-1)^3 + r(-2) + r(-3) \leftarrow r(-2) + r(-3)$ , given by a matrix, with degrees:
    ↪      ..1 ..2 ....
    ↪      --- --- +...
    ↪      1 :  1   - |..1
    ↪      1 :  1   2 |..2
    ↪      1 :  1   - |..3
    ↪      2 :  0   1 |..4
    ↪      3 :  -   0 |..5
    ↪      === ===
    ↪      2   3
print(B);
↪ x, 0,

```

```

⇒ -y,y2,
⇒ -y,0,
⇒ 1, -x-2y,
⇒ 0, 1
module D = grprod( A, B );
grview(D);
⇒ Graded homomorphism: r^3 + r(-1) <- r(-2) + r(-3), given by zero (4 x 2) \
  matrix.
print(D); // must be all zeroes due to syzygy property!
⇒ 0,0,
⇒ 0,0,
⇒ 0,0,
⇒ 0,0
ASSUME(0, size(D) == 0);

```

### D.15.6.22 grgroebner

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\], page 2346](#)).

**Usage:** `grgroebner(M)`, graded object `M`

**Return:** graded object

**Purpose:** compute graded groebner basis of `M`

**Example:**

```

LIB "gradedModules.lib";
ring r=32003,(x,y,z),dp;
module A = grobj( module([x+y, x, 0, 0], [0, x+y, y, 0]), intvec(0,0,0,1) );
grview(A);
⇒ Graded homomorphism: r^3 + r(-1) <- r(-1)^2, given by a matrix, with degr\
  ees:
⇒      ..1 ..2 ....
⇒      --- --- +...
⇒  0 :  1  -  |..1
⇒  0 :  1   1 |..2
⇒  0 :  -   1 |..3
⇒  1 :  -   - |..4
⇒      === ===
⇒      1   1
module B = grgroebner(A);
grview(B);
⇒ Graded homomorphism: r^3 + r(-1) <- r(-1)^2, given by a matrix, with degr\
  ees:
⇒      ..1 ..2 ....
⇒      --- --- +...
⇒  0 :  1  -  |..1
⇒  0 :  1   1 |..2
⇒  0 :  1   1 |..3
⇒  1 :  -   - |..4
⇒      === ===
⇒      1   1

```

### D.15.6.23 grconcat

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\], page 2346](#)).

**Usage:** `grconcat(A, B)`, graded objects A and B,  $\text{dst}(A) == \text{dst}(B) =: \text{dst}$

**Return:** graded object

**Purpose:** construct  $\text{src}(A) + \text{src}(B) \longrightarrow \text{dst}$  given by  $(A|B)$

**Example:**

```
LIB "gradedModules.lib";
ring r;
module R=grobj(module([x,y,z]),intvec(0:3));
grview(R);
⇒ Graded homomorphism:  $r^3 \leftarrow r(-1)$ , given by a matrix, with degrees:
⇒      .1 ...
⇒      -- +..
⇒      0 : 1 |.1
⇒      0 : 1 |.2
⇒      0 : 1 |.3
⇒      ==
⇒      1
module S=grobj(module([x,0,y],[xy,zy+x2,0]),intvec(0:3));
grview(S);
⇒ Graded homomorphism:  $r^3 \leftarrow r(-1) + r(-2)$ , given by a matrix, with degree\
s:
⇒      ..1 ..2 ....
⇒      --- --- +...
⇒      0 : 1 2 |..1
⇒      0 : - 2 |..2
⇒      0 : 1 - |..3
⇒      === ===
⇒      1 2
def Q=grconcat(R,S);
grview(Q);
⇒ Graded homomorphism:  $r^3 \leftarrow r(-1)^2 + r(-2)$ , given by a square matrix, wi\
th degrees:
⇒      ..1 ..2 ..3 ....
⇒      --- --- --- +...
⇒      0 : 1 1 2 |..1
⇒      0 : 1 - 2 |..2
⇒      0 : 1 1 - |..3
⇒      === === ===
⇒      1 1 2
```

#### D.15.6.24 grrndmat

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\]](#), page 2346).

**Usage:** `grrndmat(src,dst[,p,b])`, `intvec src`, `dst[, int p, b]`

**Return:** matrix of polynomials

**Purpose:** generate random matrix compatible with `src` and `dst` gradings

**Note:** optional arguments `p, b` are for 'sparsepoly' (by default: 75%, 30000).

**Todo:** this is experimental at the moment!

**Example:**

```

LIB "gradedModules.lib";
ring r=32003,(x,y,z),dp;
print( grrndmat( intvec(0, 1), intvec(1, 2, 3) ) );
⇒ 6204y, -7833,
⇒ -12498xz+12773z2, 5664z,
⇒ -10041x3-11973x2y-5644xy2,-7740xy-5361z2

```

### D.15.6.25 grrndmap

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\], page 2346](#)).

**Usage:** `grrndmap(S,D)`, graded objects `S` and `D`

**Return:** graded object

**Purpose:** construct a random 0-deg graded homomorphism  $\text{src}(S) \rightarrow \text{src}(D)$

**Example:**

```

LIB "gradedModules.lib";
ring r=32003,(x,y,z),dp;
module D=grobj(module([y,0,z],[x2+y2,z,0]),intvec(0,1,0));
grview(D);
⇒ Graded homomorphism: r + r(-1) + r <- r(-1) + r(-2), given by a matrix, w\
ith degrees:
⇒      ..1 ..2 ....
⇒      --- --- +...
⇒  0 :  1  2 |..1
⇒  1 :  -  1 |..2
⇒  0 :  1  - |..3
⇒      === ===
⇒      1  2
module S=grobj(module([x,0,y],[xy,zy+x2,0]),intvec(0,0,0));
grview(S);
⇒ Graded homomorphism: r^3 <- r(-1) + r(-2), given by a matrix, with degree\
s:
⇒      ..1 ..2 ....
⇒      --- --- +...
⇒  0 :  1  2 |..1
⇒  0 :  -  2 |..2
⇒  0 :  1  - |..3
⇒      === ===
⇒      1  2
def H=grrndmap(D,S);
grview(H);
⇒ Graded homomorphism: r(-1) + r(-2) <- r(-1) + r(-2), given by a square ma\
trix, with degrees:
⇒      ..1 ..2 ....
⇒      --- --- +...
⇒  1 :  0  1 |..1
⇒  2 :  -  0 |..2
⇒      === ===
⇒      1  2

```

### D.15.6.26 grrndmap2

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\], page 2346](#)).

**Usage:** `grrndmap2(D,S)`, graded objects S and D

**Return:** graded object

**Purpose:** construct a random 0-deg graded homomorphism between target of D and S.

**Example:**

```
LIB "gradedModules.lib";
ring r=32003,(x,y,z),dp;
module D=grobj(module([y,0,z],[x2+y2,z,0]),intvec(0,1,0));
grview(D);
↳ Graded homomorphism:  $r + r(-1) + r \leftarrow r(-1) + r(-2)$ , given by a matrix, w\
ith degrees:
↳      ..1 ..2 ....
↳      --- --- +...
↳    0 :  1  2 |..1
↳    1 :  -  1 |..2
↳    0 :  1  - |..3
↳      === ===
↳      1  2
module S=grobj(module([x,0,y],[xy,zy+x2,0]),intvec(0,0,0));
grview(S);
↳ Graded homomorphism:  $r^3 \leftarrow r(-1) + r(-2)$ , given by a matrix, with degree\
s:
↳      ..1 ..2 ....
↳      --- --- +...
↳    0 :  1  2 |..1
↳    0 :  -  2 |..2
↳    0 :  1  - |..3
↳      === ===
↳      1  2
def G=grrndmap2(D,S);
grview(G);
↳ Graded homomorphism:  $r^3 \leftarrow r + r(-1) + r$ , given by a square matrix, with\
degrees:
↳      .1 .2 .3 ...
↳      -- -- -- +..
↳    0 :  0  1  0 |.1
↳    0 :  0  1  0 |.2
↳    0 :  0  1  0 |.3
↳      == == ==
↳      0  1  0
```

### D.15.6.27 grlifting

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\]](#), page 2346).

**Usage:** `grlifting(M,N)`, graded objects M and N

**Return:** map of chain complexes (as a list)

**Purpose:** construct a map of chain complexes between free resolutions of  $\text{Img}(M)$  and  $\text{Img}(N)$ .

**Example:**

```
LIB "gradedModules.lib";
/*
```

```

ring r=32003,(x,y,z),dp;
module P=grobj(module([xy,0,xz]),intvec(0,1,0));
grview(P);
module D=grobj(module([y,0,z],[x2+y2,z,0]),intvec(0,1,0));
grview(D);
def G=grlifting(D,P);
grview(G);
kill r;
ring r=32003,(x,y,z),dp;
module D=grobj(module([y,0,z],[x2+y2,z,0],[z3,xy,xy2]),intvec(0,1,0));
D = grgroebner(D);
grview( grres(D, 0));
def G=grlifting(D, D);
grview(G);
*/
ring S = 0, (x(0..3)), dp;
list kos = grres(grobj(maxideal(1), intvec(0)), 0);
print( betti(kos), "betti");
↳
      0      1      2      3      4
↳ -----
0:      1      4      6      4      1
↳ -----
total:   1      4      6      4      1
↳
grview(kos);
↳ Graded resolution:
↳ S <-- d_1 --
↳ S(-1)^4 <-- d_2 --
↳ S(-2)^6 <-- d_3 --
↳ S(-3)^4 <-- d_4 --
↳ S(-4) <-- d_5 --
↳ 0, given by maps:
↳ d_1 :
↳ Graded homomorphism: S <- S(-1)^4, given by a matrix, with degrees:
↳      .1 .2 .3 .4 ...
↳      -- -- -- -- +..
↳ 0 : 1  1  1  1 |.1
↳      == == == ==
↳      1  1  1  1
↳ d_2 :
↳ Graded homomorphism: S(-1)^4 <- S(-2)^6, given by a matrix, with degrees:
↳      ..1 ..2 ..3 ..4 ..5 ..6 ....
↳      --- --- --- --- --- --- +...
↳ 1 : 1  1  -  1  -  - |..1
↳ 1 : 1  -  1  -  1  - |..2
↳ 1 : -  1  1  -  -  1 |..3
↳ 1 : -  -  -  1  1  1 |..4
↳      === === === === === ===
↳      2  2  2  2  2  2
↳ d_3 :
↳ Graded homomorphism: S(-2)^6 <- S(-3)^4, given by a matrix, with degrees:
↳      ..1 ..2 ..3 ..4 ....
↳      --- --- --- --- +...

```

```

⇒ 2 : 1 1 - - |..1
⇒ 2 : 1 - 1 - |..2
⇒ 2 : 1 - - 1 |..3
⇒ 2 : - 1 1 - |..4
⇒ 2 : - 1 - 1 |..5
⇒ 2 : - - 1 1 |..6
⇒      === === === ===
⇒      3 3 3 3
⇒ d_4 :
⇒ Graded homomorphism: S(-3)^4 <- S(-4), given by a matrix, with degrees:
⇒      .1 ...
⇒      -- +..
⇒ 3 : 1 |.1
⇒ 3 : 1 |.2
⇒ 3 : 1 |.3
⇒ 3 : 1 |.4
⇒      ==
⇒      4
⇒ d_5 :
⇒ Graded homomorphism: S(-4) <- 0, given by zero (1 x 0) matrix.
// module M = grshift(kos[4], 2); // phi, Syz_3(K(2))
def M = KeneshlouMatrixPresentation(intvec(0,0,1,0));
grview( gres(M, 0) );
⇒ Graded resolution:
⇒ S(-1)^4 <-- d_1 --
⇒ S(-2) <-- d_2 --
⇒ 0, given by maps:
⇒ d_1 :
⇒ Graded homomorphism: S(-1)^4 <- S(-2), given by a matrix, with degrees:
⇒      .1 ...
⇒      -- +..
⇒ 1 : 1 |.1
⇒ 1 : 1 |.2
⇒ 1 : 1 |.3
⇒ 1 : 1 |.4
⇒      ==
⇒      2
⇒ d_2 :
⇒ Graded homomorphism: S(-2) <- 0, given by zero (1 x 0) matrix.
// module N = grshift(kos[3], 1); // psi, Syz_2(K(1))
def N = KeneshlouMatrixPresentation(intvec(0,1,0,0));
grview( gres(N, 0) );
⇒ Graded resolution:
⇒ S(-1)^6 <-- d_1 --
⇒ S(-2)^4 <-- d_2 --
⇒ S(-3) <-- d_3 --
⇒ 0, given by maps:
⇒ d_1 :
⇒ Graded homomorphism: S(-1)^6 <- S(-2)^4, given by a matrix, with degrees:
⇒      ..1 ..2 ..3 ..4 ....
⇒      --- --- --- --- +...
⇒ 1 : 1 1 - - |..1
⇒ 1 : 1 - 1 - |..2

```



```

⇒ 1 : 1 - - 1 |..3
⇒ 1 : - 1 1 - |..4
⇒ 1 : - 1 - 1 |..5
⇒ 1 : - - 1 1 |..6
⇒      === === === ===
⇒      2 2 2 2
⇒ d_2 :
⇒ Graded homomorphism: S(-2)^4 <- S(-3), given by a matrix, with degrees:
⇒      .1 ...
⇒      -- +..
⇒ 2 : 1 |.1
⇒ 2 : 1 |.2
⇒ 2 : 1 |.3
⇒ 2 : 1 |.4
⇒      ==
⇒      3
⇒ d_3 :
⇒ Graded homomorphism: S(-3) <- 0, given by zero (1 x 0) matrix.
grlifting(M, N); // grview(G);
⇒ t: 2
⇒ _[1]=26642*gen(4)+24263*gen(3)+5664*gen(2)+24170*gen(1)
// def G=grlifting( grgens(M), grgens(N) ); grview(G);

```

### D.15.6.28 grlifting2

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\]](#), page 2346).

**Usage:** `grlifting2(A,B)`, graded objects A and B (matrices defining maps)

**Return:** map of chain complexes (as a list)

**Purpose:** construct a map of chain complexes between free resolution of  $M=\text{coker}(A)$  and  $N=\text{coker}(B)$ .

**Example:**

```

LIB "gradedModules.lib";
ring r;
module P=grobj(module([xy,0,xz]),intvec(0,1,0));
grview(P);
⇒ Graded homomorphism: r + r(-1) + r <- r(-2), given by a matrix, with degr\
ees:
⇒      ..1 ....
⇒      --- +...
⇒ 0 : 2 |..1
⇒ 1 : - |..2
⇒ 0 : 2 |..3
⇒      ===
⇒      2
module D=grobj(module([y,0,z],[x2+y2,z,0]),intvec(0,1,0));
grview(D);
⇒ Graded homomorphism: r + r(-1) + r <- r(-1) + r(-2), given by a matrix, w\
ith degrees:
⇒      ..1 ..2 ....
⇒      --- --- +...
⇒ 0 : 1 2 |..1

```

```

⇒ 1 : - 1 |..2
⇒ 0 : 1 - |..3
⇒    === ===
⇒      1 2
module PP = grpres(P);
grview(PP);
⇒ Graded homomorphism: r(-2) <- 0, given by zero (1 x 0) matrix.
module DD = grpres(D);
grview(DD);
⇒ Graded homomorphism: r(-1) + r(-2) <- 0, given by zero (2 x 0) matrix.
def T=grlifting2(DD,PP); T;
⇒ T[1]=0
⇒ T[2]=-5361*gen(1)
// def Z=grlifting2(P,D); Z; // WRONG!!!

```

### D.15.6.29 mappingcone

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\]](#), page 2346).

**Usage:** mappingcone(M,N), M,N graded objects

**Return:** chain complex (as a list)

**Purpose:** construct a free resolution of the cokernel of a random map between  $\text{Img}(M)$ , and  $\text{Img}(N)$ .

**Example:**

```

LIB "gradedModules.lib";
ring r=32003, (x(0..4)),dp;
def A=KeneshlouMatrixPresentation(intvec(0,0,0,0,3));
def M=grgens(A);
grview(M);
⇒ Graded homomorphism: r(-1)^3 <- r(-1)^3, given by a diagonal matrix, with\
degrees:
⇒ ..1 ..2 ..3 ....
⇒ --- --- --- +...
⇒ 1 : 0 - - |..1
⇒ 1 : - 0 - |..2
⇒ 1 : - - 0 |..3
⇒    === === ===
⇒      1 1 1
def B=KeneshlouMatrixPresentation(intvec(0,1,0,0,0));
def N=grgens(B);
grview(N);
⇒ Graded homomorphism: r^5 <- r(-1)^10, given by a matrix, with degrees:
⇒ ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 ....
⇒ --- --- --- --- --- --- --- --- --- --- +...
⇒ 0 : - - - - - - 1 1 1 1 |..1
⇒ 0 : 1 1 - 1 - - 1 - - - |..2
⇒ 0 : 1 - 1 - 1 - - 1 - - |..3
⇒ 0 : - 1 1 - - 1 - - 1 - |..4
⇒ 0 : - - - 1 1 1 - - - 1 |..5
⇒    === === === === === === === === === ===
⇒      1 1 1 1 1 1 1 1 1 1
def R=grlifting(M,N);

```

```

⇒ t: 2
grview(R);
⇒ Graded homomorphism: r(-1)^10 <- r(-1)^3, given by a matrix, with degrees\
:
⇒      ..1 ..2 ..3 ....
⇒      --- --- --- +...
⇒  1 :  0  0  0 |..1
⇒  1 :  0  0  0 |..2
⇒  1 :  0  0  0 |..3
⇒  1 :  0  0  0 |..4
⇒  1 :  0  0  0 |..5
⇒  1 :  0  0  0 |..6
⇒  1 :  0  0  0 |..7
⇒  1 :  0  0  0 |..8
⇒  1 :  0  0  0 |..9
⇒  1 :  0  0  0 |.10
⇒      === === ===
⇒      1  1  1
def T=mappingcone(M,N);
⇒ t: 2
grview(T);
⇒ Graded resolution:
⇒ r(-1)^10 <-- d_1 --
⇒ r(-1)^3 + r(-2)^10, given by maps:
⇒ d_1 :
⇒ Graded homomorphism: r(-1)^10 <- r(-1)^3 + r(-2)^10, given by a matrix, w\
ith degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 .11 .12 .13 ....
⇒      --- --- --- --- --- --- --- --- --- --- --- --- --- +...
⇒  1 :  0  0  0  1  1  -  1  -  -  -  -  -  - |..1
⇒  1 :  0  0  0  1  -  1  -  1  -  -  -  -  - |..2
⇒  1 :  0  0  0  -  1  1  -  -  1  -  -  -  - |..3
⇒  1 :  0  0  0  -  -  -  1  1  1  -  -  -  - |..4
⇒  1 :  0  0  0  1  -  -  -  -  -  1  1  -  - |..5
⇒  1 :  0  0  0  -  1  -  -  -  -  1  -  1  - |..6
⇒  1 :  0  0  0  -  -  -  1  -  -  -  1  1  - |..7
⇒  1 :  0  0  0  -  -  1  -  -  -  1  -  -  1 |..8
⇒  1 :  0  0  0  -  -  -  -  1  -  -  1  -  1 |..9
⇒  1 :  0  0  0  -  -  -  -  -  1  -  -  1  1 |.10
⇒      === === === === === === === === === === === === ===
⇒      1  1  1  2  2  2  2  2  2  2  2  2  2
def U=grtranspose(T[1]);
resolution G=mres(U,0);
print(betti(G),"beti");
⇒      0      1      2
⇒  -----
⇒  -2:      10      7      -
⇒  -1:      -      -      -
⇒   0:      -      -      1
⇒  -----
⇒ total:      10      7      1
⇒
ideal I=groebner(flatten(G[2]));

```

```

resolution GI=mres(I,0);
print(betti(GI),"betti");
↳          0      1      2      3      4
↳ -----
↳    0:      1      -      -      -      -
↳    1:      -      -      -      -      -
↳    2:      -      7     10      5      1
↳ -----
↳ total:      1      7     10      5      1
↳

```

### D.15.6.30 grlifting3

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\]](#), page 2346).

**Todo:** `grlifting4` was newer and had more documentation than this proc, but was removed...  
Please verify and update!

**Example:**

```

LIB "gradedModules.lib";
ring r=32003, x(0..4),dp;
def A=grtwist(3,1);
grview(A);
↳ Graded homomorphism: r(1)^3 <- 0, given by zero (3 x 0) matrix.
def T=KeneshlouMatrixPresentation(intvec(0,1,0,0,0));
grview(T);
↳ Graded homomorphism: r(-1)^10 <- r(-2)^10, given by a square matrix, with\
degrees:
↳    ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 ....
↳    --- --- --- --- --- --- --- --- --- --- +...
↳    1 :  1   1   -   -   1   -   -   -   -   - |..1
↳    1 :  1   -   1   -   -   1   -   -   -   -   - |..2
↳    1 :  1   -   -   1   -   -   -   1   -   -   - |..3
↳    1 :  -   1   1   -   -   -   -   1   -   -   - |..4
↳    1 :  -   1   -   1   -   -   -   -   1   -   - |..5
↳    1 :  -   -   1   1   -   -   -   -   -   1   - |..6
↳    1 :  -   -   -   -   1   1   -   1   -   -   - |..7
↳    1 :  -   -   -   -   1   -   1   -   1   -   - |..8
↳    1 :  -   -   -   -   -   1   1   -   -   1   - |..9
↳    1 :  -   -   -   -   -   -   -   1   1   1   1 |.10
↳    === === === === === === === === === ===
↳          2   2   2   2   2   2   2   2   2   2   2
def F=grlifting3(T,A);
↳          0      1      2      3
↳ -----
↳    1:      10      10      5      1
↳ -----
↳ total:      10      10      5      1
↳
↳          0
↳ -----
↳   -1:      3
↳ -----
↳ total:      3

```

```

↳
↳ t: 1
↳ Graded homomorphism: r(1)^3 <- r(-1)^10, given by a matrix, with degrees:
↳   ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 ....
↳   --- --- --- --- --- --- --- --- --- --- +...
↳ -1 : 2  2  2  2  2  2  2  2  2  2 |..1
↳ -1 : 2  2  2  2  2  2  2  2  2  2 |..2
↳ -1 : 2  2  2  2  2  2  2  2  2  2 |..3
↳   === === === === === === === === === ===
↳   1  1  1  1  1  1  1  1  1  1
grview(F);
↳ Graded resolution:
↳ r(1)^3 <-- d_1 --
↳ r(-1)^10, given by maps:
↳ d_1 :
↳ Graded homomorphism: r(1)^3 <- r(-1)^10, given by a matrix, with degrees:
↳   ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 ....
↳   --- --- --- --- --- --- --- --- --- --- +...
↳ -1 : 2  2  2  2  2  2  2  2  2  2 |..1
↳ -1 : 2  2  2  2  2  2  2  2  2  2 |..2
↳ -1 : 2  2  2  2  2  2  2  2  2  2 |..3
↳   === === === === === === === === === ===
↳   1  1  1  1  1  1  1  1  1  1
def R=KeneshlouMatrixPresentation(intvec(0,0,0,2,0));
def S=KeneshlouMatrixPresentation(intvec(1,2,0,0,0));
def H=grlifting3(R, S);
↳           0      1
↳ -----
↳ 1:      10      2
↳ -----
↳ total:   10      2
↳
↳           0      1      2      3
↳ -----
↳ 0:       1      -      -      -
↳ 1:      20     20     10     2
↳ -----
↳ total:   21     20     10     2
↳
↳ t: 2
↳ Graded homomorphism: r(-2)^20 <- r(-2)^2, given by a matrix, with degrees\
:
↳   ..1 ..2 ....
↳   --- --- +...
↳ 2 : 0  0 |..1
↳ 2 : 0  0 |..2
↳ 2 : 0  0 |..3
↳ 2 : 0  0 |..4
↳ 2 : 0  0 |..5
↳ 2 : 0  0 |..6
↳ 2 : 0  0 |..7
↳ 2 : 0  0 |..8
↳ 2 : 0  0 |..9

```

```

⇒ 2 : 0 0 |.10
⇒ 2 : 0 0 |.11
⇒ 2 : 0 0 |.12
⇒ 2 : 0 0 |.13
⇒ 2 : 0 0 |.14
⇒ 2 : 0 0 |.15
⇒ 2 : 0 0 |.16
⇒ 2 : 0 0 |.17
⇒ 2 : 0 0 |.18
⇒ 2 : 0 0 |.19
⇒ 2 : 0 0 |.20
⇒      === ===
⇒      2 2
⇒ k: 1
⇒ Graded homomorphism: r + r(-1)^20 <- r(-1)^10, given by a matrix, with de\
grees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 ....
⇒      --- --- --- --- --- --- --- --- --- --- +...
⇒ 0 : - - - - - - - - - - |..1
⇒ 1 : 0 0 0 - - 0 0 0 - - |..2
⇒ 1 : 0 0 - 0 - 0 0 - 0 - |..3
⇒ 1 : 0 0 - - 0 0 0 - - 0 |..4
⇒ 1 : 0 - 0 0 - 0 - 0 0 - |..5
⇒ 1 : 0 - 0 - 0 0 - 0 - 0 |..6
⇒ 1 : 0 - - 0 0 0 - - 0 0 |..7
⇒ 1 : - 0 0 0 - - 0 0 0 - |..8
⇒ 1 : - 0 0 - 0 - 0 0 - 0 |..9
⇒ 1 : - 0 - 0 0 - 0 - 0 0 |.10
⇒ 1 : - - 0 0 0 - - 0 0 0 |.11
⇒ 1 : 0 0 0 - - 0 0 0 - - |.12
⇒ 1 : 0 0 - - 0 0 0 - - 0 |.13
⇒ 1 : 0 0 - - 0 0 0 - - 0 |.14
⇒ 1 : 0 - 0 0 - 0 - 0 0 - |.15
⇒ 1 : 0 - 0 - 0 0 0 - 0 - 0 |.16
⇒ 1 : 0 - - 0 0 0 - - 0 0 |.17
⇒ 1 : - 0 0 0 - - 0 0 0 - |.18
⇒ 1 : - 0 0 - 0 - 0 0 - 0 |.19
⇒ 1 : - 0 - 0 0 - 0 - 0 0 |.20
⇒ 1 : - - 0 0 0 - - 0 0 0 |.21
⇒      === === === === === === === === === ===
⇒      1 1 1 1 1 1 1 1 1 1
// grview(H);
// 2nd module does not lie in the first:
// def H=grlifting3(S, R);
//def I=KeneshlouMatrixPresentation(intvec(2,3,0,6,2));
//def J=KeneshlouMatrixPresentation(intvec(4,0,1,2,1));
//def N=grlifting3(I,J); grview(N);

```

### D.15.6.31 mappingcone3

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\]](#), page 2346).

**Usage:**      `mappingcone3(A,B)`, graded objects A and B (matrices defining maps)

**Return:** chain complex (as a list)

**Purpose:** construct a free resolution of the cokernel of a random map between  $M = \text{coker}(A)$ , and  $N = \text{coker}(B)$

**Example:**

```
LIB "gradedModules.lib";
ring r=32003,x(0..4),dp;
def A=KeneshlouMatrixPresentation(intvec(0,0,0,0,3));
grview(A);
↳ Graded homomorphism: r(-1)^3 <- 0, given by zero (3 x 0) matrix.
def T= KeneshlouMatrixPresentation(intvec(0,1,0,0,0));
grview(T);
↳ Graded homomorphism: r(-1)^10 <- r(-2)^10, given by a square matrix, with\
degrees:
↳      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 ....
↳      --- --- --- --- --- --- --- --- --- --- +...
↳      1 :  1  1  -  -  1  -  -  -  -  - |..1
↳      1 :  1  -  1  -  -  1  -  -  -  - |..2
↳      1 :  1  -  -  1  -  -  1  -  -  - |..3
↳      1 :  -  1  1  -  -  -  -  1  -  - |..4
↳      1 :  -  1  -  1  -  -  -  -  1  - |..5
↳      1 :  -  -  1  1  -  -  -  -  -  1 |..6
↳      1 :  -  -  -  -  1  1  -  1  -  - |..7
↳      1 :  -  -  -  -  1  -  1  -  1  - |..8
↳      1 :  -  -  -  -  -  1  1  -  -  1 |..9
↳      1 :  -  -  -  -  -  -  -  1  1  1 |..10
↳      === === === === === === === === === ===
↳      2  2  2  2  2  2  2  2  2  2
def F=grlifting3(A,T); grview(F);
↳      0
↳      -----
↳      1:      3
↳      -----
↳      total:      3
↳
↳      0      1      2      3
↳      -----
↳      1:      10      10      5      1
↳      -----
↳      total:      10      10      5      1
↳
↳      t:  1
↳      Graded homomorphism: r(-1)^10 <- r(-1)^3, given by a matrix, with degrees\
:
↳      ..1 ..2 ..3 ....
↳      --- --- --- +...
↳      1 :  0  0  0 |..1
↳      1 :  0  0  0 |..2
↳      1 :  0  0  0 |..3
↳      1 :  0  0  0 |..4
↳      1 :  0  0  0 |..5
↳      1 :  0  0  0 |..6
↳      1 :  0  0  0 |..7
```

```

⇒ 1 : 0 0 0 |..8
⇒ 1 : 0 0 0 |..9
⇒ 1 : 0 0 0 |..10
⇒      === === ===
⇒      1 1 1
⇒ Graded resolution:
⇒ r(-1)^10 <-- d_1 --
⇒ r(-1)^3, given by maps:
⇒ d_1 :
⇒ Graded homomorphism: r(-1)^10 <- r(-1)^3, given by a matrix, with degrees\
:
⇒      ..1 ..2 ..3 ....
⇒      --- --- --- +...
⇒ 1 : 0 0 0 |..1
⇒ 1 : 0 0 0 |..2
⇒ 1 : 0 0 0 |..3
⇒ 1 : 0 0 0 |..4
⇒ 1 : 0 0 0 |..5
⇒ 1 : 0 0 0 |..6
⇒ 1 : 0 0 0 |..7
⇒ 1 : 0 0 0 |..8
⇒ 1 : 0 0 0 |..9
⇒ 1 : 0 0 0 |..10
⇒      === === ===
⇒      1 1 1
// BUG in the proc
def G=mappingcone3(A,T); grview(G);
⇒      0
⇒ -----
⇒      1:      3
⇒ -----
⇒ total:      3
⇒
⇒      0      1      2      3
⇒ -----
⇒      1:      10      10      5      1
⇒ -----
⇒ total:      10      10      5      1
⇒
⇒ t: 1
⇒ Graded homomorphism: r(-1)^10 <- r(-1)^3, given by a matrix, with degrees\
:
⇒      ..1 ..2 ..3 ....
⇒      --- --- --- +...
⇒ 1 : 0 0 0 |..1
⇒ 1 : 0 0 0 |..2
⇒ 1 : 0 0 0 |..3
⇒ 1 : 0 0 0 |..4
⇒ 1 : 0 0 0 |..5
⇒ 1 : 0 0 0 |..6
⇒ 1 : 0 0 0 |..7
⇒ 1 : 0 0 0 |..8
⇒ 1 : 0 0 0 |..9

```



```

⇒ 1 : 0 0 0 |.10
⇒      === === ===
⇒      1 1 1
⇒ Graded resolution:
⇒ r(-1)^10 <-- d_1 --
⇒ r(-1)^3 + r(-2)^10, given by maps:
⇒ d_1 :
⇒ Graded homomorphism: r(-1)^10 <- r(-1)^3 + r(-2)^10, given by a matrix, w\
ith degrees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 .11 .12 .13 ....
⇒      --- --- --- --- --- --- --- --- --- --- --- --- --- +...
⇒ 1 : 0 0 0 1 1 - - 1 - - - - - |..1
⇒ 1 : 0 0 0 1 - 1 - - 1 - - - - |..2
⇒ 1 : 0 0 0 1 - - 1 - - 1 - - - |..3
⇒ 1 : 0 0 0 - 1 1 - - - - 1 - - |..4
⇒ 1 : 0 0 0 - - 1 - 1 - - - - 1 |..5
⇒ 1 : 0 0 0 - - - 1 1 - - - - 1 |..6
⇒ 1 : 0 0 0 - - - - 1 1 - 1 - - |..7
⇒ 1 : 0 0 0 - - - - - 1 - 1 - 1 - |..8
⇒ 1 : 0 0 0 - - - - - - 1 1 - - 1 |..9
⇒ 1 : 0 0 0 - - - - - - - 1 1 1 |.10
⇒      === === === === === === === === === === === ===
⇒      1 1 1 2 2 2 2 2 2 2 2 2 2
/*
module W=grtranspose(G[1]);
resolution U=mres(W,0);
print(betti(U,0),"betti"); // ?
ideal P=groebner(flatten(U[2]));
resolution L=mres(P,0);
print(betti(L),"betti");
*/
def R=KeneshlouMatrixPresentation(intvec(0,0,0,2,0));
grview(R);
⇒ Graded homomorphism: r(-1)^10 <- r(-2)^2, given by a matrix, with degrees\
:
⇒      ..1 ..2 ....
⇒      --- --- +...
⇒ 1 : 1 - |..1
⇒ 1 : 1 - |..2
⇒ 1 : 1 - |..3
⇒ 1 : 1 - |..4
⇒ 1 : 1 - |..5
⇒ 1 : - 1 |..6
⇒ 1 : - 1 |..7
⇒ 1 : - 1 |..8
⇒ 1 : - 1 |..9
⇒ 1 : - 1 |.10
⇒      === ===
⇒      2 2
def S=KeneshlouMatrixPresentation(intvec(1,2,0,0,0));
grview(S);
⇒ Graded homomorphism: r + r(-1)^20 <- r(-2)^20, given by a matrix, with de\
grees:

```

```

⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 .11 .12 .13 .14 .15 .16 .17 \
      .18 .19 .20 ....
⇒      -----
⇒      0 : - - - - - - - - - - - - - - - - - - - - \
          - - - |..1
⇒      1 : 1 1 - - 1 - - - - - - - - - - - - - - \
          - - - |..2
⇒      1 : 1 - 1 - - 1 - - - - - - - - - - - - - - \
          - - - |..3
⇒      1 : 1 - - 1 - - 1 - - - - - - - - - - - - - - \
          - - - |..4
⇒      1 : - 1 1 - - - - 1 - - - - - - - - - - - - - - \
          - - - |..5
⇒      1 : - 1 - 1 - - - - 1 - - - - - - - - - - - - - - \
          - - - |..6
⇒      1 : - - 1 1 - - - - 1 - - - - - - - - - - - - - - \
          - - - |..7
⇒      1 : - - - - - 1 1 - 1 - - - - - - - - - - - - - - \
          - - - |..8
⇒      1 : - - - - - 1 - 1 - 1 - - - - - - - - - - - - - - \
          - - - |..9
⇒      1 : - - - - - - 1 1 - - 1 - - - - - - - - - - - - - - \
          - - - |.10
⇒      1 : - - - - - - - 1 1 1 - - - - - - - - - - - - - - \
          - - - |.11
⇒      1 : - - - - - - - - - 1 1 - - - 1 - - - - - - - - - \
          - - - |.12
⇒      1 : - - - - - - - - - 1 - 1 - - - 1 - - - - - - - - \
          - - - |.13
⇒      1 : - - - - - - - - - 1 - - 1 - - - 1 - - - 1 \
          - - - |.14
⇒      1 : - - - - - - - - - - - 1 1 - - - - - - - - - \
          1 - - - |.15
⇒      1 : - - - - - - - - - - - 1 - 1 - - - - - - - - \
          - 1 - - |.16
⇒      1 : - - - - - - - - - - - - - 1 1 - - - - - - - - \
          - - 1 |.17
⇒      1 : - - - - - - - - - - - - - - - 1 1 - - - - - \
          1 - - - |.18
⇒      1 : - - - - - - - - - - - - - - - - 1 - 1 \
          - 1 - - |.19
⇒      1 : - - - - - - - - - - - - - - - - - 1 1 \
          - - 1 |.20
⇒      1 : - - - - - - - - - - - - - - - - - - - \
          1 1 1 |.21
⇒      =====
⇒      2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 \
          2 2 2
def H=grlifting3(R,S); grview(H);
⇒      0 1
⇒      -----
⇒      1: 10 2
⇒      -----

```

```

⇒ total:      10      2
⇒
⇒           0      1      2      3
⇒ -----
⇒    0:      1      -      -      -
⇒    1:     20     20     10     2
⇒ -----
⇒ total:     21     20     10     2
⇒
⇒ t:  2
⇒ Graded homomorphism: r(-2)^20 <- r(-2)^2, given by a matrix, with degrees\
:
⇒      ..1 ..2 ....
⇒      --- --- +...
⇒  2 :  0   0 |..1
⇒  2 :  0   0 |..2
⇒  2 :  0   0 |..3
⇒  2 :  0   0 |..4
⇒  2 :  0   0 |..5
⇒  2 :  0   0 |..6
⇒  2 :  0   0 |..7
⇒  2 :  0   0 |..8
⇒  2 :  0   0 |..9
⇒  2 :  0   0 |.10
⇒  2 :  0   0 |.11
⇒  2 :  0   0 |.12
⇒  2 :  0   0 |.13
⇒  2 :  0   0 |.14
⇒  2 :  0   0 |.15
⇒  2 :  0   0 |.16
⇒  2 :  0   0 |.17
⇒  2 :  0   0 |.18
⇒  2 :  0   0 |.19
⇒  2 :  0   0 |.20
⇒      === ===
⇒      2   2
⇒ k:  1
⇒ Graded homomorphism: r + r(-1)^20 <- r(-1)^10, given by a matrix, with de\
grees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 ....
⇒      --- --- --- --- --- --- --- --- --- --- +...
⇒  0 :  -   -   -   -   -   -   -   -   -   - |..1
⇒  1 :  0   0   0   -   -   0   0   0   -   - |..2
⇒  1 :  0   0   -   0   -   0   0   -   0   - |..3
⇒  1 :  0   0   -   -   0   0   0   -   -   0 |..4
⇒  1 :  0   -   0   0   -   0   -   0   0   - |..5
⇒  1 :  0   -   0   -   0   0   -   0   -   0 |..6
⇒  1 :  0   -   -   0   0   0   -   -   0   0 |..7
⇒  1 :  -   0   0   0   -   -   0   0   0   - |..8
⇒  1 :  -   0   0   -   0   -   0   0   -   0 |..9
⇒  1 :  -   0   -   0   0   -   0   -   0   0 |.10
⇒  1 :  -   -   0   0   0   -   -   0   0   0 |.11
⇒  1 :  0   0   0   -   -   0   0   0   -   - |.12

```

```

⇒ 1 : 0 0 - 0 - 0 0 - 0 - |.13
⇒ 1 : 0 0 - - 0 0 0 - - 0 |.14
⇒ 1 : 0 - 0 0 - 0 - 0 0 - |.15
⇒ 1 : 0 - 0 - 0 0 0 - - 0 |.16
⇒ 1 : 0 - - 0 0 0 - - 0 0 |.17
⇒ 1 : - 0 0 0 - - 0 0 0 - |.18
⇒ 1 : - 0 0 - 0 - 0 0 - 0 |.19
⇒ 1 : - 0 - 0 0 - 0 - 0 0 |.20
⇒ 1 : - - 0 0 0 - - 0 0 0 |.21
⇒      === === === === === === === === ===
⇒      1 1 1 1 1 1 1 1 1 1
⇒ Graded-object collection, given by the following maps (named here as o_[1\
  .. 2]):
⇒ o_1 :
⇒ Graded homomorphism:  $r + r(-1)^{20} \leftarrow r(-1)^{10}$ , given by a matrix, with de\
  grees:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 ....
⇒      --- --- --- --- --- --- --- --- --- +...
⇒ 0 : - - - - - - - - - |..1
⇒ 1 : 0 0 0 - - 0 0 0 - - |..2
⇒ 1 : 0 0 - 0 - 0 0 0 - 0 - |..3
⇒ 1 : 0 0 - - 0 0 0 - - 0 |..4
⇒ 1 : 0 - 0 0 - 0 0 - 0 0 - |..5
⇒ 1 : 0 - 0 - 0 0 0 - 0 - 0 |..6
⇒ 1 : 0 - - 0 0 0 - - 0 0 |..7
⇒ 1 : - 0 0 0 - - 0 0 0 - |..8
⇒ 1 : - 0 0 - 0 - 0 0 - 0 |..9
⇒ 1 : - 0 - 0 0 - 0 - 0 0 |.10
⇒ 1 : - - 0 0 0 - - 0 0 0 |.11
⇒ 1 : 0 0 0 - - 0 0 0 - - |.12
⇒ 1 : 0 0 - 0 - 0 0 0 - 0 - |.13
⇒ 1 : 0 0 - - 0 0 0 - - 0 |.14
⇒ 1 : 0 - 0 0 - 0 0 - 0 0 - |.15
⇒ 1 : 0 - 0 - 0 0 0 - 0 - 0 |.16
⇒ 1 : 0 - - 0 0 0 - - 0 0 |.17
⇒ 1 : - 0 0 0 - - 0 0 0 - |.18
⇒ 1 : - 0 0 - 0 - 0 0 - 0 |.19
⇒ 1 : - 0 - 0 0 - 0 - 0 0 |.20
⇒ 1 : - - 0 0 0 - - 0 0 0 |.21
⇒      === === === === === === === === ===
⇒      1 1 1 1 1 1 1 1 1 1
⇒ o_2 :
⇒ Graded homomorphism:  $r(-2)^{20} \leftarrow r(-2)^2$ , given by a matrix, with degrees\
  :
⇒      ..1 ..2 ....
⇒      --- --- +...
⇒ 2 : 0 0 |..1
⇒ 2 : 0 0 |..2
⇒ 2 : 0 0 |..3
⇒ 2 : 0 0 |..4
⇒ 2 : 0 0 |..5
⇒ 2 : 0 0 |..6
⇒ 2 : 0 0 |..7

```

```

↳ 2 : 0 0 |..8
↳ 2 : 0 0 |..9
↳ 2 : 0 0 |.10
↳ 2 : 0 0 |.11
↳ 2 : 0 0 |.12
↳ 2 : 0 0 |.13
↳ 2 : 0 0 |.14
↳ 2 : 0 0 |.15
↳ 2 : 0 0 |.16
↳ 2 : 0 0 |.17
↳ 2 : 0 0 |.18
↳ 2 : 0 0 |.19
↳ 2 : 0 0 |.20
↳      === ===
↳      2 2
// BUG in the proc
def G=mappingcone3(R,S);
↳ // ** redefining G (def G=mappingcone3(R,S);) ./examples/mappingcone3.sin\
g:24
↳      0 1
↳ -----
↳ 1: 10 2
↳ -----
↳ total: 10 2
↳
↳      0 1 2 3
↳ -----
↳ 0: 1 - - -
↳ 1: 20 20 10 2
↳ -----
↳ total: 21 20 10 2
↳
↳ t: 2
↳ Graded homomorphism: r(-2)^20 <- r(-2)^2, given by a matrix, with degrees\
:
↳ ..1 ..2 ....
↳ --- --- +...
↳ 2 : 0 0 |..1
↳ 2 : 0 0 |..2
↳ 2 : 0 0 |..3
↳ 2 : 0 0 |..4
↳ 2 : 0 0 |..5
↳ 2 : 0 0 |..6
↳ 2 : 0 0 |..7
↳ 2 : 0 0 |..8
↳ 2 : 0 0 |..9
↳ 2 : 0 0 |.10
↳ 2 : 0 0 |.11
↳ 2 : 0 0 |.12
↳ 2 : 0 0 |.13
↳ 2 : 0 0 |.14
↳ 2 : 0 0 |.15
↳ 2 : 0 0 |.16

```

```

↳ 2 : 0 0 |.17
↳ 2 : 0 0 |.18
↳ 2 : 0 0 |.19
↳ 2 : 0 0 |.20
↳      === ===
↳      2 2
↳ k: 1
↳ Graded homomorphism: r + r(-1)^20 <- r(-1)^10, given by a matrix, with de\
grees:
↳      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 ....
↳      --- --- --- --- --- --- --- --- --- --- +...
↳ 0 : - - - - - - - - - - |..1
↳ 1 : 0 0 0 - - 0 0 0 - - |..2
↳ 1 : 0 0 - 0 - 0 0 0 - 0 - |..3
↳ 1 : 0 0 - - 0 0 0 - - 0 |..4
↳ 1 : 0 - 0 0 - 0 0 - 0 0 - |..5
↳ 1 : 0 - 0 - 0 0 0 - 0 0 - |..6
↳ 1 : 0 - - 0 0 0 0 - 0 0 - |..7
↳ 1 : - 0 0 0 - - 0 0 0 - |..8
↳ 1 : - 0 0 - 0 - 0 0 0 - 0 |..9
↳ 1 : - 0 - 0 0 - 0 - 0 0 0 |..10
↳ 1 : - - 0 0 0 - - 0 0 0 |..11
↳ 1 : 0 0 0 - - 0 0 0 - - |..12
↳ 1 : 0 0 - 0 - 0 0 0 - 0 - |..13
↳ 1 : 0 0 - - 0 0 0 - - 0 |..14
↳ 1 : 0 - 0 0 - 0 0 - 0 0 - |..15
↳ 1 : 0 - 0 - 0 0 0 - 0 - 0 |..16
↳ 1 : 0 - - 0 0 0 - - 0 0 |..17
↳ 1 : - 0 0 0 - - 0 0 0 - |..18
↳ 1 : - 0 0 - 0 - 0 0 0 - 0 |..19
↳ 1 : - 0 - 0 0 - 0 - 0 0 |..20
↳ 1 : - - 0 0 0 - - 0 0 0 |..21
↳      === === === === === === === === ===
↳      1 1 1 1 1 1 1 1 1 1
↳ // ** redefining A ( module A=grconcat(P[i],rN[i]);) gradedModules.lib\
::mappingcone3:2370
↳ // ** redefining B ( module B=grobj(zero,v,w);) gradedModules.lib::map\
pingcone3:2371
def I=KeneshlouMatrixPresentation(intvec(2,3,0,6,2));
def J=KeneshlouMatrixPresentation(intvec(4,0,1,2,1));
// def N=grlifting3(I,J);
// 2nd module does not lie in the first:
// def NN=mappingcone3(I,J); // ?????????

```

### D.15.6.32 grrange

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\]](#), page 2346).

**Usage:** `grrange(M)`, graded object `M`

**Return:** `intvec`

**Purpose:** get weights of module units, thus describing the target of `M`

**Example:**

```

LIB "gradedModules.lib";
ring r=32003,(x,y,z),dp;
module Z = grobj(freemodule(0),intvec(0:0),intvec(0:0));
grrange(Z);
↳
grdeg(Z);
↳
grview(Z);
↳ Graded homomorphism: 0 <- 0, given by zero (0^2) matrix.
module P=grojb(module([xy,0,xz]),intvec(0,1,0));
grrange(P);
↳ 0,1,0
grdeg(P);
↳ 2
grview(P);
↳ Graded homomorphism: r + r(-1) + r <- r(-2), given by a matrix, with degr\
ees:
↳      ..1 ....
↳      --- +...
↳      0 : 2 |..1
↳      1 : - |..2
↳      0 : 2 |..3
↳      ===
↳      2

```

### D.15.6.33 grneg

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\]](#), page 2346).

**Usage:** `grneg(A)`, graded object `A`

**Return:** graded object

**Purpose:** graded map defined by `-A`

**Example:**

```

LIB "gradedModules.lib";
ring r=0,(x,y,z),dp;
def A=grojb([x2,yz,xyz],intvec(1,1,0));
grview(A);
↳ Graded homomorphism: r(-1)^2 + r <- r(-3), given by a matrix, with degree\
s:
↳      .1 ...
↳      -- +..
↳      1 : 2 |.1
↳      1 : 2 |.2
↳      0 : 3 |.3
↳      ==
↳      3
def F=grneg(A);
grview(F);
↳ Graded homomorphism: r(-1)^2 + r <- r(-3), given by a matrix, with degree\
s:
↳      .1 ...
↳      -- +..

```

```

↳ 1 : 2 |.1
↳ 1 : 2 |.2
↳ 0 : 3 |.3
↳ ==
↳ 3

```

### D.15.6.34 matrixpres

Procedure from library `gradedModules.lib` (see [Section D.15.6 \[gradedModules.lib\]](#), page 2346).

**Usage:** `matrixpres(a), intvec a`

**Return:** graded object

**Purpose:** matrix presentation for direct sum of  $\omega^a[i](i)$  in form of a graded object

**Example:**

```

LIB "gradedModules.lib";
ring r = 32003,(x(0..4)),dp;
def R=matrixpres(intvec(1,4,0,0,0));
↳ // ** redefining j (      int j=size(a)-i;) gradedModules.lib::matrixpres\
:2480
grview(R);
↳ Graded homomorphism: r(-1)^21 <- r(-2)^4, given by a matrix, with degrees\
:
↳      ..1 ..2 ..3 ..4 ....
↳      --- --- --- --- +...
↳ 1 : - - - - |..1
↳ 1 : 1 - - - |..2
↳ 1 : 1 - - - |..3
↳ 1 : 1 - - - |..4
↳ 1 : 1 - - - |..5
↳ 1 : 1 - - - |..6
↳ 1 : - 1 - - |..7
↳ 1 : - 1 - - |..8
↳ 1 : - 1 - - |..9
↳ 1 : - 1 - - |.10
↳ 1 : - 1 - - |.11
↳ 1 : - - 1 - |.12
↳ 1 : - - 1 - |.13
↳ 1 : - - 1 - |.14
↳ 1 : - - 1 - |.15
↳ 1 : - - 1 - |.16
↳ 1 : - - - 1 |.17
↳ 1 : - - - 1 |.18
↳ 1 : - - - 1 |.19
↳ 1 : - - - 1 |.20
↳ 1 : - - - 1 |.21
↳      === === === ===
↳      2 2 2 2
def S=matrixpres(intvec(0,0,3,0,0));
↳ // ** redefining j (      int j=size(a)-i;) gradedModules.lib::matrixpres\
:2480
grview(S);
↳ Graded homomorphism: r(-1)^30 <- r(-2)^15, given by a matrix, with degree\

```



```

s:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 .11 .12 .13 .14 .15 ....
⇒      --- --- --- --- --- --- --- --- --- --- --- --- --- --- --- +...
⇒      1 : 1  1  -  -  -  -  -  -  -  -  -  -  -  -  - |..1
⇒      1 : 1  -  1  -  -  -  -  -  -  -  -  -  -  -  - |..2
⇒      1 : 1  -  -  1  -  -  -  -  -  -  -  -  -  -  - |..3
⇒      1 : 1  -  -  -  1  -  -  -  -  -  -  -  -  -  - |..4
⇒      1 : -  1  1  -  -  -  -  -  -  -  -  -  -  -  - |..5
⇒      1 : -  1  -  1  -  -  -  -  -  -  -  -  -  -  - |..6
⇒      1 : -  1  -  -  1  -  -  -  -  -  -  -  -  -  - |..7
⇒      1 : -  -  1  1  -  -  -  -  -  -  -  -  -  -  - |..8
⇒      1 : -  -  1  -  1  -  -  -  -  -  -  -  -  -  - |..9
⇒      1 : -  -  -  1  1  -  -  -  -  -  -  -  -  -  - |.10
⇒      1 : -  -  -  -  -  1  1  -  -  -  -  -  -  -  - |.11
⇒      1 : -  -  -  -  -  1  -  1  -  -  -  -  -  -  - |.12
⇒      1 : -  -  -  -  -  1  -  -  1  -  -  -  -  -  - |.13
⇒      1 : -  -  -  -  -  1  -  -  -  1  -  -  -  -  - |.14
⇒      1 : -  -  -  -  -  -  1  1  -  -  -  -  -  -  - |.15
⇒      1 : -  -  -  -  -  -  1  -  1  -  -  -  -  -  - |.16
⇒      1 : -  -  -  -  -  -  1  -  -  1  -  -  -  -  - |.17
⇒      1 : -  -  -  -  -  -  -  1  1  -  -  -  -  -  - |.18
⇒      1 : -  -  -  -  -  -  -  1  -  1  -  -  -  -  - |.19
⇒      1 : -  -  -  -  -  -  -  -  1  1  -  -  -  -  - |.20
⇒      1 : -  -  -  -  -  -  -  -  -  1  1  -  -  -  - |.21
⇒      1 : -  -  -  -  -  -  -  -  -  -  1  1  -  -  - |.22
⇒      1 : -  -  -  -  -  -  -  -  -  -  1  -  1  -  - |.23
⇒      1 : -  -  -  -  -  -  -  -  -  -  -  1  -  1  - |.24
⇒      1 : -  -  -  -  -  -  -  -  -  -  -  1  1  -  - |.25
⇒      1 : -  -  -  -  -  -  -  -  -  -  -  1  -  1  - |.26
⇒      1 : -  -  -  -  -  -  -  -  -  -  -  1  -  -  1 |.27
⇒      1 : -  -  -  -  -  -  -  -  -  -  -  -  1  1  - |.28
⇒      1 : -  -  -  -  -  -  -  -  -  -  -  -  1  -  1 |.29
⇒      1 : -  -  -  -  -  -  -  -  -  -  -  -  1  1 |.30
⇒      === === === === === === === === === === === === ===
⇒      2  2  2  2  2  2  2  2  2  2  2  2  2  2  2  2
def N1 = matrixpres(intvec(2,0,0,0,0));
grview(N1);
⇒ Graded homomorphism: r(-1)^2 <- 0, given by zero (2 x 0) matrix.
def N2 = matrixpres(intvec(0,0,0,0,3));
grview(N2);
⇒ Graded homomorphism: r^3 <- 0, given by zero (3 x 0) matrix.
def N = matrixpres(intvec(2,0,0,0,3));
grview(N);
⇒ Graded homomorphism: r(-1)^2 + r^3 <- 0, given by zero (5 x 0) matrix.
def M1 = matrixpres(intvec(0,1,0,0,0));
⇒ // ** redefining j (      int j=size(a)-i;) gradedModules.lib::matrixpres\
:2480
grview(M1);
⇒ Graded homomorphism: r(-1)^5 <- r(-2), given by a matrix, with degrees:
⇒      .1 ...
⇒      -- +..
⇒      1 : 1 |.1
⇒      1 : 1 |.2

```

```

⇒ 1 : 1 |.3
⇒ 1 : 1 |.4
⇒ 1 : 1 |.5
⇒ ==
⇒ 2
def M2 = matrixpres(intvec(0,1,1,0,0));
⇒ // ** redefining j (      int j=size(a)-i;) gradedModules.lib::matrixpres\
:2480
⇒ // ** redefining j (      int j=size(a)-i;) gradedModules.lib::matrix\
pres:2488
grview(M2);
⇒ Graded homomorphism: r(-1)^15 <- r(-2)^6, given by a matrix, with degrees\
:
⇒ ..1 ..2 ..3 ..4 ..5 ..6 ....
⇒ --- --- --- --- --- --- +...
⇒ 1 : 1 - - - - - |..1
⇒ 1 : 1 - - - - - |..2
⇒ 1 : 1 - - - - - |..3
⇒ 1 : 1 - - - - - |..4
⇒ 1 : 1 - - - - - |..5
⇒ 1 : - 1 1 - - - |..6
⇒ 1 : - 1 - 1 - - - |..7
⇒ 1 : - 1 - - 1 - - |..8
⇒ 1 : - 1 - - - 1 |..9
⇒ 1 : - - 1 1 - - |..10
⇒ 1 : - - 1 - 1 - |..11
⇒ 1 : - - 1 - - 1 |..12
⇒ 1 : - - - 1 1 - |..13
⇒ 1 : - - - 1 - 1 |..14
⇒ 1 : - - - - 1 1 |..15
⇒ === === === === ===
⇒ 2 2 2 2 2 2
def M3 = matrixpres(intvec(0,0,0,1,0));
⇒ // ** redefining j (      int j=size(a)-i;) gradedModules.lib::matrixpres\
:2480
grview(M3);
⇒ Graded homomorphism: r(-1)^10 <- r(-2)^10, given by a square matrix, with\
degrees:
⇒ ..1 ..2 ..3 ..4 ..5 ..6 ..7 ..8 ..9 .10 ....
⇒ --- --- --- --- --- --- --- --- --- --- +...
⇒ 1 : 1 1 - - 1 - - - - - |..1
⇒ 1 : 1 - 1 - - 1 - - - - |..2
⇒ 1 : 1 - - 1 - - 1 - - - |..3
⇒ 1 : - 1 1 - - - 1 - - - |..4
⇒ 1 : - 1 - 1 - - - - 1 - |..5
⇒ 1 : - - 1 1 - - - - - 1 |..6
⇒ 1 : - - - 1 1 - 1 - - - |..7
⇒ 1 : - - - - 1 - 1 - 1 - |..8
⇒ 1 : - - - - - 1 1 - - 1 |..9
⇒ 1 : - - - - - - 1 1 1 |..10
⇒ === === === === ===
⇒ 2 2 2 2 2 2 2 2 2 2
def M = matrixpres(intvec(1,1,1,0,0));

```

```

⇒ // ** redefining j (      int j=size(a)-i;) gradedModules.lib::matrixpres\
:2480
⇒ // ** redefining j (      int j=size(a)-i;) gradedModules.lib::matrix\
pres:2488
grview(M);
⇒ Graded homomorphism: r(-1)^16 <- r(-2)^6, given by a matrix, with degrees\
:
⇒      ..1 ..2 ..3 ..4 ..5 ..6 ....
⇒      --- --- --- --- --- --- +...
⇒  1 :  -   -   -   -   -   - |..1
⇒  1 :  1   -   -   -   -   - |..2
⇒  1 :  1   -   -   -   -   - |..3
⇒  1 :  1   -   -   -   -   - |..4
⇒  1 :  1   -   -   -   -   - |..5
⇒  1 :  1   -   -   -   -   - |..6
⇒  1 :  -   1   1   -   -   - |..7
⇒  1 :  -   1   -   1   -   - |..8
⇒  1 :  -   1   -   -   1   - |..9
⇒  1 :  -   1   -   -   -   1 |.10
⇒  1 :  -   -   1   1   -   - |.11
⇒  1 :  -   -   1   -   1   - |.12
⇒  1 :  -   -   1   -   -   1 |.13
⇒  1 :  -   -   -   1   1   - |.14
⇒  1 :  -   -   -   1   -   1 |.15
⇒  1 :  -   -   -   -   1   1 |.16
⇒      === === === === === ===
⇒      2   2   2   2   2   2

```

### D.15.7 hodge.lib

**Library:** hodge.lib

**Purpose:** Algorithms for Hodge ideals

**Authors:** Guillem Blanco, email: guillem.blanco@kuleuven.be

**Overview:** A library for computing the Hodge ideals [MP19] of  $\mathbb{Q}$ -divisors associated to any reduced hypersurface  $\text{fin}R$ .

The implemented algorithm [Bla21] is based on the characterization of the Hodge ideals in terms of the  $V$ -filtration of Malgrange and Kashiwara on  $R_f f^s$ , see [MP20].

As a consequence, this library provides also an algorithm to compute the multiplier ideals and the jumping numbers of any hypersurface, see [BS05].

**References:**

- [Bla21] G. Blanco, An algorithm for Hodge ideals, to appear.
- [BS05] N. Budur, M. Saito, Multiplier ideals,  $V$ -filtration, and spectrum, J. Algebraic Geom. 14 (2005), no. 2, 269-282. 2, 4
- [MP19] M. Mustata, M. Popa: Hodge ideals, Mem. Amer. Math. Soc. 262 (2019), no. 1268
- [MP20] M. Mustata, M. Popa: Hodge ideals for  $\mathbb{Q}$ -divisors,  $V$ -filtration, and minimal exponent, Forum Math. Sigma 8 (2020), no. e19, 41 pp.

**Procedures:** See also: [Section 7.5.5 \[dmodapp.lib\]](#), [page 421](#).

### D.15.7.1 Vfiltration

Procedure from library `hodge.lib` (see [Section D.15.7 \[hodge.lib\]](#), page 2421).

**Usage:** `Vfiltration(f, p [, eng]);`  $f$  a poly,  $p$  a non-negative integer,  $eng$  an optional integer.

**Return:** ring

**Purpose:** compute  $R$ -generators for the  $V$ -filtration on  $R_f f^s$  truncated up to degree  $p$  in  $\partial_t$ .

**Note:** activate the output ring with the `setring` command.

In the output ring, the list `Vfilt` contains the  $V$ -filtration.

The value of `eng` controls the algorithm used for Groebner basis computations.

See the `engine` procedure from [Section 7.5.5 \[dmodapp.lib\]](#), page 421 for the available algorithms.

**Display:** If `printlevel=1`, progress debug messages will be printed.

**Example:**

```
LIB "hodge.lib";
ring R = 0,(x,y),dp;
poly f = y^2-x^3;
def D = Vfiltration(f, 1);
setring D; Vfilt;
⇒ [1]:
⇒ [1]:
⇒ [1]:
⇒ -1
⇒ [2]:
⇒ -Dt*y
⇒ [3]:
⇒ -Dt*x
⇒ [2]:
⇒ 1/6
⇒ [3]:
⇒ 1
⇒ [2]:
⇒ [1]:
⇒ [1]:
⇒ -Dt*y
⇒ [2]:
⇒ -1
⇒ [3]:
⇒ -Dt*x^2
⇒ [2]:
⇒ 5/6
⇒ [3]:
⇒ 1
⇒ [3]:
⇒ [1]:
⇒ [1]:
⇒ -2*Dt*y^2+1
⇒ [2]:
⇒ -Dt*x*y
⇒ [3]:
⇒ -y
```

```

↳      [4]:
↳      -x
↳      [5]:
↳      -3*Dt*x^3-1
↳      [2]:
↳      1
↳      [3]:
↳      1

```

### D.15.7.2 hodgeIdeals

Procedure from library `hodge.lib` (see [Section D.15.7 \[hodge.lib\]](#), page 2421).

**Usage:** `hodgeIdeals(f, p [, eng]);`  $f$  a reduced poly,  $p$  a non-negative integer,  $eng$  an optional integer.

**Return:** ring

**Purpose:** compute the Hodge ideals of  $f^\alpha$  up to level  $p$ , for a reduced hypersurface  $f$ .

**Note:** activate the output ring with the `setring` command.  
 In the output ring, the list of ideals `hodge` contains the Hodge ideals of  $f$ .  
 The value of `eng` controls the algorithm used for Groebner basis computations.  
 See the `engine` procedure from [Section 7.5.5 \[dmodapp.lib\]](#), page 421 for the available algorithms.

**Display:** If `printlevel=1`, progress debug messages will be printed.

**Example:**

```

LIB "hodge.lib";
ring R = 0,(x,y),dp;
poly f = y^2-x^3;
def Ra = hodgeIdeals(f, 2);
setring Ra; hodge;
↳ [1]:
↳      [1]:
↳      _[1]=1
↳      [2]:
↳      _[1]=y
↳      _[2]=x
↳      [3]:
↳      _[1]=y^2
↳      _[2]=x*y
↳      _[3]=x^3
↳      [2]:
↳      1/6
↳      [3]:
↳      1
↳ [2]:
↳      [1]:
↳      _[1]=1
↳      [2]:
↳      _[1]=y
↳      _[2]=x^2

```

```

⇒      [3]:
⇒      _[1]=y^3
⇒      _[2]=x*y^2
⇒      _[3]=x^2*y
⇒      _[4]=x^3+(2*a+1)*y^2
⇒      [2]:
⇒      5/6
⇒      [3]:
⇒      1
⇒      [3]:
⇒      [1]:
⇒      [1]:
⇒      _[1]=y
⇒      _[2]=x
⇒      [2]:
⇒      _[1]=y^2
⇒      _[2]=x*y
⇒      _[3]=x^3
⇒      [3]:
⇒      _[1]=y^3
⇒      _[2]=x^2*y^2
⇒      _[3]=x^3*y
⇒      _[4]=x^4+(2*a+1)*x*y^2
⇒      [2]:
⇒      1
⇒      [3]:
⇒      1

```

### D.15.7.3 multIdeals

Procedure from library `hodge.lib` (see [Section D.15.7 \[hodge.lib\]](#), page 2421).

**Usage:** `multIdeals(f, [, eng]);`  $f$  a reduced poly, `eng` an optional integer.

**Return:** list

**Purpose:** compute the multiplier ideals of a hypersurface  $f \in R$ .

**Note:** The value of `eng` controls the algorithm used for Groebner basis computations. See the `engine` procedure from [Section 7.5.5 \[dmodapp.lib\]](#), page 421 for the available algorithms.

**Display:** If `printlevel=1`, progress debug messages will be printed.

**Example:**

```

LIB "hodge.lib";
ring R = 0,(x,y),dp;
poly f = y^2-x^3;
multIdeals(f);
⇒ [1]:
⇒ [1]:
⇒ _[1]=1
⇒ [2]:
⇒ 5/6
⇒ [3]:
⇒ 1

```

```

↳ [2] :
↳      [1] :
↳          _[1]=y
↳          _[2]=x
↳      [2] :
↳          1
↳      [3] :
↳          1

```

#### D.15.7.4 nextHodgeIdeal

Procedure from library `hodge.lib` (see [Section D.15.7 \[hodge.lib\], page 2421](#)).

**Usage:** `nextHodgeIdeal(f, I, p);`  $f$  a poly,  $I$  an ideal,  $p$  a non-negative integer

**Return:** ideal

**Purpose:** given the  $p$ -th Hodge ideal  $I$  of  $f^\alpha$  compute the  $p+1$ -th Hodge ideal assuming that the Hodge filtration of the underlying mixed Hodge module is generated at level less than or equal to  $p$ .

**Example:**

```

LIB "hodge.lib";
ring R = 0,(x,y),dp;
poly f = y^2-x^3;
def Ra = hodgeIdeals(f, 2);
setring(Ra);
int p = 1;
nextHodgeIdeal(y^2-x^3, hodge[3][1][p+1], p);
↳ _[1]=x^2*y^2
↳ _[2]=y^3
↳ _[3]=x^3*y
↳ _[4]=x^4+(2*a+1)*x*y^2

```

#### D.15.8 lrcalc\_lib

**Library:** `lrcalc.lib`

**Purpose:** An interface to the Littlewood-Richardson Calculator by Anders Buch

**Author:** Oleksandr Iena, o.g.yena@gmail.com

**Overview:** An interface to the documented functions of the Littlewood-Richardson Calculator by Anders Buch is implemented.

The library requires the Littlewood-Richardson Calculator by Anders Buch, which is available at <http://math.rutgers.edu/~asbuch/lrcalc/>

**References:**

[1] <http://math.rutgers.edu/~asbuch/lrcalc/>  
<http://math.rutgers.edu/~asbuch/lrcalc/lrcalc-1.2/README>

**Procedures:**

### D.15.8.1 LRinstall

Procedure from library `lrcalc.lib` (see [Section D.15.8 \[lrcalc.lib\]](#), page 2425).

**Usage:** LRinstall();

**Return:** int (exit status of the shell)

**Purpose:** installs the Littlewood-Richardson Calculator

**Note:**

**Example:**

```
LIB "lrcalc.lib";
// In order to install the Littlewood-Richardson Calculator
// type "LRinstall();"
// This will execute the following commands:
// wget math.rutgers.edu/~asbuch/lrcalc/lrcalc-1.2.tar.gz
// tar zxvf lrcalc-1.2.tar.gz
// cd lrcalc-1.2
// ./configure
// make
// sudo make install
```

### D.15.8.2 LRcoef

Procedure from library `lrcalc.lib` (see [Section D.15.8 \[lrcalc.lib\]](#), page 2425).

**Usage:** LRcoef(z, x, y); z, x, y lists of integers (partitions)

**Return:** bigint

**Purpose:** computes the Littlewood-Richardson coefficient  $c^z_{x,y}$

**Note:**

**Example:**

```
LIB "lrcalc.lib";
// Compute the Littlewood-Richardson coefficient  $c^z_{x,y}$ 
// for z= (3, 2, 1), x=(2, 1), y=(2, 1)
list z = 3, 2, 1;
list x = 2, 1;
list y = 2, 1;
LRcoef(z, x, y);
↦ 2
```

### D.15.8.3 LRskew

Procedure from library `lrcalc.lib` (see [Section D.15.8 \[lrcalc.lib\]](#), page 2425).

**Usage:** LRskew(z, x [,s, r]); z, x lists of integers (partitions) s string equal to 'r', r non-negative integer

**Return:** list of lists

**Purpose:** computes the partitions y for which the Littlewood-Richardson coefficient  $c^z_{x,y}$  is non-zero together with that coefficient; only partitions up to length r are computed if the optional parameters age given

**Note:**



**Example:**

```

LIB "lrcalc.lib";
// Compute the partitions y for which the Littlewood-Richardson coefficient
// c^z_{x,y} is non-zero together with that coefficient
// for z= (3, 2, 1), x=(2, 1)
list z = 3, 2, 1;
list x = 2, 1;
LRskew(z, x);
↳ [1]:
↳   [1]:
↳     1
↳   [2]:
↳     [1]:
↳       3
↳ [2]:
↳   [1]:
↳     2
↳   [2]:
↳     [1]:
↳       2
↳     [2]:
↳       1
↳ [3]:
↳   [1]:
↳     1
↳   [2]:
↳     [1]:
↳       1
↳     [2]:
↳       1
↳     [3]:
↳       1
// Now compute only the partitions with at most 2 entries
LRskew(z, x, "r", 2);
↳ [1]:
↳   [1]:
↳     1
↳   [2]:
↳     [1]:
↳       3
↳ [2]:
↳   [1]:
↳     2
↳   [2]:
↳     [1]:
↳       2
↳     [2]:
↳       1

```

**D.15.8.4 LRmult**

Procedure from library `lrcalc.lib` (see [Section D.15.8 \[lrcalc.lib\]](#), page 2425).

**Usage:** LRmult(x, y); x, y lists of integers (partitions)  
 LRmult(x, y [, s, r]); x, y lists of integers (partitions), s string equal to 'r', r integer  
 LRmult(x, y [, s, m, k]); x, y lists of integers (partitions), s string equal to 'q' or 'f',  
 m, k integers

**Return:** list of lists

**Purpose:** computes the partitions z for which the Littlewood-Richardson coefficient  $c^z_{x,y}$  is non-zero together with that coefficient; partitions up to length r

**Note:**

**Example:**

```
LIB "lrcalc.lib";
// Compute the partitions z for which the Littlewood-Richardson coefficient
//  $c^z_{x,y}$  is non-zero together with that coefficient
// for x= (2, 1), y=(2, 1)
list x = 2, 1;
list y = 2, 1;
LRmult(x, y);
⇨ [1]:
⇨   [1]:
⇨     1
⇨   [2]:
⇨     [1]:
⇨       3
⇨     [2]:
⇨       3
⇨ [2]:
⇨   [1]:
⇨     1
⇨   [2]:
⇨     [1]:
⇨       4
⇨     [2]:
⇨       2
⇨ [3]:
⇨   [1]:
⇨     1
⇨   [2]:
⇨     [1]:
⇨       2
⇨     [2]:
⇨       2
⇨     [3]:
⇨       1
⇨     [4]:
⇨       1
⇨ [4]:
⇨   [1]:
⇨     1
⇨   [2]:
⇨     [1]:
⇨       2
⇨     [2]:
```

```

↳      2
↳      [3]:
↳      2
↳ [5]:
↳      [1]:
↳      2
↳      [2]:
↳      [1]:
↳      3
↳      [2]:
↳      2
↳      [3]:
↳      1
↳ [6]:
↳      [1]:
↳      1
↳      [2]:
↳      [1]:
↳      4
↳      [2]:
↳      1
↳      [3]:
↳      1
↳ [7]:
↳      [1]:
↳      1
↳      [2]:
↳      [1]:
↳      3
↳      [2]:
↳      1
↳      [3]:
↳      1
↳      [4]:
↳      1
// Now compute only the partitions with at most 2 entries
LRmult(x, y, "r", 2);
↳ [1]:
↳      [1]:
↳      1
↳      [2]:
↳      [1]:
↳      3
↳      [2]:
↳      3
↳ [2]:
↳      [1]:
↳      1
↳      [2]:
↳      [1]:
↳      4
↳      [2]:
↳      2

```

```

// Now compute the product in the quantum cohomology ring of the Grassmannian Gr(3,3)
LRmult(x, y, "q", 3, 2);
⇨ [1]:
⇨   [1]:
⇨     1
⇨   [2]:
⇨     [1]:
⇨       2
⇨     [2]:
⇨       2
⇨     [3]:
⇨       2
⇨ [2]:
⇨   [1]:
⇨     1
⇨   [2]:
⇨     [1]:
⇨       1
// Compute the same product with the output given in fusion ring notation
LRmult(x, y, "f", 3, 2);
⇨ [1]:
⇨   [1]:
⇨     1
⇨   [2]:
⇨     [1]:
⇨       2
⇨     [2]:
⇨       2
⇨     [3]:
⇨       2
⇨ [2]:
⇨   [1]:
⇨     1
⇨   [2]:
⇨     [1]:
⇨       3
⇨     [2]:
⇨       2
⇨     [3]:
⇨       1

```

### D.15.8.5 LRcoprod

Procedure from library `lrcalc.lib` (see [Section D.15.8 \[lrcalc.lib\]](#), page 2425).

**Usage:** LRcoprod(z); z list of integers (partition)

**Return:** list of lists

**Purpose:** computes the pairs of partitions x and y for which the Littlewood-Richardson coefficient  $c^{z}_{x,y}$  is non-zero together with that coefficient

**Note:**

**Example:**

```

LIB "lrcalc.lib";
// Compute the pairs of partitions x and y for which the Littlewood-Richardson
// coefficient  $c^z_{x,y}$  is non-zero together with that coefficient
// for  $z = (3, 2, 1)$ 
list z = 3, 2, 1;
LRcoprod(z);
⇨ [1]:
⇨   [1]:
⇨     1
⇨   [2]:
⇨     [1]:
⇨       3
⇨     [2]:
⇨       1
⇨   [3]:
⇨     [1]:
⇨       2
⇨ [2]:
⇨   [1]:
⇨     1
⇨   [2]:
⇨     [1]:
⇨       2
⇨     [2]:
⇨       2
⇨   [3]:
⇨     [1]:
⇨       1
⇨     [2]:
⇨       1
⇨ [3]:
⇨   [1]:
⇨     1
⇨   [2]:
⇨     [1]:
⇨       3
⇨     [2]:
⇨       1
⇨   [3]:
⇨     [1]:
⇨       1
⇨     [2]:
⇨       1
⇨ [4]:
⇨   [1]:
⇨     1
⇨   [2]:
⇨     [1]:
⇨       2
⇨     [2]:
⇨       1
⇨   [3]:
⇨     [1]:

```

```

↳      3
↳ [5]:
↳   [1]:
↳      2
↳   [2]:
↳   [1]:
↳      2
↳   [2]:
↳      1
↳   [3]:
↳   [1]:
↳      2
↳   [2]:
↳      1
↳ [6]:
↳   [1]:
↳      1
↳   [2]:
↳   [1]:
↳      2
↳   [2]:
↳      2
↳   [3]:
↳      1
↳   [3]:
↳   [1]:
↳      1
↳ [7]:
↳   [1]:
↳      1
↳   [2]:
↳   [1]:
↳      3
↳   [2]:
↳      2
↳   [3]:
↳      1
↳   [3]:
↳   [1]:
↳      0
↳ [8]:
↳   [1]:
↳      1
↳   [2]:
↳   [1]:
↳      3
↳   [2]:
↳      2
↳   [3]:
↳   [1]:
↳      1
↳ [9]:
↳   [1]:

```

```

↳      1
↳      [2]:
↳      [1]:
↳      2
↳      [2]:
↳      1
↳      [3]:
↳      1
↳      [3]:
↳      [1]:
↳      2
↳ [10]:
↳      [1]:
↳      1
↳      [2]:
↳      [1]:
↳      3
↳      [2]:
↳      1
↳      [3]:
↳      1
↳      [3]:
↳      [1]:
↳      1
↳ [11]:
↳      [1]:
↳      1
↳      [2]:
↳      [1]:
↳      1
↳      [2]:
↳      1
↳      [3]:
↳      1
↳      [3]:
↳      [1]:
↳      2
↳      [2]:
↳      1
↳ [12]:
↳      [1]:
↳      1
↳      [2]:
↳      [1]:
↳      2
↳      [2]:
↳      1
↳      [3]:
↳      1
↳      [3]:
↳      [1]:
↳      1
↳      [2]:

```

```

⇒          1
⇒ [13]:
⇒   [1]:
⇒       1
⇒   [2]:
⇒   [1]:
⇒       2
⇒   [2]:
⇒       2
⇒   [3]:
⇒   [1]:
⇒       2

```

### D.15.8.6 LRschubmult

Procedure from library `lrcalc.lib` (see [Section D.15.8 \[lrcalc.lib\]](#), page 2425).

**Usage:** LRschubmult(x, y); x, y lists of integers

**Return:** list of lists

**Purpose:** computes the expansion of a product  
of two Schubert polynomials in the basis of Schubert polynomials

**Note:**

**Example:**

```

LIB "lrcalc.lib";
// Compute the expansion of a square of the Schubert polynomial
// corresponding to (1 3 2) in the basis of Schubert polynomials
list x = 1, 3, 2;
LRschubmult(x, x);
⇒ [1]:
⇒   [1]:
⇒       1
⇒   [2]:
⇒   [1]:
⇒       1
⇒   [2]:
⇒       4
⇒   [3]:
⇒       2
⇒   [4]:
⇒       3
⇒ [2]:
⇒   [1]:
⇒       1
⇒   [2]:
⇒   [1]:
⇒       2
⇒   [2]:
⇒       3
⇒   [3]:
⇒       1

```



## D.15.9 maxlike.lib

**Library:** maxlike.lib

**Purpose:** Procedures to compute maximum likelihood estimates

**Author:** Adrian Koch (kocha at rhrk.uni-kl.de)

**References:**

Lior Pachter, Bernd Sturmfels; Algebraic Statistics for Computational Biology; published by Cambridge University Press

**Procedures:**

### D.15.9.1 likeIdeal

Procedure from library maxlike.lib (see [Section D.15.9 \[maxlike.lib\]](#), page 2435).

**Usage:** likeIdeal(I,u); ideal I, intvec u  
I represents the algebraic statistical model and u is the data vector under consideration.

**Return:** ideal: the likelihood ideal with respect to I and u

**Example:**

```
LIB "maxlike.lib";
ring r = 0,(x,y),dp;
poly pA = -10x+2y+25;
poly pC = 8x-y+25;
poly pG = 11x-2y+25;
poly pT = -9x+y+25;
intvec u = 10,14,15,10;
ideal I = pA,pC,pG,pT;
ideal L = likeIdeal(I,u); L;
↪ L[1]=2744y2+13003050x-2116125y-6290625
↪ L[2]=9114xy-98y2+5828550x-891375y-2833125
↪ L[3]=847602x2-98y2+89716875x-12967425y-43985625
```

### D.15.9.2 logHessian

Procedure from library maxlike.lib (see [Section D.15.9 \[maxlike.lib\]](#), page 2435).

**Usage:** logHessian(I,u); ideal I, intvec u  
I represents the algebraic statistical model and u is the data vector under consideration.

**Return:** matrix: a modified version of the Hessian matrix of the loglikelihood function defined by u and (the given generators of) I.

**Note:** This matrix has the following property: if it is negative definite at a point, then the actual Hessian is also negative definite at that point. The same holds for positive definiteness.

**Example:**

```
LIB "maxlike.lib";
ring r = 0,(x,y),dp;
poly pA = -10x+2y+25;
poly pC = 8x-y+25;
poly pG = 11x-2y+25;
```

```

poly pT = -9x+y+25;
intvec u = 10,14,15,10;
ideal I = pA,pC,pG,pT;
matrix H = logHessian(I,u); H;
⇒ H[1,1]=-689040x3+314898x2y-44808xy2+1974y3+9619350x2-2949075xy+240975y2+6\
151875x-1072500y-70640625
⇒ H[1,2]=110880x3-51936x2y+7596xy2-348y3-1489200x2+431400xy-33450y2-1072500\
x+176250y+11437500
⇒ H[2,1]=110880x3-51936x2y+7596xy2-348y3-1489200x2+431400xy-33450y2-1072500\
x+176250y+11437500
⇒ H[2,2]=-16580x3+7972x2y-1192xy2+56y3+243150x2-66800xy+4900y2+176250x-2750\
0y-1937500

```

### D.15.9.3 getMaxPoints

Procedure from library `maxlike.lib` (see [Section D.15.9 \[maxlike.lib\]](#), page 2435).

**Usage:** `getMaxPoints(Iu, H, prec [, "nodisplay"])`; ideal `Iu`, matrix `H`, int `prec`, int `k` `Iu` the likelihood ideal, `H` the (modified) Hessian of the considered algebraic statistical model, `prec` the precision with which to compute the maximum likelihood estimates

**Return:** ring: a complex ring `R` in which you can find the following two lists: - `MPOINTS`, points in which the loglikelihood function has a local maximum, and - `LHESSIANS`, the (modified) Hessians at those points  
also prints out the points in `MPOINTS`, unless a fourth argument is given

**Note:** it is assumed that the likelihood ideal is 0-dimensional

**Example:**

```

LIB "maxlike.lib";
ring r = 0,(x,y),dp;
poly pA = -10x+2y+25;
poly pC = 8x-y+25;
poly pG = 11x-2y+25;
poly pT = -9x+y+25;
intvec u = 10,14,15,10;
ideal I = pA,pC,pG,pT;
ideal L = likeIdeal(I,u);
matrix H = logHessian(I,u);
def R = getMaxPoints(L, H, 50);
⇒ [1]:
⇒ [1]:
⇒ 0.51912639453217837465463128685404418932771758896637
⇒ [2]:
⇒ 0.21725133256396491722887792998009835426225610459149
⇒
⇒ // In the ring created by getmaxpoints you can find the lists
⇒ // MPOINTS, containing points in which the loglikelihood function has a\
local maximum, and
⇒ // LHESSIANS, containing the (modified) Hessians at those points.
⇒
setring R;
MPOINTS;
⇒ [1]:
⇒ [1]:

```

```

⇒ 0.51912639453217837465463128685404418932771758896637
⇒ [2]:
⇒ 0.21725133256396491722887792998009835426225610459149
LHESSIANS;
⇒ [1]:
⇒ _[1,1]=-65487950.391931360088969690060635799847590217779318
⇒ _[1,2]=10577428.579689959415257134363650588464921754723022
⇒ _[2,1]=10577428.579689959415257134363650588464921754723022
⇒ _[2,2]=-1795635.2877514452321365400508526832132830781483598

```

#### D.15.9.4 maxPoints

Procedure from library `maxlike.lib` (see [Section D.15.9 \[maxlike.lib\]](#), page 2435).

**Usage:** `maxPoints(I,u,prec [, "nodisplay"])`; ideal `I`, intvec `u`, int `prec` `I` represents the algebraic statistical model, `u` is the data vector under consideration, and `prec` is the precision to be used in the computations

**Return:** ring: a complex ring `R` in which you can find the following two lists: - `MPOINTS`, points in which the loglikelihood function has a local maximum, and - `LHESSIANS`, the (modified) Hessians at those points  
also prints out the points in `MPOINTS`, unless a fourth argument is given

**Note:** Just uses `likeideal`, `loghessian` and `getmaxpoints`.

**Example:**

```

LIB "maxlike.lib";
ring r = 0,(x,y),dp;
poly pA = -10x+2y+25;
poly pC = 8x-y+25;
poly pG = 11x-2y+25;
poly pT = -9x+y+25;
intvec u = 10,14,15,10;
ideal I = pA,pC,pG,pT;
def R = maxPoints(I, u, 50);
⇒ [1]:
⇒ [1]:
⇒ 0.51912639453217837465463128685404418932771758896637
⇒ [2]:
⇒ 0.21725133256396491722887792998009835426225610459149
⇒
⇒ // In the ring created by getmaxpoints you can find the lists
⇒ // MPOINTS, containing points in which the loglikelihood function has a\
    local maximum, and
⇒ // LHESSIANS, containing the (modified) Hessians at those points.
⇒
setring R;
MPOINTS;
⇒ [1]:
⇒ [1]:
⇒ 0.51912639453217837465463128685404418932771758896637
⇒ [2]:
⇒ 0.21725133256396491722887792998009835426225610459149
LHESSIANS;
⇒ [1]:

```

```

⇒ _[1,1]=-65487950.391931360088969690060635799847590217779318
⇒ _[1,2]=10577428.579689959415257134363650588464921754723022
⇒ _[2,1]=10577428.579689959415257134363650588464921754723022
⇒ _[2,2]=-1795635.2877514452321365400508526832132830781483598

```

### D.15.9.5 maxPointsProb

Procedure from library `maxlike.lib` (see [Section D.15.9 \[maxlike.lib\]](#), page 2435).

**Usage:** `maxPointsProb(I,u,prec [, "nodisplay"])`; ideal `I`, intvec `u`, int `prec` `I` represents the algebraic statistical model, `u` is the data vector under consideration, and `prec` is the precision to be used in the computations

**Return:** ring: a complex ring `R` in which you can find the following two lists: - `MPOINTS`, points in which the loglikelihood function has a local maximum, - `LHESSIANS`, the (modified) Hessians at those points, and - `VALS`, the resulting probability distributions (that is, the values of the polynomials given by `I` at the points in `MPOINTS`). Also prints out the points in `MPOINTS`, unless a fourth argument is given.

**Note:** Does not compute the likelihood ideal via elimination, but rather computes the critical points by projection.

**Example:**

```

LIB "maxlike.lib";
ring r = 0,(x,y),dp;
poly pA = -10x+2y+25;
poly pC = 8x-y+25;
poly pG = 11x-2y+25;
poly pT = -9x+y+25;
intvec u = 10,14,15,10;
ideal I = pA,pC,pG,pT;
def R = maxPointsProb(I, u, 50);
⇒ [1]:
⇒ [1]:
⇒ 0.51912639453217837465463128685404418932771758896637
⇒ [2]:
⇒ 0.21725133256396491722887792998009835426225610459149
⇒
⇒ // In the ring created by getmaxpoints you can find the lists
⇒ // MPOINTS, containing points in which the loglikelihood function has a\
    local maximum,
⇒ // LHESSIANS, containing the (modified) Hessians at those points, and
⇒ // VALS, containing the probability distributions at those points.
⇒
setring R;
MPOINTS;
⇒ [1]:
⇒ [1]:
⇒ 0.51912639453217837465463128685404418932771758896637
⇒ [2]:
⇒ 0.21725133256396491722887792998009835426225610459149
LHESSIANS;
⇒ [1]:
⇒ _[1,1]=-65487950.391931360088969690060635799847590217779318

```

```

⇒  _[1,2]=10577428.579689959415257134363650588464921754723022
⇒  _[2,1]=10577428.579689959415257134363650588464921754723022
⇒  _[2,2]=-1795635.2877514452321365400508526832132830781483598
VALS;
⇒  [1]:
⇒    [1]:
⇒      20.243238719806146087911442991419754815247336319519
⇒    [2]:
⇒      28.935759823693462080008172364852255160359484607139
⇒    [3]:
⇒      30.275887674726032286743188295434289374080381269447
⇒    [4]:
⇒      20.545113781774359545337196348293700650312797803894

```

### D.15.10 modfinduni.lib

**Library:** modfinduni.lib

**Purpose:** finduni using modular methods

**Authors:** H. Schoenemann hannes@mathematik.uni-kl.de

**Overview:** parallel version of finduni via modular.lib

**Procedures:**

#### D.15.10.1 modFinduni

Procedure from library `modfinduni.lib` (see [Section D.15.10 \[modfinduni.lib\]](#), page 2439).

**Usage:** modFinduni(I); I ideal

**Return:**

**Note:** The procedure computes a standard basis of I (over the rational numbers) by using modular methods.

See also: [Section D.2.6.1 \[modular\]](#), page 884.

### D.15.11 modwalk.lib

**Library:** modwalk.lib

**Purpose:** Groebner basis conversion

**Authors:** S. Oberfranz oberfran@mathematik.uni-kl.de

**Overview:** A library for converting Groebner bases of an ideal in the polynomial ring over the rational numbers using modular methods. The procedures are inspired by the following paper:  
Elizabeth A. Arnold: Modular algorithms for computing Groebner bases. *Journal of Symbolic Computation* 35, 403-419 (2003).

**Procedures:** See also: [Section D.4.10 \[grwalk.lib\]](#), page 1088; [Section D.15.17 \[rwalk.lib\]](#), page 2537; [Section D.15.22 \[swalk.lib\]](#), page 2553.

### D.15.11.1 modWalk

Procedure from library `modwalk.lib` (see [Section D.15.11 \[modwalk.lib\]](#), page 2439).

**Return:** a standard basis of  $I$

**Note:** The procedure computes a standard basis of  $I$  (over the rational numbers) by using modular methods.

**Example:**

```
LIB "modwalk.lib";
ring R1 = 0, (x,y,z,t), dp;
ideal I = 3x3+x2+1, 11y5+y3+2, 5z4+z2+4;
I = std(I);
ring R2 = 0, (x,y,z,t), lp;
ideal I = fetch(R1, I);
ideal J = modWalk(I);
J;
⇒ J[1]=x3+1/3x2+1/3
⇒ J[2]=z4+1/5z2+4/5
⇒ J[3]=y5+1/11y3+2/11
ring S1 = 0, (a,b,c,d), Dp;
ideal I = 5b2, ac2+9d3+3a2+5b, 2a2c+7abd+bcd+4a2, 2ad2+6b2d+7c3+8ad+4c;
I = std(I);
ring S2 = 0, (c,d,b,a), lp;
ideal I = fetch(S1,I);
// I is assumed to be a Dp-Groebner basis.
// We compute a lp-Groebner basis.
ideal J = modWalk(I,"Dp");
J;
⇒ J[1]=a25+16a24+96a23+256a22+256a21+256/9a20+1024/3a19+2048a18+65536/9a17+
32768/3a16+16384/81a15+131072/81a14+1048576/81a13+1048576/27a12+1048576/9\
a11
⇒ J[2]=ba11+1522867351997104938459/91668001658017308797687087104a24+4293036\
9782248629690765/91668001658017308797687087104a23+80925218629630777478637\
/22917000414504327199421771776a22+7108535670237178684767/2864625051813040\
899927721472a21-3255817194541612658349/89519532869157528122741296a20+5380\
8965391546362724459/358078131476630112490965184a19+1534729815590907963215\
01/358078131476630112490965184a18-260815719913165309506063/44759766434578\
764061370648a17-1485276141860757031491027/89519532869157528122741296a16-4\
92332725360316960775/22379883217289382030685324a15+7423992361030571232440\
/16784912412967036523013993a14-17640364913371983121693/167849124129670365\
23013993a13-37723213977586186442564/5594970804322345507671331a12+92047580\
41857159721472414/5594970804322345507671331a11
⇒ J[3]=b2a6-63/2ba10+41087306587333357057895823883/924013456712814472680685\
83800832a24+93915562116924232413944264677/1320019223875449246686694054297\
6a23+1314662746341964624103002499857/30800448557093815756022861266944a22+\
2628268795042931967685617407557/23100336417820361817017145950208a21+32861\
7969148352577032114618159/2887542052227545227127143243776a20+175631724829\
284757906915538269/12993939235023953522072144596992a19+303812212296039043\
29090032467/206253003730538944794795945984a18+581332950565269518541458030\
9/6445406366579342024837373312a17+1392577308804410648719627124495/4060606\
01094498547564754518656a16+728489619081836101063651608791/135353533698166\
182521584839552a15+80182830319998353431517995037/913636352462621732020697\
```

```

666976a14+22597043807001043905240513127/32629869730807919000739202392a13+\
2699689128255025271541196980091/456818176231310866010348833488a12+6291773\
289016174025274735397/352483160672307766983293853a11+4/7a6
↪ J[4]=b3a5+4/7ba5-5398327059462849163101023479/739210765370251578144548670\
406656a24+3500067651845908053488406611/92401345671281447268068583800832a2\
3+81936880336538263803253409291/46200672835640723634034291900416a22+85587\
506267677081700930548967/6600096119377246233433470271488a21+4349147060430\
84846670081811347/11550168208910180908508572975104a20+1021713731964491721\
016405426709/25987878470047907044144289193984a19+189087309367688338214840\
3875/812121202188997095129509037312a18+3873208423822126582196454031/10312\
6501865269472397397972992a17+104243440049859097996976327663/4060606010944\
98547564754518656a16+52530075262606982469983545909/5075757513681231844559\
4314832a15+6154556265260978917193662164647/365454540985048692808279066790\
4a14+138400848395446486358821423/5639730570756924271732701648a13+68577767\
53456717192310999393/38068181352609238834195736124a12+3439955547980759942\
57860510037/228409088115655433005174416744a11+9/2a10+18a9
↪ J[5]=b4-63/2b3a4+4/7b2+3859043113737/128ba10-61261515/8ba9-525086793/32ba\
8+3969/16ba7-19845/4ba6+9/2ba5-317530772199391516703685862633925319890715\
5/9853890705461273608546303647569412096a24-889000025950798002413919214249\
17407047623059/17244308734557228814956031383246471168a23-4443358152931308\
1841158459544460024461570519/1437025727879769067913002615270539264a22-443\
94201884268758058296712245149507940575423/5388846479549134004673759807264\
52224a21-88794706503699382133468723144783169844707611/1077769295909826800\
934751961452904448a20-1763218978464215084468461907410740543335425/1732129\
22556936450150227993804931072a19-4436403904696107325808495191699219229446\
85/4210036312147760941151374849425408a18-21791665871565292166347154127869\
734967876143/33680290497182087529210998795403264a17-382244791239220889094\
021792022832016196677083/151561307237319393881449494579314688a16-25381669\
887869257097552146559952295821945667/6315054468221641411727062274138112a1\
5-3081988985408645474889243822837753592680251/487161344691383766047516232\
57636864a14-42991086686519981772739427292780808208480959/8525323532099215\
9058315340700864512a13-44791324614077928834488691100822888689474979/10656\
654415124019882289417587608064a12-250969820105414481651193596059252574282\
0133/197345452131926294116470696066816a11+771901337907/64a10-1701/4a9+437\
53689/4a8-19845/8a7-19845/2a6
↪ J[6]=da-1323/800b3a4+63/20b2a4+2701701074331/1280ba10-61269453/320ba9+306\
291699/160ba8-27783/16ba6-189/200ba4-685883099436497069901143509465258298\
0621641/123173633818265920106828795594617651200a24-5486790468650205133145\
8584531975294116209709/61586816909132960053414397797308825600a23-60954530\
98260341081175657932145681344303369/1140496609428388149137303662913126400\
a22-54839253809480536502430419967744127237862429/384917605682081000333839\
9862331801600a21-6855741428155530311932580353074238311128479/481147007102\
601250417299982791475200a20-2878029541644786739384025698911163129301521/1\
732129225569364501502279938049310720a19-539637460347386518746735121876349\
5466944867/288688204261560750250379989674885120a18-1360851075145362976080\
2370233473935440353147/120286751775650312604324995697868800a17-2261317453\
3404843047583509526614944559001629/54129038299042640671946248064040960a16\
-19899168362053216404911620136404443756991/309379505595808417192193918976\
00a15-3443170203569948104100043669593479708510001/30447584043211485377969\
7645360230400a14-27203224432640761938740031635681832222207391/30447584043\
2114853779697645360230400a13-11007706754707314979975768772869745826795967\
3/152237920216057426889848822680115200a12-1380727908059280736173048219943\
2331439121223/6343246675669059453743700945004800a11-1929592445193/640a10+\

```



```

437570343/1600a9+164090367/100a8+3969/8a6+9/5a4
⇒ J[7]=db2+4/7d+81/160b3a4-189/200b3a3+9/5b2a3+250047/32ba10+4750893/320ba9\
-567/40ba8-567/10ba7+81/280ba4-27/50ba3-627430952592078879073202920720376\
153131/5173292620367168644486809414973941350400a24-2906859466655459274817\
4299538370243493/18476045072739888016024319339192647680a23-69048240138740\
571289193103124930690109/11975214398998075565941688460587827200a22+488085\
08639174871490992992380443024047/11547528170462430010015199586995404800a2\
1+5093069077149293873574182379116672811143/808326971932370100701063971089\
67833600a20+3286117544936595300844164893345508176887/36374713736956654531\
547878699035525120a19-57124126734867585149642894515011901057/189451634046\
6492423518118682241433600a18-296407013643924816106633846582502767991/2526\
021787288656564690824909655244800a17-193831832299525900077669479610197531\
4379/11367098042798954541108712093448601600a16+49045157132492520042165362\
0398209599/433131307834131784069071486566400a15+9565278519988924726047202\
0815175898220317/25575970596297647717494602210259353600a14-51800242230268\
556716933919846651112061/456713760648172280669546468040345600a13-15498981\
08498974500694541229046869490537/1598498162268602982343412638141209600a12\
-700479417253540849801700529315888259/26641636037810049705723543969020160\
a11+20611017/1600a10+21504771/320a9+243/50a8-1701/100a7+36/35a3
⇒ J[8]=d2
⇒ J[9]=ca-1630191535/8275968d2ba2-36060437345/409660416d2ba+38107475/568972\
8d2a2+4391588006225/10115072d2a+92575/2844864d2+91285075/182071296db4a+35\
/9216db4-1088785/5689728db3a2+91259683/91035648db3a+24027/2528768db3-8162\
999/79656192db2a3-6539063/79656192db2a2+25495277/45517824db2a+5095/126438\
4db2-5243/158048dba4-109809/632192dba3-173005/632192dba2+265684003/159312\
384dba+24027/4425344db-7/256da5+5/96da3+5/24da2+35/32da+2645/2489256d+861\
7/1896576b5a-25921/158048b5+7/256b4a2-2163/79024b4a+2163/19756b4+529/3160\
96b3a3+7/128b3a2-12811/59268b3a+15435/19756b3+311787/632192b2a4+529/15804\
8b2a3+1/64b2a2-309/19756b2a+309/4939b2+19467/9878ba4+529/553168ba3+1/32ba\
2-1/8ba+1/2b+3/32a6+3/8a5+63/32a4+529/276584a3
⇒ J[10]=cb-452831001/4597760d2ba2-3340189909/75863040d2ba-3176523/158048d2a\
2+2194845076291/10115072d2a+5145/316096d2+25404981/101150720db4a-7203/102\
4db4+453789/790240db3a2+32184817/10115072db3a-25399101/12643840db3-324051\
/6321920db2a3+194523/790240db2a2+11921259/25287680db2a-9921275/1264384db2\
-27783/632192dba4-966217/3160960dba3-43211/395120dba2+1278655/2528768dba-\
31286843/3160960db-5/48da2-5/12da-172823/79024d+64827/3160960b5a-1351/148\
170b5+64827/1580480b4a-64827/395120b4+1323/1580480b3a3+9261/790240b3a-386\
/74085b3-583443/395120b2a4-10887849/1580480b2a3+9261/395120b2a-9261/98780\
b2-583443/197560ba4-6222951/395120ba3-3/16a5-3/4a4-1555407/395120a3
⇒ J[11]=cd+66199/8d2ba2+6818809/2304d2ba+35/64d2b+47/96d2a2-7004232463/384d\
2a-7/192db5+7/256db4a+7/320db4-1/80db3a2+3/640db3a-23/96db3+441/128db2a3+\
1/64db2a+1/80db2+63/8dba3+1/32dba-1/8db+141/160da5+141/40da4+63/32da3-63/\
80b4a2-63/40b3a2-9/20b2a2-9/10ba2
⇒ J[12]=c2+1/3cb2+5/3d+3a3
intvec w = 3,2,1,2;
ring S3 = 0, (c,d,b,a), (a(w),lp);
ideal I = fetch(S1,I);
// I is assumed to be a Dp-Groebner basis.
// We compute a (a(w),lp)-Groebner basis.
ideal J = modWalk(I,"Dp",w);
J;
⇒ J[1]=d2
⇒ J[2]=c2+3a3+1/3cb2+5/3d

```



$\mapsto J[3]=ca^2+4ca+7/2b^3+2b$   
 $\mapsto J[4]=cda-6/7ba^3+4/7c^2-2/21cb^3+1/7dba-10/21db$   
 $\mapsto J[5]=db^4a-192/49b^2a^3+128/49c^2b-64/7cda-64/147cb^4+4/63d^2ba+2db^3a-4db^4+60/4\backslash$   
 $9db^2a-8db^3+8/7dba-656/147db^2-32/7db$   
 $\mapsto J[6]=db^6+333576da^3-189/4b^8+15876b^4a^2-5186640/2401b^2a^3-2860453892752/17294\backslash$   
 $403c^2b+119097010895/311299254cd^2+1815960/49cdb^2-1455130425704/2470629cda+\backslash$   
 $8395442084/7203cb^4-2304/49cb^2a-826877376/2401ca^2+6069139793/155649627d^2ba\backslash$   
 $-43215/7db^5+199982018/49db^3a-4664160/7dba^2-126b^7+72/7b^5a+31752b^3a^2+1728/2\backslash$   
 $401ba^3-5770728573920/17294403c^2-1012272/49cdb+5598114168/2401cb^3-4608/49c\backslash$   
 $ba+604630/21609d^2b^2-129552/49db^4+864216/2401db^2a-117b^6+144/7b^4a+9072b^2a^2-\backslash$   
 $20160cd+4663152/7cb^2-3307509504/2401ca-7757580/16807d^2b-172796/49db^3+3888\backslash$   
 $0529598704/17294403dba-1296/7b^5+288/49b^3a+18144ba^2+9326304/7cb+185320d^2-1\backslash$   
 $9529120/7203db^2-1188/7b^4+576/49b^2a+2752/2401db-413460864/343b^3-3312/49b^2-\backslash$   
 $1653754752/2401b$   
 $\mapsto J[7]=cb^3a-5/8da^3-245/1024b^8-7/32b^6a+2/343b^2a^3-945270062561/1067311728c^2b+\backslash$   
 $595485054475/307385777664cd^2+5/112cdb^2-945263061245/304946208cda-108013/4\backslash$   
 $9392cb^4+25/7cb^2a-966391/1372ca^2+63120825485/76846444416d^2ba-3/256db^5-1543\backslash$   
 $1/2016db^3a+5/4dba^2-245/256b^7-25/32b^5a+5/1372ba^3-945267210173/533655864c^2+\backslash$   
 $5/84cdb-108035/24696cb^3+16/7cba+1512605/3556224d^2b^2-47/2688db^4+53/24696db\backslash$   
 $2a-91/256b^6-5/8b^4a-5/84cd-5/4cb^2-966979/343ca-3232325/1382976d^2b+1/192db^3\backslash$   
 $-954609818105/2134623456dba+65/32b^5-1/14b^3a-5/2cb-25/72d^2-667/98784db^2+85\backslash$   
 $/64b^4-2/7b^2a+109/12348db-1932803/784b^3+3/14ba+93/112b^2-966979/686b$   
 $\mapsto J[8]=cb^5-4/7db^2a^2+96/7b^2a^3-64/7c^2b-4/441cdb^2+32cda+116/21cb^4+1152/7ca^2-8/\backslash$   
 $7dba^2-8/441cdb+32/7cb^3+14db^4-16/7db^2a+16/7cb^2+4608/7ca+28db^3+16/7cb+328/2\backslash$   
 $1db^2+16db+576b^3+2304/7b$   
 $\mapsto J[9]=cdb^3+18da^3+2cdb^2+63cb^4+441/2db^3a-36dba^2+126cb^3+36cb^2+126dba+72cb+10d\backslash$   
 $2$   
 $\mapsto J[10]=ba^4-2/3c^2a+1/9cb^3a+14/3cda-1/6dba^2+49/12db^3+5/9dba+7/3db$   
 $\mapsto J[11]=b^3a^3+2744/5c^2d+382/15c^2b^2-1127/15cdb^3+7357/45cdba+129659/45cb^5+2352\backslash$   
 $/5cb^3a+906269/90db^4a-8232/5db^2a^2-1029/10b^6a+236/5c^2b+161/5cdb^2+5474/45cda\backslash$   
 $+28812/5cb^4+1680cb^2a+55174/405d^2ba-98/15db^5-2549/90db^3a-735/2b^5a-152/15c^2\backslash$   
 $-224/15cdb+8232/5cb^3+5376/5cba+2156/45db^4+259588/45db^2a+2058/5b^6-294b^4a-2\backslash$   
 $8cd+16464/5cb^2-4032/5ca+1069/9db^3-898/45dba+1470b^5-168/5b^3a+1232/45db^2+11\backslash$   
 $76b^4-672/5b^2a+3136/45db+672/5b^3+504/5ba+2688/5b^2-2016/5b$   
 $\mapsto J[12]=b^5a^2+182/3c^2d+32/9c^2b^2-71/9cdb^3+28204/1323cdba+931/2cb^5+64cb^3a-116/\backslash$   
 $7cba^2+17/1134d^2b^4+34/200120949d^2b^2a-7/18db^6+175759/108db^4a-117310/441db^2a\backslash$   
 $2-14b^6a+4b^4a^2+16480960/3176523c^2b-340/200120949cd^2+194/63cdb^2+4999816/453\backslash$   
 $789cda+1225cb^4+1600/7cb^2a-128/7ca^2+8819321851/400241898d^2ba-11/6db^5+19380\backslash$   
 $5/189db^3a-74096/441dba^2-50b^5a+32/7b^3a^2-12215296/3176523c^2-124/21cdb+854cb\backslash$   
 $3+80cba+34/3969d^2b^2+1079/189db^4+1232881/1323db^2a+56b^6-40b^4a+16/7b^2a^2-400/\backslash$   
 $63cd+700cb^2-1280/7ca+406/27db^3+1857442660/3176523dba+200b^5-32/7b^3a+16/7ba\backslash$   
 $2+336cb+4484/1323db^2+102b^4-128/7b^2a+12160/1323db-320/7b^3+96/7ba+40b^2-640/\backslash$   
 $7b$   
 $\mapsto J[13]=b^7a-13606662/245c^2d+20584/15435c^2b^2+23004293/2205cdb^3+865156/5145cd\backslash$   
 $ba+758/5cb^5+3904/245cb^3a+12618cba^2+45280/9261d^2b^2a+4/63db^6+8178503/15435d\backslash$   
 $b^4a-3032/35db^2a^2-52/35b^6a-1008b^4a^2+8917192/108045c^2b+20176/15435cdb^2+3119\backslash$   
 $812/5145cda+20430577/35cb^4+26496/343cb^2a+9216ca^2+147043/19845d^2ba+863872/\backslash$   
 $2205db^5+7003823691/3430db^3a-16335524/49dba^2-4b^7-772/49b^5a-2016b^3a^2+172580\backslash$   
 $32/108045c^2+2892256/2205cdb+40842962/35cb^3+12382792/245cba+53656/315db^4+5\backslash$   
 $20568/1715db^2a+208/35b^6-736/49b^4a-576b^2a^2+3946912/3087cd+81722308/245cb^2+\backslash$   
 $63150592/1715ca+100/441d^2b+709600/3087db^3+42025695302/36015dba+3088/49b^5-\backslash$   
 $432/1715b^3a-1152ba^2+32671944/49cb+1502048/15435db^2+2166931/49b^4-1984/245b\backslash$   
 $2a+7552/2205db+55320768/1715b^3+8896/1715ba+6190756/245b^2+31575296/1715b$

```

⇒ J[14]=b9-48392128/5145c2d+60497909109056/16343210835c2b2-238194021790/294\
17779503cd2b+8065888/5145cdb3+30249615617056/2334744405cdba+384/35cb5+307\
2/1715cb3a-11648320/16807cba2-232497742664/29417779503d2b2a-4/441db6+1380\
928/36015db4a-1536/245db2a2+4b8-96/245b6a-256/16807b2a3+119801054187904/1\
6343210835c2b+7936/36015cdb2-3065728/12005cda+74726439712/756315cb4+16384\
/2401cb2a-2420168/1361367d2b3+87904/138915d2ba-664/15435db5+12451636912/3\
6015db3a-19361152/343dba2+36/7b7-512/343b5a-110595584/756315c2+256/108045\
cdb+48404416/245cb3-232478976/84035cba+10343440/1058841d2b2+304/1715db4+3\
0607186827808/16343210835db2a+1504/245b6-512/343b4a-1024/21609cd+96827264\
/1715cb2-16384/12005ca-640/3087d2b+1056/2401db3+49797330496/252105dba+372\
8/343b5-6656/12005b3a+38722304/343cb+72448/756315db2-5806688/2401b4-8704\
12005b2a+4096/15435db+42304/12005b3+2048/12005ba-116239488/84035b2-8192/1\
2005b
⇒ J[15]=dba3-2/3c2d+1/9cdb3-1/6d2ba+5/9d2b
⇒ J[16]=db3a2-147/2cdb3-10cdba-288cba2+63b4a2-32c2b-132cda-576ca2-49/2db5+4\
db3a+126b3a2-64c2-82cdb-1152cba-21/2db4+36b2a2-80cd-2304ca-14db3-16dba+72\
ba2-6db2-1008b4-2016b3-576b2-1152b
⇒ J[17]=a5+1/9cb2a2-4/3c2a-7/6cb3+5/9da2-2/3cb
⇒ J[18]=da4-4/21c3+11/63cdba+8/63c2b-1/63db3a+10/63cdb+32/7cba+5/9d2a+2/63d\
b2a-b4a+4b4-4/7b2a+16/7b2

```

See also: [Section D.2.6.1 \[modular\]](#), [page 884](#).

### D.15.11.2 modrWalk

Procedure from library `modwalk.lib` (see [Section D.15.11 \[modwalk.lib\]](#), [page 2439](#)).

**Return:** a standard basis of  $I$

**Note:** The procedure computes a standard basis of  $I$  (over the rational numbers) by using modular methods.

**Example:**

```

LIB "modwalk.lib";
ring R1 = 0, (x,y,z,t), dp;
ideal I = 3x3+x2+1, 11y5+y3+2, 5z4+z2+4;
I = std(I);
ring R2 = 0, (x,y,z,t), lp;
ideal I = fetch(R1, I);
int radius = 2;
ideal J = modrWalk(I,radius);
J;
⇒ J[1]=x3+1/3x2+1/3
⇒ J[2]=z4+1/5z2+4/5
⇒ J[3]=y5+1/11y3+2/11
ring S1 = 0, (a,b,c,d), Dp;
ideal I = 5b2, ac2+9d3+3a2+5b, 2a2c+7abd+bcd+4a2, 2ad2+6b2d+7c3+8ad+4c;
I = std(I);
ring S2 = 0, (c,d,b,a), lp;
ideal I = fetch(S1,I);
// I is assumed to be a Dp-Groebner basis.
// We compute a lp-Groebner basis.
ideal J = modrWalk(I,radius,"Dp");
J;
⇒ J[1]=a25+16a24+96a23+256a22+256a21+256/9a20+1024/3a19+2048a18+65536/9a17+\

```

```

32768/3a16+16384/81a15+131072/81a14+1048576/81a13+1048576/27a12+1048576/9\
a11
⇒ J[2]=ba11+1522867351997104938459/91668001658017308797687087104a24+4293036\
9782248629690765/91668001658017308797687087104a23+80925218629630777478637\
/22917000414504327199421771776a22+7108535670237178684767/2864625051813040\
899927721472a21-3255817194541612658349/89519532869157528122741296a20+5380\
8965391546362724459/358078131476630112490965184a19+1534729815590907963215\
01/358078131476630112490965184a18-260815719913165309506063/44759766434578\
764061370648a17-1485276141860757031491027/89519532869157528122741296a16-4\
92332725360316960775/22379883217289382030685324a15+7423992361030571232440\
/16784912412967036523013993a14-17640364913371983121693/167849124129670365\
23013993a13-37723213977586186442564/5594970804322345507671331a12+92047580\
41857159721472414/5594970804322345507671331a11
⇒ J[3]=b2a6-63/2ba10+41087306587333357057895823883/924013456712814472680685\
83800832a24+93915562116924232413944264677/1320019223875449246686694054297\
6a23+1314662746341964624103002499857/30800448557093815756022861266944a22+\
2628268795042931967685617407557/23100336417820361817017145950208a21+32861\
7969148352577032114618159/2887542052227545227127143243776a20+175631724829\
284757906915538269/12993939235023953522072144596992a19+303812212296039043\
29090032467/206253003730538944794795945984a18+581332950565269518541458030\
9/6445406366579342024837373312a17+1392577308804410648719627124495/4060606\
01094498547564754518656a16+728489619081836101063651608791/135353533698166\
182521584839552a15+80182830319998353431517995037/913636352462621732020697\
666976a14+22597043807001043905240513127/32629869730807919000739202392a13+\
2699689128255025271541196980091/456818176231310866010348833488a12+6291773\
289016174025274735397/352483160672307766983293853a11+4/7a6
⇒ J[4]=b3a5+4/7ba5-5398327059462849163101023479/739210765370251578144548670\
406656a24+3500067651845908053488406611/92401345671281447268068583800832a2\
3+81936880336538263803253409291/46200672835640723634034291900416a22+85587\
506267677081700930548967/6600096119377246233433470271488a21+4349147060430\
84846670081811347/11550168208910180908508572975104a20+1021713731964491721\
016405426709/25987878470047907044144289193984a19+189087309367688338214840\
3875/812121202188997095129509037312a18+3873208423822126582196454031/10312\
6501865269472397397972992a17+104243440049859097996976327663/4060606010944\
98547564754518656a16+52530075262606982469983545909/5075757513681231844559\
4314832a15+6154556265260978917193662164647/365454540985048692808279066790\
4a14+138400848395446486358821423/5639730570756924271732701648a13+68577767\
53456717192310999393/38068181352609238834195736124a12+3439955547980759942\
57860510037/228409088115655433005174416744a11+9/2a10+18a9
⇒ J[5]=b4-63/2b3a4+4/7b2+3859043113737/128ba10-61261515/8ba9-525086793/32ba\
8+3969/16ba7-19845/4ba6+9/2ba5-317530772199391516703685862633925319890715\
5/9853890705461273608546303647569412096a24-889000025950798002413919214249\
17407047623059/17244308734557228814956031383246471168a23-4443358152931308\
1841158459544460024461570519/1437025727879769067913002615270539264a22-443\
94201884268758058296712245149507940575423/5388846479549134004673759807264\
52224a21-88794706503699382133468723144783169844707611/1077769295909826800\
934751961452904448a20-1763218978464215084468461907410740543335425/1732129\
22556936450150227993804931072a19-4436403904696107325808495191699219229446\
85/4210036312147760941151374849425408a18-21791665871565292166347154127869\
734967876143/33680290497182087529210998795403264a17-382244791239220889094\
021792022832016196677083/151561307237319393881449494579314688a16-25381669\
887869257097552146559952295821945667/6315054468221641411727062274138112a1\
5-3081988985408645474889243822837753592680251/487161344691383766047516232\

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57636864a14-42991086686519981772739427292780808208480959/8525323532099215\
9058315340700864512a13-44791324614077928834488691100822888689474979/10656\
654415124019882289417587608064a12-250969820105414481651193596059252574282\
0133/197345452131926294116470696066816a11+771901337907/64a10-1701/4a9+437\
53689/4a8-19845/8a7-19845/2a6
⇒ J[6]=da-1323/800b3a4+63/20b2a4+2701701074331/1280ba10-61269453/320ba9+306\
291699/160ba8-27783/16ba6-189/200ba4-685883099436497069901143509465258298\
0621641/123173633818265920106828795594617651200a24-5486790468650205133145\
8584531975294116209709/61586816909132960053414397797308825600a23-60954530\
98260341081175657932145681344303369/1140496609428388149137303662913126400\
a22-54839253809480536502430419967744127237862429/384917605682081000333839\
9862331801600a21-6855741428155530311932580353074238311128479/481147007102\
601250417299982791475200a20-2878029541644786739384025698911163129301521/1\
732129225569364501502279938049310720a19-539637460347386518746735121876349\
5466944867/288688204261560750250379989674885120a18-1360851075145362976080\
2370233473935440353147/120286751775650312604324995697868800a17-2261317453\
3404843047583509526614944559001629/54129038299042640671946248064040960a16\
-19899168362053216404911620136404443756991/309379505595808417192193918976\
00a15-3443170203569948104100043669593479708510001/30447584043211485377969\
7645360230400a14-2720322443264076193874003163568183222207391/30447584043\
2114853779697645360230400a13-11007706754707314979975768772869745826795967\
3/152237920216057426889848822680115200a12-1380727908059280736173048219943\
2331439121223/6343246675669059453743700945004800a11-1929592445193/640a10+\
437570343/1600a9+164090367/100a8+3969/8a6+9/5a4
⇒ J[7]=db2+4/7d+81/160b3a4-189/200b3a3+9/5b2a3+250047/32ba10+4750893/320ba9\
-567/40ba8-567/10ba7+81/280ba4-27/50ba3-627430952592078879073202920720376\
153131/5173292620367168644486809414973941350400a24-2906859466655459274817\
4299538370243493/18476045072739888016024319339192647680a23-69048240138740\
571289193103124930690109/11975214398998075565941688460587827200a22+488085\
08639174871490992992380443024047/11547528170462430010015199586995404800a2\
1+5093069077149293873574182379116672811143/808326971932370100701063971089\
67833600a20+3286117544936595300844164893345508176887/36374713736956654531\
547878699035525120a19-57124126734867585149642894515011901057/189451634046\
6492423518118682241433600a18-296407013643924816106633846582502767991/2526\
021787288656564690824909655244800a17-193831832299525900077669479610197531\
4379/11367098042798954541108712093448601600a16+49045157132492520042165362\
0398209599/433131307834131784069071486566400a15+9565278519988924726047202\
0815175898220317/25575970596297647717494602210259353600a14-51800242230268\
556716933919846651112061/456713760648172280669546468040345600a13-15498981\
08498974500694541229046869490537/1598498162268602982343412638141209600a12\
-700479417253540849801700529315888259/26641636037810049705723543969020160\
a11+20611017/1600a10+21504771/320a9+243/50a8-1701/100a7+36/35a3
⇒ J[8]=d2
⇒ J[9]=ca-1630191535/8275968d2ba2-36060437345/409660416d2ba+38107475/568972\
8d2a2-456425375/60690432d2a+92575/2844864d2+91285075/182071296db4a+35/921\
6db4-1088785/5689728db3a2+2258752616171/182071296db3a+24027/2528768db3-81\
62999/79656192db2a3-6539063/79656192db2a2+25495277/45517824db2a+5095/1264\
384db2-5243/158048dba4-753114969703/1486915584dba3-753467060575/223037337\
6dba2+30870844232377/2230373376dba+24027/4425344db-7/256da5+5/96da3+34222\
8711385/101380608da2+342235311685/25345152da+2645/2489256d-68441518085/57\
931776b6+8617/1896576b5a-752908955671/318624768b5+7/256b4a2-2163/79024b4a\
-752734601911/1115186688b4+529/316096b3a3+7/128b3a2-12811/59268b3a-752421\
061495/557593344b3+311787/632192b2a4+529/158048b2a3+1/64b2a2-309/19756b2a\

```



```

+309/4939b2+68441518085/22529024ba5+752978795959/61954816ba4+529/553168ba\
3+1/32ba2-1/8ba+1/2b+3/32a6+68445742277/11264512a5+68447062337/2816128a4+\
529/276584a3
⇒ J[10]=cb-452831001/4597760d2ba2-3340189909/75863040d2ba-3176523/158048d2a\
2-76214943/20230144d2a+5145/316096d2+25404981/101150720db4a-7203/1024db4+\
453789/790240db3a2+125486265849/20230144db3a-25399101/12643840db3-324051/\
6321920db2a3+194523/790240db2a2+11921259/25287680db2a-9921275/1264384db2-\
27783/632192dba4-12813612147/50575360dba3-12806449187/75863040dba2+104952\
523865/15172608dba-31286843/3160960db+1163396585/689664da2+1163396585/172\
416da-172823/79024d-1628855795/2758656b6+64827/3160960b5a-89587760437/758\
63040b5+64827/1580480b4a-12804376067/37931520b4+1323/1580480b3a3+9261/790\
240b3a-12798251491/18965760b3-583443/395120b2a4-10887849/1580480b2a3+9261\
/395120b2a-9261/98780b2+698081055/459776ba5+38375787849/6321920ba4-622295\
1/395120ba3+698037951/229888a5+698037951/57472a4-1555407/395120a3
⇒ J[11]=cd+66199/8d2ba2+6818809/2304d2ba+35/64d2b+47/96d2a2-155/768d2a-7/19\
2db5+7/256db4a+7/320db4-1/80db3a2-133413949/256db3a-23/96db3+441/128db2a3\
+1/64db2a+1/80db2+667316711/31360dba3+667069751/47040dba2-27349858321/470\
40dba-1/8db+141/160da5+141/40da4+63/32da3-667069751/4704da2-667069751/117\
6da+667069751/13440b6+667069751/6720b5-63/80b4a2+667069751/23520b4-63/40b\
3a2+667069751/11760b3-9/20b2a2-2001209253/15680ba5-2001209253/3920ba4-9/1\
0ba2-2001209253/7840a5-2001209253/1960a4
⇒ J[12]=c2+1/3cb2+5/3d+3a3
intvec w = 3,2,1,2;
ring S3 = 0, (c,d,b,a), (a(w),lp);
ideal I = fetch(S1,I);
// I is assumed to be a Dp-Groebner basis.
// We compute a (a(w),lp)-Groebner basis.
ideal J = modrWalk(I,radius,"Dp",w);
J;
⇒ J[1]=d2
⇒ J[2]=c2+3a3+1/3cb2+5/3d
⇒ J[3]=ca2+4ca+7/2b3+2b
⇒ J[4]=cda-6/7ba3+4/7c2-2/21cb3+1/7dba-10/21db
⇒ J[5]=db4a-192/49b2a3+128/49c2b-64/7cda-64/147cb4+4/63d2ba+2db3a-4db4+60/4\
9db2a-8db3+8/7dba-656/147db2-32/7db
⇒ J[6]=db6+333576da3-189/4b8+15876b4a2-5186640/2401b2a3+44514592/2470629c2b\
+772122956311640/155649627cd2+1815960/49cdb2-3535382608/352947cda+8395442\
084/7203cb4-2304/49cb2a-826877376/2401ca2+308820941902337/311299254d2ba-4\
3215/7db5+199982018/49db3a-4664160/7dba2-126b7+72/7b5a+31752b3a2+1728/240\
1ba3-7028226304/2470629c2-1012272/49cdb+5598114168/2401cb3-4608/49cba-191\
05043/21609d2b2-129552/49db4+864216/2401db2a-117b6+144/7b4a+9072b2a2-2016\
0cd+4663152/7cb2-3307509504/2401ca-128772920/352947d2b-172796/49db3+57587\
01763736/2470629dba-1296/7b5+288/49b3a+18144ba2+9326304/7cb+185320d2-1952\
9120/7203db2-1188/7b4+576/49b2a+2752/2401db-413460864/343b3-3312/49b2-165\
3754752/2401b
⇒ J[7]=cb3a-5/8da3-245/1024b8-7/32b6a+2/343b2a3-1833054229/38118276c2b-3473\
26678687775/9605805552cd2+5/112cdb2-916402091/5445468cda-108013/49392cb4+\
25/7cb2a-966391/1372ca2-2222840154809345/307385777664d2ba-3/256db5-15431/\
2016db3a+5/4dba2-245/256b7-25/32b5a+5/1372ba3-916476179/9529569c2+5/84cdb\
-108035/24696cb3+16/7cba-3313805/790272d2b2-47/2688db4+53/24696db2a-91/25\
6b6-5/8b4a-5/84cd-5/4cb2-966979/343ca-80483075/43563744d2b+1/192db3-21666\
16927/76236552dba+65/32b5-1/14b3a-5/2cb-25/72d2-667/98784db2+85/64b4-2/7b\
2a+109/12348db-1932803/784b3+3/14ba+93/112b2-966979/686b

```

- ⇒  $J[8] = \text{cb5} - 4/7\text{db2a2} + 96/7\text{b2a3} - 64/7\text{c2b} - 4/441\text{cdb2} + 32\text{cda} + 116/21\text{cb4} + 1152/7\text{ca2} - 8/7\text{dba2} - 8/441\text{cdb} + 32/7\text{cb3} + 14\text{db4} - 16/7\text{db2a} + 16/7\text{cb2} + 4608/7\text{ca} + 28\text{db3} + 16/7\text{cb} + 328/2/1\text{db2} + 16\text{db} + 576\text{b3} + 2304/7\text{b}$
- ⇒  $J[9] = \text{cdb3} + 18\text{da3} + 2\text{cdb2} + 63\text{cb4} + 441/2\text{db3a} - 36\text{dba2} + 126\text{cb3} + 36\text{cb2} + 126\text{dba} + 72\text{cb} + 10\text{d}^2$
- ⇒  $J[10] = \text{ba4} - 2/3\text{c2a} + 1/9\text{cb3a} + 14/3\text{cda} - 1/6\text{dba2} + 49/12\text{db3} + 5/9\text{dba} + 7/3\text{db}$
- ⇒  $J[11] = \text{b3a3} + 2744/5\text{c2d} + 382/15\text{c2b2} - 1127/15\text{cdb3} + 693167/15\text{cdba} + 129659/45\text{cb5} + 23/52/5\text{cb3a} - 8647/27\text{d2b2a} - 49/15\text{db4a} - 8232/5\text{db2a2} - 1029/10\text{b6a} + 616/15\text{c2b} + 86470/27/cd2 + 161/5\text{cdb2} + 1382932/15\text{cda} + 28812/5\text{cb4} + 1680\text{cb2a} + 686/5\text{d2ba} - 98/15\text{db5} - 121045/6\text{db3a} - 735/2\text{b5a} - 112/5\text{c2} - 224/15\text{cdb} + 8232/5\text{cb3} + 5376/5\text{cba} + 201698/5\text{db4} + 38/3\text{db2/a} + 2058/5\text{b6} - 294\text{b4a} - 28\text{cd} + 16464/5\text{cb2} - 4032/5\text{ca} + 3631597/45\text{db3} - 173024/15\text{dba} + 147/0\text{b5} - 168/5\text{b3a} + 115256/5\text{db2} + 1176\text{b4} - 672/5\text{b2a} + 138352/3\text{db} + 672/5\text{b3} + 504/5\text{ba} + 2688/5\text{b2} - 2016/5\text{b}$
- ⇒  $J[12] = \text{b5a2} + 2420474/3\text{c2d} + 32/9\text{c2b2} - 1210217/9\text{cdb3} - 689428/147\text{cdba} + 931/2\text{cb5} + 64/cb3a - 345872/7\text{cba2} - 6173/81\text{d2b4} - 1867616522/3969\text{d2b2a} - 7/18\text{db6} + 669815/252\text{db4a} - 117310/441\text{db2a2} - 14\text{b6a} + 4\text{b4a2} + 395552/63\text{c2b} + 18676165220/3969\text{cd2} + 194/63\text{cdb2/a} - 1844168/147\text{cda} - 8469797\text{cb4} + 1600/7\text{cb2a} - 128/7\text{ca2} + 7471160423/7938\text{d2ba} - 11/6\text{db5} - 622555298/21\text{db3a} + 2134623448/441\text{dba2} - 50\text{b5a} + 32/7\text{b3a2} + 790208/63\text{c2} - 124/21\text{cdb/a} - 16941190\text{cb3} - 1382464/7\text{cba} - 24692/567\text{d2b2} - 259351/63\text{db4} + 74519/49\text{db2a} + 56\text{b6} - 40/b4a + 16/7\text{b2a2} - 400/63\text{cd} - 4839884\text{cb2} - 1280/7\text{ca} - 518474/63\text{db3} - 7469280824/441\text{dba/a} - 200\text{b5} - 32/7\text{b3a} + 16/7\text{ba2} - 9680832\text{cb} - 1037348/441\text{db2} - 172776\text{b4} - 128/7\text{b2a} - 2073632/441\text{db} - 320/7\text{b3} + 96/7\text{ba} - 691232/7\text{b2} - 640/7\text{b}$
- ⇒  $J[13] = \text{b7a} - 13606662/245\text{c2d} + 1024/735\text{c2b2} - 864700/27783\text{cd2b} + 23004293/2205\text{cdb3/a} + 29189312/12005\text{cdba} + 758/5\text{cb5} + 3904/245\text{cb3a} + 12618\text{cba2} + 12600644807/388962\text{d2b/a} - 2\text{a} + 4/63\text{db6} + 537472/15435\text{db4a} - 3032/35\text{db2a2} - 52/35\text{b6a} - 1008\text{b4a2} + 1266752/15435\text{c/a} - 2\text{b} - 21001940345/64827\text{cd2} + 20176/15435\text{cdb2} + 554322208/108045\text{cda} + 20430577/35\text{cb/a} - 4 + 26496/343\text{cb2a} + 9216\text{ca2} - 2334414076/36015\text{d2ba} + 863872/2205\text{db5} + 12600769819/6/174\text{db3a} - 16335524/49\text{dba2} - 4\text{b7} - 772/49\text{b5a} - 2016\text{b3a2} + 2447488/15435\text{c2} + 2892256/22/05\text{cdb} + 40842962/35\text{cb3} + 12382792/245\text{cba} + 33193268/15435\text{db4} + 49664/2401\text{db2a} + 208/35\text{b6} - 736/49\text{b4a} - 576\text{b2a2} + 3946912/3087\text{cd} + 81722308/245\text{cb2} + 63150592/1715\text{ca} + 10/0/441\text{d2b} + 9239464/2205\text{db3} + 126015926254/108045\text{dba} + 3088/49\text{b5} - 432/1715\text{b3a} - 115/2\text{ba2} + 32671944/49\text{cb} + 44256944/36015\text{db2} + 2166931/49\text{b4} - 1984/245\text{b2a} + 16325536/72/03\text{db} + 55320768/1715\text{b3} + 8896/1715\text{ba} + 6190756/245\text{b2} + 31575296/1715\text{b}$
- ⇒  $J[14] = \text{b9} - 48392128/5145\text{c2d} + 469223949568/2334744405\text{c2b2} + 4446618588746560/29/417779503\text{cd2b} + 8065888/5145\text{cdb3} + 293190604544/333534915\text{cdba} + 384/35\text{cb5} + 3072/1715\text{cb3a} - 11648320/16807\text{cba2} + 1050576605312618/29417779503\text{d2b2a} - 4/441\text{db6} - 15/04/108045\text{db4a} - 1536/245\text{db2a2} + 4\text{b8} - 96/245\text{b6a} - 256/16807\text{b2a3} + 767602156544/2334/744405\text{c2b} - 10670463200/194481\text{cd2} + 7936/36015\text{cdb2} + 23968768/252105\text{cda} + 7472643/9712/756315\text{cb4} + 16384/2401\text{cb2a} + 2651044/151263\text{d2b3} - 74705512256/6806835\text{d2ba/a} - 664/15435\text{db5} + 7469324432/21609\text{db3a} - 19361152/343\text{dba2} + 36/7\text{b7} - 512/343\text{b5a} - 1757/184/12005\text{c2} + 256/108045\text{cdb} + 48404416/245\text{cb3} - 232478976/84035\text{cba} + 515091680/66/706983\text{d2b2} + 16596304/108045\text{db4} + 46922869120/466948881\text{db2a} + 1504/245\text{b6} - 512/34/3\text{b4a} - 1024/21609\text{cd} + 96827264/1715\text{cb2} - 16384/12005\text{ca} - 640/3087\text{d2b} + 33201824/108/045\text{db3} + 49786270144/252105\text{dba} + 3728/343\text{b5} - 6656/12005\text{b3a} + 38722304/343\text{cb} + 9483/008/108045\text{db2} - 5806688/2401\text{b4} - 8704/12005\text{b2a} + 8854528/50421\text{db} + 42304/12005\text{b3/a} - 2048/12005\text{ba} - 116239488/84035\text{b2} - 8192/12005\text{b}$
- ⇒  $J[15] = \text{dba3} - 2/3\text{c2d} + 1/9\text{cdb3} - 1/6\text{d2ba} + 5/9\text{d2b}$
- ⇒  $J[16] = \text{db3a2} - 147/2\text{cdb3} - 10\text{cdba} - 288\text{cba2} + 63\text{b4a2} - 32\text{c2b} - 132\text{cda} - 576\text{ca2} - 49/2\text{db5} + 4/db3a + 126\text{b3a2} - 64\text{c2} - 82\text{cdb} - 1152\text{cba} - 21/2\text{db4} + 36\text{b2a2} - 80\text{cd} - 2304\text{ca} - 14\text{db3} - 16\text{dba} + 72/ba2 - 6\text{db2} - 1008\text{b4} - 2016\text{b3} - 576\text{b2} - 1152\text{b}$
- ⇒  $J[17] = \text{a5} + 1/9\text{cb2a2} - 4/3\text{c2a} - 7/6\text{cb3} + 5/9\text{da2} - 2/3\text{cb}$
- ⇒  $J[18] = \text{da4} - 4/21\text{c3} + 11/63\text{cdba} + 8/63\text{c2b} - 1/63\text{db3a} + 10/63\text{cdb} + 32/7\text{cba} + 5/9\text{d2a} + 2/63\text{d/b2a} - \text{b4a} + 4\text{b4} - 4/7\text{b2a} + 16/7\text{b2}$

See also: [Section D.2.6.1 \[modular\]](#), page 884.

### D.15.11.3 modfWalk

Procedure from library `modwalk.lib` (see [Section D.15.11 \[modwalk.lib\]](#), page 2439).

**Return:** a standard basis of  $I$

**Note:** The procedure computes a standard basis of  $I$  (over the rational numbers) by using modular methods.

**Example:**

```
LIB "modwalk.lib";
ring R1 = 0, (x,y,z,t), dp;
ideal I = 3x3+x2+1, 11y5+y3+2, 5z4+z2+4;
I = std(I);
ring R2 = 0, (x,y,z,t), lp;
ideal I = fetch(R1, I);
ideal J = modfWalk(I);
J;
⇒ J[1]=z4+1/5z2+4/5
⇒ J[2]=y5+1/11y3+2/11
⇒ J[3]=x3+1/3x2+1/3
ring S1 = 0, (a,b,c,d), Dp;
ideal I = 5b2, ac2+9d3+3a2+5b, 2a2c+7abd+bcd+4a2, 2ad2+6b2d+7c3+8ad+4c;
I = std(I);
ring S2 = 0, (c,d,b,a), lp;
ideal I = fetch(S1,I);
// I is assumed to be a Dp-Groebner basis.
// We compute a lp-Groebner basis.
ideal J = modfWalk(I,"Dp");
J;
⇒ J[1]=a25+16a24+96a23+256a22+256a21+256/9a20+1024/3a19+2048a18+65536/9a17+\
32768/3a16+16384/81a15+131072/81a14+1048576/81a13+1048576/27a12+1048576/9\
a11
⇒ J[2]=ba11+1522867351997104938459/91668001658017308797687087104a24+4293036\
9782248629690765/91668001658017308797687087104a23+80925218629630777478637\
/22917000414504327199421771776a22+7108535670237178684767/2864625051813040\
899927721472a21-3255817194541612658349/89519532869157528122741296a20+5380\
8965391546362724459/358078131476630112490965184a19+1534729815590907963215\
01/358078131476630112490965184a18-260815719913165309506063/44759766434578\
764061370648a17-1485276141860757031491027/89519532869157528122741296a16-4\
92332725360316960775/22379883217289382030685324a15+7423992361030571232440\
/16784912412967036523013993a14-17640364913371983121693/167849124129670365\
23013993a13-37723213977586186442564/5594970804322345507671331a12+92047580\
41857159721472414/5594970804322345507671331a11
⇒ J[3]=b2a6-1275460856846934902527/2619112706933760ba13-9203207900045442436\
039/4583447237134080ba12-86229451659721876411/6261539941440ba11-63/2ba10-\
176103105371/907641397248a21-306836641361771/103471119286272a20-556810728\
2265313/181074458750976a19+463053737062394039071/32593402575175680a18+233\
493474751858876283/2037087660948480a17+66864105293320847653/2910125229926\
40a16-2870620980805387/101854383047424a15-9651151354818523/25463595761856\
a14-471264260028168789475/611126298284544a13-489585210177544137565/152781\
574571136a12-4323891804194718805849/190976968213920a11+4/7a6
```

$\mapsto J[4]=b3a5+1165198340059361507/60324030330315296b2a8+2834258341382482451/3\backslash$   
 $3932267060802354b2a7-3535202147650393/45243022747736472b2a6-1743263488518\backslash$   
 $66272086637/108100662351925010432ba13-180877903989284763838711/2702516558\backslash$   
 $7981252608ba12-2783949606995368438959121/60806622572957818368ba11+7423924\backslash$   
 $5100658253/30162015165157648ba10+4/7ba5+82576115266283735625891/172961059\backslash$   
 $7630800166912a18+2949131532722419989261/7721475882280357888a17+8257374787\backslash$   
 $8379091193883/108100662351925010432a16-194476798666289267/270251655879812\backslash$   
 $52608a15+10816123814459571297/6756291396995313152a14-13707944038216024518\backslash$   
 $9169/54050331175962505216a13-143935810252288492483523/1351258279399062630\backslash$   
 $4a12-108976370407053801898907/1447776727927567104a11+9/2a10+18a9+11651983\backslash$   
 $40059361507/105567053078051768a8+5668516682764964902/118762934712808239a7\backslash$   
 $-3535202147650393/79175289808538826a6$

$\mapsto J[5]=b4-63/2b3a4+53590985045967705613474269749/1407238979545595225088b2a8\backslash$   
 $+11625419776309240542435260807/58634957481066467712b2a7+12739505342603800\backslash$   
 $3554252983797/175904872443199403136b2a6+4/7b2-761164231440237277536357328\backslash$   
 $493047/700492292040474067599360ba13-104521491576210733019159653207691/250\backslash$   
 $17581858588359557120ba12-11826338279466524349922531923473911/394026914272\backslash$   
 $766663024640ba11+286749359828001390655523145503/39089971654044311808ba10-\backslash$   
 $61261515/8ba9-525086793/32ba8+3969/16ba7-19845/4ba6+9/2ba5+45295478141475\backslash$   
 $95860677850296017/138368847810464013352960a18+247345844501521328455296898\backslash$   
 $6663/9729059611673250938880a17+324025940290811430696988658749687/70049229\backslash$   
 $2040474067599360a16-4286410117811597590771586510563/350246146020237033799\backslash$   
 $68a15+4054710919870705842300540149/2918717883501975281664a14-394300742086\backslash$   
 $35887294045314648189/23349743068015802253312a13-1139723514755969617170188\backslash$   
 $24182045/17512307301011851689984a12-3226716701597602362857429388525683/65\backslash$   
 $671152378794443837440a11+771901337907/64a10-1701/4a9+80528689834593164587\backslash$   
 $767860213/2462668214204791643904a8+11625165236461787969048941667/10261117\backslash$   
 $5591866318496a7+127391998947868572673617154117/307833526775598955488a6$

$\mapsto J[6]=da+13042589597338317401292876694323783641/32769687632608435019205956\backslash$   
 $084751360b8-339347956624707128686162624470779591/136540365135868479246691\backslash$   
 $48368646400b7a+44776090034814931064318387941081165483/3276968763260843501\backslash$   
 $9205956084751360b7+72711086789965200830753309360861163/341350912839671198\backslash$   
 $1167287092161600b6a2+17945571338576304647577664032291913/1092322921086947\backslash$   
 $833973531869491712b6a+1005252013115630519751071456639664155699/5734695335\backslash$   
 $70647612836104231483148800b6+9615448034722990303485382435238901/136540365\backslash$   
 $13586847924669148368646400b5a3+663547630395349792235158387078347149/13654\backslash$   
 $036513586847924669148368646400b5a2-10588152408989285294663152330377654/37\backslash$   
 $3352560918390372940172025705175b5a+8954607782583032024373974051595847951/\backslash$   
 $5734695335706476128361042314831488b5-16768783292422657909931339309776827/\backslash$   
 $6827018256793423962334574184323200b4a4+507941899552434600769579126963767/\backslash$   
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⇒ J[7]=db2-52682490423871082296629464283815234/5824767849546495553439269268\
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⇒ J[8]=d2
⇒ J[9]=ca-1630191535/8275968d2ba2-10498813705/148967424d2ba-456425375/60690\
432d2a+92575/2844864d2+91285075/182071296db4a+35/9216db4-1088785/5689728d\
b3a2+2258752616171/182071296db3a+24027/2528768db3-8162999/79656192db2a3-6\
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688b4+529/316096b3a3+7/128b3a2-12811/59268b3a-752421061495/557593344b3+31\
1787/632192b2a4+529/158048b2a3+1/64b2a2-309/19756b2a+309/4939b2+684415180\
85/22529024ba5+752978795959/61954816ba4+529/553168ba3+1/32ba2-1/8ba+1/2b+\
3/32a6+68445742277/11264512a5+68447062337/2816128a4+529/276584a3
⇒ J[10]=cb-452831001/4597760d2ba2-972112421/27586560d2ba-76214943/20230144d\
2a+5145/316096d2+25404981/101150720db4a-7203/1024db4+453789/790240db3a2+1\
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1/229888a5+698037951/57472a4-1555407/395120a3
⇒ J[11]=cd+1148200/372848203cb-1524024/27175525ca+5764250385873/695693440d2\
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69640600a5-38118165490546077/37284820300a4-664713/54351050a3
⇒ J[12]=c2+1/3cb2+5/3d+3a3
intvec w = 3,2,1,2;
ring S3 = 0, (c,d,b,a), (a(w),lp);
ideal I = fetch(S1,I);
// I is assumed to be a Dp-Groebner basis.
// We compute a (a(w),lp)-Groebner basis.
ideal J = modfWalk(I,"Dp",w);
J;
⇒ J[1]=d2
⇒ J[2]=c2+3a3+1/3cb2+5/3d

```

$\mapsto J[3]=ca2+4ca+7/2b3+2b$   
 $\mapsto J[4]=cda-6/7ba3-2/21cb3-12/7a3-4/21cb2+1/7dba-10/21db-20/21d$   
 $\mapsto J[5]=db4a-192/49b2a3-64/147cb4+2db3a-768/49ba3-256/147cb3-4db4+60/49db2a-\backslash$   
 $768/49a3-256/147cb2-8db3+120/49dba-656/147db2-1952/147db-1280/147d$   
 $\mapsto J[6]=db6+333576da3-189/4b8+15876b4a2-5186640/2401b2a3+1815960/49cdb2+8395\backslash$   
 $442084/7203cb4-2304/49cb2a-43215/7db5+199982018/49db3a-4664160/7dba2-126b\backslash$   
 $7+72/7b5a+31752b3a2-20742528/2401ba3-1012272/49cdb+16787427752/7203cb3-46\backslash$   
 $08/49cba-129552/49db4+864216/2401db2a-117b6+144/7b4a+9072b2a2-20738496/24\backslash$   
 $01a3-20160cd+4791470576/7203cb2-172796/49db3+5599841736/2401dba-1296/7b5+\backslash$   
 $288/49b3a+18144ba2+9326304/7cb-19529120/7203db2-1188/7b4+576/49b2a-345655\backslash$   
 $04/7203db-3168/49b3-34564160/7203d-3312/49b2$   
 $\mapsto J[7]=cb3a-5/8da3-245/1024b8-7/32b6a+2/343b2a3+5/112cdb2-108013/49392cb4+2\backslash$   
 $5/7cb2a-3/256db5-15431/2016db3a+5/4dba2-245/256b7-25/32b5a+8/343ba3+5/84c\backslash$   
 $db-107981/24696cb3+16/7cba-47/2688db4+53/24696db2a-91/256b6-5/8b4a+8/343a\backslash$   
 $3-5/84cd-15403/12348cb2-12/7ca+1/192db3-12015/2744dba+65/32b5-1/14b3a-5/2\backslash$   
 $cb-667/98784db2+85/64b4-2/7b2a+61/3087db-3/112b3+3/14ba+40/3087d+93/112b2\backslash$   
 $-6/7b$   
 $\mapsto J[8]=cb5-4/7db2a2+96/7b2a3-4/441cdb2+116/21cb4-8/7dba2+384/7ba3-8/441cdb+\backslash$   
 $32/3cb3+14db4-16/7db2a+384/7a3+176/21cb2+28db3-32/7dba+16/7cb+328/21db2+9\backslash$   
 $76/21db+640/21d$   
 $\mapsto J[9]=cdb3+18da3+2cdb2+63cb4+441/2db3a-36dba2+126cb3+36cb2+126dba+72cb$   
 $\mapsto J[10]=ba4+5/72da3+245/9216b8+7/288b6a-2/3087b2a3+2a4-5/1008cdb2+108013/44\backslash$   
 $4528cb4-11/63cb2a+1/768db5+15431/18144db3a-11/36dba2+245/2304b7+25/288b5a\backslash$   
 $+12340/3087ba3-5/756cdb+206765/222264cb3-16/63cba+47/24192db4-53/222264db\backslash$   
 $2a+91/2304b6+5/72b4a+24688/3087a3+5/756cd+114187/111132cb2+4/21ca+7055/17\backslash$   
 $28db3+9271/24696dba-65/288b5+1/126b3a+5/18cb+667/889056db2+10/9da-85/576b\backslash$   
 $4+2/63b2a+126506/27783db+1/336b3-1/42ba+123440/27783d-31/336b2+2/21b$   
 $\mapsto J[11]=b3a3+4/63db2a2+7203/64b8+52/21b2a3+18526/3969cdb2-32/189cb4-49/48db\backslash$   
 $5-1/6db3a+8/63dba2+7203/16b7-44/21ba3+37052/3969cdb-20/27cb3-259/72db4-5/\backslash$   
 $63db2a+9261/16b6-128/21a3-176/189cb2-113/36db3+32/63dba+1029/2b5-16/63cb-\backslash$   
 $257/378db2+2205/4b4-556/189db+147b3-640/189d+147b2$   
 $\mapsto J[12]=b5a2-4/441db2a2+129724da3-49/16b8+6178b4a2-864464/1029b2a3+302660/2\backslash$   
 $1cdb2+4197721018/9261cb4-128/7cb2a-28823/12db5+99991009/63db3a-114271928/\backslash$   
 $441dba2+49/4b7+4b5a+86468/7b3a2-3457184/1029ba3-506216/63cdb+8393713780/9\backslash$   
 $261cb3-256/7cba-129625/126db4+432008/3087db2a+133/4b6+8b4a+24712/7b2a2-34\backslash$   
 $56512/1029a3-494080/63cd+2395735192/9261cb2-86405/63db3+2799920668/3087db\backslash$   
 $a-2b5+16/7b3a+49408/7ba2+518128cb-9766570/9261db2+9b4+32/7b2a-17280544/92\backslash$   
 $61db-36/7b3-17282560/9261d-44/7b2$   
 $\mapsto J[13]=b7a-296432/7da3+109/16b8+2b6a-2016b4a2+4610560/16807b2a3-14521796/3\backslash$   
 $087cdb2-22401156488/151263cb4+8000/343cb2a+1382867/1764db5-1600806884/308\backslash$   
 $7db3a+4148384/49dba2+77/4b7-194/49b5a-4032b3a2+18438656/16807ba3+8103808/\backslash$   
 $3087cdb-44793095440/151263cb3+13568/343cba+230189/686db4-6908840/151263db\backslash$   
 $2a+533/28b6-312/49b4a-1152b2a2+18435072/16807a3+7901200/3087cd-4261620128\backslash$   
 $/50421cb2-4864/343ca+1382435/3087db3-4980595264/16807dba+2078/49b5+176/34\backslash$   
 $3b3a-2304ba2-8294976/49cb+52056614/151263db2+2435/49b4-1472/343b2a+921811\backslash$   
 $84/151263db+1900/343b3+608/343ba+30725120/50421d+8044/343b2-2432/343b$   
 $\mapsto J[14]=b9+1037536/343da3+4b8+144b4a2-329408/16807b2a3+148256/441cdb2+15987\backslash$   
 $36592/151263cb4-24704/441db5+799730504/21609db3a-296064/49dba2+40/7b7+288\backslash$   
 $b3a2-1317376/16807ba3-4047040/21609cdb+3196814624/151263cb3+2048/2401cba-\backslash$   
 $24db4+164704/50421db2a+40/7b6-64/343b4a+576/7b2a2-3840/49a3-3950080/21609\backslash$   
 $cd+886720/147cb2+4096/2401ca-98816/3087db3+1036320/49dba+272/49b5-128/343\backslash$   
 $b3a+1152/7ba2+592000/49cb-3721504/151263db2+1152/343b4-256/2401b2a-658688\backslash$   
 $0/151263db+1024/343b3-512/2401ba-6400/147d+1920/2401b2+2048/2401b$

```

⇒ J[15]=dba3-7cb4-49/2db3a+4dba2-14cb3-4cb2-14dba-8cb
⇒ J[16]=db3a2+1323da3+63b4a2-60/7b2a3+147cdb2+194441/42cb4-49/2db5+64843/4d\
b3a-2646dba2+126b3a2-240/7ba3-82cdb+194401/21cb3-21/2db4+10/7db2a+36b2a2-\
240/7a3-80cd+55486/21cb2-14db3+64847/7dba+72ba2+5292cb-226/21db2-400/21db\
-400/21d
⇒ J[17]=a5+4a4-7/6cb3+5/9da2-7/18b5-2/3cb+20/9da-2/9b3
⇒ J[18]=da4-2/49b2a3-2/441cb4-1/63db3a-4/49ba3+10/63cdb-4/441cb3+32/7cba+1/\
147db2a-b4a+20/63cd+64/7ca-2b3a-10/441db2+4b4-4/7b2a-20/441db+8b3-8/7ba+1\
6/7b2+32/7b

```

See also: [Section D.2.6.1 \[modular\]](#), page 884.

#### D.15.11.4 modfrWalk

Procedure from library `modwalk.lib` (see [Section D.15.11 \[modwalk.lib\]](#), page 2439).

**Return:** a standard basis of  $I$

**Note:** The procedure computes a standard basis of  $I$  (over the rational numbers) by using modular methods.

**Example:**

```

LIB "modwalk.lib";
ring R1 = 0, (x,y,z,t), dp;
ideal I = 3x3+x2+1, 11y5+y3+2, 5z4+z2+4;
I = std(I);
ring R2 = 0, (x,y,z,t), lp;
ideal I = fetch(R1, I);
int radius = 2;
ideal J = modfrWalk(I,radius);
J;
⇒ J[1]=z4+1/5z2+4/5
⇒ J[2]=y5+1/11y3+2/11
⇒ J[3]=x3+1/3x2+1/3
ring S1 = 0, (a,b,c,d), Dp;
ideal I = 5b2, ac2+9d3+3a2+5b, 2a2c+7abd+bcd+4a2, 2ad2+6b2d+7c3+8ad+4c;
I = std(I);
ring S2 = 0, (c,d,b,a), lp;
ideal I = fetch(S1,I);
// I is assumed to be a Dp-Groebner basis.
// We compute a lp-Groebner basis.
ideal J = modfrWalk(I,radius,"Dp");
J;
⇒ J[1]=a25+16a24+96a23+256a22+256a21+256/9a20+1024/3a19+2048a18+65536/9a17+\
32768/3a16+16384/81a15+131072/81a14+1048576/81a13+1048576/27a12+1048576/9\
a11
⇒ J[2]=ba11+1522867351997104938459/91668001658017308797687087104a24+4293036\
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↪ J[12]=c2+1/3cb2+5/3d+3a3
intvec w = 3,2,1,2;
ring S3 = 0, (c,d,b,a), (a(w),lp);
ideal I = fetch(S1,I);
// I is assumed to be a Dp-Groebner basis.
// We compute a (a(w),lp)-Groebner basis.
ideal J = modfrWalk(I,radius,"Dp",w);
J;
↪ J[1]=d2
↪ J[2]=c2+3a3+1/3cb2+5/3d
↪ J[3]=ca2+4ca+7/2b3+2b
↪ J[4]=cda-6/7ba3-2/21cb3-12/7a3-4/21cb2+1/7dba-10/21db-20/21d
↪ J[5]=db4a-192/49b2a3-64/147cb4+2db3a-768/49ba3-256/147cb3-4db4+60/49db2a-\
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↪ J[9]=cdb3+18da3+2cdb2+63cb4+441/2db3a-36dba2+126cb3+36cb2+126dba+72cb
↪ J[10]=ba4+5/72da3+245/9216b8+7/288b6a-2/3087b2a3+2a4-5/1008cdb2+108013/44\
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63db2a+9261/16b6-128/21a3-176/189cb2-113/36db3+32/63dba+1029/2b5-16/63cb-\
257/378db2+2205/4b4-556/189db+147b3-640/189d+147b2
↪ J[12]=b5a2-4/441db2a2+129724da3-49/16b8+6178b4a2-864464/1029b2a3+302660/2\
1cdb2+4197721018/9261cb4-128/7cb2a-28823/12db5+99991009/63db3a-114271928/\
441dba2+49/4b7+4b5a+86468/7b3a2-3457184/1029ba3-506216/63cdb+8393713780/9\
261cb3-256/7cba-129625/126db4+432008/3087db2a+133/4b6+8b4a+24712/7b2a2-34\
56512/1029a3-494080/63cd+2395735192/9261cb2-86405/63db3+2799920668/3087db\

```



```

a-2b5+16/7b3a+49408/7ba2+518128cb-9766570/9261db2+9b4+32/7b2a-17280544/92\
61db-36/7b3-17282560/9261d-44/7b2
⇒ J[13]=b7a-296432/7da3+109/16b8+2b6a-2016b4a2+4610560/16807b2a3-14521796/3\
087cdb2-22401156488/151263cb4+8000/343cb2a+1382867/1764db5-1600806884/308\
7db3a+4148384/49dba2+77/4b7-194/49b5a-4032b3a2+18438656/16807ba3+8103808/\
3087cdb-44793095440/151263cb3+13568/343cba+230189/686db4-6908840/151263db\
2a+533/28b6-312/49b4a-1152b2a2+18435072/16807a3+7901200/3087cd-4261620128\
/50421cb2-4864/343ca+1382435/3087db3-4980595264/16807dba+2078/49b5+176/34\
3b3a-2304ba2-8294976/49cb+52056614/151263db2+2435/49b4-1472/343b2a+921811\
84/151263db+1900/343b3+608/343ba+30725120/50421d+8044/343b2-2432/343b
⇒ J[14]=b9+1037536/343da3+4b8+144b4a2-329408/16807b2a3+148256/441cdb2+15987\
36592/151263cb4-24704/441db5+799730504/21609db3a-296064/49dba2+40/7b7+288\
b3a2-1317376/16807ba3-4047040/21609cdb+3196814624/151263cb3+2048/2401cba-\
24db4+164704/50421db2a+40/7b6-64/343b4a+576/7b2a2-3840/49a3-3950080/21609\
cd+886720/147cb2+4096/2401ca-98816/3087db3+1036320/49dba+272/49b5-128/343\
b3a+1152/7ba2+592000/49cb-3721504/151263db2+1152/343b4-256/2401b2a-658688\
0/151263db+1024/343b3-512/2401ba-6400/147d+1920/2401b2+2048/2401b
⇒ J[15]=dba3-7cb4-49/2db3a+4dba2-14cb3-4cb2-14dba-8cb
⇒ J[16]=db3a2+1323da3+63b4a2-60/7b2a3+147cdb2+194441/42cb4-49/2db5+64843/4d\
b3a-2646dba2+126b3a2-240/7ba3-82cdb+194401/21cb3-21/2db4+10/7db2a+36b2a2-\
240/7a3-80cd+55486/21cb2-14db3+64847/7dba+72ba2+5292cb-226/21db2-400/21db\
-400/21d
⇒ J[17]=a5+4a4-7/6cb3+5/9da2-7/18b5-2/3cb+20/9da-2/9b3
⇒ J[18]=da4-2/49b2a3-2/441cb4-1/63db3a-4/49ba3+10/63cdb-4/441cb3+32/7cba+1/\
147db2a-b4a+20/63cd+64/7ca-2b3a-10/441db2+4b4-4/7b2a-20/441db+8b3-8/7ba+1\
6/7b2+32/7b

```

See also: [Section D.2.6.1 \[modular\]](#), page 884.

## D.15.12 multigrading\_lib

Todos/Issues:

See <http://code.google.com/p/convex-singular/wiki/Multigrading>

**Library:** multigrading.lib

**Purpose:** Multigraded Rings

**Authors:** Benjamin Bechtold, [benjamin.bechtold@gmail.com](mailto:benjamin.bechtold@gmail.com)  
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 Simon Keicher, [keicher@mail.mathematik.uni-tuebingen.de](mailto:keicher@mail.mathematik.uni-tuebingen.de)  
 Oleksandr Motsak, U@D, where U={motsak}, D={mathematik.uni-kl.de}  
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**Overview:** This library allows one to virtually add multigradings to Singular: grade multi-variate polynomial rings with arbitrary (fin. gen. Abelian) groups. For more see <http://code.google.com/p/convex-singular/wiki/Multigrading> For theoretical references see:

E. Miller, B. Sturmfels: 'Combinatorial Commutative Algebra' and  
 M. Kreuzer, L. Robbiano: 'Computational Commutative Algebra'.

**Note:** 'multiDegBasis' relies on 4ti2 for computing Hilbert Bases. All groups are finitely generated Abelian

**Procedures:**

### D.15.12.1 setBaseMultigrading

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:** `setBaseMultigrading(M[, G]);` M is an integer matrix, G is a group (or lattice)

**Purpose:** attaches weights of variables and grading group to the basering.

**Note:** M encodes the weights of variables column-wise.

**Return:** nothing

**Example:**

```
LIB "multigrading.lib";
ring R = 0, (x, y, z), dp;
// Weights of variables
intmat M[3][3] =
1, 0, 0,
0, 1, 0,
0, 0, 1;
// GradingGroup:
intmat L[3][2] =
1, 1,
1, 3,
1, 5;
// attaches M & L to R (==basing):
setBaseMultigrading(M, L); // Grading: Z^3/L
// Weights are accessible via "getVariableWeights()":
getVariableWeights();
⇒ 1,0,0,
⇒ 0,1,0,
⇒ 0,0,1
// Test all possible usages:
(getVariableWeights() == M) && (getVariableWeights(R) == M) && (getVariableWeights(basing) == M)
⇒ 1
// Grading group is accessible via "getLattice()":
getLattice();
⇒ 1,1,
⇒ 1,3,
⇒ 1,5
// Test all possible usages:
(getLattice() == L) && (getLattice(R) == L) && (getLattice(basing) == L);
⇒ 1
// And its hermite NF via getLattice("hermite"):
getLattice("hermite");
⇒ 1,0,
⇒ 1,2,
⇒ 1,4
// Test all possible usages:
intmat H = hermiteNormalForm(L);
(getLattice("hermite") == H) && (getLattice(R, "hermite") == H) && (getLattice(basing, "hermite") == H)
⇒ 1
kill L, M;
// ----- isomorphic multigrading ----- //
// Weights of variables
intmat M[2][3] =
```

```

1, -2, 1,
1, 1, 0;
// Torsion:
intmat L[2][1] =
0,
2;
// attaches M & L to R (==basing):
setBaseMultigrading(M, L); // Grading: Z + (Z/2Z)
// Weights are accessible via "getVariableWeights()":
getVariableWeights() == M;
⇒ 1
// Torsion is accessible via "getLattice()":
getLattice() == L;
⇒ 1
kill L, M;
// ----- extreme case ----- //
// Weights of variables
intmat M[1][3] =
1, -1, 10;
// Torsion:
intmat L[1][1] =
0;
// attaches M & L to R (==basing):
setBaseMultigrading(M); // Grading: Z^3
// Weights are accessible via "getVariableWeights()":
getVariableWeights() == M;
⇒ 1
// Torsion is accessible via "getLattice()":
getLattice() == L;
⇒ 1

```

### D.15.12.2 getVariableWeights

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:** `getVariableWeights([R])`

**Purpose:** get associated multigrading matrix for the basering [or R]

**Return:** intmat, matrix of multidegrees of variables

**Example:**

```

LIB "multigrading.lib";
ring R = 0, (x, y, z), dp;
// Weights of variables
intmat M[3][3] =
1, 0, 0,
0, 1, 0,
0, 0, 1;
// Grading group:
intmat L[3][2] =
1, 1,
1, 3,
1, 5;
// attaches M & L to R (==basing):

```

```

setBaseMultigrading(M, L); // Grading:  $Z^3/L$ 
// Weights are accessible via "getVariableWeights()":
getVariableWeights() == M;
↪ 1
kill L, M;
// ----- isomorphic multigrading ----- //
// Weights of variables
intmat M[2][3] =
1, -2, 1,
1, 1, 0;
// Grading group:
intmat L[2][1] =
0,
2;
// attaches M & L to R (==basing):
setBaseMultigrading(M, L); // Grading:  $Z + (Z/2Z)$ 
// Weights are accessible via "getVariableWeights()":
getVariableWeights() == M;
↪ 1
kill L, M;
// ----- extreme case ----- //
// Weights of variables
intmat M[1][3] =
1, -1, 10;
// Grading group:
intmat L[1][1] =
0;
// attaches M & L to R (==basing):
setBaseMultigrading(M); // Grading:  $Z^3$ 
// Weights are accessible via "getVariableWeights()":
getVariableWeights() == M;
↪ 1

```

### D.15.12.3 getGradingGroup

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:**     `getGradingGroup([R])`

**Purpose:**    get associated grading group

**Return:**   group, the grading group

**Example:**

```

LIB "multigrading.lib";
ring R = 0, (x, y, z), dp;
// Weights of variables
intmat M[3][3] =
1, 0, 0,
0, 1, 0,
0, 0, 1;
// Torsion:
intmat L[3][2] =
1, 1,
1, 3,

```

```

1, 5;
// attaches M & L to R (==basing):
setBaseMultigrading(M, L); // Grading:  $Z^3/L$ 
def G = getGradingGroup();
printGroup( G );
↳ Generators:
↳      1      0      0
↳      0      1      0
↳      0      0      1
↳ Relations:
↳      1      1
↳      1      3
↳      1      5
G[1] == M; G[2] == L;
↳ 1
↳ 1
kill L, M, G;
// ----- isomorphic multigrading ----- //
// Weights of variables
intmat M[2][3] =
1, -2, 1,
1, 1, 0;
// Torsion:
intmat L[2][1] =
0,
2;
// attaches M & L to R (==basing):
setBaseMultigrading(M, L); // Grading:  $Z + (Z/2Z)$ 
def G = getGradingGroup();
printGroup( G );
↳ Generators:
↳      1      -2      1
↳      1      1      0
↳ Relations:
↳      0
↳      2
G[1] == M; G[2] == L;
↳ 1
↳ 1
kill L, M, G;
// ----- extreme case ----- //
// Weights of variables
intmat M[1][3] =
1, -1, 10;
// Torsion:
intmat L[1][1] =
0;
// attaches M & L to R (==basing):
setBaseMultigrading(M); // Grading:  $Z^3$ 
def G = getGradingGroup();
printGroup( G );
↳ Generators:
↳      1      -1      10

```

```

    ↪ Relations:
    ↪      0
    G[1] == M; G[2] == L;
    ↪ 1
    ↪ 1
    kill L, M, G;

```

#### D.15.12.4 getLattice

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:** `getLattice([R[,opt]])`

**Purpose:** get associated grading group matrix, i.e. generators (cols) of the grading group

**Return:** intmat, the grading group matrix, or  
its hermite normal form if an optional argument ("hermiteNormalForm") is given or  
smith normal form if an optional argument ("smith") is given

**Example:**

```

LIB "multigrading.lib";
ring R = 0, (x, y, z), dp;
// Weights of variables
intmat M[3][3] =
1, 0, 0,
0, 1, 0,
0, 0, 1;
// Torsion:
intmat L[3][2] =
1, 1,
1, 3,
1, 5;
// attaches M & L to R (==basing):
setBaseMultigrading(M, L); // Grading: Z^3/L
// Torsion is accessible via "getLattice()":
getLattice() == L;
↪ 1
// its hermite NF:
print(getLattice("hermite"));
↪      1      0
↪      1      2
↪      1      4
kill L, M;
// ----- isomorphic multigrading ----- //
// Weights of variables
intmat M[2][3] =
1, -2, 1,
1, 1, 0;
// Torsion:
intmat L[2][1] =
0,
2;
// attaches M & L to R (==basing):
setBaseMultigrading(M, L); // Grading: Z + (Z/2Z)
// Torsion is accessible via "getLattice()":

```

```

getLattice() == L;
↳ 1
// its hermite NF:
print(getLattice("hermite"));
↳ 0
↳ 2
kill L, M;
// ----- extreme case ----- //
// Weights of variables
intmat M[1][3] =
1, -1, 10;
// Torsion:
intmat L[1][1] =
0;
// attaches M & L to R (==basing):
setBaseMultigrading(M); // Grading: Z^3
// Torsion is accessible via "getLattice()":
getLattice() == L;
↳ 1
// its hermite NF:
print(getLattice("hermite"));
↳ 0

```

### D.15.12.5 createGroup

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:** `createGroup(S, L)`; S, L are integer matrices

**Purpose:** create the group of the form  $(S+L)/L$ , i.e.  
S specifies generators, L specifies relations.

**Return:** group

**Example:**

```

LIB "multigrading.lib";
intmat S[3][3] =
1, 0, 0,
0, 1, 0,
0, 0, 1;
intmat L[3][2] =
1, 1,
1, 3,
1, 5;
def G = createGroup(S, L); // (S+L)/L
printGroup(G);
↳ Generators:
↳ 1 0 0
↳ 0 1 0
↳ 0 0 1
↳ Relations:
↳ 1 1
↳ 1 3
↳ 1 5
kill S, L, G;

```

```

////////////////////////////////////
intmat S[2][3] =
1, -2, 1,
1, 1, 0;
intmat L[2][1] =
0,
2;
def G = createGroup(S, L); // (S+L)/L
printGroup(G);
⇒ Generators:
⇒      1      -2      1
⇒      1       1      0
⇒ Relations:
⇒      0
⇒      2
kill S, L, G;
// ----- extreme case ----- //
intmat S[1][3] =
1, -1, 10;
// Torsion:
intmat L[1][1] =
0;
def G = createGroup(S, L); // (S+L)/L
printGroup(G);
⇒ Generators:
⇒      1      -1      10
⇒ Relations:
⇒      0

```

### D.15.12.6 createQuotientGroup

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:** `createGroup(L)`;  $L$  is an integer matrix

**Purpose:** create the group of the form  $(I+L)/L$ ,  
where  $I$  is the square identity matrix of size  $\text{nrows}(L)$  x  $\text{nrows}(L)$

**Note:**  $L$  specifies relations between free generators of  $Z^{\text{nrows}(L)}$

**Return:** group

**Example:**

```

LIB "multigrading.lib";
intmat I[3][3] =
1, 0, 0,
0, 1, 0,
0, 0, 1;
intmat L[3][2] =
1, 1,
1, 3,
1, 5;
// The group  $Z^3 / L$  can be constructed as follows:
// shortcut:
def G = createQuotientGroup(L);

```



```

printGroup(G);
↳ Generators:
↳      1      0      0
↳      0      1      0
↳      0      0      1
↳ Relations:
↳      1      1
↳      1      3
↳      1      5
// the general way:
def GG = createGroup(I, L); // (I+L)/L
printGroup(GG);
↳ Generators:
↳      1      0      0
↳      0      1      0
↳      0      0      1
↳ Relations:
↳      1      1
↳      1      3
↳      1      5

```

### D.15.12.7 createTorsionFreeGroup

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:** `createTorsionFreeGroup(S)`; S is an integer matrix

**Purpose:** create the free subgroup generated by S within the free Abelian group of rank `nrows(S)`

**Return:** group

**Example:**

```

LIB "multigrading.lib";
// ----- extreme case ----- //
intmat S[1][3] =
1, -1, 10;
// Torsion:
intmat L[1][1] =
0;
// The free subgroup generated by elements of S within  $\mathbb{Z}^1$ 
// can be constructed as follows:
// shortcut:
def G = createTorsionFreeGroup(S);
printGroup(G);
↳ Generators:
↳      1      -1      10
↳ Relations:
↳      0
// the general way:
def GG = createGroup(S, L); // (S+L)/L
printGroup(GG);
↳ Generators:
↳      1      -1      10
↳ Relations:

```

$\mapsto$  0

### D.15.12.8 printGroup

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:** `printGroup(G)`; G is a group

**Purpose:** prints the group G

**Return:** nothing

**Example:**

```
LIB "multigrading.lib";
intmat S[3][3] =
1, 0, 0,
0, 1, 0,
0, 0, 1;
intmat L[3][2] =
1, 1,
1, 3,
1, 5;
def G = createGroup(S, L); // (S+L)/L
printGroup(G);
 $\mapsto$  Generators:
 $\mapsto$       1      0      0
 $\mapsto$       0      1      0
 $\mapsto$       0      0      1
 $\mapsto$  Relations:
 $\mapsto$       1      1
 $\mapsto$       1      3
 $\mapsto$       1      5
```

### D.15.12.9 isGroup

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:** `isGroup(G)`; G a list

**Purpose:** checks whether G is a valid group

**Note:** G should be created by `createGroup`  
(or `createQuotientGroup`, `createTorsionFreeGroup`)

**Return:** int, 1 if G is a valid group and 0 otherwise

**Example:**

```
LIB "multigrading.lib";
intmat S[3][3] =
1, 0, 0,
0, 1, 0,
0, 0, 1;
intmat L[3][2] =
1, 1,
1, 3,
1, 5;
def G = createGroup(S, L); // (S+L)/L
```

```

isGroup(G);
↳ 1
printGroup(G);
↳ Generators:
↳      1      0      0
↳      0      1      0
↳      0      0      1
↳ Relations:
↳      1      1
↳      1      3
↳      1      5

```

### D.15.12.10 isGroupHomomorphism

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\], page 2459](#)).

**Usage:** `isGoupHomomorphism(L1,L2,A)`; L1 and L2 are groups, A is an integer matrix

**Purpose:** checks whether A defines a group homomorphism  $\phi: L1 \rightarrow L2$

**Return:** int, 1 if A defines the homomorphism and 0 otherwise

**Example:**

```

LIB "multigrading.lib";
intmat L1[4][1]=
0,
0,
0,
2;
intmat L2[3][2]=
0, 0,
2, 0,
0, 3;
intmat A[3][4] =
1, 2, 3, 0,
7, 0, 0, 0,
1, 2, 0, 3;
print( A );
↳      1      2      3      0
↳      7      0      0      0
↳      1      2      0      3
isGroupHomomorphism(L1, L2, A);
↳ 1
intmat B[3][4] =
1, 2, 3, 0,
7, 0, 0, 0,
1, 2, 0, 2;
print( B );
↳      1      2      3      0
↳      7      0      0      0
↳      1      2      0      2
isGroupHomomorphism(L1, L2, B); // Not a homomorphism!
↳ 0

```

### D.15.12.11 isGradedRingHomomorphism

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:** `isGradedRingHomomorphism(R, f, A)`; ring R, ideal f, group homomorphism A

**Purpose:** test a multigraded group ring homomorphism defined by a ring map from R to the current ring, given by generators images f and a group homomorphism A between grading groups

**Return:** int, 1 for TRUE, 0 otherwise

**Example:**

```
LIB "multigrading.lib";
ring r = 0, (x, y, z), dp;
intmat S1[3][3] =
1, 0, 0,
0, 1, 0,
0, 0, 1;
intmat L1[3][1] =
0,
0,
0;
def G1 = createGroup(S1, L1); // (S1 + L1)/L1
printGroup(G1);
⇨ Generators:
⇨      1      0      0
⇨      0      1      0
⇨      0      0      1
⇨ Relations:
⇨      0
⇨      0
⇨      0
setBaseMultigrading(S1, L1); // to change...
ring R = 0, (a, b, c), dp;
intmat S2[2][3] =
1, 0,
0, 1;
intmat L2[2][1] =
0,
2;
def G2 = createGroup(S2, L2);
printGroup(G2);
⇨ Generators:
⇨      1      0      0
⇨      1      0      0
⇨ Relations:
⇨      0
⇨      2
setBaseMultigrading(S2, L2); // to change...
map F = r, a, b, c;
intmat A[nrows(L2)][nrows(L1)] =
1, 0, 0,
3, 2, -6;
// graded ring homomorphism is given by (compatible):
```

```

print(F);
↳ F[1]=a
↳ F[2]=b
↳ F[3]=c
print(A);
↳      1      0      0
↳      3      2     -6
isGradedRingHomomorphism(r, ideal(F), A);
↳ 1
def h = createGradedRingHomomorphism(r, ideal(F), A);
print(h);
↳ [1]:
↳      // coefficients: QQ
↳      // number of vars : 3
↳      //      block   1 : ordering dp
↳      //      : names   x y z
↳      //      block   2 : ordering C
↳ [2]:
↳      _[1]=a
↳      _[2]=b
↳      _[3]=c
↳ [3]:
↳      1,0,0,
↳      3,2,-6
// not a homo..
intmat B[nrows(L2)][nrows(L1)] =
1, 1, 1,
0, 0, 0;
print(B);
↳      1      1      1
↳      0      0      0
isGradedRingHomomorphism(r, ideal(F), B); // FALSE: there is no such homomorphism!
↳ 0
// Therefore: the following command should return an error
// createGradedRingHomomorphism(r, ideal(F), B);

```

#### D.15.12.12 createGradedRingHomomorphism

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:** `createGradedRingHomomorphism(R, f, A);` ring R, ideal f, group homomorphism A

**Purpose:** create a multigraded group ring homomorphism defined by a ring map from R to the current ring, given by generators images f and a group homomorphism A between grading groups

**Return:** graded ring homomorphism

**Example:**

```

LIB "multigrading.lib";
ring r = 0, (x, y, z), dp;
intmat S1[3][3] =
1, 0, 0,
0, 1, 0,
0, 0, 1;

```

```

intmat L1[3][1] =
0,
0,
0;
def G1 = createGroup(S1, L1); // (S1 + L1)/L1
printGroup(G1);
↳ Generators:
↳      1      0      0
↳      0      1      0
↳      0      0      1
↳ Relations:
↳      0
↳      0
↳      0
setBaseMultigrading(S1, L1); // to change...
ring R = 0, (a, b, c), dp;
intmat S2[2][3] =
1, 0,
0, 1;
intmat L2[2][1] =
0,
2;
def G2 = createGroup(S2, L2);
printGroup(G2);
↳ Generators:
↳      1      0      0
↳      1      0      0
↳ Relations:
↳      0
↳      2
setBaseMultigrading(S2, L2); // to change...
map F = r, a, b, c;
intmat A[nrows(L2)][nrows(L1)] =
1, 0, 0,
3, 2, -6;
// graded ring homomorphism is given by (compatible):
print(F);
↳ F[1]=a
↳ F[2]=b
↳ F[3]=c
print(A);
↳      1      0      0
↳      3      2     -6
isGradedRingHomomorphism(r, ideal(F), A);
↳ 1
def h = createGradedRingHomomorphism(r, ideal(F), A);
print(h);
↳ [1]:
↳      // coefficients: QQ
↳      // number of vars : 3
↳      //      block 1 : ordering dp
↳      //      : names      x y z
↳      //      block 2 : ordering C

```

```

↳ [2]:
↳   _[1]=a
↳   _[2]=b
↳   _[3]=c
↳ [3]:
↳   1,0,0,
↳   3,2,-6

```

### D.15.12.13 setModuleGrading

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:** `setModuleGrading(m, G)`, `m` module/vector, `G` intmat

**Purpose:** attaches the multiweights of free module generators to 'm'

**Warning:** The method does not verify whether the multigrading makes the module/vector homogeneous. One can do that using `isHomogeneous(m)`.

**Example:**

```

LIB "multigrading.lib";
ring R = 0, (x,y), dp;
intmat M[2][2]=
1, 1,
0, 2;
intmat T[2][5]=
1, 2, 3, 4, 0,
0, 10, 20, 30, 1;
setBaseMultigrading(M, T);
ideal I = x, y, xy^5;
intmat V = multiDeg(I);
// V == M; modulo T
print(V);
↳      1      1      10
↳      0      2      10
module S = syz(I);
S = setModuleGrading(S, V);
getModuleGrading(S) == V;
↳ 1
print(S);
↳ -y,x4y5,
↳ x, 0,
↳ 0, -1
vector v = getGradedGenerator(S, 1);
getModuleGrading(v) == V;
↳ 1
print( multiDeg(v) );
↳ 2,
↳ 2
isHomogeneous(S);
↳ 1
print( multiDeg(S) );
↳      2      10
↳      2      10

```

### D.15.12.14 getModuleGrading

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:** `getModuleGrading(m)`, 'm' module/vector

**Return:** integer matrix of the multiweights of free module generators attached to 'm'

**Example:**

```
LIB "multigrading.lib";
ring R = 0, (x,y), dp;
intmat M[2][2]=
1, 1,
0, 2;
intmat T[2][5]=
1, 2, 3, 4, 0,
0, 10, 20, 30, 1;
setBaseMultigrading(M, T);
ideal I = x, y, xy^5;
isHomogeneous(I);
↳ 1
intmat V = multiDeg(I); print(V);
↳      1      1      10
↳      0      2      10
module S = syz(I); print(S);
↳ -y,x4y5,
↳ x, 0,
↳ 0, -1
S = setModuleGrading(S, V);
getModuleGrading(S) == V;
↳ 1
vector v = getGradedGenerator(S, 1);
getModuleGrading(v) == V;
↳ 1
isHomogeneous(v);
↳ 1
print( multiDeg(v) );
↳ 2,
↳ 2
isHomogeneous(S);
↳ 1
print( multiDeg(S) );
↳      2      10
↳      2      10
```

### D.15.12.15 isSublattice

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:** `isSublattice(L, S)`; L, S are of type `intmat`

**Purpose:** checks whether the lattice created by L is a sublattice of the lattice created by S.  
The procedure checks whether each generator of L is contained in S.



**Return:** integer, 0 if false, 1 if true

**Example:**

```
LIB "multigrading.lib";
//ring R = 0,(x,y),dp;
intmat S2[3][3]=
0, 2, 3,
0, 1, 1,
3, 0, 2;
intmat S1[3][2]=
0, 6,
0, 2,
3, 4;
isSublattice(S1,S2); // Yes!
⇒ 1
intmat S3[3][1] =
0,
0,
1;
not(isSublattice(S3,S2)); // Yes!
⇒ 1
```

#### D.15.12.16 imageLattice

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:** `imageLattice(Q,L)`; Q and L are of type `intmat`

**Purpose:** compute an integral basis for the image of the lattice L under the homomorphism of lattices Q.

**Return:** `intmat`

**Example:**

```
LIB "multigrading.lib";
intmat Q[2][3] =
1,2,3,
3,2,1;
intmat L[3][2] =
1,4,
2,5,
3,6;
// should be a 2x2 matrix with columns
// [2,-14], [0,36]
imageLattice(Q,L);
⇒ 2,0,
⇒ 22,36
```

#### D.15.12.17 intRank

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:** `intRank(A)`; `intmat` A

**Purpose:** compute the rank of the integral matrix A by computing a hermite normalform.

**Returns:** int

**Example:**

```
LIB "multigrading.lib";
intmat A[3][4] =
1,0,1,0,
1,2,0,0,
0,0,0,0;
int r = intRank(A);
print(A);
↪      1      0      1      0
↪      1      2      0      0
↪      0      0      0      0
print(r); // Should be 2
↪ 2
// another example
intmat B[2][2] =
1,2,
1,2;
int d = intRank(B);
print(B);
↪      1      2
↪      1      2
print(d); // Should be 1
↪ 1
kill A, B, r, d;
```

#### D.15.12.18 kernelLattice

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:** kernelLattice(P); intmat P

**Purpose:** compute a integral basis for the kernel of the homomorphism of lattices defined by the intmat P.

**Returns:** intmat

**Example:**

```
LIB "multigrading.lib";
intmat LL[3][4] =
1,4,7,10,
2,5,8,11,
3,6,9,12;
// should be a 4x2 matrix whose columns
// generate the same lattice as [-1,2,-1,0],[2,-3,0,1]
intmat B = kernelLattice(LL);
print(B);
↪      -1      2
↪       2     -3
↪      -1      0
↪       0      1
// another example
intmat C[2][4] =
1,0,2,0,
```

```

0,1,2,0;
// should result in a matrix whose
// columns create the same lattice as
// [-2,-2,1,0], [0,0,0,1]
intmat D = kernelLattice(C);
print(D);
↪      -2      0
↪      -2      0
↪       1      0
↪       0      1
kill B;

```

### D.15.12.19 latticeBasis

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:**      `latticeBasis(B); intmat B`

**Purpose:**     compute an integral basis for the lattice defined by the columns of B.

**Returns:**    `intmat`

**Example:**

```

LIB "multigrading.lib";
intmat L[3][3] =
1,4,8,
2,5,10,
3,6,12;
intmat B = latticeBasis(L);
print(B); // should result in a matrix whose columns generate the same lattice as [
↪       1      0
↪       2      3
↪       3      6
// another example
intmat C[2][4] =
1,1,2,0,
2,3,4,0;
// should result in a matrix whose
// columns create the same lattice as
// [0,1],[1,0]
intmat D = latticeBasis(C);
print(D);
↪       1      0
↪       0      1
kill B,L;

```

### D.15.12.20 preimageLattice

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:**      `preimageLattice(P, B); intmat P, intmat B`

**Purpose:**     compute an integral basis for the preimage of B under the homomorphism of lattices defined by the `intmat` P.

**Returns:**    `intmat`

**Example:**

```

LIB "multigrading.lib";
intmat P[2][3] =
2,6,10,
4,8,12;
intmat B[2][1] =
1,
0;
// should be a (3x2)-matrix with columns e.g. [1,1,-1] and [0,3,-2] (the generated lattice)
print(preimageLattice(P,B));
↪      1      0
↪      1      3
↪     -1     -2
// another example
intmat W[3][3] =
1,0,0,
0,1,1,
0,2,0;
intmat Z[3][2] =
1,0,
0,1,
0,0;
// should be a (3x2)-matrix with columns e.g. [1,0,0] and [0,0,-1] (the generated lattice)
print(preimageLattice(W,Z));
↪      1      0
↪      0      0
↪      0      1

```

**D.15.12.21 projectLattice**

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:** `projectLattice(B); intmat B`

**Purpose:** A set of vectors in  $\mathbb{Z}^m$  is given as the columns of B. Then this function provides a linear map  $\mathbb{Z}^m \rightarrow \mathbb{Z}^n$  having the primitive span of B its kernel.

**Returns:** `intmat`

**Example:**

```

LIB "multigrading.lib";
intmat B[4][2] =
1,5,
2,6,
3,7,
4,8;
// should result in a (2x4)-matrix such that the corresponding lattice is created by
// [-1, 2], [-2, 3], [-1, 0] and [0, 1]
print(projectLattice(B));
↪     -1      2     -1      0
↪      2     -3      0      1
// another example
intmat BB[4][2] =
1,0,
0,1,

```

```

0,0,
0,0;
// should result in a (2x4)-matrix such that the corresponding lattice is created by
// [0,0],[0,0],[1,0],[0,1]
print(projectLattice(BB));
↪      0      0      1      0
↪      0      0      0      1
// another example
intmat BBB[3][4] =
1,0,1,2,
1,1,0,0,
3,0,0,3;
// should result in the (1x3)-matrix that consists of just zeros
print(projectLattice(BBB));
↪      0      0      0

```

### D.15.12.22 intersectLattices

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:**      `intersectLattices(A, B);` intmat A, intmat B

**Purpose:**     compute an integral basis for the intersection of the lattices A and B.

**Returns:**    intmat

**Example:**

```

LIB "multigrading.lib";
intmat A[3][2] =
1,4,
2,5,
3,6;
intmat B[3][2] =
6,9,
7,10,
8,11;
// should result in a (3x2)-matrix with columns
// e.g. [0, 3, 6], [-3, 0, 3] (the lattice should be the same)
print(intersectLattices(A,B));
↪      3      0
↪      0      3
↪     -3      6
// another example
intmat C[2][3] =
1,0,0,
3,2,5;
intmat D[2][3] =
4,5,0,
0,5,0;
// should result in a (3x2)-matrix whose columns generate the
// same lattice as [1,5], [0, 20]
print(intersectLattices(C,D));
↪      1      0
↪      5     20

```

### D.15.12.23 isIntegralSurjective

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\], page 2459](#)).

**Purpose:** test whether the given linear map  $P$  of lattices is surjective.

**Returns:** int, where 0 is false and 1 is true.

**Example:**

```
LIB "multigrading.lib";
intmat A[2][3] =
1,3,5,
2,4,6;
// should be 0
int b = isIntegralSurjective(A);
print(b);
⇒ 0
// another example
intmat B[2][3] =
1,1,5,
2,3,6;
// should be 1
int c = isIntegralSurjective(B);
print(c);
⇒ 1
kill A, b, B, c;
```

### D.15.12.24 isPrimitiveSublattice

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\], page 2459](#)).

**Purpose:** check whether the given set of integral vectors in  $\mathbb{Z}^m$ , i.e. the columns of  $A$ , generate a primitive sublattice in  $\mathbb{Z}^m$  (a direct summand of  $\mathbb{Z}^m$ ).

**Returns:** int, where 0 is false and 1 is true.

**Example:**

```
LIB "multigrading.lib";
intmat A[3][2] =
1,4,
2,5,
3,6;
// should be 0
int b = isPrimitiveSublattice(A);
print(b);
⇒ 0
// another example
intmat B[2][2] =
1,0,
0,1;
// should be 1
int c = isPrimitiveSublattice(B);
print(c);
⇒ 1
kill A, b, B, c;
```

**D.15.12.25 intInverse**

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\], page 2459](#)).

**Purpose:** compute the integral inverse of the intmat A.  
If  $\det(A)$  is neither 1 nor -1 an error is returned.

**Returns:** intmat

**Example:**

```
LIB "multigrading.lib";
intmat A[3][3] =
1,1,3,
3,2,0,
0,0,1;
intmat B = intInverse(A);
// should be the unit matrix
print(A * B);
↪      1      0      0
↪      0      1      0
↪      0      0      1
// another example
intmat C[2][2] =
2,1,
3,2;
intmat D = intInverse(C);
// should be the unit matrix
print(C * D);
↪      1      0
↪      0      1
kill A, B, C, D;
```

**D.15.12.26 integralSection**

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\], page 2459](#)).

**Purpose:** for a given linear surjective map P of lattices  
this procedure returns an integral section of P.

**Returns:** intmat

**Example:**

```
LIB "multigrading.lib";
intmat P[2][4] =
1,3,4,6,
2,4,5,7;
// should be a matrix with two columns
// for example: [-2, 1, 0, 0], [3, -3, 0, 1]
intmat U = integralSection(P);
print(U);
↪      -1      1
↪      -2      1
↪       2     -1
↪       0      0
print(P * U);
↪       1      0
```

```

    ↪      0      1
    kill U;

```

### D.15.12.27 primitiveSpan

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\], page 2459](#)).

**Purpose:** compute an integral basis for the minimal primitive sublattice that contains the given vectors, i.e. the columns of  $V$ .

**Returns:** int, where 0 is false and 1 is true.

**Example:**

```

LIB "multigrading.lib";
intmat V[3][2] =
1,4,
2,5,
3,6;
// should return a (3x2)-matrix whose columns
// generate the same lattice as [1, 2, 3] and [0, 1, 2]
intmat R = primitiveSpan(V);
print(R);
↪      1      0
↪      2      1
↪      3      2
// another example
intmat W[2][2] =
1,0,
0,1;
// should return a (2x2)-matrix whose columns
// generate the same lattice as [1, 0] and [0, 1]
intmat S = primitiveSpan(W);
print(S);
↪      1      0
↪      0      1
kill V, R, S, W;

```

### D.15.12.28 factorgroup

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\], page 2459](#)).

**Usage:** `factorgroup(G,H);` list  $G$ , list  $H$

**Purpose:** returns a representation of the factor group  $G \bmod H$  using the first isomorphism thm

**Returns:** list

**Example:**

```

LIB "multigrading.lib";
intmat S1[2][2] =
1,0,
0,1;
intmat L1[2][1] =
2,
0;
intmat S2[2][1] =

```



```

1,
0;
intmat L2[2][1] =
2,
0;
list G = createGroup(S1,L1);
list H = createGroup(S2,L2);
list N = factorgroup(G,H);
print(N);
⇒ [1]:
⇒ 1,0,
⇒ 0,1
⇒ [2]:
⇒ 2,1,
⇒ 0,0
kill G,H,N,S1,L1,S2,L2;

```

### D.15.12.29 productgroup

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:**     `productgroup(G,H);` list G, list H

**Purpose:**     Returns a representation of the group  $G \times H$

**Returns:**    list

**Example:**

```

LIB "multigrading.lib";
intmat S1[2][2] =
1,0,
0,1;
intmat L1[2][1] =
2,
0;
intmat S2[2][2] =
1,0,
0,2;
intmat L2[2][1] =
0,
3;
list G = createGroup(S1,L1);
list H = createGroup(S2,L2);
list N = productgroup(G,H);
print(N);
⇒ [1]:
⇒ 1,0,0,0,
⇒ 0,1,0,0,
⇒ 0,0,1,0,
⇒ 0,0,0,2
⇒ [2]:
⇒ 2,0,
⇒ 0,0,
⇒ 0,0,
⇒ 0,3

```

```
kill G,H,N,S1,L1,S2,L2;
```

### D.15.12.30 multiDeg

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:** `multiDeg(A);` polynomial/vector/ideal/module A

**Purpose:** compute multidegree

**Example:**

```
LIB "multigrading.lib";
ring r = 0,(x, y), dp;
intmat A[2][2] = 1, 0, 0, 1;
print(A);
  ↪      1      0
  ↪      0      1
intmat Ta[2][1] = 0, 3;
print(Ta);
  ↪      0
  ↪      3
// attrib(A, "gradingGroup", Ta); // to think about
// "poly:";
setBaseMultigrading(A);
multiDeg( x*x, A );
  ↪ 2,0
multiDeg( y*y*y, A );
  ↪ 0,3
setBaseMultigrading(A, Ta);
multiDeg( x*x*y );
  ↪ 2,1
multiDeg( y*y*y*x );
  ↪ 1,3
multiDeg( x*y + x + 1 );
  ↪ 1,1
multiDegPartition(x*y + x + 1);
  ↪ _[1]=xy
  ↪ _[2]=x
  ↪ _[3]=1
print ( multiDeg(0) );
  ↪ 0,
  ↪ 0
poly zero = 0;
print ( multiDeg(zero) );
  ↪ 0,
  ↪ 0
// "ideal:";
ideal I = y*x*x, x*y*y*y;
print( multiDeg(I) );
  ↪      2      1
  ↪      1      3
print ( multiDeg(ideal(0)) );
  ↪ 0,
  ↪ 0
```

```

print ( multiDeg(ideal(0,0,0)) );
↳      0      0      0
↳      0      0      0
// "vectors:";
intmat B[2][2] = 0, 1, 1, 0;
print(B);
↳      0      1
↳      1      0
multiDeg( setModuleGrading(y*y*y*gen(2), B ));
↳ 1,3
multiDeg( setModuleGrading(x*x*gen(1), B ));
↳ 2,1
vector V = x*gen(1) + y*gen(2);
V = setModuleGrading(V, B);
multiDeg( V );
↳ 1,1
vector v1 = setModuleGrading([0, 0, 0], B);
print( multiDeg( v1 ) );
↳ 0,
↳ 0
vector v2 = setModuleGrading([0], B);
print( multiDeg( v2 ) );
↳ 0,
↳ 0
// "module:";
module D = x*gen(1), y*gen(2);
D;
↳ D[1]=x*gen(1)
↳ D[2]=y*gen(2)
D = setModuleGrading(D, B);
print( multiDeg( D ) );
↳      1      1
↳      1      1
module DD = [0, 0],[0, 0, 0];
DD = setModuleGrading(DD, B);
print( multiDeg( DD ) );
↳      0      0
↳      0      0
module DDD = [0, 0];
DDD = setModuleGrading(DDD, B);
print( multiDeg( DDD ) );
↳ 0,
↳ 0

```

### D.15.12.31 multiDegBasis

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:**      `multiDegBasis(d), multidegree: intvec d`

**Assume:**    current ring is multigraded, monomial ordering is global

**Purpose:**     compute all monomials of multidegree `d`

**Example:**

```

LIB "multigrading.lib";
ring R = 0, (x, y), dp;
intmat g1[2][2]=1,0,0,1;
intmat l[2][1]=2,0;
intmat g2[2][2]=1,1,1,1;
intvec v1=4,0;
intvec v2=4,4;
intmat g3[1][2]=1,1;
setBaseMultigrading(g3);
intvec v3=4:1;
v3;
↳ 4
multiDegBasis(v3);
↳ _[1]=x4
↳ _[2]=y4
↳ _[3]=xy3
↳ _[4]=x2y2
↳ _[5]=x3y
setBaseMultigrading(g1,l);
multiDegBasis(v1);
↳ _[1]=1
setBaseMultigrading(g2);
multiDegBasis(v2);
↳ _[1]=y4
↳ _[2]=x4
↳ _[3]=x3y
↳ _[4]=x2y2
↳ _[5]=xy3
intmat M[2][2] = 1, -1, -1, 1;
intvec d = -2, 2;
setBaseMultigrading(M);
multiDegBasis(d);
↳ _[1]=y2
attrib(_, "ZeroPart");
↳ _[1]=xy
kill R, M, d;
ring R = 0, (x, y, z), dp;
intmat M[2][3] = 1, -2, 1,      1, 1, 0;
intmat L[2][1] = 0, 2;
intvec d = 4, 1;
setBaseMultigrading(M, L);
multiDegBasis(d);
↳ _[1]=x3z
↳ _[2]=x6y
↳ _[3]=yz6
↳ _[4]=xz3
attrib(_, "ZeroPart");
↳ _[1]=xyz
↳ _[2]=x4y2
↳ _[3]=y2z4
kill R, M, d;
ring R = 0, (x, y, z), dp;
qring Q = std(ideal( y^6+ x*y^3*z-x^2*z^2 ));

```

```

intmat M[2][3] = 1, 1, 2,      2, 1, 1;
// intmat T[2][1] = 0, 2;
setBaseMultigrading(M); // BUG????
intvec d = 6, 6;
multiDegBasis(d);
↪ _[1]=x2z2
↪ _[2]=xy3z
attrib(_, "ZeroPart");
↪ _[1]=0
kill R, Q, M, d;
ring R = 0, (x, y, z), dp;
qring Q = std(ideal( x*z^3 - y *z^6, x*y*z - x^4*y^2 ));
intmat M[2][3] = 1, -2, 1,      1, 1, 0;
intmat T[2][1] = 0, 2;
intvec d = 4, 1;
setBaseMultigrading(M, T); // BUG????
multiDegBasis(d);
↪ _[1]=x3z
↪ _[2]=x6y
↪ _[3]=xz3
attrib(_, "ZeroPart");
↪ _[1]=xyz
↪ _[2]=y2z4

```

### D.15.12.32 multiDegPartition

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:**      `multiDegPartition(def p)`, `p` polynomial/vector

**Returns:**    an ideal/module consisting of multigraded-homogeneous parts of `p`

**Example:**

```

LIB "multigrading.lib";
ring r = 0, (x,y,z), dp;
intmat g[2][3]=
1,0,1,
0,1,1;
intmat t[2][1]=
-2,
1;
setBaseMultigrading(g,t);
poly f = x10yz+x8y2z-x4z2+y5+x2y2-z2+x17z3-y6;
multiDegPartition(f);
↪ _[1]=x17z3
↪ _[2]=x10yz+x8y2z
↪ _[3]=-y6
↪ _[4]=-x4z2+y5
↪ _[5]=x2y2-z2
vector v = xy*gen(1)-x3y2*gen(2)+x4y*gen(3);
intmat B[2][3]=1,-1,-2,0,0,1;
v = setModuleGrading(v,B);
getModuleGrading(v);
↪ 1,-1,-2,

```

```

    ↦ 0,0,1
    multiDegPartition(v, B);
    ↦ _[1]=x4y*gen(3)-x3y2*gen(2)
    ↦ _[2]=xy*gen(1)

```

### D.15.12.33 isTorsionFree

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:**      `isTorsionFree()`

**Purpose:**     Determines whether the multigrading attached to the current ring is free.

**Return:**     boolean, the result of the test

**Example:**

```

LIB "multigrading.lib";
ring R = 0,(x,y),dp;
intmat M[2][2]=
1,0,
0,1;
intmat T[2][5]=
1, 2, 3, 4, 0,
0,10,20,30, 1;
setBaseMultigrading(M,T);
// Is the resulting group free?
isTorsionFree();
↦ 1
kill R, M, T;
////////////////////////////////////
ring R=0,(x,y,z),dp;
intmat A[3][3] =
1,0,0,
0,1,0,
0,0,1;
intmat B[3][4]=
3,3,3,3,
2,1,3,0,
1,2,0,3;
setBaseMultigrading(A,B);
// Is the resulting group free?
isTorsionFree();
↦ 0
kill R, A, B;

```

### D.15.12.34 isPositive

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:**      `isPositive()`

**Purpose:**     Computes whether the multigrading of the ring is positive. For computation theorem 8.6 of the Miller/Sturmfels book is used.

**Returns:**     true if the multigrading is positive

**Example:**

```

LIB "multigrading.lib";
printlevel = 3;
ring r = 0,(x,y),dp;
intmat A[1][2]=-1,1;
setBaseMultigrading(A);
isPositive();
↳ 0
intmat B[1][2]=1,1;
setBaseMultigrading(B);
isPositive(B);
↳ 1

```

### D.15.12.35 isZeroElement

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:** `isZeroElement(d, [T]);` intvec `d`, group `T`

**Purpose:** For a integer vector 'd' representing the multidegree of some polynomial or vector this method computes if the multidegree is contained in the grading group `group` (either set globally or given as an optional argument), i.e. if it is zero in the multigrading.

**Example:**

```

LIB "multigrading.lib";
ring r = 0,(x,y,z),dp;
intmat g[2][3]=
1,0,1,
0,1,1;
intmat t[2][1]=
-2,
1;
intmat tt[2][1]=
1,
-1;
setBaseMultigrading(g,t);
poly a = x10yz;
poly b = x8y2z;
poly c = x4z2;
poly d = y5;
poly e = x2y2;
poly f = z2;
intvec v1 = multiDeg(a) - multiDeg(b);
v1;
↳ 2,-1
isZeroElement(v1);
↳ 1
isZeroElement(v1, tt);
↳ 0
intvec v2 = multiDeg(a) - multiDeg(c);
v2;
↳ 5,0
isZeroElement(v2);
↳ 0
isZeroElement(v2, tt);

```

```

↳ 0
intvec v3 = multiDeg(e) - multiDeg(f);
v3;
↳ 0,0
isZeroElement(v3);
↳ 1
isZeroElement(v3, tt);
↳ 1
intvec v4 = multiDeg(c) - multiDeg(d);
v4;
↳ 6,-3
isZeroElement(v4);
↳ 1
isZeroElement(v4, tt);
↳ 0

```

### D.15.12.36 areZeroElements

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:** `areZeroElements(D, [T]);` intmat D, group T

**Purpose:** For a integer matrix D, considered column-wise as a set of integer vecors representing the multidegree of some polynomial or vector this method checks whether all these multidegrees are contained in the grading group group (either set globally or given as an optional argument), i.e. if they all are zero in the multigrading.

**Example:**

```

LIB "multigrading.lib";
ring r = 0,(x,y,z),dp;
intmat S[2][3]=
1,0,1,
0,1,1;
intmat L[2][1]=
2,
2;
setBaseMultigrading(S,L);
poly a = 1;
poly b = xyz;
ideal I = a, b;
print(multiDeg(I));
↳      0      2
↳      0      2
intmat m[5][2]=multiDeg(a),multiDeg(b); m=transpose(m);
print(multiDeg(a));
↳ 0,
↳ 0
print(multiDeg(b));
↳ 2,
↳ 2
print(m);
↳      0      2      0      0      0
↳      0      2      0      0      0

```



```

areZeroElements(m);
↳ 1
intmat LL[2][1]=
1,
-1;
areZeroElements(m,LL);
↳ 0

```

### D.15.12.37 isHomogeneous

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:** `isHomogeneous(a[, f]);` a polynomial/vector/ideal/module

**Return:** boolean, TRUE if a is (multi)homogeneous, and FALSE otherwise

**Example:**

```

LIB "multigrading.lib";
ring r = 0,(x,y,z),dp;
//Grading and Torsion matrices:
intmat M[3][3] =
1,0,0,
0,1,0,
0,0,1;
intmat T[3][1] =
1,2,3;
setBaseMultigrading(M,T);
attrib(r);
↳ attr:cf_class, type int
↳ attr:global, type int
↳ attr:maxExp, type int
↳ attr:ring_cf, type int
↳ attr:isLetterplaceRing, type int
↳ attr:gradingGroup, type list
↳ attr:mgrad, type intmat
poly f = x-yz;
multiDegPartition(f);
↳ _[1]=-yz
↳ _[2]=x
print(multiDeg(_));
↳      0      1
↳      1      0
↳      1      0
isHomogeneous(f); // f: is not homogeneous
↳ 0
poly g = 1-xy2z3;
isHomogeneous(g); // g: is homogeneous
↳ 1
multiDegPartition(g);
↳ _[1]=-xy2z3+1
kill T;
////////////////////////////////////
// new Torsion matrix:
intmat T[3][4] =

```

```

3,3,3,3,
2,1,3,0,
1,2,0,3;
setBaseMultigrading(M,T);
f;
↳ -yz+x
isHomogeneous(f);
↳ 0
multiDegPartition(f);
↳ _[1]=-yz
↳ _[2]=x
// -----
g;
↳ -xy2z3+1
isHomogeneous(g);
↳ 0
multiDegPartition(g);
↳ _[1]=-xy2z3
↳ _[2]=1
kill r, T, M;
ring R = 0, (x,y,z), dp;
intmat A[2][3] =
0,0,1,
3,2,1;
intmat T[2][1] =
-1,
4;
setBaseMultigrading(A, T);
isHomogeneous(ideal(x2 - y3 -xy +z, x*y-z, x^3 - y^2*z + x^2 -y^3)); // 1
↳ 1
isHomogeneous(ideal(x2 - y3 -xy +z, x*y-z, x^3 - y^2*z + x^2 -y^3), "checkGens");
↳ 0
isHomogeneous(ideal(x+y, x2 - y2)); // 0
↳ 0
// Degree partition:
multiDegPartition(x2 - y3 -xy +z);
↳ _[1]=-y3+x2
↳ _[2]=-xy+z
multiDegPartition(x3 -y2z + x2 -y3 + z + 1);
↳ _[1]=x3-y2z
↳ _[2]=-y3+x2
↳ _[3]=z
↳ _[4]=1
module N = gen(1) + (x+y) * gen(2), z*gen(3);
intmat V[2][3] = 0; // 1, 2, 3, 4, 5, 6; // column-wise weights of components!??
vector v1, v2;
v1 = setModuleGrading(N[1], V); v1;
↳ x*gen(2)+y*gen(2)+gen(1)
multiDegPartition(v1);
↳ _[1]=x*gen(2)
↳ _[2]=y*gen(2)
↳ _[3]=gen(1)
print( multiDeg(_) );

```

```

↳      0      0      0
↳      3      2      0
v2 = setModuleGrading(N[2], V); v2;
↳ z*gen(3)
multiDegPartition(v2);
↳ _[1]=z*gen(3)
print( multiDeg(_) );
↳      1
↳      1
N = setModuleGrading(N, V);
isHomogeneous(N);
↳ 0
print( multiDeg(N) );
↳      0      1
↳      3      1
////////////////////////////////////
V =
1, 2, 3,
4, 5, 6;
v1 = setModuleGrading(N[1], V); v1;
↳ x*gen(2)+y*gen(2)+gen(1)
multiDegPartition(v1);
↳ _[1]=x*gen(2)
↳ _[2]=y*gen(2)
↳ _[3]=gen(1)
print( multiDeg(_) );
↳      2      2      1
↳      8      7      4
v2 = setModuleGrading(N[2], V); v2;
↳ z*gen(3)
multiDegPartition(v2);
↳ _[1]=z*gen(3)
print( multiDeg(_) );
↳      4
↳      7
N = setModuleGrading(N, V);
isHomogeneous(N);
↳ 0
print( multiDeg(N) );
↳      2      4
↳      8      7
////////////////////////////////////
V =
0, 0, 0,
4, 1, 0;
N = gen(1) + x * gen(2), z*gen(3);
N = setModuleGrading(N, V); print(N);
↳ 1,0,
↳ x,0,
↳ 0,z
isHomogeneous(N);
↳ 1
print( multiDeg(N) );

```

```

↳      0      1
↳      4      1
v1 = getGradedGenerator(N,1); print(v1);
↳ [1,x]
multiDegPartition(v1);
↳ _[1]=x*gen(2)+gen(1)
print( multiDeg(_) );
↳      0
↳      4
N = setModuleGrading(N, V); print(N);
↳ 1,0,
↳ x,0,
↳ 0,z
isHomogeneous(N);
↳ 1
print( multiDeg(N) );
↳      0      1
↳      4      1

```

### D.15.12.38 equalMultiDeg

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:** `equalMultiDeg(exp1, exp2[, V]);` intvec exp1, exp2, intmat V

**Purpose:** Tests if the exponent vectors of two monomials (given by exp1 and exp2) represent the same multidegree.

**Note:** the integer matrix V encodes multidegrees of module components, if module component is present in exp1 and exp2

**Example:**

```

LIB "multigrading.lib";
printlevel=3;
ring r = 0,(x,y,z),dp;
intmat g[2][3]=
1,0,1,
0,1,1;
intmat t[2][1]=
-2,
1;
setBaseMultigrading(g,t);
poly a = x10yz;
poly b = x8y2z;
poly c = x4z2;
poly d = y5;
poly e = x2y2;
poly f = z2;
equalMultiDeg(leadexp(a), leadexp(b));
↳ 1
equalMultiDeg(leadexp(a), leadexp(c));
↳ 0
equalMultiDeg(leadexp(a), leadexp(d));
↳ 0
equalMultiDeg(leadexp(a), leadexp(e));

```

```

↳ 0
equalMultiDeg(leadexp(a), leadexp(f));
↳ 0
equalMultiDeg(leadexp(b), leadexp(c));
↳ 0
equalMultiDeg(leadexp(b), leadexp(d));
↳ 0
equalMultiDeg(leadexp(b), leadexp(e));
↳ 0
equalMultiDeg(leadexp(b), leadexp(f));
↳ 0
equalMultiDeg(leadexp(c), leadexp(d));
↳ 1
equalMultiDeg(leadexp(c), leadexp(e));
↳ 0
equalMultiDeg(leadexp(c), leadexp(f));
↳ 0
equalMultiDeg(leadexp(d), leadexp(e));
↳ 0
equalMultiDeg(leadexp(d), leadexp(f));
↳ 0
equalMultiDeg(leadexp(e), leadexp(f));
↳ 1

```

### D.15.12.39 multiDegGroebner

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:**      `multiDegGroebner(I)`; I is a poly/vector/ideal/module

**Purpose:**     computes the multigraded standard/groebner basis of I

**Note:**       I must be multigraded homogeneous

**Returns:**    ideal/module, the computed basis

**Example:**

```

LIB "multigrading.lib";
ring r = 0,(x,y,z,w),dp;
intmat MM[2][4]=
1,1,1,1,
0,1,3,4;
setBaseMultigrading(MM);
module M = ideal( xw-yz, x2z-y3, xz2-y2w, yw2-z3);
intmat v[2][nrows(M)]=
1,
0;
M = setModuleGrading(M, v);
////////////////////////////////////
// GB:
M = multiDegGroebner(M); M;
↳ M[1]=yz*gen(1)-xw*gen(1)
↳ M[2]=z3*gen(1)-yw2*gen(1)
↳ M[3]=xz2*gen(1)-y2w*gen(1)
↳ M[4]=y3*gen(1)-x2z*gen(1)

```

```

"Module Units Multigrading: "; print( getModuleGrading(M) );
↳ Module Units Multigrading:
↳      1
↳      0
"Multidegrees: "; print(multiDeg(M));
↳ Multidegrees:
↳      3      4      4      4
↳      4      9      6      3
isHomogeneous(M);
↳ 1
////////////////////////////////////
// Let's compute Syzygy!
def S = multiDegSyzygy(M); S;
↳ S[1]=yw*gen(1)-x*gen(2)+z*gen(3)
↳ S[2]=z2*gen(1)-y*gen(2)+w*gen(3)
↳ S[3]=xz*gen(1)-y*gen(3)-w*gen(4)
↳ S[4]=y2*gen(1)-x*gen(3)-z*gen(4)
"Module Units Multigrading: "; print( getModuleGrading(S) );
↳ Module Units Multigrading:
↳      3      4      4      4
↳      4      9      6      3
"Multidegrees: "; print(multiDeg(S));
↳ Multidegrees:
↳      5      5      5      5
↳      9     10      7      6
isHomogeneous(S);
↳ 1
////////////////////////////////////
// GB:
S = multiDegGroebner(S); S;
↳ S[1]=yw*gen(1)-x*gen(2)+z*gen(3)
↳ S[2]=z2*gen(1)-y*gen(2)+w*gen(3)
↳ S[3]=xz*gen(1)-y*gen(3)-w*gen(4)
↳ S[4]=y2*gen(1)-x*gen(3)-z*gen(4)
↳ S[5]=xy*gen(2)-yz*gen(3)-xw*gen(3)-zw*gen(4)
↳ S[6]=xz2*gen(2)-z3*gen(3)-y2w*gen(2)+yw2*gen(3)
↳ S[7]=x2z*gen(2)-xz2*gen(3)-y2w*gen(3)-yw2*gen(4)
↳ S[8]=y3*gen(2)-xz2*gen(3)-z3*gen(4)-y2w*gen(3)
↳ S[9]=y3*gen(3)-x2z*gen(3)-xz2*gen(4)+y2w*gen(4)
"Module Units Multigrading: "; print( getModuleGrading(S) );
↳ Module Units Multigrading:
↳      3      4      4      4
↳      4      9      6      3
"Multidegrees: "; print(multiDeg(S));
↳ Multidegrees:
↳      5      5      5      5      6      7      7      7      7
↳      9     10      7      6     10     15     12     12     9
isHomogeneous(S);
↳ 1

```

#### D.15.12.40 multiDegSyzygy

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:**      `multiDegSyzygy(I)`; I is a ideal or a module  
**Purpose:**     computes the multigraded syzygy module of I  
**Returns:**    module, the syzygy module of I  
**Note:**       generators of I must be multigraded homogeneous

**Example:**

```
LIB "multigrading.lib";
ring r = 0,(x,y,z,w),dp;
intmat MM[2][4]=
1,1,1,1,
0,1,3,4;
setBaseMultigrading(MM);
module M = ideal( xw-yz, x2z-y3, xz2-y2w, yw2-z3);
intmat v[2][nrows(M)]=
1,
0;
M = setModuleGrading(M, v);
isHomogeneous(M);
↳ 1
"Multidegrees: "; print(multiDeg(M));
↳ Multidegrees:
↳      3      4      4      4
↳      4      3      6      9
// Let's compute syzygies!
def S = multiDegSyzygy(M); S;
↳ S[1]=yw*gen(1)-x*gen(4)-z*gen(3)
↳ S[2]=z2*gen(1)-y*gen(4)-w*gen(3)
↳ S[3]=xz*gen(1)+y*gen(3)-w*gen(2)
↳ S[4]=y2*gen(1)+x*gen(3)-z*gen(2)
"Module Units Multigrading: "; print( getModuleGrading(S) );
↳ Module Units Multigrading:
↳      3      4      4      4
↳      4      3      6      9
"Multidegrees: "; print(multiDeg(S));
↳ Multidegrees:
↳      5      5      5      5
↳      9     10      7      6
isHomogeneous(S);
↳ 1
```

#### D.15.12.41 multiDegModulo

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:**      `multiDegModulo(I)`; I, J are ideals or modules  
**Purpose:**     computes the multigraded 'modulo' module of I and J  
**Returns:**    module, see 'modulo' command  
**Note:**       I and J should have the same multigrading, and their  
                generators must be multigraded homogeneous

**Example:**

```

LIB "multigrading.lib";
ring r = 0,(x,y,z),dp;
intmat MM[2][3]=
-1,1,1,
0,1,3;
setBaseMultigrading(MM);
ideal h1 = x, y, z;
ideal h2 = x;
"Multidegrees: "; print(multiDeg(h1));
↪ Multidegrees:
↪      -1      1      1
↪      0      1      3
// Let's compute modulo(h1, h2):
def K = multiDegModulo(h1, h2); K;
↪ K[1]=gen(1)
↪ K[2]=y*gen(3)-z*gen(2)
↪ K[3]=x*gen(2)
↪ K[4]=x*gen(3)
"Module Units Multigrading: "; print( getModuleGrading(K) );
↪ Module Units Multigrading:
↪      -1      1      1
↪      0      1      3
"Multidegrees: "; print(multiDeg(K));
↪ Multidegrees:
↪      -1      2      0      0
↪      0      4      1      3
isHomogeneous(K);
↪ 1

```

### D.15.12.42 multiDegResolution

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:** `multiDegResolution(I,l,[f]);` I is poly/vector/ideal/module; l,f are integers

**Purpose:** computes the multigraded resolution of I of the length l, or the whole resolution if l is zero. Returns minimal resolution if an optional argument 1 is supplied

**Note:** input must have multigraded-homogeneous generators.  
The returned list is truncated beginning with the first zero differential.

**Returns:** list, the computed resolution

**Example:**

```

LIB "multigrading.lib";
ring r = 0,(x,y,z,w),dp;
intmat M[2][4]=
1,1,1,1,
0,1,3,4;
setBaseMultigrading(M);
module m= ideal( xw-yz, x2z-y3, xz2-y2w, yw2-z3);
isHomogeneous(ideal( xw-yz, x2z-y3, xz2-y2w, yw2-z3), "checkGens");
↪ 1
ideal A = xw-yz, x2z-y3, xz2-y2w, yw2-z3;
int j;

```



```

for(j=1; j<=ncols(A); j++)
{
multiDegPartition(A[j]);
}
↳ _[1]=-yz+xw
↳ _[1]=-y3+x2z
↳ _[1]=xz2-y2w
↳ _[1]=-z3+yw2
intmat v[2][1]=
1,
0;
m = setModuleGrading(m, v);
// Let's compute Syzygy!
def S = multiDegSyzygy(m); S;
↳ S[1]=yw*gen(1)-x*gen(4)-z*gen(3)
↳ S[2]=z2*gen(1)-y*gen(4)-w*gen(3)
↳ S[3]=xz*gen(1)+y*gen(3)-w*gen(2)
↳ S[4]=y2*gen(1)+x*gen(3)-z*gen(2)
"Module Units Multigrading: "; print( getModuleGrading(S) );
↳ Module Units Multigrading:
↳      3      4      4      4
↳      4      3      6      9
"Multidegrees: "; print(multiDeg(S));
↳ Multidegrees:
↳      5      5      5      5
↳      9     10      7      6
////////////////////////////////////
S = multiDegGroebner(S); S;
↳ S[1]=yw*gen(1)-x*gen(4)-z*gen(3)
↳ S[2]=z2*gen(1)-y*gen(4)-w*gen(3)
↳ S[3]=xz*gen(1)+y*gen(3)-w*gen(2)
↳ S[4]=y2*gen(1)+x*gen(3)-z*gen(2)
↳ S[5]=xy*gen(4)+yz*gen(3)+xw*gen(3)-zw*gen(2)
↳ S[6]=xz2*gen(4)+z3*gen(3)-y2w*gen(4)-yw2*gen(3)
↳ S[7]=x2z*gen(4)+xz2*gen(3)+y2w*gen(3)-yw2*gen(2)
↳ S[8]=y3*gen(3)-x2z*gen(3)+xz2*gen(2)-y2w*gen(2)
↳ S[9]=y3*gen(4)+xz2*gen(3)-z3*gen(2)+y2w*gen(3)
"Module Units Multigrading: "; print( getModuleGrading(S) );
↳ Module Units Multigrading:
↳      3      4      4      4
↳      4      3      6      9
"Multidegrees: "; print(multiDeg(S));
↳ Multidegrees:
↳      5      5      5      6      7      7      7      7
↳      9     10      7      6     10     15     12      9     12
////////////////////////////////////
list L = multiDegResolution(m, 0, 1);
for( j =1; j<=size(L); j++)
{
"----- ", j, " -----";
L[j];
"Module Multigrading: "; print( getModuleGrading(L[j]) );
"Multigrading: "; print(multiDeg(L[j]));

```

```

}
↳ ----- 1 -----
↳ _[1]=yz*gen(1)-xw*gen(1)
↳ _[2]=z3*gen(1)-yw2*gen(1)
↳ _[3]=xz2*gen(1)-y2w*gen(1)
↳ _[4]=y3*gen(1)-x2z*gen(1)
↳ Module Multigrading:
↳      1
↳      0
↳ Multigrading:
↳      3      4      4      4
↳      4      9      6      3
↳ ----- 2 -----
↳ _[1]=yw*gen(1)-x*gen(2)+z*gen(3)
↳ _[2]=z2*gen(1)-y*gen(2)+w*gen(3)
↳ _[3]=xz*gen(1)-y*gen(3)-w*gen(4)
↳ _[4]=y2*gen(1)-x*gen(3)-z*gen(4)
↳ Module Multigrading:
↳      3      4      4      4
↳      4      9      6      3
↳ Multigrading:
↳      5      5      5      5
↳      9     10      7      6
↳ ----- 3 -----
↳ _[1]=x*gen(2)-y*gen(1)-z*gen(3)+w*gen(4)
↳ Module Multigrading:
↳      5      5      5      5
↳      9     10      7      6
↳ Multigrading:
↳      6
↳     10
////////////////////////////////////
L = multiDegResolution(maxideal(1), 0, 1);
for( j =1; j<=size(L); j++)
{
  "----- ", j, " -----";
  L[j];
  "Module Multigrading: "; print( getModuleGrading(L[j]) );
  "Multigrading: "; print(multiDeg(L[j]));
}
↳ ----- 1 -----
↳ _[1]=w
↳ _[2]=z
↳ _[3]=y
↳ _[4]=x
↳ Module Multigrading:
↳      0
↳      0
↳ Multigrading:
↳      1      1      1      1
↳      4      3      1      0
↳ ----- 2 -----
↳ _[1]=-z*gen(1)+w*gen(2)

```

```

⇒ _[2]=-y*gen(1)+w*gen(3)
⇒ _[3]=-y*gen(2)+z*gen(3)
⇒ _[4]=-x*gen(1)+w*gen(4)
⇒ _[5]=-x*gen(2)+z*gen(4)
⇒ _[6]=-x*gen(3)+y*gen(4)
⇒ Module Multigrading:
⇒      1      1      1      1
⇒      4      3      1      0
⇒ Multigrading:
⇒      2      2      2      2      2      2
⇒      7      5      4      4      3      1
⇒ ----- 3 -----
⇒ _[1]=y*gen(1)-z*gen(2)+w*gen(3)
⇒ _[2]=x*gen(1)-z*gen(4)+w*gen(5)
⇒ _[3]=x*gen(2)-y*gen(4)+w*gen(6)
⇒ _[4]=x*gen(3)-y*gen(5)+z*gen(6)
⇒ Module Multigrading:
⇒      2      2      2      2      2      2
⇒      7      5      4      4      3      1
⇒ Multigrading:
⇒      3      3      3      3
⇒      8      7      5      4
⇒ ----- 4 -----
⇒ _[1]=-x*gen(1)+y*gen(2)-z*gen(3)+w*gen(4)
⇒ Module Multigrading:
⇒      3      3      3      3
⇒      8      7      5      4
⇒ Multigrading:
⇒      4
⇒      8
kill v;
def h = hilbertSeries(m);
⇒ -----
⇒ This proc returns a ring with polynomials called 'numerator1/2' and 'deno\
  minator1/2'!
⇒ They represent the first and the second Hilbert Series.
⇒ The s_(i)-variables are defined to be the inverse of the t_(i)-variables.
⇒ -----
setring h;
numerator1;
⇒ -t_(1)^6*t_(2)^10+t_(1)^5*t_(2)^10+t_(1)^5*t_(2)^9-t_(1)^4*t_(2)^9+t_(1)^\
  5*t_(2)^7+t_(1)^5*t_(2)^6-t_(1)^4*t_(2)^6-t_(1)^4*t_(2)^3-t_(1)^3*t_(2)^4\
  +t_(1)
factorize(numerator1);
⇒ [1]:
⇒      _[1]=-1
⇒      _[2]=t_(1)
⇒      _[3]=t_(1)*t_(2)-1
⇒      _[4]=t_(1)*t_(2)^3-1
⇒      _[5]=t_(1)^3*t_(2)^6-t_(1)^2*t_(2)^6-t_(1)^2*t_(2)^2-t_(1)*t_(2)^3-t_(\
  1)*t_(2)-1
⇒ [2]:
⇒      1,1,1,1,1

```

```

denominator1;
⇒  $t_1^4 t_2^8 - t_1^3 t_2^8 - t_1^3 t_2^7 + t_1^2 t_2^7 - t_1^3 t_2^5 - t_1^3 t_2^4 + t_1^2 t_2^5 + 2 t_1^2 t_2^4 + t_1^2 t_2^3 - t_1 t_2^4 - t_1 t_2^3 + t_1^2 t_2 - t_1 t_2 - t_1 + 1$ 
factorize(denominator1);
⇒ [1]:
⇒  $1$ 
⇒  $t_1 - 1$ 
⇒  $t_1 t_2 - 1$ 
⇒  $t_1 t_2^3 - 1$ 
⇒  $t_1 t_2^4 - 1$ 
⇒ [2]:
⇒  $1, 1, 1, 1, 1$ 
numerator2;
⇒  $-t_1^4 t_2^6 + t_1^3 t_2^6 + t_1^3 t_2^2 + t_1^2 t_2^3 + t_1^2 t_2 + t_1 t_2^3 + t_1$ 
factorize(numerator2);
⇒ [1]:
⇒  $-1$ 
⇒  $t_1$ 
⇒  $t_1^3 t_2^6 - t_1^2 t_2^6 - t_1^2 t_2^2 - t_1 t_2^3 - t_1 t_2 - 1$ 
⇒ [2]:
⇒  $1, 1, 1$ 
denominator2;
⇒  $t_1^2 t_2^4 - t_1 t_2^4 - t_1 + 1$ 
factorize(denominator2);
⇒ [1]:
⇒  $1$ 
⇒  $t_1 - 1$ 
⇒  $t_1 t_2^4 - 1$ 
⇒ [2]:
⇒  $1, 1, 1$ 

```

### D.15.12.43 multiDegTensor

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:** `multiDegTensor(m, n)`,  $m, n$  modules or matrices.

**Purpose:** Computes the multigraded tensor product of  $m$  to multigraded modules.

**Return:** A module.

**Example:**

```

LIB "multigrading.lib";
ring r = 0, (x), dp;
intmat g[2][1]=1,1;
setBaseMultigrading(g);
matrix m[5][3]=1,2,3,4,5,6,7,8,9,10,11,12,13,14,15;
matrix n[3][2]=x,x2,x3,x4,x5,x6;
module mm = m;
module nn = n;
intmat gm[2][5]=1,2,3,4,5,0,0,0,0,0;
intmat gn[2][3]=0,0,0,1,2,3;

```

```

mm = setModuleGrading(mm, gm);
nn = setModuleGrading(nn, gn);
module mmtnn = multiDegTensor(mm, nn);
print(mmtnn);
↳ x, x2,0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 2, 0, 0, 3, 0, 0,
↳ x3,x4,0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 2, 0, 0, 3, 0,
↳ x5,x6,0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 2, 0, 0, 3,
↳ 0, 0, x, x2,0, 0, 0, 0, 0, 0, 0, 4, 0, 0, 5, 0, 0, 6, 0, 0,
↳ 0, 0, x3,x4,0, 0, 0, 0, 0, 0, 0, 0, 4, 0, 0, 5, 0, 0, 6, 0,
↳ 0, 0, x5,x6,0, 0, 0, 0, 0, 0, 0, 0, 0, 4, 0, 0, 5, 0, 0, 6,
↳ 0, 0, 0, 0, x, x2,0, 0, 0, 0, 0, 7, 0, 0, 8, 0, 0, 9, 0, 0,
↳ 0, 0, 0, 0, x3,x4,0, 0, 0, 0, 0, 0, 7, 0, 0, 8, 0, 0, 9, 0,
↳ 0, 0, 0, 0, x5,x6,0, 0, 0, 0, 0, 0, 0, 7, 0, 0, 8, 0, 0, 9,
↳ 0, 0, 0, 0, 0, 0, x, x2,0, 0, 10,0, 0, 11,0, 0, 12,0, 0,
↳ 0, 0, 0, 0, 0, 0, x3,x4,0, 0, 0, 10,0, 0, 11,0, 0, 12,0,
↳ 0, 0, 0, 0, 0, 0, x5,x6,0, 0, 0, 0, 10,0, 0, 11,0, 0, 12,
↳ 0, 0, 0, 0, 0, 0, 0, 0, x, x2,13,0, 0, 14,0, 0, 15,0, 0,
↳ 0, 0, 0, 0, 0, 0, 0, 0, x3,x4,0, 13,0, 0, 14,0, 0, 15,0,
↳ 0, 0, 0, 0, 0, 0, 0, 0, x5,x6,0, 0, 13,0, 0, 14,0, 0, 15
getModuleGrading(mmtnn);
↳ 1,1,1,2,2,2,3,3,3,4,4,4,5,5,5,
↳ 1,2,3,1,2,3,1,2,3,1,2,3,1,2,3
LIB "homolog.lib";
module tt = tensorMod(mm,nn);
print(tt);
↳ x, x2,0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 2, 0, 0, 3, 0, 0,
↳ x3,x4,0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 2, 0, 0, 3, 0,
↳ x5,x6,0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 2, 0, 0, 3,
↳ 0, 0, x, x2,0, 0, 0, 0, 0, 0, 0, 4, 0, 0, 5, 0, 0, 6, 0, 0,
↳ 0, 0, x3,x4,0, 0, 0, 0, 0, 0, 0, 0, 4, 0, 0, 5, 0, 0, 6, 0,
↳ 0, 0, x5,x6,0, 0, 0, 0, 0, 0, 0, 0, 0, 4, 0, 0, 5, 0, 0, 6,
↳ 0, 0, 0, 0, x, x2,0, 0, 0, 0, 0, 7, 0, 0, 8, 0, 0, 9, 0, 0,
↳ 0, 0, 0, 0, x3,x4,0, 0, 0, 0, 0, 0, 7, 0, 0, 8, 0, 0, 9, 0,
↳ 0, 0, 0, 0, x5,x6,0, 0, 0, 0, 0, 0, 0, 7, 0, 0, 8, 0, 0, 9,
↳ 0, 0, 0, 0, 0, 0, x, x2,0, 0, 10,0, 0, 11,0, 0, 12,0, 0,
↳ 0, 0, 0, 0, 0, 0, x3,x4,0, 0, 0, 10,0, 0, 11,0, 0, 12,0,
↳ 0, 0, 0, 0, 0, 0, x5,x6,0, 0, 0, 0, 10,0, 0, 11,0, 0, 12,
↳ 0, 0, 0, 0, 0, 0, 0, 0, x, x2,13,0, 0, 14,0, 0, 15,0, 0,
↳ 0, 0, 0, 0, 0, 0, 0, 0, x3,x4,0, 13,0, 0, 14,0, 0, 15,0,
↳ 0, 0, 0, 0, 0, 0, 0, 0, x5,x6,0, 0, 13,0, 0, 14,0, 0, 15
kill m, mm, n, nn, gm, gn;
matrix m[7][3] = x, x-1,x+2, 3x, 4x, x5, x6, x-7, x-8, 9, 10, 11x, 12 -x, 13x, 14x, x;
matrix n[2][4] = 1, 2, 3, 4, x, x2, x3, x4;
module mm = m;
module nn = n;
print(mm);
↳ x,      x-1,      x+2,
↳ 3x,     4x,       x5,
↳ x6,     x-7,      x-8,
↳ 9,      10,       11x,
↳ -x+12,13x,      14x,
↳ x15,    x2-8x+16,x17,
↳ 18x,    19x,     20x

```

```

print(nn);
⇨ 1,2, 3, 4,
⇨ x,x2,x3,x4
intmat gm[2][7] = 1, 2, 3, 4, 5, 6, 7, 0, 0, 0, 0, 0, 0, 0;
intmat gn[2][2] = 0, 0, 1, 2;
mm = setModuleGrading(mm, gm);
nn = setModuleGrading(nn, gn);
module mmtnn = multiDegTensor(mm, nn);
⇨ // ** redefining mmtnn (module mmtnn = multiDegTensor(mm, nn);) ./example\
  s/multiDegTensor.sing:30
print(mmtnn);
⇨ 1,2, 3, 4, 0,0, 0, 0, 0,0, 0, 0, 0,0, 0, 0, 0,0, 0, 0, 0,0, 0, \
  0,
⇨ x, 0, x-1, 0, x+2,0,
⇨ x,x2,x3,x4,0,0, 0, 0, 0,0, 0, 0, 0,0, 0, 0, 0,0, 0, 0, 0,0, 0, \
  0,
⇨ 0, x, 0, x-1, 0, x+2,
⇨ 0,0, 0, 0, 1,2, 3, 4, 0,0, 0, 0, 0,0, 0, 0, 0,0, 0, 0, 0,0, 0, \
  0,
⇨ 3x, 0, 4x, 0, x5, 0,
⇨ 0,0, 0, 0, x,x2,x3,x4,0,0, 0, 0, 0,0, 0, 0, 0,0, 0, 0, 0,0, 0, \
  0,
⇨ 0, 3x, 0, 4x, 0, x5,
⇨ 0,0, 0, 0, 0,0, 0, 0, 1,2, 3, 4, 0,0, 0, 0, 0,0, 0, 0, 0,0, 0, \
  0,
⇨ x6, 0, x-7, 0, x-8,0,
⇨ 0,0, 0, 0, 0,0, 0, 0, x,x2,x3,x4,0,0, 0, 0, 0,0, 0, 0, 0,0, 0, \
  0,
⇨ 0, x6, 0, x-7, 0, x-8,
⇨ 0,0, 0, 0, 0,0, 0, 0, 0,0, 0, 0, 1,2, 3, 4, 0,0, 0, 0, 0,0, 0, \
  0,
⇨ 9, 0, 10, 0, 11x,0,
⇨ 0,0, 0, 0, 0,0, 0, 0, 0,0, 0, 0, x,x2,x3,x4,0,0, 0, 0, 0,0, 0, \
  0,
⇨ 0, 9, 0, 10, 0, 11x,
⇨ 0,0, 0, 0, 0,0, 0, 0, 0,0, 0, 0, 0,0, 0, 0, 1,2, 3, 4, 0,0, 0, \
  0,
⇨ -x+12,0, 13x, 0, 14x,0,
⇨ 0,0, 0, 0, 0,0, 0, 0, 0,0, 0, 0, 0,0, 0, 0, x,x2,x3,x4,0,0, 0, \
  0,
⇨ 0, -x+12,0, 13x, 0, 14x,
⇨ 0,0, 0, 0, 0,0, 0, 0, 0,0, 0, 0, 0,0, 0, 0, 1,2, 3, 4, 0,0, 0, \
  0,
⇨ x15, 0, x2-8x+16,0, x17,0,
⇨ 0,0, 0, 0, 0,0, 0, 0, 0,0, 0, 0, 0,0, 0, 0, 0,0, 0, 0, x,x2,x3,x4,0,0, \
  0,
⇨ 0, x15, 0, x2-8x+16,0, x17,
⇨ 0,0, 0, 0, 0,0, 0, 0, 0,0, 0, 0, 0,0, 0, 0, 0,0, 0, 0, 1,2, 3, \
  4,
⇨ 18x, 0, 19x, 0, 20x,0,
⇨ 0,0, 0, 0, 0,0, 0, 0, 0,0, 0, 0, 0,0, 0, 0, 0,0, 0, 0, x,x2,x3\
  ,x4,
⇨ 0, 18x, 0, 19x, 0, 20x

```

[illegible]

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading\\_lib\]](#), page 2459).

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading\\_lib\]](#), page 2459).

**Purpose:** Yields a matrix which has to be appended to the grading group matrix to make the polynomial  $f$  homogeneous in the grading by  $\text{grad}$ .

```
LIB "multigrading.lib";
ring r =0,(x,y,z),dp;
intmat grad[2][3] =
1,0,1,
0,1,1;
setBaseMultigrading(grad);
poly f = x2y3-z5+x-3zx;
intmat M = defineHomogeneous(f);
M;
⇒ 3,0,-1,
⇒ 2,-2,-3
defineHomogeneous(f, grad) == M;
⇒ 1
isHomogeneous(f);
⇒ 0
setBaseMultigrading(grad, M);
isHomogeneous(f);
⇒ 1
```

### D.15.12.46 pushForward

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:**     `pushForward(f);`

**Purpose:**    Computes the finest grading of the image ring which makes the map  $f$  a map of graded rings. The group map between the two grading groups is given by `transpose( (Id, 0) )`. Pay attention that the group spanned by the columns of the grading group matrix may not be a subgroup of the grading group. Still all columns are needed to find the correct image of the preimage gradings.

**Example:**

```
LIB "multigrading.lib";
ring r = 0,(x,y,z),dp;
// Setting degrees for preimage ring.;
intmat grad[3][3] =
1,0,0,
0,1,0,
0,0,1;
setBaseMultigrading(grad);
// grading on r:
getVariableWeights();
⇒ 1,0,0,
⇒ 0,1,0,
⇒ 0,0,1
getLattice();
⇒ 0,0,0
// only for the purpose of this example
if( voice > 1 ){ /*keepring(r);*/ export(r); }
ring R = 0,(a,b),dp;
ideal i = a2-b2+a6-b5+ab3,a7b+b15-ab6+a6b6;
// The quotient ring by this ideal will become our image ring.;
qring Q = std(i);
listvar();
⇒ // Q                [0]  *ring
⇒ // R                [0]  ring
⇒ // grad             [0]  intmat 3 x 3
⇒ // r                [0]  ring
map f = r,-a2b6+b5+a3b+a2+ab,-a2b7-3a2b5+b4+a,a6-b6-b3+a2; f;
⇒ f[1]=-a2b6+b5+a3b+a2+ab
⇒ f[2]=-a2b7-3a2b5+b4+a
⇒ f[3]=a6-b6-b3+a2
// TODO: Unfortunately this is not a very spectacular example...:
// Pushing forward f:
pushForward(f);
// due to pushForward we have got new grading on Q
getVariableWeights();
⇒ 0,0,
⇒ 0,0,
⇒ 0,0,
⇒ 1,0,
⇒ 0,1
getLattice();
```



```

⇒ 1,0,0,0,0,
⇒ 0,1,0,0,0,
⇒ 0,0,1,0,0,
⇒ 1,1,1,1,0,
⇒ 1,1,1,1,1
// only for the purpose of this example
if( voice > 1 ){ kill r; }

```

#### D.15.12.47 gradiator

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Purpose:** coarsens the grading of the basering until the polynom or ideal `h` becomes homogeneous.

**Example:**

```

LIB "multigrading.lib";
ring r = 0,(x,y,z),dp;
intmat g[2][3] = 1,0,1,0,1,1;
intmat l[2][1] = 3,0;
setBaseMultigrading(g,l);
getLattice();
⇒ 3,0
ideal i = -y5+x4,
y6+xz,
x2y;
gradiator(i);
⇒ 1
getLattice();
⇒ 3,4,2,
⇒ 0,-5,-5
isHomogeneous(i);
⇒ 1

```

#### D.15.12.48 hermiteNormalForm

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:** `hermiteNormalForm( A );`

**Purpose:** Computes the (lower triangular) Hermite Normal Form of the matrix `A` by column operations.

**Return:** `intmat`, the Hermite Normal Form of `A`

**Example:**

```

LIB "multigrading.lib";
intmat M[2][5] =
1, 2, 3, 4, 0,
0,10,20,30, 1;
// Hermite Normal Form of M:
print(hermiteNormalForm(M));
⇒      1      0      0      0      0
⇒      0      1      0      0      0
intmat T[3][4] =
3,3,3,3,
2,1,3,0,

```

```

1,2,0,3;
// Hermite Normal Form of T:
print(hermiteNormalForm(T));
⇒      3      0      0      0
⇒      0      1      0      0
⇒      3     -1      0      0
intmat A[4][5] =
1,2,3,2,2,
1,2,3,4,0,
0,5,4,2,1,
3,2,4,0,2;
// Hermite Normal Form of A:
print(hermiteNormalForm(A));
⇒      1      0      0      0      0
⇒      1      2      0      0      0
⇒      0      0      1      0      0
⇒      0      1      0      1      0

```

#### D.15.12.49 smithNormalForm

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:** `smithNormalForm(A[,opt]); intmat A`

**Purpose:** Computes the Smith Normal Form of A

**Return:** if no optional argument is given: intmat, the Smith Normal Form of A, otherwise: a list of 3 integer matrices P, D Q, such that  $D == P \cdot A \cdot Q$ .

**Example:**

```

LIB "multigrading.lib";
intmat A[5][7] =
1,0,1,0,-2,9,-71,
0,-24,248,-32,-96,448,-3496,
0,4,-42,4,-8,30,-260,
0,0,0,18,-90,408,-3168,
0,0,0,-32,224,-1008,7872;
print( smithNormalForm(A) );
⇒      1      0      0      0      0      0      0
⇒      0      2      2      6     12      0      0
⇒      0      0      2      0      0      0      0
⇒      0      0      0      8      0      0      0
⇒      0      0      0      0     48      0      0
list l = smithNormalForm(A, 5);
l;
⇒ [1]:
⇒      1,0,0,0,0,
⇒      0,0,1,0,0,
⇒      0,-1,4,-1,0,
⇒      0,3,-12,4,-1,
⇒      0,-6,24,-8,3
⇒ [2]:
⇒      1,0,0,0,0,0,0,
⇒      0,2,2,6,12,0,0,
⇒      0,0,2,0,0,0,0,

```

```

⇒ 0,0,0,8,0,0,0,
⇒ 0,0,0,0,48,0,0
⇒ [3]:
⇒ 1,5,45,187,384,8,18591,
⇒ 0,-52,-461,-1914,-3929,-82,-190555,
⇒ 0,-5,-53,-218,-445,-10,-23236,
⇒ 0,0,-28,-109,-215,-7,-16260,
⇒ 0,0,221,871,1729,53,123084,
⇒ 0,0,50,197,391,12,27876,
⇒ 0,0,0,0,0,0,1
l[1]*A*l[3];
⇒ 1,0,0,0,0,0,0,
⇒ 0,2,2,6,12,0,0,
⇒ 0,0,2,0,0,0,0,
⇒ 0,0,0,8,0,0,0,
⇒ 0,0,0,0,48,0,0
det(l[1]);
⇒ 1
det(l[3]);
⇒ 1

```

### D.15.12.50 hilbertSeries

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

**Usage:** `hilbertSeries(I)`; I is poly/vector/ideal/module

**Purpose:** computes the multigraded Hilbert Series of I

**Note:** input must have multigraded-homogeneous generators.  
Multigrading should be positive.

**Returns:** a ring in variables  $t_{(i)}$ ,  $s_{(i)}$ , with polynomials  
numerator1 and denominator1 and mutually prime numerator2 and denominator2, quo-  
tients of which give the series.

**Example:**

```

LIB "multigrading.lib";
ring r = 0,(x,y,z,w),dp;
intmat g[2][4]=
1,1,1,1,
0,1,3,4;
setBaseMultigrading(g);
module M = ideal(xw-yz, x2z-y3, xz2-y2w, yw2-z3);
intmat V[2][1]=
1,
0;
M = setModuleGrading(M, V);
def h = hilbertSeries(M); setring h;
⇒ -----
⇒ This proc returns a ring with polynomials called 'numerator1/2' and 'deno\
  minator1/2'!
⇒ They represent the first and the second Hilbert Series.
⇒ The s_(i)-variables are defined to be the inverse of the t_(i)-variables.
⇒ -----

```

```

factorize(numerator2);
⇒ [1]:
⇒   _[1]=-1
⇒   _[2]=t_(1)
⇒   _[3]=t_(1)^3*t_(2)^6-t_(1)^2*t_(2)^6-t_(1)^2*t_(2)^2-t_(1)*t_(2)^3-t_(1)*t_(2)-1
⇒ [2]:
⇒   1,1,1
factorize(denominator2);
⇒ [1]:
⇒   _[1]=1
⇒   _[2]=t_(1)-1
⇒   _[3]=t_(1)*t_(2)^4-1
⇒ [2]:
⇒   1,1,1
kill g, h; setring r;
intmat g[2][4]=
1,2,3,4,
0,0,5,8;
setBaseMultigrading(g);
ideal I = x^2, y, z^3;
I = std(I);
list L = multiDegResolution(I, 0, 1);
for( int j = 1; j<=size(L); j++)
{
"-----", j, "-----";
L[j];
"Module Multigrading: "; print( getModuleGrading(L[j]) );
"Multigrading: "; print(multiDeg(L[j]));
}
⇒ ----- 1 -----
⇒ _[1]=y
⇒ _[2]=x2
⇒ _[3]=z3
⇒ Module Multigrading:
⇒   0
⇒   0
⇒ Multigrading:
⇒   2   2   9
⇒   0   0  15
⇒ ----- 2 -----
⇒ _[1]=-x2*gen(1)+y*gen(2)
⇒ _[2]=-z3*gen(1)+y*gen(3)
⇒ _[3]=-z3*gen(2)+x2*gen(3)
⇒ Module Multigrading:
⇒   2   2   9
⇒   0   0  15
⇒ Multigrading:
⇒   4  11  11
⇒   0  15  15
⇒ ----- 3 -----
⇒ _[1]=z3*gen(1)-x2*gen(2)+y*gen(3)
⇒ Module Multigrading:

```

```

⇒      4      11      11
⇒      0      15      15
⇒ Multigrading:
⇒      13
⇒      15
multiDeg(I);
⇒ 2,2,9,
⇒ 0,0,15
def h = hilbertSeries(I); setring h;
⇒ -----
⇒ This proc returns a ring with polynomials called 'numerator1/2' and 'deno\
  minator1/2'!
⇒ They represent the first and the second Hilbert Series.
⇒ The s_(i)-variables are defined to be the inverse of the t_(i)-variables.
⇒ -----
factorize(numerator2);
⇒ [1]:
⇒   _[1]=-1
⇒   _[2]=t_(1)+1
⇒   _[3]=t_(1)^6*t_(2)^10+t_(1)^3*t_(2)^5+1
⇒ [2]:
⇒   1,1,1
factorize(denominator2);
⇒ [1]:
⇒   _[1]=1
⇒   _[2]=t_(1)*t_(2)^2-1
⇒   _[3]=t_(1)*t_(2)^2+1
⇒   _[4]=t_(1)^2*t_(2)^4+1
⇒ [2]:
⇒   1,1,1,1
kill r, h, g, V;
////////////////////////////////////
ring R = 0,(x,y,z),dp;
intmat W[2][3] =
1,1, 1,
0,0,-1;
setBaseMultigrading(W);
ideal I = x3y,yz2,y2z,z4;
def h = hilbertSeries(I); setring h;
⇒ -----
⇒ This proc returns a ring with polynomials called 'numerator1/2' and 'deno\
  minator1/2'!
⇒ They represent the first and the second Hilbert Series.
⇒ The s_(i)-variables are defined to be the inverse of the t_(i)-variables.
⇒ -----
factorize(numerator2);
⇒ [1]:
⇒   _[1]=1
⇒   _[2]=-t_(1)^5*s_(2)+t_(1)^3*s_(2)^3+t_(1)^2*s_(2)^2+t_(1)^3+t_(1)^2*s_\
  (2)+t_(1)^2+t_(1)*s_(2)+t_(1)+1
⇒ [2]:
⇒   1,1
factorize(denominator2);

```

```

⇒ [1]:
⇒   _[1]=-1
⇒   _[2]=t_(1)-1
⇒ [2]:
⇒   1,1
kill R, W, h;
////////////////////////////////////
ring R = 0,(x,y,z,a,b,c),dp;
intmat W[2][6] =
1,1, 1,1,1,1,
0,0,-1,0,0,0;
setBaseMultigrading(W);
ideal I = x3y,yz2,y2z,z4;
def h = hilbertSeries(I); setring h;
⇒ -----
⇒ This proc returns a ring with polynomials called 'numerator1/2' and 'deno\
  minator1/2'!
⇒ They represent the first and the second Hilbert Series.
⇒ The s_(i)-variables are defined to be the inverse of the t_(i)-variables.
⇒ -----
factorize(numerator2);
⇒ [1]:
⇒   _[1]=1
⇒   _[2]=-t_(1)^5*s_(2)+t_(1)^3*s_(2)^3+t_(1)^2*s_(2)^2+t_(1)^3+t_(1)^2*s_\
  (2)+t_(1)^2+t_(1)*s_(2)+t_(1)+1
⇒ [2]:
⇒   1,1
factorize(denominator2);
⇒ [1]:
⇒   _[1]=1
⇒   _[2]=t_(1)-1
⇒ [2]:
⇒   1,4
kill R, W, h;
////////////////////////////////////
// This is example 5.3.9. from Robbianos book.
ring R = 0,(x,y,z,w),dp;
intmat W[1][4] =
1,1, 1,1;
setBaseMultigrading(W);
ideal I = z3,y3zw2,x2y4w2xyz2;
hilb(std(I));
⇒ (t8-t6-t3+1) / (1-t)^4
⇒ (-t7-t6+t2+t+1) / (1-t)^3
⇒ // dimension (proj.) = 2
⇒ // degree (proj.) = 1
def h = hilbertSeries(I); setring h;
⇒ -----
⇒ This proc returns a ring with polynomials called 'numerator1/2' and 'deno\
  minator1/2'!
⇒ They represent the first and the second Hilbert Series.
⇒ The s_(i)-variables are defined to be the inverse of the t_(i)-variables.
⇒ -----

```

```

numerator1;
⇒  $t_{(1)}^8 - t_{(1)}^6 - t_{(1)}^3 + 1$ 
denominator1;
⇒  $t_{(1)}^4 - 4t_{(1)}^3 + 6t_{(1)}^2 - 4t_{(1)} + 1$ 
factorize(numerator2);
⇒ [1]:
⇒   _[1]=1
⇒   _[2]= $t_{(1)}^7 + t_{(1)}^6 - t_{(1)}^2 - t_{(1)} - 1$ 
⇒ [2]:
⇒   1,1
factorize(denominator2);
⇒ [1]:
⇒   _[1]=1
⇒   _[2]= $t_{(1)} - 1$ 
⇒ [2]:
⇒   1,3
kill h;
////////////////////////////////////
setring R;
ideal I2 = x2,y2,z2; I2;
⇒ I2[1]=x2
⇒ I2[2]=y2
⇒ I2[3]=z2
hilb(std(I2));
⇒  $(-t^6 + 3t^4 - 3t^2 + 1) / (1-t)^4$ 
⇒  $(t^3 + 3t^2 + 3t + 1) / (1-t)^1$ 
⇒ // dimension (proj.) = 0
⇒ // degree (proj.) = 8
def h = hilbertSeries(I2); setring h;
⇒ -----
⇒ This proc returns a ring with polynomials called 'numerator1/2' and 'deno\
  minator1/2'!
⇒ They represent the first and the second Hilbert Series.
⇒ The s_(i)-variables are defined to be the inverse of the t_(i)-variables.
⇒ -----
numerator1;
⇒  $-t_{(1)}^6 + 3t_{(1)}^4 - 3t_{(1)}^2 + 1$ 
denominator1;
⇒  $t_{(1)}^4 - 4t_{(1)}^3 + 6t_{(1)}^2 - 4t_{(1)} + 1$ 
kill h;
////////////////////////////////////
setring R;
W = 2,2,2,2;
setBaseMultigrading(W);
getVariableWeights();
⇒ 2,2,2,2
intvec w = 2,2,2,2;
hilb(std(I2), 1, w);
⇒ 1,0,0,0,-3,0,0,0,3,0,0,0,-1,0
kill w;
def h = hilbertSeries(I2); setring h;
⇒ -----
⇒ This proc returns a ring with polynomials called 'numerator1/2' and 'deno\

```

```

    minator1/2'!
    ↪ They represent the first and the second Hilbert Series.
    ↪ The s_(i)-variables are defined to be the inverse of the t_(i)-variables.
    ↪ -----
    numerator1; denominator1;
    ↪ -t_(1)^12+3*t_(1)^8-3*t_(1)^4+1
    ↪ t_(1)^8-4*t_(1)^6+6*t_(1)^4-4*t_(1)^2+1
    kill h;
    kill R, W;
    //////////////////////////////////////
    ring R = 0,(x),dp;
    intmat W[1][1] =
    1;
    setBaseMultigrading(W);
    ideal I;
    I = 1; I;
    ↪ I[1]=1
    hilb(std(I));
    ↪ (0) / (1-t)^1
    ↪ (0) / (1-t)^0
    ↪ // dimension (affine) = 0
    ↪ // degree (affine) = 0
    def h = hilbertSeries(I); setring h;
    ↪ -----
    ↪ This proc returns a ring with polynomials called 'numerator1/2' and 'deno\
    minator1/2'!
    ↪ They represent the first and the second Hilbert Series.
    ↪ The s_(i)-variables are defined to be the inverse of the t_(i)-variables.
    ↪ -----
    numerator1; denominator1;
    ↪ 0
    ↪ -t_(1)+1
    kill h;
    //////////////////////////////////////
    setring R;
    I = x; I;
    ↪ I[1]=x
    hilb(std(I));
    ↪ (-t+1) / (1-t)^1
    ↪ (1) / (1-t)^0
    ↪ // dimension (affine) = 0
    ↪ // degree (affine) = 1
    def h = hilbertSeries(I); setring h;
    ↪ -----
    ↪ This proc returns a ring with polynomials called 'numerator1/2' and 'deno\
    minator1/2'!
    ↪ They represent the first and the second Hilbert Series.
    ↪ The s_(i)-variables are defined to be the inverse of the t_(i)-variables.
    ↪ -----
    numerator1; denominator1;
    ↪ -t_(1)+1
    ↪ -t_(1)+1
    kill h;

```



```

////////////////////////////////////
setring R;
I = x^5; I;
⇨ I[1]=x5
hilb(std(I));
⇨ (-t5+1) / (1-t)^1
⇨ (t4+t3+t2+t+1) / (1-t)^0
⇨ // dimension (affine) = 0
⇨ // degree (affine) = 5
hilb(std(I), 1);
⇨ 1,0,0,0,0,-1,0
def h = hilbertSeries(I); setring h;
⇨ -----
⇨ This proc returns a ring with polynomials called 'numerator1/2' and 'deno\
  minator1/2'!
⇨ They represent the first and the second Hilbert Series.
⇨ The s_(i)-variables are defined to be the inverse of the t_(i)-variables.
⇨ -----
numerator1; denominator1;
⇨ -t_(1)^5+1
⇨ -t_(1)+1
kill h;
////////////////////////////////////
setring R;
I = x^10; I;
⇨ I[1]=x10
hilb(std(I));
⇨ (-t10+1) / (1-t)^1
⇨ (t9+t8+t7+t6+t5+t4+t3+t2+t+1) / (1-t)^0
⇨ // dimension (affine) = 0
⇨ // degree (affine) = 10
def h = hilbertSeries(I); setring h;
⇨ -----
⇨ This proc returns a ring with polynomials called 'numerator1/2' and 'deno\
  minator1/2'!
⇨ They represent the first and the second Hilbert Series.
⇨ The s_(i)-variables are defined to be the inverse of the t_(i)-variables.
⇨ -----
numerator1; denominator1;
⇨ -t_(1)^10+1
⇨ -t_(1)+1
kill h;
////////////////////////////////////
setring R;
module M = 1;
M = setModuleGrading(M, W);
hilb(std(M));
⇨ (0) / (1-t)^1
⇨ (0) / (1-t)^0
⇨ // dimension (affine) = 0
⇨ // degree (affine) = 0
def h = hilbertSeries(M); setring h;
⇨ -----

```

```

⇒ This proc returns a ring with polynomials called 'numerator1/2' and 'deno\
  minator1/2'!
⇒ They represent the first and the second Hilbert Series.
⇒ The s_(i)-variables are defined to be the inverse of the t_(i)-variables.
⇒ -----
numerator1; denominator1;
⇒ 0
⇒ -t_(1)+1
kill h;
////////////////////////////////////
setring R;
kill M; module M = x^5*gen(1);
// intmat V[1][3] = 0; // TODO: this would lead to a wrong result!!!?
intmat V[1][1] = 0; // all gen(i) of degree 0!
M = setModuleGrading(M, V);
hilb(std(M));
⇒ (-t5+1) / (1-t)^1
⇒ (t4+t3+t2+t+1) / (1-t)^0
⇒ // dimension (affine) = 0
⇒ // degree (affine) = 5
def h = hilbertSeries(M); setring h;
⇒ -----
⇒ This proc returns a ring with polynomials called 'numerator1/2' and 'deno\
  minator1/2'!
⇒ They represent the first and the second Hilbert Series.
⇒ The s_(i)-variables are defined to be the inverse of the t_(i)-variables.
⇒ -----
numerator1; denominator1;
⇒ -t_(1)^5+1
⇒ -t_(1)+1
kill h;
////////////////////////////////////
setring R;
module N = x^5*gen(3);
kill V;
intmat V[1][3] = 0; // all gen(i) of degree 0!
N = setModuleGrading(N, V);
hilb(std(N));
⇒ (-t5+3) / (1-t)^1
⇒ (-t5+3) / (1-t)^1
⇒ // dimension (proj.) = 0
⇒ // degree (proj.) = 2
def h = hilbertSeries(N); setring h;
⇒ -----
⇒ This proc returns a ring with polynomials called 'numerator1/2' and 'deno\
  minator1/2'!
⇒ They represent the first and the second Hilbert Series.
⇒ The s_(i)-variables are defined to be the inverse of the t_(i)-variables.
⇒ -----
numerator1; denominator1;
⇒ -t_(1)^5+3
⇒ -t_(1)+1
kill h;

```

```

////////////////////////////////////////
setring R;
module S = M + N;
S = setModuleGrading(S, V);
hilb(std(S));
⇒ (-2t5+3) / (1-t)1
⇒ (-2t5+3) / (1-t)1
⇒ // dimension (proj.) = 0
⇒ // degree (proj.) = 1
def h = hilbertSeries(S); setring h;
⇒ -----
⇒ This proc returns a ring with polynomials called 'numerator1/2' and 'deno\
  minator1/2'!
⇒ They represent the first and the second Hilbert Series.
⇒ The s(i)-variables are defined to be the inverse of the t(i)-variables.
⇒ -----
numerator1; denominator1;
⇒ -2*t(1)5+3
⇒ -t(1)+1
kill h;
kill V;
kill R, W;

```

### D.15.12.51 III

Procedure from library `multigrading.lib` (see [Section D.15.12 \[multigrading.lib\]](#), page 2459).

#### Example:

```

LIB "multigrading.lib";
ring R = 0,x,dp;
intmat m[5][5] =
13,25,37,83,294,
12,-33,9,0,64,
77,12,34,6,1,
43,2,88,91,100,
-46,32,37,42,15;
l1l(m);
⇒ 0,-9,8,43,20,
⇒ 12,-33,9,0,64,
⇒ -46,41,29,-1,-5,
⇒ 77,12,34,6,1,
⇒ 23,101,-35,-2,67
list l =
intvec(13,25,37, 83, 294),
intvec(12, -33, 9,0,64),
intvec (77,12,34,6,1),
intvec (43,2,88,91,100),
intvec (-46,32,37,42,15);
l1l(l);
⇒ [1]:
⇒ 0,-9,8,43,20
⇒ [2]:
⇒ 12,-33,9,0,64

```

```

⇒ [3] :
⇒      -46, 41, 29, -1, -5
⇒ [4] :
⇒      77, 12, 34, 6, 1
⇒ [5] :
⇒      23, 101, -35, -2, 67

```

### D.15.13 pfd\_lib

**Library:** pfd.lib

**Purpose:** Multivariate Partial Fraction Decomposition

**Author:** Marcel Wittmann, e-mail: mwittman@mathematik.uni-kl.de

**Overview:** This Library implements an algorithm based on the work of E. K. Leinartas to write rational functions in multiple variables as a sum of functions with "smaller" numerators and denominators.

This can be used to shorten the IBP reduction coefficients of multi-loop Feynman integrals. For this application,

we also provide a procedure that applies the algorithm to all entries of a matrix of rational functions given as one (possibly very big) txt-file. If you use the library pfd.lib, please cite the corresponding paper [J. Boehm, M. Wittmann, Z. Wu, Y. Xu, Y. Zhang: 'IBP reduction coefficients made simple' (preprint 2020)].

**Procedures:**

#### D.15.13.1 pfd

Procedure from library `pfd.lib` (see [Section D.15.13 \[pfd\\_lib\]](#), page 2518).

**Usage:** `pfd(f,g[,debug]); f,g poly, debug int`  
`pfd(f,g[,debug]); f poly, g list, debug int`  
`pfd(arguments[, parallelize]); arguments list, parallelize int`

**Return:** a partial fraction decomposition of  $f/g$  as a list `l` where `l[1]` is an ideal generated by irreducible polynomials and `l[2]` is a list of fractions. Each fraction is represented by a list of

- 1) the numerator polynomial
- 2) an intvec of indices `i` for which `l[1][i]` occurs as a factor in the denominator
- 3) an intvec containing the exponents of those irreducible factors.

Setting `debug` to a positive integer measures runtimes and creates a log file (default: `debug=0`).

The denominator `g` can also be given in factorized form as a list of an ideal of irreducible non constant polynomials and an intvec of exponents. This can save time since the first step in the algorithm is to factorize `g`. (A list of the zero-ideal and an empty intvec represents a denominator of 1.)

If instead of `f` and `g`, the input is a single list (or even a list of lists) containing elements of the form `list(f,g[,debug])` (`f,g,debug` as above), the algorithm is applied to all entries in parallel (using [Section D.2.7 \[parallel.lib\]](#), page 885), if `parallelize=1` (default) and in sequence if `parallelize=0`. A list (or list of lists) of the results is returned.

**Note:** The result depends on the monomial ordering. For "small" results use `dp`.

**Example:**

```

LIB "pfd.lib";
ring R = 0,(x,y),dp;
poly f = x^3+3*x^2*y+2*y^2-x^2+4*x*y;
poly g = x^2*y*(x-1)*(x-y)^2;
list dec = pfd(f,g);
 $\mapsto (2) / (q_3*q_4)$ 
 $\mapsto + (-2) / (q_1*q_4)$ 
 $\mapsto + (-6) / (q_3*q_4^2)$ 
 $\mapsto + (1) / (q_2*q_4^2)$ 
 $\mapsto + (9) / (q_1*q_4^2)$ 
 $\mapsto + (2) / (q_3^2*q_4)$ 
 $\mapsto$  where
 $\mapsto q_1 = x-1$ 
 $\mapsto q_2 = y$ 
 $\mapsto q_3 = x$ 
 $\mapsto q_4 = x-y$ 
 $\mapsto$  (6 terms)
 $\mapsto$ 
displaypfd_long(dec); // display result
 $\mapsto (2)/((x)*(x-y))$ 
 $\mapsto + (-2)/((x-1)*(x-y))$ 
 $\mapsto + (-6)/((x)*(x-y)^2)$ 
 $\mapsto + (1)/((y)*(x-y)^2)$ 
 $\mapsto + (9)/((x-1)*(x-y)^2)$ 
 $\mapsto + (2)/((x)^2*(x-y))$ 
 $\mapsto$  (6 terms)
 $\mapsto$ 
checkpfd(list(f,g),dec); // check for equality to f/g
 $\mapsto 1$ 
// calculate decompositions of a 2x2 matrix of rational functions at once:
list arguments = list(list(f, g), list(1, f) ),
list(list(x*y, y+1), list(1, x^2-y^2));
dec = pfd(arguments);
// the result has the same shape as the
// input (2x2 matrix as list of lists):
displaypfd_long(dec[1][1]);
 $\mapsto (2)/((x)*(x-y))$ 
 $\mapsto + (-2)/((x-1)*(x-y))$ 
 $\mapsto + (-6)/((x)*(x-y)^2)$ 
 $\mapsto + (1)/((y)*(x-y)^2)$ 
 $\mapsto + (9)/((x-1)*(x-y)^2)$ 
 $\mapsto + (2)/((x)^2*(x-y))$ 
 $\mapsto$  (6 terms)
 $\mapsto$ 
displaypfd_long(dec[1][2]);
 $\mapsto (1)/((x^3+3*x^2*y-x^2+4*x*y+2*y^2))$ 
 $\mapsto$  (1 terms)
 $\mapsto$ 
displaypfd_long(dec[2][1]);
 $\mapsto (x)$ 
 $\mapsto + (-x)/((y+1))$ 
 $\mapsto$  (2 terms)
 $\mapsto$ 

```

```

displaypdf_long(dec[2][2]);
↳ (1)/((x-y)*(x+y))
↳ (1 terms)
↳
// a more complicated example
ring S = 0,(s12,s15,s23,s34,s45),dp;
poly f = 7*s12^4*s15^2 + 11*s12^3*s15^3 + 4*s12^2*s15^4 - 10*s12^4*s15*s23
- 14*s12^3*s15^2*s23 - 4*s12^2*s15^3*s23 + 3*s12^4*s23^2 + 3*s12^3*s15*s23^2
+ 13*s12^4*s15*s34 + 12*s12^3*s15^2*s34 + 2*s12^2*s15^3*s34
- 5*s12^4*s23*s34 + 33*s12^3*s15*s23*s34 + 49*s12^2*s15^2*s23*s34
+ 17*s12*s15^3*s23*s34 - 17*s12^3*s23^2*s34 - 19*s12^2*s15*s23^2*s34
- 5*s12*s15^2*s23^2*s34 - 24*s12^3*s15*s34^2 - 15*s12^2*s15^2*s34^2
+ 2*s12*s15^3*s34^2 + 15*s12^3*s23*s34^2 - 34*s12^2*s15*s23*s34^2
- 31*s12*s15^2*s23*s34^2 + 2*s15^3*s23*s34^2 + 33*s12^2*s23^2*s34^2
+ 29*s12*s15*s23^2*s34^2 + 5*s15^2*s23^2*s34^2 + 9*s12^2*s15*s34^3
- 4*s12*s15^2*s34^3 - 15*s12^2*s23*s34^3 + 9*s12*s15*s23*s34^3
- 4*s15^2*s23*s34^3 - 27*s12*s23^2*s34^3 - 13*s15*s23^2*s34^3
+ 2*s12*s15*s34^4 + 5*s12*s23*s34^4 + 2*s15*s23*s34^4 + 8*s23^2*s34^4
- 6*s12^3*s15^2*s45 - 9*s12^2*s15^3*s45 - 2*s12*s15^4*s45
+ 30*s12^3*s15*s23*s45 + 56*s12^2*s15^2*s23*s45 + 24*s12*s15^3*s23*s45
- 12*s12^3*s23^2*s45 - 23*s12^2*s15*s23^2*s45 - 10*s12*s15^2*s23^2*s45
- 30*s12^3*s15*s34*s45 - 32*s12^2*s15^2*s34*s45 - 6*s12*s15^3*s34*s45
+ 7*s12^3*s23*s34*s45 - 86*s12^2*s15*s23*s34*s45 - 104*s12*s15^2*s23*s34*s45
- 15*s15^3*s23*s34*s45 + 41*s12^2*s23^2*s34*s45 + 51*s12*s15*s23^2*s34*s45
+ 10*s15^2*s23^2*s34*s45 - 5*s12^3*s34^2*s45 + 33*s12^2*s15*s34^2*s45
+ 14*s12*s15^2*s34^2*s45 - 2*s15^3*s34^2*s45 - 21*s12^2*s23*s34^2*s45
+ 62*s12*s15*s23*s34^2*s45 + 28*s15^2*s23*s34^2*s45 - 46*s12*s23^2*s34^2*s45
- 28*s15*s23^2*s34^2*s45 + 10*s12^2*s34^3*s45 - s12*s15*s34^3*s45
+ 4*s15^2*s34^3*s45 + 21*s12*s23*s34^3*s45 - 6*s15*s23*s34^3*s45
+ 17*s23^2*s34^3*s45 - 5*s12*s34^4*s45 - 2*s15*s34^4*s45 - 7*s23*s34^4*s45
- 6*s12^2*s15^2*s45^2 - 5*s12*s15^3*s45^2 - 2*s15^4*s45^2
- 28*s12^2*s15*s23*s45^2 - 42*s12*s15^2*s23*s45^2 - 10*s15^3*s23*s45^2
+ 9*s12^2*s23^2*s45^2 + 10*s12*s15*s23^2*s45^2 + 24*s12^2*s15*s34*s45^2
+ 36*s12*s15^2*s34*s45^2 + 10*s15^3*s34*s45^2 - 11*s12^2*s23*s34*s45^2
+ 31*s12*s15*s23*s34*s45^2 + 25*s15^2*s23*s34*s45^2
- 18*s12*s23^2*s34*s45^2 - 10*s15*s23^2*s34*s45^2 + 4*s12^2*s34^2*s45^2
- 29*s12*s15*s34^2*s45^2 - 17*s15^2*s34^2*s45^2 + 27*s12*s23*s34^2*s45^2
+ 2*s15*s23*s34^2*s45^2 + 9*s23^2*s34^2*s45^2 - 3*s12*s34^3*s45^2
+ 10*s15*s34^3*s45^2 - 16*s23*s34^3*s45^2 - s34^4*s45^2 + 6*s12*s15^2*s45^3
+ 3*s15^3*s45^3 + 8*s12*s15*s23*s45^3 + 10*s15^2*s23*s45^3
- 8*s12*s15*s34*s45^3 - 10*s15^2*s34*s45^3 + 9*s12*s23*s34*s45^3
+ s12*s34^2*s45^3 + 8*s15*s34^2*s45^3 - 9*s23*s34^2*s45^3 - s34^3*s45^3
- s15^2*s45^4 + s15*s34*s45^4;
poly g = 4*s12*s15*(s12 + s15 - s34)*(s15 - s23 - s34)*(s12 + s23 - s45)
*(s12 - s34 - s45)*(s12 + s15 - s34 - s45)*s45;
list dec = pfd(f,g);
↳ (5/4) / (q7*q8)
↳ + (-11/4) / (q6*q8)
↳ + (3/4) / (q5*q8)
↳ + (-11/4) / (q4*q8)
↳ + (1/4) / (q3*q8)
↳ + (5/4) / (q2*q8)
↳ + (1) / (q1*q8)

```

```

⇒ + (1/2) / (q6*q7)
⇒ + (1) / (q5*q6)
⇒ + (-3/2) / (q2*q6)
⇒ + (-1/4) / (q3*q4)
⇒ + (3) / (q1*q4)
⇒ + (1/4) / (q1*q2)
⇒ + (5/4*s45) / (q1*q7*q8)
⇒ + (11/4*s45) / (q1*q6*q8)
⇒ + (-3/4*s34) / (q1*q5*q8)
⇒ + (11/4*s45) / (q1*q4*q8)
⇒ + (-1/4*s34) / (q1*q3*q8)
⇒ + (1/4*s45) / (q1*q2*q8)
⇒ + (s34) / (q5*q6*q7)
⇒ + (-1/2*s45) / (q1*q6*q7)
⇒ + (19/4*s34) / (q4*q5*q7)
⇒ + (-3/4*s23) / (q2*q5*q6)
⇒ + (-7/4*s34) / (q1*q4*q5)
⇒ + (2*s34) / (q1*q2*q5)
⇒ + (2*s34^2) / (q1*q2*q4*q5)
⇒ where
⇒ q1 = s12
⇒ q2 = s15
⇒ q3 = s12+s15-s34
⇒ q4 = s15-s23-s34
⇒ q5 = s45
⇒ q6 = s12+s23-s45
⇒ q7 = s12-s34-s45
⇒ q8 = s12+s15-s34-s45
⇒ (26 terms)
⇒
displaypf(dec);
⇒ (5/4) / (q7*q8)
⇒ + (-11/4) / (q6*q8)
⇒ + (3/4) / (q5*q8)
⇒ + (-11/4) / (q4*q8)
⇒ + (1/4) / (q3*q8)
⇒ + (5/4) / (q2*q8)
⇒ + (1) / (q1*q8)
⇒ + (1/2) / (q6*q7)
⇒ + (1) / (q5*q6)
⇒ + (-3/2) / (q2*q6)
⇒ + (-1/4) / (q3*q4)
⇒ + (3) / (q1*q4)
⇒ + (1/4) / (q1*q2)
⇒ + (5/4*s45) / (q1*q7*q8)
⇒ + (11/4*s45) / (q1*q6*q8)
⇒ + (-3/4*s34) / (q1*q5*q8)
⇒ + (11/4*s45) / (q1*q4*q8)
⇒ + (-1/4*s34) / (q1*q3*q8)
⇒ + (1/4*s45) / (q1*q2*q8)
⇒ + (s34) / (q5*q6*q7)
⇒ + (-1/2*s45) / (q1*q6*q7)
⇒ + (19/4*s34) / (q4*q5*q7)

```

```

⇒ + (-3/4*s23) / (q2*q5*q6)
⇒ + (-7/4*s34) / (q1*q4*q5)
⇒ + (2*s34) / (q1*q2*q5)
⇒ + (2*s34^2) / (q1*q2*q4*q5)
⇒ where
⇒ q1 = s12
⇒ q2 = s15
⇒ q3 = s12+s15-s34
⇒ q4 = s15-s23-s34
⇒ q5 = s45
⇒ q6 = s12+s23-s45
⇒ q7 = s12-s34-s45
⇒ q8 = s12+s15-s34-s45
⇒ (26 terms)
⇒
checkpfd(list(f,g),dec);
⇒ 1
// size comparison:
size(string(f)) + size(string(g));
⇒ 4368
size(getStringpfd(dec));
⇒ 1055

```

See also: [Section D.15.13.2 \[checkpfd\]](#), page 2522; [Section D.15.13.4 \[displaypfd\]](#), page 2524; [Section D.15.13.5 \[displaypfd\\_long\]](#), page 2525; [Section D.15.13.3 \[evaluatepfd\]](#), page 2524; [Section D.15.13.9 \[pfdMat\]](#), page 2528.

### D.15.13.2 checkpfd

Procedure from library `pfd.lib` (see [Section D.15.13 \[pfd.lib\]](#), page 2518).

**Usage:** `checkpfd(list(f,g),dec[,N,C]);`  $f, g$  poly,  $dec$  list,  $N, C$  int

**Return:** 0 or 1

**Purpose:** test for (mathematical) equality of  $f/g$  and a partial fraction decomposition  $dec$ . The list  $dec$  has to have the same structure as the output of [Section D.15.13.1 \[pfd\]](#), page 2518.

The denominator  $g$  can also be given in factorized form as a list of an ideal of irreducible non constant polynomials and an intvec of exponents. This can save time since the first step in the algorithm is to factorize  $g$ . (a list of the zero-ideal and an empty intvec represents a denominator of 1.)

By default the test is done (exactly) by bringing all terms of the decomposition on the same denominator and comparing to  $f/g$ .

If additional parameters  $N$  [,  $C$ ] are given and if  $N > 0$ , a probabilistic method is chosen: evaluation at  $N$  random points with coordinates between  $-C$  and  $C$ . This may be faster for big polynomials.

**Example:**

```

LIB "pfd.lib";
ring R = 0,(x,y),dp;
poly f = x^3+3*x^2*y+2*y^2-x^2+4*x*y;
poly g = x^2*y*(x-1)*(x-y)^2;
// partial fraction decomposition of f/g:
list dec = pfd(f,g);

```



```

⇒ (2) / (q3*q4)
⇒ + (-2) / (q1*q4)
⇒ + (-6) / (q3*q4^2)
⇒ + (1) / (q2*q4^2)
⇒ + (9) / (q1*q4^2)
⇒ + (2) / (q3^2*q4)
⇒ where
⇒ q1 = x-1
⇒ q2 = y
⇒ q3 = x
⇒ q4 = x-y
⇒ (6 terms)
⇒
// some other decomposition (not equal to f/g):
list wrong_dec = pfd(f+1,g);
⇒ (1) / (q3*q4)
⇒ + (-1) / (q1*q4)
⇒ + (-1) / (q2*q3)
⇒ + (1) / (q1*q2)
⇒ + (-7) / (q3*q4^2)
⇒ + (1) / (q2*q4^2)
⇒ + (10) / (q1*q4^2)
⇒ + (1) / (q3^2*q4)
⇒ + (-1) / (q2*q3^2)
⇒ + (1) / (q3*q4^3)
⇒ + (-1) / (q2*q4^3)
⇒ + (1) / (q3*q4^4)
⇒ + (-1) / (q2*q4^4)
⇒ + (1) / (q3^2*q4^3)
⇒ where
⇒ q1 = x-1
⇒ q2 = y
⇒ q3 = x
⇒ q4 = x-y
⇒ (14 terms)
⇒
displaypfd_long(dec);
⇒ (2)/((x)*(x-y))
⇒ + (-2)/((x-1)*(x-y))
⇒ + (-6)/((x)*(x-y)^2)
⇒ + (1)/((y)*(x-y)^2)
⇒ + (9)/((x-1)*(x-y)^2)
⇒ + (2)/((x)^2*(x-y))
⇒ (6 terms)
⇒
list fraction = f,g;
// exact test:
checkpfd(fraction,dec);
⇒ 1
checkpfd(fraction,wrong_dec);
⇒ 0
// probabilistic test (evaluation at 10 random points):
checkpfd(fraction,dec,10);

```

```

    ↪ 1
    checkpfd(fraction,wrong_dec,10);
    ↪ 0

```

See also: [Section D.15.13.1 \[pfd\]](#), page 2518.

### D.15.13.3 evaluatepfd

Procedure from library `pfd.lib` (see [Section D.15.13 \[pfd.lib\]](#), page 2518).

**Usage:** `evaluatepfd(dec, values[, mode]);` dec list, values ideal, mode int

**Return:** the number gotten by substituting the numbers generating the ideal `values` for the variables in the partial fraction decomposition `dec`. The list `dec` has to have the same structure as the output of [Section D.15.13.1 \[pfd\]](#), page 2518.  
`mode=1`: raise Error in case of division by 0 (default)  
`mode=2`: return a second integer which is 1 if the denominator becomes 0, and 0 otherwise.

**Example:**

```

LIB "pfd.lib";
ring R = 0,(x,y),dp;
poly f = x+2*y;
poly g = x^2-y^2;
// partial fraction decomposition of f/g:
list dec = pfd(f,g);
↪ (-1/2) / (q2)
↪ + (3/2) / (q1)
↪ where
↪ q1 = x-y
↪ q2 = x+y
↪ (2 terms)
↪
displaypfd_long(dec);
↪ (-1/2)/((x+y))
↪ + (3/2)/((x-y))
↪ (2 terms)
↪
// evaluation at x=2, y=1:
ideal values = 2,1;
evaluatepfd(dec,values);
↪ 4/3
// compare: f(2,1)/g(2,1) = (2+2*1)/(2^2-1^1) = 4/3

```

See also: [Section D.15.13.1 \[pfd\]](#), page 2518.

### D.15.13.4 displaypfd

Procedure from library `pfd.lib` (see [Section D.15.13 \[pfd.lib\]](#), page 2518).

**Usage:** `displaypfd(dec);` dec list

**Purpose:** print a partial fraction decomposition `dec` in a readable way. The list `dec` has to have the same structure as the output of [Section D.15.13.1 \[pfd\]](#), page 2518.

**Example:**

```

LIB "pfd.lib";
ring R = 0,(x,y),dp;
poly f = x^3+3*x^2*y+2*y^2-x^2+4*x*y;
poly g = x^2*y*(x-1)*(x-y)^2;
list dec = pfd(f,g);
 $\mapsto$  (2) / (q3*q4)
 $\mapsto$  + (-2) / (q1*q4)
 $\mapsto$  + (-6) / (q3*q4^2)
 $\mapsto$  + (1) / (q2*q4^2)
 $\mapsto$  + (9) / (q1*q4^2)
 $\mapsto$  + (2) / (q3^2*q4)
 $\mapsto$  where
 $\mapsto$  q1 = x-1
 $\mapsto$  q2 = y
 $\mapsto$  q3 = x
 $\mapsto$  q4 = x-y
 $\mapsto$  (6 terms)
 $\mapsto$ 
displaypfd(dec);
 $\mapsto$  (2) / (q3*q4)
 $\mapsto$  + (-2) / (q1*q4)
 $\mapsto$  + (-6) / (q3*q4^2)
 $\mapsto$  + (1) / (q2*q4^2)
 $\mapsto$  + (9) / (q1*q4^2)
 $\mapsto$  + (2) / (q3^2*q4)
 $\mapsto$  where
 $\mapsto$  q1 = x-1
 $\mapsto$  q2 = y
 $\mapsto$  q3 = x
 $\mapsto$  q4 = x-y
 $\mapsto$  (6 terms)
 $\mapsto$ 

```

See also: [Section D.15.13.5 \[displaypfd\\_long\]](#), page 2525; [Section D.15.13.6 \[getStringpfd\]](#), page 2526; [Section D.15.13.7 \[getStringpfd\\_indexed\]](#), page 2527; [Section D.15.13.1 \[pfd\]](#), page 2518.

### D.15.13.5 displaypfd\_long

Procedure from library `pfd.lib` (see [Section D.15.13 \[pfd.lib\]](#), page 2518).

**Usage:** `displaypfd_long(dec); dec list`

**Purpose:** like [Section D.15.13.4 \[displaypfd\]](#), page 2524, but denominators are written out, not indexed.

**Example:**

```

LIB "pfd.lib";
ring R = 0,(x,y),dp;
poly f = x^3+3*x^2*y+2*y^2-x^2+4*x*y;
poly g = x^2*y*(x-1)*(x-y)^2;
list dec = pfd(f,g);
 $\mapsto$  (2) / (q3*q4)
 $\mapsto$  + (-2) / (q1*q4)
 $\mapsto$  + (-6) / (q3*q4^2)
 $\mapsto$  + (1) / (q2*q4^2)

```

```

⇒ + (9) / (q1*q4^2)
⇒ + (2) / (q3^2*q4)
⇒ where
⇒ q1 = x-1
⇒ q2 = y
⇒ q3 = x
⇒ q4 = x-y
⇒ (6 terms)
⇒
displaypfd_long(dec);
⇒ (2)/((x)*(x-y))
⇒ + (-2)/((x-1)*(x-y))
⇒ + (-6)/((x)*(x-y)^2)
⇒ + (1)/((y)*(x-y)^2)
⇒ + (9)/((x-1)*(x-y)^2)
⇒ + (2)/((x)^2*(x-y))
⇒ (6 terms)
⇒

```

See also: [Section D.15.13.4 \[displaypfd\]](#), page 2524; [Section D.15.13.6 \[getStringpfd\]](#), page 2526; [Section D.15.13.7 \[getStringpfd\\_indexed\]](#), page 2527; [Section D.15.13.1 \[pfd\]](#), page 2518.

### D.15.13.6 getStringpfd

Procedure from library `pfd.lib` (see [Section D.15.13 \[pfd.lib\]](#), page 2518).

**Usage:**        `getStringpfd(dec); dec list`

**Purpose:**      turn a partial fraction decomposition `dec` into one string. The list `dec` has to have the same structure as the output of [Section D.15.13.1 \[pfd\]](#), page 2518.

**Example:**

```

LIB "pfd.lib";
ring R = 0,(x,y),dp;
poly f = x^3+3*x^2*y+2*y^2-x^2+4*x*y;
poly g = x^2*y*(x-1)*(x-y)^2;
list dec = pfd(f,g);
⇒ (2) / (q3*q4)
⇒ + (-2) / (q1*q4)
⇒ + (-6) / (q3*q4^2)
⇒ + (1) / (q2*q4^2)
⇒ + (9) / (q1*q4^2)
⇒ + (2) / (q3^2*q4)
⇒ where
⇒ q1 = x-1
⇒ q2 = y
⇒ q3 = x
⇒ q4 = x-y
⇒ (6 terms)
⇒
displaypfd_long(dec);
⇒ (2)/((x)*(x-y))
⇒ + (-2)/((x-1)*(x-y))
⇒ + (-6)/((x)*(x-y)^2)
⇒ + (1)/((y)*(x-y)^2)

```

```

 $\mapsto + (9)/((x-1)*(x-y)^2)$ 
 $\mapsto + (2)/((x)^2*(x-y))$ 
 $\mapsto (6 \text{ terms})$ 
 $\mapsto$ 
getStringpfd(dec);
 $\mapsto (2)/((x)*(x-y)) + (-2)/((x-1)*(x-y)) + (-6)/((x)*(x-y)^2) + (1)/((y)*(x-y)^2) + (9)/((x-1)*(x-y)^2) + (2)/((x)^2*(x-y))$ 

```

See also: [Section D.15.13.4 \[displaypfd\]](#), page 2524; [Section D.15.13.5 \[displaypfd\\_long\]](#), page 2525; [Section D.15.13.7 \[getStringpfd\\_indexed\]](#), page 2527; [Section D.15.13.1 \[pfd\]](#), page 2518.

### D.15.13.7 getStringpfd\_indexed

Procedure from library pfd.lib (see [Section D.15.13 \[pfd\\_lib\]](#), page 2518).

**Usage:**      getStringpfd\_indexed(dec); dec list

**Purpose:**     turn a partial fraction decomposition **dec** into one string, writing the denominator factors just as q1,q2,... . The list **dec** has to have the same structure as the output of [Section D.15.13.1 \[pfd\]](#), page 2518.

**Example:**

```

LIB "pfd.lib";
ring R = 0,(x,y),dp;
poly f = x^3+3*x^2*y+2*y^2-x^2+4*x*y;
poly g = x^2*y*(x-1)*(x-y)^2;
list dec = pfd(f,g);
 $\mapsto (2) / (q3*q4)$ 
 $\mapsto + (-2) / (q1*q4)$ 
 $\mapsto + (-6) / (q3*q4^2)$ 
 $\mapsto + (1) / (q2*q4^2)$ 
 $\mapsto + (9) / (q1*q4^2)$ 
 $\mapsto + (2) / (q3^2*q4)$ 
 $\mapsto$  where
 $\mapsto q1 = x-1$ 
 $\mapsto q2 = y$ 
 $\mapsto q3 = x$ 
 $\mapsto q4 = x-y$ 
 $\mapsto (6 \text{ terms})$ 
 $\mapsto$ 
displaypfd(dec);
 $\mapsto (2) / (q3*q4)$ 
 $\mapsto + (-2) / (q1*q4)$ 
 $\mapsto + (-6) / (q3*q4^2)$ 
 $\mapsto + (1) / (q2*q4^2)$ 
 $\mapsto + (9) / (q1*q4^2)$ 
 $\mapsto + (2) / (q3^2*q4)$ 
 $\mapsto$  where
 $\mapsto q1 = x-1$ 
 $\mapsto q2 = y$ 
 $\mapsto q3 = x$ 
 $\mapsto q4 = x-y$ 
 $\mapsto (6 \text{ terms})$ 
 $\mapsto$ 
getStringpfd_indexed(dec);

```

$$\mapsto (2)/(q_3*q_4) + (-2)/(q_1*q_4) + (-6)/(q_3*q_4^2) + (1)/(q_2*q_4^2) + (9)/(q_1*q_4^2) + (2)/(q_3^2*q_4)$$

See also: [Section D.15.13.4 \[displaypfd\]](#), page 2524; [Section D.15.13.5 \[displaypfd\\_long\]](#), page 2525; [Section D.15.13.6 \[getStringpfd\]](#), page 2526; [Section D.15.13.1 \[pfd\]](#), page 2518.

### D.15.13.8 readInputTXT

Procedure from library `pfd.lib` (see [Section D.15.13 \[pfd.lib\]](#), page 2518).

**Usage:** `readInputTXT(file[, mode])`, file string, mode int  
`readInputTXT(filelist[, mode])`, filelist list, mode int

**Purpose:** read matrix of rational functions from a txt-file and turn it into a matrix (i.e. a list of lists) of pairs of polynomials (numerators and denominators). The string `file` should be the [directory +] name of the file in the form "`<path-to-file>/<filename>.txt`". The input file should be a list of lists separated by the characters "{", "}" and ",". Example:

```
"{{(x+y)/(x^2-x*y), -(x^2*y+1)/(y), x^2}, {(x+1)/y, y/x, 0}}"
```

Each rational function has to be an expression of the form "`a`", "`(a)/(b)`", "`(b)^(-n)`" or "`(a)*(b)^(-n)`" where "`a`", "`b`" stand for polynomials (i.e. strings, that can be parsed as a polynomial with the `execute` command) and "`n`" stands for a positive integer. A minus sign "-" followed by such an expression is also allowed.

IMPORTANT: The strings "`a`", "`b`" must NOT contain the symbol "/". (So in case the coefficient field is the rationals, all denominators in the coefficients of numerator and denominator polynomials should be cleared.)

The file should contain less than  $2^{31}$  characters (filesize < 2 GB). For bigger files the matrix should be split row-wise into multiple matrices and saved in different files (each smaller than 2 GB). A list of the filenames (in the right order) can then be given as first argument instead.

Also the basering has to match the variable names used in the input file(s).

`mode=1` (default): save result to an ssi-file of the same name

`mode=2`: return result

`mode=3`: save to ssi-file AND return result

See also: [Section D.15.13.9 \[pfdMat\]](#), page 2528.

### D.15.13.9 pfdMat

Procedure from library `pfd.lib` (see [Section D.15.13 \[pfd.lib\]](#), page 2518).

**Usage:** `pfdMat(file[, dotest, ignore_nonlin, output_mode, parallelize])`; file string, dotest, ignore\_nonlin, output\_mode, parallelize int

**Purpose:** apply `pfd` to all entries of a matrix of rational functions saved in a txt-file. The string `file` should be the [directory +] name of the file.

The input file can either be a txt-file or an ssi-file created with `readInputTXT`. In case of a txt-file, the base ring has to match and the matrix has to be in the same format specified in [Section D.15.13.8 \[readInputTXT\]](#), page 2528. Also, txt-files that are bigger than 2 GB should be split as described for `readInputTXT` and a list of the filenames can be given as first argument instead.

The result is saved in multiple txt- (and ssi-) files (see below) within the directory of the input file.

Also a logfile is created, which protocols the memory used and the runtimes of `pfd` for each matrix entry in real-time.

There are also 4 optional arguments:

If `dotest` is nonzero, test the results with `checkpfd`:

`dotest<0` (default): exact test (may be slow),

`dotest>0`: do this amount of probabilistic tests for each entry (see [Section D.15.13.2 \[checkpfd\]](#), page 2522).

If `ignore_nonlin` is nonzero (default), for each denominator, the nonlinear factors in the factorization are removed before applying `pfd` (and added back in in the output files).

If `parallelize` is nonzero (default), the decompositions are calculated in parallel using [Section D.2.7 \[parallel\\_lib\]](#), page 885.

The parameter `output_mode` controls the output files created:

`output_mode=1` (default): The result consists of two files: `<filename>_pfd_indexed.txt` contains the matrix of all decompositions (as list of lists separated by the characters "{", "}" and ",") where all the denominators are written in factorized form depending on irreducible factors `q1, q2, ...`. The file `<filename>_denominator_factors.txt` lists all the polynomials `q1, q2, ...`.

`output_mode=2`: Additionally to mode 1, the file `<filename>_pfd.txt` is created, which also contains the matrix of decompositions but the factors in the denominators are written out.

`output_mode=3`: Additionally to mode 2, the result and some intermediate results are saved as SINGULAR objects in ssi-files:

`<filename>.ssi`: contains the result of `readInputTXT` in case a txt-file was given as input.

`<filename>_factorized_denominators.ssi`: like the first file, but the denominators are saved in factorized form, that is as a list of an ideal of irreducible non constant polynomials and an intvec of exponents.

`<filename>_linear_part.ssi` (only if `ignore_nonlin` is nonzero): like the previous file, but all the irreducible denominator factors are removed

`<filename>_non_linear_factors.ssi` (only if `ignore_nonlin` is nonzero): a list of an ideal `p` generated by irreducible polynomials and a matrix (list of lists) of the nonlinear denominator factors of each entry of the input matrix. These are represented as lists of an intvec of indices `i` for which `p[i]` occurs as a (nonlinear) factor in the denominator and an intvec containing the exponents of those factors.

`<filename>_pfd.ssi`: a list, where the first entry is an ideal `q` of denominator factors and the second entry is a matrix (as list of lists) containing the decompositions, each of which is a list of terms, where a term is represented as in the result of [Section D.15.13.1 \[pfd\]](#), page 2518 by a list containing

1) the numerator polynomial

2) an intvec of indices `i` for which `q[i]` occurs as a factor in the denominator

3) an intvec containing the exponents of those irreducible factors.

IMPORTANT: If `ignore_nonlin` is nonzero, this file contains the decompositions of the entries of the matrix in `<filename>_linear_part.ssi`. Thus the nonlinear factors, are NOT contained in this file.

`output_mode=4`: Additionally to mode 3, the direct output of each call of `pfd` is saved in separate ssi-files called `pfd_results_i_j.ssi` where `i,j` are the matrix indices. This creates a lot of files, but may be useful in case the algorithm does not terminate

in time for every matrix entry. Other than the files created in mode 1-3, these files are saved in the current directory, rather than the directory of the input file.

See also: [Section D.15.13.2 \[checkpfd\]](#), page 2522; [Section D.15.13.10 \[checkpfdMat\]](#), page 2530; [Section D.15.13.1 \[pfd\]](#), page 2518; [Section D.15.13.8 \[readInputTXT\]](#), page 2528.

### D.15.13.10 checkpfdMat

Procedure from library `pfd.lib` (see [Section D.15.13 \[pfd.lib\]](#), page 2518).

**Usage:** `checkpfdMat(input, output, denomFactors[, N, parallelize]);` `input, output, denomFactors` string, `N, parallelize` int

**Purpose:** test the output files of `pfdMat` for correctness. Input and output (indexed) txt-files have to be given as strings in the form "`<path-to-file>/<filename>.txt`". The output should be indexed (that is the output file ending in `..._pfd_indexed.txt`) and `denomFactors` has to be the file containing the denominator factors `q1, q2, ...` (the txt-file ending in `..._denominator_factors.txt`).

As for `readInputTXT` and `pfdMat`, the basering has to match the variable names used in the input file, which has to be in the same format specified in [Section D.15.13.8 \[readInputTXT\]](#), page 2528. Also, files bigger than 2 GB have to be split as described for `readInputTXT` and a list of filenames can be given as first argument instead.

If a positive integer `N` is given, the test is done probabilistically by evaluation at `N` random points for each entry of the matrix. If `N` is nonpositive (default), the fractions in the decompositions will be expanded symbolically and compared to the input (may be slower).

If `parallelize` is nonzero (default), the tests are run in parallel using [Section D.2.7 \[parallel.lib\]](#), page 885.

The result is printed and as in `pfdMat` a logfile is created showing the results for each matrix entry.

See also: [Section D.15.13.2 \[checkpfd\]](#), page 2522; [Section D.15.13.1 \[pfd\]](#), page 2518; [Section D.15.13.9 \[pfdMat\]](#), page 2528; [Section D.15.13.8 \[readInputTXT\]](#), page 2528.

### D.15.14 polyclass\_lib

**Library:** `polyclass.lib`

**Purpose:** Data types for normal form equations

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**Overview:** This library implements a ring independent polynomial type used for the return value in `classify2.lib` and `realclassify.lib`. You can use `+`, `*` and `==` for addition, multiplication and comparison. The key over contains the base ring of the polynomial, the key value its value as a polynomial of type `poly`. The constructor can be called by assigning a polynomial of type `poly` to a polynomial of type `Poly` via `=`.

Moreover the library implements a class `NormalformEquation` consisting out of a string type, an integer `milnorNumber`, a `Poly normalFormEquation`, and integer modality, a list of numbers parameters, a list variables, an integer `corank`, in the real case, an integer `inertiaIndex`, a list of open intervals represented as lists consisting out of two rationals used to select a real root of the minimal polynomial (which is stored in



the variable `minpoly` of the polynomial ring containing `normalFormEquation`, that is, in `normalFormEquation.in`), or if no minimal polynomial is defined then an interval containing the rational parameter value.

Acknowledgements: This research was supported by the Staff Exchange Bursary Programme of the University of Pretoria, DFG SPP 1489, DFG TRR 195. The financial assistance of the National Research Foundation (NRF), South Africa, towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at are those of the author and are not necessarily to be attributed to the National Research Foundation, South Africa.

**Procedures:** See also: [Section D.6.5 \[classify2\\_lib\], page 1655](#); [Section D.6.19 \[realclassify\\_lib\], page 1746](#).

### D.15.14.1 makePoly

Procedure from library `polyclass.lib` (see [Section D.15.14 \[polyclass\\_lib\], page 2530](#)).

**Usage:** `makePoly(f); f poly`

**Return:** make a ring independent Poly from a poly in the basering

**Example:**

```
LIB "polyclass.lib";
ring R=0,(x,y),dp;
Poly f=3*x^2+x*y+1;
Poly g=2*x+y^3;
f*g;
⇒ 3x2y3+xy4+6x3+2x2y+y3+2x
⇒
f+g;
⇒ y3+3x2+xy+2x+1
⇒
f^3;
⇒ 27x6+27x5y+9x4y2+x3y3+27x4+18x3y+3x2y2+9x2+3xy+1
⇒
```

### D.15.14.2 printPoly

Procedure from library `polyclass.lib` (see [Section D.15.14 \[polyclass\\_lib\], page 2530](#)).

### D.15.14.3 printNormalFormEquation

Procedure from library `polyclass.lib` (see [Section D.15.14 \[polyclass\\_lib\], page 2530](#)).

**Usage:** `printNormalFormEquation(F); F NormalFormEquation`

**Return:** print a normal form equation

**Example:**

```
LIB "polyclass.lib";
ring R=(0,a),(x,y,z,w),ds;
minpoly = a^2-2;
Poly f=x^4+x^2*y^2+a*y^8+z^2-w^2;
NormalFormEquation F;
F.vars = ringlist(R)[2];
```

```

F.realCase = 1;
F.normalFormEquation = f;
F.modality = 1;
F.corank = 2;
F.inertiaIndex = 1;
F.determinacy = 8;
F.milnorNumber = milnor(f.value);
F.parameters = list(list(a*y^8,list(0,2)));
F.singularityType = "X[13]";
F;
⇒ Corank = 2
⇒ Inertia index = 1
⇒ Normalform equation of type = X[13]
⇒ Normalform equation = z2-w2+x4+x2y2+(a)*y8
⇒ Milnor number = 13
⇒ Modality = 1
⇒ Parameter term = (a)*y8
⇒ Minimal polynomial = (a2-2)
⇒ Interval = [0, 2]
⇒ Determinacy <= 8
⇒
ring R=(0),(x,y,z,w),ds;
⇒ // ** redefining R (ring R=(0),(x,y,z,w),ds;) ./examples/printNormalFormE\
    quation.sing:17
Poly f=x^4+x^2*y^2+7*y^8+z^2-w^2;
⇒ // ** redefining f (Poly f=x^4+x^2*y^2+7*y^8+z^2-w^2;) ./examples/printNo\
    rmalFormEquation.sing:18
NormalFormEquation F;
⇒ // ** redefining F (NormalFormEquation F;) ./examples/printNormalFormEqua\
    tion.sing:19
F.vars = ringlist(R)[2];
F.realCase = 1;
F.normalFormEquation = f;
F.modality = 1;
F.corank = 2;
F.inertiaIndex = 1;
F.determinacy = 8;
F.milnorNumber = milnor(f.value);
F.parameters = list(list(7*y^8,list(-6,8)));
F.singularityType = "X[13]";
F;
⇒ Corank = 2
⇒ Inertia index = 1
⇒ Normalform equation of type = X[13]
⇒ Normalform equation = z2-w2+x4+x2y2+7y8
⇒ Milnor number = 13
⇒ Modality = 1
⇒ Parameter term = 7y8
⇒ Determinacy <= 8
⇒

```

### D.15.15 puseuxexpansions\_lib

**Library:** `puiseuxexpansion.lib`

**Purpose:** Puiseux expansions over algebraic extensions

**Authors:** J. Boehm, j.boehm at mx.uni-saarland.de  
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**Overview:** This library implements the Newton-Puiseux algorithm to compute Puiseux expansions and provides a class and procedures to work with Puiseux expansions in  $K[X][Y]$ , where  $K$  is the field  $\mathbb{Q}$  of rational numbers or an algebraic extension of  $\mathbb{Q}$ .

**Procedures:**

### D.15.15.1 `puiseuxList`

Procedure from library `puiseuxexpansions.lib` (see [Section D.15.15 \[puiseuxexpansions.lib\]](#), [page 2532](#)).

**Usage:** `puiseuxList(PP, maxDeg, iVarX, iVarY)`; Puiseux expansion PP, int maxDeg is the integer up to which the Puiseux expansions will be computed (if maxDeg = -1 computes the singular part),  
 int iVarX is the index of the X variable for the ring  $C\{X\}[Y]$ , int iVarY, the index of the Y variable for the ring  $C\{X\}[Y]$ .

**Assume:** basering has exactly two variables;  
 f is convenient, that is,  $f(x,0) \neq 0 \neq f(0,y)$ .

**Return:** a list with the Puiseux expansions of PP.

**Example:**

```
LIB "puiseuxexpansions.lib";
ring r=0,(x,y),dp;
Puisseux PP = x5+2x3y-x2y2+3xy5+y6+y7;
puiseuxList(PP, 5, 1, 2);
⇒ [1]:
⇒ ( 6326x5+634x4+71x3+10x2+3x-1 ) / 1
⇒ Denominator of exponent : 1
⇒
⇒ [2]:
⇒ ( -1876641/256x10-2694181/32768x9-38313/64x8+27513/1024x7-909/16x6+751\
  /128x5-27/4x4+9/8x3-3/2x2+x ) / 1
⇒ Denominator of exponent : 2
⇒
⇒ [3]:
⇒ ( -1876641/256x10+2694181/32768x9-38313/64x8-27513/1024x7-909/16x6-751\
  /128x5-27/4x4-9/8x3-3/2x2-x ) / 1
⇒ Denominator of exponent : 2
⇒
⇒ [4]:
⇒ ( -746719/256*x^10+(-12549/16*@a)*x^9+13801/64*x^8+(989/16*@a)*x^7-299\
  /16*x^6+(-45/8*@a)*x^5+7/4*x^4+(@a)*x^3-x^2+(@a)*x ) / 1
⇒ Denominator of exponent : 2
⇒ Minimal polynomial: (@a^2+1)
⇒
```

```

⇒ [5]:
⇒ ( 1813627/128x5+2113/16x4+639/8x3+1/2x2+2x ) / 1
⇒ Denominator of exponent : 1
⇒
⇒ [6]:
⇒ ( 5/128x5-1/16x4+1/8x3-1/2x2 ) / 1
⇒ Denominator of exponent : 1
⇒

```

### D.15.15.2 makePuisseux

Procedure from library `puisseuxexpansions.lib` (see [Section D.15.15 \[puisseuxexpansions.lib\]](#), [page 2532](#)).

**Usage:**      `makePuisseux(f, denom, fr);`  $f$  polynomial in two variables,  $\text{denom}$  polynomial in the first variable of the ring,  $\text{int}$   $\text{fr}$

**Return:**    make a ring independent polynomial over Puiseux series

**Example:**

```

LIB "puisseuxexpansions.lib";
ring R=0,(x,y),dp;
poly f=3*x^2+x*y+1;
makePuisseux(f,x^2,3);
⇒ ( 3x2+xy+1 ) / x2
⇒ Denominator of exponent : 3
⇒

```

### D.15.15.3 makePuisseuxFromPoly

Procedure from library `puisseuxexpansions.lib` (see [Section D.15.15 \[puisseuxexpansions.lib\]](#), [page 2532](#)).

**Usage:**      `makePuisseuxFromPoly(f);`  $f$  polynomial in two variables

**Return:**    make a ring independent polynomial over Puiseux series

**Example:**

```

LIB "puisseuxexpansions.lib";
ring R=0,(x,y),dp;
poly f=3*x^2+x+1;
makePuisseuxFromPoly(f);
⇒ ( 3x2+x+1 ) / 1
⇒ Denominator of exponent : 1
⇒

```

### D.15.15.4 printPuisseux

Procedure from library `puisseuxexpansions.lib` (see [Section D.15.15 \[puisseuxexpansions.lib\]](#), [page 2532](#)).

**Usage:**      `printPuisseux(f);`  $f$  Puiseux expansion

**Return:**    prints information for Puiseux elements

**Example:**

```

LIB "puiseuxexpansions.lib";
ring R=0,(x,y),dp;
poly f=3*x^2+x*y+1;
Puisseux F = makePuisseux(f,x^2,3);
printPuisseux(F);
 $\mapsto (3x^2+xy+1) / x^2$ 
 $\mapsto$  Denominator of exponent : 3

```

### D.15.15.5 puiseux

Procedure from library `puiseuxexpansions.lib` (see [Section D.15.15 \[puiseuxexpansions.lib\]](#), [page 2532](#)).

**Usage:** `puiseux(f, maxDeg, atOrigin);`  $f$  polynomial in two variables, `int maxDeg`, `int atOrigin`

**Return:** the Puiseux expansions of  $f$  developed up to degree `maxDeg`. If `atOrigin = 1`, only the expansions passing through the origin will be returned.

**Example:**

```

LIB "puiseuxexpansions.lib";
ring R=0,(x,y),dp;
poly f=y^3 + x^2 + x^8;
puiseux(f,3,0);
 $\mapsto$  [1]:
 $\mapsto$  [1]:
 $\mapsto$   $-1/3x^{20}-x^2$ 
 $\mapsto$  [2]:
 $\mapsto$  3
 $\mapsto$  [6]:
 $\mapsto$  [1]:
 $\mapsto$  1
 $\mapsto$  [2]:
 $\mapsto$  1
 $\mapsto$  [7]:
 $\mapsto$  [1]:
 $\mapsto$  2

```

### D.15.16 ringgb.lib

**Library:** `ringgb.lib`

**Purpose:** Functions for coefficient rings

**Author:** Oliver Wienand, email: [wienand@mathematik.uni-kl.de](mailto:wienand@mathematik.uni-kl.de)

**Procedures:**

#### D.15.16.1 findZeroPoly

Procedure from library `ringgb.lib` (see [Section D.15.16 \[ringgb.lib\]](#), [page 2535](#)).

**Usage:** `findZeroPoly(f);`  $f$  - a polynomial

**Return:** zero polynomial with the same leading term as  $f$  if exists, otherwise 0

**Example:**

```
LIB "ringgb.lib";
ring r = (integer, 65536), (y,x), dp;
poly f = 1024*x^8*y^2+11264*x^8*y+28672*x^8+45056*x^7*y^2+36864*x^7*y+16384*x^7+40960
findZeroPoly(f);
↪ 1024y2x8+36864y2x7+64512yx8+2048y2x6+28672yx7+24576y2x5+63488yx6+50176y2x\
   4+40960yx5+53248y2x3+15360yx4+12288y2x2+12288yx3+16384y2x+53248yx2+49152y\
   x
```

### D.15.16.2 zeroReduce

Procedure from library `ringgb.lib` (see [Section D.15.16 \[ringgb.lib\]](#), page 2535).

**Usage:** `zeroReduce(f, [i = 0]);`  $f$  - a polynomial,  $i$  - noise level (if  $i \neq 0$  prints all steps)

**Return:** reduced normal form of  $f$  modulo zero polynomials

**Example:**

```
LIB "ringgb.lib";
ring r = (integer, 65536), (y,x), dp;
poly f = 1024*x^8*y^2+11264*x^8*y+28672*x^8+45056*x^7*y^2+36864*x^7*y+16384*x^7+40960
zeroReduce(f);
↪ 0
kill r;
ring r = (integer, 2, 32), (x,y,z), dp;
// Polynomial 1:
poly p1 = 3795162112*x^3+587202566*x^2*y+2936012853*x*y*z+2281701376*x+548767119*y^3+
// Polynomial 2:
poly p2 = 1647678464*x^3+587202566*x^2*y+2936012853*x*y*z+134217728*x+548767119*y^3+
zeroReduce(p1-p2);
↪ 0
```

### D.15.16.3 testZero

Procedure from library `ringgb.lib` (see [Section D.15.16 \[ringgb.lib\]](#), page 2535).

**Usage:** `testZero(f);`  $f$  - a polynomial

**Return:** returns 1 if  $f$  is zero as a function and otherwise a counterexample as a list  $[f(x_1, \dots, x_n), x_1, \dots, x_n]$

**Example:**

```
LIB "ringgb.lib";
ring r = (integer, 12), (y,x), dp;
poly f = 1024*x^8*y^2+11264*x^8*y+28672*x^8+45056*x^7*y^2+36864*x^7*y+16384*x^7+40960
//zeroReduce(f);
testZero(f);
↪ Teste 144 Belegungen ...
↪ bisher: 11
↪ [1]:
↪ 1
↪ [2]:
↪ 1
↪ [3]:
↪ 4
poly g = findZeroPoly(x2y3);
g;
```

```

    ↪ y3x2+11y3x+9y2x2+3y2x+2yx2+10yx
    testZero(g);
    ↪ Teste 144 Belegungen ...
    ↪ bisher: 11
    ↪ bisher: 23
    ↪ bisher: 35
    ↪ bisher: 47
    ↪ bisher: 59
    ↪ bisher: 71
    ↪ bisher: 83
    ↪ bisher: 95
    ↪ bisher: 107
    ↪ bisher: 119
    ↪ bisher: 131
    ↪ bisher: 143
    ↪ 1

```

#### D.15.16.4 noElements

Procedure from library `ringgb.lib` (see [Section D.15.16 \[ringgb.lib\]](#), page 2535).

**Usage:**      `noElements(r)`; `r` - a ring with a finite coefficient ring of type integer

**Return:**     returns the number of elements of the coefficient ring of `r`

**Example:**

```

LIB "ringgb.lib";
ring r = (integer, 233,6), (y,x), dp;
noElements(r);
↪ 160005726539569

```

#### D.15.17 rwalk.lib

**Library:**    `rwalk.lib`

**Purpose:**     Groebner Walk Conversion Algorithms

**Author:**    Stephan Oberfranz

**Procedures:** See also: [Section D.4.10 \[grwalk.lib\]](#), page 1088; [Section D.15.22 \[swalk.lib\]](#), page 2553.

##### D.15.17.1 prwalk

Procedure from library `rwalk.lib` (see [Section D.15.17 \[rwalk.lib\]](#), page 2537).

**Syntax:**     `rwalk(ideal i, int radius)`;  
               if `size(#)>0` then `rwalk(ideal i, int radius, intvec v, intvec w)`;

**Type:**       ideal

**Purpose:**     compute the standard basis of the ideal, calculated via the Random Perturbation Walk algorithm from the ordering "`a(v),lp`", "`dp`", "`Dp`" or "`M`" to the ordering "`a(w),lp`", "`a(1,0,...,0),lp`" or "`M`".

**Example:**

```

LIB "rwalk.lib";
// compute a Groebner basis of I w.r.t. lp.
ring r = 32003,(z,y,x), lp;
ideal I = y3+xyz+y2z+xz3, 3+xy+x2y+y2z;
int radius = 1;
int o_perturb_deg = 2;
int t_perturb_deg = 2;
prwalk(I,radius,o_perturb_deg,t_perturb_deg);
⇒ _[1]=y9-y7x2-y7x-y6x3-y6x2-3y6-3y5x-y3x7-3y3x6-3y3x5-y3x4-9y2x5-18y2x4-9y\
2x3-27yx3-27yx2-27x
⇒ _[2]=zx+8297y8x2+8297y8x+3556y7-8297y6x4+15409y6x3-8297y6x2-8297y5x5+1540\
9y5x4-8297y5x3+3556y5x2+3556y5x+3556y4x3+3556y4x2-10668y4-10668y3x-8297y2\
x9-1185y2x8+14224y2x7-1185y2x6-8297y2x5-14223yx7-10666yx6-10666yx5-14223y\
x4+x5+2x4+x3
⇒ _[3]=zy2+yx2+yx+3

```

See also: [Section D.4.10.3 \[awalk1\]](#), page 1089; [Section D.4.10.4 \[awalk2\]](#), page 1090; [Section D.4.10.1 \[fwalk\]](#), page 1088; [Section 5.1.53 \[groebner\]](#), page 191; [Section D.4.10.6 \[gwalk\]](#), page 1091; [Section D.4.10.5 \[pwalk\]](#), page 1090; [Section 5.1.151 \[std\]](#), page 271; [Section 5.1.152 \[stdfglm\]](#), page 272; [Section D.4.10.2 \[twalk\]](#), page 1089.

### D.15.17.2 rwalk

Procedure from library `rwalk.lib` (see [Section D.15.17 \[rwalk.lib\]](#), page 2537).

**Syntax:** `rwalk(ideal i, int radius);`  
if `size(#)>0` then `rwalk(ideal i, int radius, intvec v, intvec w);` intermediate Groebner bases are not reduced if `reduction = 0`

**Type:** ideal

**Purpose:** compute the standard basis of the ideal, calculated via the Random walk algorithm from the ordering  
`"(a(v),lp)"`, `"dp"`, `"Dp"` or `"M"`  
to the ordering `"(a(w),lp)"`, `"(a(1,0,...,0),lp)"` or `"M"`.

**Example:**

```

LIB "rwalk.lib";
// compute a Groebner basis of I w.r.t. lp.
ring r = 32003,(z,y,x), lp;
ideal I = y3+xyz+y2z+xz3, 3+xy+x2y+y2z;
int radius = 1;
int perturb_deg = 2;
rwalk(I,radius,perturb_deg);
⇒ _[1]=y9-y7x2-y7x-y6x3-y6x2-3y6-3y5x-y3x7-3y3x6-3y3x5-y3x4-9y2x5-18y2x4-9y\
2x3-27yx3-27yx2-27x
⇒ _[2]=zx+8297y8x2+8297y8x+3556y7-8297y6x4+15409y6x3-8297y6x2-8297y5x5+1540\
9y5x4-8297y5x3+3556y5x2+3556y5x+3556y4x3+3556y4x2-10668y4-10668y3x-8297y2\
x9-1185y2x8+14224y2x7-1185y2x6-8297y2x5-14223yx7-10666yx6-10666yx5-14223y\
x4+x5+2x4+x3
⇒ _[3]=zy2+yx2+yx+3

```

See also: [Section D.4.10.3 \[awalk1\]](#), page 1089; [Section D.4.10.4 \[awalk2\]](#), page 1090; [Section D.4.10.1 \[fwalk\]](#), page 1088; [Section 5.1.53 \[groebner\]](#), page 191; [Section D.4.10.6 \[gwalk\]](#), page 1091; [Section D.4.10.5 \[pwalk\]](#), page 1090; [Section 5.1.151 \[std\]](#), page 271; [Section 5.1.152 \[stdfglm\]](#), page 272; [Section D.4.10.2 \[twalk\]](#), page 1089.



### D.15.17.3 frandwalk

Procedure from library `rwalk.lib` (see [Section D.15.17 \[rwalk.lib\]](#), page 2537).

**Syntax:** `frwalk(ideal i, int radius);`  
`frwalk(ideal i, int radius, intvec v, intvec w);`

**Type:** `ideal`

**Purpose:** compute the standard basis of the ideal w.r.t. the lexicographical ordering or a weighted-lex ordering, calculated via the Random Fractal walk algorithm.

**Example:**

```
LIB "rwalk.lib";
ring r = 0,(z,y,x), lp;
ideal I = y3+xyz+y2z+xz3, 3+xy+x2y+y2z;
int reduction = 0;
frandwalk(I,2);
⇒ _[1]=y9-y7x2-y7x-y6x3-y6x2-3y6-3y5x-y3x7-3y3x6-3y3x5-y3x4-9y2x5-18y2x4-9y\
2x3-27yx3-27yx2-27x
⇒ _[2]=zx-2/27y8x2-2/27y8x+1/9y7+2/27y6x4+4/27y6x3+2/27y6x2+2/27y5x5+4/27y5\
x4+2/27y5x3+1/9y5x2+1/9y5x+1/9y4x3+1/9y4x2-1/3y4-1/3y3x+2/27y2x9+8/27y2x8\
+4/9y2x7+8/27y2x6+2/27y2x5+5/9yx7+5/3yx6+5/3yx5+5/9yx4+x5+2x4+x3
⇒ _[3]=zy2+yx2+yx+3
```

See also: [Section D.4.10.3 \[awalk1\]](#), page 1089; [Section D.4.10.4 \[awalk2\]](#), page 1090; [Section 5.1.53 \[groebner\]](#), page 191; [Section D.4.10.6 \[gwalk\]](#), page 1091; [Section D.4.10.5 \[pwalk\]](#), page 1090; [Section 5.1.151 \[std\]](#), page 271; [Section 5.1.152 \[stdfglm\]](#), page 272; [Section D.4.10.2 \[twalk\]](#), page 1089.

### D.15.18 sagbigrob.lib

**Library:** `sagbigrob.lib`

**Purpose:** Compute Sagbi-Groebner basis of an ideal of a subalgebra

**Authors:** Nazish Kanwal, Lecturer of Mathematics, School of Mathematics and Computer Science, Institute of Business Administration, Karachi, Pakistan  
 Junaid Alam Khan, Associate Professor of Mathematics, School of Mathematics and Computer Science, Institute of Business Administration,  
 Karachi, Pakistan

**Procedures:** See also: [Section D.4.32 \[sagbi.lib\]](#), page 1377.

#### D.15.18.1 LTGS

Procedure from library `sagbigrob.lib` (see [Section D.15.18 \[sagbigrob.lib\]](#), page 2539).

**Usage:** `LTGS(I,A);` I ideal of subalgebra A, A subalgebra (which is a finite sagbi basis).

**Return:** a module M.

**Example:**

```
LIB "sagbigrob.lib";
// Example 1:
ring r=ZZ,(x,y),Dp;
ideal A=2x2+xy,2y2,3xy;
ideal I=4x2y2+2xy3,18x2y4,36xy5;
```

```

LTGS(I,A);
⇨ _[1]=9y2*gen(1)-2*gen(2)
⇨ _[2]=9y4*gen(1)-xy*gen(3)
⇨ _[3]=-xy*gen(3)+2y2*gen(2)
⇨ _[4]=xy3*gen(1)
⇨ _[5]=-18xy3*gen(1)+2x2*gen(3)+xy*gen(3)
// Example 2:
ring r2=QQ,(x,y),Dp;
ideal A=x2,xy;
ideal I=x3y+x2,x4+x2y2,-x3y3-x2y2;
LTGS(I,A);
⇨ _[1]=x2*gen(1)-xy*gen(2)
⇨ _[2]=-x2y2*gen(1)-x2*gen(3)
⇨ _[3]=-x3y3*gen(2)-x4*gen(3)

```

### D.15.18.2 SGNF

Procedure from library `sagbigrob.lib` (see [Section D.15.18 \[sagbigrob.lib\]](#), page 2539).

**Usage:**      `SGNF(f,I,A)`;  $f$  polynomial,  $I$  ideal of subalgebra  
                   $A$ ,  $A$  ideal (which is a finite Sagbi bases).

**Return:**     a polynomial  $h$ : the normalform of  $f$  wrt.  $I$  in  $A$

**Example:**

```

LIB "sagbigrob.lib";
ring r=0,(x,y),Dp;
ideal A=x2,xy;
ideal I=x3y+x2,x4+x2y2,-x3y3-x2y2;
poly f=x6y6+x2;
poly g=x12+x6y6-x8+x5y;
SGNF(f,I,A);
⇨ x2y2+x2
SGNF(g,I,A);
⇨ 0

```

### D.15.18.3 SPOLY

Procedure from library `sagbigrob.lib` (see [Section D.15.18 \[sagbigrob.lib\]](#), page 2539).

**Usage:**      `SPOLY(I,A)`;  $I$  ideal of subalgebra  $A$ ,  $A$  subalgebra (which is a finite sagbi basis).

**Return:**     an ideal  $S$ :  $S$ -polynomials of ideal  $I$

**Example:**

```

LIB "sagbigrob.lib";
// Example 1:
ring r=ZZ,(x,y),Dp;
ideal A=2x2+xy,2y2,3xy;
ideal I=4x2y2+2xy3,18x2y4;
SPOLY(I,A);
⇨ _[1]=18xy5
⇨ _[2]=4x3y5+2x2y6
// Example 2:
ring r2=QQ,(x,y),Dp;
ideal A=x2,xy;

```

```

ideal I=x3y+x2,x4+x2y2,-x3y3-x2y2;
SPOLY(I,A);
↪ _[1]=-x3y3+x4
↪ _[2]=-x5y5+x6y2

```

#### D.15.18.4 SGB

Procedure from library `sagbigrob.lib` (see [Section D.15.18 \[sagbigrob.lib\]](#), page 2539).

**Usage:**     `SGB(I,A)`; I ideal of subalgebra A, A subalgebra (which is a finite sagbi basis).

**Return:**    an ideal SB.

**Example:**

```

LIB "sagbigrob.lib";
// Example 1:
ring r=ZZ,(x,y),Dp;
ideal A=2x2+xy,2y2,3xy;
ideal I=4x2y2+2xy3,18x2y4;
SGB(I,A);
↪ 18xy5
↪ 0
↪ 0
↪ -2x2y6+8xy7
↪ 0
↪ 0
↪ 0
↪ _[1]=4x2y2+2xy3
↪ _[2]=18x2y4
↪ _[3]=18xy5
↪ _[4]=-2x2y6+8xy7
// Example 2:
ring r2=QQ,(w,x,y,z),lp;
ideal A=wxy+2z2, y2-4z, x+3y;
ideal I= wxy-y2+2z2+4z, x+y2+3y-4z, x2+6xy+9y2;
SGB(I,A);
↪ 0
↪ -y6+12y4z-48y2z2+64z3
↪ 0
↪ _[1]=wxy-y2+2z2+4z
↪ _[2]=x+y2+3y-4z
↪ _[3]=x2+6xy+9y2
↪ _[4]=-y6+12y4z-48y2z2+64z3

```

#### D.15.19 sagbiNormaliz.lib

**Library:**    `sagbiNormaliz.lib`

**Purpose:**     Provides an interface for the computation of Sagbi bases. It uses `normaliz.lib` (version 4... or higher) for combinatorial computations.

**Authors:**    Winfried Bruns, [wbruns@uos.de](mailto:wbruns@uos.de)

**Overview:**   The library `sagbiNormaliz.lib` provides functions for the computations of Sagbi bases. It is based on `normaliz.lib`. Its functions compute Sagbi bases with or without the

control by Hilbert functions and/orv degrees. Hilbert functions and degrees require that the ambient polynomial ring is standard graded. (An extension to general positive gradings would not be difficult.)

Use of this library requires the program Normaliz to be installed and the availability of normaliz.lib. You can download both from [@uref{https://github.com/Normaliz/Normaliz/releases}](https://github.com/Normaliz/Normaliz/releases). Please make sure that the executable is in the search path or use setNmzExecPath (defined in normaliz.lib).

The computations of this library require reading Normaliz output files and therefore a file name must be set. The standard file name chosen by the library is NmzSagbiExchange in the current directory. The user can overwrite it by setting another name.

## Procedures:

### D.15.19.1 sagbiGeneral

Procedure from library `sagbiNormaliz.lib` (see [Section D.15.19 \[sagbiNormaliz.lib\]](#), page 2541).

**Usage:** `sagbiGeneral(ideal Q [, int sagbiMaxRounds, int sorting, int verb]);`

**Return:** An ideal whose generators form the Sagbi basis of the algebra generated by the polynomials in  $Q$  as far as computed in at most `sagbiMaxRounds`.

If sorting is set, the computed elements are degrevlex sorted before a round of the algorithm. The optional parameter `verb` sets the terminal output. Default is 1 = on.

The component has a second integer component. Its possible values are: 0, if the full Sagbi basis has not been reached, 1 if this is unknown, and 2 if the full Sagbi basis has been computed.

## Example:

```
LIB "sagbiNormaliz.lib";
ring R = 0, (x,y,z), dp;
ideal P = x^6+y^6+z^6, x^7+y^7+z^7;
ideal Q;
int success;
// at most 3 rounds, no sorting, no terminal output
(Q,success) = sagbiGeneral(P,3,0,0);
Q;
⇒ Q[1]=x6+y6+z6
⇒ Q[2]=x7+y7+z7
⇒ Q[3]=x36y6-6/7x35y7+3x30y12-15/7x28y14+5x24y18-20/7x21y21+5x18y24-15/7x14\
y28+3x12y30-6/7x7y35+x6y36+x36z6+6x30y6z6+15x24y12z6+20x18y18z6+15x12y24z\
6+6x6y30z6+y36z6-6/7x35z7-30/7x28y7z7-60/7x21y14z7-60/7x14y21z7-30/7x7y28\
z7-6/7y35z7+3x30z12+15x24y6z12+30x18y12z12+30x12y18z12+15x6y24z12+3y30z12\
-15/7x28z14-60/7x21y7z14-90/7x14y14z14-60/7x7y21z14-15/7y28z14+5x24z18+20\
x18y6z18+30x12y12z18+20x6y18z18+5y24z18-20/7x21z21-60/7x14y7z21-60/7x7y14\
z21-20/7y21z21+5x18z24+15x12y6z24+15x6y12z24+5y18z24-15/7x14z28-30/7x7y7z\
28-15/7y14z28+3x12z30+6x6y6z30+3y12z30-6/7x7z35-6/7y7z35+x6z36+y6z36
"Note: success = 2 <==> Sagbi basi complete";
⇒ Note: success = 2 <==> Sagbi basi complete
"success",success;
⇒ success 2
```

### D.15.19.2 sagbiByDegree

Procedure from library `sagbiNormaliz.lib` (see [Section D.15.19 \[sagbiNormaliz.lib\], page 2541](#)).

**Usage:** `sagbiByDegree(ideal Q, int sagbiDegreeBound [,int sorting, int verb]);`

**Return:** RETURN: An ideal whose generators form the Sagbi basis of the algebra generated by the polynomials `inin` `degrwees` `<= Sagbi_degree_bound`.

If sorting is set, the computed elements are degrevlex sorted before a round of the algorithm. The optional parameter `verb` sets the terminal output. Fefault is 1 = on.

The component has a second integer component. Its possible values are: 0, if the full Sagbi basis has not been reached, 1 if this is unknown, and 2 if the full Sagbi basis has been computed.

**Example:**

```
LIB "sagbiNormaliz.lib";
ring R = 0, (x,y,z),dp;
ideal P = x^6+y^6+z^6, x^7+y^7+z^7, x^8+y^8+z^8;
ideal Q;
int success;
// degree bound 40, no sorting, no terminal output
(Q,success) = sagbiByDegree(P,40,0,0);
lead(Q);
⇒ _[1]=x6
⇒ _[2]=x7
⇒ _[3]=x8
⇒ _[4]=x8y6
⇒ _[5]=x18y6
⇒ _[6]=x23y7
⇒ _[7]=x25y7
>Note: success = 1 <==> Unknown whether complete";
⇒ Note: success = 1 <==> Unknown whether complete
"success",success;
⇒ success 1
```

### D.15.19.3 sagbiHilbControlled

Procedure from library `sagbiNormaliz.lib` (see [Section D.15.19 \[sagbiNormaliz.lib\], page 2541](#)).

**Usage:** `sagbiHilbControlled(ideal Q, intvec HS_num_algebra, intvec HS_denom_algebra, int Sagbi_degree_bound [,int finalChweck, int sorting, int verb]);`

**Return:** An ideal whose generators form the Sagbi basis of the algebra generated by the polynomials `inin` `degrwees` `<= Sagbi_degree_bound`. The computation is controlled by the Hilbert series given by the numerator and the denominator.

If sorting is set, the computed elements are degrevlex sorted before a round of the algorithm. The optional parameter `verb` sets the terminal output. Fefault is 1 = on.

The component has a second integer component. Its possible values are: 0, if the full Sagbi basis has not been reached, 1 if this is unknown, and 2 if the full Sagbi basis has been computed.

**Example:**

```
LIB "sagbiNormaliz.lib";
ring R = 0, (x,y,z),dp;
```

```

ideal P = x^6+y^6+z^6, x^7+y^7+z^7, x^8+y^8+z^8;
ideal Q;
intvec HS_num = 1;
intvec HS_denom = 6,7,8;
int success;
// degree bound 40, final check, ,no sorting, no terminal output
(Q,success) = sagbiHilbControlled(P,HS_num, HS_denom,40, 1,0,0);
lead(Q);
↳ _[1]=x6
↳ _[2]=x7
↳ _[3]=x8
↳ _[4]=x8y6
↳ _[5]=x18y6
↳ _[6]=x23y7
↳ _[7]=x25y7
"Note: success = 0 <==> Sagbi basis incomplete";
↳ Note: success = 0 <==> Sagbi basis incomplete
"success",success;
↳ success 0

```

### D.15.20 stanleyreisner\_lib

**Library:** stanleyreisner.lib

**Purpose:** Deformations of Stanley-Reiser ideals

**Authors:**

**Overview:** Firstly, we implement the graded pieces has certain degree of cotangent modules  $T_1$  and  $T_2$  for a general Stanley-Reiser ring. And the graded pieces of homomorphisms are represented by lists of integers.

**Types:** Homomorphism class of homomorphisms

**Procedures:**

#### D.15.20.1 T1

Procedure from library `stanleyreisner.lib` (see [Section D.15.20 \[stanleyreisner\\_lib\]](#), page 2544).

**Usage:**  $T_1(h)$ ;  $h$  = ideal.

**Assume:**  $I$  is the ideal generated by the monomials w.r.t. faces.

**Return:** first order deformation on ideal  $I$ .

**Theory:** The procedure will compute the first order deformation.

**Example:**

```

LIB "stanleyreisner.lib";
ring R=0,(x(1..4)),lp;
ideal i=x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3);
list L=T1(i);
L;
↳ [1]:
↳ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0
↳ phi.source -> phi.over / phi.source

```

```

⇒
⇒ [2]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [3]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [4]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [5]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [6]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [7]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [8]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [9]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [10]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [11]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [12]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [13]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 1
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [14]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 1
⇒ phi.source -> phi.over / phi.source
⇒

```

```

⇒ [15]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0,0
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [16]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0,0
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [17]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0,0
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [18]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0,0
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [19]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0,0
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [20]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0,0
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [21]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0,0
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [22]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0,0
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [23]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0,0
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [24]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0,0
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [25]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0,0
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [26]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0,0
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [27]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 1
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [28]:

```



```

⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 1
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [29]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0,0
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [30]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0,0
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [31]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0,0
⇒ phi.source -> phi.over / phi.source
⇒
⇒ [32]:
⇒ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3) -> 0,0
⇒ phi.source -> phi.over / phi.source
⇒

```

### D.15.20.2 T2

Procedure from library `stanleyreisner.lib` (see [Section D.15.20 \[stanleyreisner.lib\]](#), page 2544).

**Usage:** T2(h); h = ideal.

**Assume:** I is the ideal generated by the monomials w.r.t. faces.

**Return:** second order deformation on ideal I.

**Theory:** The procedure will compute the second order deformation.

**Example:**

```

LIB "stanleyreisner.lib";
ring R=0,(x(1..4)),lp;
ideal i=x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3);
list L=T2(i);
L;
⇒ [1]:
⇒ phi.relations -> phi.over/phi.source
⇒
⇒ [2]:
⇒ phi.relations -> phi.over/phi.source
⇒
⇒ [3]:
⇒ phi.relations -> phi.over/phi.source
⇒
⇒ [4]:
⇒ phi.relations -> phi.over/phi.source
⇒
⇒ [5]:
⇒ phi.relations -> phi.over/phi.source
⇒
⇒ [6]:
⇒ phi.relations -> phi.over/phi.source

```

```

↳
↳ [7]:
↳   phi.relations -> phi.over/phi.source
↳
↳ [8]:
↳   phi.relations -> phi.over/phi.source
↳
↳ [9]:
↳   phi.relations -> phi.over/phi.source
↳
↳ [10]:
↳   phi.relations -> phi.over/phi.source
↳
↳ [11]:
↳   phi.relations -> phi.over/phi.source
↳
↳ [12]:
↳   phi.relations -> phi.over/phi.source
↳
↳ [13]:
↳   phi.relations -> phi.over/phi.source
↳
↳ [14]:
↳   phi.relations -> phi.over/phi.source
↳
↳ [15]:
↳   phi.relations -> phi.over/phi.source
↳
↳ [16]:
↳   phi.relations -> phi.over/phi.source
↳
↳ [17]:
↳   phi.relations -> phi.over/phi.source
↳
↳ [18]:
↳   phi.relations -> phi.over/phi.source
↳
↳ [19]:
↳   phi.relations -> phi.over/phi.source
↳
↳ [20]:
↳   phi.relations -> phi.over/phi.source
↳
↳ [21]:
↳   phi.relations -> phi.over/phi.source
↳
↳ [22]:
↳   phi.relations -> phi.over/phi.source
↳
↳ [23]:
↳   phi.relations -> phi.over/phi.source
↳
↳ [24]:

```

```

⇒ phi.relations -> phi.over/phi.source
⇒
⇒ [25]:
⇒ phi.relations -> phi.over/phi.source
⇒
⇒ [26]:
⇒ phi.relations -> phi.over/phi.source
⇒
⇒ [27]:
⇒ phi.relations -> phi.over/phi.source
⇒
⇒ [28]:
⇒ phi.relations -> phi.over/phi.source
⇒
⇒ [29]:
⇒ phi.relations -> phi.over/phi.source
⇒
⇒ [30]:
⇒ phi.relations -> phi.over/phi.source
⇒
⇒ [31]:
⇒ phi.relations -> phi.over/phi.source
⇒
⇒ [32]:
⇒ phi.relations -> phi.over/phi.source
⇒

```

### D.15.20.3 makeQPoly

Procedure from library `stanleyreisner.lib` (see [Section D.15.20 \[stanleyreisner.lib\]](#), page 2544).

**Usage:**     `makeQPoly(p)`;  $p$  = polynomial.

**Assume:**    $p$  is a polynomial in basering.

**Return:**    a QPoly which is in a quotient ring.

**Theory:**    The procedure will convert a polynomial in basering to a polynomial in a quotient ring.

**Example:**

```

LIB "stanleyreisner.lib";
ring r=0,(x,y,z),lp;
ideal i=xy;
qring Q=std(i);
poly p=x*y+z;
QPoly q=makeQPoly(p);
q.over;
⇒ // coefficients: QQ
⇒ // number of vars : 3
⇒ //          block 1 : ordering lp
⇒ //          : names  x y z
⇒ //          block 2 : ordering C
⇒ // quotient ring from ideal
⇒ _[1]=xy
q.value;
⇒ z

```

### D.15.20.4 fPiece

Procedure from library `stanleyreisner.lib` (see [Section D.15.20 \[stanleyreisner.lib\]](#), page 2544).

**Usage:** `fPiece(I,a,b)`;  $I$  = ideal,  $a$  = poly,  $b$  = poly.

**Assume:**  $I$  is the ideal generated by the monomials w.r.t. faces,  $a-b$  is the degree of the graded piece.

**Return:** graded piece (of degree  $a-b$ ) of first order deformation on ideal  $I$ .

**Theory:** The procedure will compute the graded pieces of the first order deformation which has degree  $a-b$ .

**Example:**

```
LIB "stanleyreisner.lib";
ring R=0,(x(1..4)),lp;
ideal i=x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3);
FirstOrderDeformation phi=fPiece(i,x(4)^2,x(1)*x(2));
phi;
→ x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3)  ->  1
→ phi.source  ->  phi.over / phi.source
→
```

### D.15.20.5 sPiece

Procedure from library `stanleyreisner.lib` (see [Section D.15.20 \[stanleyreisner.lib\]](#), page 2544).

**Assume:**  $I$  is the ideal generated by the monomials w.r.t. faces,  $a-b$  is the degree of the graded piece.

**Return:** graded piece (of degree  $a-b$ ) of second order deformation on ideal  $I$ .

**Theory:** The procedure will compute the graded pieces of the second order deformation which has degree  $(a-b)$ .

**Example:**

```
LIB "stanleyreisner.lib";
ring R=0,(x(1..4)),lp;
ideal i=x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3);
SecondOrderDeformation psi=sPiece(i,x(4)^2,x(1)*x(2));
psi;
→ phi.relations  ->  phi.over/phi.source
→
```

### D.15.20.6 makeLinks

Procedure from library `stanleyreisner.lib` (see [Section D.15.20 \[stanleyreisner.lib\]](#), page 2544).

**Usage:** `makeLinks(I,p,q)`;  $I$  = ideal,  $p$  = poly,  $p$ =poly.

**Assume:**  $I$  is the ideal generated by the monomials w.r.t. faces.

**Return:** links of  $p$ , graded pieces of first and second order deformation of links on ideal  $I$ .

**Example:**

```

LIB "stanleyreisner.lib";
ring R=0,(x(1..4)),lp;
ideal i=x(1),x(2),x(3),x(4),x(1)*x(4),x(2)*x(4),x(1)*x(3),x(2)*x(3);
LinksDeformation L=makeLinks(i,x(4)^2,x(1)*x(2));
L;
⇒ The links of x(4)^2 is :
⇒ _[1]=x(1)
⇒ _[2]=x(2)
⇒ The first order deformation of the links is :
⇒ x(1),x(2) -> 10
⇒ phi.source -> phi.over / phi.source
⇒
⇒ The second order deformation of the links is :
⇒ phi.relations -> phi.over/phi.source
⇒
⇒

```

### D.15.21 stdmodule.lib

**Library:** stdmodule.lib

**Purpose:** Compute Standard Bases of submodule of free module over polynomial subalgebra w.r.t Natural Grading.

**Authors:** Nazish Kanwal, Assistant Professor of Mathematics, Department of Sciences & Humanities, FAST-National University of Computer & Emerging Sciences-Karachi, Pakistan  
Email: nkanwal@iba.edu.pk, nazish.kanwal@nu.edu.pk

**Procedures:**

**Reference:**

N, Kanwal, N, Jabeen & J. A, Khan (2021) Standard bases for modules over polynomial subalgebras, Communications in Algebra, 49:1, 58-72, DOI: 10.1080/00927872.2020.1793160

See also: [Section D.4.32 \[sagbi.lib\]](#), page 1377.

#### D.15.21.1 MPGS

Procedure from library stdmodule.lib (see [Section D.15.21 \[stdmodule.lib\]](#), page 2551).

**Usage:** G free module over subalgebra A, A subalgebra(which is finite sagbi basis).

**Return:** a list W: generating set for syzygies of G over subalgebra A."

**Example:**

```

LIB "stdmodule.lib";
// Example 1:
ring r=QQ,(x,y),Dp;
ideal A=x2,y2;
module G=[x4+2x2y2-1,y4],[-x2y2,x2y2],[2x2y4-y2,x4y2+y6],[-x2y2,x2y6+x6y2+2x4y4];
MPGS(G,A);
⇒ [1]:
⇒ _[1]=x2*gen(2)-x2*gen(1)+y2*gen(1)-gen(3)+gen(2)
⇒ _[2]=2y4*gen(1)-x2*gen(3)+x2*gen(1)-2y2*gen(3)+2y2*gen(2)-y2*gen(1)+ge\
n(4)-gen(2)

```

```
// Example 2:
ring r2=QQ,(x,y),Dp;
ideal A=x2-xy+y, y2+x;
module G=x2y2-xy3+x3-x2y+y3+x2+y, y4+2xy2+x2-y2-x, 2xy-2x2+2xy-2y;
MPGS(G,A);
↳ [1]:
↳ _[1]=gen(3)-gen(2)
↳ _[2]=x2*gen(2)-x2*gen(1)-xy*gen(2)+xy*gen(1)-y2*gen(1)-x*gen(1)+y*gen(\
2)-y*gen(1)
```

### D.15.21.2 SNF

Procedure from library `stdmodule.lib` (see [Section D.15.21 \[stdmodule.lib\]](#), page 2551).

**Usage:** SNF(f,G,A); f vector, G free module over A, A subalgebra (which is a finite Sagbi bases).

**Return:** a vector g: normal form of a vector g wrt G over subalgebra A.

**Example:**

```
LIB "stdmodule.lib";
ring r=0,(x,y,z),dp;
ideal A=x2-y,xz+z,y2-z;
module G=[x3z+x2z-xyz-yz,y2+xz],[y4-2y2z+z2,xy2z+y2z-xz2-z2],[x2y2-y3-x2z+yz,x4-2x2y-
vector v=[x2y4-y5-2x2y2z+2y3z+x2z2-yz2-xz-z, x4y2-2x2y3-x4z+y4+2x2yz-y2z+2];
vector u=[x2y4-y5-2x2y2z+2x3z+2y3z+x2z2+2x2z-2xyz-yz2-xz-2yz-z, x4y2-2x2y3-x4z+y4+2x
SNF(v,G,A);
↳ -xz*gen(1)-z*gen(1)+2*gen(2)
SNF(u,G,A);
↳ y2*gen(2)-2xz*gen(2)-xz*gen(1)-3z*gen(2)-z*gen(1)+7*gen(2)
```

### D.15.21.3 MSPOLY

Procedure from library `stdmodule.lib` (see [Section D.15.21 \[stdmodule.lib\]](#), page 2551).

**Example:**

```
LIB "stdmodule.lib";
// Example 1:
ring r=QQ,(x,y,z),Dp;
ideal A=x2-yz,xy+x,xz;
module G=[x2yz+x2z-2,xz+xyz2-x3z],[x3y-xy2z+x3-xyz,-1],[xy+x,x2z2-x2+yz];
MSPOLY(G,A);
↳ _[1]=-x7y*gen(1)+3x5y2z*gen(1)-x5z3*gen(2)+x4yz3*gen(1)-3x3y3z2*gen(1)+x3\
yz4*gen(2)+xy4z3*gen(1)-x7*gen(1)+3x5yz*gen(1)+x4z3*gen(1)-3x3y2z2*gen(1)\
+xy3z3*gen(1)-x6*gen(2)+x5y*gen(1)+3x4yz*gen(2)+x4z2*gen(2)-2x3y2z*gen(1)\
+x3z3*gen(2)-3x2y2z2*gen(2)-x2yz3*gen(2)+xy3z2*gen(1)+y3z3*gen(2)+x5*gen(\
1)-2x3yz*gen(1)+xy2z2*gen(1)+x4*gen(2)-2x3z*gen(1)-2x2yz*gen(2)+x2z2*gen(\
2)-2x2z2*gen(1)+2xyz2*gen(1)+y2z2*gen(2)
// Example 2:
ring r2=QQ,(x,y,z),dp;
ideal A=x2-y,xz+z,y2-z;
module G=[x2yz+x2z-2,xyz2+xz-x3z],[x3y+x3-xy2z-xyz,xz-xz-1],[xy+x,x2z2-x2+yz];
MSPOLY(G,A);
↳ _[1]=-x3z*gen(2)+x2yz*gen(1)+xyz2*gen(2)+x2z*gen(1)+xz*gen(2)-2*gen(1)
↳ _[2]=x3y*gen(1)-xy2z*gen(1)+x3*gen(1)-xyz*gen(1)-gen(2)
```

### D.15.21.4 MSGB

Procedure from library `stdmodule.lib` (see [Section D.15.21 \[stdmodule.lib\]](#), page 2551).

**Usage:** `MSGB(G,A)`; `G` submodule over subalgebra `A`, `A` subalgebra (which is a finite sagbi basis).

**Return:** a module `MB`: standard bases of `G` over subalgebra `A`.

**Example:**

```
LIB "stdmodule.lib";
// Example 1:
ring r=ZZ,(x,y),Dp;
ideal A=2x2+xy,2y2,3xy;
ideal I=4x2y2+2xy3,18x2y4;
MSGB(I,A);
↪ _[1]=4x2y2*gen(1)+2xy3*gen(1)
↪ _[2]=18x2y4*gen(1)
// Example 2:
ring r2=QQ,(x,y),Dp;
ideal A= x2-xy+y, y2+x;
module G=x2y2-xy3+x3-x2y+y3+x2+y, y4+2xy2+x2-y2-x;
MSGB(G,A);
↪ _[1]=x2y2*gen(1)-xy3*gen(1)+x3*gen(1)-x2y*gen(1)+y3*gen(1)+x2*gen(1)+y*ge\
n(1)
↪ _[2]=y4*gen(1)+2xy2*gen(1)+x2*gen(1)-y2*gen(1)-x*gen(1)
↪ _[3]=2x2*gen(1)-2xy*gen(1)+2y*gen(1)
// Example 3:
ring r3=QQ,(x,y,z),Dp;
ideal A= x3-yz+2x, x2y2+xz2+3z;
module G=[x3-yz+2x, x2y2+xz2+3z, 2],[x2y2-x3+xz2+yz-2x-3z, 2x3-2yz+4x, -x2y2-xz2-3z+
MSGB(G,A);
↪ _[1]=x2y2*gen(2)+x3*gen(1)+xz2*gen(2)-yz*gen(1)+2x*gen(1)+3z*gen(2)+2*gen\
(3)
↪ _[2]=-x2y2*gen(3)+x2y2*gen(1)+2x3*gen(2)-x3*gen(1)-xz2*gen(3)+xz2*gen(1)-\
2yz*gen(2)+yz*gen(1)+4x*gen(2)-2x*gen(1)-3z*gen(3)-3z*gen(1)+3*gen(3)
↪ _[3]=x6y6*gen(2)+3x5y4z2*gen(2)+3x4y2z4*gen(2)+9x4y4z*gen(2)+x3z6*gen(2)+\
18x3y2z3*gen(2)+x5y2*gen(3)+x5y2*gen(2)+9x2z5*gen(2)+x6*gen(1)+x4z2*gen(3)\
-x2y3z*gen(3)+27x2y2z2*gen(2)+2x3y2*gen(3)-2x3yz*gen(1)-xyz3*gen(3)+27xz\
4*gen(2)+4x4*gen(1)+3x3z*gen(3)+2x2z2*gen(3)+y2z2*gen(1)-4xyz*gen(1)-3yz2\
*gen(3)+27z3*gen(2)+4x2*gen(1)+6xz*gen(3)
↪ _[4]=x7y4*gen(1)+2x6y2z2*gen(1)-x4y5z*gen(1)+2x5y4*gen(1)+x5z4*gen(1)-2x3\
y3z3*gen(1)+6x5y2z*gen(1)+2x4y4*gen(3)+4x4y2z2*gen(1)-x2yz5*gen(1)-x5y2*g\
en(3)-x5y2*gen(2)+6x4z3*gen(1)+4x3y2z2*gen(3)+2x3z4*gen(1)-6x2y3z2*gen(1)\
-x6*gen(1)-x4z2*gen(3)+12x3y2z*gen(1)+x2y3z*gen(3)+2x2z4*gen(3)-6xyz4*gen\
(1)-2x3y2*gen(3)+2x3yz*gen(1)+9x3z2*gen(1)+12x2y2z*gen(3)+12x2z3*gen(1)+x\
yz3*gen(3)-4x4*gen(1)-3x3z*gen(3)-x2y2*gen(3)+x2y2*gen(1)-2x2z2*gen(3)+12\
xz3*gen(3)-y2z2*gen(1)-9yz3*gen(1)+2x3*gen(2)-x3*gen(1)+4xyz*gen(1)-xz2*g\
en(3)+19xz2*gen(1)+3yz2*gen(3)-4x2*gen(1)-6xz*gen(3)-2yz*gen(2)+yz*gen(1)\
+18z2*gen(3)+4x*gen(2)-2x*gen(1)-3z*gen(3)-3z*gen(1)+3*gen(3)
```

### D.15.22 swalk\_lib

**Library:** `swalk.lib`

**Purpose:** Sagbi Walk Conversion Algorithm

**Author:** Junaid Alam Khan [junaidalamkhan@gmail.com](mailto:junaidalamkhan@gmail.com)

**Overview:** A library for computing the Sagbi basis of subalgebra through Sagbi walk algorithm.

**Theory:** The concept of SAGBI ( Subalgebra Analog to Groebner Basis for Ideals) is defined in [L. Robbiano, M. Sweedler: Subalgebra Bases, volume 42, volume 1430 of Lectures Note in Mathematics series, Springer-Verlag (1988),61-87]. The Sagbi Walk algorithm is the subalgebra analogue to the Groebner Walk algorithm which has been proposed in [S. Collart, M. Kalkbrener and D.Mall: Converting bases with the Grobner Walk. J. Symbolic Computation 24 (1997), 465-469].

**Procedures:** See also: [Section D.4.10 \[grwalk\\_lib\]](#), page 1088; [Section D.15.17 \[rwalk\\_lib\]](#), page 2537.

### D.15.22.1 swalk

Procedure from library `swalk.lib` (see [Section D.15.22 \[swalk\\_lib\]](#), page 2553).

**Usage:** `swalk(i[v,w]);` i ideal, v,w int vectors

**Return:** The sagbi basis of the subalgebra defined by the generators of i, calculated via the Sagbi walk algorithm from the ordering dp to lp if v,w are not given (resp. from the ordering (a(v),lp) to the ordering (a(w),lp) if v and w are given).

**Example:**

```
LIB "swalk.lib";
ring r = 0,(x,y), lp;
ideal I =x2,y2,xy+y,2xy2+y3;
swalk(I);
↪ _[1]=x2
↪ _[2]=y2
↪ _[3]=xy+y
↪ _[4]=xy2
↪ _[5]=y3
```

### D.15.22.2 rswalk

Procedure from library `swalk.lib` (see [Section D.15.22 \[swalk\\_lib\]](#), page 2553).

**Usage:** `rswalk(i,weight_rad,p_deg[v,w]);` i ideal, v,w int vectors

**Return:** The sagbi basis of the subalgebra defined by the generators of i, calculated via the Sagbi walk algorithm from the ordering dp to lp if v,w are not given (resp. from the ordering (a(v),lp) to the ordering (a(w),lp) if v and w are given).

**Example:**

```
LIB "swalk.lib";
ring r = 0,(x,y), lp;
ideal I =x2,y2,xy+y,2xy2+y3;
rswalk(I,2,2);
↪ _[1]=x2
↪ _[2]=y2
↪ _[3]=xy+y
↪ _[4]=xy2
↪ _[5]=y3
```



### D.15.23 `systhreads.lib`

**Library:** `systhreads.lib`

**Purpose:** Primitives for Singular's multi-threaded objects

**Author:** Reimer Behrends

**Overview:** This library implements basic functionality for shared objects in a multi-threaded system, such as channels, shared tables & lists, and synchronization variables.

### D.15.24 `tateProdCplxNegGrad.lib`

**Library:** `tateProdCplxNegGrad.lib`

**Purpose:** for computing sheaf cohomology on product of projective spaces

**Author:** Clara Petroll ([petroll@mathematik.uni-kl.de](mailto:petroll@mathematik.uni-kl.de))

**Overview:** In this library, we use Tate resolutions for computing sheaf cohomology of coherent sheaves on products of projective spaces. The algorithms can be used for arbitrary products. We work over the multigraded Cox ring and the corresponding exterior algebra. Multigraded complexes are realized as the newstruct `multigradedcomplex`. The main algorithm is the one for computing subquotient complexes of a Tate resolution. It allows to compute cohomology tables, respectively hash table of the dimensions of sheaf cohomology groups.

**References:**

- [1] Eisenbud, Erman, Schreyer: Tate Resolutions for Products of Projective Spaces, *Acta Mathematica Vietnamica* (2015)
- [2] Eisenbud, Erman, Schreyer: Tate Resolutions on Products of Projective Spaces: Cohomology and Direct Image Complexes (2019)

**Procedures:**

#### D.15.24.1 `productOfProjectiveSpaces`

Procedure from library `tateProdCplxNegGrad.lib` (see [Section D.15.24 \[tateProdCplxNegGrad.lib\]](#), page 2555).

**Usage:** `productOfProjectiveSpaces(c); c intvec`

**Purpose:** creates two rings  $S$  and  $E$  corresponding to the product of projective spaces  $P^{c_1} \times \dots \times P^{c_t}$

**Assume:** input are two integers or an intvec

**Return:** two rings  $S, E$  (homogeneous coordinate ring and the exterior algebra of  $P^{c_1} \times P^{c_2} \times \dots$ )

**Example:**

```
LIB "tateProdCplxNegGrad.lib";
intvec c = 1,2;
def (S,E) = productOfProjectiveSpaces(c);
print(S);
⇒ polynomial ring, over a field, global ordering
⇒ // coefficients: QQ
⇒ // number of vars : 5
⇒ //          block 1 : ordering C
```

```

⇒ //          block  2 : ordering dp
⇒ //          : names  x(0)(0) x(0)(1) x(1)(0) x(1)(1) x(1)(2)
print(E);
⇒ polynomial ring, over a field, global ordering
⇒ // coefficients: QQ
⇒ // number of vars : 5
⇒ //          block  1 : ordering C
⇒ //          block  2 : ordering dp
⇒ //          : names  e(0)(0) e(0)(1) e(1)(0) e(1)(1) e(1)(2)
⇒ // noncommutative relations:
⇒ //      e(0)(1)e(0)(0)=-e(0)(0)*e(0)(1)
⇒ //      e(1)(0)e(0)(0)=-e(0)(0)*e(1)(0)
⇒ //      e(1)(1)e(0)(0)=-e(0)(0)*e(1)(1)
⇒ //      e(1)(2)e(0)(0)=-e(0)(0)*e(1)(2)
⇒ //      e(1)(0)e(0)(1)=-e(0)(1)*e(1)(0)
⇒ //      e(1)(1)e(0)(1)=-e(0)(1)*e(1)(1)
⇒ //      e(1)(2)e(0)(1)=-e(0)(1)*e(1)(2)
⇒ //      e(1)(1)e(1)(0)=-e(1)(0)*e(1)(1)
⇒ //      e(1)(2)e(1)(0)=-e(1)(0)*e(1)(2)
⇒ //      e(1)(2)e(1)(1)=-e(1)(1)*e(1)(2)
⇒ // quotient ring from ideal
⇒ _[1]=e(1)(2)^2
⇒ _[2]=e(1)(1)^2
⇒ _[3]=e(1)(0)^2
⇒ _[4]=e(0)(1)^2
⇒ _[5]=e(0)(0)^2
intvec d = 2,1,2;
def (S2,E2) = productOfProjectiveSpaces(d);
print(S2);
⇒ polynomial ring, over a field, global ordering
⇒ // coefficients: QQ
⇒ // number of vars : 8
⇒ //          block  1 : ordering C
⇒ //          block  2 : ordering dp
⇒ //          : names  x(0)(0) x(0)(1) x(0)(2) x(1)(0) x(1)(1) x(\
2)(0) x(2)(1) x(2)(2)
print(E2);
⇒ polynomial ring, over a field, global ordering
⇒ // coefficients: QQ
⇒ // number of vars : 8
⇒ //          block  1 : ordering C
⇒ //          block  2 : ordering dp
⇒ //          : names  e(0)(0) e(0)(1) e(0)(2) e(1)(0) e(1)(1) e(\
2)(0) e(2)(1) e(2)(2)
⇒ // noncommutative relations:
⇒ //      e(0)(1)e(0)(0)=-e(0)(0)*e(0)(1)
⇒ //      e(0)(2)e(0)(0)=-e(0)(0)*e(0)(2)
⇒ //      e(1)(0)e(0)(0)=-e(0)(0)*e(1)(0)
⇒ //      e(1)(1)e(0)(0)=-e(0)(0)*e(1)(1)
⇒ //      e(2)(0)e(0)(0)=-e(0)(0)*e(2)(0)
⇒ //      e(2)(1)e(0)(0)=-e(0)(0)*e(2)(1)
⇒ //      e(2)(2)e(0)(0)=-e(0)(0)*e(2)(2)
⇒ //      e(0)(2)e(0)(1)=-e(0)(1)*e(0)(2)

```

```

⇒ //      e(1)(0)e(0)(1)=-e(0)(1)*e(1)(0)
⇒ //      e(1)(1)e(0)(1)=-e(0)(1)*e(1)(1)
⇒ //      e(2)(0)e(0)(1)=-e(0)(1)*e(2)(0)
⇒ //      e(2)(1)e(0)(1)=-e(0)(1)*e(2)(1)
⇒ //      e(2)(2)e(0)(1)=-e(0)(1)*e(2)(2)
⇒ //      e(1)(0)e(0)(2)=-e(0)(2)*e(1)(0)
⇒ //      e(1)(1)e(0)(2)=-e(0)(2)*e(1)(1)
⇒ //      e(2)(0)e(0)(2)=-e(0)(2)*e(2)(0)
⇒ //      e(2)(1)e(0)(2)=-e(0)(2)*e(2)(1)
⇒ //      e(2)(2)e(0)(2)=-e(0)(2)*e(2)(2)
⇒ //      e(1)(1)e(1)(0)=-e(1)(0)*e(1)(1)
⇒ //      e(2)(0)e(1)(0)=-e(1)(0)*e(2)(0)
⇒ //      e(2)(1)e(1)(0)=-e(1)(0)*e(2)(1)
⇒ //      e(2)(2)e(1)(0)=-e(1)(0)*e(2)(2)
⇒ //      e(2)(0)e(1)(1)=-e(1)(1)*e(2)(0)
⇒ //      e(2)(1)e(1)(1)=-e(1)(1)*e(2)(1)
⇒ //      e(2)(2)e(1)(1)=-e(1)(1)*e(2)(2)
⇒ //      e(2)(1)e(2)(0)=-e(2)(0)*e(2)(1)
⇒ //      e(2)(2)e(2)(0)=-e(2)(0)*e(2)(2)
⇒ //      e(2)(2)e(2)(1)=-e(2)(1)*e(2)(2)
⇒ // quotient ring from ideal
⇒ _[1]=e(2)(2)^2
⇒ _[2]=e(2)(1)^2
⇒ _[3]=e(2)(0)^2
⇒ _[4]=e(1)(1)^2
⇒ _[5]=e(1)(0)^2
⇒ _[6]=e(0)(2)^2
⇒ _[7]=e(0)(1)^2
⇒ _[8]=e(0)(0)^2

```

### D.15.24.2 truncateM

Procedure from library `tateProdCplxNegGrad.lib` (see [Section D.15.24 \[tateProdCplxNegGrad.lib\]](#), page 2555).

**Usage:** `truncateM(M,c);` M module, c intvec

**Purpose:** truncate M at c

**Assume:** M is multigraded S-module with S multigraded ring, c is an intvec of the right length

**Return:** module, the truncated module  $M_{\geq c}$

**Note:** Output is the truncated module (multigraded, grading is not shifted), works for arbitrary products

**Example:**

```

LIB "tateProdCplxNegGrad.lib";
intvec c = 1,1,1;
def(S,E) = productOfProjectiveSpaces(c);
setring(S);
intmat grading[3][2] = 0,0,0,0,0,0;
module te = freemodule(2);
te = setModuleGrading(te,grading);
intvec c = 1,1,1;
⇒ // ** redefining c (intvec c = 1,1,1;) ./examples/truncateM.sing:8

```

```

module Mtrunc = truncateM(te,c);
Mtrunc;
↳ Mtrunc[1]=x(0)(1)*x(1)(1)*x(2)(1)*gen(1)
↳ Mtrunc[2]=x(0)(0)*x(1)(1)*x(2)(1)*gen(1)
↳ Mtrunc[3]=x(0)(1)*x(1)(0)*x(2)(1)*gen(1)
↳ Mtrunc[4]=x(0)(0)*x(1)(0)*x(2)(1)*gen(1)
↳ Mtrunc[5]=x(0)(1)*x(1)(1)*x(2)(0)*gen(1)
↳ Mtrunc[6]=x(0)(0)*x(1)(1)*x(2)(0)*gen(1)
↳ Mtrunc[7]=x(0)(1)*x(1)(0)*x(2)(0)*gen(1)
↳ Mtrunc[8]=x(0)(0)*x(1)(0)*x(2)(0)*gen(1)
↳ Mtrunc[9]=x(0)(1)*x(1)(1)*x(2)(1)*gen(2)
↳ Mtrunc[10]=x(0)(0)*x(1)(1)*x(2)(1)*gen(2)
↳ Mtrunc[11]=x(0)(1)*x(1)(0)*x(2)(1)*gen(2)
↳ Mtrunc[12]=x(0)(0)*x(1)(0)*x(2)(1)*gen(2)
↳ Mtrunc[13]=x(0)(1)*x(1)(1)*x(2)(0)*gen(2)
↳ Mtrunc[14]=x(0)(0)*x(1)(1)*x(2)(0)*gen(2)
↳ Mtrunc[15]=x(0)(1)*x(1)(0)*x(2)(0)*gen(2)
↳ Mtrunc[16]=x(0)(0)*x(1)(0)*x(2)(0)*gen(2)
getModuleGrading(Mtrunc);
↳ 0,0,
↳ 0,0,
↳ 0,0
multiDeg(Mtrunc);
↳ 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
↳ 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
↳ 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1

```

### D.15.24.3 truncateCoker

Procedure from library `tateProdCplxNegGrad.lib` (see [Section D.15.24 \[tateProdCplxNegGrad.lib\]](#), [page 2555](#)).

**Usage:** `truncateCoker(M,c)`;  $M$  module,  $c$  intvec

**Purpose:** truncate cokernel `coker(M)` at the multidegree  $c$

**Return:** module, which is a presentation matrix of the truncation of `coker(M)` at  $c$

**Example:**

```

LIB "tateProdCplxNegGrad.lib";
// example 1
intvec c1 = 1,1,1;
def(S1,E1) = productOfProjectiveSpaces(c1);
setring(S1);
module M1= 0;
intmat grading1[3][1] = 0,0,0;
M1 = setModuleGrading(M1,grading1);
truncateCoker(M1,c1);
↳ _[1]=x(0)(1)*gen(2)-x(0)(0)*gen(1)
↳ _[2]=x(1)(1)*gen(3)-x(1)(0)*gen(1)
↳ _[3]=x(1)(1)*gen(4)-x(1)(0)*gen(2)
↳ _[4]=x(0)(1)*gen(4)-x(0)(0)*gen(3)
↳ _[5]=x(2)(1)*gen(5)-x(2)(0)*gen(1)
↳ _[6]=x(2)(1)*gen(6)-x(2)(0)*gen(2)
↳ _[7]=x(0)(1)*gen(6)-x(0)(0)*gen(5)

```

```

⇒ _[8]=x(2)(1)*gen(7)-x(2)(0)*gen(3)
⇒ _[9]=x(1)(1)*gen(7)-x(1)(0)*gen(5)
⇒ _[10]=x(2)(1)*gen(8)-x(2)(0)*gen(4)
⇒ _[11]=x(1)(1)*gen(8)-x(1)(0)*gen(6)
⇒ _[12]=x(0)(1)*gen(8)-x(0)(0)*gen(7)
// example 2
intvec c2 = 1,1;
def (S2,E2) = productOfProjectiveSpaces(c2);
setring(S2);
module M2 = 0;
intmat grading2[2][1] = 0,0;
M2 = setModuleGrading(M2,grading2);
truncateCoker(M2,c2);
⇒ _[1]=x(0)(1)*gen(2)-x(0)(0)*gen(1)
⇒ _[2]=x(1)(1)*gen(3)-x(1)(0)*gen(1)
⇒ _[3]=x(1)(1)*gen(4)-x(1)(0)*gen(2)
⇒ _[4]=x(0)(1)*gen(4)-x(0)(0)*gen(3)

```

#### D.15.24.4 symExt

Procedure from library `tateProdCplxNegGrad.lib` (see [Section D.15.24 \[tateProdCplxNegGrad.lib\]](#), page 2555).

**Usage:** `symExt(m)`; `m` matrix

**Purpose:** computes differential  $R(M_0) \rightarrow R(M_1)$  for the module `M` over `S` corresponding to the linear presentation matrix `m`, however, in order to get the result, `m` has to be fetched to the exterior algebra `E`

**Assume:** `m` a matrix, linear presentation matrix over `S`; Note: also works for nonlinear matrices, but makes no sense to use it in this case

**Return:** matrix `B` representing  $R(M_0) \rightarrow R(M_1)$

**Note:** output lives in `S` (not as in Macaulay2 in the ring `E`, to get the same result, just fetch the matrix to `E`)

**Example:**

```

LIB "tateProdCplxNegGrad.lib";
intvec c = 1,2;
def (S,E) = productOfProjectiveSpaces(c);
setring(S);
matrix m[4][2] = x(0)(0), x(1)(0), x(0)(1), 0, 0, x(1)(1), 0, x(1)(2);
matrix A = symExt(m);
print(A);
⇒ 0,      x(0)(0), 0,      0,
⇒ 0,      0,      x(0)(0), 0,
⇒ 0,      0,      0,      x(0)(0),
⇒ x(0)(1), 0,      0,      0,
⇒ -x(0)(0), x(0)(1), 0,      0,
⇒ 0,      0,      x(0)(1), 0,
⇒ 0,      0,      0,      x(0)(1),
⇒ 0,      x(1)(0), 0,      0,
⇒ 0,      0,      x(1)(0), 0,
⇒ 0,      0,      0,      x(1)(0),
⇒ x(1)(1), 0,      0,      0,

```

```

↳ 0,      x(1)(1),0,      0,
↳ -x(1)(0),0,      x(1)(1), 0,
↳ 0,      0,      0,      x(1)(1),
↳ x(1)(2), 0,      0,      0,
↳ 0,      x(1)(2),0,      0,
↳ 0,      0,      x(1)(2), 0,
↳ 0,      0,      -x(1)(1),x(1)(2)
setring(E);
print(fetch(S,A));
↳ 0,      e(0)(0),0,      0,
↳ 0,      0,      e(0)(0), 0,
↳ 0,      0,      0,      e(0)(0),
↳ e(0)(1), 0,      0,      0,
↳ -e(0)(0),e(0)(1),0,      0,
↳ 0,      0,      e(0)(1), 0,
↳ 0,      0,      0,      e(0)(1),
↳ 0,      e(1)(0),0,      0,
↳ 0,      0,      e(1)(0), 0,
↳ 0,      0,      0,      e(1)(0),
↳ e(1)(1), 0,      0,      0,
↳ 0,      e(1)(1),0,      0,
↳ -e(1)(0),0,      e(1)(1), 0,
↳ 0,      0,      0,      e(1)(1),
↳ e(1)(2), 0,      0,      0,
↳ 0,      e(1)(2),0,      0,
↳ 0,      0,      e(1)(2), 0,
↳ 0,      0,      -e(1)(1),e(1)(2)

```

#### D.15.24.5 sufficientlyPositiveMultidegree

Procedure from library `tateProdCplxNegGrad.lib` (see [Section D.15.24 \[tateProdCplxNegGrad.lib\]](#), page 2555).

**Usage:** `sufficientlyPositiveMultidegree(M);` M module

**Purpose:** computes a sufficiently positive multidegree for `coker(M)`

**Assume:** M is multigraded S-module

**Return:** intvec that is sufficiently positive for M

**Example:**

```

LIB "tateProdCplxNegGrad.lib";
// example 1
intvec c1 = 1,2;
def (S1,E1) = productOfProjectiveSpaces(c1);
setring(S1);
module M1 = x(0)(0),x(1)(0)^3 + x(1)(1)^3 +x(1)(2)^3;
intmat grading1[2][1] = 0,0;
M1 = setModuleGrading(M1,grading1);
sufficientlyPositiveMultidegree(M1);
↳ 1,3
// example 2
intvec c2 = 1,1;
def (S2,E2) = productOfProjectiveSpaces(c2);

```

```

setring(S2);
intmat grading2[2][1] = -1,-1;
module M2 = 0;
M2 = setModuleGrading(M2,grading2);
sufficientlyPositiveMultidegree(M2);
↪ -1,-1
// example 3
intvec c3 = 1,1,1;
def (S3,E3) = productOfProjectiveSpaces(c3);
setring(S3);
module M3 = 0;
intmat grading3[3][1] = -1,-1,-1;
M3 = setModuleGrading(M3,grading3);
sufficientlyPositiveMultidegree(M3);
↪ -1,-1,-1

```

#### D.15.24.6 tateResolution

Procedure from library `tateProdCplxNegGrad.lib` (see [Section D.15.24 \[tateProdCplxNegGrad.lib\]](#), page 2555).

**Usage:** `tateResolution(M,low,high)`; M module, L list, low intvec, high intvec

**Purpose:** compute tate resolution of `coker(M)` where M is  $\mathbb{Z}^t$ -graded S-module

**Assume:** M a module over multigraded ring S

**Return:** (E,tate), tate a multigradedcomplex, E the ring in which tate has to be viewed, however note that tate is not ring dependent

**Example:**

```

LIB "tateProdCplxNegGrad.lib";
// example 1
intvec c1 = 1,1,1;
intvec low1 = 0,0,0;
intvec high1 = 0,1,0;
def (S1,E1) = productOfProjectiveSpaces(c1);
setring(S1);
module M1 = 0;
intmat grading1[3][1] = -1,-1,-1;
M1 = setModuleGrading(M1,grading1);
multigradedcomplex tate1;
(E1,tate1) = tateResolution(M1,low1,high1);
setring(E1);
tate1;
↪ E^12 <-- E^20
↪ -1      0
↪
tate1.differentials;
↪ [1]:
↪ _[1]=-e(0)(0)*gen(2)-e(0)(1)*gen(1)
↪ _[2]=-e(1)(0)*gen(3)-e(1)(1)*gen(1)
↪ _[3]=-e(1)(0)*gen(4)-e(1)(1)*gen(2)
↪ _[4]=-e(0)(0)*gen(4)-e(0)(1)*gen(3)
↪ _[5]=-e(1)(0)*gen(5)-e(1)(1)*gen(3)

```

```

↳ _[6]=-e(1)(0)*gen(6)-e(1)(1)*gen(4)
↳ _[7]=-e(0)(0)*gen(6)-e(0)(1)*gen(5)
↳ _[8]=-e(2)(0)*gen(7)-e(2)(1)*gen(1)
↳ _[9]=-e(2)(0)*gen(8)-e(2)(1)*gen(2)
↳ _[10]=-e(0)(0)*gen(8)-e(0)(1)*gen(7)
↳ _[11]=-e(2)(0)*gen(9)-e(2)(1)*gen(3)
↳ _[12]=-e(1)(0)*gen(9)-e(1)(1)*gen(7)
↳ _[13]=-e(2)(0)*gen(10)-e(2)(1)*gen(4)
↳ _[14]=-e(1)(0)*gen(10)-e(1)(1)*gen(8)
↳ _[15]=-e(0)(0)*gen(10)-e(0)(1)*gen(9)
↳ _[16]=-e(2)(0)*gen(11)-e(2)(1)*gen(5)
↳ _[17]=-e(1)(0)*gen(11)-e(1)(1)*gen(9)
↳ _[18]=-e(2)(0)*gen(12)-e(2)(1)*gen(6)
↳ _[19]=-e(1)(0)*gen(12)-e(1)(1)*gen(10)
↳ _[20]=-e(0)(0)*gen(12)-e(0)(1)*gen(11)
// example 2
intvec c2 = 1,2;
def (S2,E2) = productOfProjectiveSpaces(c2);
setring(S2);
intvec low2 = -3,-3;
intvec high2 = 0,0;
module M2 = x(0)(0),x(1)(0)^3 + x(1)(1)^3 +x(1)(2)^3;;
intmat grading2[2][1] = 0,0;
M2 = setModuleGrading(M2,grading2);
multigradedcomplex tate2;
(E2,tate2) = tateResolution(M2,low2,high2);
setring(E2);
tate2;
↳ E^6 <-- E^10 <-- E^14 <-- E^20 <-- E^29 <-- E^41 <-- E^56 <-\
  - E^74 <-- E^95
↳ -2      -1      0      1      2      3      4      \
      5      6
↳
// example 3
intvec c3 = 1,1;
def (S3,E3) = productOfProjectiveSpaces(c3);
intvec low3 = -3,-3;
intvec high3 = 3,3;
setring(S3);
module M3 = 0;
intmat grading3[2][1] = -1,-1;
M3 = setModuleGrading(M3,grading3);
multigradedcomplex tate3;
(E3,tate3) = tateResolution(M3,low3,high3);
setring(E3);
tate3;
↳ E^25 <-- E^40 <-- E^46 <-- E^44 <-- E^35 <-- E^30 <-- E^38 <-\
  -- E^56 <-- E^81 <-- E^110 <-- E^141 <-- E^174 <-- E^210
↳ -6      -5      -4      -3      -2      -1      0      \
      1      2      3      4      5      6
↳

```



### D.15.24.7 cohomologyMatrix

Procedure from library `tateProdCplxNegGrad.lib` (see [Section D.15.24 \[tateProdCplxNegGrad.lib\]](#), page 2555).

**Usage:** `cohomologyMatrix(M,L,low,high)`; M module, L list, low intvec, high intvec

**Purpose:** computes the cohomology matrix of the sheaf corresponding to `coker(M)`

**Assume:** M module over S, L list of two rings S and E (e.g. result of `productOfProjectiveSpaces`) first entry `L[1] = S` and `L[2] = E`, integer vectors `low <= high`

**Return:** ring Z in which cohomology matrix lives, it is exported in the variable `cohomologymat`, `cohomologymat` covers all cohomology groups of twists in the range between low and high

**Example:**

```
LIB "tateProdCplxNegGrad.lib";
// example 1
intvec c1 = 1,1;
def (S1,E1) = productOfProjectiveSpaces(c1);
intvec low1 = -3,-3;
intvec high1 = 3,3;
setring(S1);
module M1 = 0;
intmat grading1[2][1] = -1,-1;
M1 = setModuleGrading(M1,grading1);
ring Z1 = cohomologyMatrix(M1,low1,high1);
setring(Z1);
print(cohomologymat);
⇒ 5h,0,5,10,15,20,25,
⇒ 4h,0,4,8, 12,16,20,
⇒ 3h,0,3,6, 9, 12,15,
⇒ 2h,0,2,4, 6, 8, 10,
⇒ h, 0,1,2, 3, 4, 5,
⇒ 0, 0,0,0, 0, 0, 0,
⇒ h2,0,h,2h,3h,4h,5h
// example 2
intvec c2 = 1,2;
def (S2,E2) = productOfProjectiveSpaces(c2);
intvec low2 = -3,-3;
intvec high2 = 0,0;
setring(S2);
module M2 = 0;
intmat grading2[2][1] = 0,0;
M2 = setModuleGrading(M2,grading2);
ring Z2 = cohomologyMatrix(M2,low2,high2);
setring(Z2);
print(cohomologymat);
⇒ 2h, h, 0,1,
⇒ 0, 0, 0,0,
⇒ 0, 0, 0,0,
⇒ 2h3,h3,0,h2
// example 3
setring(S2);
```

```

module M3 = x(0)(0),x(1)(0)^3 + x(1)(1)^3 +x(1)(2)^3;
intmat grading3[2][1] = 0,0;
M3 = setModuleGrading(M3,grading3);
ring Z3 = cohomologyMatrix(M3,low2,high2);
setring(Z3);
print(cohomologymat);
↪ h+1,h+1,h+1,h+1,
↪ 3h, 3h, 3h, 3h,
↪ 6h, 6h, 6h, 6h,
↪ 9h, 9h, 9h, 9h

```

#### D.15.24.8 cohomologyMatrixFromResolution

Procedure from library `tateProdCplxNegGrad.lib` (see [Section D.15.24 \[tateProdCplxNegGrad.lib\]](#), page 2555).

- Usage:** `cohomologyMatrixFromResolution(T,low,high)`; `T` multigradedcomplex, `low` intvec, `high` intvec
- Purpose:** computes the cohomology matrix corresponding to the multigraded complex `T` (part of a Tate resolution)
- Assume:** `T` is a multigraded complex representing a part of a Tate resolution (for example output of `tateResolution`), basering is `E`
- Return:** ring `Z` in which cohomology matrix lives, it is exported in the variable `cohomologymat`, `cohomologymat` stores information in the range between `low` and `high`

**Example:**

```

LIB "tateProdCplxNegGrad.lib";
intvec c = 1,1;
def (S,E) = productOfProjectiveSpaces(c);
intvec low = -3,-3;
intvec high = 3,3;
setring(S);
module M = 0;
intmat grading[2][1] = -1,-1;
M = setModuleGrading(M,grading);
multigradedcomplex tate;
(E,tate) = tateResolution(M,low,high);
setring(E);
ring Z = cohomologyMatrixFromResolution(tate,low,high);
setring(Z);
print(cohomologymat);
↪ 5h,0,5,10,15,20,25,
↪ 4h,0,4,8, 12,16,20,
↪ 3h,0,3,6, 9, 12,15,
↪ 2h,0,2,4, 6, 8, 10,
↪ h, 0,1,2, 3, 4, 5,
↪ 0, 0,0,0, 0, 0, 0,
↪ h2,0,h,2h,3h,4h,5h

```

#### D.15.24.9 eulerPolynomialTable

Procedure from library `tateProdCplxNegGrad.lib` (see [Section D.15.24 \[tateProdCplxNegGrad.lib\]](#), page 2555).

- Usage:** `eulerPolynomialTable(M,low,high)`; M module, L list, low intvec, high intvec
- Purpose:** computes hash table of euler polynomials of twists of `coker(M)` in the range between low and high
- Assume:** M module, note that at the moment M is a module over S,
- Return:** `(Z,eulerpolynomialtable)`, where `eulerpolynomialtable` is a hash table with entries in the ring  $Z = \mathbb{Z}[h]$
- Note:** this function works for arbitrary products  $P^{\{n_1\}} \times \dots \times P^{\{n_t\}}$  and corresponding  $Z^t$ -gradings, entries can be accessed via `eulerpolynomialtable*(a_1,...,a_t)` where  $a=(a_1,...,a_t)$  is a multidegree between low and high

**Example:**

```
LIB "tateProdCplxNegGrad.lib";
// example 1
intvec c1 = 1,1;
def (S1,E1) = productOfProjectiveSpaces(c1);
intvec low1 = -3,-3;
intvec high1 = 3,3;
setring(S1);
module M1 = 0;
intmat grading1[2][1] = -1,-1;
M1 = setModuleGrading(M1,grading1);
def (Z1,eulerTable1) = eulerPolynomialTable(M1,low1,high1);
setring(Z1);
print(eulerTable1);
↪ -3,-3 => h2
↪ -2,-3 => 0
↪ -1,-3 => h
↪ 0,-3 => 2h
↪ 1,-3 => 3h
↪ 2,-3 => 4h
↪ 3,-3 => 5h
↪ -3,-2 => 0
↪ -3,-1 => h
↪ -3,0 => 2h
↪ -3,1 => 3h
↪ -3,2 => 4h
↪ -3,3 => 5h
↪ -2,-2 => 0
↪ -2,-1 => 0
↪ -2,0 => 0
↪ -2,1 => 0
↪ -2,2 => 0
↪ -2,3 => 0
↪ -1,-2 => 0
↪ -1,-1 => 1
↪ -1,0 => 2
↪ -1,1 => 3
↪ -1,2 => 4
↪ -1,3 => 5
↪ 0,-2 => 0
↪ 0,-1 => 2
```

```

⇒ 0,0 => 4
⇒ 0,1 => 6
⇒ 0,2 => 8
⇒ 0,3 => 10
⇒ 1,-2 => 0
⇒ 1,-1 => 3
⇒ 1,0 => 6
⇒ 1,1 => 9
⇒ 1,2 => 12
⇒ 1,3 => 15
⇒ 2,-2 => 0
⇒ 2,-1 => 4
⇒ 2,0 => 8
⇒ 2,1 => 12
⇒ 2,2 => 16
⇒ 2,3 => 20
⇒ 3,-2 => 0
⇒ 3,-1 => 5
⇒ 3,0 => 10
⇒ 3,1 => 15
⇒ 3,2 => 20
⇒ 3,3 => 25
eulerTable1*low1;
⇒ h2
setring(S1);
ring Z = cohomologyMatrix(M1,low1,high1);
setring(Z);
print(cohomologymat);
⇒ 5h,0,5,10,15,20,25,
⇒ 4h,0,4,8, 12,16,20,
⇒ 3h,0,3,6, 9, 12,15,
⇒ 2h,0,2,4, 6, 8, 10,
⇒ h, 0,1,2, 3, 4, 5,
⇒ 0, 0,0,0, 0, 0, 0,
⇒ h2,0,h,2h,3h,4h,5h
// example 2
intvec c2 = 1,1,1;
def (S2,E2) = productOfProjectiveSpaces(c2);
setring(S2);
intvec low2 = 0,0,0;
intvec high2 = 1,1,1;
module M2 = 0;
intmat grading2[3][1] = -1,-1,-1;
M2 = setModuleGrading(M2,grading2);
def (Z2,eulerTable2) = eulerPolynomialTable(M2,low2,high2);
setring(Z2);
print(eulerTable2);
⇒ 0,0,0 => 8
⇒ 1,0,0 => 12
⇒ 0,1,0 => 12
⇒ 1,1,0 => 18
⇒ 0,0,1 => 12
⇒ 1,0,1 => 18

```

$\mapsto 0,1,1 \Rightarrow 18$   
 $\mapsto 1,1,1 \Rightarrow 27$

### D.15.24.10 cohomologyHashTable

Procedure from library `tateProdCplxNegGrad.lib` (see [Section D.15.24 \[tateProdCplxNegGrad.lib\]](#), page 2555).

- Usage:** `cohomologyHashTable(M,L,low,high)`; M module, low intvec, high intvec
- Purpose:** computes hashtable of sheaf cohomology groups of twists in the range between low and high corresponding to `coker(M)`
- Assume:** M module representing a sheaf F on the product of t projective spaces, note that at the moment M is a module over S,
- Return:** `cohomologytable` where `cohomologytable` is a hash table with integer vectors in  $\mathbb{Z}^{t+1}$  as keys, entries can be accessed via `cohomologytable*(c_1,...,c_t,i) = dim(Hi(F(c_1,...,c_t)))`
- Note:** this function works for arbitrary products  $P^{n_1} \times \dots \times P^{n_t}$  and corresponding  $\mathbb{Z}^t$ -gradings

**Example:**

```

LIB "tateProdCplxNegGrad.lib";
intvec c = 1,1;
def (S,E) = productOfProjectiveSpaces(c);
intvec low = -3,-3;
intvec high = 3,3;
setring(S);
module M = 0;
intmat grading[2][1] = -1,-1;
M = setModuleGrading(M,grading);
def cohomologytable = cohomologyHashTable(M,low,high);
print(cohomologytable);
 $\mapsto -3,-3,0 \Rightarrow 0$ 
 $\mapsto -3,-3,1 \Rightarrow 0$ 
 $\mapsto -3,-3,2 \Rightarrow 1$ 
 $\mapsto -2,-3,0 \Rightarrow 0$ 
 $\mapsto -2,-3,1 \Rightarrow 0$ 
 $\mapsto -2,-3,2 \Rightarrow 0$ 
 $\mapsto -1,-3,0 \Rightarrow 0$ 
 $\mapsto -1,-3,1 \Rightarrow 1$ 
 $\mapsto -1,-3,2 \Rightarrow 0$ 
 $\mapsto 0,-3,0 \Rightarrow 0$ 
 $\mapsto 0,-3,1 \Rightarrow 2$ 
 $\mapsto 0,-3,2 \Rightarrow 0$ 
 $\mapsto 1,-3,0 \Rightarrow 0$ 
 $\mapsto 1,-3,1 \Rightarrow 3$ 
 $\mapsto 1,-3,2 \Rightarrow 0$ 
 $\mapsto 2,-3,0 \Rightarrow 0$ 
 $\mapsto 2,-3,1 \Rightarrow 4$ 
 $\mapsto 2,-3,2 \Rightarrow 0$ 
 $\mapsto 3,-3,0 \Rightarrow 0$ 
 $\mapsto 3,-3,1 \Rightarrow 5$ 

```

```

 $\mapsto 3, -3, 2 \Rightarrow 0$ 
 $\mapsto -3, -2, 0 \Rightarrow 0$ 
 $\mapsto -3, -2, 1 \Rightarrow 0$ 
 $\mapsto -3, -2, 2 \Rightarrow 0$ 
 $\mapsto -3, -1, 0 \Rightarrow 0$ 
 $\mapsto -3, -1, 1 \Rightarrow 1$ 
 $\mapsto -3, -1, 2 \Rightarrow 0$ 
 $\mapsto -3, 0, 0 \Rightarrow 0$ 
 $\mapsto -3, 0, 1 \Rightarrow 2$ 
 $\mapsto -3, 0, 2 \Rightarrow 0$ 
 $\mapsto -3, 1, 0 \Rightarrow 0$ 
 $\mapsto -3, 1, 1 \Rightarrow 3$ 
 $\mapsto -3, 1, 2 \Rightarrow 0$ 
 $\mapsto -3, 2, 0 \Rightarrow 0$ 
 $\mapsto -3, 2, 1 \Rightarrow 4$ 
 $\mapsto -3, 2, 2 \Rightarrow 0$ 
 $\mapsto -3, 3, 0 \Rightarrow 0$ 
 $\mapsto -3, 3, 1 \Rightarrow 5$ 
 $\mapsto -3, 3, 2 \Rightarrow 0$ 
 $\mapsto -2, -2, 0 \Rightarrow 0$ 
 $\mapsto -2, -2, 1 \Rightarrow 0$ 
 $\mapsto -2, -2, 2 \Rightarrow 0$ 
 $\mapsto -2, -1, 0 \Rightarrow 0$ 
 $\mapsto -2, -1, 1 \Rightarrow 0$ 
 $\mapsto -2, -1, 2 \Rightarrow 0$ 
 $\mapsto -2, 0, 0 \Rightarrow 0$ 
 $\mapsto -2, 0, 1 \Rightarrow 0$ 
 $\mapsto -2, 0, 2 \Rightarrow 0$ 
 $\mapsto -2, 1, 0 \Rightarrow 0$ 
 $\mapsto -2, 1, 1 \Rightarrow 0$ 
 $\mapsto -2, 1, 2 \Rightarrow 0$ 
 $\mapsto -2, 2, 0 \Rightarrow 0$ 
 $\mapsto -2, 2, 1 \Rightarrow 0$ 
 $\mapsto -2, 2, 2 \Rightarrow 0$ 
 $\mapsto -2, 3, 0 \Rightarrow 0$ 
 $\mapsto -2, 3, 1 \Rightarrow 0$ 
 $\mapsto -2, 3, 2 \Rightarrow 0$ 
 $\mapsto -1, -2, 0 \Rightarrow 0$ 
 $\mapsto -1, -2, 1 \Rightarrow 0$ 
 $\mapsto -1, -2, 2 \Rightarrow 0$ 
 $\mapsto -1, -1, 0 \Rightarrow 1$ 
 $\mapsto -1, -1, 1 \Rightarrow 0$ 
 $\mapsto -1, -1, 2 \Rightarrow 0$ 
 $\mapsto -1, 0, 0 \Rightarrow 2$ 
 $\mapsto -1, 0, 1 \Rightarrow 0$ 
 $\mapsto -1, 0, 2 \Rightarrow 0$ 
 $\mapsto -1, 1, 0 \Rightarrow 3$ 
 $\mapsto -1, 1, 1 \Rightarrow 0$ 
 $\mapsto -1, 1, 2 \Rightarrow 0$ 
 $\mapsto -1, 2, 0 \Rightarrow 4$ 
 $\mapsto -1, 2, 1 \Rightarrow 0$ 
 $\mapsto -1, 2, 2 \Rightarrow 0$ 
 $\mapsto -1, 3, 0 \Rightarrow 5$ 

```

```

 $\mapsto -1,3,1 \Rightarrow 0$ 
 $\mapsto -1,3,2 \Rightarrow 0$ 
 $\mapsto 0,-2,0 \Rightarrow 0$ 
 $\mapsto 0,-2,1 \Rightarrow 0$ 
 $\mapsto 0,-2,2 \Rightarrow 0$ 
 $\mapsto 0,-1,0 \Rightarrow 2$ 
 $\mapsto 0,-1,1 \Rightarrow 0$ 
 $\mapsto 0,-1,2 \Rightarrow 0$ 
 $\mapsto 0,0,0 \Rightarrow 4$ 
 $\mapsto 0,0,1 \Rightarrow 0$ 
 $\mapsto 0,0,2 \Rightarrow 0$ 
 $\mapsto 0,1,0 \Rightarrow 6$ 
 $\mapsto 0,1,1 \Rightarrow 0$ 
 $\mapsto 0,1,2 \Rightarrow 0$ 
 $\mapsto 0,2,0 \Rightarrow 8$ 
 $\mapsto 0,2,1 \Rightarrow 0$ 
 $\mapsto 0,2,2 \Rightarrow 0$ 
 $\mapsto 0,3,0 \Rightarrow 10$ 
 $\mapsto 0,3,1 \Rightarrow 0$ 
 $\mapsto 0,3,2 \Rightarrow 0$ 
 $\mapsto 1,-2,0 \Rightarrow 0$ 
 $\mapsto 1,-2,1 \Rightarrow 0$ 
 $\mapsto 1,-2,2 \Rightarrow 0$ 
 $\mapsto 1,-1,0 \Rightarrow 3$ 
 $\mapsto 1,-1,1 \Rightarrow 0$ 
 $\mapsto 1,-1,2 \Rightarrow 0$ 
 $\mapsto 1,0,0 \Rightarrow 6$ 
 $\mapsto 1,0,1 \Rightarrow 0$ 
 $\mapsto 1,0,2 \Rightarrow 0$ 
 $\mapsto 1,1,0 \Rightarrow 9$ 
 $\mapsto 1,1,1 \Rightarrow 0$ 
 $\mapsto 1,1,2 \Rightarrow 0$ 
 $\mapsto 1,2,0 \Rightarrow 12$ 
 $\mapsto 1,2,1 \Rightarrow 0$ 
 $\mapsto 1,2,2 \Rightarrow 0$ 
 $\mapsto 1,3,0 \Rightarrow 15$ 
 $\mapsto 1,3,1 \Rightarrow 0$ 
 $\mapsto 1,3,2 \Rightarrow 0$ 
 $\mapsto 2,-2,0 \Rightarrow 0$ 
 $\mapsto 2,-2,1 \Rightarrow 0$ 
 $\mapsto 2,-2,2 \Rightarrow 0$ 
 $\mapsto 2,-1,0 \Rightarrow 4$ 
 $\mapsto 2,-1,1 \Rightarrow 0$ 
 $\mapsto 2,-1,2 \Rightarrow 0$ 
 $\mapsto 2,0,0 \Rightarrow 8$ 
 $\mapsto 2,0,1 \Rightarrow 0$ 
 $\mapsto 2,0,2 \Rightarrow 0$ 
 $\mapsto 2,1,0 \Rightarrow 12$ 
 $\mapsto 2,1,1 \Rightarrow 0$ 
 $\mapsto 2,1,2 \Rightarrow 0$ 
 $\mapsto 2,2,0 \Rightarrow 16$ 
 $\mapsto 2,2,1 \Rightarrow 0$ 
 $\mapsto 2,2,2 \Rightarrow 0$ 

```

```

↳ 2,3,0 => 20
↳ 2,3,1 => 0
↳ 3,-2,0 => 0
↳ 3,-2,1 => 0
↳ 3,-2,2 => 0
↳ 3,-1,0 => 5
↳ 3,-1,1 => 0
↳ 3,-1,2 => 0
↳ 3,0,0 => 10
↳ 3,0,1 => 0
↳ 3,0,2 => 0
↳ 3,1,0 => 15
↳ 3,1,1 => 0
↳ 3,1,2 => 0
↳ 3,2,0 => 20
↳ 3,2,1 => 0
↳ 3,3,0 => 25
intvec d = 3,3,0;
cohomologytable*d;
↳ 25
def (Z,eulerTable) = eulerPolynomialTable(M,low,high);
setring(Z);
print(eulerTable);
↳ -3,-3 => h2
↳ -2,-3 => 0
↳ -1,-3 => h
↳ 0,-3 => 2h
↳ 1,-3 => 3h
↳ 2,-3 => 4h
↳ 3,-3 => 5h
↳ -3,-2 => 0
↳ -3,-1 => h
↳ -3,0 => 2h
↳ -3,1 => 3h
↳ -3,2 => 4h
↳ -3,3 => 5h
↳ -2,-2 => 0
↳ -2,-1 => 0
↳ -2,0 => 0
↳ -2,1 => 0
↳ -2,2 => 0
↳ -2,3 => 0
↳ -1,-2 => 0
↳ -1,-1 => 1
↳ -1,0 => 2
↳ -1,1 => 3
↳ -1,2 => 4
↳ -1,3 => 5
↳ 0,-2 => 0
↳ 0,-1 => 2
↳ 0,0 => 4
↳ 0,1 => 6
↳ 0,2 => 8

```



```

⇒ 0,3 => 10
⇒ 1,-2 => 0
⇒ 1,-1 => 3
⇒ 1,0 => 6
⇒ 1,1 => 9
⇒ 1,2 => 12
⇒ 1,3 => 15
⇒ 2,-2 => 0
⇒ 2,-1 => 4
⇒ 2,0 => 8
⇒ 2,1 => 12
⇒ 2,2 => 16
⇒ 2,3 => 20
⇒ 3,-2 => 0
⇒ 3,-1 => 5
⇒ 3,0 => 10
⇒ 3,1 => 15
⇒ 3,2 => 20
⇒ 3,3 => 25

```

#### D.15.24.11 twist

Procedure from library `tateProdCplxNegGrad.lib` (see [Section D.15.24 \[tateProdCplxNegGrad.lib\]](#), page 2555).

**Usage:** `twist(M,c)`; `M` module, `c` intvec

**Purpose:** twists the module `M` by `c`

**Assume:** `M` is a multigraded module

**Return:** `M` with the new grading

**Example:**

```

LIB "tateProdCplxNegGrad.lib";
intvec c = 1,1;
def (S,E) = productOfProjectiveSpaces(c);
setring(S);
module M = freemodule(2);
intmat gradeM[2][2] = 0,1,0,1;
M = setModuleGrading(M,gradeM);
getModuleGrading(M);
⇒ 0,1,
⇒ 0,1
intvec c = 2,2;
⇒ // ** redefining c (intvec c = 2,2;) ./examples/twist.sing:9
module Mtwist = twist(M,c);
getModuleGrading(Mtwist);
⇒ -2,-1,
⇒ -2,-1

```

#### D.15.24.12 beilinsonWindow

Procedure from library `tateProdCplxNegGrad.lib` (see [Section D.15.24 \[tateProdCplxNegGrad.lib\]](#), page 2555).

- Usage:** `beilinsonWindow(T);`  $T$  multigradedcomplex
- Purpose:** compute the subquotient complex of  $T$  consisting of summands generated in degrees  $0 \leq a \leq n$
- Assume:**  $T$  is a multigraded complex of free  $E$ -modules
- Return:** multigradedcomplex, the Beilinson window of  $T$
- Note:** The returned summands are the only ones that contribute to the Beilinson monad.

**Example:**

```
LIB "tateProdCplxNegGrad.lib";
intvec f = 1,1;
def (S,E) = productOfProjectiveSpaces(f);
intvec low = -3,-3;
intvec high = 3,3;
setring(S);
module M = 0;
intmat MGrading[2][1] = -1,-1;
M = setModuleGrading(M,MGrading);
multigradedcomplex tate;
(E,tate) = tateResolution(M,low,high);
setring(E);
multigradedcomplex W = beilinsonWindow(tate);
W;
↪ 0 <-- E^4 <-- E^4 <-- E^1 <-- 0
↪ -1      0      1      2      3
↪
intvec c = 1,1,1;
intvec low2 = 0,0,0;
intvec high2 = 0,1,0;
def (S2,E2) = productOfProjectiveSpaces(c);
setring(S2);
module M2 = 0;
intmat gradeM[3][1] = -1,-1,-1;
M2 = setModuleGrading(M2,gradeM);
multigradedcomplex tate2;
(E2,tate2) = tateResolution(M2,low2,high2);
setring(E2);
multigradedcomplex W2 = beilinsonWindow(tate2);
W2;
↪ 0 <-- E^8
↪ -1      0
↪
```

**D.15.24.13 regionComplex**

Procedure from library `tateProdCplxNegGrad.lib` (see [Section D.15.24 \[tateProdCplxNegGrad.lib\]](#), page 2555).

- Usage:** `regionComplex(T,d,I,J,K);`  $T$  multigradedcomplex,  $d$  intvec,  $I$  intvec,  $J$  intvec,  $K$  intvec
- Purpose:** compute the region complex of  $T$  w.r.t. the sets  $I,J,K$  and the vector  $d$
- Assume:**  $I,J,K$  are intvecs representing disjoint subsets of  $\{1,\dots,t\}$ ,  $T$  is a complex in ring  $E$ , zero represents the empty set

**Return:** multigraded complex which is the region complex  $T_d(I,J,K)$  of  $T$

**Example:**

```
LIB "tateProdCplxNegGrad.lib";
intvec f = 1,1;
def (S,E) = productOfProjectiveSpaces(f);
intvec low = -3,-3;
intvec high = 3,3;
setring(S);
module M = 0;
intmat MGrading[2][1] = -1,-1;
M = setModuleGrading(M,MGrading);
multigradedcomplex tate;
(E,tate) = tateResolution(M,low,high);
setring(E);
tate;
⇨ E^25 <-- E^40 <-- E^46 <-- E^44 <-- E^35 <-- E^30 <-- E^38 <\
    -- E^56 <-- E^81 <-- E^110 <-- E^141 <-- E^174 <-- E^210
⇨ -6      -5      -4      -3      -2      -1      0      \
    1          2          3          4          5          6
⇨
ring Z = cohomologyMatrixFromResolution(tate,low,high);
setring(Z);
print(cohomologymat);
⇨ 5h,0,5,10,15,20,25,
⇨ 4h,0,4,8, 12,16,20,
⇨ 3h,0,3,6, 9, 12,15,
⇨ 2h,0,2,4, 6, 8, 10,
⇨ h, 0,1,2, 3, 4, 5,
⇨ 0, 0,0,0, 0, 0, 0,
⇨ h2,0,h,2h,3h,4h,5h
setring(E);
intvec c= 0,-3;
intvec I = 0;
intvec J = 0,1;
intvec K = 0,2;
multigradedcomplex U = regionComplex(tate,c,I,J,K);
U;
⇨ 0 <-- E^10 <-- E^8 <-- E^6 <-- E^4 <-- E^2 <-- E^2 <-- 0
⇨ -4      -3      -2      -1      0      1      2      3
⇨
Z = cohomologyMatrixFromResolution(U,low,high);
setring(Z);
print(cohomologymat);
⇨ 0,0,0,10,0,0,0,
⇨ 0,0,0,8, 0,0,0,
⇨ 0,0,0,6, 0,0,0,
⇨ 0,0,0,4, 0,0,0,
⇨ 0,0,0,2, 0,0,0,
⇨ 0,0,0,0, 0,0,0,
⇨ 0,0,0,2h,0,0,0
setring(E);
multigradedcomplex V = regionComplex(tate,c,I,J,J);
```

$\mapsto$  ? I,J,K have to be disjoint.  
 $\mapsto$  ? leaving `tateProdCplxNegGrad.lib::regionComplex (0)`

### D.15.24.14 strand

Procedure from library `tateProdCplxNegGrad.lib` (see [Section D.15.24 \[tateProdCplxNegGrad.lib\]](#), page 2555).

**Usage:** `strand(T,c,J)`

**Purpose:** compute the strand of T w.r.t. the set J and the vector c

**Return:** subquotient complex of T which is the strand of T

**Example:**

```

LIB "tateProdCplxNegGrad.lib";
intvec f = 1,1;
def (S,E) = productOfProjectiveSpaces(f);
intvec low = -3,-3;
intvec high = 3,3;
setring(S);
module M = 0;
intmat MGrading[2][1] = -1,-1;
M = setModuleGrading(M,MGrading);
multigradedcomplex tate;
(E,tate) = tateResolution(M,low,high);
setring(E);
ring Z = cohomologyMatrixFromResolution(tate,low,high);
setring(Z);
print(cohomologymat);
 $\mapsto$  5h,0,5,10,15,20,25,
 $\mapsto$  4h,0,4,8, 12,16,20,
 $\mapsto$  3h,0,3,6, 9, 12,15,
 $\mapsto$  2h,0,2,4, 6, 8, 10,
 $\mapsto$  h, 0,1,2, 3, 4, 5,
 $\mapsto$  0, 0,0,0, 0, 0, 0,
 $\mapsto$  h2,0,h,2h,3h,4h,5h
setring(E);
intvec c= 0,-3;
intvec J = 1;
multigradedcomplex U = strand(tate,c,J);
U;
 $\mapsto$  0 <-- E^10 <-- E^8 <-- E^6 <-- E^4 <-- E^2 <-- E^2 <-- E^4 \
      <-- E^6 <-- E^8 <-- E^10
 $\mapsto$  -4      -3      -2      -1      0      1      2      3 \
      4      5      6
 $\mapsto$ 
Z = cohomologyMatrixFromResolution(U,low,high);
setring(Z);
print(cohomologymat);
 $\mapsto$  0,0,0,10,0,0,0,
 $\mapsto$  0,0,0,8, 0,0,0,
 $\mapsto$  0,0,0,6, 0,0,0,
 $\mapsto$  0,0,0,4, 0,0,0,
 $\mapsto$  0,0,0,2, 0,0,0,
```

$\mapsto 0,0,0,0, 0,0,0,$   
 $\mapsto 0,0,0,2h,0,0,0$

### D.15.24.15 firstQuadrantComplex

Procedure from library `tateProdCplxNegGrad.lib` (see [Section D.15.24 \[tateProdCplxNegGrad.lib\]](#), page 2555).

**Usage:** `firstQuadrantComplex(T,c);` T multigradedcomplex, c intvec  
**Purpose:** compute the first quadrant complex of T w.r.t. the set J and the vector c  
**Return:** subquotient complex of T which is the first quadrant complex of T

### D.15.24.16 lastQuadrantComplex

Procedure from library `tateProdCplxNegGrad.lib` (see [Section D.15.24 \[tateProdCplxNegGrad.lib\]](#), page 2555).

**Usage:** `lastQuadrantComplex(T,c);` T multigradedcomplex, c intvec  
**Purpose:** compute the last quadrant complex of T w.r.t. the set J and the vector c  
**Return:** subquotient complex of T which is the strand of T

## D.15.25 tropicalEllipticCovers.lib

**Library:** `tropicalEllipticCovers.lib`

**Purpose:** Gromov-Witten numbers of tropical elliptic curves and their covers

**Authors:** J. Boehm, boehm @ mathematik.uni-kl.de  
 F. Dastur, dastur.firoozeh @ gmail.com

**Overview:** We implement a formula for computing the number of covers of (abstract) elliptic curves with and without tangency conditions, and for the surface `ExP1`. For the first case, we have used methods discussed in [BGM1]. For the covers without tangency conditions, we obtain the covers of an elliptic curve using tropical methods in [BBM], and for `ExP1`, we have used the concepts from [BGM2]. A Feynman graph  $G$  of genus  $g$  is a trivalent, connected graph (of genus  $g$ ) with  $n$  number of vertices and  $r$  number of edges (in case of abstract elliptic curve in [BBM] we have  $2g-2$  vertices and  $3g-3$  edges, since we consider only trivalent graphs). A pearl chain  $P$  of type  $(d_2, g)$  is a genus  $g$ , connected graph of degree  $d_1$  such that it has two different types of vertices, say vertices of type  $w$  and  $b$  (where  $w$  represent the white vertices and  $b$  the black ones). Every edge in  $P$  connect vertex of type  $w$  to vertex of type  $b$  such that the vertices adjacent to each edge are not of the same type. There are precisely  $d_2$  vertices of type  $w$  and  $d_2+g-1$  vertices of type  $b$ . The vertices of type  $b$  should be 2-valent and the vertices of type  $w$  can have any valency.  
 For each case we compute the respective propagator function from which we later use terms to compute the Gromov-Witten invariants  $N_-(g,d)$ ,  $N_-(g,d,\Omega)$  and  $N_-(P,d_1,d_2,g)$ .

**References:**

- [BBM] J. Boehm, A. Buchholz, H. Markwig: Tropical mirror symmetry for elliptic curves, arXiv:1309.5893 (2013).
- [BGM1] J. Boehm, C. Goldner, H. Markwig: Tropical mirror symmetry in dimension

one, arXiv:1809.10659 (2018).

[BGM2] J. Boehm, C. Goldner, H. Markwig, Counts of (tropical) curves in  $\text{Exp1}$  and Feynman integrals, arXiv:1812.04936 (2018).

**Types:** graph,number,list,poly,string

**Procedures:**

### D.15.25.1 cf\_of

Procedure from library `tropicalEllipticCovers.lib` (see [Section D.15.25 \[tropicalEllipticCovers.lib\]](#), page 2575).

**Usage:** cf\_of(f,g); f poly, g poly

**Assume:** f is a polynomial, g is a monomial

**Return:** poly.

**Theory:** For a given polynomial f, it computes the coefficient of g in f

**Example:**

```
LIB "tropicalEllipticCovers.lib";
ring r=0,(x1,x2,x3,x4),dp;
poly f=3*x1^6+11*x1^4*x2^2+x1^2*x2^4+3*x2^6;
cf_of(f,x1^4*x2^2);
```

### D.15.25.2 preim

Procedure from library `tropicalEllipticCovers.lib` (see [Section D.15.25 \[tropicalEllipticCovers.lib\]](#), page 2575).

**Usage:** preim(O, x); O list, x poly/number

**Assume:** O is a list of ordering of the set of vertices (or indices of the vertices), x is a monomial representing a vertex of a graph (or the index of the vertex we are interested in).

**Return:** int.

**Theory:** This gives the entry number of x in list O.

**Example:**

```
LIB "tropicalEllipticCovers.lib";
ring r=0,(x1,x2,x3,x4),dp;
preim(list(x1,x3,x4,x2),x2);
```

### D.15.25.3 coverTuple

Procedure from library `tropicalEllipticCovers.lib` (see [Section D.15.25 \[tropicalEllipticCovers.lib\]](#), page 2575).

**Usage:** coverTuple(G,aa,O,t[,l]); G graph, aa list, O list, t int, l list

**Assume:**  $G$  is a Feynman graph,  $aa$  is a partition of the degree of the graph  $G$ ,  $O$  is a given ordering of the vertices of  $G$ ,  $t$  is an integer and  $l$  is a list of leaks in  $G$ .

**Return:** list.

**Theory:** Computes a list of tuples (depending on the int  $t$ ) of the form  $(a_k, N_k, D_k)$  for all  $k=1, \dots, r$ , where  $r$  is the number of edges,  $a_k$  is the  $k$ -th entry in the partition  $aa$  of the degree of  $G$ ,  $N_k$  and  $D_k$  are the corresponding numerator and denominator (respectively) of the propagator function whose product would return the coefficient of the Feynman integral.

Every list of the tuples corresponds to a cover of the elliptic curve  $E$ . (For  $t=0$ , we get the results for simply ramified covers [BBM].  $t=1, 2$  we get results for graphs w/o vertex contributions respectively w/ vertex contributions as in [BGM1], and for  $t=3$  we get the results for pearl chains as in [BGM2])

**Example:**

```
LIB "tropicalEllipticCovers.lib";
ring r=0, (x1,x2,x3,z1,z2,z3), dp;
graph F=makeGraph(list(1,2,3),list(list(1,1),list(1,2),list(2,3),list(1,3)));
coverTuple(F,list(1,0,0,2),list(x1,x2,x3),2,list(0,0,0));
```

#### D.15.25.4 CoverMult

Procedure from library `tropicalEllipticCovers.lib` (see [Section D.15.25 \[tropicalEllipticCovers.lib\]](#), page 2575).

**Usage:** `CoverMult(G,aa,O,t[, l]);`  $G$  graph,  $aa$  list,  $O$  list,  $t$  int,  $l$  list

**Assume:**  $G$  is a Feynman graph,  $aa$  is a partition of the degree of the graph  $G$ ,  $O$  is a given ordering of the vertices of  $G$ ,  $l$  is a list of leaks of  $G$  and  $t$  is some integer between 0 and 3.

**Return:** list.

**Theory:** For a given ordering of the vertices of  $G$ , this function computes a list of multiplicity for each cover of  $G$ . Each entry in the list is a multiplicity of a cover of  $G$ . (For  $t=0$ , we get the results for simply ramified covers [BBM].  $t=1, 2$  we get results for graphs w/o vertex contributions respectively w/ vertex contributions as in [BGM1], and for  $t=3$  we get the results for pearl chains as in [BGM2])

**Example:**

```
LIB "tropicalEllipticCovers.lib";
ring r=0, (x1,x2,x3,x4), dp;
graph G=makeGraph(list(1,2,3,4),list(list(1,3),list(1,2),list(1,2),list(2,4),list(3,4)));
CoverMult(G,list(0,2,1,0,0,1), list(x2,x4,x3,x1),0);
```

#### D.15.25.5 Tropicalcover

Procedure from library `tropicalEllipticCovers.lib` (see [Section D.15.25 \[tropicalEllipticCovers.lib\]](#), page 2575).

**Usage:** `Tropicalcover(G,aa,O,t[, l]);`  $G$  graph,  $aa$  list,  $O$  list,  $t$  int,  $l$  list

**Assume:**  $G$  is a Feynman graph,  $aa$  is a partition of the degree of the graph  $G$ ,  $O$  is a given ordering of the vertices of  $G$ ,  $t$  is an integer and  $l$  is a list of leaks of  $G$ .

**Return:** list of matrices.

**Theory:** For a given ordering of the vertices of  $G$ , this function computes a list of matrices such that each matrix in the list represents a cover of the elliptic curve  $E$ . The first column of each matrix shows how many points on each edge of the cover (of type  $G$ ) has a preimage of the base point  $p_0$ . The rows of the matrix represents an edge of the cover and each non-zero entry in a row represents a vertex connected to that edge. (For  $t=0$ , we get the results for simply ramified covers [BBM].  $t=1, 2$  we get results for graphs w/o vertex contributions respectively w/ vertex contributions as in [BGM1], and for  $t=3$  we get the results for pearl chains as in [BGM2])

**Example:**

```
LIB "tropicalEllipticCovers.lib";
ring r=0, (x1,x2,x3,x4),dp;
graph G=makeGraph(list(1,2,3,4),list(list(1,3),list(1,2),list(1,2),list(2,4),list(3,4)),
Tropicalcover(G,list(0,2,1,0,0,1), list(x2,x4,x3,x1),0);
```

### D.15.25.6 sinh

Procedure from library `tropicalEllipticCovers.lib` (see [Section D.15.25 \[tropicalEllipticCovers.lib\]](#), page 2575).

**Usage:** `sinh(t,n);`  $t$  poly,  $n$  int

**Assume:**  $t$  is a polynomial and  $n$  is a non-zero integer

**Return:** poly

**Theory:** Returns the power series expansion of the hyperbolic sine function up to  $n$  terms.

**Example:**

```
LIB "tropicalEllipticCovers.lib";
ring R=0,(x,y),dp;
sinh(x,5); // expands sine hyperbolic function for variable x to first 5 terms
```

### D.15.25.7 Sfunction

Procedure from library `tropicalEllipticCovers.lib` (see [Section D.15.25 \[tropicalEllipticCovers.lib\]](#), page 2575).

**Usage:** `Sfunction(z,k);`  $z$  poly,  $k$  int

**Assume:**  $z$  is a polynomial and  $k$  is an integer

**Return:** poly

**Theory:** Returns the S-function in the form of  $(1+X)$ , where  $X$  is a Taylor series expansion of the sinh function upto  $k$  terms.

**Example:**

```
LIB "tropicalEllipticCovers.lib";
ring R=0,(x,y),dp;
Sfunction(x,3);
```



### D.15.25.8 FloorDiagrams

Procedure from library `tropicalEllipticCovers.lib` (see [Section D.15.25 \[tropicalEllipticCovers.lib\]](#), page 2575).

**Usage:** `FloorDiagrams(G,aa,O,Lw,n)`; `G` graph, `aa` list, `O` list, `Lw` list, `n` int

**Assume:** `G` is a pearl chain of the degree `d`, `aa` is a partition of degree `d`, `O` is an ordering of the vertices of `G`, `Lw` is a list where the first `n` terms represent the leaks in `G` and the next `d2` terms is a list of white pearls (assuming the first pearl is always white). NOTE: Requires `pdflatex` to be installed.

**Return:** Latex file.

**Theory:** If `n=0`: the procedure generates a Latex file for all the tropical stable maps to `Exp1` and if `n=1`: the procedure generates the source file that can be executed in Latex to get all the stable maps of `Exp1` (in the form of floor diagrams) for every curled pearl chain of an elliptic curve `E` provided similar information.

**Example:**

```
LIB "tropicalEllipticCovers.lib";
ring R=0,(x1,x2,x3,x4,x5),dp;
graph P=makeGraph(list(1,2,3,4,5),list(list(1,2),list(1,3),list(3,4),list(2,4),list(4,5)),list(1,2,3,4,5));
//graph P is a pearl chain
FloorDiagrams(P,list(0,0,2,0,0,1),list(x5,x4,x2,x1,x3),list(0,0,0,0,0,1,4),0);
```

### D.15.25.9 PropagatorFunction

Procedure from library `tropicalEllipticCovers.lib` (see [Section D.15.25 \[tropicalEllipticCovers.lib\]](#), page 2575).

**Usage:** `PropagatorFunction(G,aa,O,t[,l])`; `G` graph, `aa` list, `O` list, `t` int, `l` list

**Assume:** `G` is a graph (a Feynman graph or a pearl chain) of the degree `d`, `aa` is a partition of degree `d`, `O` is an ordering of the vertices of `G`, `l` is a list representing the leaky edges of `G` and `t` is one of 0,1,2 or 3.

**Return:** list `P_t` of propagator function for each edge of `G` as follows: `P_0`: Propagator for a Feynman graph `G` as in [BBM], i.e., a graph without any self-looping edges, leaks or vertex contributions.  
`P_1`: Propagator for a Feynman graph `G` without vertex contributions as in [BGM1], i.e., A graph that may have self-looping edges and leaks. `P_2`: Propagator for a Feynman graph `G` with vertex contributions as in [BGM1] possibly with self-looping edges and leaks.  
`P_3`: Propagator for a pearl chain `G` as in [BGM2], i.e., graph `G` may have leaks.

**Theory:** Computes the numerator and the denominator of the propagator function for every edge of graph `G` depending on the integer `t`.

**Example:**

```
LIB "tropicalEllipticCovers.lib";
ring r1=0, (x1,x2,x3,x4),dp;
graph G = makeGraph(list(1,2,3,4),list(list(1,3),list(1,2),list(1,2),list(2,4),list(4,5)),list(1,2,3,4));
PropagatorFunction(G,list(0,2,1,0,0,1),list(x1,x3,x4,x2),0);
//This is another example:
```

```

ring r2=0,(x1,x2,x3,z1,z2,z3),dp;
graph GG = makeGraph(list(1,2,3),list(list(1,1),list(1,2),list(2,3),list(3,1)));
PropagatorFunction(GG,list(1,0,0,3),list(x1,x2,x3),1);
//Yet another example
ring r3=0,(x1,x2,x3,x4,x5),dp;
graph P=makeGraph(list(1,2,3,4),list(list(1,2),list(2,3),list(3,4),list(4,1)));
//graph P is a pearl chain
PropagatorFunction(P, list (1,0,0,1),list (x1,x2,x3,x4),3);

```

### D.15.25.10 FeynmanIntegralo

Procedure from library `tropicalEllipticCovers.lib` (see [Section D.15.25 \[tropicalEllipticCovers.lib\]](#), page 2575).

**Usage:** `FeynmanIntegralo(G,aa,O,l,k,t[,gg]);` G graph, aa list, O list, l list, k int, t int, gg list

**Assume:** G is a graph (a Feynman graph or a pearl chain) of the degree d, aa is a partition of degree d, O is an ordering of the vertices of G, l is a list representing the leaky edges of G, k is any integer, gg is a list representing the genus function and t is one of 0,1,2 or 3.

**Return:** number or list Q\_t (depending on k) of Feynman integral for a fixed ordering of vertices of G, the results are as follows:

Q\_0: Feynman integral for a fixed ordering of a Feynman graph G as in [BBM], i.e., a graph without any self-looping edges, leaks or vertex contributions. Q\_1: Feynman integral for a fixed ordering of a Feynman graph G without vertex contributions as in [BGM1], i.e. A graph that may have self-looping edges and leaks.

Q\_2: Feynman integral for a fixed ordering of a Feynman graph G with vertex contributions as in [BGM1] possibly with self-looping edges and leaks. Q\_3: Feynman integral for a fixed ordering of a pearl chain G as in [BGM2], i.e., graph G may have leaks.

**Theory:** If k is zero it returns the coefficient of the Feynman integral for a given ordering of the vertices of the graph G. Otherwise, returns a list showing the ordering and the coefficient of the Feynman integral for the corresponding ordering.

**Example:**

```

LIB "tropicalEllipticCovers.lib";
ring r1=0, (x1,x2,x3,x4),dp;
graph G = makeGraph(list(1,2,3,4),list(list(1,3),list(1,2),list(1,2),list(2,4),list(3,4)));
FeynmanIntegralo(G,list (0,2,1,0,0,1), list (x1,x3,x4,x2),list(0,0,0,0,0,0),1,0);
//This is another example:
ring r2=0,(x1,x2,x3,z1,z2,z3),dp;
graph GG = makeGraph(list(1,2,3),list(list(1,1),list(1,2),list(2,3),list(3,1)));
FeynmanIntegralo(GG, list (2,0,0,1), list (x1,x2,x3),list (0,0,0), 0,1);
//Yet another example
ring r3=0,(x1,x2,x3,x4,x5),dp;
graph P=makeGraph(list(1,2,3,4),list(list(1,2),list(2,3),list(3,4),list(4,1)));
//graph P is a pearl chain
FeynmanIntegralo(P,list (1,0,0,1),list (x1,x2,x3,x4), list (0,0,0,0), 0,3);

```

### D.15.25.11 FeynmanIntegralO

Procedure from library `tropicalEllipticCovers.lib` (see [Section D.15.25 \[tropicalEllipticCovers.lib\]](#), page 2575).

**Usage:** `FeynmanIntegralO(G,aa,l,k,t[gg]);` G graph, aa list, l list, k int, t int, gg list

**Assume:** G is a graph (a Feynman graph or a pearl chain) of the degree d, aa is a partition of degree d, l is a list representing the leaky edges of G, k is any integer, gg is a list representing the genus function and t is one of 0,1,2 or 3.

**Return:** number or list  $Q_t$  (depending on k) of Feynman integral for all possible orderings of vertices of G, the results are as follows:  $Q_0$ : Feynman integral for all orderings of a Feynman graph G as in [BBM], i.e., a graph without any self-looping edges, leaks or vertex contributions.  $Q_1$ : Feynman integral for all orderings of a Feynman graph G without vertex contributions as in [BGM1], i.e. A graph that may have self-looping edges and leaks.  $Q_2$ : Feynman integral for all orderings of a Feynman graph G with vertex contributions as in [BGM1] possibly with self-looping edges and leaks.  $Q_3$ : Feynman integral for all orderings of a pearl chain G as in [BGM2], i.e., a graph G may have leaks.

**Theory:** If k is zero it returns the coefficient which is a sum of Feynman integrals over all orderings of the vertices of the graph G. Otherwise, returns a list showing the ordering, the coefficient of the Feynman integral for the corresponding ordering and the sum.

**Example:**

```
LIB "tropicalEllipticCovers.lib";
ring r1=0, (x1,x2,x3,x4),dp;
graph G = makeGraph(list(1,2,3,4),list(list(1,3),list(1,2),list(1,2),list(2,4),list(3,4)),list(0,0,0,0,0,0,0,0));
FeynmanIntegralO(G,list (0,2,1,0,0,1),list (0,0,0,0,0,0),1,0);
```

### D.15.25.12 FeynmanIntegralA

Procedure from library `tropicalEllipticCovers.lib` (see [Section D.15.25 \[tropicalEllipticCovers.lib\]](#), page 2575).

**Usage:** `FeynmanIntegralA(G,O,d,l,k,t[gg]);` G graph, O list, d int, l list, k int, t int, gg list

**Assume:** G is a graph (a Feynman graph or a pearl chain) of the degree d, O is an ordering of vertices of G, d is the degree of G, l is a list representing the leaky edges of G, k is any integer, gg is a list representing the genus function and t is one of 0,1,2 or 3.

**Return:** number or list  $Q_t$  (depending on k) of Feynman integral for a fixed ordering of vertices of G, the results are as follows:  
 $Q_0$ : sum of Feynman integrals (over all partitions of d) for a fixed ordering of a Feynman graph G as in [BBM], i.e. a graph without any self-looping edges, leaks or vertex contributions.  
 $Q_1$ : sum of Feynman integrals (over all partitions of d) for a fixed ordering of a Feynman graph G without vertex contributions as in [BGM1], i.e. A graph that may have self-looping edges and leaks.  
 $Q_2$ : sum of Feynman integrals (over all partitions of d) for a fixed ordering of a Feynman graph G with vertex contributions as in [BGM1] possibly with self-looping edges and leaks.  
 $Q_3$ : sum of Feynman integrals (over all partitions of d) for a fixed ordering of a pearl chain G as in [BGM2], i.e. graph G may have leaks.

**Theory:** If k is zero it returns the coefficient which is a sum of Feynman integrals over all partitions of d. Otherwise, returns a list showing the partition from which the Feynman

integral is computed, the respective coefficient of the Feynman integral for the given ordering and the sum over all partitions.

**Example:**

```
LIB "tropicalEllipticCovers.lib";
ring r1=0, (x1,x2,x3,x4),dp;
graph G = makeGraph(list(1,2,3,4),list(list(1,3),list(1,2),list(1,2),list(2,4),list(3,4)),list(0,0,0,0,0,0,0,0));
FeynmanIntegralA(G,list(x1,x3,x4,x2),4,list(0,0,0,0,0,0,0,0));
```

### D.15.25.13 DrawCovers

Procedure from library `tropicalEllipticCovers.lib` (see [Section D.15.25 \[tropicalEllipticCovers.lib\]](#), page 2575).

**Usage:** `DrawCovers(G,aa,O,t[, lw]);` G graph, aa list, O list, t int,n int and lw list

**Assume:** G is a Feynman graph/pearl chain, aa is a partition of the degree of the graph G, O is a given ordering of the vertices of G, t is an integer between 0 and 3 and lw is a list where the first n entries represent the leaks in G and the remaining terms tells us about the white pearls of G. NOTE: Requires `pdflatex` to be installed.

**Return:** if n=0: C.t is a Latex file generating all the covers of an elliptic curve and if n=1: C.t is the source file that can be executed in Latex to get the covers of an elliptic curve (given an ordering O and a partition aa). The results for C.t are as follows:  
C.0: returns a Latex file drawing all simply ramified covers of an elliptic curve given a fixed ordering of the vertices of G and a fixed partition of its degree. C.1: returns a Latex file drawing all covers of an elliptic curve such that G has no vertex contributions, given a fixed ordering of the vertices of G and a fixed partition of its degree.  
C.2: returns a Latex file drawing all covers of an elliptic curve such that G has vertex contributions, given a fixed ordering of the vertices of G and a fixed partition of its degree.  
C.3: returns a Latex file drawing all curled pearl chains of an elliptic curve given a fixed ordering of the vertices of G and a fixed partition of its degree.

**Theory:** For a given ordering of the vertices of G, this function computes Latex file drawing all the covers of an elliptic curve E. If G is a pearl chain then this procedure draws the black and white pearls of G accordingly using information from list lw.  
(For t=0, we get the results for simply ramified covers as in [BBM]. For t=1, 2 we get results for graphs w/o vertex contributions respectively w/ vertex contributions as in [BGM1], and for t=3 we get the results for pearl chains as in [BGM2])

**Example:**

```
LIB "tropicalEllipticCovers.lib";
ring r1=0, (x1,x2,x3,x4),dp;
graph G=makeGraph(list(1,2,3,4),list(list(1,3),list(1,2),list(1,2),list(2,4),list(3,4)),list(0,0,0,0,0,0,0,0));
DrawCovers(G,list(0,2,1,0,0,1), list(x2,x4,x3,x1),0,0);
//This is another example:
ring r2=0,(x1,x2,x3,z1,z2,z3),dp;
//F is graph with loops
graph F=makeGraph(list(1,2,3),list(list(1,1),list(1,2),list(2,3),list(1,3)));
DrawCovers(F,list(1,0,0,2),list(x1,x2,x3),2,0,list(0,0,0));
//Yet another example
ring r3=0,(x1,x2,x3,x4,x5),dp;
```

```
//in P vertices 1 and 3 are white, genus 2
graph P=makeGraph(list(1,2,3,4,5),list(list(1,2),list(2,3),list(3,4),list(1,4),list(
DrawCovers(P,list(0,0,0,1,2,0),list(x1,x2,x3,x4,x5),3,1, list(0,0,0,0,0,1,3));
```

### D.15.25.14 TropCovandMaps

Procedure from library `tropicalEllipticCovers.lib` (see [Section D.15.25 \[tropicalEllipticCovers.lib\]](#), [page 2575](#)).

**Usage:** TropCovandMaps(G,aa,O,Lw,n); G graph, aa list, O list, Lw list, n int

**Assume:** G is a pearl chain, aa is a partition of the degree of the graph G, O is a given ordering of the vertices of G, Lw is a list of leaks and white vertices in G. NOTE: Requires `pdflatex` to be installed.

**Return:** Depending on n this procedure returns either a pdf or a latex file for the curled pearl chains of an elliptic curve and their corresponding stable maps of  $\text{Exp1}$  for a fixed ordering of the vertices of the source curve and a fixed partition of its degree.

**Theory:** For a given ordering of the vertices of G, this function computes a list of commands that can be executed in TikZ. Each list represents a cover of the elliptic curve E and its corresponding stable map of  $\text{Exp1}$ . Since G is a pearl chain this procedure draws the black and white pearls of G accordingly using information from list lw.

**Example:**

```
LIB "tropicalEllipticCovers.lib";
ring r=0,(x1,x2,x3,x4,x5),dp;
graph P=makeGraph(list(1,2,3,4,5),list(list(1,2),list(2,3),list(3,4),list(4,1),list(
graph P1=makeGraph(list(1,2,3,4),list(list(1,2),list(2,3),list(3,4),list(4,1)));
//graphs P and P1 are pearl chains
TropCovandMaps(P,list(0,0,0,1,2,0),list(x1,x2,x3,x4,x5),list(0,0,0,0,0,1,3), 0);
```

### D.15.26 VecField.lib

**Library:** VecField.lib

**Purpose:** vector fields, with algorithms for jordan and diagonal forms

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**References:**

[1] Kyoji Saito, Quasihomogene isolierte Singularitaeten von Hyperflaechen, 1971

**Overview:** Implements a class `VecField`, represented by a vector. For example, `'VecField V = [x3,xy]'` declares the vector field  $v = x^3 dx + xy dy$ . Instead of a vector, an  $n \times 1$  matrix is also accepted. The vector can be recovered as `V.vec`.

Supports coordinate transformations (via maps), which are represented by tracking a map `'V.coordinates'` which maps the standard coordinates to those in which V is currently represented. `V.dimension` stores the vector field's dimension, which is just `nvars(basering)`, and `V.lin` yields the linear part of V. You may set an additional parameter `V.precision`, which dictates the degree to which operations on the vector field should be exact.

The default precision is 1. Precision is preserved across transformations, additions, and all other manipulations of vector fields.

**Procedures:****D.15.26.1 applyVecField**

Procedure from library `VecField.lib` (see [Section D.15.26 \[VecField.lib\]](#), page 2583).

**Usage:** `applyVecField(VecField V, poly/ideal p [,int n])`

**Return:** if  $p$  is a polynomial, return a polynomial  $V(p)$ , if  $p$  is an ideal, the ideal  $V(p)$ . If an optional  $n$  is given, only the parts of  $V$  up to and including degree  $n$  are applied to  $p$ .

**Example:**

```
LIB "VecField.lib";
int oldp = printlevel;
printlevel = 1;
ring r = 0, (x, y, z), ds;
vector v = [-1,-1,-1];
VecField V = v;
poly f = x3+2y3+xz2;
poly g = V*f;
ideal I = (x2,y);
ideal J = V*I;
printlevel = oldp;
```

**D.15.26.2 changeCoordinates**

Procedure from library `VecField.lib` (see [Section D.15.26 \[VecField.lib\]](#), page 2583).

**Usage:** `changeCoordinates(vecField V, map psi)`, where  $\psi$  is an algebra morphism  $k[x_1, \dots, x_n] \rightarrow k[y_1, \dots, y_m]$  expressing the new basis in terms of the old.

**Return:** A new vecfield in the new coordinates.

**Note:** the coordinate change necessitates inverting  $\psi$ .  
The inversion will be correct up to the precision of  $V$ .

**Example:**

```
LIB "VecField.lib";
ring r = 0, (x, y, z), ds;
vector v = [-1,-1,-1];
VecField V = v;
V.precision = 4;
map phi = r, x-2y2+z3, 2y+y3+z, z;
VecField W = changeCoordinates(V, phi);
```

**D.15.26.3 jordanVecField**

Procedure from library `VecField.lib` (see [Section D.15.26 \[VecField.lib\]](#), page 2583).

**Usage:** `jordanVecField(VecField V)`

**Return:** new vecfield  $W$  in coordinates s.t.  $W.lin$  is in Jordan normal form.

**Assume:** eigenvalues of  $V.lin$  in basefield.

**Example:**

```

LIB "VecField.lib";
ring r = 0, (x, y, z), ds;
vector v = [-1,-1,-1];
VecField V = v;
V.precision = 4;
map phi = r, x-2y2+z3, 2y+y3+z, z;
VecField W = changeCoordinates(V, phi);
VecField X = jordanVecField(W);

```

#### D.15.26.4 diagonalizeVecFieldLin

Procedure from library `VecField.lib` (see [Section D.15.26 \[VecField.lib\]](#), page 2583).

**Usage:** `diagonalizeVecFieldLin(list l)`, where `l` is a list of `VecFields`

**Return:** list `W` of the same `VecFields` in new coords s.t. the linear parts are diagonal, all in the same coordinates

**Assume:** All linear parts of the entries of `l` are simultaneously diagonalizable.

**Example:**

```

LIB "VecField.lib";
ring r = 0, (x, y, z), ds;
vector v = [-1,-1,-1];
VecField V = v;
V.precision = 4;
map phi = r, x-2y2+z3, 2y+y3+z, z;
VecField W = changeCoordinates(V, phi);
VecField X = [-2,1,0];
list d = diagonalizeVecFieldLin(W,X);

```

#### D.15.26.5 SaitoBase

Procedure from library `VecField.lib` (see [Section D.15.26 \[VecField.lib\]](#), page 2583).

**Usage:** `SaitoBase(VecField V)`

**Return:** new `VecField W` in the base from [1] thm. 3.1, where semisimple and nilpotent components are easily read off.

**Note:** the algorithm requires inversions of algebra morphisms. These will be exact to the precision of `V`. Warning: The algorithm assumes standard coordinates. If `V` is not given in standard coordinates, it will be converted to standard coordinates, but not converted back at the end, so the resulting transformation is from standard coordinates to the Saito coordinates. If you want the entire transformation from your original coordinates to the Saito ones, add the inverse of your original coordinate transformation manually. Note that weight vectors only take `Int64`, making the algorithm fail for very large entries in `V.vec`.

**Example:**

```

LIB "VecField.lib";
ring r = 0, (x, y, z), ds;
vector v = [-1,-1,-1];
VecField V = v;
V.precision = 4;
map phi = r, x-2y2+z3, 2y+y3+z, z;
VecField W = changeCoordinates(V, phi);
VecField WS = SaitoBase(W);

```



### D.15.26.6 diagonalizeVecField

Procedure from library `VecField.lib` (see [Section D.15.26 \[VecField.lib\]](#), page 2583).

**Usage:** `diagonalizeVecField(list l)`, `l` a list of `VecFields`

**Return:** list of `VecFields` `W` such that they are all simultaneously diagonal.

**Assume:** all input `VecFields` have to be simultaneously diagonalizable.

**Note:** the algorithm requires inversions of algebra morphisms. These will be exact to the precision of `l[1]`. Warning: The algorithm assumes standard coordinates. If `V` is not given in standard coordinates, it will be converted to standard coordinates, but not converted back at the end, so the resulting transformation is from standard coordinates to the Saito coordinates. If you want the entire transformation from your original coordinates to the Saito ones, add the inverse of your original coordinate transformation manually.

Note that weight vectors only take `Int64`, making the algorithm fail for very large entries in `V.vec`.

**Example:**

```
LIB "VecField.lib";
int oldp = printlevel;
printlevel = 1;
ring r = 0, (x, y, z), ds;
vector v = [-1,-1,-1];
VecField V = v;
V.precision = 4;
map phi = r, x-2y2+z3, 2y+y3+z, z;
VecField W = changeCoordinates(V, phi);
VecField X = [-2, 1, 0];
VecField WS = diagonalizeVecField(W, X);
⇒ ? error in diagonalizeVecField: no solutions found
⇒ ? leaving VecField.lib::diagonalizeVecField (0)
printlevel = oldp;
```

### D.15.26.7 vecFieldToMatrix

Procedure from library `VecField.lib` (see [Section D.15.26 \[VecField.lib\]](#), page 2583).

**Usage:** `vecFieldToMatrix(VecField V, ideal W)`

**Return:** the matrix representation of `V` restricted to `span(W)`, in the basis specified by `W`.

**Example:**

```
LIB "VecField.lib";
int oldp = printlevel;
printlevel = 1;
ring r = 0, (x, y, z), ds;
vector v = [-1,-1,-1];
VecField V = v;
V.precision = 4;
map phi = r, x-2y2+z3, 2y+y3+z, z;
VecField W = changeCoordinates(V, phi);
matrix m = vecFieldToMatrix(W, maxideal(2));
printlevel = oldp;
```



### D.15.26.8 decomposeVecField

Procedure from library `VecField.lib` (see [Section D.15.26 \[VecField.lib\]](#), page 2583).

- Usage:** `decomposeVecField(VecField V)` or `decomposeVecField(list s)`, `s` a list of `VecFields`
- Return:** list `l` containing two `VecFields`, where `l[1]` is the semisimple part of `V`, `l[2]` the nilpotent part. If called with a list, `l[2n-1]` is the semisimple part of the `n`-th `VecField`, `l[2n]` its nilpotent part.
- Assume:** The algorithm assumes standard coordinates. If `V` is not given in standard coordinates, it will be converted to standard coordinates, but not converted back at the end, so the resulting transformation is from standard coordinates to the new coordinates. If you want the entire transformation from your original coordinates to the new ones, add the inverse of your original coordinate transformation manually.

**Example:**

```
LIB "VecField.lib";
ring r = 0, (x, y, z), ds;
vector v = [-1,-1,-1];
VecField V = v;
V.precision = 4;
map phi = r, x-2y2+z3, 2y+y3+z, z;
VecField W = changeCoordinates(V, phi);
list l = decomposeVecField(W);
```

### D.15.26.9 diagonalizeMatrixSimul

Procedure from library `VecField.lib` (see [Section D.15.26 \[VecField.lib\]](#), page 2583).

- Usage:** `diagonalizeMatrixSimul(list l)`, where `l` is a list of quadratic matrices of same dimensions
- Return:** trafo matrix `U` s.t.  $U \cdot A \cdot \text{inv}(U)$  is diagonal for each `A` in `l`
- Assume:** matrices are simultaneously diagonalizable, `nvars(basering) >= dimension of the matrices`

**Example:**

```
LIB "VecField.lib";
ring r = 0, (x(1), x(2), x(3), x(4)), dp;
matrix a[4][4] = 2, 0, 0, 0, 0, 3, 0, 0, 0, 0, 3, 0, 0, 0, 0, 5;
matrix b[4][4] = 6, 0, 0, 0, 0, 6, 0, 0, 0, 0, 2, 0, 0, 0, 0, 7;
matrix trafo[4][4] = 3, 3, 0, 0, 0, 1, 3, 0, 2, 0, 1, 0, 0, 2, 2, 3;
matrix invtrafo = inverse(trafo);
matrix at = invtrafo * a * trafo;
matrix bt = invtrafo * b * trafo;
list l;
l = insert(l, at);
l = insert(l, bt);
matrix diagtrafo = diagonalizeMatrixSimul(l);
```

### D.15.26.10 invertAlgebraMorphism

Procedure from library `VecField.lib` (see [Section D.15.26 \[VecField.lib\]](#), page 2583).

- Usage:** `invertAlgebraMorphism(map p, int n)`, where `p` is an algebra morphism  $k[x_1, \dots, x_n] \rightarrow k[y_1, \dots, y_m]$ ,  
or `invertAlgebraMorphism(ideal p, int n)` if you represent the map as an ideal

**Return:** the inverse of  $p$  mod  $(\text{maxideal})^n$ .  
If  $n=-1$ , try for infinite precision. If input was an ideal, return an ideal.

**Note:** the algorithm used is described in [1], chapter 3.2

**Example:**

```
LIB "VecField.lib";
ring r = 0, (x, y, z), ds;
vector v = [-1,-1,-1];
VecField V = v;
V.precision = 4;
map phi = r, x-2y^2+z^3, 2y+y^3+z, z;
map phiinv = invertAlgebraMorphism(phi,4);
```

## 8 Release Notes

### 8.1 News and changes

## NEWS in SINGULAR 4.4.0

### News for version 4.4.0

New libraries:

`modfinduni.lib`: parallel computation of `finduni` ([Section D.15.10 \[modfinduni.lib\]](#), page 2439 [Section 5.1.43 \[finduni\]](#), page 185)

`sagbiNormaliz.lib`: computation of Sagbi bases via `normaliz` ([Section D.15.19 \[sagbiNormaliz.lib\]](#), page 2541)

`stdmodule.lib`: Compute Standard Bases of submodule of free module over polynomial subalgebra ([Section D.15.21 \[stdmodule.lib\]](#), page 2551)

`tropicalEllipticCovers.lib`: Gromov-Witten numbers of tropical elliptic curves and their covers ([Section D.15.25 \[tropicalEllipticCovers.lib\]](#), page 2575)

Changed libraries:

`normal.lib`: optimized, integrated `locnorma.lib` and `modnormal.lib` ([Section D.4.23 \[normal.lib\]](#), page 1184)

`normaliz.lib`: update to `normaliz` 3.10 ([Section D.4.24 \[normaliz.lib\]](#), page 1210)

`hess.lib`: fixes ([Section D.5.7 \[hess.lib\]](#), page 1449)

`sheafcoh.lib`: new routines [Section D.5.18.5 \[sheafCohBGGregul\]](#), page 1610, [Section D.5.18.6 \[sheafCohBGGregul\\_w\]](#), page 1611, [Section D.5.18.9 \[sheafCoh\\_w\]](#), page 1618, ([Section D.5.18 \[sheafcoh.lib\]](#), page 1605)

`sheafcoh.lib`: display of the tables is moved to [Section D.5.18.12 \[displayCohom\]](#), page 1620 ([Section D.5.18 \[sheafcoh.lib\]](#), page 1605)

`elim.lib`: rename `sat` to `sat_with_exp`, `sat` now return the saturation `i9deal` (only).

`primdec.lib`: fixed a bug in absolute primary decomposition ([Section D.4.26 \[primdec.lib\]](#), page 1240)

New type:

`bigintvec`: [Section 4.4 \[bigintvec\]](#), page 78

New commands:

new command [Section 5.1.123 \[prune\\_map\]](#), page 245

new command [Section 5.1.99 \[mres\\_map\]](#), page 226

New/renamed monomial orderings:

To avoid confusion (with the degree reverse lexicographic ordering) the orderings `rp` and `rs` are renamed:

renamed orderings `rp` -> `ip` `rs` -> `is`

new ordering `Ip`

Changes in the kernel/build system:

- new algorithm for [Section 5.1.151 \[std\]](#), page 271 in the case of ordering `ds` and zero dimensional ideal over `QQ`
- new algorithm for lift function ([Section 5.1.80 \[lift\]](#), page 211)
- `mstd` for local rings ([Section 5.1.100 \[mstd\]](#), page 227)
- use the new Hilbert function algorithm for [Section 5.1.56 \[hilb\]](#), page 195. (Different output format/ diffren return type)
- renamed orderings `ringorder_rp` -> `ringorder_ip` `ringorder_rs` -> `ringorder_is`
- new ordering `ringorder_Ip`
- changes for FLINT 3.0.x/3.1.x

## News for version 4-3-2

Changed libraries:

- many: call `std` in many places only if attribute `isSB` is not set
- many: change many calls to `execute` by better variants

New commands:

- `coeffs` for `ring` ([Section 5.1.12 \[coeffs\]](#), page 165)

Changes in the kernel/build system:

- new algorithm for Hilbert function ([Section 5.1.56 \[hilb\]](#), page 195)

## News for version 4-3-1

New libraries:

- `normal.lib`: new command `isNormal`, option `normalCheck` for `normal` ([Section D.4.23 \[normal.lib\]](#), page 1184)

Changed libraries:

- `chern.lib`: fixed `symNsym` ([Section D.5.2 \[chern.lib\]](#), page 1391)
- `homolog.lib`: fixed corner cases for `depth` ([Section D.4.11 \[homolog.lib\]](#), page 1091)

New commands:

- `chinrem`, `farey` for `smatrix` ([Section 4.21 \[smatrix\]](#), page 129)

Changed commands:

- overflow check for [Section 5.1.168 \[vdim\]](#), page 286
- Hilbert function ([Section 5.1.56 \[hilb\]](#), page 195) cannot overflow (different return type)
- `delete` accepts `intvec` for the indices to delete ([Section 5.1.21 \[delete\]](#), page 171)
- several GB based commands accept the choice of the algorithm: `"std"`, `"slimgb"`, `"sba"`, `"modstd"`, `"singmatic"`, `"groebner"`, `"ffmod"`, `"nfmod"` ([Section 5.1.156 \[syz\]](#), page 280, [Section 5.1.28 \[eliminate\]](#), page 176, [Section 5.1.65 \[intersect\]](#), page 202, [Section 5.1.94 \[modulo\]](#), page 223, [Section 5.1.81 \[liftstd\]](#), page 212, [Section 5.1.80 \[lift\]](#), page 211)
- `reduce` for non-field coefficients: search best reduction

Changes in the kernel/build system:

- changes for gcc 12

new option `--log LOGFILE`  
 configure can change the path to hml documentation (`--htmldir=...`)  
 different format for `doc.tbz2`  
 new algorithm for Hilbert function ([Section 5.1.56 \[hilb\]](#), page 195)

## News for version 4-3-0

New libraries:

`enumpoints.lib`: enumerating rational points ([Section D.15.3 \[enumpoints\\_lib\]](#), page 2329)  
`sagbigrob.lib`: Sagbi-Groebner basis of an ideal of a subalgebra ([Section D.15.18 \[sagbigrob\\_lib\]](#), page 2539)  
`puiseuxexpansion.lib`: Puiseux expansions over algebraic extensions ([Section D.15.15 \[puiseux-expansions\\_lib\]](#), page 2532)  
`integralbasis.lib`: Integral basis in algebraic function fields: new version ([Section D.4.12 \[integralbasis\\_lib\]](#), page 1115)

Changes in the kernel/build system:

input history is stored by default in `.singularhistory` ([Section 3.1.5 \[Editing input\]](#), page 18).  
 ABI change: all number routines (`n_...`) have only `coeffs` as last argument, functions with `ring` as last argument are removed  
 PATH is not changed for `system("sh",...)` (use [\[system, SingularBin\]](#), page [\[undefined\]](#))  
`hilb` avoids int overflow (also in `degree`, `stdhilb`)  
`liftstd` (with 2 arguments) improved ([Section 5.1.81 \[liftstd\]](#), page 212)  
`noether` improved ([Section 5.3.5 \[noether\]](#), page 304), use in `groebner(I,"HC")` for faster results for local orderings, 0-dimensional ideals ([Section 5.1.53 \[groebner\]](#), page 191).  
 letterplace routines improved ([Section 7.7 \[LETTERPLACE\]](#), page 617)  
 info file is now `singular.info` instead of `singular.hlp`  
 update for using FLINT 2.8.x

## News for version 4-2-1

New commands:

Letterplace: `dim`, `rightStd` for q rings ([Section 7.7 \[LETTERPLACE\]](#), page 617)  
 Letterplace: `map`, `fetch`, `imap` ([Section 4.12 \[map\]](#), page 106, [Section 5.1.38 \[fetch\]](#), page 182, [Section 5.1.59 \[imap\]](#), page 198)

New libraries:

`decomp.lib`: functional decomposition of polynomials ([Section D.4.6 \[decomp\\_lib\]](#), page 1052)  
`hodge.lib`: algorithms for Hodge ideals ([Section D.15.7 \[hodge\\_lib\]](#), page 2421)  
`tateProdCplxNegGrad.lib`: sheaf cohomology on product of projective spaces ([Section D.15.24 \[tateProdCplxNegGrad\\_lib\]](#), page 2555)

Changes in the kernel/build system:

`liftstd` (with 2 arguments) improved ([Section 5.1.81 \[liftstd\]](#), page 212)  
 building on Cygwin with shared libraries  
 building the manual via `--enable-doc-build`

## News for version 4-2-0

Syntax changes:

renamed `poly.lib` to `polylib.lib` ([Section D.2.8 \[polylib.lib\]](#), page 888)

New libraries:

`interval.lib`: interval arithmetic ([Section D.8.2 \[interval.lib\]](#), page 1825)

`maxlike.lib`: algebraic statistics ([Section D.15.9 \[maxlike.lib\]](#), page 2435)

`nchilbert.lib`: Hilbert series for LetterPlace algebras ([Section 7.5.13 \[nchilbert.lib\]](#), page 518)

`polyclass.lib`: class of polynomials ([Section D.15.14 \[polyclass.lib\]](#), page 2530)

`recover.lib`: Hybrid numerical/symbolical algorithms ([Section D.8.7 \[recover.lib\]](#), page 1862)

`redcgs.lib`: Reduced Comprehensive Groebner Systems ([Section D.2.9 \[redcgs.lib\]](#), page 898)

`ringgb.lib`: coefficient rings ([Section D.15.16 \[ringgb.lib\]](#), page 2535)

`sets.lib`: Sets ([Section D.14.8 \[sets.lib\]](#), page 2269)

`stanleyreisner.lib`: T1 and T2 for a general Stanley-Reiser ring ([Section D.15.20 \[stanleyreisner.lib\]](#), page 2544)

`systhreads.lib`: multi-threaded objects ([Section D.15.23 \[systhreads.lib\]](#), page 2555)

Changed libraries:

`classify_aeq.lib`: new procedure `classSpaceCurve` ([Section D.6.6 \[classify\\_aeq.lib\]](#), page 1657)

`grobcov.lib`: new version ([Section D.2.4 \[grobcov.lib\]](#), page 818)

`modular.lib`: parallel version for verification via `system("verifyGB",I)`

New commands:

`system("verifyGB",I)`: test, if `I` is a Groebner basis (using parallel processes)

Letterplace: `modulo,syz,lift,liftstd, rightStd` ([Section 7.7 \[LETTERPLACE\]](#), page 617)

Changes in the kernel/build system:

update for using FLINT 2.6.x and for FLINT 2.7.0

Singular can be build with NTL or FLINT or both (if none is available, `factorize` and `gcd` will not work.)

## News for version 4-1-3

New libraries:

`invar.lib`: Invariant theory [Section D.7.4 \[invar.lib\]](#), page 1813

`moddiq.lib`: ideal quotient and saturation [Section D.4.14 \[moddiq.lib\]](#), page 1121

`ncModslimgb.lib`: modular Groebner bases for G-algebras [Section 7.5.18 \[ncModslimgb.lib\]](#), page 554

Changed libraries:

`chern.lib`: new version ([Section D.5.2 \[chern.lib\]](#), page 1391)

`grobcov.lib`: new version ([Section D.2.4 \[grobcov.lib\]](#), page 818), new functions [Section D.2.4.22 \[ConsLevels\]](#), page 870, [Section D.2.4.23 \[Levels\]](#), page 871, [Section D.2.4.24 \[Grob1Levels\]](#), page 873, [Section D.2.4.25 \[DifConsLCSets\]](#), page 876

Changes in the kernel/build system:

improved `gcd` and multiplication via FLINT

improved lift (and related)  
 port to polymake 3.5.x  
 rational functions via flint ([Section 5.1.44 \[flintQ\]](#), page 186)  
 free algebra over  $\mathbb{Z}$  ([Section 7.7 \[LETTERPLACE\]](#), page 617)  
 adaptations/functions for `Singular.jl` (<https://github.com/oscar-system/Singular.jl>)

## News for version 4-1-2

### New libraries:

`arnoldclassify.lib`: Arnol'd Classifier of Singularities ([Section D.6.3 \[arnoldclassify.lib\]](#), page 1640)  
`diffform.lib`: Procedures for differential forms ([Section D.15.2 \[diffform.lib\]](#), page 2291)  
`dmodideal.lib`: Algorithms for Bernstein-Sato ideals of morphisms ([Section 7.5.6 \[dmodideal.lib\]](#), page 447)  
`fpalgebras.lib`: Generation of various algebras in the letterplace case ([Section 7.10.2 \[fpalgebras.lib\]](#), page 646)  
`ncrat.lib`: non-commutative rational functions ([Section 7.10.6 \[ncrat.lib\]](#), page 676)

### Changed libraries:

`freegb.lib`: `lpDivision`, `lpPrint` ([Section 7.10.4 \[freegb.lib\]](#), page 666)  
`fpadim.lib` ([Section 7.10.1 \[fpadim.lib\]](#), page 640)  
`schreyer.lib`: deprecated  
`goettsche.lib`: new, extended version (The Nakajima-Yoshioka formula up to  $n$ -th degree, Poincaré Polynomial of the punctual Quot-scheme of rank  $r$  on  $n$  planar points, Betti numbers of the punctual Quot-scheme of rank  $r$  on  $n$  planar points) ([Section D.5.5 \[goettsche.lib\]](#), page 1438)  
`grobcov.lib`: small bug fix ([Section D.2.4 \[grobcov.lib\]](#), page 818)

### Changes in the kernel/build system:

integrated `xalloc` into `omalloc`: (`./configure --disable-omalloc`)  
 improved heuristic for `det` ([Section 5.1.23 \[det\]](#), page 172)  
 improved reading of long polynomials  
 improved groebner bases over  $\mathbb{Z}$  coefficients  
 code for free algebras (letterplace rings) rewritten (using now the standard `+, -, *, ^, std, ...`) ([Section 7.7 \[LETTERPLACE\]](#), page 617)  
 new commands `rightstd` ([Section 7.8.10 \[rightstd \(letterplace\)\]](#), page 633)  
 extended `twostd` to LETTERPLACE ([Section 7.8.14 \[twostd \(letterplace\)\]](#), page 635, [Section 7.3.29 \[twostd \(plural\)\]](#), page 364)  
 pseudo type `polyBucket`  
 new type `smatrix`: sparse matrix (experimental) ([Section 4.21 \[smatrix\]](#), page 129).  
 extended `coef` to ideals ([Section 5.1.11 \[coef\]](#), page 164).  
 error and signal handling in `libSingular` ([Section 8.3 \[libSingular\]](#), page 2600).  
 updated `gfanlib` to version 0.6.2  
 port to NTL 11 (needs C++11: gcc6 or `-std=c++11`), which does not conflict with polymake (needs C++14)

## News for version 4-1-1

### New syntax:

**alias**: may be used as a prefix to a variable declaration. Can only be used in procedure headings. ([Section 3.5.1 \[General command syntax\]](#), page 41).

### New command:

**fres**: improved version of **sres**: computes a (not necessarily minimal) free resolution of the input ideal/module, using Schreyer's algorithm. ([Section 5.1.48 \[fres\]](#), page 188, [Section 5.1.149 \[sres\]](#), page 269).

### Extended commands:

pseudo ordering **L** allows setting of limits for exponents in polynomials ([Section B.2.9 \[Pseudo ordering L\]](#), page 774, [Section 5.1.2 \[attrib\]](#), page 156 for **maxExp**)

**%,mod**: also for poly operands ([Section 4.17.3 \[poly operations\]](#), page 121).

**delete**: extended to intvec, ideal, module ([Section 5.1.21 \[delete\]](#), page 171).

**syz** ([Section 5.1.156 \[syz\]](#), page 280), **lift** ([Section 5.1.80 \[lift\]](#), page 211), **liftstd** ([Section 5.1.81 \[liftstd\]](#), page 212), **intersect** ([Section 5.1.65 \[intersect\]](#), page 202): with a specified GB algorithm

### New libraries:

**classify2.lib**: Classification of isolated singularities of corank  $\leq 2$  and modality  $\leq$  wrt. right equivalence over the complex numbers according to Arnold's list. ([Section D.6.5 \[classify2.lib\]](#), page 1655)

**goettsche.lib**: Goettsche's formula for the Betti numbers of the Hilbert scheme of points on a surface, Macdonald's formula for the symmetric product ([Section D.5.5 \[goettsche.lib\]](#), page 1438)

**combinat.lib**, **modules.lib**, **methods.lib**, **nets.lib**: a more mathematical view of modules ([Section D.14.2 \[combinat.lib\]](#), page 2233: combinatorics), ([Section D.14.4 \[methods.lib\]](#), page 2239: construct procedures), ([Section D.4.15 \[modules.lib\]](#), page 1123: free resolutions), ([Section D.14.5 \[nets.lib\]](#), page 2240: pretty printing)

**ncHilb.lib**: Hilbert series of non-commutative monomial algebras ([Section 7.10.5 \[ncHilb.lib\]](#), page 673)

**realclassify.lib**: Classification of real singularities ([Section D.6.19 \[realclassify.lib\]](#), page 1746)

**rootisolation.lib**: real root isolation using interval arithmetic ([Section D.8.8 \[rootisolation.lib\]](#), page 1890)

**rstandard.lib**: Janet bases and border bases for ideals ([Section D.4.31 \[rstandard.lib\]](#), page 1266)

### Changed libraries:

**chern.lib**: new version ([Section D.5.2 \[chern.lib\]](#), page 1391)

**gitfan.lib**: new (incompatible) version ([Section D.13.3 \[gitfan.lib\]](#), page 2090)

**grobcov.lib**: new version ([Section D.2.4 \[grobcov.lib\]](#), page 818)

### Changes in the kernel/build system:

port to polymake 3.x.x

port to NTL 10 with threads (needs also C++11: gcc6 or -std=c++11)

**p\_Invers** is only a helper for **p\_Series**: now static

**p\_Divide** is now **p\_MDivide**, **pDivide**/**p\_Divide** is a new routine



## News for version 4-1-0

### Syntax changes:

- new (additional) form of ring definitions: (for example `ring R=QQ[x,y,z];`) ([Section 3.3.2 \[General syntax of a ring declaration\]](#), page 32)
- new (additional) form of multi-indicies: (for example `i(1,2,3,4,5)`) ([Section 3.5.3 \[Names\]](#), page 44)
- changed behaviour of `charstr` ([Section 5.1.7 \[charstr\]](#), page 162)
- new data type `cring` to describe the coefficient rings, to be used for the new definitions for (polynomial) rings ([Section 3.3.2 \[General syntax of a ring declaration\]](#), page 32)
- new command `ring_list` to access the parts used to construct polynomial rings ([Section 5.1.138 \[ring\\_list\]](#), page 257, [Section 5.1.137 \[ringlist\]](#), page 255)
- extended polynomial ring construction: also from lists produced by `ring_list`
- new attribute `ring_cf` for `ring` ([Section 5.1.2 \[attrib\]](#), page 156)
- printing of rings changed to match `cring` names ([Section 5.1.7 \[charstr\]](#), page 162)

### New libraries:

- new library: `classifyMapGerm.lib`: standard basis of the tangent space at the orbit of an algebraic group action ([Section D.6.9 \[classifyMapGerm.lib\]](#), page 1663)
- new library: `ffmodstd.lib`: Groebner bases of ideals in polynomial rings over algebraic function fields ([Section D.4.9 \[ffmodstd.lib\]](#), page 1081)
- new library: `nfmodsyz.lib`: syzygy modules of submodules of free modules over algebraic number fields ([Section D.4.21 \[nfmodsyz.lib\]](#), page 1178)
- new library: `curveInv.lib`: invariants of curves ([Section D.4.5 \[curveInv.lib\]](#), page 1049)
- new library: `gfan.lib`: interface to `gfanlib` ([Section D.13.2 \[gfan.lib\]](#), page 2050)
- extended library: interface to `polymake` merged into [Section D.13.4 \[polymake.lib\]](#), page 2129
- new library: `tropicalNewton.lib`: Newton polygon methods in tropical geometry ([Section D.13.7 \[tropicalNewton.lib\]](#), page 2196)
- new library: `schubert.lib`: some procedures for intersection theory ([Section D.5.17 \[schubert.lib\]](#), page 1587)

### Changed libraries:

- `classify_aeq.lib`: new procedures ([Section D.6.6 \[classify\\_aeq.lib\]](#), page 1657)
- `grobcov.lib`: new version ([Section D.2.4 \[grobcov.lib\]](#), page 818)
- `ncfactor.lib`: factorization in some noncommutative algebras ([Section 7.5.12 \[ncfactor.lib\]](#), page 487) with new routine `ncfactor` ([Section 7.5.12.1 \[ncfactor\]](#), page 487)
- `primdec.lib`: new option "subsystem" ([Section D.4.26 \[primdec.lib\]](#), page 1240)

### Changes in the kernel:

- improved mapping of polynomials/ideals/...
- port to gcc 6
- port to `gfanlib` 0.6 (requires C++11, i.e. gcc >=4.3)
- port to NTL 10
- port to `polymake` 3.0
- port to readline 7

[Section 5.1.140 \[sba\]](#), page 259 works for global orderings, also for coefficient types  $\mathbb{Z}$  and  $\mathbb{Z}/m$

[Section 5.1.151 \[std\]](#), [page 271](#) works for all orderings, also for coefficient types  $\mathbb{Z}$  and  $\mathbb{Z}/m$  with local/mixed orderings

[Section 5.1.36 \[factorize\]](#), [page 180](#) works for polynomial rings over  $\mathbb{Z}\mathbb{Z}$

Experimental stuff:

module [Section D.14.3 \[customstd.lib\]](#), [page 2238](#): modify `std` ([Section D.14.3.2 \[satstd\]](#), [page 2239](#))

## News for version 4-0-3

New libraries:

new library: `brillnoether.lib`: Riemann-Roch spaces of divisors on curves ([Section D.5.1 \[brillnoether.lib\]](#), [page 1390](#))

new library: `chern.lib`: Chern classes ([Section D.5.2 \[chern.lib\]](#), [page 1391](#))

new library: `ffmodstd.lib`: Groebner bases of ideals in polynomial rings over algebraic function fields ([Section D.4.9 \[ffmodstd.lib\]](#), [page 1081](#))

new library: `GND.lib`: General Neron Desingularization ([Section D.15.5 \[GND.lib\]](#), [page 2342](#))

new library: `graal.lib`: localization at prime ideals ([Section D.5.6 \[graal.lib\]](#), [page 1444](#))

new library: `hess.lib`: Riemann-Roch space of divisors ([Section D.5.7 \[hess.lib\]](#), [page 1449](#))

Changed libraries:

renamed `algemodstd.lib` to [Section D.4.20 \[nfmodstd.lib\]](#), [page 1175](#), extended to module

renamed `derham.lib` to [Section D.5.3 \[deRham.lib\]](#), [page 1427](#)

`grobcov.lib` (`grobcovK`): Groebner Cover for parametric ideals ([Section D.2.4 \[grobcov.lib\]](#), [page 818](#)) with new routine `ConsLevels` ([Section D.2.4.22 \[ConsLevels\]](#), [page 870](#)), removed `AddCons` `AddConsP`.

## News for version 4-0-2

New commands:

`align` ([Section 5.1.1 \[align\]](#), [page 156](#))

`branchTo` ([Section 4.18.3 \[procs with different argument types\]](#), [page 125](#))

`->` ([Section 4.18.2 \[proc expression\]](#), [page 125](#))

Change in ring handling:

`typeof( qring )` returns `"ring"`

New libraries:

`algemodstd.lib`: Groebner bases of ideals in polynomial rings over algebraic number fields (renamed to [Section D.4.20 \[nfmodstd.lib\]](#), [page 1175](#))

`arr.lib`: arrangements of hyperplanes ([Section D.14.1 \[arr.lib\]](#), [page 2205](#))

`brillnoether.lib`: Riemann-Roch spaces of divisors on curve ([Section D.5.1 \[brillnoether.lib\]](#), [page 1390](#))

`hess.lib`: Riemann-Roch space of divisors on function fields and curves ([Section D.5.7 \[hess.lib\]](#), [page 1449](#))

`gradedModules.lib`: graded modules/matrices/resolutions ([Section D.15.6 \[gradedModules.lib\]](#), [page 2346](#))

Changed libraries:

revised polymake interface (polymake.so)  
 revised gfanlib interface (gfanlib.so)  
 Presolve::findvars ([Section D.8.3.6 \[findvars\]](#), page 1834, [Section 5.1.166 \[variables\]](#), page 285)  
 Ring::addvarsTo ([Section D.2.12.26 \[addvarsTo\]](#), page 957)  
 Ring::addNvarsTo ([Section D.2.12.27 \[addNvarsTo\]](#), page 958)  
 Ring::hasAlgExtensionCoefficient ([Section D.2.12.21 \[hasAlgExtensionCoefficient\]](#), page 956)  
 Schreyer::s\_res ([s\\_res](#))  
 grobcov.lib (grobcovK) ([Section D.2.4 \[grobcov\\_lib\]](#), page 818) with new routines AddCons  
 AddConsP.  
 normaliz.lib (for normaliz  $\geq 2.8$ ) ([Section D.4.24 \[normaliz\\_lib\]](#), page 1210)  
 renamed groebnerFan to groebnerFanP in polymake.lib ([Section D.13.4 \[polymake\\_lib\]](#),  
[page 2129](#))  
 renamed fVector to fVectorP in polymake.lib ([Section D.13.4 \[polymake\\_lib\]](#),  
[page 2129](#), [polymakeInterface\\_lib](#))

## News for version 4-0-1

Version 4-0-1 is a bug fix release.

New feature: attribute `ring_cf` for `ring` ([Section 5.1.2 \[attrib\]](#), page 156)

## News for version 4-0-0

Version 4-0-0 is a milestone release of Singular. The new release series 4 aims for an entirely modularized architecture simplifying connectivity with other systems and paving the way for parallel computations. As a first step in modularization, the new release features an internal structural separation of coefficient rings and polynomial rings. This allows for a flexible integration of new coefficient rings.

SINGULAR 4-0-0's list of new functionality and significant improvements further extends that of the 3-1-6/7 prerelease series.

New functionality

de Rham cohomology of complements of algebraic varieties ([Section D.5.3 \[deRham\\_lib\]](#),  
[page 1427](#))  
 Gromov-Witten numbers of elliptic curves ([Section D.4.8 \[ellipticcovers\\_lib\]](#), page 1068)  
 classification of isolated complete intersection singularities in characteristic 0 ([Section D.6.8 \[classifyci\\_lib\]](#),  
[page 1662](#))  
 parametrization of orbits of unipotent actions ([Section D.5.10 \[orbitparam\\_lib\]](#), page 1459)  
 F5-like Groebner basis algorithm ([Section 5.1.140 \[sba\]](#), page 259)  
 element-wise application of functions to data structures ([Section 5.2.1 \[apply\]](#), page 290)  
 support for debugging libraries ([Section 3.9.1 \[ASSUME\]](#), page 68)

Improved functionality

Groebner cover for parametric ideals ([Section D.2.4 \[grobcov\\_lib\]](#), page 818)  
 normalization of affine rings ([Section D.4.23 \[normal\\_lib\]](#), page 1184)  
 classification of real singularities ([Section D.6.19 \[realclassify\\_lib\]](#), page 1746)  
 GIT-fans ([Section D.13.3 \[gitfan\\_lib\]](#), page 2090)

algebraic/transcendental field extensions

[Chapter 7 \[Non-commutative subsystem\]](#), page 318

an abstraction layer for parallel computations ([Section D.2.7 \[parallel\\_lib\]](#), page 885)

run-time loading of supplementary kernel code ([Section A.1.9 \[Dynamic modules\]](#), page 709)

interpreter language support for name spaces ([Section 4.16 \[package\]](#), page 119)

#### Availability

SINGULAR is available as source code and for Linux, Mac OS X, Windows, FreeBSD and SunOS-5.

## 8.2 Singular 3 and Singular 4

The purpose of this section is to describe new features and changes between Singular 3-1-7 and Singular 4.\* (formerly known as Spielwiese) both for developers and Singular users. In what follows we will refer to the systems as Singular 3 and Singular 4.

### 8.2.1 Version schema for Singular

SINGULAR version is of the form **a.b.c.d** which may also be written as **a-b-c-d** where a,b,c and d are numbers:

a is changed with major, incompatible changes

b is changed with incompatible changes (of some commands/libraries)

c is changed with compatible changes (i.e. new commands, extended options, new algorithms, etc.)

d is changed with each release (i.e. with bug fixes, etc.)

SINGULAR does also have "unofficial" build originating from a code version between "official" version: such builds display "Development version a.b.c" in the header while "official" versions show "version a.b.c". Also the manual describes version a-b-c. To get the complete version number, use `system("version");` or use `SINGULAR_VERSION` in C.

### 8.2.2 Notes for Singular users

#### Coefficient rings

To allow for easy integration of new coefficient rings into Singular, the the way coefficient rings are being handled has been redesigned.

In general, the user syntax has not changed, however there are some changes in the behaviour of Singular:

- setting `minpoly` results in changing the current coefficient domain and clears all previously defined variables of that ring
- Minor changes in the output of coefficient ring description. Moreover the output of elements of certain rings has been improved (for example, reals).
- Algebraic and transcendental extensions of rationals and finite fields have been reimplemented. In particular, the heuristics for clearing denominators and factoring out content have been changed. In some cases this leads to a different, mathematically equivalent results of Groebner bases and related computations. For example a Groebner basis element may differ by a unit.
- Most notably, due to the redesign of the coefficient rings, if the user sets the minimal polynomial all variables dependent on the current ring are deleted.

## Ring-dependent options

Formally global Singular [Section 5.1.111 \[option\]](#), [page 234](#) now belong to individual polynomial rings. This includes:

- `intStrategy`
- `redTail`
- `redThrough`

Also the following settings now belong to individual (currently active) polynomial rings:

- `short`
- `minpoly`
- `noether`

Hence setting these options only affects the current ring. Be aware of this when switching between different rings, since the options affect the result of various computations (in particular Groebner bases).

## Path names

- The tree structure of the binary Singular distribution has been changed. The typical tree now looks as show at <https://github.com/Singular/Singular/wiki/Sw-tree>
- Accordingly Singular search paths (where Singular searches for libraries, dynamic modules, etc.) have been changed. You can display them by calling Singular by `Singular -v`.
- currently, multi-arch installations of Singular 4 aere not possible.

## Library versioning

Due to switching from Subversion to GIT revision control system for the Singular source code, library version variables (displayed when loading a library) have changed.

## New orderings for modules

The now can assign weights to module components, when defining a monomial ordering. For example

```
ring R = 0, (x,y,z), (am(1,2,3, 10,20,30,40), dp, C);
deg(x*gen(1));
↳ 11
```

will assign weights 1,2,3 to x,y,z respectively, and weights 10,20,30,40,0,0,... to components of any free module defined over R. This ordering will first sort by this weighted degree, then by dp on the ring monomials and then will give priority to the large component index.

## Future benefits of Singular 4

The redesign of Singular will allow us to provide new features in the future, for example:

- Interpreter type for coefficient rings.
- User defined coefficient rings.
- Improved syntax for defining polynomial rings.

### 8.2.3 Notes for developers

There has been an intensive process of refactoring, redesign and modularization of Singular to facilitate easier maintenance and future development:

- Build System : automake, libfac has been integrated into Factory
- Removed MP (Multi protocol) in favor of SSI links.
- Separation/modularization into libraries and packages
- For easy integration of new coefficient rings, we defined a generic interface for coefficient rings and a supporting framework for making them accessible to the user.

In particular we have separated everything related to coefficient rings into a separate library `libcoeffs`. Dependency tree between restructured packages is show at <https://www.singular.uni-kl.de/dox/singular.png>

In order to use `libSingular` as a C++ library, see [Section 8.3 \[libSingular\]](#), page 2600.

### 8.2.4 Building Singular

The user can build and install Singular with the following standard UNIX-like procedure:

- Download and extract the latest official source package (.tar.gz).
- Run the configure script, for instance, `./configure`.
- Build Singular by running `make`.
- Install Singular by running `make install`.

In contrast to Singular 3, there are now many more configuration options.

All possible options for configure can be seen by running the configure script with option `--help`. On a multicore compute consider running make with the option `-j [cores]`.

### 8.2.5 Side-by-side installation

Due to choosing paths according to FS standards it is no longer possible to have a side-by-side installation of different Singular versions or versions for different architectures.

## 8.3 libSingular

`libSingular` is the C++-library version of SINGULAR.

`Singular/libsingular.h` is the main include file, `-lSingular` the link parameter, `lib/pkgconfig/Singular.pc` provides all parameters in the `pkconfig` format.

It contains all parts of SINGULAR with the following exceptions:

1. memory allocation functions for GMP (see `mmInit` in `Singular/testths.cc`)
2. signal handlers (see `init_signals` in `Singular/cntrlc.cc`).

At least a handler for `SIGCHLD` must be installed for the commands [Section 5.1.170 \[waitfirst\]](#), page 287, [Section 5.1.169 \[waitall\]](#), page 286 and the routines from [Section D.2.7 \[parallel\\_lib\]](#), page 885, [Section D.4.16 \[modstd\\_lib\]](#), page 1147, [Section D.4.23 \[normal\\_lib\]](#), page 1184, [Section D.2.13 \[tasks\\_lib\]](#), page 959.

If the child was started by `libSingular` the handler has to call `sig_chld_hdl` from `Singular/links/ssiLink.cc` or implement something similar (call `slClose(1)` for ssi links).

3. error handlers for factory, NTL (see `init_signals` in `Singular/cntrlc.cc`).

## 8.4 Download instructions

SINGULAR is available as source and binary program for most common hard- and software platforms. Instructions to download and install SINGULAR can be found at

<https://www.singular.uni-kl.de/index.php/singular-download.html>.

Sources of release versions of SINGULAR are also available from our FTP site

<ftp://jim.mathematik.uni-kl.de/pub/Math/Singular/src/>.

or

<https://www.singular.uni-kl.de/ftp/pub/Math/Singular/src/>.

## 8.5 Used environment variables

SINGULAR needs to find some files (dynamic modules, libraries, help files). Usually they are found relative to the location of the (main) executable (after following symlinks). This can be changed by setting the following environment variables.

SINGULAR\_EXECUTABLE (should usually not be set)

the complete filename of the main executable, usually derived from the command line (inspecting also PATH, following symlinks).

If SINGULAR\_EXECUTABLE cannot be found, \$prefix/bin/Singular is assumed.

For libSingular: SINGULAR\_EXECUTABLE is set to the argument of `siInit` (it must exist).

SINGULAR\_BIN\_DIR

the directory of the main executable, usually derived from \$SINGULAR\_EXECUTABLE

SINGULAR\_ROOT\_DIR

the root of the singular tree, default: \$SINGULAR\_BIN\_DIR/..

SINGULAR\_DATA\_DIR

the root of the singular data files, default: \$SINGULAR\_BIN\_DIR/./share

SINGULARPATH

the directories for libraries and optional dynamic modules (separated by ;), default:

\$SINGULAR\_DATA\_DIR/singular/LIB

\$SINGULAR\_ROOT\_DIR/share/singular/LIB

\$SINGULAR\_BIN\_DIR/./share/singular/LIB

\$SINGULAR\_DATA\_DIR/factory

\$SINGULAR\_ROOT\_DIR/share/factory

\$SINGULAR\_BIN\_DIR/LIB

\$SINGULAR\_BIN\_DIR/./factory

\$SINGULAR\_BIN\_DIR/MOD

\$SINGULAR\_ROOT\_DIR/lib/singular/MOD

\$SINGULAR\_ROOT\_DIR/libexec/singular/MOD

\$prefix/lib/singular/MOD

\$prefix/libexec/singular/MOD

\$SINGULAR\_BIN\_DIR

SINGULAR\_PROCS\_DIR

the directories for dynamic modules (separated by ;), default:

\$SINGULAR\_BIN\_DIR/MOD



```

$SINGULAR_ROOT_DIR/lib/singular/MOD
$SINGULAR_ROOT_DIR/libexec/singular/MOD
$prefix/lib/singular/MOD
$prefix/libexec/singular/MOD
SINGULAR_INFO_FILE
singular.info, default: $SINGULAR_DATA_DIR/info/singular.info
SINGULAR_IDX_FILE
the help index, default: $SINGULAR_DATA_DIR/singular/singular.idx
SINGULAR_HTML_DIR
the directory of the manual as html files, default: $SINGULAR_DATA_DIR/singular/html
SINGULAR_URL
the URL of the manual, default: https://www.singular.uni-kl.de/Manual/

```

The effective list of directories/files can be printed by `Singular -v`, see [Section 3.1.6 \[Command line options\]](#), page 19.

Depending on the used functions, these environment variables apply also to `libSingular`.

## 8.6 Unix installation instructions

Install binaries: <https://www.singular.uni-kl.de/index.php/singular-download/install-linuxunix> or build it yourself:

Install the necessary packages:

```

libtool
gnu make
gcc, g++
libreadline
gmp, mpfr
ntl
libcdd

```

Install flint 2.5 (or newer): `./configure --with-gmp=/usr --prefix=$HOME/tmp`  
`make && make install`

Install Singular `./configure --with-flint=$HOME/tmp --enable-gfanlib --prefix=$HOME/Singular4`  
`make && make install`

(`$prefix/bin/Singular` is the main executable)

(optional) install 4ti2

(optional) install surf/surfer

(optional) install normaliz 2.8 (or newer)

See also <https://github.com/Singular/Singular/wiki/Step-by-Step-Installation-Instructions-for-Singular> which includes instructions adapted for debian and fedora based systems.

## 8.7 Windows installation instructions

Singular relies on Cygwin as its environment under Windows. There is a 32bit and a 64bit version of Cygwin.

<https://www.singular.uni-kl.de/index.php/singular-download/install-windows.html>



## 8.8 Macintosh installation instructions

**Installation of the provided binaries** <https://www.singular.uni-kl.de/index.php/singular-download/install-os-x.html>

If your Mac refuses to open Singular because of an "unidentified developer": Open System Preferences. Go to the Security & Privacy tab. Click on the lock and enter your password so you can make changes. Change the setting for 'Allow apps downloaded from' to 'App Store and identified developers'.

You may also check <https://support.apple.com/en-en/guide/mac-help/mh40616/mac>

## 9 Index

<b>!</b>		
! .....	44	
!= .....	43, 89	
<b>#</b>		
# .....	44	
<b>\$</b>		
\$ .....	44	
<b>%</b>		
% .....	43, 75, 87, 94, 118, 121	
<b>&amp;</b>		
&& .....	44, 90	
<b>(</b>		
( .....	43	
() .....	107	
<b>)</b>		
) .....	43	
<b>*</b>		
* .....	43, 75, 77, 79, 82, 87, 92, 94, 110, 113, 118, 121	
** .....	43	
<b>-</b>		
- .....	43, 75, 77, 79, 87, 92, 94, 110, 118, 121, 135	
- .....	43	
-allow-net .....	19	
-batch .....	21	
-browser .....	19	
-echo .....	19	
-emacs .....	21	
-emacs-dir .....	21	
-emacs-load .....	21	
-execute .....	20	
-help .....	19	
-min-time .....	20	
-MPhost .....	21	
-MPport .....	21	
-no-out .....	20	
-no-rc .....	20	
-no-shell .....	20	
-no-stdlib .....	20	
-no-tty .....	20	
-no-warn .....	20	
-quiet .....	20	
-random .....	20	
-sdb .....	19	
-singular .....	21	
-ticks-per-sec .....	21	
-user-option .....	20	
-> .....	125	
-b .....	21	
-c .....	20	
-d .....	19	
-e .....	19	
-h .....	19	
-q .....	20	
-r .....	20	
-u .....	20	
-v .....	20	
<b>•</b>		
.....	44	
.singular .....	18	
.singularrc file .....	22	
.singularrc file, no loading .....	20	
<b>/</b>		
/ .....	43, 73, 94, 110, 118, 121, 135	
// .....	44	
<b>:</b>		
: .....	44, 94	
:: .....	44, 119	
<b>;</b>		
;; .....	44	
<b>=</b>		
= .....	43	
== .....	43, 89, 110, 118, 121, 128, 132, 135	
<b>?</b>		
? .....	44, 194	
<b>[</b>		
[ .....	43	
[] .....	113, 121, 132, 135	
<b>]</b>		
] .....	43	

'	
'	44
-	
-	44
{	
{	43
	44, 90
}	
}	43
~	
~	44, 302
"	
"	44
+	
+	43, 75, 77, 79, 82, 87, 92, 94, 105, 110, 113, 118, 121, 128, 132, 135
++	43
>	
>	43, 121, 135
>=	43, 89, 118, 121, 132, 135
^	
^	43, 75, 82, 118, 121
\	
\	44
<	
<	44, 121, 135, 184
<=	43, 89, 118, 121, 132, 135
<>	43, 89, 110, 118, 121, 128, 132, 135

## A

a, ordering	773
A_L	1652
A_Z	806
A_Z_L	807
absfact.lib	1002
absfact.lib	1002
absFactorize	1002
absFactorizeBCG	1003
absolute factorization	1002
absPrimdecGTZ	1251
absPrimdecGTZE	1252
abstract variety, projective space, Grassmannian	1587, 1588
abstractR	1546
absValue	808
Access to elements of a user defined type	137
actionIsProper	1804
add	2302, 2318
addcol	977
addcores	942
addElement	2273
addLeftFractions	593
addModules	1135
addNmzOption	1226
addnondegeneratevariables	1755
addNvarsTo	958
addRat	537
addrow	978
addSheaf	1597
addvarsTo	957
ademRelations	648
ADGT	865
Adj_div	1923
adjoint	992
Adjoint ideal	1465
adjointIdeal	1465
adjunction divisor	1924
admissibleSub	558
afaces	2091
affine code	789
AG codes	764
AGcode_L	1931
AGcode_Omega	1932
ainvar.lib	1802
ainvar.lib	1802
aksaka.lib	1987
aksaka.lib	1987
Alexander polynomial	1627
alexanderpolynomial	1629
alexpoly.lib	1627
alexpoly.lib	1627
alg_kernel	1008
algDependent	1007
algebra.lib	1004
algebra_containment	1005

algebra.lib	1004	apply	290
algebraic dependence	600	applyAdF	396
Algebraic dependence	733	applyMatrix	1827
algebraic field extension	1262	applyPermutationToIntvec	2126
Algebraic geometry	1390	applyVecField	2584
Algebraic Geometry codes	1923	arcpoint.lib	1637
algebraic statistics	2435	arcpoint_lib	1637
algebraicDependence	1381	areEqualLeftFractions	595
algorithm of Bigatti, La Scala and Robbiano	784	areZeroElements	2490
algorithm of Conti and Traverso	782	argument, default	52
algorithm of Di Biase and Urbanke	783	argument, optional	52
algorithm of Hosten and Sturmfels	783	Arnol'd	1640
algorithm of Pottier	783	Arnold, classification	1656
alias	41	ArnoldAction	1742
align	156	arnoldClassify	1642
all.lib	795	arnoldclassify.lib	1640
all.lib	795	arnoldclassify_lib	1640
allDoubleExt	607	arnoldClassify_to_string	1644
allExtOfLeft	605	arnoldCorank	1645
allExtOfRight	606	arnoldDeterminacy	1645
allNmzOptions	1224	arnoldListAllSeries	1640
allowing net access	19	arnoldMilnorCode	1645
allPositive	386	arnoldMorseSplit	1646
allprint	878	arnoldNormalForm	1642
allreal	2044	arnoldShowSeries	1641
allrealst	2043	arr.lib	2205
allsquarefree	1735	arr_lib	2205
AltVarEnd	578	arr2multarr	2224
AltVarStart	577	arrange	418
ambientDimension	2051	arrangement	2206, 2224, 2225
and	90, 311	arrBoolean	2217
Ann	1240	arrBoundedChambers	2231
annfalphaI	452	arrBraid	2217
annfs	403	arrCenter	2210
annfs0	414	arrCentered	2211
annfs2	415	arrCentral	2211
annfsBMI	411	arrCentralize	2212
annfsLogIdeal	448	arrChambers	2231
annfsParamBM	410	arrCharPoly	2230
annfspecial	403	arrCone	2213
annfspecialOld	404	arrCoordChange	2212
annfsRB	416	arrCoordNormalize	2213
annihilator of polynomial	422	arrDecone	2214
annihilator of rational function	422, 523	arrDer	2222
annihilatorMultiFs	449	arrEdelmanReiner	2220
annil	1166	arrEssentialize	2216
annPoly	422	arrExponents	2224
annRat	423	arrFlats	2227
annRatSyz	530	arrIsEssential	2216
Appel function	422	arrIsFree	2223
Appel hypergeometric function	422	arrLastVar	2209
appelF1	439	arrLattice	2228
appelF2	440	arrLocalize	2214
appelF4	440	arrOrlikSolomon	2221
appendWeight2Ord	560	arrPoincare	2231
application	2325	arrPrintMatrix	2207
Applications	760	arrRandom	2219

[illegible]

biszygies ..... 372  
 Biszygy ..... 639  
 bitrinity ..... 373  
 blackbox ..... 140, 146  
 block ..... 48, 290  
 BlowingUp ..... 1682  
 blowUp ..... 1529  
 blowup0 ..... 1060  
 blowUp2 ..... 1530  
 blowUpBO ..... 1533  
 Blowupcenter ..... 1488  
 boolean ..... 2217  
 boolean expressions ..... 89  
 boolean operations ..... 90  
 boolean\_constant ..... 2261  
 boolean\_ideal ..... 2263  
 boolean\_poly ..... 2262  
 boolean\_poly\_ring ..... 2261  
 boolean\_set ..... 2264  
 boolean\_std ..... 2260  
 border basis ..... 1267  
 borderBasis ..... 1267  
 BorelCheck ..... 1624  
 Bott's formula ..... 1587  
 boundBuFou ..... 2042  
 boundDes ..... 2043  
 boundposDes ..... 2042  
 bounds ..... 1891  
 bounds2 ..... 1826  
 boxSet ..... 1892  
 bracket ..... 275, 312, 336  
 braid ..... 2217  
 Branches of space curve singularities ..... 747  
 branchTo ..... 125  
 break ..... 291  
 break point ..... 302  
 breakpoint ..... 291  
 Briancon-Maisonobe algorithm ..... 402  
 Brieskorn lattice... 1703, 1704, 1705, 1706, 1707, 1708,  
 1709, 1710, 1711, 1713, 1718, 1719, 1739  
 Brill-Noether algorithm ..... 1923  
 BrillNoether ..... 1927  
 brillnoether.lib ..... 1390  
 brillnoether.lib ..... 1390  
 brnoeth.lib ..... 1922  
 brnoeth.lib ..... 1922  
 browser, command line option ..... 19  
 browser, setting the ..... 275  
 browsers ..... 16  
 browsers, setting the ..... 275  
 BSidealFromAnn ..... 450  
 bubblesort ..... 1990  
 Buchberger algorithm for toric ideals ..... 784  
 Budur-Mustata-Saito approach ..... 454  
 Building Singular ..... 2600  
 buildtree ..... 904  
 buildtree, buildtreetoMaple, CGS ..... 909

buildtree, Maple ..... 906  
 buildtreetoMaple ..... 906  
 busadj ..... 991  
 bvar ..... 2267

## C

C programming language ..... 311  
 c, module ordering ..... 771  
 C, module ordering ..... 771  
 calculate\_max\_sum ..... 2014  
 calculate\_ordering ..... 2002  
 calculateI ..... 1512  
 cancelunit, option ..... 236  
 canonicial form ..... 833, 834  
 canonical ..... 822  
 canonical form ..... 870, 872, 874, 876  
 canonical, ..... 837  
 canonicalizeCone ..... 2051  
 canonize ..... 1974  
 canonMap ..... 1092  
 cantodiffcgs ..... 931  
 cantoradd ..... 2020  
 cantormult ..... 2022  
 cantorred ..... 2021  
 cardGroup ..... 1048  
 case ..... 312  
 Castelnuovo-Mumford regularity ..... 1609  
 Category string ..... 56  
 catNets ..... 2240  
 cccMatrixToPositiveIntvec ..... 2198  
 cdd ..... 1  
 cddlib ..... 1  
 CenCharDec ..... 485  
 center ..... 387, 391, 2210, 2211, 2212  
 Center ..... 1531  
 CenterBO ..... 1536  
 centerRed ..... 390  
 centerVS ..... 389  
 central ..... 2207, 2211, 2212, 2220  
 central.lib ..... 387  
 central.lib ..... 387  
 central1st ..... 2333  
 central2nd ..... 2333  
 centralize ..... 387  
 centralizer ..... 387, 392  
 centralizerRed ..... 389  
 centralizerVS ..... 388  
 centralizeSet ..... 388  
 CentralQuot ..... 484  
 CentralSaturation ..... 485  
 cf\_class ..... 157  
 cf\_of ..... 2576  
 cgs ..... 798  
 CGS ..... 830  
 CGS, disjoint, reduced, comprehensive Groebner  
 system ..... 904

cgsdr	828	Chow ring, abstract variety, projective space, Grassmannian	1589
chaincrit	2026	ChowRing	1589
chAll	1397	chProd	1401
chAllInv	1398	chProdE	1402
chambers	2231	chProdL	1403
Change of rings	11	chProdLP	1404
changechar	944	chProdM	1404
changeCoordinates	2584	chProdMP	1405
changeDenominator	1207	chProj	1415
changeord	944	chSum	1399
changeordTo	957	chSymm	1408
changes	2589	chSymm2L	1409
changevar	945	chSymm2LP	1409
char	161	chWedge	1409
char_series	161	chWedge2L	1410
characteristic exponents	1634, 1635, 1636, 1727	chWedge2LP	1411
characteristic polynomial	2230	cimonom.lib	2048
Characteristic sets	777	cimonom.lib	2048
characteristic variety	422, 523	cisimplicial.lib	1046
CharacteristicExponents	1689	cisimplicial_lib	1046
charexp2conductor	1636	classification	1640, 1655
charexp2generators	1635	Classification of hypersurface singularities	750
charexp2inter	1635	Classification of singularities	1747
charexp2multseq	1634	classify	1647
charexp2poly	1636	classify.lib	1646
charInfo	437	classify_aeq.lib	1657
charpoly	992	classify_aeq_lib	1657
charstr	162	classify_lib	1646
charVariety	436	classify2.lib	1655
chDual	1400	classify2.lib	1655
chebyshev	1057	classifyCeq	1662
checkFactor	417	classifyceq.lib	1661
checkpfd	2522	classifyceq_lib	1661
checkpfdMat	2530	classifyci.lib	1662
checkRoot	412	classifyci.lib	1662
chern	1394	classifyicis	1662
chern.lib	1391	classifyMapGerms.lib	1663
chern_lib	1391	classifyMapGerms_lib	1663
chernCharPoly	1418	classifySimpleMaps	1681
ChernClass	1594	classifySimpleMaps1	1681
chernPoly	1418	classifyUnimodalMaps	1682
ChernRootsDual	1400	classpoly	1418
ChernRootsHom	1405	classSpaceCurve	1661
ChernRootsProd	1401	cleanTmp	2176
ChernRootsSum	1399	cleanunit	1510
ChernRootsSymm	1407	cleardenom	163
ChernRootsWedge	1407	close	163
Chevalley-Rosenlicht theorem	1460	closed_points	1935
chHE	1398	closetex	1908
chHom	1406	closureFrac	1203
chineseRem	1992	CM_regularity	1609
chinrem	162	CMtype	1768
chinrempoly	1176	Code	786
chinrestp	2017	Codes and the decoding problem	786
chNum	1395	codim	1757
chNumbers	1395	coDim	1664
chNumbersProj	1417		

- codimension . . . . . 2052
- coDimMap . . . . . 1663
- coding theory . . . . . 764
- Coding theory . . . . . 1922
- coef . . . . . 164
- coefficient . . . . . 2207, 2208
- coefficient field . . . . . 115
- coefficient of a polynomial . . . . . 2576
- Coefficient rings . . . . . 2598
- coefficient rings, ring of integers, zero divisors, p-adic numbers . . . . . 37
- coefficients . . . . . 2297
- coefficients, long . . . . . 705
- coeffmod . . . . . 1988
- coeffs . . . . . 165
- coHom . . . . . 549
- cohomologyHashTable . . . . . 2567
- cohomologyMatrix . . . . . 2563
- cohomologyMatrixFromResolution . . . . . 2564
- coker . . . . . 1133
- colength of derivations . . . . . 1052
- collectDiv . . . . . 1541
- colrank . . . . . 1973
- colred . . . . . 981
- combinat.lib . . . . . 2233
- combinat\_lib . . . . . 2233
- comma . . . . . 312
- Command line options . . . . . 19
- command,custom . . . . . 138
- command-line option, setting value of . . . . . 279
- command-line option, value of . . . . . 279
- command-line options, print all values of . . . . . 279
- command-line options, short help . . . . . 19
- Commands . . . . . 156
- commands (letterplace) . . . . . 625
- commands (plural) . . . . . 335
- Commands for user defined types . . . . . 138
- comment . . . . . 44
- commRing . . . . . 538
- Commutative algebra . . . . . 1002
- Commutative Algebra . . . . . 715
- compare . . . . . 2305, 2306, 2307, 2323, 2324
- compareMatrix . . . . . 1138
- compareModules . . . . . 1134
- compareTasks . . . . . 961
- compareVectors . . . . . 1146
- CompDecomp . . . . . 375
- CompInt . . . . . 2049
- complement . . . . . 2271
- CompleteHomog . . . . . 1393
- completeReduction . . . . . 1805
- complex . . . . . 30
- complexClassify . . . . . 1655
- complexSingType . . . . . 1654
- complexType . . . . . 1656
- ComplexValue . . . . . 2338
- compose . . . . . 1058
- compregb.lib . . . . . 798
- compregb\_lib . . . . . 798
- comprehensive Groebner system . . . . . 798
- Comprehensive Groebner System . . . . . 830
- Comprehensive Groebner Systems . . . . . 818, 898
- compress . . . . . 968
- computeAfaceOrbits . . . . . 2099
- computeConstant . . . . . 1070
- computeGromovWitten . . . . . 1071
- computemcm . . . . . 1515
- computeN . . . . . 1567
- computeV . . . . . 1547
- Computing Groebner and Standard Bases . . . . . 710
- concat . . . . . 969
- conductor . . . . . 1050, 1209, 1636
- conductor, degree . . . . . 1727
- cone . . . . . 140, 2213, 2214
- coneLink . . . . . 2057
- coneViaInequalities . . . . . 2055
- coneViaPoints . . . . . 2053
- conic, parametrization, rational point . . . . . 1474
- conicWithTangents . . . . . 2159
- ConsLevels . . . . . 870
- constructblwup . . . . . 1516
- constructH . . . . . 1516
- constructible set . . . . . 870, 874
- constructible sets . . . . . 872, 876
- constructlastblwup . . . . . 1517
- constructor . . . . . 2224, 2225, 2296, 2314, 2315, 2316
- contact matrix . . . . . 1635
- ContactMatrix . . . . . 1690
- containedQ . . . . . 1904
- containment . . . . . 2310
- containsAsFace . . . . . 2057
- containsInCollection . . . . . 2076
- containsInSupport . . . . . 2058
- containsPositiveVector . . . . . 2058
- containsRelatively . . . . . 2059
- content . . . . . 275, 895
- contentSB, option . . . . . 236
- Conti-Traverso algorithm . . . . . 782
- continue . . . . . 291, 312
- contract . . . . . 167
- contraction . . . . . 2326, 2327, 2328
- contraHom . . . . . 550
- contribution bundle, graph, multiple cover, rational curve, . . . . . 1600
- contributionBundle . . . . . 1600
- contributors . . . . . 275
- Contributors . . . . . 3
- control . . . . . 1967
- Control structures . . . . . 290
- Control theory . . . . . 1966
- control.lib . . . . . 1967
- control\_lib . . . . . 1967
- control\_Matrix . . . . . 1703
- controlDim . . . . . 1968



- controlExample . . . . . 1977
  - convertdata . . . . . 1514
  - convertLeftToRightFraction . . . . . 591
  - convertRightToLeftFraction . . . . . 590
  - convexHull . . . . . 2059
  - convexIntersection . . . . . 2060
  - convloc . . . . . 419
  - Cooper philosophy . . . . . 787
  - coordinate change . . . . . 2212, 2213
  - coordinates . . . . . 2036
  - coords . . . . . 2036
  - copyright . . . . . 1
  - copyTask . . . . . 960
  - corank . . . . . 1648
  - corner, highest . . . . . 304
  - cornerMonomials . . . . . 1235
  - countPoints . . . . . 2000
  - coverings . . . . . 1071, 1072
  - CoverMult . . . . . 2577
  - coverTuple . . . . . 2576
  - cProj . . . . . 1414
  - cpu . . . . . 276, 278
  - crcgs . . . . . 925
  - create\_ring . . . . . 168
  - createBO . . . . . 1535
  - createGradedRingHomomorphism . . . . . 2471
  - createGroup . . . . . 2465
  - createlist . . . . . 1524
  - createQuotientGroup . . . . . 2466
  - createTask . . . . . 959
  - createTorsionFreeGroup . . . . . 2467
  - Crep . . . . . 832
  - CRHT-ideal . . . . . 787
  - cring . . . . . 73
  - cring declarations . . . . . 73
  - cring expressions . . . . . 73
  - cring operations . . . . . 73
  - cring related functions . . . . . 74
  - Critical points . . . . . 737
  - crossprod . . . . . 168
  - crypto.lib . . . . . 1989
  - crypto.lib . . . . . 1989
  - crystallographicGroupC2MM . . . . . 653
  - crystallographicGroupCM . . . . . 652
  - crystallographicGroupP1 . . . . . 649
  - crystallographicGroupP2 . . . . . 651
  - crystallographicGroupP2GG . . . . . 651
  - crystallographicGroupP2MM . . . . . 650
  - crystallographicGroupP3 . . . . . 655
  - crystallographicGroupP31M . . . . . 655
  - crystallographicGroupP3M1 . . . . . 656
  - crystallographicGroupP4 . . . . . 653
  - crystallographicGroupP4GM . . . . . 654
  - crystallographicGroupP4MM . . . . . 654
  - crystallographicGroupP6 . . . . . 656
  - crystallographicGroupP6MM . . . . . 657
  - crystallographicGroupPG . . . . . 650
  - crystallographicGroupPM . . . . . 649
  - CSMA . . . . . 1425
  - cup . . . . . 1093
  - cupproduct . . . . . 1095
  - curve singularities . . . . . 1627, 1719
  - curve singularity . . . . . 1049
  - curveColengthDerivations . . . . . 1051
  - curveConductorMult . . . . . 1050
  - curveDeligneNumber . . . . . 1051
  - curveDeltaInv . . . . . 1049
  - curveInv.lib . . . . . 1049
  - curveInv.lib . . . . . 1049
  - curvepar.lib . . . . . 1682
  - curvepar.lib . . . . . 1682
  - CurveParam . . . . . 1686
  - CurveRes . . . . . 1683
  - Curves . . . . . 1465
  - custom assignment . . . . . 139
  - custom command . . . . . 138
  - custom type . . . . . 136
  - Customization of the Emacs interface . . . . . 27
  - customstd.lib . . . . . 2238
  - customstd.lib . . . . . 2238
  - cycleLength . . . . . 2131
  - cyclePoints . . . . . 2135
  - cyclic . . . . . 889
  - cyclic code . . . . . 786
  - Cyclic code . . . . . 1938
  - Cyclic roots . . . . . 707
  - cyclotomic . . . . . 1781
  - CYGWIN and ESingular . . . . . 26
- ## D
- D-integration . . . . . 422
  - D-localization . . . . . 422, 523
  - D-module . . . . . 377, 402, 422, 448, 454, 523, 601
  - D-module structure . . . . . 402, 448, 454
  - D-restriction . . . . . 422
  - Data types . . . . . 73
  - Data types (plural) . . . . . 319
  - datetime . . . . . 169
  - DBM links . . . . . 102
  - dbprint . . . . . 169
  - debug.log . . . . . 1653
  - debugger . . . . . 69
  - debugging library code . . . . . 69
  - Debugging tools . . . . . 68
  - debugLib, option . . . . . 236
  - dec1var . . . . . 1165
  - decimal . . . . . 1991
  - Decker, Wolfram . . . . . 3
  - Declaration of objects of a user defined type . . . . . 137
  - decode . . . . . 1951
  - decodeCode . . . . . 1956
  - decdegb.lib . . . . . 1938

- decodegb\_lib ..... 1938
- decodeRandom ..... 1952
- decodeRandomFL ..... 1964
- decodeSV ..... 1934
- Decoding ..... 1938
- Decoding codes with Groebner bases ..... 785
- Decoding method based on quadratic equations ... 790
- decoding, decoding problem ..... 786
- decoef ..... 1983
- decomp.lib ..... 1052
- decomp\_lib ..... 1052
- decompopts ..... 1054
- decompose ..... 1055
- decomposeVecField ..... 2587
- decomposition ..... 2297, 2518
- decomposition of modules ..... 1161
- decomposition, numerical ..... 1454
- decone ..... 2213, 2214
- def ..... 79
- def declarations ..... 80
- default argument ..... 52
- defined ..... 170
- defineHomogeneous ..... 2505
- definingBinomialIdeal ..... 1216
- Definition of a user defined type ..... 136
- defl ..... 1459
- deform ..... 1757
- deform.lib ..... 1692
- deform\_lib ..... 1692
- Deformations ..... 742
- Deformations, T1 and T2 ..... 740
- defRes, option ..... 236
- defring ..... 946
- defringp ..... 947
- defrings ..... 946
- deg ..... 170
- Deg ..... 1131
- degBound ..... 302
- degree ..... 171, 315, 2307, 2309
- Degree ..... 1132
- degree lexicographical ordering ..... 770
- degree of a polynomial ..... 315
- degree reverse lexicographical ordering ..... 770
- degreeDivisor ..... 1433
- degreeFormalDivisor ..... 1437
- degreepart ..... 1829
- DegreePure ..... 1453
- Degrees ..... 1132
- delete ..... 105, 171, 206
- deleteGenerator ..... 446
- deleteSublist ..... 809
- deligne number ..... 1049, 1051, 1052
- delta ..... 1731
- Delta ..... 1536
- delta invariant ..... 1049, 1204, 1732
- delta invariant ..... 1186, 1193, 1196
- DeltaList ..... 1537
- deltaLoc ..... 1204
- Demo mode ..... 27
- denom\_list ..... 276
- denominator ..... 172
- depth ..... 1099
- Depth ..... 728
- depthIdeal ..... 1172
- deRham.lib ..... 1427
- deRham.lib ..... 1427
- deRhamCohom ..... 434
- deRhamCohomIdeal ..... 435
- deRhamCohomology ..... 1427
- derivate ..... 1803
- derivation ... 2223, 2312, 2313, 2314, 2315, 2316, 2325
- derivationAdd ..... 2318
- derivationCheckList ..... 2315
- derivationConstructor ..... 2316
- derivationContraction ..... 2326
- derivationContractionGen ..... 2326
- derivationEqu ..... 2323
- derivationEval ..... 2325
- derivationFromList ..... 2314
- derivationFromPoly ..... 2315
- derivationLie ..... 2328
- derivationMul ..... 2322
- derivationNeg ..... 2321
- derivationNeq ..... 2324
- derivationPrint ..... 2317
- derivationSub ..... 2320
- derivationToString ..... 2317
- desingularization ..... 2342
- det ..... 172
- det\_B ..... 993
- detadj ..... 1739
- determinacy ..... 1755
- Determinacy ..... 1755
- determinecenter ..... 1481
- detropicalise ..... 2174
- develop ..... 1722
- Di Biase-Urbanke algorithm ..... 783
- diag ..... 970
- diag\_test ..... 991
- diagInvariants ..... 1220
- diagonalizeMatrixSimul ..... 2587
- diagonalizeVecField ..... 2586
- diagonalizeVecFieldLin ..... 2585
- DifConsLCSets ..... 876
- diff ..... 173
- diffAlgebra ..... 2292
- diffAlgebraChangeOrd ..... 2294
- diffAlgebraGens ..... 2294
- diffAlgebraListGen ..... 2295
- diffAlgebraStructure ..... 2293
- diffAlgebraUnivDerIdeal ..... 2294
- difference ..... 2302, 2320, 2321
- differential ..... 2296, 2313
- differential algebra ..... 2292, 2293, 2294

- differential forms . . . . . 2292
- differentials . . . . . 2292
- difform.lib . . . . . 2291
- difform\_lib . . . . . 2291
- difformAdd . . . . . 2302
- difformCoef . . . . . 2296
- difformDeg . . . . . 2307
- difformDiff . . . . . 2313
- difformDiv . . . . . 2304
- difformEqu . . . . . 2305
- difformFromPoly . . . . . 2296
- difformGenToString . . . . . 2299
- difformHomogDecomp . . . . . 2299
- difformIsBigger . . . . . 2306
- difformIsGen . . . . . 2301
- difformIsHomog . . . . . 2308
- difformIsHomogDeg . . . . . 2309
- difformIsSmaller . . . . . 2306
- difformListCont . . . . . 2310
- difformListSort . . . . . 2310
- difformMul . . . . . 2303
- difformNeg . . . . . 2303
- difformNeq . . . . . 2305
- difformPrint . . . . . 2300
- difformSub . . . . . 2302
- difformToString . . . . . 2300
- difformUnivDer . . . . . 2312
- diffRat . . . . . 537
- difpoly2tex . . . . . 1983
- dim . . . . . 174
- dim (letterplace) . . . . . 625
- dim (plural) . . . . . 337
- dim\_slocus . . . . . 1757
- dimension . . . . . 2061
- dimension, moduli space of stable maps, rational curves . . . . . 1599
- dimensionOfLocalization . . . . . 1445
- dimGradedPart . . . . . 1620
- dimH . . . . . 1619
- dimMon . . . . . 1155
- dimStack . . . . . 1599
- direct\_boolean\_poly . . . . . 2262
- direct\_from\_boolean\_poly . . . . . 2265
- discr . . . . . 1770
- discrepancy . . . . . 1540
- discrim . . . . . 862
- discussion of parametric ideal . . . . . 822
- disjoint . . . . . 830
- disp\_zdd . . . . . 2268
- DISPLAY environment variable . . . . . 17
- displayCohom . . . . . 1620
- displayHNE . . . . . 1726
- displayInvariants . . . . . 1728
- displayMultsequence . . . . . 1730
- displaypfd . . . . . 2524
- displaypfd\_long . . . . . 2525
- displayPuisseuxExpansion . . . . . 2150
- displayTropicalLifting . . . . . 2146
- distributed computing . . . . . 885, 942, 959
- Distributed computing . . . . . 884
- div . . . . . 75, 87, 121, 314
- dividelist . . . . . 1523
- divideUnits . . . . . 1980
- division . . . . . 174, 831, 832, 2304
- division (plural) . . . . . 338
- division, pdivi2, reduce . . . . . 903
- division, reduce . . . . . 902
- divisor . . . . . 2018
- divisorplus . . . . . 1430
- divisors . . . . . 1430, 1431, 1432, 1433, 1434, 1435, 1436, 1437
- divisors, polyhedra . . . . . 1437
- divisors.lib . . . . . 1429
- divisors\_lib . . . . . 1429
- DLoc . . . . . 424
- DLoc0 . . . . . 425
- Dlocalization . . . . . 523
- dmod.lib . . . . . 401
- dmod\_lib . . . . . 401
- dmodAction . . . . . 535
- dmodActionRat . . . . . 535
- dmodapp.lib . . . . . 421
- dmodapp\_lib . . . . . 421
- dmodGeneralAssumptionCheck . . . . . 531
- dmodideal.lib . . . . . 447
- dmodideal\_lib . . . . . 447
- dmodloc.lib . . . . . 523
- dmodloc\_lib . . . . . 523
- dmodoublext . . . . . 550
- dmodvar.lib . . . . . 454
- dmodvar\_lib . . . . . 454
- Documentation Tool . . . . . 63
- double . . . . . 2021
- doubleExt . . . . . 607
- downloading . . . . . 2601
- dp, global ordering . . . . . 770
- Dp, global ordering . . . . . 770
- DrawCovers . . . . . 2582
- drawNewtonSubdivision . . . . . 2155
- drawTropicalCurve . . . . . 2153
- ds, local ordering . . . . . 770
- Ds, local ordering . . . . . 770
- DsingularLocus . . . . . 527
- dsum . . . . . 970
- dual\_code . . . . . 1936
- dualCone . . . . . 2062
- dualConic . . . . . 2175
- dualPart . . . . . 1421
- dualPartition . . . . . 1604
- dualPolytope . . . . . 2088
- dualSheaf . . . . . 1595
- dump . . . . . 175
- dyckGroup1 . . . . . 658
- dyckGroup2 . . . . . 658

dyckGroup3 . . . . . 658  
dynamic geometry . . . . . 846, 849  
Dynamic loading . . . . . 71  
Dynamic modules . . . . . 709

## E

ecart . . . . . 2022  
echo . . . . . 303  
ECoef . . . . . 1502  
ECP . . . . . 2001  
Edatalist . . . . . 1499  
Edelman-Reiner . . . . . 2220  
edge, highest . . . . . 304  
Editing input . . . . . 18  
Editing SINGULAR input files with Emacs . . . . . 28  
eexgcdN . . . . . 1991  
effective . . . . . 1434  
egcdMain . . . . . 1903  
ehrhartRing . . . . . 1214  
eigenvals . . . . . 998  
eigenvalue . . . . . 246, 998  
eigenvalues . . . . . 1385  
elemSymmId . . . . . 889  
elim . . . . . 1063  
elim.lib . . . . . 1059  
elim.lib . . . . . 1059  
elim1 . . . . . 1064  
elim2 . . . . . 1065  
eliminate . . . . . 176  
eliminate (plural) . . . . . 339  
eliminateNC . . . . . 556  
elimination . . . . . 556  
Elimination . . . . . 718  
elimlinearpart . . . . . 1830  
elimpart . . . . . 1830  
elimpartanyr . . . . . 1832  
elimrep . . . . . 1508  
elimRing . . . . . 1061  
elimWeight . . . . . 561  
elliptic curve . . . . . 1070  
elliptic curve cover multiplicities . . . . . 2577  
elliptic curve covers . . . . . 2578  
elliptic curve draw covers . . . . . 2582, 2583  
elliptic curves . . . . . 1068, 1071, 1072, 2576, 2578, 2579,  
2580, 2581, 2582  
ellipticAdd . . . . . 1998  
ellipticAllPoints . . . . . 2000  
ellipticcovers.lib . . . . . 1068  
ellipticcovers.lib . . . . . 1068  
ellipticMult . . . . . 1998  
ellipticNF . . . . . 2137  
ellipticRandomCurve . . . . . 1999  
ellipticRandomPoint . . . . . 1999  
else . . . . . 296  
Emacs . . . . . 22

Emacs, a quick guide . . . . . 23  
Emacs, customization of Singular mode . . . . . 27  
Emacs, editing Singular input files . . . . . 28  
Emacs, important commands . . . . . 29  
Emacs, overview . . . . . 23  
Emacs, running Singular under . . . . . 25  
Emacs, Singular demo mode . . . . . 27  
Emacs, user interface . . . . . 22  
Emaxcont . . . . . 1509  
embedMat . . . . . 581  
emptyFan . . . . . 2076  
encode . . . . . 1946  
endomorphism filtration . . . . . 1713  
endvfil . . . . . 1712  
ensureLeftNcfrac . . . . . 543  
ensureRightNcfrac . . . . . 545  
Enumerative geometry . . . . . 1587  
enumpoints.lib . . . . . 2329  
enumpoints.lib . . . . . 2329  
envelop . . . . . 850, 851, 855, 857, 859  
envelope . . . . . 340  
enveloping algebra . . . . . 276, 372  
environment variable, DISPLAY . . . . . 17  
environment variables . . . . . 2601  
EOrdlist . . . . . 1501  
equal . . . . . 2305, 2323, 2324  
Equal . . . . . 1452  
equalJinI . . . . . 1639  
equalMultiDeg . . . . . 2494  
equations . . . . . 2063  
equidim . . . . . 1249  
equidimMax . . . . . 1250  
equidimMaxEHV . . . . . 1250  
equidimZ . . . . . 1258  
equiRadical . . . . . 1248  
equisg.lib . . . . . 1697  
equisg.lib . . . . . 1697  
equisingular Tjurina number . . . . . 1627  
equisingularity ideal . . . . . 1698  
equisingularity stratum . . . . . 1700  
Eresol . . . . . 1480  
ERROR . . . . . 177  
error recovery . . . . . 15  
errorInsert . . . . . 1948  
errormap . . . . . 2335  
errorRand . . . . . 1949  
esIdeal . . . . . 1698  
ESingular, CYGWIN . . . . . 26  
essential . . . . . 2216  
esStratum . . . . . 1699  
eta . . . . . 2133  
euler . . . . . 1988  
EulerAff . . . . . 1426  
eulerChProj . . . . . 1417  
eulerPolynomialTable . . . . . 2564  
EulerProj . . . . . 1426  
eval . . . . . 177

- evalJacobianAtBox ..... 1894
  - evalPolyAtBox ..... 1894
  - evalPolyAtBox2 ..... 1828
  - evaluate\_reynolds ..... 1785
  - evaluateFormalDivisor ..... 1435
  - evaluatepfd ..... 2524
  - evaluateProduct ..... 2122
  - evaluation ..... 2325
  - Evaluation of logical expressions ..... 311
  - evalutateIntegral ..... 1070
  - exactness recovery ..... 1862
  - example ..... 178, 2217, 2218, 2219, 2220
  - Examples ..... 699
  - Examples of ring declarations ..... 31
  - Examples of use of LETTERPLACE ..... 617
  - Examples of use of LETTERPLACE over  $\mathbb{Z}$  ..... 620
  - exclusionTest ..... 1828
  - execute ..... 178
  - exit ..... 300
  - Exp for matrices ..... 1461
  - exp2pt ..... 1984
  - Experimental libraries ..... 2274
  - exponents ..... 2224
  - export ..... 292
  - exportNuminvs ..... 1223
  - exportto ..... 293
  - expp ..... 1604
  - expression list ..... 73, 319
  - Ext ..... 1101
  - Ext, computation of ..... 725
  - ext-module ..... 601
  - Ext\_R ..... 1099
  - extcurve ..... 1930
  - extdevelop ..... 1723
  - extendedTensor ..... 561
  - extendGC ..... 836
  - extendpoly ..... 835
  - extendingring ..... 947
  - extendWeyl ..... 532
  - extension of rings ..... 1262
  - Exterior ..... 568
  - exterior basis ..... 985
  - exterior power ..... 986
  - exteriorBasis ..... 984
  - exteriorPower ..... 986
  - extgcd ..... 178
  - Extra weight vector ..... 773
  - extractS ..... 453
- F**
- faceContaining ..... 2063
  - facets ..... 2064
  - facFirstShift ..... 496
  - facFirstWeyl ..... 493
  - facGBIdeal ..... 1236
  - facShift ..... 495
  - facstd ..... 179
  - facSubWeyl ..... 494
  - factmodd ..... 180
  - factor ..... 241, 902
  - factorgroup ..... 2482
  - factorH ..... 817
  - factorial ..... 809
  - factorization ..... 241, 1002
  - Factorization ..... 729
  - factorize ..... 180, 241
  - factorLenstraECM ..... 2001
  - factorMain ..... 1903
  - factory ..... 1
  - facvar ..... 902
  - facWeyl ..... 492
  - FamElementsAtEnvCompPoints ..... 858
  - fan ..... 140, 2129
  - fanViaCones ..... 2077
  - farey ..... 181, 310
  - fareypoly ..... 1082
  - fastelim ..... 1833
  - fastExpt ..... 1987
  - fastHC, option ..... 235
  - fetch ..... 182
  - fetch (letterplace) ..... 626
  - fetch (plural) ..... 341
  - fetchall ..... 949
  - Feynman graph ..... 1068, 2576
  - Feynman integral ..... 2576
  - Feynman integral of a graph for a fixed ordering ..... 2580
  - Feynman integrals ..... 2582
  - FeynmanIntegralA ..... 2581
  - FeynmanIntegralo ..... 2580
  - FeynmanIntegralO ..... 2580
  - ffmodStd ..... 1087
  - ffmodstd.lib ..... 1081
  - ffmodstd.lib ..... 1081
  - ffsolve ..... 1821
  - ffsolve.lib ..... 1821
  - ffsolve.lib ..... 1821
  - fglm ..... 183, 272
  - fglm.solve ..... 1850
  - fglmquot ..... 184
  - fibonacci ..... 810
  - fibonacciGroup ..... 659
  - field ..... 115
  - file, .singularrc ..... 22
  - filecmd ..... 184
  - filtration ..... 601
  - finalcases ..... 908
  - finalCharts ..... 1525
  - find ..... 185
  - find\_first\_primitive\_root ..... 2003
  - find\_index ..... 2004
  - findAuto ..... 460
  - findifs.lib ..... 1981
  - findifs\_example ..... 1981

- findifs.lib . . . . . 1981
  - findimAlgebra . . . . . 568
  - findInvo . . . . . 458
  - findInvoDiag . . . . . 459
  - findOrientedBoundary . . . . . 2134
  - findTorsion . . . . . 1976
  - finduni . . . . . 185
  - findvars . . . . . 1834
  - findZeroPoly . . . . . 2535
  - finite field . . . . . 115, 1821, 2329
  - Finite fields . . . . . 716
  - finiteDiagInvariants . . . . . 1219
  - finitediff.lib . . . . . 2330
  - finitediff.lib . . . . . 2330
  - finitely presented algebra . . . . . 641, 646, 661, 666, 673
  - finitely presented algebra, standard finitely presented algebra . . . . . 637
  - finitely presented group . . . . . 646
  - finitenessTest . . . . . 1012
  - finiterep . . . . . 1815
  - finvar.lib . . . . . 1775
  - finvar.lib . . . . . 1775
  - first index is 1 . . . . . 313
  - first order deformation . . . . . 2544, 2550
  - First steps . . . . . 6
  - firstoct . . . . . 1898
  - firstQuadrantComplex . . . . . 2575
  - fitting . . . . . 1104
  - Fitzgerald-Lax method . . . . . 789
  - fixed points, moduli space of stable maps, graph . . . . . 1599
  - fixedPoints . . . . . 1599
  - fl2poly . . . . . 444
  - flatten . . . . . 971
  - flatteningStrat . . . . . 1104
  - flintQ . . . . . 186
  - Float . . . . . 186
  - floating point . . . . . 954
  - floor diagrams . . . . . 2579
  - FloorDiagrams . . . . . 2579
  - Flow control . . . . . 48
  - for . . . . . 295
  - Formal Checker . . . . . 63
  - formal linear representations . . . . . 676
  - formaldivisorplus . . . . . 1435
  - Formatting output . . . . . 707
  - forward . . . . . 2332
  - fourier . . . . . 441
  - fouriersystem . . . . . 2337
  - fpadim.lib . . . . . 640
  - fpadim.lib . . . . . 640
  - fpalgebras.lib . . . . . 646
  - fpalgebras.lib . . . . . 646
  - fpaprops.lib . . . . . 660
  - fpaprops.lib . . . . . 660
  - fPiece . . . . . 2550
  - fprintf . . . . . 186
  - fracStatus . . . . . 586
  - framed sheaves . . . . . 1438
  - frandwalk . . . . . 2539
  - free . . . . . 2223, 2224
  - free associative algebra . . . . . 646, 666
  - free associative algebra, tensor algebra . . . . . 636
  - Free associative algebras . . . . . 636
  - free noncommutative Groebner basis . . . . . 666
  - Free resolution . . . . . 720
  - Free resolution, graded . . . . . 723
  - freeAlgebra (letterplace) . . . . . 627
  - freegb.lib . . . . . 666
  - freegb.lib . . . . . 666
  - freemodule . . . . . 188
  - freeModule . . . . . 1124
  - freeModule2Module . . . . . 1138
  - freerank . . . . . 890
  - fres . . . . . 188
  - from\_boolean\_constant . . . . . 2264
  - from\_boolean\_ideal . . . . . 2266
  - from\_boolean\_poly . . . . . 2264
  - from\_boolean\_set . . . . . 2266
  - frwalk . . . . . 189
  - full representation . . . . . 835, 837
  - fullDimImages . . . . . 2092
  - fullFan . . . . . 2079
  - fullSerreRelations . . . . . 647
  - fullSpace . . . . . 2050
  - FultonA . . . . . 1424
  - Functional decomposition . . . . . 1054
  - Functionality and release notes of LETTERPLACE . . . . . 623
  - Functions . . . . . 156
  - Functions (letterplace) . . . . . 625
  - further\_hn\_proc . . . . . 1735
  - furtherInvar . . . . . 1806
  - Future benefits of Singular 4 . . . . . 2599
  - fVector . . . . . 2080
  - fwalk . . . . . 1088
- ## G
- G-algebra . . . . . 366, 518
  - G-algebra, setup . . . . . 367
  - G\_a -Invariants . . . . . 753
  - galois field . . . . . 115
  - Gamma . . . . . 1513
  - Gauss-Manin connection . . . . . 778, 1739, 1773
  - Gauss-Manin system . . . . . 1703, 1704, 1705, 1706, 1707, 1708, 1709, 1710, 1711, 1713, 1718, 1719
  - gauss\_col . . . . . 975
  - gauss\_nf . . . . . 995
  - gauss\_row . . . . . 976
  - gaussColWithoutPerm . . . . . 1865
  - gaussred . . . . . 993
  - gaussred\_pivot . . . . . 994
  - gaussRowWithoutPerm . . . . . 1864
  - GBsolve . . . . . 1823

- GBWeight ..... 426
- gcd ..... 190
- gcddivisor ..... 2019
- gcdMon ..... 1151
- Gelfand-Kirillov dimension ..... 661
- Gelfand-Kirillov dimension, G-Algebra ..... 522
- gen ..... 191
- General command syntax ..... 41
- General concepts ..... 15
- general error-locator polynomial ..... 787
- general neron ..... 2342
- General purpose ..... 795
- General syntax of a ring declaration ..... 32
- general weighted lexicographical ordering ..... 770
- general weighted reverse lexicographical ordering .. 770
- general.lib ..... 806
- general.lib ..... 806
- Generalized Hilbert Syzygy Theorem ..... 369
- Generalized Newton identities ..... 788
- generalOrder ..... 2032
- generateG ..... 2001
- generator ..... 2294, 2301, 2326
- generators ..... 1635, 2295, 2297, 2299
- generatorsOfLinealitySpace ..... 2065
- generatorsOfSpan ..... 2065
- generic representation ..... 835
- genericid ..... 936
- genericity ..... 1973
- genericmat ..... 971
- genMDSMat ..... 1950
- genoutput ..... 1517
- genSymId ..... 1384
- genus ..... 1201
- Geometric genus ..... 1465
- Geometric Invariant Theory ..... 754
- Geometric Theorems ..... 866
- geometrical locus .... 846, 849, 851, 855, 856, 857, 859
- German Umlaute ..... 310
- getArguments ..... 967
- getCommand ..... 966
- getCone ..... 2081
- getcores ..... 943
- getdump ..... 191
- getenv ..... 276
- getGradingGroup ..... 2462
- getLattice ..... 2464
- getLinearForms ..... 2066
- getMaxPoints ..... 2436
- getModuleGrading ..... 2474
- getMultiplicity ..... 2066
- getOneVar ..... 1207
- getRelations ..... 1863
- getRelationsRadical ..... 1864
- getResult ..... 967
- getSmallest ..... 1207
- getState ..... 968
- getStringpfd ..... 2526
- getStringpfd\_indexed ..... 2527
- Getting started ..... 6
- getVariableWeights ..... 2461
- getWitnessSet ..... 1865
- gfan.lib ..... 2050
- gfan.lib ..... 2050
- gfanlib ..... 1
- GIT-Fans ..... 754
- gitCone ..... 2128
- GITcone ..... 2094
- GITfan ..... 2095
- gitfan.lib ..... 2090
- gitfan.lib ..... 2090
- GITfanFromOrbitCones ..... 2096
- GITfanParallel ..... 2097
- GITfanParallelSymmetric ..... 2124
- GITfanSymmetric ..... 2122
- GKdim ..... 465
- gkdim.lib ..... 464
- gkdim.lib ..... 464
- GKExp ..... 522
- GKZfan ..... 2098
- GKZsystem ..... 482
- global ..... 157
- global Bernstein-Sato ideal ..... 448
- global Bernstein-Sato polynomial ..... 377, 402
- global Bernstein-Sato polynomial for variety ..... 454
- global homological dimension ..... 661
- Global orderings ..... 770
- globalSections ..... 1432
- GMP ..... 1, 2600
- gmscoeffs ..... 1705
- gmsnf ..... 1705
- gmspoly.lib ..... 1718
- gmspoly.lib ..... 1718
- gmsring ..... 1704
- gmssing.lib ..... 1703
- gmssing.lib ..... 1703
- GND.lib ..... 2342
- GND.lib ..... 2342
- Goettsche's formula ..... 1438
- goettsche.lib ..... 1438
- goettsche.lib ..... 1438
- GoettscheF ..... 1438
- good basis ..... 1703, 1711, 1718, 1719
- goodBasis ..... 1719
- graal.lib ..... 1444
- graal.lib ..... 1444
- graalMixed ..... 1444
- graded ..... 2295
- graded algebra ..... 2274
- Graded commutative algebras ..... 615
- graded Hilbert series ..... 673
- graded module, graded piece ..... 1620
- graded modules, graded homomorphisms, syzygies ..... 2346
- graded modules, handling of ..... 723



graded piece, second order piece .....	2550	Gromov-Witten invariants, multiple covers .....	1601
graded Weyl algebra .....	377	Gromov-Witten invariants, rational curves on	
graded-module, graded-resolution,		Calabi-Yau threefolds .....	1602
homogenous-matrix, R-module-homomorphism		Gromov-Witten invariants. ....	1587
.....	1123	gromovWitten .....	1071
gradedModules.lib .....	2346	grorder .....	2362
gradedModules_lib .....	2346	ground field .....	115
gradeNumber .....	604	group_reynolds .....	1781
gradiator .....	2507	groupActionOnHashes .....	2106
graph .....	1069, 1597, 1598	groupActionOnQImage .....	2105
graphics.lib .....	1905	growth of algebra .....	661
graphics_lib .....	1905	grpowers .....	2357
graphviz .....	1	grpres .....	2360
Grassmannian .....	1589	grprod .....	2396
Grassmannian, abstract variety, Schubert calculus		grrange .....	2416
.....	1589	gres .....	2393
graver4ti2 .....	1383	grndmap .....	2399
grconcat .....	2397	grndmap2 .....	2399
grdeg .....	2348	grndmat .....	2398
Greuel, Gert-Martin .....	3	grshift .....	2352
grgens .....	2359	grsum .....	2354
grgroebner .....	2397	grsyz .....	2392
grlift .....	2395	grtest .....	2347
grlifting .....	2400	grtranspose .....	2358
grlifting2 .....	2403	grtransposel .....	2364
grlifting3 .....	2406	grtwist .....	2353
grneg .....	2417	grtwists .....	2354
Grob1Levels .....	873	grview .....	2350
grobcov .....	820	grwalk.lib .....	1088
grobcov.lib .....	818	grwalk_lib .....	1088
grobcov_lib .....	818	grzero .....	2353
grobj .....	2346	GTZmod .....	1164
groebner .....	191, 711	GTZopt .....	1169
Groebner Bases .....	710	gwalk .....	1091
Groebner bases for two-sided ideals in free associative		Gweights .....	563
algebras .....	638		
Groebner bases in free associative algebras .....	638		
Groebner bases in G-algebras .....	367		
Groebner bases, decodeGB .....	1938		
Groebner bases, slim .....	715		
groebner basis computations .....	192		
Groebner basis conversion .....	713		
Groebner cover .....	822, 835, 837		
Groebner fan .....	2129		
Groebner walk .....	1090, 1091, 2537, 2538		
Groebner-Shirshov bases .....	638		
Groebner-Shirshov bases in free associative algebras			
.....	638		
groebnerBasis .....	1217		
groebnerComplex .....	2187		
groebnerCone .....	2176		
groebnerFan .....	2185		
Groebnerwalk .....	1088, 2439, 2537, 2554		
Gromov-Witten invariants ....	1068, 1071, 1072, 2576,		
2582			
Gromov-Witten invariants, lines on hypersurfaces			
.....	1601		

## H

h_increasing_knapsack .....	2010
H2basis .....	1741
Hamburger-Noether expansion .....	1627, 1719
Hamburger-Noether expansions .....	1924
Handling graded modules .....	723
hardware platform .....	278
hasAlgExtensionCoefficient .....	956
hasCommutativeVars .....	955
hasGlobalOrdering .....	955
hashesToFan .....	2127
hashToCone .....	2127
hasLeftDenom .....	540
hasMixedOrdering .....	955
hasNumericCoeffs .....	954
hasRightDenom .....	541
hasTransExtensionCoefficient .....	956
Hcode .....	1648
headStand .....	983



- heightZ ..... 1258
  - help ..... 193
  - help browsers ..... 16
  - help browsers, dummy ..... 16
  - help browsers, emacs ..... 16
  - help browsers, html ..... 16
  - help browsers, setting command to use ..... 17
  - help browsers, setting the ..... 275
  - Help string ..... 54
  - help, accessing over the net ..... 19
  - help, online help system ..... 15
  - Hensel ..... 180
  - hensel lifting ..... 1118
  - henselGlobal ..... 1118
  - hermiteNormalForm ..... 2507
  - hess.lib ..... 1449
  - hess.lib ..... 1449
  - hessenberg ..... 276, 998
  - HHnormalForm ..... 1660
  - high corner method ..... 192
  - highcorner ..... 192, 194
  - highest corner ..... 304
  - highest edge ..... 304
  - hilb ..... 195
  - Hilbert function ..... 274, 518, 775
  - Hilbert polynomial ..... 518
  - Hilbert scheme ..... 1438
  - Hilbert series ..... 518, 641, 775
  - Hilbert-driven GB algorithm ..... 713
  - hilbert4ti2 ..... 1383
  - HilbertClassPoly ..... 1756
  - hilbertSeries ..... 2509
  - HilbertSeries ..... 1807
  - HilbertWeights ..... 1807
  - hilbPoly ..... 898
  - hilbvec ..... 1834
  - history ..... 18
  - hnexpansion ..... 1720
  - hnoether.lib ..... 1719
  - hnoether.lib ..... 1719
  - Hodge ideals ..... 2421
  - hodge.lib ..... 2421
  - hodge.lib ..... 2421
  - hodgeIdeals ..... 2423
  - holonomic rank ..... 523
  - holonomicRank ..... 526
  - hom ..... 1143
  - Hom ..... 1105
  - hom\_kernel ..... 1110
  - HomJJ ..... 1200
  - homog ..... 196, 2308
  - homog\_part ..... 1603
  - homog\_parts ..... 1603
  - homogeneitySpace ..... 2178
  - homogeneous ..... 196, 2299, 2308, 2309
  - homogeneous decomposition ..... 2299
  - homogfacFirstQWeyl ..... 498
  - homogfacFirstQWeyl.all ..... 517
  - homogfacNthQWeyl ..... 497
  - homogfacNthQWeyl.all ..... 499
  - homogfacNthWeyl ..... 497
  - homolog.lib ..... 1091
  - homolog.lib ..... 1091
  - homology ..... 1107
  - homomorphism ..... 1136
  - Hosten-Sturmfels algorithm ..... 783
  - How to enter and exit ..... 15
  - How to use this manual ..... 4
  - howto, download ..... 2601
  - howto, install on Macintosh ..... 2603
  - howto, install on Unix ..... 2602
  - howto, install on Windows ..... 2602
  - hres ..... 197
  - html, default help ..... 16
  - Hurwitz numbers ..... 1071, 1072
  - hybrid algorithms ..... 1862
  - hyperbolic sine series expansion ..... 2578
  - hyperel.lib ..... 2016
  - hyperel.lib ..... 2016
  - hyperplane ..... 2206
  - hyperplane arrangement ..... 402
  - Hypersurface singularities, classification of ..... 750
  - hypersurface singularity ..... 1739
- ## I
- I-regular function ..... 835
  - id ..... 1123
  - id2mod ..... 895
  - ideal ..... 80
  - ideal (plural) ..... 319
  - ideal declarations ..... 81
  - ideal declarations (plural) ..... 319
  - ideal expressions ..... 81
  - ideal expressions (plural) ..... 320
  - Ideal membership ..... 775
  - ideal operations ..... 82
  - ideal operations (plural) ..... 321
  - ideal related functions ..... 83
  - ideal related functions (plural) ..... 322
  - ideal, toric ..... 782
  - ideals ..... 315
  - idealsimplify ..... 1639
  - idealSplit ..... 1841
  - identifier ..... 316
  - Identifiers, syntax of ..... 44
  - identifyvar ..... 1499
  - if ..... 296
  - image ..... 1134
  - image\_of\_variety ..... 1802
  - ImageGroup ..... 1808
  - imageLattice ..... 2475
  - ImageVariety ..... 1807

- imap ..... 198
- imap (letterplace) ..... 628
- imap (plural) ..... 342
- Imap, option ..... 236
- imapall ..... 950
- impart ..... 199
- Implemented algorithms ..... 37
- importfrom ..... 296
- iMult ..... 1204
- IN ..... 209
- inCenter ..... 394
- inCentralizer ..... 394
- Incl. .... 1452
- indepSet ..... 199
- Index ..... 2604
- indexed names ..... 43, 44
- indices, multi ..... 44
- indSet ..... 1168
- inequalities ..... 2066
- infinitely presented algebra ..... 673
- info ..... 17
- Info string ..... 56
- inForm ..... 429
- infRedTail, option ..... 235
- iniD ..... 1521
- init\_debug ..... 1649
- initial ..... 2178
- initial form ..... 377
- initial ideal ..... 377
- initial ideal approach ..... 377, 454
- initialForm ..... 2166
- initialIdeal ..... 2166
- initialIdealW ..... 428
- initialMalgrange ..... 427
- injective\_knapsack ..... 2011
- inout.lib ..... 878
- inout.lib ..... 878
- input ..... 48
- insert ..... 105, 200
- insertCone ..... 2081
- insertGenerator ..... 445
- instructions, downloading ..... 2601
- instructions, Macintosh installation ..... 2603
- instructions, Unix installation ..... 2602
- instructions, Windows installation ..... 2602
- inSubring ..... 1007
- int ..... 85
- int declarations ..... 85
- int expressions ..... 85
- int operations ..... 87
- int related functions ..... 88
- intclMonIdeal ..... 1215
- intclToricRing ..... 1211
- integer division ..... 314
- integer programming ..... 784
- integral ..... 1591
- integral basis ..... 1116, 1466
- integral closure ..... 1186, 1193, 1196, 1210
- integralBasis ..... 1115
- integralbasis.lib ..... 1115
- integralbasis.lib ..... 1115
- integralIdeal ..... 432
- integralModule ..... 433
- integralSection ..... 2481
- integration of D-module ..... 422
- Interactive use ..... 15
- InterDiv ..... 1524
- interface, Emacs ..... 22
- internalfunctions ..... 1649
- interpolate ..... 1849
- interpolation ..... 200
- interpret ..... 1144
- interpretElem ..... 1146
- interpretInv ..... 1145
- interpretList ..... 1146
- interred ..... 201
- Interrupting SINGULAR ..... 18
- intersect ..... 202
- intersect (plural) ..... 343
- intersectElim, option ..... 236
- intersection ..... 1731, 2271
- intersection lattice ..... 2227, 2228
- intersection multiplicity ..... 1635, 1731
- Intersection theory ..... 1587
- intersectionDiv ..... 1538
- IntersectionMatrix ..... 1690
- intersectionSet ..... 2271
- intersectionValRingIdeals ..... 1221
- intersectionValRings ..... 1221
- intersectLattices ..... 2479
- intersectLists ..... 2233
- intersectMon ..... 1152
- intersectOrbitsWithMovingCone ..... 2103
- intersectpar ..... 865
- intersectSyz, option ..... 236
- IntersectWithSub ..... 486
- intersectZ ..... 1259
- interval arithmetic ..... 1891
- interval.lib ..... 1825
- interval.lib ..... 1825
- intervalmatrixInit ..... 1826
- intInverse ..... 2481
- intmat ..... 90
- intmat declarations ..... 90
- intmat expressions ..... 91
- intmat operations ..... 92
- intmat related functions ..... 93
- intmat type cast ..... 92
- intmat2binomials ..... 1233
- intmat2mons ..... 1232
- intPart ..... 1993
- intprog.lib ..... 1119
- intprog.lib ..... 1119
- intRank ..... 2475

Introduction	4	is_cenSubbimodule	551
intRoot	1993	is_ci	1758
intRoots	443	is_complex	972
intStrategy, option	235	is_composite	1056
intvec	93	is_fix_injective	2015
intvec declarations	94	is_h_injective	2015
intvec expressions	94	is_injective	1009
intvec operations	95	is_irred	1731
intvec related functions	96	is_is	1759
invar	1817	is_nested	1175
invar.lib	1813	is_NND	1732
invar_lib	1813	is_NP	1174
invariant	1051	is_prime	2004
invariant ring minimal generating set matrix group	1779	is_primitive_root	2003
invariant ring minimal generating set permutation group	1780	is_pure	609
Invariant theory	1775	is_reg	1759
Invariant Theory	753	is_regs	1760
invariant_algebra_perm	1780	is_surjective	1010
invariant_algebra_reynolds	1779	is_zero	891
invariant_basis	1785	isAface	2091
invariant_basis_reynolds	1786	isAntiEndo	464
invariant_ring	1776	isartinianMon	1155
invariant_ring_random	1777	isCartan	395
InvariantQ	1810	isCentral	574
invariantRing	1802	isCI	157, 1048
InvariantRing	1809	isCM	157, 1108
invariants	1049, 1052, 1727	isCMcod2	1768
Invariants of a finite group	753	isCommutative	576
Invariants of plane curve singularities	745	isCompatible	2083
inverse	988	isDenom	552
inverse lexicographical ordering	770	isElement	2272
inverse of a matrix via its LU-decomposition	216	isEqualDivisor	1432
inverse_B	989	isEquising	1702
inverse_L	990	isFlat	1108
inverse_modulus	2004	isFreeAlgebra	666
inverseFourier	441	isFsat	439
invertAlgebraMorphism	2587	isFullSpace	2067
invertBirMap	1466	isgenericMon	1155
invertLeftFraction	596	isGradedRingHomomorphism	2470
invertNcfrac	547	isGroup	2468
invertNumberMain	1903	isGroupHomomorphism	2469
involut.lib	457	isHolonomic	419
involut_lib	457	isHomog	157
involution	462	isHomogeneous	2491
invunit	1739	ishyper	2016
iostruct	1975	isInS	585
irred_secondary_char0	1793	isInt	446
irred_secondary_no_molien	1796	isIntegralSurjective	2480
irreddecMon	1156	isInvertibleLeftFraction	596
irreducible power series	1731	isInvertibleNcfrac	546
irreducible secondary invariant	1794	isInvolution	463
irrRealizationDim	2139	isirreducibleMon	1154
is_active	1758	isLocalizationRegular	1446
is_bijective	1010	isLocallyFree	1109
is_cenBimodule	551	isMonomial	1151
		isNC	575
		isNormal	1210

isoncurve	2017
isOnCurve	1998
isOneFraction	598
isOneNcfrac	542
isOrderingShiftInvariant	669
isOrigin	2067
isparam	2047
isPositive	2488
isprimaryMon	1154
isprimeMon	1154
isPrimitiveSublattice	2480
isPure	2084
isPureTensor	375
isQuotientRing	956
isRational	421
isReg	1109
isSB	157
IsSCA	579
isSimplicial	2068
isSublattice	2474
isSubModule	957
isSubset	2272
isSuperset	2273
isSymmetric	1385
isTame	1718
isTorsionFree	2488
isTwoSidedGB	376
isuni	2041
isUpperTriangular	559
isWeyl	576
isZeroElement	2489
isZeroFraction	597
isZeroNcfrac	541
ivmatGaussian	1893
ivmatGaussian2	1827
ivmatInit	1892
ivmatSet	1893

## J

J-marked schemes	1621
J-marked schemes, Borel ideals	1624
jacob	203
Jacobi	1992
jacoblift	1740
jacobson	1979
Jacobson form	1978
Jacobson normal form	1978
jacobson.lib	1978
jacobson.lib	1978
janet	203
janet basis	1267
Janet basis	1376
Janet division, Janet basis	1306
jet	204
JMarkedScheme	1624
JMBTest.lib	1621

JMBTest.lib	1621
JMSConst.lib	1623
JMSConst.lib	1623
jordan	1000
jordanbasis	1000
jordanmatrix	1001
jordannf	1001
jordanVecField	2584
Jumping numbers	2421
jungfib	1528
jungnormal	1527
jungresolve	1526
JuReTopDim	1456
JuReZeroDim	1456

## K

K-basis	641
K-dimension	641
katsura	889
kbase	205
kbase (plural)	343
keeping	298
KeneshlouMatrixPresentation	2390
Ker	1134
kerHom	1144
kernel	206
Kernel of module homomorphisms	733
kernelLattice	2476
kill	206
killall	810
killattrib	207
killTask	960
kmemory	810
kohom	1110
kontrahom	1111
koszul	207
KoszulHomology	1111
Kronecker module	1438
Kronecker product	280
KScoef	1738
KSconvert	1737
KSkerr	1736
kskernel.lib	1736
kskernel.lib	1736
KSlinear	1737
KSpencerKernel	1772

**L**

- L, ordering . . . . . 774
- l-adic numbers . . . . . 37
- laguerre . . . . . 208
- laguerre\_solve . . . . . 1843
- lastQuadrantComplex . . . . . 2575
- lastvarGeneral . . . . . 2031
- latex.lib . . . . . 1907
- latex\_lib . . . . . 1907
- lattice . . . . . 2228
- latticeArea . . . . . 2136
- latticeBasis . . . . . 2477
- latticeIdeal . . . . . 1217
- Laurent series . . . . . 1070
- laxfrT . . . . . 2331
- laxfrX . . . . . 2332
- lazy, option . . . . . 235
- lcm . . . . . 891
- lcmMon . . . . . 1152
- lcmN . . . . . 1991
- lcmofall . . . . . 1515
- lead . . . . . 209
- leadcoef . . . . . 209
- leadexp . . . . . 210
- leadmonom . . . . . 210
- leadmonomial . . . . . 2023
- Left and two-sided Groebner bases . . . . . 757
- left annihilator ideal . . . . . 402, 448
- Left ideal membership (plural) . . . . . 368
- Left normal form . . . . . 368
- leftInverse . . . . . 1972
- leftKernel . . . . . 1971
- leftOre . . . . . 588
- Leinartas . . . . . 2518
- length . . . . . 1892
- length, option . . . . . 235
- length2 . . . . . 1826
- LengthSym . . . . . 612
- LengthSymElement . . . . . 612
- letplaceGBasis . . . . . 670
- letterplace . . . . . 219, 251
- Letterplace . . . . . 617
- LETTERPLACE . . . . . 617
- Letterplace correspondence . . . . . 640
- Letterplace Groebner basis . . . . . 641, 661, 666
- Letterplace libraries . . . . . 640
- LETTERPLACE libraries . . . . . 640
- Letterplace ring . . . . . 271
- Levels . . . . . 871
- lex\_solve . . . . . 1851
- Lexicographic Groebner bases, computation of . . . . . 713
- lexicographical ordering . . . . . 770
- LIB . . . . . 211
- LIB commands . . . . . 57
- lib2doc . . . . . 63
- libcdd . . . . . 1
- libfac . . . . . 1
- libparse . . . . . 70
- Libraries . . . . . 55, 700
- Libraries in the SINGULAR Documentation . . . . . 55
- library . . . . . 2260
- Library versioning . . . . . 2599
- library, gitfan, GIT, geometric invariant theory, quotients . . . . . 2090
- library, info string . . . . . 61
- library, polybori.lib . . . . . 2260
- library, ringgb.lib . . . . . 2535
- library, template . . . . . 59
- library, template.lib . . . . . 61
- LIBs . . . . . 794
- libSingular . . . . . 2600
- Lie . . . . . 2328
- Lie algebra . . . . . 1460
- Lie bracket . . . . . 336
- Lie group . . . . . 1462, 1464
- lieBracket . . . . . 671
- lift . . . . . 211, 372
- lift (letterplace) . . . . . 629
- lift (plural) . . . . . 344
- lift\_kbase . . . . . 1696
- lift\_rel\_kb . . . . . 1697
- liftenvelope . . . . . 374
- liftstd . . . . . 212
- liftstd (letterplace) . . . . . 630
- liftstd (plural) . . . . . 345
- likeIdeal . . . . . 2435
- likelihood ideal . . . . . 2435
- Limitations . . . . . 310
- linalg.lib . . . . . 988
- linalg\_lib . . . . . 988
- linealityDimension . . . . . 2069
- linealitySpace . . . . . 2069
- linear algebra . . . . . 158, 216, 217
- Linear algebra . . . . . 968
- Linear code . . . . . 1938
- linear code, dual . . . . . 1936
- linear code, systematic . . . . . 1937
- linear interreduction . . . . . 377
- linear\_relations . . . . . 982
- LinearActionQ . . . . . 1811
- LinearCombinationQ . . . . . 1811
- linearCombinations . . . . . 398
- LinearizeAction . . . . . 1810
- linearlyEquivalent . . . . . 1433
- linearMapKernel . . . . . 397
- linearpart . . . . . 1835
- linesHypersurface . . . . . 1601
- link . . . . . 96, 708
- link declarations . . . . . 96
- link expressions . . . . . 97
- link related functions . . . . . 97
- links . . . . . 2550
- linReduce . . . . . 384

linReduceIdeal	385
linSyzSolve	386
list	103, 2310, 2311
list declarations	103
list expressions	104
list operations	105
list related functions	105
lists	2233
lists, elements	2237
listvar	213
lll	2517
LLL	276
load	299
Loading a library	67
loadLib, option	236
loadProc, option	236
local methods	1191
local names	54
Local orderings	770
local rings, computing in	703
local weighted lexicographical ordering	770
local weighted reverse lexicographical ordering	770
localInvar	1805
localization	703, 2215
localization of D-module	422, 523
locally closed set	833, 834, 835, 870, 874
localstd	2029
locAtZero	1205
locNormal	1190
locStatus	582
locstd	1760
locus	844, 846, 849, 851, 855, 856, 857, 859
locusdg	849
locusto	855
Log for matrices	1462
log2	1987
logarithmic annihilator ideal	402, 448
logg	1603
logHessian	2435
Long coefficients	705
LOT algorithm	402
lp, global ordering	770
lpCalcSubstDegBound	665
lpDegBound	666
lpDivision	667
lpGBPres2Poly	668
lpGkDim	662
lpGlDimBound	664
lpHilbert	644
lpIsPrime	662
lpIsSemiPrime	661
lpKDim	642
lpMonomialBasis	643
lpNgenCount	667
lpNoetherian	661
lprint	879
lpSubstitute	664

lpVarBlockSize	667
lrcalc.lib	2425
lrcalc.lib	2425
LRcoef	2426
LRcoprod	2430
lres	215
LRinstall	2426
LRmult	2427
LRschubmult	2434
LRskew	2426
ls, local ordering	770
lsum	1080
LTGS	2539
LU-decomposition of a matrix of numbers	216
ludecomp	216
luinverse	216
lusolve	217

## M

M, ordering	771
m_merkle_hellman_decryption	2008
m_merkle_hellman_encryption	2008
m_merkle_hellman_transformation	2008
Macdonald's formula	1438
MacdonaldF	1441
Macintosh installation	2603
magnitude	1985
makedistinguished	1058
makeDivisor	1430
makeFormalDivisor	1434
makeGraph	1069
makeGraphVE	1597
makeHeisenberg	567
makeIdeal	1125
makeLetterplaceRing	669
makeLinks	2550
makeMalgrange	457
makeMatrix	1124
makeModElimRing	581
makePDivisor	1437
makePoly	2531
makePuisseux	2534
makePuisseuxFromPoly	2534
makeQPoly	2549
makeQsl2	480
makeQsl3	480
makeQso3	479
makeSheaf	1592
makeUe6	477
makeUe7	478
makeUe8	478
makeUf4	476
makeUg2	476
makeUgl	467
makeUsl	466
makeUsl2	466

makeUso10	471	max	218
makeUso11	471	Max	1745
makeUso12	472	maxabs	2044
makeUso5	468	maxcoef	892
makeUso6	469	maxdeg	892
makeUso7	469	maxdeg1	892
makeUso8	470	maxEord	1501
makeUso9	470	maxExp	157
makeUsp1	473	maxideal	219
makeUsp2	473	maximalGroebnerCone	2177
makeUsp3	474	maximum likelihood estimate	2435
makeUsp4	475	maximum number of zeroes	1464
makeUsp5	475	Maximus	1622
makeVariety	1587	maxIntRoot	534
makeVector	1139	maxlike.lib	2435
makeWeyl	566	maxlike.lib	2435
map	106	Maxord	1513
map (plural)	323	maxPoints	2437
map declarations	106	maxPointsProb	2438
map declarations (plural)	324	maxZeros	1463
map expressions	107	mdouble	61
map expressions (plural)	325	mem, option	236
map operations	107	member	2237
map operations (plural)	325	memberpos	901
map related functions	108	membershipMon	1152
map related functions (plural)	325	memory	219
mapall	950	memory management	219
mapIsFinite	1012	merkle_hellman_decryption	2009
mappingcone	2404	merkle_hellman_encryption	2009
mappingcone3	2408	methods, hashtables	2240
mapToRatNormCurve	1471	methods.lib	2239
markov4ti2	1382	methods.lib	2239
mat_rk	996	midpoint	2334
mat2arr	2206	MillerRabin	1995
mat2carr	2207	milnor	1761
matbil	2034	Milnor code	1640
Mathematical background	775	Milnor number	735, 1755
Mathematical background (letterplace)	636	milnorcode	1650
Mathematical background (plural)	366	milnornumber	1755
mathematical objects	768	min	220
mathinit	1906	Min	1746
matmult	2035	minAssChar	1245
matrix	108, 2207, 2208	minAssCharE	1246
matrix declarations	108	minAssGTZ	1244
matrix diagonalization	1978	minAssGTZE	1245
matrix expressions	109	minAssZ	1256
matrix operations	110	minbase	221
Matrix orderings	771	minbaseMon	1151
matrix related functions	112	mindeg	893
matrix type cast	110	mindeg1	893
matrix.lib	968	mindist	1951
matrix.lib	968	minEcart	2025
matrixExp	1461	minimal display time, setting the	279
matrixLog	1462	minimal representations	676
matrixpres	2418	minimalAfaceOrbits	2100
matrixsystem	2336	minimalAfaces	2093
matrixT1	1769	MinimalDecomposition	1812

- minimalOrbitConeOrbits . . . . . 2102
- Minimum distance . . . . . 1938
- Minimus . . . . . 1621
- minIntRoot . . . . . 420
- minIntRoot2 . . . . . 534
- minipoly . . . . . 999
- minMult . . . . . 1046
- MinMult . . . . . 2049
- minor . . . . . 221
- minpoly . . . . . 303
- minres . . . . . 223
- minres (plural) . . . . . 346
- minus . . . . . 2302, 2303, 2320, 2321
- mirror symmetry . . . . . 1068, 2576
- Miscellaneous libraries . . . . . 2205
- mixed Hodge structure . . 1703, 1708, 1709, 1710, 1711,  
1713, 1718, 1719
- mod . . . . . 75, 87, 118, 121
- mod\_versal . . . . . 1695
- mod2id . . . . . 895
- modality . . . . . 1654
- modberlekampMassey . . . . . 1086
- modBorder . . . . . 1271
- modDec . . . . . 1163
- moddiq.lib . . . . . 1121
- moddiq\_lib . . . . . 1121
- ModEqn . . . . . 1742
- modFinduni . . . . . 2439
- modfinduni.lib . . . . . 2439
- modfinduni\_lib . . . . . 2439
- modfrWalk . . . . . 2454
- modfWalk . . . . . 2449
- modGB . . . . . 1148
- modIntersect . . . . . 1149
- modJanet . . . . . 1376
- modNormal . . . . . 1191
- modNPos . . . . . 1182
- modNpos\_test . . . . . 1181
- modQuotient . . . . . 1121
- modrationalInterpolation . . . . . 1084
- modregCM . . . . . 1184
- modrWalk . . . . . 2444
- modSagbiAlg . . . . . 1660
- modSat . . . . . 1122
- modsatiety . . . . . 1183
- modStd . . . . . 1147
- modstd.lib . . . . . 1147
- modstd\_lib . . . . . 1147
- modSyz . . . . . 1149
- modular . . . . . 884
- modular methods . . . . . 1191
- Modular techniques . . . . . 884
- modular techniques . . . . . 1192
- modular.lib . . . . . 884
- modular\_lib . . . . . 884
- module . . . . . 112
- module (plural) . . . . . 326
- module declarations . . . . . 113
- module declarations (plural) . . . . . 326
- module expressions . . . . . 113
- module expressions (plural) . . . . . 326
- module operations . . . . . 113
- module operations (plural) . . . . . 327
- module ordering c . . . . . 771
- module ordering C . . . . . 771
- Module orderings . . . . . 770
- module related functions . . . . . 114
- module related functions (plural) . . . . . 327
- module\_containment . . . . . 1006
- Modules and and their annihilator . . . . . 12
- modules.lib . . . . . 1123
- modules\_lib . . . . . 1123
- moduliSpace . . . . . 1598
- modulo . . . . . 223
- modulo (letterplace) . . . . . 631
- modulo (plural) . . . . . 347
- moduloSlim . . . . . 573
- modVStd . . . . . 1668
- modVStd0 . . . . . 1675
- modWalk . . . . . 2440
- modwalk.lib . . . . . 2439
- modwalk\_lib . . . . . 2439
- moebius . . . . . 2229
- moebius function . . . . . 2229
- molien . . . . . 1782
- mondim . . . . . 522
- mondromy.lib . . . . . 1738
- mondromy\_lib . . . . . 1738
- monic . . . . . 1118
- monitor . . . . . 224
- monodromy . . . . . 1703, 1707, 1711, 1718, 1719
- Monodromy . . . . . 1739
- monodromyB . . . . . 1740
- monoideal, dimension . . . . . 522
- monomial . . . . . 225
- monomial ordering . . . . . 955
- monomial orderings . . . . . 703, 769
- Monomial orderings . . . . . 769
- monomial orderings introduction . . . . . 769
- Monomial orderings on free algebras . . . . . 637
- Monomial orderings, Term orderings . . . . . 34
- monomial output . . . . . 305
- monomial, write . . . . . 305
- monomialabortstd . . . . . 2238
- monomialideal.lib . . . . . 1150
- monomialideal\_lib . . . . . 1150
- monomialInIdeal . . . . . 533
- monomialLcm . . . . . 2024
- monomials and precedence . . . . . 314
- mons2intmat . . . . . 1231
- Mori dream spaces . . . . . 2274
- Morse lemma . . . . . 1754, 1755
- morsesplit . . . . . 1651
- movingCone . . . . . 2098



mp\_res\_mat ..... 1848  
 MPGS ..... 2551  
 mplot ..... 1906  
 mpresmat ..... 225  
 mprimdec.lib ..... 1161  
 mprimdec.lib ..... 1161  
 mrcgs ..... 910  
 mrcgs, rcgs, buildtree, cantreetoMaple, cantodiffcgs  
 ..... 927  
 mrcgs, rcgs, crcgs, Maple ..... 931  
 mregular.lib ..... 1171  
 mregular.lib ..... 1171  
 mres ..... 225  
 mRes ..... 1128  
 mres (plural) ..... 348  
 mres\_map ..... 226  
 MSGB ..... 2553  
 MSPOLY ..... 2552  
 mstd ..... 227  
 msum ..... 62  
 mtriple ..... 62  
 mult ..... 228, 315  
 multarr2arr ..... 2225  
 multarrMultRestrict ..... 2226  
 multarrRestrict ..... 2225  
 multBound ..... 304  
 multcol ..... 978  
 multdivisor ..... 1431  
 multformaldivisor ..... 1436  
 multi ..... 2018  
 multi indices ..... 44  
 multi-graded Hilbert series ..... 673  
 multiarrangement ..... 2223, 2224, 2225, 2226  
 multiIdeals ..... 2424  
 multiDeg ..... 2484  
 multiDegBasis ..... 2485  
 multiDegGroebner ..... 2495  
 multiDegModulo ..... 2497  
 multiDegPartition ..... 2487  
 multiDegResolution ..... 2498  
 multiDegSyzygy ..... 2496  
 multiDegTensor ..... 2502  
 multiDegTor ..... 2505  
 multidimensional\_knapsack ..... 2006  
 multigrading, multidegree, multiweights,  
 multigraded-homogeneous, integral linear algebra  
 ..... 2459  
 multigrading.lib ..... 2459  
 multigrading.lib ..... 2459  
 multipleCover ..... 1600  
 multiplication ..... 2303, 2322  
 multiplicities, sequence of ..... 1727  
 multiplicity ..... 518, 1050  
 multiplicity sequence ..... 1634, 1636, 1729  
 MultiplicitySequence ..... 1689  
 Multiplier ideals ..... 2421  
 multiplyLeftFractions ..... 594

multiplylist ..... 1523  
 multivariate equations ..... 1821, 2329  
 multRat ..... 537  
 multrow ..... 979  
 multseq2charexp ..... 1634  
 multsequence ..... 1728  
 MVComplex ..... 1428

## N

naccache\_stern\_decryption ..... 2007  
 naccache\_stern\_encryption ..... 2007  
 naccache\_stern\_generation ..... 2007  
 NakYoshF ..... 1440  
 nameof ..... 228  
 names ..... 229  
 Names ..... 44  
 Names in procedures ..... 54  
 Names, indexed ..... 44  
 nashmult ..... 1637  
 nblocks ..... 276  
 nc\_algebra ..... 349  
 nc\_hilb ..... 277  
 ncalg.lib ..... 465  
 ncalg.lib ..... 465  
 ncalgebra ..... 351  
 ncdecomp.lib ..... 484  
 ncdecomp.lib ..... 484  
 ncdetection ..... 462  
 ncExt\_R ..... 548  
 ncfactor ..... 487  
 ncfactor.lib ..... 487  
 ncfactor.lib ..... 487  
 ncfrac.lib ..... 539  
 ncfrac.lib ..... 539  
 ncgen ..... 631  
 nchilb ..... 673  
 ncHilb ..... 518  
 ncHilb.lib ..... 673  
 ncHilb.lib ..... 673  
 nchilbert.lib ..... 518  
 nchilbert.lib ..... 518  
 ncHilbertMultiplicity ..... 521  
 ncHilbertPolynomial ..... 520  
 ncHilbertSeries ..... 519  
 ncHom ..... 549  
 nchomolog.lib ..... 548  
 nchomolog.lib ..... 548  
 ncInit ..... 676  
 ncloc.lib ..... 552  
 ncloc.lib ..... 552  
 ncmodslimgb ..... 554  
 ncModslimgb.lib ..... 554  
 ncModslimgb.lib ..... 554  
 ncols ..... 231  
 ncones ..... 2085

ncpreim.lib . . . . .	555	netIntMatShort . . . . .	2243
ncpreim.lib . . . . .	555	netIntVector . . . . .	2244
ncrat.lib . . . . .	676	netIntVectorShort . . . . .	2244
ncrat.lib . . . . .	676	netList . . . . .	2245
ncratAdd . . . . .	678	netMap . . . . .	2246
ncratDefine . . . . .	677	netMap2 . . . . .	2246
ncratEvaluateAt . . . . .	682	netmatrix . . . . .	2247
ncratFromPoly . . . . .	681	netMatrix . . . . .	1139
ncratFromString . . . . .	681	netmatrixShort . . . . .	2247
ncratInvert . . . . .	680	netNumber . . . . .	2245
ncratMultiply . . . . .	679	netPoly . . . . .	2248
ncratPower . . . . .	682	netPrimePower . . . . .	2252
ncratPrint . . . . .	681	netRing . . . . .	2252
ncratSPrint . . . . .	680	nets.lib . . . . .	2240
ncratSubtract . . . . .	679	nets.lib . . . . .	2240
ncRelations . . . . .	574	netString . . . . .	2254
ncrepAdd . . . . .	683	netvector . . . . .	2255
ncrepDim . . . . .	687	netVector . . . . .	1139
ncrepEvaluate . . . . .	689	netvectorShort . . . . .	2255
ncrepEvaluateAt . . . . .	689	New orderings for modules . . . . .	2599
ncrepGet . . . . .	682	newline . . . . .	130
ncrepGetRegularMinimal . . . . .	695	news . . . . .	2589
ncrepGetRegularZeroMinimal . . . . .	694	newstruct . . . . .	136, 2260
ncrepInvert . . . . .	686	Newton non-degenerate . . . . .	1732
ncrepIsDefined . . . . .	691	Newton polygon . . . . .	1732
ncrepIsDefinedDim . . . . .	690	Newton polygons . . . . .	2196
ncrepIsRegular . . . . .	692	Newton polytope . . . . .	2129
ncrepMultiply . . . . .	685	Newton step . . . . .	1891
ncrepPencilCombine . . . . .	696	newtonDiag . . . . .	897
ncrepPencilGet . . . . .	696	newtonpoly . . . . .	1732
ncrepPrint . . . . .	686	newtonPolygonNegSlopes . . . . .	2197
ncrepRegularMinimize . . . . .	693	newtonPolytope . . . . .	2089
ncrepRegularZeroMinimize . . . . .	693	nextHodgeIdeal . . . . .	2425
ncrepSubstitute . . . . .	688	NF . . . . .	251
ncrepSubtract . . . . .	684	NF (letterplace) . . . . .	632
nctools.lib . . . . .	563	NF (plural) . . . . .	357
nctools.lib . . . . .	563	nf_icis . . . . .	1761
ncVarsAdd . . . . .	677	nfmmodStd . . . . .	1177
ncVarsGet . . . . .	677	nfmmodstd.lib . . . . .	1175
ndcond . . . . .	565	nfmmodstd.lib . . . . .	1175
negatedCone . . . . .	2070	nfmmodSyz . . . . .	1179
negateNcfrac . . . . .	546	nfmmodsyz.lib . . . . .	1178
negation . . . . .	2303	nfmmodsyz.lib . . . . .	1178
negative degree lexicographical ordering . . . . .	770	NFMora . . . . .	2025
negative degree reverse lexicographical ordering . . . . .	770	nilpotent Lie algebras . . . . .	1460
negative lexicographical ordering . . . . .	770	nmaxcones . . . . .	2084
negativedivisor . . . . .	1431	noElements . . . . .	2537
negativeformaldivisor . . . . .	1436	noether . . . . .	304
net . . . . .	2240, 2241	Noether position . . . . .	1181
net access . . . . .	19	noether.lib . . . . .	1180
netBigInt . . . . .	2243	noether.lib . . . . .	1180
netBigIntMat . . . . .	2241	noetherNormal . . . . .	1011
netBigIntMatShort . . . . .	2241	NoetherPosition . . . . .	1173
netCoefficientRing . . . . .	2242	Non-commutative algebra . . . . .	757, 785
netIdeal . . . . .	2242	Non-commutative subsystem . . . . .	318
netInt . . . . .	2242	non-english special characters . . . . .	310
netIntMat . . . . .	2243	noncommutative, rational expressions . . . . .	676

- none, option . . . . . 234
  - Nonhyp . . . . . 1495
  - nonMonomials . . . . . 1234
  - nonZeroEntry . . . . . 1013
  - norm . . . . . 2017
  - normaform (up to a bound) . . . . . 277
  - normal . . . . . 1184
  - normal bundle, graph, rational curves, multiple covers,  
  lines on . . . . . 1600
  - normal form . . . . . 775
  - normal integral closure . . . . . 1210
  - normal.lib . . . . . 1184
  - normal.lib . . . . . 1184
  - normalBundle . . . . . 1600
  - normalC . . . . . 1195
  - normalConductor . . . . . 1208
  - normalform . . . . . 1653, 1660
  - normalForm . . . . . 1432
  - normalI . . . . . 1266
  - normaliz . . . . . 1, 1227
  - normaliz.lib . . . . . 1210
  - normaliz.lib . . . . . 1210
  - normalization . . . . . 1049, 1186, 1191, 1192, 1193, 1196,  
  1209, 1210
  - Normalization . . . . . 732
  - normalization . . . . . 1116, 1466
  - normalize . . . . . 894
  - normalizeMonoidal . . . . . 598
  - normalizeRational . . . . . 599
  - normalP . . . . . 1192
  - normalToricRing . . . . . 1212
  - normalToricRingFromBinomials . . . . . 1213
  - norTest . . . . . 1206
  - not . . . . . 90
  - not equal . . . . . 2305, 2324
  - notBuckets, option . . . . . 235
  - Notes for developers . . . . . 2600
  - Notes for Singular users . . . . . 2598
  - notRegularity, option . . . . . 235
  - notSugar, option . . . . . 235
  - notWarnSB, option . . . . . 236
  - npar . . . . . 1985
  - npars . . . . . 231
  - NPos . . . . . 1181
  - NPos\_test . . . . . 1180
  - nres . . . . . 232
  - nres (plural) . . . . . 351
  - nrows . . . . . 232
  - nrroots . . . . . 2047
  - nrRootsDeterm . . . . . 2033
  - nrRootsProbab . . . . . 2032
  - nsatiety . . . . . 1182
  - nselect . . . . . 1065
  - NSplaces . . . . . 1926
  - nt\_solve . . . . . 1861
  - NTL . . . . . 1
  - ntsolve.lib . . . . . 1861
  - ntsolve.lib . . . . . 1861
  - NullCone . . . . . 1812
  - num\_elim . . . . . 1885
  - num\_elim1 . . . . . 1887
  - num\_prime\_decom . . . . . 1868
  - num\_prime\_decom1 . . . . . 1870
  - num\_radical\_via\_decom . . . . . 1871
  - num\_radical\_via\_randlincom . . . . . 1875
  - num\_radical1 . . . . . 1878
  - num\_radical2 . . . . . 1881
  - number . . . . . 115, 2208
  - number declarations . . . . . 115
  - number expressions . . . . . 117
  - number operations . . . . . 118
  - number related functions . . . . . 119
  - number\_e . . . . . 811
  - number\_pi . . . . . 811
  - numberOfConesOfDimension . . . . . 2085
  - numerAlg.lib . . . . . 1451
  - numerAlg.lib . . . . . 1451, 1454
  - numerator . . . . . 233
  - numerDecom.lib . . . . . 1454
  - numerDecom.lib . . . . . 1452, 1454
  - numerical algebraic geometry . . . . . 1862
  - numerical irreducible decomposition . . . . . 1454
  - NumIrrDecom . . . . . 1458
  - NumLocalDim . . . . . 1453
  - NumPrimDecom . . . . . 1459
  - nvars . . . . . 233
- ## O
- Oaku-Takayama algorithm . . . . . 402
  - Objects . . . . . 46
  - olga.lib . . . . . 582
  - olga.lib . . . . . 582
  - oneDimBelongSemigroup . . . . . 1047
  - oneNcfrac . . . . . 543
  - onesVector . . . . . 2090
  - online help . . . . . 15
  - open . . . . . 234
  - opentex . . . . . 1908
  - operatorAlgebra . . . . . 646
  - operatorBM . . . . . 408
  - operatorModulo . . . . . 409
  - oppose . . . . . 353
  - opposite . . . . . 354
  - opposite polynomial . . . . . 277
  - opposite ring . . . . . 277
  - option . . . . . 234
  - option Imap . . . . . 236
  - option(warn) . . . . . 71
  - optionIsSet . . . . . 954
  - or . . . . . 90, 311
  - orbit . . . . . 1460, 1462, 1464
  - orbit\_variety . . . . . 1799
  - orbitConeOrbits . . . . . 2101

orbitCones . . . . . 2094  
 orbitparam.lib . . . . . 1459  
 orbitparam.lib . . . . . 1459  
 ord . . . . . 238  
 ord\_test . . . . . 951  
 order ideal, border basis . . . . . 1267, 1271  
 ordering . . . . . 2294, 2306, 2307  
 orderings . . . . . 769  
 orderings introduction . . . . . 769  
 orderings, a . . . . . 773  
 orderings, global . . . . . 770  
 orderings, L . . . . . 774  
 orderings, local . . . . . 770  
 orderings, M . . . . . 771  
 orderings, product . . . . . 773  
 ordstr . . . . . 239  
 origin . . . . . 2050  
 Orlik-Solomon . . . . . 2221  
 orthogonalize . . . . . 991  
 outer . . . . . 972  
 output . . . . . 48  
 Output . . . . . 1139, 2240, 2241, 2242, 2243, 2244, 2245,  
 2246, 2247, 2248, 2252, 2255, 2256  
 Output, formatting of . . . . . 707  
 outputting monomials . . . . . 305

## P

p-adic numbers . . . . . 37  
 package . . . . . 119  
 package declarations . . . . . 119  
 package related functions . . . . . 120  
 pairset . . . . . 2026  
 par . . . . . 239  
 par2varRing . . . . . 794  
 paraConic . . . . . 1474  
 parallel computing . . . . . 2555  
 parallel skeletons . . . . . 885  
 parallel.lib . . . . . 885  
 parallel.lib . . . . . 885  
 parallelization . . . . . 885, 942, 959  
 Parallelization . . . . . 708, 884  
 parallelTestAND . . . . . 887  
 parallelTestOR . . . . . 888  
 parallelWaitAll . . . . . 886  
 parallelWaitFirst . . . . . 886  
 parallelWaitN . . . . . 885  
 param . . . . . 1724  
 Parameter list . . . . . 52  
 parameter, as numbers . . . . . 115  
 parameter, default . . . . . 52  
 parameter, optional . . . . . 52  
 Parameters . . . . . 706  
 parameterSubstitute . . . . . 2175  
 parametric annihilator . . . . . 402  
 parametric annihilator for variety . . . . . 454  
 parametric ideal . . . . . 822, 835, 837  
 parametrization . . . . . 1460, 1462, 1725  
 Parametrization . . . . . 1465  
 Parametrization, image . . . . . 1475  
 parametrizeOrbit . . . . . 1462  
 paraPlaneCurve . . . . . 1467  
 paraplanecurves.lib . . . . . 1464  
 paraplanecurves.lib . . . . . 1464  
 pardeg . . . . . 239  
 parstr . . . . . 240  
 part . . . . . 1420  
 PartC . . . . . 1422  
 partial fraction . . . . . 2518  
 partial\_molien . . . . . 1784  
 partitions . . . . . 1074  
 PartitionVar . . . . . 2338  
 ParToVar . . . . . 2340  
 partOver . . . . . 1422  
 partUnder . . . . . 1423  
 path . . . . . 2601  
 path integral . . . . . 1068  
 Path names . . . . . 2599  
 paths . . . . . 2601  
 pause . . . . . 883  
 PBW . . . . . 387  
 PBW basis . . . . . 366  
 PBW\_eqDeg . . . . . 400  
 PBW\_maxDeg . . . . . 400  
 PBW\_maxMonom . . . . . 401  
 pdivi . . . . . 831, 832  
 pdivi2 . . . . . 902  
 PerfectPowerTest . . . . . 1988  
 permcol . . . . . 979  
 permrow . . . . . 979  
 permutationFromIntvec . . . . . 2121  
 permutations . . . . . 1076  
 permutations, sum, max, min . . . . . 2233  
 permutationToIntvec . . . . . 2121  
 permute . . . . . 1076  
 permute\_L . . . . . 1937  
 perron . . . . . 600  
 perron.lib . . . . . 600  
 perron.lib . . . . . 600  
 Perturbation walk . . . . . 1090  
 PEsolve . . . . . 1821  
 pFactor . . . . . 1997  
 pfd . . . . . 2518  
 pfd.lib . . . . . 2518  
 pfd.lib . . . . . 2518  
 pfdMat . . . . . 2528  
 Pfister, Gerhard . . . . . 3  
 PH\_ais . . . . . 2258  
 PH\_nais . . . . . 2259  
 phindex.lib . . . . . 2256  
 phindex.lib . . . . . 2256  
 picksFormula . . . . . 2136  
 pid . . . . . 277

- pIntersect . . . . . 382
- pIntersectSyz . . . . . 383
- Pipe links . . . . . 101
- plainInvariants . . . . . 1690
- planeCur . . . . . 1659
- plot . . . . . 1918
- plotRot . . . . . 1920
- plotRotated . . . . . 1920
- plotRotatedDirect . . . . . 1921
- plotRotatedList . . . . . 1921
- plotRotatedListFromSpecifyList . . . . . 1922
- plural . . . . . 955
- PLURAL . . . . . 318
- PLURAL libraries . . . . . 371
- PLURAL LIBs . . . . . 371
- pmat . . . . . 879
- pnormalf . . . . . 832
- pnormalform . . . . . 903
- PocklingtonLehmer . . . . . 1995
- Poincare polynomial . . . . . 2231
- pointid.lib . . . . . 1233
- pointid.lib . . . . . 1233
- points . . . . . 2329
- Polar curves . . . . . 738
- PollardRho . . . . . 1996
- pollTask . . . . . 965
- polSol . . . . . 527
- polSolFiniteRank . . . . . 528
- poly . . . . . 120
- poly (plural) . . . . . 328
- poly declarations . . . . . 120
- poly declarations (plural) . . . . . 328
- poly expressions . . . . . 121
- poly expressions (plural) . . . . . 329
- poly operations . . . . . 121
- poly operations (plural) . . . . . 330
- poly related functions . . . . . 123
- poly related functions (plural) . . . . . 330
- poly2list . . . . . 444
- poly2zdd . . . . . 2267
- polybori . . . . . 2260
- Polybori, Boolean Groebner Basis . . . . . 2268
- PolyBoRi, Boolean Groebner Basis . . . . . 2261, 2262, 2263, 2264, 2265, 2266, 2267
- PolyBoRi, zero-suppressed decision diagram . . . . . 2267
- polybori.lib . . . . . 2259, 2260
- polybori.lib . . . . . 2259
- polyclass.lib . . . . . 2530
- polyclass.lib . . . . . 2530
- polyDK . . . . . 1117
- polyInterpolation . . . . . 1083
- polylib.lib . . . . . 888
- polylib.lib . . . . . 888
- polymake.lib . . . . . 2129
- polymake.lib . . . . . 2129
- polynomial . . . . . 2315
- Polynomial data . . . . . 768
- polynomial functions . . . . . 2535
- polynomial solutions . . . . . 523
- polynomial, opposite . . . . . 277
- polynomials . . . . . 2531
- polytope . . . . . 140, 2129
- polytopeViaInequalities . . . . . 2070
- polytopeViaPoints . . . . . 2071
- polyVars . . . . . 533
- pos\_def . . . . . 997
- poset . . . . . 2228
- positiveOrthant . . . . . 2050
- posweight . . . . . 1771
- Pottier algorithm . . . . . 783
- power . . . . . 973
- power set . . . . . 2234
- power\_products . . . . . 1791
- powerN . . . . . 1991
- powerpolyX . . . . . 1989
- powersums . . . . . 2038
- powerX . . . . . 1994
- powSumSym . . . . . 1396
- PPolyH . . . . . 1439
- PPolyN . . . . . 1443
- PPolyQp . . . . . 1440
- PPolyS . . . . . 1441
- preComp . . . . . 1167
- Preface . . . . . 1
- preim . . . . . 2576
- preimage . . . . . 240, 556, 2576
- preimage (plural) . . . . . 355
- preimage under a map between local rings, map  
    between local rings, map between local and global  
    rings . . . . . 953
- preimageLattice . . . . . 2477
- preimageLoc . . . . . 953
- preimageNC . . . . . 557
- Prep . . . . . 833
- prepareAss . . . . . 1249
- prepEmbDiv . . . . . 1544
- prepMat . . . . . 1818
- prepRealclassify . . . . . 1654
- prepSV . . . . . 1933
- presentation . . . . . 1140
- presentTree . . . . . 1533
- presolve.lib . . . . . 1829
- presolve.lib . . . . . 1829
- primary decomposition . . . . . 1240, 1253
- Primary decomposition . . . . . 730
- primary decomposition of modules . . . . . 1161
- primary decomposition, numerical . . . . . 1454
- primary\_char0 . . . . . 1786
- primary\_char0\_no\_molien . . . . . 1787
- primary\_char0\_no\_molien\_random . . . . . 1790
- primary\_char0\_random . . . . . 1789
- primary\_charp . . . . . 1787
- primary\_charp\_no\_molien . . . . . 1788
- primary\_charp\_no\_molien\_random . . . . . 1790

- primary\_charp\_random . . . . . 1789
- primary\_charp\_without . . . . . 1788
- primary\_charp\_without\_random . . . . . 1791
- primary\_invariants . . . . . 1777
- primary\_invariants\_random . . . . . 1778
- primdec.lib . . . . . 1240
- primdec.lib . . . . . 1240
- PrimdecA . . . . . 1162
- PrimdecB . . . . . 1162
- primdecGTZ . . . . . 1241
- primdecGTZE . . . . . 1241
- primdecint.lib . . . . . 1253
- primdecint.lib . . . . . 1253
- primdecMon . . . . . 1158
- primdecSY . . . . . 1242
- primdecSYE . . . . . 1243
- primdecZ . . . . . 1253
- primdecZM . . . . . 1255
- prime . . . . . 241
- primeClosure . . . . . 1201
- primecoeffs . . . . . 815
- primefactors . . . . . 241
- primes . . . . . 812
- primitiv.lib . . . . . 1260
- primitiv.lib . . . . . 1260
- primitive . . . . . 1260
- primitive element . . . . . 1261
- primitive\_extra . . . . . 1261
- primitiveSpan . . . . . 2482
- primL . . . . . 1992
- primList . . . . . 1992
- primparam . . . . . 1687
- primRoot . . . . . 1385
- primTest . . . . . 1167
- principal intersection . . . . . 377
- print . . . . . 242, 2208, 2299, 2300, 2317, 2318
- printBetti . . . . . 1131
- printf . . . . . 244
- printFreeModule . . . . . 1126
- printGraph . . . . . 1069
- printGraphG . . . . . 1598
- printGroup . . . . . 2468
- printHom . . . . . 1127
- printlevel . . . . . 304
- printMatrix . . . . . 1126
- printModule . . . . . 1127
- printMoebius . . . . . 2232
- printNormalFormEquation . . . . . 2531
- printPoly . . . . . 2531
- printPuisseux . . . . . 2534
- printResolution . . . . . 1126
- printSheaf . . . . . 1592
- printStack . . . . . 1598
- printTask . . . . . 962
- printVariety . . . . . 1588
- proc . . . . . 124
- proc declaration . . . . . 124
- proc expression . . . . . 125
- Procedure definition . . . . . 51
- procedure, ASCII help . . . . . 61
- procedure, ASCII/Texinfo help . . . . . 62
- procedure, texinfo help . . . . . 62
- Procedure-specific commands . . . . . 55
- Procedures . . . . . 50
- Procedures and libraries . . . . . 10, 700
- Procedures in a library . . . . . 58
- procedures, help string . . . . . 54
- procedures, static . . . . . 51
- procs with different argument types . . . . . 125
- prodcrit . . . . . 2026
- product . . . . . 812, 2303, 2322
- Product orderings . . . . . 773
- product variety, abstract variety . . . . . 1588
- productgroup . . . . . 2483
- productOfProjectiveSpaces . . . . . 2555
- productVariety . . . . . 1588
- Programming . . . . . 699
- progress watch . . . . . 235
- projective bundle, abstract variety, sheaf, enumeration  
of conics . . . . . 1591
- projective dimension . . . . . 601
- projective lines . . . . . 37
- projective space, abstract variety . . . . . 1590
- projectiveBundle . . . . . 1591
- projectiveDimension . . . . . 601
- projectiveSpace . . . . . 1590
- projectLattice . . . . . 2478
- projPoints . . . . . 2329
- prompt . . . . . 15
- prompt, option . . . . . 236
- propagator . . . . . 1068, 1069, 2576
- propagator of a graph . . . . . 2579
- PropagatorFunction . . . . . 2579
- prot, option . . . . . 235
- protocol of computations . . . . . 235
- proximitymatrix . . . . . 1632
- prune . . . . . 245
- prune\_map . . . . . 245
- pruneModule . . . . . 1143
- prwalk . . . . . 2537
- Pseudo ordering L . . . . . 774
- psigncnd . . . . . 1898
- PtoCrep . . . . . 834
- puiseux . . . . . 2535
- Puiseux expansion . . . . . 1627, 1719
- Puiseux expansions . . . . . 2533
- Puiseux pairs . . . . . 745, 1727
- puiseux2generators . . . . . 1734
- puiseuxExpansion . . . . . 2147
- puiseuxexpansions.lib . . . . . 2532
- puiseuxexpansions.lib . . . . . 2532
- puiseuxList . . . . . 2533
- pure tensor . . . . . 372
- purelist . . . . . 609

purity ..... 601  
 purityFiltration ..... 602  
 purityfiltration.lib ..... 601  
 purityfiltration.lib ..... 601  
 purityTriang ..... 603  
 pushForward ..... 2506  
 pwalk ..... 1090  
 pyobject ..... 140, 2260  
 pyobject declarations ..... 141  
 pyobject expressions ..... 141  
 pyobject operations ..... 142  
 pyobject related functions ..... 143  
 pyramid ..... 2334  
 python\_eval ..... 145  
 python\_import ..... 145  
 python\_run ..... 146

## Q

qbase ..... 2040  
 qepcad ..... 2340  
 qepcadsystem ..... 2341  
 qhmatrix ..... 1770  
 qhmoduli.lib ..... 1742  
 qhmoduli.lib ..... 1742  
 qhspectrum ..... 1762  
 qhweight ..... 246  
 qmatrix.lib ..... 610  
 qmatrix.lib ..... 610  
 qminor ..... 611  
 QPoly, quotient ring ..... 2549  
 QQ ..... 30, 73  
 qlds ..... 246  
 qring ..... 127, 315  
 qring (plural) ..... 330  
 qring declaration ..... 129  
 qring declaration (plural) ..... 331  
 qring related functions (plural) ..... 331  
 qring ring ideal 'factor ring' ..... 956  
 qringNF, option ..... 235  
 qslimgb ..... 794  
 Qso3Casimir ..... 482  
 quadraticSieve ..... 1997  
 quantMat ..... 610  
 quickclass ..... 1651  
 quit ..... 300  
 Quot-scheme ..... 1438  
 quote ..... 247  
 quotient ..... 248  
 Quotient ..... 1899  
 quotient (plural) ..... 356  
 QuotientEquations ..... 1743  
 quotientLatticeBasis ..... 2072  
 quotientMain ..... 1903  
 quotientMon ..... 1153  
 quotSheaf ..... 1596

## R

rad\_con ..... 894  
 radical ..... 1247  
 radicalEHV ..... 1248  
 radicalMemberShip ..... 2172  
 radicalMon ..... 1153  
 radicalZ ..... 1257  
 randcharpoly ..... 2037  
 randlinpoly ..... 2038  
 random ..... 249, 277, 2219, 2220  
 random number generator, seed ..... 277  
 random.lib ..... 936  
 random.lib ..... 936  
 randomBinomial ..... 941  
 randomCheck ..... 1949  
 randomid ..... 936  
 randomLast ..... 941  
 randommat ..... 937  
 randomPoint ..... 2073  
 randomPolyInT ..... 2176  
 rank ..... 157, 249  
 rankSheaf ..... 1593  
 ratgb.lib ..... 613  
 ratgb.lib ..... 613  
 Rational curves ..... 1465  
 rational curves, rational parametrization of rational  
     curves ..... 1468  
 rational functions ..... 676  
 rational normal curve, projection ..... 1469, 1472, 1473  
 rational reconstruction ..... 181  
 rational solutions ..... 523  
 rational univariate projection ..... 2037  
 rationalCurve ..... 1602  
 rationalPointConic ..... 1470  
 ratSol ..... 529  
 ratstd ..... 613  
 rays ..... 2073  
 rcgs ..... 919  
 rcgs, crcgs, buildtree, cantreetoMaple, cantodiffcgs  
     ..... 912, 920  
 rcolon ..... 675  
 re2squ ..... 1454  
 read ..... 250  
 reading, option ..... 236  
 readInputTXT ..... 2528  
 readline ..... 1  
 readNmzData ..... 1229  
 real ..... 30  
 real root isolation ..... 1891  
 real roots, univariate polynomial ..... 2041  
 real roots, univariate projection ..... 2032  
 real roots, sign conditions ..... 1897  
 realclassify ..... 1747  
 realclassify.lib ..... 1746  
 realclassify.lib ..... 1746  
 realizationDim ..... 2139



realizationDimPoly	2140	removeCone	2086
realizationMatroids.lib	2138	removepower	1638
realizationMatroids.lib	2138	repart	253
realLLL	1890	replace	1986
realmorsesplit	1754	representation, math objects	768
realpoly	1263	res	253
realrad	1264	Res	1129
realrad.lib	1263	resbinomial.lib	1476
realrad.lib	1263	resbinomial.lib	1476
realzero	1264	reservedName	254
recover.lib	1862	resetNmzOptions	1226
recover.lib	1862	resfunction	1510
recursive_boolean_poly	2263	resgraph.lib	1524
recursive_from_boolean_poly	2265	resgraph.lib	1524
redcgs.lib	898	resjung.lib	1526
redcgs.lib	898	resjung.lib	1526
redefine, option	236	reslist	1522
redSB, option	235	resolution	125, 601
redspec	903	Resolution	13
redTail, option	235	resolution (plural)	332
redThrough, option	235	resolution declarations	126
reduce	251, 387, 831, 832	resolution declarations (plural)	332
reduce (letterplace)	632	resolution expressions	126
reduce (plural)	357	resolution expressions (plural)	333
reduce (up to a bound)	277	resolution graph	1627
reduced	830	Resolution of singularities	751
Reduced Comprehensive Groebner Systems	898	resolution related functions	126
reduced standard basis	235	resolution related functions (plural)	333
reduceIntChain	1145	resolution, computation of	253
reduction	1804	Resolution, free	720
ReesAlgebra	1265	resolution, hilbert-driven	197
reesclos.lib	1265	resolution, La Scala's method	215
reesclos.lib	1265	resolutiongraph	1627
reference	146	resolutionInLocalization	1448
reference declarations	148	resolve	1531
reference expressions	149	resolve.lib	1528
reference operations	151	resolve.lib	1528
reference related functions	153	resources.lib	942
References	792	resources.lib	942
References (plural)	370	ResTree	1525
References and history of LETTERPLACE	624	restriction	2215, 2225, 2226
regCM	1183	restriction of	422
regIdeal	1171	restrictionIdeal	430
regionComplex	2572	restrictionModule	431
regMonCurve	1173	resultant	225, 255
regularity	252, 777, 1183, 1184	reszeta.lib	1538
reiffen	418	reszeta.lib	1538
ReJunkUseHomo	1455	return	301
rel_orbit_variety	1799	return type of procedures	313
relations	600	returnSB, option	234
relative_orbit_variety	1801	reverse	2046
relativeInteriorPoint	2073	reynolds_molien	1783
Release Notes	2589	ReynoldsImage	1813
Release Notes (letterplace)	697	ReynoldsOperator	1813
relweight	1771	rho	1994
remainder	1900	rHRR	1419
remainderMain	1904	RiemannRochBN	1391



RiemannRochHess	1449	rootsMain	1904
Right Groebner bases and syzygies	758	rootsmr.lib	2032
Right ideal membership (plural)	368	rootsmr_lib	2032
rightInverse	1972	rootsur.lib	2040
rightKernel	1971	rootsur_lib	2040
rightModulo	572	round	1990
rightNF	572	rowred	980
rightNFWeyl	539	rowShift	157, 159
rightOre	589	rp, global ordering	770
rightStd	571	rstandard.lib	1266
rightstd (letterplace)	633	rstandard_lib	1266
ring	127	rswalk	2554
ring (plural)	333	rtimer	308
ring coefficients	956	Running SINGULAR under Emacs	25
ring declarations	127	rvalue	311
ring declarations (plural)	333	rvar	258
ring operations	128	rwalk	2538
ring operations (plural)	334	rwalk.lib	2537
ring related functions	128	rwalk_lib	2537
ring related functions (plural)	334		
ring theory	661		
ring, opposite	277	<b>S</b>	
ring, rings	900	S-function	2578
Ring-dependent options	2599	sa_poly_reduce	393
ring.lib	943	sa_reduce	393
ring_cf	157	sagbi	1379
ring.lib	943	sagbi basis	1660
ring.list	257	Sagbi basis, initial algebra	2542
ring_variable	2208, 2209	sagbi.lib	1377
ringgb.lib	2535	sagbi_lib	1377
ringgb.lib, functions for coefficient rings	2535	sagbiAlg	1657
ringgb_lib	2535	sagbiByDegree	2543
ringlist	255	sagbiGeneral	2542
ringlist (plural)	358	sagbigrob.lib	2539
Rings and orderings	30	sagbigrob_lib	2539
Rings and standard bases	7	sagbiHilbControlled	2543
ringtensor	951	sagbiMod	1658
ringweights	953	sagbiNormaliz.lib	2541
rinvar.lib	1807	sagbiNormaliz_lib	2541
rinvar_lib	1807	sagbiPart	1380
rJanet	1306	sagbiReduce	1378
rm_unitcol	983	sagbiSPoly	1378
rm_unitrow	982	SaitoBase	2585
rMacaulay	880	salida	1521
rmNmzFiles	1231	sameComponent	2023
rmx	1916	sameQ	1905
rncAntiCanonicalMap	1469	Sannfs	405
rncItProjEven	1473	SannfsBFCT	413
rncItProjOdd	1472	Sannfslog	406
root of Bernstein-Sato polynomial	402	SannfsVar	456
rootIsolation	1895	sat	1066
rootisolation.lib	1890	sat_with_exp	1066
rootisolation_lib	1890	satiety	1172
rootIsolationNoPreprocessing	1895	satstd	2239
rootIsolationPrimdec	1896	saturation	1066, 1122, 1183
rootofUnity	953	Saturation	715
roots	1900	sba	259

SCA	615	setModuleGrading	2473
scalarProd	387	setMultiplicity	2075
scheme	2331	setNmzDataPath	1230
Schönemann, Hans	3	setNmzExecPath	1228
Schoof	2000	setNmzFilename	1230
Schubert calculus	1587	setNmzOption	1225
schubert.lib	1587	setring	260
schubert.lib	1587	sets.lib	2269
SchubertClass	1604	sets.lib	2269
SchurCh	1420	Setting up a G-algebra	367
SchurS	1419	setUniformizingParameter	2196
SDB breakpoint	69	Sfunction	2578
SDB debugger	69	SGB	2541
sdb, source code debugger	69	SGNF	2540
SDLoc	425	sh	278
second degree	862	ShanksMestre	2000
second order deformation	2547	shared	146
secondary fan	2129	shared declarations	150
secondary polytope	2129	shared expressions	150
secondary_and_irreducibles_no_molien	1797	shared operations	151
secondary_char0	1792	shared related functions	153
secondary_charp	1794	sheaf cohomology	1605
secondary_no_molien	1795	sheafCoh	1616
secondary_not_cohen_macaulay	1798	sheafcoh.lib	1605
secondaryPolytope	2130	sheafcoh.lib	1605
segre	1393	sheafCoh_w	1618
SegreA	1423	sheafCohBGG	1609
select	1067	sheafCohBGG2	1612
select1	1067	sheafCohBGGregul	1610
semaphore	277, 943	sheafCohBGGregul_w	1611
semaphores	942	short	305
semi-prime ideal	661	show	881
semiCMcod2	1769	showBO	1532
semidiv	2019	showDataTypes	1533
semigroup	1630, 1635	showgrades	605
semiGroup	1658	showNmzOptions	1226
Semigroup	1663	showNuminvs	1222
semigroup of values	1727	showrecursive	882
semigroupGenerator	2074	signal handler	2600
semiMod	1659	signatureBrieskorn	1774
sep	1253	signatureL	2257
separateHNE	1734	signatureLqf	2257
separator	1161	signatureNemethi	1775
Serre relations	646	signaturePuisseux	1775
serreRelations	646	signcnd	1897
set	2206, 2269	signcond.lib	1897
set, intersectionSet, union, complement, equality,		signcond.lib	1897
isEqual	2269	simplePrune	1147
set.is_injective	2014	simplesolver	1823
setBaseMultigrading	2460	simplex	261
setcores	942	simplexOut	1852
SetDeg	1131	simplify	263
setenv	278	SimplifyIdeal	1813
setglobalrings	900	simplifyRat	536
setinitials	2335	sing.lib	1756
setLetterplaceAttributes	671	sing.lib	1756
setLinearForms	2074	sing4ti2.lib	1382

- sing4ti2.lib . . . . . 1382
- Singular . . . . . 278
- Singular 3 and Singular 4 . . . . . 2598
- SINGULAR libraries . . . . . 794
- singular locus of D-module . . . . . 523
- Singular, customization of Emacs user interface . . . . . 27
- Singular, demo mode . . . . . 27
- Singular, editing input files with Emacs . . . . . 28
- Singular, important commands of Emacs interface . . . . . 29
- Singular, running within Emacs . . . . . 25
- Singular2bertini . . . . . 1455
- SingularBin . . . . . 278
- SINGULARHIST . . . . . 18
- singularities . . . . . 1640, 1655, 1703, 1704, 1705, 1706,  
1707, 1708, 1709, 1710, 1711, 1713, 1773
- Singularities . . . . . 1626
- singularities ordinary multiple point . . . . . 1117
- singularities, resolution of . . . . . 751
- singularity . . . . . 1652
- Singularity Theory . . . . . 735
- SingularityDBM.lib . . . . . 1640
- SingularityDBM.lib . . . . . 1640
- SingularLib . . . . . 278
- singularrc . . . . . 22
- sinh . . . . . 2578
- size . . . . . 264, 315
- Skeletons for parallelization . . . . . 884
- skewmat . . . . . 973
- SL . . . . . 1814
- sleep . . . . . 270
- slimgb . . . . . 265, 715
- slimgb (plural) . . . . . 360
- slocus . . . . . 1762
- SLreynolds . . . . . 1818
- smaller . . . . . 2307
- smatrix . . . . . 129
- smith . . . . . 1979
- Smith form . . . . . 1978
- Smith normal form . . . . . 1978
- smithNormalForm . . . . . 2508
- SNF . . . . . 2552
- SolowayStrassen . . . . . 1995
- solutionsMod2 . . . . . 1993
- solve . . . . . 208, 862, 1845
- solve a linear equation system  $A*x = b$  via the  
LU-decomposition of A . . . . . 217
- solve.lib . . . . . 1843
- solve\_IP . . . . . 1119
- solve\_lib . . . . . 1843
- solvelinearpart . . . . . 1836
- solveTInitialFormPar . . . . . 2173
- solving . . . . . 1862
- Solving systems of polynomial equations . . . . . 760
- sort . . . . . 813, 2311
- sortandmap . . . . . 1837
- sortier . . . . . 1806
- sortIntvec . . . . . 446
- sortvars . . . . . 1838
- sortvec . . . . . 266
- source . . . . . 1137
- Source . . . . . 1125
- Source code debugger, invocation . . . . . 19
- source code debugger, sdb . . . . . 69
- Space curve singularities, branches of . . . . . 747
- spaceCur . . . . . 1659
- spadd . . . . . 1714
- span . . . . . 2075
- sparseHomogIdeal . . . . . 940
- sparseid . . . . . 937
- sparseInterpolation . . . . . 1087
- sparsemat . . . . . 938
- sparsematrix . . . . . 938
- sparsepoly . . . . . 939
- sparsetriag . . . . . 939
- spcurve.lib . . . . . 1768
- spcurve.lib . . . . . 1768
- Special characters . . . . . 43
- special characters, non-english . . . . . 310
- specification . . . . . 903
- spectral pairs . . . . . 1703, 1708, 1710, 1711, 1718, 1719
- spectralNeg . . . . . 1539
- spectrum . . . . . 1703, 1707, 1708, 1709, 1710, 1711, 1718,  
1719, 1773
- spectrum.lib . . . . . 1773
- spectrum.lib . . . . . 1773
- spectrumnd . . . . . 1773
- spgamma . . . . . 1717
- spgeomgenus . . . . . 1717
- sPiece . . . . . 2550
- spissemicont . . . . . 1716
- split . . . . . 882
- splitPolygon . . . . . 2132
- splitring . . . . . 1262
- splitting . . . . . 1166
- Splitting lemma . . . . . 1754, 1755
- spmilnor . . . . . 1717
- spmul . . . . . 1715
- spnf . . . . . 999
- spoly . . . . . 2024
- SPOLY . . . . . 2540
- sppairs . . . . . 1708
- sppnf . . . . . 1714
- sppprint . . . . . 1714
- spprint . . . . . 999
- sprintf . . . . . 267
- spsemicont . . . . . 1716
- spsub . . . . . 1715
- sqfrNorm . . . . . 1901
- sqfrNormMain . . . . . 1904
- sqrfree . . . . . 266
- squarefree . . . . . 1735
- squareRoot . . . . . 1993
- sres . . . . . 269
- sRes . . . . . 1129

ssi	708	sturmseq	2045
Ssi file links	99	sublists	2233
Ssi links	99	submat	974
Ssi tcp links	100	subquotient	1133
StabEqn	1743	subrInterred	896
StabEqnId	1744	subset	901
stabilizer	2282	subset_sum01	2004
StabOrder	1745	subset_sum02	2005
stack, moduli space of stable maps	1598	subst	274
stack, moduli space of stable maps, rational curves	1598	subst (letterplace)	633
stackNets	2256	subst (plural)	363
staircase	1905	substAll	1862
standard	2028	substitute	896
Standard bases	775	sufficientlyPositiveMultidegree	2560
Standard Bases	710	sugarCrit, option	236
standard bases, saturation	2239	sum	814, 2302, 2318
standard bases	2238	sum of Feynman integrals	2581
standard.lib	794	sum_of_powers	1396
standard.lib	794	sumlist	1522
Stanely-Reisner ideals, Stanley-Reisner rings, deformations	2544	sumofquotients	1602
stanleyreisner.lib	2544	super-commutative algebras	615
stanleyreisner.lib	2544	super_increasing_knapsack	2010
startNmz	1231	superCommutative	569
StartOrderingV	1622	surf.lib	1918
startTasks	962	surf.lib	1918
Startup sequence	22	surfacesignature.lib	1773
static procedures	51	surfacesignature.lib	1773
status	270	surfer	1, 1919
std	271, 711	surfex	1
std (letterplace)	633	surfex.lib	1919
std (plural)	361	surfex.lib	1919
stdfglm	272	suspend	270
stdhilb	273	SV-decoding algorithm	1934
stdlocus	856	SV-decoding algorithm, preprocessing	1933
stdmodule.lib	2551	swalk	2554
stdmodule.lib	2551	swalk.lib	2553
stopTask	963	swalk.lib	2553
storeActionOnOrbitConeIndices	2121	swap	1653, 2209
strand	2574	switch	312
stratify	1819	switchRingsAndComputeInitialIdeal	2200
stratify.lib	1818	sym_gauss	990
stratify.lib	1818	Symbolic-numerical solving	760, 1821
string	130, 315, 2299, 2300, 2317, 2318	symExt	2559
string declarations	130	SymGroup	612
string expressions	131	symm	1392
string operations	132	symmat	974
string related functions	133	symmetric basis	984
string type cast	131	symmetric power	985
StringF	1738	symmetric product	1438
stripHNE	1733	symmetricBasis	984
structure	2315	symmetricPower	985
sturm	2044	symmetricPowerSheaf	1596
sturmha	2045	symmetries	2274
sturmhaseq	2046	symmfunc	2039
sturmquery	2034	symmStd	1386
		symNsym	1392
		syModStd	1387

symodstd.lib	1384	system, semaphore	277
symodstd_lib	1384	system, semic	277
sympower	1816	system, setenv	278
symsignature	2033	system, sh	278
syndrome	1946	system, shrinktest	278
sys_code	1936	system, Singular	278
sysBin	1945	system, SingularBin	278
sysCRHT	1938	system, SingularLib	278
sysCRHTMindist	1940	system, spadd	278
sysFL	1960	system, spectrum	278
sysNewton	1942	system, spmul	278
sysQE	1947	system, std_syz	278
system	275	system, tensorModuleMult	278
System and Control theory	1966	system, twostd	278
System dependent limitations	310	system, uname	278
System variables	302	system, verifyGB	278
system, -	279	system, version	278
system, -long_option_name	279	system, with	278
system, -long_option_name=value	279	systemOfParametersOfLocalization	1446
system, absFact	275	systhreads.lib	2555
system, alarm	275	systhreads_lib	2555
system, blackbox	275	syz	280
system, bracket	275	syz (letterplace)	634
system, browsers	275	syz (plural)	363
system, complexNearZero	275	Syzygies and resolutions	776
system, content	275	Syzygies and resolutions (plural)	368
system, contributors	275	Syzygy bimodule	639
system, cpu	276, 278		
system, denom_list	276		
system, eigenvals	276		
system, env	276		
system, executable	276		
system, getenv	276		
system, getPrecDigits	276		
system, gmsnf	276		
system, HC	276		
system, hessenberg	276		
system, install	276		
system, LLL	276		
system, nblocks	276		
system, nc_hilb	277		
system, neworder	277		
system, newstruct	277		
system, opp	277		
system, oppose	277		
system, pcvBasis	277		
system, pcvCV2P	277		
system, pcvDim	277		
system, pcvLAddL	277		
system, pcvMinDeg	277		
system, pcvP2CV	277		
system, pcvPMulL	277		
system, pid	277		
system, random	277		
system, reserve	277		
system, reservedLink	277		
system, rref	277		
		<b>T</b>	
		T_1	1765
		T_12	1767
		T_2	1766
		T1	740, 2544
		T2	740, 2547
		tab	883
		tail	2023
		tame polynomial	1718, 1719
		tangentcone	1767
		tangentGens	1460
		target	1136
		Target	1125
		task	959
		tasks.lib	959
		tasks_lib	959
		tateProdCplxNegGrad.lib	2555
		tateProdCplxNegGrad_lib	2555
		tateResolution	2561
		tau_es	1698
		tau_es2	1636
		tdCf	1413
		tDetropicalise	2174
		tdFactor	1414
		tdProj	1416
		tdTerms	1414
		teach_lpGkDim	663

teach_lpKDim	642	the first alternative algorithm	1089
teach_lpKDimCheck	642	The fractal walk algorithm	1088, 2539
teach_lpSickleDim	645	The online help system	15
Teaching	1987	The SINGULAR language	41
teachstd.lib	2022	The Tran algorithm	1089
teachstd.lib	2022	three_elements	2015
Template for writing a library	59	timeFactorize	816
template.lib	61	timer	305
template.lib	59, 61	timer resolution, setting the	279
tensor	280, 372	timeStd	816
tensor algebra	666	timestep	2336
tensorFreemodMod	1142	tInitialForm	2165
tensorFreeModule	1142	tInitialFormPar	2172
tensorMatrix	1140	tInitialFormParMax	2173
tensorMod	1112	tInitialIdeal	2165
tensorModFreemod	1142	tjurina	1764
tensorModule	1140	Tjurina	1763
tensorProduct	1143	Tjurina number	735, 1204
tensorSheaf	1595	tmatrix	1711
term orderings	769	todd	1411
term orderings introduction	769	toddE	1412
testFraction	587	toddPoly	1419
TestGRRes	2368	tolessvars	1835
TestJMark	1623	Top 20 Emacs commands	29
testLift	672	topChernClass	1594
testLocData	585	topological invariants	1627, 1634, 1635, 1636
testNCfac	493	Tor	1113
testNcfrac	548	toric ideals	782
testNcfracExamples	548	Toric ideals and integer programming	782
testNcloc	553	toric ring	1210
testNclocExamples	554	toric.lib	1388
testOlga	599	toric_ideal	1388
testOlgaExamples	599	toric_lib	1388
testParametrization	1475	toric_std	1389
testPointConic	1476	toricRingFromBinomials	1214
testPrimary	1247	torus	1814
testPrimaryE	1247	torusInvariants	1218
testSyz	672	torusrep	1815
testZero	2536	torusreynolds	1818
tetrahedronGroup	659	total divisors	372
tex	1908	total multiplicities	1627
texcoef	1985	totalChernClass	1593
texdemo	1909	totalmultiplicities	1628
texDrawBasic	2167	totalSegreClass	1595
texDrawNewtonSubdivision	2170	trace	281
texDrawTriangulation	2171	TRACE	68, 306
texDrawTropical	2168	tracemult	2035
texfactorize	1909	transpose	281
texmap	1910	trapezoid	2334
texMatrix	2167	triagmatrix	940
texname	1911	triang.lib	1856
texNumber	2166	triang_lib	1856
texobj	1911	triang_solve	1855
texpoly	1913	triangL	1856
texPolynomial	2167	triangL_solve	1854
texproc	1913	triangLf_solve	1852
texring	1915	triangLfak	1857

triangM . . . . . 1859  
 triangM.solve . . . . . 1853  
 triangMH . . . . . 1860  
 triangularGroup . . . . . 660  
 triangulations . . . . . 2129  
 Tricks and pitfalls . . . . . 310  
 triMNewton . . . . . 1861  
 TropCovandMaps . . . . . 2583  
 tropical covers . . . . . 2576  
 tropical curves . . . . . 2142  
 tropical geometry . . . . . 1068, 2196, 2576  
 Tropical Geometry . . . . . 2048  
 tropical mirror symmetry . . . . . 1068, 2576  
 tropical polynomials . . . . . 2142  
 tropical stable maps . . . . . 2576, 2579  
 tropical varieties . . . . . 2196  
 tropical.lib . . . . . 2140  
 tropical.lib . . . . . 2140  
 Tropicalcover . . . . . 2577  
 tropicalCurve . . . . . 2151  
 tropicalEllipticCovers.lib . . . . . 2575  
 tropicalEllipticCovers.lib . . . . . 2575  
 tropicalise . . . . . 2164  
 tropicaliseSet . . . . . 2164  
 tropicalJInvariant . . . . . 2158  
 tropicalLifting . . . . . 2142  
 tropicalLinkNewton . . . . . 2202  
 tropicalNewton.lib . . . . . 2196  
 tropicalNewton.lib . . . . . 2196  
 tropicalPointNewton . . . . . 2198  
 tropicalSubst . . . . . 2175  
 tropicalVariety . . . . . 2179  
 tropicalVarietyNewton . . . . . 2201  
 truncate . . . . . 1605  
 truncateCoker . . . . . 2558  
 truncated module . . . . . 1605, 1606  
 truncateFast . . . . . 1606  
 truncateM . . . . . 2557  
 tuple to compute a cover of elliptic curve . . . . . 2577  
 twalk . . . . . 1089  
 twist . . . . . 2571  
 two-sided . . . . . 372  
 two-sided Groebner basis . . . . . 372  
 Two-sided ideal membership (plural) . . . . . 368  
 twostd . . . . . 278  
 twostd (letterplace) . . . . . 635  
 twostd (plural) . . . . . 364  
 type . . . . . 282  
 type B . . . . . 2218  
 Type casting . . . . . 47  
 Type conversion . . . . . 46  
 type D . . . . . 2219  
 type,custom . . . . . 136  
 type2arr . . . . . 2206  
 typecasting . . . . . 2206, 2207  
 typeof . . . . . 282  
 Typesetting of help and info strings . . . . . 64

## U

u . . . . . 2330  
 U\_D\_O . . . . . 996  
 Ufnarovski graph . . . . . 661  
 uname . . . . . 278  
 unbounded\_knapsack . . . . . 2006  
 union . . . . . 2270  
 unipotent groups . . . . . 1460  
 uniquePoint . . . . . 2075  
 unitmat . . . . . 975  
 unitMatrix . . . . . 1893  
 unitMatrix2 . . . . . 1826  
 univariate . . . . . 283  
 univarpoly . . . . . 2039  
 universal . . . . . 2312, 2313  
 universal derivation . . . . . 2294  
 Unix installation . . . . . 2602  
 unknown syndrome . . . . . 790  
 unset . . . . . 206  
 untyped definitions . . . . . 79, 319  
 updatePairs . . . . . 2027  
 UpOneMatrix . . . . . 576  
 UpperMonomials . . . . . 1745  
 ures.solve . . . . . 1847  
 uressolve . . . . . 283  
 usage, option . . . . . 237  
 UseBertini . . . . . 1455  
 used environment variables . . . . . 2601  
 User defined types . . . . . 136  
 user interface, Emacs . . . . . 22

## V

V-filtration . . . . . 1703, 1708, 1709, 1710, 1711, 1713, 1718,  
 1719, 2421  
 val . . . . . 2197  
 valvars . . . . . 1839  
 vandermonde . . . . . 284  
 vanishId . . . . . 1959  
 vanishing polynomial . . . . . 2535  
 var . . . . . 284  
 variables . . . . . 285, 2208, 2209  
 Variables, indexed . . . . . 44  
 variablesSorted . . . . . 399  
 variablesStandard . . . . . 399  
 varMat . . . . . 2208  
 varNum . . . . . 2208  
 vars2pars . . . . . 533  
 varsigns . . . . . 2042  
 varstr . . . . . 285  
 VarToPar . . . . . 2339  
 vdim . . . . . 286  
 vdim (letterplace) . . . . . 635  
 vdim (plural) . . . . . 365  
 vec2poly . . . . . 387  
 VecField.lib . . . . . 2583



VecField.lib	2583
vecFieldToMatrix	2586
vector	134
vector declarations	134
vector expressions	134
vector operations	135
vector related functions	135
verify	2037
verify Groebner base	278
verifyGB	278, 310
veronese	1863
versal	1693
version	278
Version number	2598
Version schema for Singular	2598
Version string	56
vertices	2089
vfilt	1708
Vfiltration	2422
view	1977
Visualization	1905
visualize	2330
voice	308
vspace	310
vStd	1665
vwfilt	1710

## W

waitall	286
waitAllTasks	965
waitfirst	287
waitTasks	964
walk, groebner	1088, 2439, 2537, 2554
warkedPreimageStd	1446
warn, option	71, 234
watchdog	815
wedge	287
weierstr.lib	2030
weierstr.lib	2030
Weierstrass	1928
Weierstrass semigroup	1923
weierstrassForm	2159
weierstrDiv	2030
weierstrPrep	2031
weight	288
weight filtration	1703, 1708, 1710, 1711, 1718, 1719
weighted lexicographical ordering	770
weighted reverse lexicographical ordering	770
weightedRing	564
weightKB	288
weightM, option	236
Weyl	566
Weyl algebra	402, 448, 454
Weyl closure	523

WeylClosure	525
WeylClosure1	525
whichvariable	2041
while	301
Wibmer's Lemma	864
Windows installation	2602
withDim	157
withHilb	157
withMult	157
withRes	157
withSB	157
WitSet	1457
WitSupSet	1457
WLCGS	865
WLemma	863
wp, global ordering	770
WP, global ordering	770
write	289
writeBertiniInput	1867
writeNmzData	1228
writeNmzPaths	1230
writing monomials	305
ws, local ordering	770
Ws, local ordering	770
WSemigroup	1687
wurzel	1988

## X

xchange	1986
xdvi	1917
XLsolve	1824

## Z

zdd	2260
zdd2poly	2267
zero	1124
zero-dimensional	2040
zerodec	1250
zeroMod	1164
zeroNcfrac	542
zeroOpt	1170
zeroRadical	1014
zeroreduce	2535
zeroReduce	2536
zeroSet	1901
zeroset.lib	1899
zeroset.lib	1899
zetaDL	1541
ZZ	30, 73
ZZ/m	30
ZZ/p	30
ZZsolve	1824



# Table of Contents

<b>1</b>	<b>Preface .....</b>	<b>1</b>
<b>2</b>	<b>Introduction .....</b>	<b>4</b>
2.1	Background .....	4
2.2	How to use this manual .....	4
2.3	Getting started .....	6
2.3.1	First steps .....	6
2.3.2	Rings and standard bases .....	7
2.3.3	Procedures and libraries .....	10
2.3.4	Change of rings .....	11
2.3.5	Modules and their annihilator .....	12
2.3.6	Resolution .....	13
<b>3</b>	<b>General concepts .....</b>	<b>15</b>
3.1	Interactive use .....	15
3.1.1	How to enter and exit .....	15
3.1.2	The SINGULAR prompt .....	15
3.1.3	The online help system .....	15
3.1.4	Interrupting SINGULAR .....	18
3.1.5	Editing input .....	18
3.1.6	Command line options .....	19
3.1.7	Startup sequence .....	22
3.2	Emacs user interface .....	22
3.2.1	A quick guide to Emacs .....	23
3.2.2	Running SINGULAR under Emacs .....	25
3.2.3	Demo mode .....	27
3.2.4	Customization of the Emacs interface .....	27
3.2.5	Editing SINGULAR input files with Emacs .....	28
3.2.6	Top 20 Emacs commands .....	29
3.3	Rings and orderings .....	30
3.3.1	Examples of ring declarations .....	31
3.3.2	General syntax of a ring declaration .....	32
3.3.3	Term orderings .....	34
3.3.4	Coefficient rings .....	37
3.4	Implemented algorithms .....	37
3.5	The SINGULAR language .....	41
3.5.1	General command syntax .....	41
3.5.2	Special characters .....	43
3.5.3	Names .....	44
3.5.4	Objects .....	46
3.5.5	Type conversion and casting .....	46
3.5.6	Flow control .....	48
3.6	Input and output .....	48
3.7	Procedures .....	50
3.7.1	Procedure definition .....	51
3.7.2	Parameter list .....	52

3.7.3	Help string	54
3.7.4	Names in procedures	54
3.7.5	Procedure-specific commands	55
3.8	Libraries	55
3.8.1	Libraries in the SINGULAR Documentation	55
3.8.2	Version string	56
3.8.3	Category string	56
3.8.4	Info string	56
3.8.5	LIB commands	57
3.8.6	Procedures in a library	58
3.8.7	template_lib	59
3.8.7.1	mdouble	61
3.8.7.2	mtriple	62
3.8.7.3	msum	62
3.8.8	Formal Checker	63
3.8.9	Documentation Tool	63
3.8.10	Typesetting of help and info strings	64
3.8.11	Loading a library	67
3.9	Debugging tools	68
3.9.1	ASSUME	68
3.9.2	Tracing of procedures	68
3.9.3	Source code debugger	69
3.9.4	Break points	70
3.9.5	Printing of data	70
3.9.6	libparse	70
3.9.7	option(warn)	71
3.10	Dynamic loading	71
<b>4</b>	<b>Data types</b>	<b>73</b>
4.1	cring	73
4.1.1	cring declarations	73
4.1.2	cring expressions	73
4.1.3	cring operations	73
4.1.4	cring related functions	74
4.2	bigint	74
4.2.1	bigint declarations	74
4.2.2	bigint expressions	74
4.2.3	bigint operations	75
4.2.4	bigint related functions	75
4.3	bigintmat	75
4.3.1	bigintmat declarations	75
4.3.2	bigintmat expressions	76
4.3.3	bigintmat type cast	76
4.3.4	bigintmat operations	77
4.4	bigintvec	78
4.4.1	bigintvec declarations	78
4.4.2	bigintvec expressions	78
4.4.3	bigintvec type cast	78
4.4.4	bigintvec operations	79
4.5	def	79
4.5.1	def declarations	80
4.6	ideal	80

4.6.1	ideal declarations	81
4.6.2	ideal expressions	81
4.6.3	ideal operations	82
4.6.4	ideal related functions	83
4.7	int	85
4.7.1	int declarations	85
4.7.2	int expressions	85
4.7.3	int operations	87
4.7.4	int related functions	88
4.7.5	boolean expressions	89
4.7.6	boolean operations	90
4.8	intmat	90
4.8.1	intmat declarations	90
4.8.2	intmat expressions	91
4.8.3	intmat type cast	92
4.8.4	intmat operations	92
4.8.5	intmat related functions	93
4.9	intvec	93
4.9.1	intvec declarations	94
4.9.2	intvec expressions	94
4.9.3	intvec operations	95
4.9.4	intvec related functions	96
4.10	link	96
4.10.1	link declarations	96
4.10.2	link expressions	97
4.10.3	link related functions	97
4.10.4	ASCII links	98
4.10.5	Ssi links	99
4.10.5.1	Ssi file links	99
4.10.5.2	Ssi tcp links	100
4.10.6	Pipe links	101
4.10.7	DBM links	102
4.11	list	103
4.11.1	list declarations	103
4.11.2	list expressions	104
4.11.3	list operations	105
4.11.4	list related functions	105
4.12	map	106
4.12.1	map declarations	106
4.12.2	map expressions	107
4.12.3	map operations	107
4.12.4	map related functions	108
4.13	matrix	108
4.13.1	matrix declarations	108
4.13.2	matrix expressions	109
4.13.3	matrix type cast	110
4.13.4	matrix operations	110
4.13.5	matrix related functions	112
4.14	module	112
4.14.1	module declarations	113
4.14.2	module expressions	113
4.14.3	module operations	113
4.14.4	module related functions	114

4.15	number .....	115
4.15.1	number declarations .....	115
4.15.2	number expressions .....	117
4.15.3	number operations .....	118
4.15.4	number related functions .....	119
4.16	package .....	119
4.16.1	package declarations .....	119
4.16.2	package related functions .....	120
4.17	poly .....	120
4.17.1	poly declarations .....	120
4.17.2	poly expressions .....	121
4.17.3	poly operations .....	121
4.17.4	poly related functions .....	123
4.18	proc .....	124
4.18.1	proc declaration .....	124
4.18.2	proc expression .....	125
4.18.3	procs with different argument types .....	125
4.19	resolution .....	125
4.19.1	resolution declarations .....	126
4.19.2	resolution expressions .....	126
4.19.3	resolution related functions .....	126
4.20	ring .....	127
4.20.1	qring .....	127
4.20.2	ring declarations .....	127
4.20.3	ring related functions .....	128
4.20.4	ring operations .....	128
4.20.5	qring declaration .....	129
4.21	smatrix .....	129
4.22	string .....	130
4.22.1	string declarations .....	130
4.22.2	string expressions .....	131
4.22.3	string type cast .....	131
4.22.4	string operations .....	132
4.22.5	string related functions .....	133
4.23	vector .....	134
4.23.1	vector declarations .....	134
4.23.2	vector expressions .....	134
4.23.3	vector operations .....	135
4.23.4	vector related functions .....	135
4.24	User defined types .....	136
4.24.1	Definition of a user defined type .....	136
4.24.2	Declaration of objects of a user defined type .....	137
4.24.3	Access to elements of a user defined type .....	137
4.24.4	Commands for user defined types .....	138
4.24.5	Assignments for user defined types .....	139
4.25	cone .....	140
4.26	fan .....	140
4.27	polytope .....	140
4.28	pyobject .....	140
4.28.1	pyobject declarations .....	141
4.28.2	pyobject expressions .....	141
4.28.3	pyobject operations .....	142
4.28.4	pyobject related functions .....	143

4.28.5	python_eval .....	145
4.28.6	python_import .....	145
4.28.7	python_run .....	146
4.29	reference and shared (experimental) .....	146
4.29.1	reference declarations .....	148
4.29.2	reference expressions .....	149
4.29.3	shared declarations .....	150
4.29.4	shared expressions .....	150
4.29.5	reference and shared operations .....	151
4.29.6	reference and shared related functions .....	153
<b>5</b>	<b>Functions and system variables .....</b>	<b>156</b>
5.1	Functions .....	156
5.1.1	align .....	156
5.1.2	attrib .....	156
5.1.3	bareiss .....	158
5.1.4	beti .....	159
5.1.5	char .....	161
5.1.6	char_series .....	161
5.1.7	charstr .....	162
5.1.8	chinrem .....	162
5.1.9	cleardenom .....	163
5.1.10	close .....	163
5.1.11	coef .....	164
5.1.12	coeffs .....	165
5.1.13	contract .....	167
5.1.14	create_ring .....	168
5.1.15	crossprod .....	168
5.1.16	datetime .....	169
5.1.17	dbprint .....	169
5.1.18	defined .....	170
5.1.19	deg .....	170
5.1.20	degree .....	171
5.1.21	delete .....	171
5.1.22	denominator .....	172
5.1.23	det .....	172
5.1.24	diff .....	173
5.1.25	dim .....	174
5.1.26	division .....	174
5.1.27	dump .....	175
5.1.28	eliminate .....	176
5.1.29	eval .....	177
5.1.30	ERROR .....	177
5.1.31	example .....	178
5.1.32	execute .....	178
5.1.33	extgcd .....	178
5.1.34	facstd .....	179
5.1.35	factmodd .....	180
5.1.36	factorize .....	180
5.1.37	farey .....	181
5.1.38	fetch .....	182
5.1.39	fglm .....	183

5.1.40	<code>fglmquot</code> . . . . .	184
5.1.41	<code>files</code> , input from . . . . .	184
5.1.42	<code>find</code> . . . . .	185
5.1.43	<code>finduni</code> . . . . .	185
5.1.44	<code>flintQ</code> . . . . .	186
5.1.45	<code>Float</code> . . . . .	186
5.1.46	<code>fprintf</code> . . . . .	186
5.1.47	<code>freemodule</code> . . . . .	188
5.1.48	<code>fres</code> . . . . .	188
5.1.49	<code>frwalk</code> . . . . .	189
5.1.50	<code>gcd</code> . . . . .	190
5.1.51	<code>gen</code> . . . . .	191
5.1.52	<code>getdump</code> . . . . .	191
5.1.53	<code>groebner</code> . . . . .	191
5.1.54	<code>help</code> . . . . .	193
5.1.55	<code>highcorner</code> . . . . .	194
5.1.56	<code>hilb</code> . . . . .	195
5.1.57	<code>homog</code> . . . . .	196
5.1.58	<code>hres</code> . . . . .	197
5.1.59	<code>imap</code> . . . . .	198
5.1.60	<code>impart</code> . . . . .	199
5.1.61	<code>indepSet</code> . . . . .	199
5.1.62	<code>insert</code> . . . . .	200
5.1.63	<code>interpolation</code> . . . . .	200
5.1.64	<code>interred</code> . . . . .	201
5.1.65	<code>intersect</code> . . . . .	202
5.1.66	<code>jacob</code> . . . . .	203
5.1.67	<code>janet</code> . . . . .	203
5.1.68	<code>jet</code> . . . . .	204
5.1.69	<code>kbase</code> . . . . .	205
5.1.70	<code>kernel</code> . . . . .	206
5.1.71	<code>kill</code> . . . . .	206
5.1.72	<code>killattrib</code> . . . . .	207
5.1.73	<code>koszul</code> . . . . .	207
5.1.74	<code>laguerre</code> . . . . .	208
5.1.75	<code>lead</code> . . . . .	209
5.1.76	<code>leadcoef</code> . . . . .	209
5.1.77	<code>leadexp</code> . . . . .	210
5.1.78	<code>leadmonom</code> . . . . .	210
5.1.79	<code>LIB</code> . . . . .	211
5.1.80	<code>lift</code> . . . . .	211
5.1.81	<code>liftstd</code> . . . . .	212
5.1.82	<code>listvar</code> . . . . .	213
5.1.83	<code>lres</code> . . . . .	215
5.1.84	<code>ludecomp</code> . . . . .	216
5.1.85	<code>luinverse</code> . . . . .	216
5.1.86	<code>lusolve</code> . . . . .	217
5.1.87	<code>max</code> . . . . .	218
5.1.88	<code>maxideal</code> . . . . .	219
5.1.89	<code>memory</code> . . . . .	219
5.1.90	<code>min</code> . . . . .	220
5.1.91	<code>minbase</code> . . . . .	221
5.1.92	<code>minor</code> . . . . .	221

5.1.93	minres	223
5.1.94	modulo	223
5.1.95	monitor	224
5.1.96	monomial	225
5.1.97	mpresmat	225
5.1.98	mres	225
5.1.99	mres_map	226
5.1.100	mstd	227
5.1.101	mult	228
5.1.102	nameof	228
5.1.103	names	229
5.1.104	ncols	231
5.1.105	npars	231
5.1.106	nres	232
5.1.107	nrows	232
5.1.108	numerator	233
5.1.109	nvars	233
5.1.110	open	234
5.1.111	option	234
5.1.112	ord	238
5.1.113	ordstr	239
5.1.114	par	239
5.1.115	pardeg	239
5.1.116	parstr	240
5.1.117	preimage	240
5.1.118	prime	241
5.1.119	primefactors	241
5.1.120	print	242
5.1.121	printf	244
5.1.122	prune	245
5.1.123	prune_map	245
5.1.124	qhweight	246
5.1.125	qrds	246
5.1.126	quote	247
5.1.127	quotient	248
5.1.128	random	249
5.1.129	rank	249
5.1.130	read	250
5.1.131	reduce	251
5.1.132	regularity	252
5.1.133	repart	253
5.1.134	res	253
5.1.135	reservedName	254
5.1.136	resultant	255
5.1.137	ringlist	255
5.1.138	ring_list	257
5.1.139	rvar	258
5.1.140	sba	259
5.1.141	setring	260
5.1.142	simplex	261
5.1.143	simplify	263
5.1.144	size	264
5.1.145	slimgb	265

5.1.146	sortvec	266
5.1.147	sqrfree	266
5.1.148	sprintf	267
5.1.149	sres	269
5.1.150	status	270
5.1.151	std	271
5.1.152	stdfglm	272
5.1.153	stdhilb	273
5.1.154	subst	274
5.1.155	system	275
5.1.156	syz	280
5.1.157	tensor	280
5.1.158	trace	281
5.1.159	transpose	281
5.1.160	type	282
5.1.161	typeof	282
5.1.162	univariate	283
5.1.163	uressolve	283
5.1.164	vandermonde	284
5.1.165	var	284
5.1.166	variables	285
5.1.167	varstr	285
5.1.168	vdim	286
5.1.169	waitall	286
5.1.170	waitfirst	287
5.1.171	wedge	287
5.1.172	weight	288
5.1.173	weightKB	288
5.1.174	write	289
5.2	Control structures	290
5.2.1	apply	290
5.2.2	break	291
5.2.3	breakpoint	291
5.2.4	continue	291
5.2.5	else	292
5.2.6	export	292
5.2.7	exportto	293
5.2.8	for	295
5.2.9	if	296
5.2.10	importfrom	296
5.2.11	keepring	298
5.2.12	load	299
5.2.13	quit	300
5.2.14	return	301
5.2.15	while	301
5.2.16	~ (break point)	302
5.3	System variables	302
5.3.1	degBound	302
5.3.2	echo	303
5.3.3	minpoly	303
5.3.4	multBound	304
5.3.5	noether	304
5.3.6	printlevel	304



5.3.7	short .....	305
5.3.8	timer .....	305
5.3.9	TRACE .....	306
5.3.10	rtimer .....	308
5.3.11	voice .....	308
<b>6</b>	<b>Tricks and pitfalls .....</b>	<b>310</b>
6.1	Limitations .....	310
6.2	System dependent limitations .....	310
6.3	Major differences to the C programming language .....	311
6.3.1	No rvalue of increments and assignments .....	311
6.3.2	Evaluation of logical expressions .....	311
6.3.3	No case or switch statement .....	312
6.3.4	Usage of commas .....	312
6.3.5	Usage of brackets .....	312
6.3.6	Behavior of continue .....	312
6.3.7	Return type of procedures .....	313
6.3.8	First index is 1 .....	313
6.4	Miscellaneous oddities .....	314
6.5	Identifier resolution .....	316
<b>7</b>	<b>Non-commutative subsystem .....</b>	<b>318</b>
7.1	PLURAL .....	318
7.2	Data types (plural) .....	319
7.2.1	ideal (plural) .....	319
7.2.1.1	ideal declarations (plural) .....	319
7.2.1.2	ideal expressions (plural) .....	320
7.2.1.3	ideal operations (plural) .....	321
7.2.1.4	ideal related functions (plural) .....	322
7.2.2	map (plural) .....	323
7.2.2.1	map declarations (plural) .....	324
7.2.2.2	map expressions (plural) .....	325
7.2.2.3	map (plural) operations .....	325
7.2.2.4	map related functions (plural) .....	325
7.2.3	module (plural) .....	326
7.2.3.1	module declarations (plural) .....	326
7.2.3.2	module expressions (plural) .....	326
7.2.3.3	module operations (plural) .....	327
7.2.3.4	module related functions (plural) .....	327
7.2.4	poly (plural) .....	328
7.2.4.1	poly declarations (plural) .....	328
7.2.4.2	poly expressions (plural) .....	329
7.2.4.3	poly operations (plural) .....	330
7.2.4.4	poly related functions (plural) .....	330
7.2.5	qring (plural) .....	330
7.2.5.1	qring declaration (plural) .....	331
7.2.5.2	qring related functions (plural) .....	331
7.2.6	resolution (plural) .....	332
7.2.6.1	resolution declarations (plural) .....	332
7.2.6.2	resolution expressions (plural) .....	333
7.2.6.3	resolution related functions (plural) .....	333
7.2.7	ring (plural) .....	333

	7.2.7.1	ring declarations (plural)	333
	7.2.7.2	ring operations (plural)	334
	7.2.7.3	ring related functions (plural)	334
7.3		Functions (plural)	335
	7.3.1	betti (plural)	335
	7.3.2	bracket	336
	7.3.3	dim (plural)	337
	7.3.4	division (plural)	338
	7.3.5	eliminate (plural)	339
	7.3.6	envelope	340
	7.3.7	fetch (plural)	341
	7.3.8	imap (plural)	342
	7.3.9	intersect (plural)	343
	7.3.10	kbase (plural)	343
	7.3.11	lift (plural)	344
	7.3.12	liftstd (plural)	345
	7.3.13	minres (plural)	346
	7.3.14	modulo (plural)	347
	7.3.15	mres (plural)	348
	7.3.16	nc_algebra	349
	7.3.17	ncalgebra	351
	7.3.18	nres (plural)	351
	7.3.19	oppose	353
	7.3.20	opposite	354
	7.3.21	preimage (plural)	355
	7.3.22	quotient (plural)	356
	7.3.23	reduce (plural)	357
	7.3.24	ringlist (plural)	358
	7.3.25	slimgb (plural)	360
	7.3.26	std (plural)	361
	7.3.27	subst (plural)	363
	7.3.28	syz (plural)	363
	7.3.29	twostd (plural)	364
	7.3.30	vdim (plural)	365
7.4		Mathematical background (plural)	366
	7.4.1	G-algebras	366
	7.4.2	Groebner bases in G-algebras	367
	7.4.3	Syzygies and resolutions (plural)	368
	7.4.4	References (plural)	370
7.5		PLURAL libraries	371
	7.5.1	bimodules.lib	371
		7.5.1.1 bistd	372
		7.5.1.2 bitrinity	373
		7.5.1.3 liftenvelope	374
		7.5.1.4 CompDecomp	375
		7.5.1.5 isPureTensor	375
		7.5.1.6 isTwoSidedGB	376
	7.5.2	bfun.lib	376
		7.5.2.1 bfct	377
		7.5.2.2 bfctSyz	378
		7.5.2.3 bfctAnn	379
		7.5.2.4 bfctOneGB	380
		7.5.2.5 bfctIdeal	381

7.5.2.6	pIntersect	382
7.5.2.7	pIntersectSyz	383
7.5.2.8	linReduce	384
7.5.2.9	linReduceIdeal	385
7.5.2.10	linSyzSolve	386
7.5.2.11	allPositive	386
7.5.2.12	scalarProd	387
7.5.2.13	vec2poly	387
7.5.3	central_lib	387
7.5.3.1	centralizeSet	388
7.5.3.2	centralizerVS	388
7.5.3.3	centralizerRed	389
7.5.3.4	centerVS	389
7.5.3.5	centerRed	390
7.5.3.6	center	391
7.5.3.7	centralizer	392
7.5.3.8	sa_reduce	393
7.5.3.9	sa_poly_reduce	393
7.5.3.10	inCenter	394
7.5.3.11	inCentralizer	394
7.5.3.12	isCartan	395
7.5.3.13	applyAdF	396
7.5.3.14	linearMapKernel	397
7.5.3.15	linearCombinations	398
7.5.3.16	variablesStandard	399
7.5.3.17	variablesSorted	399
7.5.3.18	PBW_eqDeg	400
7.5.3.19	PBW_maxDeg	400
7.5.3.20	PBW_maxMonom	401
7.5.4	dmod_lib	401
7.5.4.1	annfs	403
7.5.4.2	annfspecial	403
7.5.4.3	annfspecialOld	404
7.5.4.4	Sannfs	405
7.5.4.5	Sannfslog	406
7.5.4.6	bernsteinBM	407
7.5.4.7	bernsteinLift	407
7.5.4.8	operatorBM	408
7.5.4.9	operatorModulo	409
7.5.4.10	annfsParamBM	410
7.5.4.11	annfsBMI	411
7.5.4.12	checkRoot	412
7.5.4.13	SannfsBFCT	413
7.5.4.14	annfs0	414
7.5.4.15	annfs2	415
7.5.4.16	annfsRB	416
7.5.4.17	checkFactor	417
7.5.4.18	arrange	418
7.5.4.19	reiffen	418
7.5.4.20	isHolonomic	419
7.5.4.21	convloc	419
7.5.4.22	minIntRoot	420
7.5.4.23	isRational	421

7.5.5	dmodapp_lib	421
7.5.5.1	annPoly	422
7.5.5.2	annRat	423
7.5.5.3	DLoc	424
7.5.5.4	SDLoc	425
7.5.5.5	DLoc0	425
7.5.5.6	GBWeight	426
7.5.5.7	initialMalgrange	427
7.5.5.8	initialIdealW	428
7.5.5.9	inForm	429
7.5.5.10	restrictionIdeal	430
7.5.5.11	restrictionModule	431
7.5.5.12	integralIdeal	432
7.5.5.13	integralModule	433
7.5.5.14	deRhamCohom	434
7.5.5.15	deRhamCohomIdeal	435
7.5.5.16	charVariety	436
7.5.5.17	charInfo	437
7.5.5.18	isFsat	439
7.5.5.19	appelF1	439
7.5.5.20	appelF2	440
7.5.5.21	appelF4	440
7.5.5.22	fourier	441
7.5.5.23	inverseFourier	441
7.5.5.24	bFactor	442
7.5.5.25	intRoots	443
7.5.5.26	poly2list	444
7.5.5.27	fl2poly	444
7.5.5.28	insertGenerator	445
7.5.5.29	deleteGenerator	446
7.5.5.30	isInt	446
7.5.5.31	sortIntvec	446
7.5.6	dmodideal_lib	447
7.5.6.1	annfsLogIdeal	448
7.5.6.2	annihilatorMultiFs	449
7.5.6.3	BSidealFromAnn	450
7.5.6.4	BernsteinSatoIdeal	451
7.5.6.5	BFBoundsBudur	452
7.5.6.6	annfalphabet	452
7.5.6.7	extractS	453
7.5.7	dmodvar_lib	454
7.5.7.1	bfctVarIn	454
7.5.7.2	bfctVarAnn	455
7.5.7.3	SannfsVar	456
7.5.7.4	makeMalgrange	457
7.5.8	involut_lib	457
7.5.8.1	findInvo	458
7.5.8.2	findInvoDiag	459
7.5.8.3	findAuto	460
7.5.8.4	ncdetection	462
7.5.8.5	involution	462
7.5.8.6	isInvolution	463
7.5.8.7	isAntiEndo	464

7.5.9	gkdim.lib	464
7.5.9.1	GKdim	465
7.5.10	ncalg.lib	465
7.5.10.1	makeUsl2	466
7.5.10.2	makeUsl	466
7.5.10.3	makeUgl	467
7.5.10.4	makeUso5	468
7.5.10.5	makeUso6	469
7.5.10.6	makeUso7	469
7.5.10.7	makeUso8	470
7.5.10.8	makeUso9	470
7.5.10.9	makeUso10	471
7.5.10.10	makeUso11	471
7.5.10.11	makeUso12	472
7.5.10.12	makeUsp1	473
7.5.10.13	makeUsp2	473
7.5.10.14	makeUsp3	474
7.5.10.15	makeUsp4	475
7.5.10.16	makeUsp5	475
7.5.10.17	makeUg2	476
7.5.10.18	makeUf4	476
7.5.10.19	makeUe6	477
7.5.10.20	makeUe7	478
7.5.10.21	makeUe8	478
7.5.10.22	makeQso3	479
7.5.10.23	makeQsl2	480
7.5.10.24	makeQsl3	480
7.5.10.25	Qso3Casimir	482
7.5.10.26	GKZsystem	482
7.5.11	ncdecomp.lib	484
7.5.11.1	CentralQuot	484
7.5.11.2	CentralSaturation	485
7.5.11.3	CenCharDec	485
7.5.11.4	IntersectWithSub	486
7.5.12	ncfactor.lib	487
7.5.12.1	ncfactor	487
7.5.12.2	facWeyl	492
7.5.12.3	facFirstWeyl	493
7.5.12.4	testNCfac	493
7.5.12.5	facSubWeyl	494
7.5.12.6	facShift	495
7.5.12.7	facFirstShift	496
7.5.12.8	homogfacNthWeyl	497
7.5.12.9	homogfacNthQWeyl	497
7.5.12.10	homogfacFirstQWeyl	498
7.5.12.11	homogfacNthQWeyl_all	499
7.5.12.12	homogfacFirstQWeyl_all	517
7.5.13	nchilbert.lib	518
7.5.13.1	ncHilb	518
7.5.13.2	ncHilbertSeries	519
7.5.13.3	ncHilbertPolynomial	520
7.5.13.4	ncHilbertMultiplicity	521
7.5.13.5	GKExp	522

	7.5.13.6	mondim	522
7.5.14		dmodloc_lib	523
	7.5.14.1	Dlocalization	523
	7.5.14.2	WeylClosure	525
	7.5.14.3	WeylClosure1	525
	7.5.14.4	holonomicRank	526
	7.5.14.5	DsingularLocus	527
	7.5.14.6	polSol	527
	7.5.14.7	polSolFiniteRank	528
	7.5.14.8	ratSol	529
	7.5.14.9	bfctBound	529
	7.5.14.10	annRatSyz	530
	7.5.14.11	dmodGeneralAssumptionCheck	531
	7.5.14.12	extendWeyl	532
	7.5.14.13	polyVars	533
	7.5.14.14	monomialInIdeal	533
	7.5.14.15	vars2pars	533
	7.5.14.16	minIntRoot2	534
	7.5.14.17	maxIntRoot	534
	7.5.14.18	dmodAction	535
	7.5.14.19	dmodActionRat	535
	7.5.14.20	simplifyRat	536
	7.5.14.21	addRat	537
	7.5.14.22	multRat	537
	7.5.14.23	diffRat	537
	7.5.14.24	commRing	538
	7.5.14.25	rightNFWeyl	539
7.5.15		ncfrac_lib	539
	7.5.15.1	hasLeftDenom	540
	7.5.15.2	hasRightDenom	541
	7.5.15.3	isZeroNcfrac	541
	7.5.15.4	isOneNcfrac	542
	7.5.15.5	zeroNcfrac	542
	7.5.15.6	oneNcfrac	543
	7.5.15.7	ensureLeftNcfrac	543
	7.5.15.8	ensureRightNcfrac	545
	7.5.15.9	negateNcfrac	546
	7.5.15.10	isInvertibleNcfrac	546
	7.5.15.11	invertNcfrac	547
	7.5.15.12	testNcfrac	548
	7.5.15.13	testNcfracExamples	548
7.5.16		nchomolog_lib	548
	7.5.16.1	ncExt_R	548
	7.5.16.2	ncHom	549
	7.5.16.3	coHom	549
	7.5.16.4	contraHom	550
	7.5.16.5	dmodoublext	550
	7.5.16.6	is_cenBimodule	551
	7.5.16.7	is_cenSubbimodule	551
7.5.17		ncloc_lib	552
	7.5.17.1	isDenom	552
	7.5.17.2	testNcloc	553
	7.5.17.3	testNclocExamples	554

7.5.18	ncModslimgb_lib .....	554
7.5.18.1	ncmodslimgb .....	554
7.5.19	ncpreim_lib .....	555
7.5.19.1	eliminateNC .....	556
7.5.19.2	preimageNC .....	557
7.5.19.3	admissibleSub .....	558
7.5.19.4	isUpperTriangular .....	559
7.5.19.5	appendWeight2Ord .....	560
7.5.19.6	elimWeight .....	561
7.5.19.7	extendedTensor .....	561
7.5.20	nctools_lib .....	563
7.5.20.1	Gweights .....	563
7.5.20.2	weightedRing .....	564
7.5.20.3	ndcond .....	565
7.5.20.4	Weyl .....	566
7.5.20.5	makeWeyl .....	566
7.5.20.6	makeHeisenberg .....	567
7.5.20.7	Exterior .....	568
7.5.20.8	findimAlgebra .....	568
7.5.20.9	superCommutative .....	569
7.5.20.10	rightStd .....	571
7.5.20.11	rightNF .....	572
7.5.20.12	rightModulo .....	572
7.5.20.13	moduloSlim .....	573
7.5.20.14	ncRelations .....	574
7.5.20.15	isCentral .....	574
7.5.20.16	isNC .....	575
7.5.20.17	isCommutative .....	576
7.5.20.18	isWeyl .....	576
7.5.20.19	UpOneMatrix .....	576
7.5.20.20	AltVarStart .....	577
7.5.20.21	AltVarEnd .....	578
7.5.20.22	IsSCA .....	579
7.5.20.23	makeModElimRing .....	581
7.5.20.24	embedMat .....	581
7.5.21	olga_lib .....	582
7.5.21.1	locStatus .....	582
7.5.21.2	testLocData .....	585
7.5.21.3	isInS .....	585
7.5.21.4	fracStatus .....	586
7.5.21.5	testFraction .....	587
7.5.21.6	leftOre .....	588
7.5.21.7	rightOre .....	589
7.5.21.8	convertRightToLeftFraction .....	590
7.5.21.9	convertLeftToRightFraction .....	591
7.5.21.10	addLeftFractions .....	593
7.5.21.11	multiplyLeftFractions .....	594
7.5.21.12	areEqualLeftFractions .....	595
7.5.21.13	isInvertibleLeftFraction .....	596
7.5.21.14	invertLeftFraction .....	596
7.5.21.15	isZeroFraction .....	597
7.5.21.16	isOneFraction .....	598
7.5.21.17	normalizeMonoidal .....	598

	7.5.21.18	normalizeRational	599
	7.5.21.19	testOlga	599
	7.5.21.20	testOlgaExamples	599
7.5.22		perron.lib	600
	7.5.22.1	perron	600
7.5.23		purityfiltration.lib	601
	7.5.23.1	projectiveDimension	601
	7.5.23.2	purityFiltration	602
	7.5.23.3	purityTriang	603
	7.5.23.4	gradeNumber	604
	7.5.23.5	showgrades	605
	7.5.23.6	allExtOfLeft	605
	7.5.23.7	allExtOfRight	606
	7.5.23.8	doubleExt	607
	7.5.23.9	allDoubleExt	607
	7.5.23.10	is_pure	609
	7.5.23.11	purelist	609
7.5.24		qmatrix.lib	610
	7.5.24.1	quantMat	610
	7.5.24.2	qminor	611
	7.5.24.3	SymGroup	612
	7.5.24.4	LengthSymElement	612
	7.5.24.5	LengthSym	612
7.5.25		ratgb.lib	613
	7.5.25.1	ratstd	613
7.6		Graded commutative algebras (SCA)	615
7.7		LETTERPLACE	617
	7.7.1	Examples of use of LETTERPLACE	617
	7.7.2	Example of use of LETTERPLACE over $\mathbb{Z}$	620
	7.7.3	Functionality and release notes of LETTERPLACE	623
	7.7.4	References and history of LETTERPLACE	624
7.8		Functions (letterplace)	625
	7.8.1	dim (letterplace)	625
	7.8.2	fetch (letterplace)	626
	7.8.3	freeAlgebra (letterplace)	627
	7.8.4	imap (letterplace)	628
	7.8.5	lift (letterplace)	629
	7.8.6	liftstd (letterplace)	630
	7.8.7	modulo (letterplace)	631
	7.8.8	ncgen	631
	7.8.9	reduce (letterplace)	632
	7.8.10	rightstd (letterplace)	633
	7.8.11	std (letterplace)	633
	7.8.12	subst (letterplace)	633
	7.8.13	syz (letterplace)	634
	7.8.14	twostd (letterplace)	635
	7.8.15	vdim (letterplace)	635
7.9		Mathematical background (letterplace)	636
	7.9.1	Free associative algebras	636
	7.9.2	Monomial orderings on free algebras	637
	7.9.3	Groebner bases for two-sided ideals in free associative algebras	638
	7.9.4	Bimodules and syzygies and lifts	639



7.9.5	Letterplace correspondence	640
7.10	LETTERPLACE libraries	640
7.10.1	fpadim_lib	640
7.10.1.1	teach_lpKDimCheck	642
7.10.1.2	lpKDim	642
7.10.1.3	teach_lpKDim	642
7.10.1.4	lpMonomialBasis	643
7.10.1.5	lpHilbert	644
7.10.1.6	teach_lpSickleDim	645
7.10.2	fpalgebras_lib	646
7.10.2.1	operatorAlgebra	646
7.10.2.2	serreRelations	646
7.10.2.3	fullSerreRelations	647
7.10.2.4	ademRelations	648
7.10.2.5	baumslagSolitar	648
7.10.2.6	baumslagGroup	649
7.10.2.7	crystallographicGroupP1	649
7.10.2.8	crystallographicGroupPM	649
7.10.2.9	crystallographicGroupPG	650
7.10.2.10	crystallographicGroupP2MM	650
7.10.2.11	crystallographicGroupP2	651
7.10.2.12	crystallographicGroupP2GG	651
7.10.2.13	crystallographicGroupCM	652
7.10.2.14	crystallographicGroupC2MM	653
7.10.2.15	crystallographicGroupP4	653
7.10.2.16	crystallographicGroupP4MM	654
7.10.2.17	crystallographicGroupP4GM	654
7.10.2.18	crystallographicGroupP3	655
7.10.2.19	crystallographicGroupP31M	655
7.10.2.20	crystallographicGroupP3M1	656
7.10.2.21	crystallographicGroupP6	656
7.10.2.22	crystallographicGroupP6MM	657
7.10.2.23	dyckGroup1	658
7.10.2.24	dyckGroup2	658
7.10.2.25	dyckGroup3	658
7.10.2.26	fibonacciGroup	659
7.10.2.27	tetrahedronGroup	659
7.10.2.28	triangularGroup	660
7.10.3	fpaprops_lib	660
7.10.3.1	lpNoetherian	661
7.10.3.2	lpIsSemiPrime	661
7.10.3.3	lpIsPrime	662
7.10.3.4	lpGkDim	662
7.10.3.5	teach_lpGkDim	663
7.10.3.6	lpGldimBound	664
7.10.3.7	lpSubstitute	664
7.10.3.8	lpCalcSubstDegBound	665
7.10.4	freegb_lib	666
7.10.4.1	isFreeAlgebra	666
7.10.4.2	lpDegBound	666
7.10.4.3	lpVarBlockSize	667
7.10.4.4	lpNcgenCount	667
7.10.4.5	lpDivision	667

7.10.4.6	lpGBPres2Poly	668
7.10.4.7	isOrderingShiftInvariant	669
7.10.4.8	makeLetterplaceRing	669
7.10.4.9	letplaceGBasis	670
7.10.4.10	lieBracket	671
7.10.4.11	setLetterplaceAttributes	671
7.10.4.12	testLift	672
7.10.4.13	testSyz	672
7.10.5	ncHilb_lib	673
7.10.5.1	nchilb	673
7.10.5.2	rcolon	675
7.10.6	ncrat_lib	676
7.10.6.1	ncInit	676
7.10.6.2	ncVarsGet	677
7.10.6.3	ncVarsAdd	677
7.10.6.4	ncratDefine	677
7.10.6.5	ncratAdd	678
7.10.6.6	ncratSubtract	679
7.10.6.7	ncratMultiply	679
7.10.6.8	ncratInvert	680
7.10.6.9	ncratSPrint	680
7.10.6.10	ncratPrint	681
7.10.6.11	ncratFromString	681
7.10.6.12	ncratFromPoly	681
7.10.6.13	ncratPower	682
7.10.6.14	ncratEvaluateAt	682
7.10.6.15	ncrepGet	682
7.10.6.16	ncrepAdd	683
7.10.6.17	ncrepSubtract	684
7.10.6.18	ncrepMultiply	685
7.10.6.19	ncrepInvert	686
7.10.6.20	ncrepPrint	686
7.10.6.21	ncrepDim	687
7.10.6.22	ncrepSubstitute	688
7.10.6.23	ncrepEvaluate	689
7.10.6.24	ncrepEvaluateAt	689
7.10.6.25	ncrepIsDefinedDim	690
7.10.6.26	ncrepIsDefined	691
7.10.6.27	ncrepIsRegular	692
7.10.6.28	ncrepRegularZeroMinimize	693
7.10.6.29	ncrepRegularMinimize	693
7.10.6.30	ncrepGetRegularZeroMinimal	694
7.10.6.31	ncrepGetRegularMinimal	695
7.10.6.32	ncrepPencilGet	696
7.10.6.33	ncrepPencilCombine	696
7.11	Release Notes (letterplace)	697

<b>Appendix A</b>	<b>Examples</b>	<b>699</b>
A.1	Programming	699
A.1.1	Basic programming	699
A.1.2	Writing procedures and libraries	700
A.1.3	Rings associated to monomial orderings	703
A.1.4	Long coefficients	705
A.1.5	Parameters	706
A.1.6	Formatting output	707
A.1.7	Cyclic roots	707
A.1.8	Parallelization with ssi links	708
A.1.9	Dynamic modules	709
A.2	Computing Groebner and Standard Bases	710
A.2.1	groebner and std	711
A.2.2	Groebner basis conversion	713
A.2.3	slim Groebner bases	715
A.3	Commutative Algebra	715
A.3.1	Saturation	715
A.3.2	Finite fields	716
A.3.3	Elimination	718
A.3.4	Free resolution	720
A.3.5	Handling graded modules	723
A.3.6	Computation of Ext	725
A.3.7	Depth	728
A.3.8	Factorization	729
A.3.9	Primary decomposition	730
A.3.10	Normalization	732
A.3.11	Kernel of module homomorphisms	733
A.3.12	Algebraic dependence	733
A.4	Singularity Theory	735
A.4.1	Milnor and Tjurina number	735
A.4.2	Critical points	737
A.4.3	Polar curves	738
A.4.4	T1 and T2	740
A.4.5	Deformations	742
A.4.6	Invariants of plane curve singularities	745
A.4.7	Branches of space curve singularities	747
A.4.8	Classification of hypersurface singularities	750
A.4.9	Resolution of singularities	751
A.5	Invariant Theory	753
A.5.1	G <sub>a</sub> -Invariants	753
A.5.2	Invariants of a finite group	753
A.6	Geometric Invariant Theory	754
A.6.1	GIT-Fans	754
A.7	Non-commutative Algebra	757
A.7.1	Left and two-sided Groebner bases	757
A.7.2	Right Groebner bases and syzygies	758
A.8	Applications	760
A.8.1	Solving systems of polynomial equations	760
A.8.2	AG codes	764

## Appendix B Polynomial data ..... 768

B.1	Representation of mathematical objects .....	768
B.2	Monomial orderings .....	769
B.2.1	Introduction to orderings .....	769
B.2.2	General definitions for orderings .....	769
B.2.3	Global orderings .....	770
B.2.4	Local orderings .....	770
B.2.5	Module orderings .....	770
B.2.6	Matrix orderings .....	771
B.2.7	Product orderings .....	773
B.2.8	Extra weight vector .....	773
B.2.9	Pseudo ordering L .....	774

## Appendix C Mathematical background ..... 775

C.1	Standard bases .....	775
C.2	Hilbert function .....	775
C.3	Syzygies and resolutions .....	776
C.4	Characteristic sets .....	777
C.5	Gauss-Manin connection .....	778
C.6	Toric ideals and integer programming .....	782
C.6.1	Toric ideals .....	782
C.6.2	Algorithms .....	782
C.6.2.1	The algorithm of Conti and Traverso .....	782
C.6.2.2	The algorithm of Pottier .....	783
C.6.2.3	The algorithm of Hosten and Sturmfels .....	783
C.6.2.4	The algorithm of Di Biase and Urbanke .....	783
C.6.2.5	The algorithm of Bigatti, La Scala and Robbiano .....	784
C.6.3	The Buchberger algorithm for toric ideals .....	784
C.6.4	Integer programming .....	784
C.6.5	Relevant References .....	785
C.7	Non-commutative algebra .....	785
C.8	Decoding codes with Groebner bases .....	785
C.8.1	Codes and the decoding problem .....	786
C.8.2	Cooper philosophy .....	787
C.8.3	Generalized Newton identities .....	788
C.8.4	Fitzgerald-Lax method .....	789
C.8.5	Decoding method based on quadratic equations .....	790
C.8.6	References for decoding with Groebner bases .....	791
C.9	References .....	792

## Appendix D SINGULAR libraries . . . . . 794

D.1	standard_lib . . . . .	794
D.1.1	qslimgb . . . . .	794
D.1.2	par2varRing . . . . .	794
D.2	General purpose . . . . .	795
D.2.1	all_lib . . . . .	795
D.2.2	compregb_lib . . . . .	798
D.2.2.1	cgs . . . . .	798
D.2.2.2	base2str . . . . .	806
D.2.3	general_lib . . . . .	806
D.2.3.1	A_Z . . . . .	806
D.2.3.2	A_Z_L . . . . .	807
D.2.3.3	ASCII . . . . .	808
D.2.3.4	absValue . . . . .	808
D.2.3.5	binomial . . . . .	809
D.2.3.6	deleteSublist . . . . .	809
D.2.3.7	factorial . . . . .	809
D.2.3.8	fibonacci . . . . .	810
D.2.3.9	kmemory . . . . .	810
D.2.3.10	killall . . . . .	810
D.2.3.11	number_e . . . . .	811
D.2.3.12	number_pi . . . . .	811
D.2.3.13	primes . . . . .	812
D.2.3.14	product . . . . .	812
D.2.3.15	sort . . . . .	813
D.2.3.16	sum . . . . .	814
D.2.3.17	watchdog . . . . .	815
D.2.3.18	primecoeffs . . . . .	815
D.2.3.19	timeStd . . . . .	816
D.2.3.20	timeFactorize . . . . .	816
D.2.3.21	factorH . . . . .	817
D.2.4	grobcov_lib . . . . .	818
D.2.4.1	grobcov . . . . .	820
D.2.4.2	cgsdr . . . . .	828
D.2.4.3	pdivi . . . . .	831
D.2.4.4	pnormalf . . . . .	832
D.2.4.5	Crep . . . . .	832
D.2.4.6	Prep . . . . .	833
D.2.4.7	PtoCrep . . . . .	834
D.2.4.8	extendpoly . . . . .	835
D.2.4.9	extendGC . . . . .	836
D.2.4.10	locus . . . . .	844
D.2.4.11	locusdg . . . . .	849
D.2.4.12	envelop . . . . .	850
D.2.4.13	locusto . . . . .	855
D.2.4.14	stdlocus . . . . .	856
D.2.4.15	AssocTanToEnv . . . . .	856
D.2.4.16	FamElementsAtEnvCompPoints . . . . .	858
D.2.4.17	discrim . . . . .	862
D.2.4.18	WLemma . . . . .	863
D.2.4.19	WLCGS . . . . .	865
D.2.4.20	intersectpar . . . . .	865

	D.2.4.21	ADGT .....	865
	D.2.4.22	ConsLevels .....	870
	D.2.4.23	Levels .....	871
	D.2.4.24	Grob1Levels .....	873
	D.2.4.25	DifConsLCSets .....	876
D.2.5	inout_lib .....	878	
	D.2.5.1	allprint .....	878
	D.2.5.2	lprint .....	879
	D.2.5.3	pmat .....	879
	D.2.5.4	rMacaulay .....	880
	D.2.5.5	show .....	881
	D.2.5.6	showrecursive .....	882
	D.2.5.7	split .....	882
	D.2.5.8	tab .....	883
	D.2.5.9	pause .....	883
D.2.6	modular_lib .....	884	
	D.2.6.1	modular .....	884
D.2.7	parallel_lib .....	885	
	D.2.7.1	parallelWaitN .....	885
	D.2.7.2	parallelWaitFirst .....	886
	D.2.7.3	parallelWaitAll .....	886
	D.2.7.4	parallelTestAND .....	887
	D.2.7.5	parallelTestOR .....	888
D.2.8	polylib_lib .....	888	
	D.2.8.1	cyclic .....	889
	D.2.8.2	elemSymmId .....	889
	D.2.8.3	katsura .....	889
	D.2.8.4	freerank .....	890
	D.2.8.5	is_zero .....	891
	D.2.8.6	lcm .....	891
	D.2.8.7	maxcoef .....	892
	D.2.8.8	maxdeg .....	892
	D.2.8.9	maxdeg1 .....	892
	D.2.8.10	mindeg .....	893
	D.2.8.11	mindeg1 .....	893
	D.2.8.12	normalize .....	894
	D.2.8.13	rad_con .....	894
	D.2.8.14	content .....	895
	D.2.8.15	mod2id .....	895
	D.2.8.16	id2mod .....	895
	D.2.8.17	substitute .....	896
	D.2.8.18	subrInterred .....	896
	D.2.8.19	newtonDiag .....	897
	D.2.8.20	hilbPoly .....	898
D.2.9	redcgs_lib .....	898	
	D.2.9.1	setglobalrings .....	900
	D.2.9.2	memberpos .....	901
	D.2.9.3	subset .....	901
	D.2.9.4	pdivi2 .....	902
	D.2.9.5	facvar .....	902
	D.2.9.6	redspec .....	903
	D.2.9.7	pnormalform .....	903
	D.2.9.8	buildtree .....	904

	D.2.9.9	buildtreetoMaple	906
	D.2.9.10	finalcases	908
	D.2.9.11	mrcgs	910
	D.2.9.12	rcgs	919
	D.2.9.13	crcgs	925
	D.2.9.14	cantodiffcgs	931
D.2.10	random_lib		936
	D.2.10.1	genericid	936
	D.2.10.2	randomid	936
	D.2.10.3	randommat	937
	D.2.10.4	sparseid	937
	D.2.10.5	sparsematrix	938
	D.2.10.6	sparsemat	938
	D.2.10.7	sparsepoly	939
	D.2.10.8	sparsetriag	939
	D.2.10.9	sparseHomogIdeal	940
	D.2.10.10	triagmatrix	940
	D.2.10.11	randomLast	941
	D.2.10.12	randomBinomial	941
D.2.11	resources_lib		942
	D.2.11.1	addcores	942
	D.2.11.2	setcores	942
	D.2.11.3	getcores	943
	D.2.11.4	semaphore	943
D.2.12	ring_lib		943
	D.2.12.1	changechar	944
	D.2.12.2	changeord	944
	D.2.12.3	changevar	945
	D.2.12.4	defring	946
	D.2.12.5	defrings	946
	D.2.12.6	defringp	947
	D.2.12.7	extendring	947
	D.2.12.8	fetchall	949
	D.2.12.9	imapall	950
	D.2.12.10	mapall	950
	D.2.12.11	ord_test	951
	D.2.12.12	ringtensor	951
	D.2.12.13	ringweights	953
	D.2.12.14	preimageLoc	953
	D.2.12.15	rootofUnity	953
	D.2.12.16	optionIsSet	954
	D.2.12.17	hasNumericCoeffs	954
	D.2.12.18	hasCommutativeVars	955
	D.2.12.19	hasGlobalOrdering	955
	D.2.12.20	hasMixedOrdering	955
	D.2.12.21	hasAlgExtensionCoefficient	956
	D.2.12.22	hasTransExtensionCoefficient	956
	D.2.12.23	isQuotientRing	956
	D.2.12.24	isSubModule	957
	D.2.12.25	changeordTo	957
	D.2.12.26	addvarsTo	957
	D.2.12.27	addNvarsTo	958
D.2.13	tasks_lib		959

	D.2.13.1	createTask	959
	D.2.13.2	killTask	960
	D.2.13.3	copyTask	960
	D.2.13.4	compareTasks	961
	D.2.13.5	printTask	962
	D.2.13.6	startTasks	962
	D.2.13.7	stopTask	963
	D.2.13.8	waitTasks	964
	D.2.13.9	waitAllTasks	965
	D.2.13.10	pollTask	965
	D.2.13.11	getCommand	966
	D.2.13.12	getArguments	967
	D.2.13.13	getResult	967
	D.2.13.14	getState	968
D.3	Linear algebra		968
	D.3.1	matrix_lib	968
		D.3.1.1	compress 968
		D.3.1.2	concat 969
		D.3.1.3	diag 970
		D.3.1.4	dsum 970
		D.3.1.5	flatten 971
		D.3.1.6	genericmat 971
		D.3.1.7	is_complex 972
		D.3.1.8	outer 972
		D.3.1.9	power 973
		D.3.1.10	skewmat 973
		D.3.1.11	submat 974
		D.3.1.12	symmat 974
		D.3.1.13	unitmat 975
		D.3.1.14	gauss_col 975
		D.3.1.15	gauss_row 976
		D.3.1.16	addcol 977
		D.3.1.17	addrow 978
		D.3.1.18	multcol 978
		D.3.1.19	multrow 979
		D.3.1.20	permc col 979
		D.3.1.21	permrow 979
		D.3.1.22	rowred 980
		D.3.1.23	colred 981
		D.3.1.24	linear_relations 982
		D.3.1.25	rm_unitrow 982
		D.3.1.26	rm_unitcol 983
		D.3.1.27	headStand 983
		D.3.1.28	symmetricBasis 984
		D.3.1.29	exteriorBasis 984
		D.3.1.30	symmetricPower 985
		D.3.1.31	exteriorPower 986
	D.3.2	linalg_lib	988
		D.3.2.1	inverse 988
		D.3.2.2	inverse_B 989
		D.3.2.3	inverse_L 990
		D.3.2.4	sym_gauss 990
		D.3.2.5	orthogonalize 991



D.3.2.6	diag_test	991
D.3.2.7	busadj	991
D.3.2.8	charpoly	992
D.3.2.9	adjoint	992
D.3.2.10	det_B	993
D.3.2.11	gaussred	993
D.3.2.12	gaussred_pivot	994
D.3.2.13	gauss_nf	995
D.3.2.14	mat_rk	996
D.3.2.15	U_D_O	996
D.3.2.16	pos_def	997
D.3.2.17	hessenberg	998
D.3.2.18	eigenvals	998
D.3.2.19	minipoly	999
D.3.2.20	spnf	999
D.3.2.21	spprint	999
D.3.2.22	jordan	1000
D.3.2.23	jordanbasis	1000
D.3.2.24	jordanmatrix	1001
D.3.2.25	jordannf	1001
D.4	Commutative algebra	1002
D.4.1	absfact_lib	1002
D.4.1.1	absFactorize	1002
D.4.1.2	absFactorizeBCG	1003
D.4.2	algebra_lib	1004
D.4.2.1	algebra_containment	1005
D.4.2.2	module_containment	1006
D.4.2.3	inSubring	1007
D.4.2.4	algDependent	1007
D.4.2.5	alg_kernel	1008
D.4.2.6	is_injective	1009
D.4.2.7	is_surjective	1010
D.4.2.8	is_bijective	1010
D.4.2.9	noetherNormal	1011
D.4.2.10	mapIsFinite	1012
D.4.2.11	finitenessTest	1012
D.4.2.12	nonZeroEntry	1013
D.4.3	assprimeszerodim_lib	1014
D.4.3.1	zeroRadical	1014
D.4.3.2	assPrimes	1014
D.4.4	cisimplicial_lib	1046
D.4.4.1	minMult	1046
D.4.4.2	belongSemigroup	1047
D.4.4.3	oneDimBelongSemigroup	1047
D.4.4.4	cardGroup	1048
D.4.4.5	isCI	1048
D.4.5	curveInv_lib	1049
D.4.5.1	curveDeltaInv	1049
D.4.5.2	curveConductorMult	1050
D.4.5.3	curveDeligneNumber	1051
D.4.5.4	curveColengthDerivations	1051
D.4.6	decomp_lib	1052
D.4.6.1	decompopts	1054

	D.4.6.2	decompose	1055
	D.4.6.3	is_composite	1056
	D.4.6.4	chebyshev	1057
	D.4.6.5	compose	1058
	D.4.6.6	makedistinguished	1058
D.4.7	elim_lib		1059
	D.4.7.1	blowup0	1060
	D.4.7.2	elimRing	1061
	D.4.7.3	elim	1063
	D.4.7.4	elim1	1064
	D.4.7.5	elim2	1065
	D.4.7.6	nselect	1065
	D.4.7.7	sat	1066
	D.4.7.8	sat_with_exp	1066
	D.4.7.9	select	1067
	D.4.7.10	select1	1067
D.4.8	ellipticcovers_lib		1068
	D.4.8.1	makeGraph	1069
	D.4.8.2	printGraph	1069
	D.4.8.3	propagator	1069
	D.4.8.4	computeConstant	1070
	D.4.8.5	evalutateIntegral	1070
	D.4.8.6	gromovWitten	1071
	D.4.8.7	computeGromovWitten	1071
	D.4.8.8	partitions	1074
	D.4.8.9	permute	1076
	D.4.8.10	lsum	1080
D.4.9	ffmodstd_lib		1081
	D.4.9.1	fareypoly	1082
	D.4.9.2	polyInterpolation	1083
	D.4.9.3	modrationalInterpolation	1084
	D.4.9.4	BerlekampMassey	1085
	D.4.9.5	modberlekampMassey	1086
	D.4.9.6	sparseInterpolation	1087
	D.4.9.7	ffmodStd	1087
D.4.10	grwalk_lib		1088
	D.4.10.1	fwalk	1088
	D.4.10.2	twalk	1089
	D.4.10.3	awalk1	1089
	D.4.10.4	awalk2	1090
	D.4.10.5	pwalk	1090
	D.4.10.6	gwalk	1091
D.4.11	homolog_lib		1091
	D.4.11.1	canonMap	1092
	D.4.11.2	cup	1093
	D.4.11.3	cupproduct	1095
	D.4.11.4	depth	1099
	D.4.11.5	Ext_R	1099
	D.4.11.6	Ext	1101
	D.4.11.7	fitting	1104
	D.4.11.8	flatteningStrat	1104
	D.4.11.9	Hom	1105
	D.4.11.10	homology	1107

D.4.11.11	isCM	1108
D.4.11.12	isFlat	1108
D.4.11.13	isLocallyFree	1109
D.4.11.14	isReg	1109
D.4.11.15	hom_kernel	1110
D.4.11.16	kohom	1110
D.4.11.17	kontrahom	1111
D.4.11.18	KoszulHomology	1111
D.4.11.19	tensorMod	1112
D.4.11.20	Tor	1113
D.4.12	integralbasis_lib	1115
D.4.12.1	integralBasis	1115
D.4.12.2	polyDK	1117
D.4.12.3	monic	1118
D.4.12.4	henselGlobal	1118
D.4.13	intprog_lib	1119
D.4.13.1	solve_IP	1119
D.4.14	moddiq_lib	1121
D.4.14.1	modQuotient	1121
D.4.14.2	modSat	1122
D.4.15	modules_lib	1123
D.4.15.1	id	1123
D.4.15.2	zero	1124
D.4.15.3	freeModule	1124
D.4.15.4	makeMatrix	1124
D.4.15.5	makeIdeal	1125
D.4.15.6	Target	1125
D.4.15.7	Source	1125
D.4.15.8	printMatrix	1126
D.4.15.9	printFreeModule	1126
D.4.15.10	printResolution	1126
D.4.15.11	printModule	1127
D.4.15.12	printHom	1127
D.4.15.13	mRes	1128
D.4.15.14	sRes	1129
D.4.15.15	Res	1129
D.4.15.16	Betti	1130
D.4.15.17	printBetti	1131
D.4.15.18	SetDeg	1131
D.4.15.19	Deg	1131
D.4.15.20	Degree	1132
D.4.15.21	Degrees	1132
D.4.15.22	subquotient	1133
D.4.15.23	coker	1133
D.4.15.24	image	1134
D.4.15.25	Ker	1134
D.4.15.26	compareModules	1134
D.4.15.27	addModules	1135
D.4.15.28	homomorphism	1136
D.4.15.29	target	1136
D.4.15.30	source	1137
D.4.15.31	compareMatrix	1138
D.4.15.32	freeModule2Module	1138

D.4.15.33	makeVector	1139
D.4.15.34	netVector	1139
D.4.15.35	netMatrix	1139
D.4.15.36	presentation	1140
D.4.15.37	tensorMatrix	1140
D.4.15.38	tensorModule	1140
D.4.15.39	tensorModFreemod	1142
D.4.15.40	tensorFreemodMod	1142
D.4.15.41	tensorFreeModule	1142
D.4.15.42	tensorProduct	1143
D.4.15.43	pruneModule	1143
D.4.15.44	hom	1143
D.4.15.45	kerHom	1144
D.4.15.46	interpret	1144
D.4.15.47	interpretInv	1145
D.4.15.48	reduceIntChain	1145
D.4.15.49	interpretElem	1146
D.4.15.50	interpretList	1146
D.4.15.51	compareVectors	1146
D.4.15.52	simplePrune	1147
D.4.16	modstd_lib	1147
D.4.16.1	modStd	1147
D.4.16.2	modGB	1148
D.4.16.3	modSyz	1149
D.4.16.4	modIntersect	1149
D.4.17	monomialideal_lib	1150
D.4.17.1	isMonomial	1151
D.4.17.2	minbaseMon	1151
D.4.17.3	gcdMon	1151
D.4.17.4	lcmMon	1152
D.4.17.5	membershipMon	1152
D.4.17.6	intersectMon	1152
D.4.17.7	quotientMon	1153
D.4.17.8	radicalMon	1153
D.4.17.9	isprimeMon	1154
D.4.17.10	isprimaryMon	1154
D.4.17.11	isirreducibleMon	1154
D.4.17.12	isartinianMon	1155
D.4.17.13	isgenericMon	1155
D.4.17.14	dimMon	1155
D.4.17.15	irreddecMon	1156
D.4.17.16	primdecMon	1158
D.4.18	mprimdec_lib	1161
D.4.18.1	separator	1161
D.4.18.2	PrimdecA	1162
D.4.18.3	PrimdecB	1162
D.4.18.4	modDec	1163
D.4.18.5	zeroMod	1164
D.4.18.6	GTZmod	1164
D.4.18.7	dec1var	1165
D.4.18.8	annil	1166
D.4.18.9	splitting	1166
D.4.18.10	primTest	1167

D.4.18.11	preComp	1167
D.4.18.12	indSet	1168
D.4.18.13	GTZopt	1169
D.4.18.14	zeroOpt	1170
D.4.19	mregular_lib	1171
D.4.19.1	regIdeal	1171
D.4.19.2	depthIdeal	1172
D.4.19.3	satiety	1172
D.4.19.4	regMonCurve	1173
D.4.19.5	NoetherPosition	1173
D.4.19.6	is_NP	1174
D.4.19.7	is_nested	1175
D.4.20	nfmodstd_lib	1175
D.4.20.1	chinrempoly	1176
D.4.20.2	nfmodStd	1177
D.4.21	nfmodsyz_lib	1178
D.4.21.1	nfmodSyz	1179
D.4.22	noether_lib	1180
D.4.22.1	NPos_test	1180
D.4.22.2	modNpos_test	1181
D.4.22.3	NPos	1181
D.4.22.4	modNPos	1182
D.4.22.5	nsatiety	1182
D.4.22.6	modsatiety	1183
D.4.22.7	regCM	1183
D.4.22.8	modregCM	1184
D.4.23	normal_lib	1184
D.4.23.1	normal	1184
D.4.23.2	locNormal	1190
D.4.23.3	modNormal	1191
D.4.23.4	normalP	1192
D.4.23.5	normalC	1195
D.4.23.6	HomJJ	1200
D.4.23.7	genus	1201
D.4.23.8	primeClosure	1201
D.4.23.9	closureFrac	1203
D.4.23.10	iMult	1204
D.4.23.11	deltaLoc	1204
D.4.23.12	locAtZero	1205
D.4.23.13	norTest	1206
D.4.23.14	getSmallest	1207
D.4.23.15	getOneVar	1207
D.4.23.16	changeDenominator	1207
D.4.23.17	normalConductor	1208
D.4.23.18	isNormal	1210
D.4.24	normaliz_lib	1210
D.4.24.1	intclToricRing	1211
D.4.24.2	normalToricRing	1212
D.4.24.3	normalToricRingFromBinomials	1213
D.4.24.4	toricRingFromBinomials	1214
D.4.24.5	ehrhartRing	1214
D.4.24.6	intclMonIdeal	1215
D.4.24.7	definingBinomialIdeal	1216

D.4.24.8	latticeIdeal	1217
D.4.24.9	groebnerBasis	1217
D.4.24.10	torusInvariants	1218
D.4.24.11	finiteDiagInvariants	1219
D.4.24.12	diagInvariants	1220
D.4.24.13	intersectionValRings	1221
D.4.24.14	intersectionValRingIdeals	1221
D.4.24.15	showNuminvs	1222
D.4.24.16	exportNuminvs	1223
D.4.24.17	allNmzOptions	1224
D.4.24.18	setNmzOption	1225
D.4.24.19	addNmzOption	1226
D.4.24.20	showNmzOptions	1226
D.4.24.21	resetNmzOptions	1226
D.4.24.22	normaliz	1227
D.4.24.23	setNmzExecPath	1228
D.4.24.24	writeNmzData	1228
D.4.24.25	readNmzData	1229
D.4.24.26	setNmzFilename	1230
D.4.24.27	setNmzDataPath	1230
D.4.24.28	writeNmzPaths	1230
D.4.24.29	startNmz	1231
D.4.24.30	rmNmzFiles	1231
D.4.24.31	mons2intmat	1231
D.4.24.32	intmat2mons	1232
D.4.24.33	binomials2intmat	1232
D.4.24.34	intmat2binomials	1233
D.4.25	pointid_lib	1233
D.4.25.1	nonMonomials	1234
D.4.25.2	cornerMonomials	1235
D.4.25.3	facGBIdeal	1236
D.4.26	primdec_lib	1240
D.4.26.1	Ann	1240
D.4.26.2	primdecGTZ	1241
D.4.26.3	primdecGTZE	1241
D.4.26.4	primdecSY	1242
D.4.26.5	primdecSYE	1243
D.4.26.6	minAssGTZ	1244
D.4.26.7	minAssGTZE	1245
D.4.26.8	minAssChar	1245
D.4.26.9	minAssCharE	1246
D.4.26.10	testPrimary	1247
D.4.26.11	testPrimaryE	1247
D.4.26.12	radical	1247
D.4.26.13	radicalEHV	1248
D.4.26.14	equiRadical	1248
D.4.26.15	prepareAss	1249
D.4.26.16	equidim	1249
D.4.26.17	equidimMax	1250
D.4.26.18	equidimMaxEHV	1250
D.4.26.19	zerodec	1250
D.4.26.20	absPrimdecGTZ	1251
D.4.26.21	absPrimdecGTZE	1252

	D.4.26.22	sep	1253
D.4.27		primdecint_lib	1253
	D.4.27.1	primdecZ	1253
	D.4.27.2	primdecZM	1255
	D.4.27.3	minAssZ	1256
	D.4.27.4	radicalZ	1257
	D.4.27.5	heightZ	1258
	D.4.27.6	equidimZ	1258
	D.4.27.7	intersectZ	1259
D.4.28		primitiv_lib	1260
	D.4.28.1	primitive	1260
	D.4.28.2	primitive_extra	1261
	D.4.28.3	splitring	1262
D.4.29		realrad_lib	1263
	D.4.29.1	realpoly	1263
	D.4.29.2	realzero	1264
	D.4.29.3	realrad	1264
D.4.30		reesclos_lib	1265
	D.4.30.1	ReesAlgebra	1265
	D.4.30.2	normalI	1266
D.4.31		rstandard_lib	1266
	D.4.31.1	borderBasis	1267
	D.4.31.2	modBorder	1271
	D.4.31.3	rJanet	1306
	D.4.31.4	modJanet	1376
D.4.32		sagbi_lib	1377
	D.4.32.1	sagbiSPoly	1378
	D.4.32.2	sagbiReduce	1378
	D.4.32.3	sagbi	1379
	D.4.32.4	sagbiPart	1380
	D.4.32.5	algebraicDependence	1381
D.4.33		sing4ti2_lib	1382
	D.4.33.1	markov4ti2	1382
	D.4.33.2	hilbert4ti2	1383
	D.4.33.3	graver4ti2	1383
D.4.34		symodstd_lib	1384
	D.4.34.1	genSymId	1384
	D.4.34.2	isSymmetric	1385
	D.4.34.3	primRoot	1385
	D.4.34.4	eigenvalues	1385
	D.4.34.5	symmStd	1386
	D.4.34.6	syModStd	1387
D.4.35		toric_lib	1388
	D.4.35.1	toric_ideal	1388
	D.4.35.2	toric_std	1389
D.5		Algebraic geometry	1390
	D.5.1	brillnoether_lib	1390
		D.5.1.1 RiemannRochBN	1391
	D.5.2	chern_lib	1391
		D.5.2.1 symm	1392
		D.5.2.2 symNsym	1392
		D.5.2.3 CompleteHomog	1393
		D.5.2.4 segre	1393

D.5.2.5	chern	1394
D.5.2.6	chNum	1395
D.5.2.7	chNumbers	1395
D.5.2.8	sum_of_powers	1396
D.5.2.9	powSumSym	1396
D.5.2.10	chAll	1397
D.5.2.11	chAllInv	1398
D.5.2.12	chHE	1398
D.5.2.13	ChernRootsSum	1399
D.5.2.14	chSum	1399
D.5.2.15	ChernRootsDual	1400
D.5.2.16	chDual	1400
D.5.2.17	ChernRootsProd	1401
D.5.2.18	chProd	1401
D.5.2.19	chProdE	1402
D.5.2.20	chProdL	1403
D.5.2.21	chProdLP	1404
D.5.2.22	chProdM	1404
D.5.2.23	chProdMP	1405
D.5.2.24	ChernRootsHom	1405
D.5.2.25	chHom	1406
D.5.2.26	ChernRootsSymm	1407
D.5.2.27	ChernRootsWedge	1407
D.5.2.28	chSymm	1408
D.5.2.29	chSymm2L	1409
D.5.2.30	chSymm2LP	1409
D.5.2.31	chWedge	1409
D.5.2.32	chWedge2L	1410
D.5.2.33	chWedge2LP	1411
D.5.2.34	todd	1411
D.5.2.35	toddE	1412
D.5.2.36	Bern	1412
D.5.2.37	tdCf	1413
D.5.2.38	tdTerms	1414
D.5.2.39	tdFactor	1414
D.5.2.40	cProj	1414
D.5.2.41	chProj	1415
D.5.2.42	tdProj	1416
D.5.2.43	eulerChProj	1417
D.5.2.44	chNumbersProj	1417
D.5.2.45	classpoly	1418
D.5.2.46	chernPoly	1418
D.5.2.47	chernCharPoly	1418
D.5.2.48	toddPoly	1419
D.5.2.49	rHRR	1419
D.5.2.50	SchurS	1419
D.5.2.51	SchurCh	1420
D.5.2.52	part	1420
D.5.2.53	dualPart	1421
D.5.2.54	PartC	1422
D.5.2.55	partOver	1422
D.5.2.56	partUnder	1423
D.5.2.57	SegreA	1423



	D.5.2.58	FultonA	1424
	D.5.2.59	CSMA	1425
	D.5.2.60	EulerAff	1426
	D.5.2.61	EulerProj	1426
D.5.3		deRham_lib	1427
	D.5.3.1	deRhamCohomology	1427
	D.5.3.2	MVComplex	1428
D.5.4		divisors_lib	1429
	D.5.4.1	makeDivisor	1430
	D.5.4.2	divisorplus	1430
	D.5.4.3	multdivisor	1431
	D.5.4.4	negativedivisor	1431
	D.5.4.5	normalForm	1432
	D.5.4.6	isEqualDivisor	1432
	D.5.4.7	globalSections	1432
	D.5.4.8	degreeDivisor	1433
	D.5.4.9	linearlyEquivalent	1433
	D.5.4.10	effective	1434
	D.5.4.11	makeFormalDivisor	1434
	D.5.4.12	evaluateFormalDivisor	1435
	D.5.4.13	formaldivisorplus	1435
	D.5.4.14	negativeformaldivisor	1436
	D.5.4.15	multformaldivisor	1436
	D.5.4.16	degreeFormalDivisor	1437
	D.5.4.17	makePDivisor	1437
D.5.5		goettsche_lib	1438
	D.5.5.1	GoettscheF	1438
	D.5.5.2	PPolyH	1439
	D.5.5.3	BettiNumsH	1439
	D.5.5.4	NakYoshF	1440
	D.5.5.5	PPolyQp	1440
	D.5.5.6	BettiNumsQp	1441
	D.5.5.7	MacdonaldF	1441
	D.5.5.8	PPolyS	1441
	D.5.5.9	BettiNumsS	1442
	D.5.5.10	PPolyN	1443
	D.5.5.11	BettiNumsN	1443
D.5.6		graal_lib	1444
	D.5.6.1	graalMixed	1444
	D.5.6.2	dimensionOfLocalization	1445
	D.5.6.3	systemOfParametersOfLocalization	1446
	D.5.6.4	isLocalizationRegular	1446
	D.5.6.5	warkedPreimageStd	1446
	D.5.6.6	resolutionInLocalization	1448
D.5.7		hess_lib	1449
	D.5.7.1	RiemannRochHess	1449
D.5.8		numerAlg_lib	1451
	D.5.8.1	Incl	1452
	D.5.8.2	Equal	1452
	D.5.8.3	DegreePure	1453
	D.5.8.4	NumLocalDim	1453
D.5.9		numerDecom_lib	1454
	D.5.9.1	re2squ	1454

D.5.9.2	UseBertini	1455
D.5.9.3	Singular2bertini	1455
D.5.9.4	bertini2Singular	1455
D.5.9.5	ReJunkUseHomo	1455
D.5.9.6	JuReTopDim	1456
D.5.9.7	JuReZeroDim	1456
D.5.9.8	WitSupSet	1457
D.5.9.9	WitSet	1457
D.5.9.10	NumIrrDecom	1458
D.5.9.11	defl	1459
D.5.9.12	NumPrimDecom	1459
D.5.10	orbitparam_lib	1459
D.5.10.1	tangentGens	1460
D.5.10.2	matrixExp	1461
D.5.10.3	matrixLog	1462
D.5.10.4	parametrizeOrbit	1462
D.5.10.5	maxZeros	1463
D.5.11	paraplanecurves_lib	1464
D.5.11.1	adjointIdeal	1465
D.5.11.2	invertBirMap	1466
D.5.11.3	paraPlaneCurve	1467
D.5.11.4	rncAntiCanonicalMap	1469
D.5.11.5	rationalPointConic	1470
D.5.11.6	mapToRatNormCurve	1471
D.5.11.7	rncItProjOdd	1472
D.5.11.8	rncItProjEven	1473
D.5.11.9	paraConic	1474
D.5.11.10	testParametrization	1475
D.5.11.11	testPointConic	1476
D.5.12	resbinomial_lib	1476
D.5.12.1	BINresol	1477
D.5.12.2	Eresol	1480
D.5.12.3	determinecenter	1481
D.5.12.4	Blowupcenter	1488
D.5.12.5	Nonhyp	1495
D.5.12.6	identifyvar	1499
D.5.12.7	Edatalist	1499
D.5.12.8	EOrdlist	1501
D.5.12.9	maxEord	1501
D.5.12.10	ECoef	1502
D.5.12.11	elimrep	1508
D.5.12.12	Emaxcont	1509
D.5.12.13	cleanunit	1510
D.5.12.14	resfunction	1510
D.5.12.15	calculateI	1512
D.5.12.16	Maxord	1513
D.5.12.17	Gamma	1513
D.5.12.18	convertdata	1514
D.5.12.19	lcmofall	1515
D.5.12.20	computemcm	1515
D.5.12.21	constructH	1516
D.5.12.22	constructblwup	1516
D.5.12.23	constructlastblwup	1517

D.5.12.24	genoutput	1517
D.5.12.25	salida	1521
D.5.12.26	iniD	1521
D.5.12.27	sumlist	1522
D.5.12.28	reslist	1522
D.5.12.29	multiplylist	1523
D.5.12.30	dividelist	1523
D.5.12.31	createlist	1524
D.5.13	resgraph_lib	1524
D.5.13.1	InterDiv	1524
D.5.13.2	ResTree	1525
D.5.13.3	finalCharts	1525
D.5.14	resjung_lib	1526
D.5.14.1	jungresolve	1526
D.5.14.2	jungnormal	1527
D.5.14.3	jungfib	1528
D.5.15	resolve_lib	1528
D.5.15.1	blowUp	1529
D.5.15.2	blowUp2	1530
D.5.15.3	Center	1531
D.5.15.4	resolve	1531
D.5.15.5	showBO	1532
D.5.15.6	presentTree	1533
D.5.15.7	showDataTypes	1533
D.5.15.8	blowUpBO	1533
D.5.15.9	createBO	1535
D.5.15.10	CenterBO	1536
D.5.15.11	Delta	1536
D.5.15.12	DeltaList	1537
D.5.16	reszeta_lib	1538
D.5.16.1	intersectionDiv	1538
D.5.16.2	spectralNeg	1539
D.5.16.3	discrepancy	1540
D.5.16.4	zetaDL	1541
D.5.16.5	collectDiv	1541
D.5.16.6	prepEmbDiv	1544
D.5.16.7	abstractR	1546
D.5.16.8	computeV	1547
D.5.16.9	computeN	1567
D.5.17	schubert_lib	1587
D.5.17.1	makeVariety	1587
D.5.17.2	printVariety	1588
D.5.17.3	productVariety	1588
D.5.17.4	ChowRing	1589
D.5.17.5	Grassmannian	1589
D.5.17.6	projectiveSpace	1590
D.5.17.7	projectiveBundle	1591
D.5.17.8	integral	1591
D.5.17.9	makeSheaf	1592
D.5.17.10	printSheaf	1592
D.5.17.11	rankSheaf	1593
D.5.17.12	totalChernClass	1593
D.5.17.13	ChernClass	1594

D.5.17.14	topChernClass	1594
D.5.17.15	totalSegreClass	1595
D.5.17.16	dualSheaf	1595
D.5.17.17	tensorSheaf	1595
D.5.17.18	symmetricPowerSheaf	1596
D.5.17.19	quotSheaf	1596
D.5.17.20	addSheaf	1597
D.5.17.21	makeGraphVE	1597
D.5.17.22	printGraphG	1598
D.5.17.23	moduliSpace	1598
D.5.17.24	printStack	1598
D.5.17.25	dimStack	1599
D.5.17.26	fixedPoints	1599
D.5.17.27	contributionBundle	1600
D.5.17.28	normalBundle	1600
D.5.17.29	multipleCover	1600
D.5.17.30	linesHypersurface	1601
D.5.17.31	rationalCurve	1602
D.5.17.32	sumofquotients	1602
D.5.17.33	homog_part	1603
D.5.17.34	homog_parts	1603
D.5.17.35	logg	1603
D.5.17.36	expp	1604
D.5.17.37	SchubertClass	1604
D.5.17.38	dualPartition	1604
D.5.18	sheafcoh_lib	1605
D.5.18.1	truncate	1605
D.5.18.2	truncateFast	1606
D.5.18.3	CM_regularity	1609
D.5.18.4	sheafCohBGG	1609
D.5.18.5	sheafCohBGGregul	1610
D.5.18.6	sheafCohBGGregul_w	1611
D.5.18.7	sheafCohBGG2	1612
D.5.18.8	sheafCoh	1616
D.5.18.9	sheafCoh_w	1618
D.5.18.10	dimH	1619
D.5.18.11	dimGradedPart	1620
D.5.18.12	displayCohom	1620
D.5.19	JMBTest_lib	1621
D.5.19.1	Minimus	1621
D.5.19.2	Maximus	1622
D.5.19.3	StartOrderingV	1622
D.5.19.4	TestJMark	1623
D.5.20	JMConst_lib	1623
D.5.20.1	BorelCheck	1624
D.5.20.2	JMarkedScheme	1624
D.6	Singularities	1626
D.6.1	alexpoly_lib	1627
D.6.1.1	resolutiongraph	1627
D.6.1.2	totalmultiplicities	1628
D.6.1.3	alexanderpolynomial	1629
D.6.1.4	semigroup	1630
D.6.1.5	proximitymatrix	1632

D.6.1.6	multseq2charexp	1634
D.6.1.7	charexp2multseq	1634
D.6.1.8	charexp2generators	1635
D.6.1.9	charexp2inter	1635
D.6.1.10	charexp2conductor	1636
D.6.1.11	charexp2poly	1636
D.6.1.12	tau_es2	1636
D.6.2	arcpoint_lib	1637
D.6.2.1	nashmult	1637
D.6.2.2	removepower	1638
D.6.2.3	idealsimplify	1639
D.6.2.4	equalJinI	1639
D.6.3	arnoldclassify_lib	1640
D.6.3.1	arnoldListAllSeries	1640
D.6.3.2	arnoldShowSeries	1641
D.6.3.3	arnoldNormalForm	1642
D.6.3.4	arnoldClassify	1642
D.6.3.5	arnoldClassify_to_string	1644
D.6.3.6	arnoldCorank	1645
D.6.3.7	arnoldDeterminacy	1645
D.6.3.8	arnoldMilnorCode	1645
D.6.3.9	arnoldMorseSplit	1646
D.6.4	classify_lib	1646
D.6.4.1	basicinvariants	1646
D.6.4.2	classify	1647
D.6.4.3	corank	1648
D.6.4.4	Hcode	1648
D.6.4.5	init_debug	1649
D.6.4.6	internalfunctions	1649
D.6.4.7	milnorcode	1650
D.6.4.8	morsesplit	1651
D.6.4.9	quickclass	1651
D.6.4.10	singularity	1652
D.6.4.11	A_L	1652
D.6.4.12	normalform	1653
D.6.4.13	debug_log	1653
D.6.4.14	swap	1653
D.6.4.15	modality	1654
D.6.4.16	complexSingType	1654
D.6.4.17	prepRealclassify	1654
D.6.5	classify2_lib	1655
D.6.5.1	complexClassify	1655
D.6.5.2	complexType	1656
D.6.6	classify_aeq_lib	1657
D.6.6.1	sagbiAlg	1657
D.6.6.2	sagbiMod	1658
D.6.6.3	semiGroup	1658
D.6.6.4	semiMod	1659
D.6.6.5	planeCur	1659
D.6.6.6	spaceCur	1659
D.6.6.7	HHnormalForm	1660
D.6.6.8	modSagbiAlg	1660
D.6.6.9	classSpaceCurve	1661

D.6.7	classifyceq_lib . . . . .	1661
	D.6.7.1 classifyCeq . . . . .	1662
D.6.8	classifyci_lib . . . . .	1662
	D.6.8.1 classifyicis . . . . .	1662
	D.6.8.2 Semigroup . . . . .	1663
D.6.9	classifyMapGerms_lib . . . . .	1663
	D.6.9.1 coDimMap . . . . .	1663
	D.6.9.2 coDim . . . . .	1664
	D.6.9.3 vStd . . . . .	1665
	D.6.9.4 modVStd . . . . .	1668
	D.6.9.5 modVStd0 . . . . .	1675
	D.6.9.6 classifySimpleMaps . . . . .	1681
	D.6.9.7 classifySimpleMaps1 . . . . .	1681
	D.6.9.8 classifyUnimodalMaps . . . . .	1682
D.6.10	curvepar_lib . . . . .	1682
	D.6.10.1 BlowingUp . . . . .	1682
	D.6.10.2 CurveRes . . . . .	1683
	D.6.10.3 CurveParam . . . . .	1686
	D.6.10.4 WSemigroup . . . . .	1687
	D.6.10.5 primparam . . . . .	1687
	D.6.10.6 MultiplicitySequence . . . . .	1689
	D.6.10.7 CharacteristicExponents . . . . .	1689
	D.6.10.8 IntersectionMatrix . . . . .	1690
	D.6.10.9 ContactMatrix . . . . .	1690
	D.6.10.10 plainInvariants . . . . .	1690
D.6.11	deform_lib . . . . .	1692
	D.6.11.1 versal . . . . .	1693
	D.6.11.2 mod_versal . . . . .	1695
	D.6.11.3 lift_kbase . . . . .	1696
	D.6.11.4 lift_rel_kb . . . . .	1697
D.6.12	equising_lib . . . . .	1697
	D.6.12.1 tau_es . . . . .	1698
	D.6.12.2 esIdeal . . . . .	1698
	D.6.12.3 esStratum . . . . .	1699
	D.6.12.4 isEquising . . . . .	1702
	D.6.12.5 control_Matrix . . . . .	1703
D.6.13	gmssing_lib . . . . .	1703
	D.6.13.1 gmsring . . . . .	1704
	D.6.13.2 gmsnf . . . . .	1705
	D.6.13.3 gmscoeffs . . . . .	1705
	D.6.13.4 bernstein . . . . .	1706
	D.6.13.5 monodromy . . . . .	1707
	D.6.13.6 spectrum . . . . .	1707
	D.6.13.7 sppairs . . . . .	1708
	D.6.13.8 vfilt . . . . .	1708
	D.6.13.9 vwfilt . . . . .	1710
	D.6.13.10 tmatrix . . . . .	1711
	D.6.13.11 endvfilt . . . . .	1712
	D.6.13.12 sppnf . . . . .	1714
	D.6.13.13 sppprint . . . . .	1714
	D.6.13.14 spadd . . . . .	1714
	D.6.13.15 spsub . . . . .	1715
	D.6.13.16 spmul . . . . .	1715

	D.6.13.17	spissemicont .....	1716
	D.6.13.18	spsemicont .....	1716
	D.6.13.19	spmilnor .....	1717
	D.6.13.20	spgeomgenus .....	1717
	D.6.13.21	spgamma .....	1717
D.6.14	gmspoly_lib .....	1718	
	D.6.14.1	isTame .....	1718
	D.6.14.2	goodBasis .....	1719
D.6.15	hnoether_lib .....	1719	
	D.6.15.1	hnexpansion .....	1720
	D.6.15.2	develop .....	1722
	D.6.15.3	extdevelop .....	1723
	D.6.15.4	param .....	1724
	D.6.15.5	displayHNE .....	1726
	D.6.15.6	invariants .....	1727
	D.6.15.7	displayInvariants .....	1728
	D.6.15.8	multsequence .....	1728
	D.6.15.9	displayMultsequence .....	1730
	D.6.15.10	intersection .....	1731
	D.6.15.11	is_irred .....	1731
	D.6.15.12	delta .....	1731
	D.6.15.13	newtonpoly .....	1732
	D.6.15.14	is_NND .....	1732
	D.6.15.15	stripHNE .....	1733
	D.6.15.16	puiseux2generators .....	1734
	D.6.15.17	separateHNE .....	1734
	D.6.15.18	squarefree .....	1735
	D.6.15.19	allsquarefree .....	1735
	D.6.15.20	further_hn_proc .....	1735
D.6.16	kskernel_lib .....	1736	
	D.6.16.1	KSkер .....	1736
	D.6.16.2	KSconvert .....	1737
	D.6.16.3	KSlinear .....	1737
	D.6.16.4	KScoef .....	1738
	D.6.16.5	StringF .....	1738
D.6.17	mondromy_lib .....	1738	
	D.6.17.1	detadj .....	1739
	D.6.17.2	invunit .....	1739
	D.6.17.3	jacoblift .....	1740
	D.6.17.4	monodromyB .....	1740
	D.6.17.5	H2basis .....	1741
D.6.18	qhmoduli_lib .....	1742	
	D.6.18.1	ArnoldAction .....	1742
	D.6.18.2	ModEqn .....	1742
	D.6.18.3	QuotientEquations .....	1743
	D.6.18.4	StabEqn .....	1743
	D.6.18.5	StabEqnId .....	1744
	D.6.18.6	StabOrder .....	1745
	D.6.18.7	UpperMonomials .....	1745
	D.6.18.8	Max .....	1745
	D.6.18.9	Min .....	1746
D.6.19	realclassify_lib .....	1746	
	D.6.19.1	realclassify .....	1747

	D.6.19.2	realmorsesplit	1754
	D.6.19.3	milnornumber	1755
	D.6.19.4	determinacy	1755
	D.6.19.5	addnondegeneratevariables	1755
	D.6.19.6	HilbertClassPoly	1756
D.6.20	sing_lib		1756
	D.6.20.1	codim	1757
	D.6.20.2	deform	1757
	D.6.20.3	dim_slocus	1757
	D.6.20.4	is_active	1758
	D.6.20.5	is_ci	1758
	D.6.20.6	is_is	1759
	D.6.20.7	is_reg	1759
	D.6.20.8	is_regs	1760
	D.6.20.9	locstd	1760
	D.6.20.10	milnor	1761
	D.6.20.11	nf_icis	1761
	D.6.20.12	slocus	1762
	D.6.20.13	qhspectrum	1762
	D.6.20.14	Tjurina	1763
	D.6.20.15	tjurina	1764
	D.6.20.16	T_1	1765
	D.6.20.17	T_2	1766
	D.6.20.18	T_12	1767
	D.6.20.19	tangentcone	1767
D.6.21	spcurve_lib		1768
	D.6.21.1	isCMcod2	1768
	D.6.21.2	CMtype	1768
	D.6.21.3	matrixT1	1769
	D.6.21.4	semiCMcod2	1769
	D.6.21.5	discr	1770
	D.6.21.6	qhmatrix	1770
	D.6.21.7	relweight	1771
	D.6.21.8	posweight	1771
	D.6.21.9	KSpencerKernel	1772
D.6.22	spectrum_lib		1773
	D.6.22.1	spectrumnd	1773
D.6.23	surfacesignature_lib		1773
	D.6.23.1	signatureBrieskorn	1774
	D.6.23.2	signaturePuisseux	1775
	D.6.23.3	signatureNemethi	1775
D.7	Invariant theory		1775
	D.7.1	finvar_lib	1775
		D.7.1.1	invariant_ring
		D.7.1.2	invariant_ring_random
		D.7.1.3	primary_invariants
		D.7.1.4	primary_invariants_random
		D.7.1.5	invariant_algebra_reynolds
		D.7.1.6	invariant_algebra_perm
		D.7.1.7	cyclotomic
		D.7.1.8	group_reynolds
		D.7.1.9	molien
		D.7.1.10	reynolds_molien



D.7.1.11	partial_molien	1784
D.7.1.12	evaluate_reynolds	1785
D.7.1.13	invariant_basis	1785
D.7.1.14	invariant_basis_reynolds	1786
D.7.1.15	primary_char0	1786
D.7.1.16	primary_charp	1787
D.7.1.17	primary_char0_no_molien	1787
D.7.1.18	primary_charp_no_molien	1788
D.7.1.19	primary_charp_without	1788
D.7.1.20	primary_char0_random	1789
D.7.1.21	primary_charp_random	1789
D.7.1.22	primary_char0_no_molien_random	1790
D.7.1.23	primary_charp_no_molien_random	1790
D.7.1.24	primary_charp_without_random	1791
D.7.1.25	power_products	1791
D.7.1.26	secondary_char0	1792
D.7.1.27	irred_secondary_char0	1793
D.7.1.28	secondary_charp	1794
D.7.1.29	secondary_no_molien	1795
D.7.1.30	irred_secondary_no_molien	1796
D.7.1.31	secondary_and_irreducibles_no_molien	1797
D.7.1.32	secondary_not_cohen_macaulay	1798
D.7.1.33	orbit_variety	1799
D.7.1.34	rel_orbit_variety	1799
D.7.1.35	relative_orbit_variety	1801
D.7.1.36	image_of_variety	1802
D.7.2	ainvar_lib	1802
D.7.2.1	invariantRing	1802
D.7.2.2	derivate	1803
D.7.2.3	actionIsProper	1804
D.7.2.4	reduction	1804
D.7.2.5	completeReduction	1805
D.7.2.6	localInvar	1805
D.7.2.7	furtherInvar	1806
D.7.2.8	sortier	1806
D.7.3	rinvar_lib	1807
D.7.3.1	HilbertSeries	1807
D.7.3.2	HilbertWeights	1807
D.7.3.3	ImageVariety	1807
D.7.3.4	ImageGroup	1808
D.7.3.5	InvariantRing	1809
D.7.3.6	InvariantQ	1810
D.7.3.7	LinearizeAction	1810
D.7.3.8	LinearActionQ	1811
D.7.3.9	LinearCombinationQ	1811
D.7.3.10	MinimalDecomposition	1812
D.7.3.11	NullCone	1812
D.7.3.12	ReynoldsImage	1813
D.7.3.13	ReynoldsOperator	1813
D.7.3.14	SimplifyIdeal	1813
D.7.4	invar_lib	1813
D.7.4.1	SL	1814
D.7.4.2	torus	1814

	D.7.4.3	torusrep	1815
	D.7.4.4	finiterep	1815
	D.7.4.5	sympower	1816
	D.7.4.6	invar	1817
	D.7.4.7	SLreynolds	1818
	D.7.4.8	torusreynolds	1818
D.7.5	stratify_lib		1818
	D.7.5.1	prepMat	1818
	D.7.5.2	stratify	1819
D.8	Symbolic-numerical solving		1821
D.8.1	ffsolve_lib		1821
	D.8.1.1	ffsolve	1821
	D.8.1.2	PEsolve	1821
	D.8.1.3	simplesolver	1823
	D.8.1.4	GBsolve	1823
	D.8.1.5	XLsolve	1824
	D.8.1.6	ZZsolve	1824
D.8.2	interval_lib		1825
	D.8.2.1	length2	1826
	D.8.2.2	bounds2	1826
	D.8.2.3	intervalmatrixInit	1826
	D.8.2.4	unitMatrix2	1826
	D.8.2.5	applyMatrix	1827
	D.8.2.6	ivmatGaussian2	1827
	D.8.2.7	evalPolyAtBox2	1828
	D.8.2.8	exclusionTest	1828
D.8.3	presolve_lib		1829
	D.8.3.1	degreepart	1829
	D.8.3.2	elimlinearpart	1830
	D.8.3.3	elimpart	1830
	D.8.3.4	elimpartanyr	1832
	D.8.3.5	fastelim	1833
	D.8.3.6	findvars	1834
	D.8.3.7	hilbvec	1834
	D.8.3.8	linearpart	1835
	D.8.3.9	tolessvars	1835
	D.8.3.10	solvelinearpart	1836
	D.8.3.11	sortandmap	1837
	D.8.3.12	sortvars	1838
	D.8.3.13	valvars	1839
	D.8.3.14	idealSplit	1841
D.8.4	solve_lib		1843
	D.8.4.1	laguerre_solve	1843
	D.8.4.2	solve	1845
	D.8.4.3	ures_solve	1847
	D.8.4.4	mp_res_mat	1848
	D.8.4.5	interpolate	1849
	D.8.4.6	fglm_solve	1850
	D.8.4.7	lex_solve	1851
	D.8.4.8	simplexOut	1852
	D.8.4.9	triangLf_solve	1852
	D.8.4.10	triangM_solve	1853
	D.8.4.11	triangL_solve	1854

	D.8.4.12	triang_solve	1855
D.8.5		triang_lib	1856
	D.8.5.1	triangL	1856
	D.8.5.2	triangLfak	1857
	D.8.5.3	triangM	1859
	D.8.5.4	triangMH	1860
D.8.6		ntsolve_lib	1861
	D.8.6.1	nt_solve	1861
	D.8.6.2	triMNewton	1861
D.8.7		recover_lib	1862
	D.8.7.1	substAll	1862
	D.8.7.2	veronese	1863
	D.8.7.3	getRelations	1863
	D.8.7.4	getRelationsRadical	1864
	D.8.7.5	gaussRowWithoutPerm	1864
	D.8.7.6	gaussColWithoutPerm	1865
	D.8.7.7	getWitnessSet	1865
	D.8.7.8	writeBertiniInput	1867
	D.8.7.9	num_prime_decom	1868
	D.8.7.10	num_prime_decom1	1870
	D.8.7.11	num_radical_via_decom	1871
	D.8.7.12	num_radical_via_randlincom	1875
	D.8.7.13	num_radical1	1878
	D.8.7.14	num_radical2	1881
	D.8.7.15	num_elim	1885
	D.8.7.16	num_elim1	1887
	D.8.7.17	realLLL	1890
D.8.8		rootisolation_lib	1890
	D.8.8.1	bounds	1891
	D.8.8.2	length	1892
	D.8.8.3	boxSet	1892
	D.8.8.4	ivmatInit	1892
	D.8.8.5	ivmatSet	1893
	D.8.8.6	unitMatrix	1893
	D.8.8.7	ivmatGaussian	1893
	D.8.8.8	evalPolyAtBox	1894
	D.8.8.9	evalJacobianAtBox	1894
	D.8.8.10	rootIsolationNoPreprocessing	1895
	D.8.8.11	rootIsolation	1895
	D.8.8.12	rootIsolationPrimdec	1896
D.8.9		signcond_lib	1897
	D.8.9.1	signcnd	1897
	D.8.9.2	psigncnd	1898
	D.8.9.3	firstoct	1898
D.8.10		zeroset_lib	1899
	D.8.10.1	Quotient	1899
	D.8.10.2	remainder	1900
	D.8.10.3	roots	1900
	D.8.10.4	sqfrNorm	1901
	D.8.10.5	zeroSet	1901
	D.8.10.6	egcdMain	1903
	D.8.10.7	factorMain	1903
	D.8.10.8	invertNumberMain	1903

	D.8.10.9	quotientMain	1903
	D.8.10.10	remainderMain	1904
	D.8.10.11	rootsMain	1904
	D.8.10.12	sqfrNormMain	1904
	D.8.10.13	containedQ	1904
	D.8.10.14	sameQ	1905
D.9	Visualization		1905
	D.9.1	graphics_lib	1905
		D.9.1.1 staircase	1905
		D.9.1.2 mathinit	1906
		D.9.1.3 mplot	1906
	D.9.2	latex_lib	1907
		D.9.2.1 closetex	1908
		D.9.2.2 opentex	1908
		D.9.2.3 tex	1908
		D.9.2.4 texdemo	1909
		D.9.2.5 texfactorize	1909
		D.9.2.6 texmap	1910
		D.9.2.7 texname	1911
		D.9.2.8 texobj	1911
		D.9.2.9 texpoly	1913
		D.9.2.10 texproc	1913
		D.9.2.11 texring	1915
		D.9.2.12 rmx	1916
		D.9.2.13 xdvi	1917
	D.9.3	surf_lib	1918
		D.9.3.1 plot	1918
		D.9.3.2 surfer	1919
	D.9.4	surfex_lib	1919
		D.9.4.1 plotRotated	1920
		D.9.4.2 plotRot	1920
		D.9.4.3 plotRotatedList	1921
		D.9.4.4 plotRotatedDirect	1921
		D.9.4.5 plotRotatedListFromSpecifyList	1922
D.10	Coding theory		1922
	D.10.1	brnoeth_lib	1922
		D.10.1.1 Adj_div	1923
		D.10.1.2 NSplaces	1926
		D.10.1.3 BrillNoether	1927
		D.10.1.4 Weierstrass	1928
		D.10.1.5 extcurve	1930
		D.10.1.6 AGcode_L	1931
		D.10.1.7 AGcode_Omega	1932
		D.10.1.8 prepSV	1933
		D.10.1.9 decodeSV	1934
		D.10.1.10 closed_points	1935
		D.10.1.11 dual_code	1936
		D.10.1.12 sys_code	1936
		D.10.1.13 permute_L	1937
	D.10.2	decodegb_lib	1938
		D.10.2.1 sysCRHT	1938
		D.10.2.2 sysCRHTMindist	1940
		D.10.2.3 sysNewton	1942

	D.10.2.4	sysBin	1945
	D.10.2.5	encode	1946
	D.10.2.6	syndrome	1946
	D.10.2.7	sysQE	1947
	D.10.2.8	errorInsert	1948
	D.10.2.9	errorRand	1949
	D.10.2.10	randomCheck	1949
	D.10.2.11	genMDSMat	1950
	D.10.2.12	mindist	1951
	D.10.2.13	decode	1951
	D.10.2.14	decodeRandom	1952
	D.10.2.15	decodeCode	1956
	D.10.2.16	vanishId	1959
	D.10.2.17	sysFL	1960
	D.10.2.18	decodeRandomFL	1964
D.11		System and Control theory	1966
	D.11.1	Control theory background	1966
	D.11.2	control_lib	1967
		D.11.2.1 control	1967
		D.11.2.2 controlDim	1968
		D.11.2.3 autonom	1969
		D.11.2.4 autonomDim	1970
		D.11.2.5 leftKernel	1971
		D.11.2.6 rightKernel	1971
		D.11.2.7 leftInverse	1972
		D.11.2.8 rightInverse	1972
		D.11.2.9 colrank	1973
		D.11.2.10 genericity	1973
		D.11.2.11 canonize	1974
		D.11.2.12 iostruct	1975
		D.11.2.13 findTorsion	1976
		D.11.2.14 controlExample	1977
		D.11.2.15 view	1977
	D.11.3	jacobson_lib	1978
		D.11.3.1 smith	1979
		D.11.3.2 jacobson	1979
		D.11.3.3 divideUnits	1980
	D.11.4	findifs_lib	1981
		D.11.4.1 findifs_example	1981
		D.11.4.2 decoef	1983
		D.11.4.3 difpoly2tex	1983
		D.11.4.4 exp2pt	1984
		D.11.4.5 texcoef	1985
		D.11.4.6 npar	1985
		D.11.4.7 magnitude	1985
		D.11.4.8 replace	1986
		D.11.4.9 xchange	1986
D.12		Teaching	1987
	D.12.1	aksaka_lib	1987
		D.12.1.1 fastExpt	1987
		D.12.1.2 log2	1987
		D.12.1.3 PerfectPowerTest	1988
		D.12.1.4 wurzel	1988

D.12.1.5	euler	1988
D.12.1.6	coeffmod	1988
D.12.1.7	powerpolyX	1989
D.12.1.8	ask	1989
D.12.2	crypto_lib	1989
D.12.2.1	round	1990
D.12.2.2	bubblesort	1990
D.12.2.3	decimal	1991
D.12.2.4	eexgcdN	1991
D.12.2.5	lcmN	1991
D.12.2.6	powerN	1991
D.12.2.7	chineseRem	1992
D.12.2.8	Jacobi	1992
D.12.2.9	primList	1992
D.12.2.10	primL	1992
D.12.2.11	intPart	1993
D.12.2.12	intRoot	1993
D.12.2.13	squareRoot	1993
D.12.2.14	solutionsMod2	1993
D.12.2.15	powerX	1994
D.12.2.16	babyGiant	1994
D.12.2.17	rho	1994
D.12.2.18	MillerRabin	1995
D.12.2.19	SolowayStrassen	1995
D.12.2.20	PocklingtonLehmer	1995
D.12.2.21	PollardRho	1996
D.12.2.22	pFactor	1997
D.12.2.23	quadraticSieve	1997
D.12.2.24	isOnCurve	1998
D.12.2.25	ellipticAdd	1998
D.12.2.26	ellipticMult	1998
D.12.2.27	ellipticRandomCurve	1999
D.12.2.28	ellipticRandomPoint	1999
D.12.2.29	countPoints	2000
D.12.2.30	ellipticAllPoints	2000
D.12.2.31	ShanksMestre	2000
D.12.2.32	Schoof	2000
D.12.2.33	generateG	2001
D.12.2.34	factorLenstraECM	2001
D.12.2.35	ECPP	2001
D.12.2.36	calculate_ordering	2002
D.12.2.37	is_primitive_root	2003
D.12.2.38	find_first_primitive_root	2003
D.12.2.39	binary_add	2003
D.12.2.40	inverse_modulus	2004
D.12.2.41	is_prime	2004
D.12.2.42	find_index	2004
D.12.2.43	subset_sum01	2004
D.12.2.44	subset_sum02	2005
D.12.2.45	unbounded_knapsack	2006
D.12.2.46	multidimensional_knapsack	2006
D.12.2.47	naccache_stern_generation	2007
D.12.2.48	naccache_stern_encryption	2007

D.12.2.49	naccache_stern_decryption . . . . .	2007
D.12.2.50	m_merkle_hellman_transformation . . . . .	2008
D.12.2.51	m_merkle_hellman_encryption . . . . .	2008
D.12.2.52	m_merkle_hellman_decryption . . . . .	2008
D.12.2.53	merkle_hellman_encryption . . . . .	2009
D.12.2.54	merkle_hellman_decryption . . . . .	2009
D.12.2.55	super_increasing_knapsack . . . . .	2010
D.12.2.56	h_increasing_knapsack . . . . .	2010
D.12.2.57	injective_knapsack . . . . .	2011
D.12.2.58	calculate_max_sum . . . . .	2014
D.12.2.59	set_is_injective . . . . .	2014
D.12.2.60	is_h_injective . . . . .	2015
D.12.2.61	is_fix_injective . . . . .	2015
D.12.2.62	three_elements . . . . .	2015
D.12.3	hyperel_lib . . . . .	2016
D.12.3.1	ishyper . . . . .	2016
D.12.3.2	isoncurve . . . . .	2017
D.12.3.3	chinrestp . . . . .	2017
D.12.3.4	norm . . . . .	2017
D.12.3.5	multi . . . . .	2018
D.12.3.6	divisor . . . . .	2018
D.12.3.7	gcddivisor . . . . .	2019
D.12.3.8	semidiv . . . . .	2019
D.12.3.9	cantoradd . . . . .	2020
D.12.3.10	cantorred . . . . .	2021
D.12.3.11	double . . . . .	2021
D.12.3.12	cantormult . . . . .	2022
D.12.4	teachstd_lib . . . . .	2022
D.12.4.1	ecart . . . . .	2022
D.12.4.2	tail . . . . .	2023
D.12.4.3	sameComponent . . . . .	2023
D.12.4.4	leadmonomial . . . . .	2023
D.12.4.5	monomialLcm . . . . .	2024
D.12.4.6	spoly . . . . .	2024
D.12.4.7	minEcart . . . . .	2025
D.12.4.8	NFMora . . . . .	2025
D.12.4.9	prodcrit . . . . .	2026
D.12.4.10	chaincrit . . . . .	2026
D.12.4.11	pairset . . . . .	2026
D.12.4.12	updatePairs . . . . .	2027
D.12.4.13	standard . . . . .	2028
D.12.4.14	localstd . . . . .	2029
D.12.5	weierstr_lib . . . . .	2030
D.12.5.1	weierstrDiv . . . . .	2030
D.12.5.2	weierstrPrep . . . . .	2031
D.12.5.3	lastvarGeneral . . . . .	2031
D.12.5.4	generalOrder . . . . .	2032
D.12.6	rootsmr_lib . . . . .	2032
D.12.6.1	nrRootsProbab . . . . .	2032
D.12.6.2	nrRootsDeterm . . . . .	2033
D.12.6.3	symsignature . . . . .	2033
D.12.6.4	sturmquery . . . . .	2034
D.12.6.5	matbil . . . . .	2034

	D.12.6.6	matmult . . . . .	2035
	D.12.6.7	tracemult . . . . .	2035
	D.12.6.8	coords . . . . .	2036
	D.12.6.9	randcharpoly . . . . .	2037
	D.12.6.10	verify . . . . .	2037
	D.12.6.11	randlinpoly . . . . .	2038
	D.12.6.12	powersums . . . . .	2038
	D.12.6.13	symmfunc . . . . .	2039
	D.12.6.14	univarpoly . . . . .	2039
	D.12.6.15	qbase . . . . .	2040
D.12.7	rootsur_lib . . . . .	2040	
	D.12.7.1	isuni . . . . .	2041
	D.12.7.2	whichvariable . . . . .	2041
	D.12.7.3	varsigns . . . . .	2042
	D.12.7.4	boundBuFou . . . . .	2042
	D.12.7.5	boundposDes . . . . .	2042
	D.12.7.6	boundDes . . . . .	2043
	D.12.7.7	allrealst . . . . .	2043
	D.12.7.8	maxabs . . . . .	2044
	D.12.7.9	allreal . . . . .	2044
	D.12.7.10	sturm . . . . .	2044
	D.12.7.11	sturmseq . . . . .	2045
	D.12.7.12	sturmha . . . . .	2045
	D.12.7.13	sturmhaseq . . . . .	2046
	D.12.7.14	reverse . . . . .	2046
	D.12.7.15	nrroots . . . . .	2047
	D.12.7.16	isparam . . . . .	2047
D.13	Tropical Geometry . . . . .	2048	
	D.13.1	cimonom_lib . . . . .	2048
		D.13.1.1	BelongSemig . . . . . 2048
		D.13.1.2	MinMult . . . . . 2049
		D.13.1.3	CompInt . . . . . 2049
	D.13.2	gfan_lib . . . . .	2050
		D.13.2.1	fullSpace . . . . . 2050
		D.13.2.2	origin . . . . . 2050
		D.13.2.3	positiveOrthant . . . . . 2050
		D.13.2.4	ambientDimension . . . . . 2051
		D.13.2.5	canonicalizeCone . . . . . 2051
		D.13.2.6	codimension . . . . . 2052
		D.13.2.7	coneViaPoints . . . . . 2053
		D.13.2.8	coneViaInequalities . . . . . 2055
		D.13.2.9	coneLink . . . . . 2057
		D.13.2.10	containsAsFace . . . . . 2057
		D.13.2.11	containsInSupport . . . . . 2058
		D.13.2.12	containsPositiveVector . . . . . 2058
		D.13.2.13	containsRelatively . . . . . 2059
		D.13.2.14	convexHull . . . . . 2059
		D.13.2.15	convexIntersection . . . . . 2060
		D.13.2.16	dimension . . . . . 2061
		D.13.2.17	dualCone . . . . . 2062
		D.13.2.18	equations . . . . . 2063
		D.13.2.19	faceContaining . . . . . 2063
		D.13.2.20	facets . . . . . 2064



D.13.2.21	generatorsOfLinealitySpace	2065
D.13.2.22	generatorsOfSpan	2065
D.13.2.23	getLinearForms	2066
D.13.2.24	getMultiplicity	2066
D.13.2.25	inequalities	2066
D.13.2.26	isFullSpace	2067
D.13.2.27	isOrigin	2067
D.13.2.28	isSimplicial	2068
D.13.2.29	linealityDimension	2069
D.13.2.30	linealitySpace	2069
D.13.2.31	negatedCone	2070
D.13.2.32	polytopeViaInequalities	2070
D.13.2.33	polytopeViaPoints	2071
D.13.2.34	quotientLatticeBasis	2072
D.13.2.35	randomPoint	2073
D.13.2.36	rays	2073
D.13.2.37	relativeInteriorPoint	2073
D.13.2.38	semigroupGenerator	2074
D.13.2.39	setLinearForms	2074
D.13.2.40	setMultiplicity	2075
D.13.2.41	span	2075
D.13.2.42	uniquePoint	2075
D.13.2.43	containsInCollection	2076
D.13.2.44	emptyFan	2076
D.13.2.45	fanViaCones	2077
D.13.2.46	fullFan	2079
D.13.2.47	fVector	2080
D.13.2.48	getCone	2081
D.13.2.49	insertCone	2081
D.13.2.50	isCompatible	2083
D.13.2.51	isPure	2084
D.13.2.52	nmaxcones	2084
D.13.2.53	ncones	2085
D.13.2.54	numberOfConesOfDimension	2085
D.13.2.55	removeCone	2086
D.13.2.56	dualPolytope	2088
D.13.2.57	newtonPolytope	2089
D.13.2.58	vertices	2089
D.13.2.59	onesVector	2090
D.13.3	gitfan.lib	2090
D.13.3.1	isAface	2091
D.13.3.2	afaces	2091
D.13.3.3	fullDimImages	2092
D.13.3.4	minimalAfaces	2093
D.13.3.5	orbitCones	2094
D.13.3.6	GITcone	2094
D.13.3.7	GITfan	2095
D.13.3.8	GITfanFromOrbitCones	2096
D.13.3.9	GITfanParallel	2097
D.13.3.10	GKZfan	2098
D.13.3.11	movingCone	2098
D.13.3.12	computeAfaceOrbits	2099
D.13.3.13	minimalAfaceOrbits	2100

D.13.3.14	orbitConeOrbits . . . . .	2101
D.13.3.15	minimalOrbitConeOrbits . . . . .	2102
D.13.3.16	intersectOrbitsWithMovingCone . . . . .	2103
D.13.3.17	groupActionOnQImage . . . . .	2105
D.13.3.18	groupActionOnHashes . . . . .	2106
D.13.3.19	storeActionOnOrbitConeIndices . . . . .	2121
D.13.3.20	permutationFromIntvec . . . . .	2121
D.13.3.21	permutationToIntvec . . . . .	2121
D.13.3.22	evaluateProduct . . . . .	2122
D.13.3.23	GITfanSymmetric . . . . .	2122
D.13.3.24	GITfanParallelSymmetric . . . . .	2124
D.13.3.25	bigintToBinary . . . . .	2126
D.13.3.26	binaryToBigint . . . . .	2126
D.13.3.27	applyPermutationToIntvec . . . . .	2126
D.13.3.28	hashToCone . . . . .	2127
D.13.3.29	hashesToFan . . . . .	2127
D.13.3.30	gitCone . . . . .	2128
D.13.4	polymake.lib . . . . .	2129
D.13.4.1	triangulations . . . . .	2129
D.13.4.2	secondaryPolytope . . . . .	2130
D.13.4.3	cycleLength . . . . .	2131
D.13.4.4	splitPolygon . . . . .	2132
D.13.4.5	eta . . . . .	2133
D.13.4.6	findOrientedBoundary . . . . .	2134
D.13.4.7	cyclePoints . . . . .	2135
D.13.4.8	latticeArea . . . . .	2136
D.13.4.9	picksFormula . . . . .	2136
D.13.4.10	ellipticNF . . . . .	2137
D.13.5	realizationMatroids.lib . . . . .	2138
D.13.5.1	realizationDim . . . . .	2139
D.13.5.2	irrRealizationDim . . . . .	2139
D.13.5.3	realizationDimPoly . . . . .	2140
D.13.6	tropical.lib . . . . .	2140
D.13.6.1	tropicalLifting . . . . .	2142
D.13.6.2	displayTropicalLifting . . . . .	2146
D.13.6.3	puiseuxExpansion . . . . .	2147
D.13.6.4	displayPuisseuxExpansion . . . . .	2150
D.13.6.5	tropicalCurve . . . . .	2151
D.13.6.6	drawTropicalCurve . . . . .	2153
D.13.6.7	drawNewtonSubdivision . . . . .	2155
D.13.6.8	tropicalJInvariant . . . . .	2158
D.13.6.9	weierstrassForm . . . . .	2159
D.13.6.10	conicWithTangents . . . . .	2159
D.13.6.11	tropicalise . . . . .	2164
D.13.6.12	tropicaliseSet . . . . .	2164
D.13.6.13	tInitialForm . . . . .	2165
D.13.6.14	tInitialIdeal . . . . .	2165
D.13.6.15	initialForm . . . . .	2166
D.13.6.16	initialIdeal . . . . .	2166
D.13.6.17	texNumber . . . . .	2166
D.13.6.18	texPolynomial . . . . .	2167
D.13.6.19	texMatrix . . . . .	2167
D.13.6.20	texDrawBasic . . . . .	2167

D.13.6.21	texDrawTropical	2168
D.13.6.22	texDrawNewtonSubdivision	2170
D.13.6.23	texDrawTriangulation	2171
D.13.6.24	radicalMemberShip	2172
D.13.6.25	tInitialFormPar	2172
D.13.6.26	tInitialFormParMax	2173
D.13.6.27	solveTInitialFormPar	2173
D.13.6.28	detropicalise	2174
D.13.6.29	tDetropicalise	2174
D.13.6.30	dualConic	2175
D.13.6.31	parameterSubstitute	2175
D.13.6.32	tropicalSubst	2175
D.13.6.33	randomPolyInT	2176
D.13.6.34	cleanTmp	2176
D.13.6.35	groebnerCone	2176
D.13.6.36	maximalGroebnerCone	2177
D.13.6.37	homogeneitySpace	2178
D.13.6.38	initial	2178
D.13.6.39	tropicalVariety	2179
D.13.6.40	groebnerFan	2185
D.13.6.41	groebnerComplex	2187
D.13.7	tropicalNewton_lib	2196
D.13.7.1	setUniformizingParameter	2196
D.13.7.2	val	2197
D.13.7.3	newtonPolygonNegSlopes	2197
D.13.7.4	cccMatrixToPositiveIntvec	2198
D.13.7.5	tropicalPointNewton	2198
D.13.7.6	switchRingsAndComputeInitialIdeal	2200
D.13.7.7	tropicalVarietyNewton	2201
D.13.7.8	tropicalLinkNewton	2202
D.14	Miscellaneous libraries	2205
D.14.1	arr_lib	2205
D.14.1.1	arrSet	2206
D.14.1.2	type2arr	2206
D.14.1.3	mat2arr	2206
D.14.1.4	mat2carr	2207
D.14.1.5	arrPrintMatrix	2207
D.14.1.6	varMat	2208
D.14.1.7	varNum	2208
D.14.1.8	arrSwapVar	2209
D.14.1.9	arrLastVar	2209
D.14.1.10	arrCenter	2210
D.14.1.11	arrCentral	2211
D.14.1.12	arrCentered	2211
D.14.1.13	arrCentralize	2212
D.14.1.14	arrCoordChange	2212
D.14.1.15	arrCoordNormalize	2213
D.14.1.16	arrCone	2213
D.14.1.17	arrDecone	2214
D.14.1.18	arrLocalize	2214
D.14.1.19	arrRestrict	2215
D.14.1.20	arrIsEssential	2216
D.14.1.21	arrEssentialize	2216

D.14.1.22	arrBoolean	2217
D.14.1.23	arrBraid	2217
D.14.1.24	arrTypeB	2218
D.14.1.25	arrTypeD	2219
D.14.1.26	arrRandom	2219
D.14.1.27	arrRandomCentral	2220
D.14.1.28	arrEdelmanReiner	2220
D.14.1.29	arrOrlikSolomon	2221
D.14.1.30	arrDer	2222
D.14.1.31	arrIsFree	2223
D.14.1.32	arrExponents	2224
D.14.1.33	arr2multarr	2224
D.14.1.34	multarr2arr	2225
D.14.1.35	multarrRestrict	2225
D.14.1.36	multarrMultRestrict	2226
D.14.1.37	arrFlats	2227
D.14.1.38	arrLattice	2228
D.14.1.39	moebius	2229
D.14.1.40	arrCharPoly	2230
D.14.1.41	arrPoincare	2231
D.14.1.42	arrChambers	2231
D.14.1.43	arrBoundedChambers	2231
D.14.1.44	printMoebius	2232
D.14.2	combinat_lib	2233
D.14.2.1	intersectLists	2233
D.14.2.2	sublists	2233
D.14.2.3	member	2237
D.14.3	customstd_lib	2238
D.14.3.1	monomialabortstd	2238
D.14.3.2	satstd	2239
D.14.4	methods_lib	2239
D.14.5	nets_lib	2240
D.14.5.1	catNets	2240
D.14.5.2	net	2241
D.14.5.3	netBigIntMat	2241
D.14.5.4	netBigIntMatShort	2241
D.14.5.5	netCoefficientRing	2242
D.14.5.6	netIdeal	2242
D.14.5.7	netInt	2242
D.14.5.8	netBigInt	2243
D.14.5.9	netIntMat	2243
D.14.5.10	netIntMatShort	2243
D.14.5.11	netIntVector	2244
D.14.5.12	netIntVectorShort	2244
D.14.5.13	netNumber	2245
D.14.5.14	netList	2245
D.14.5.15	netMap	2246
D.14.5.16	netMap2	2246
D.14.5.17	netmatrix	2247
D.14.5.18	netmatrixShort	2247
D.14.5.19	netPoly	2248
D.14.5.20	netPrimePower	2252
D.14.5.21	netRing	2252

	D.14.5.22	netString	2254
	D.14.5.23	netvector	2255
	D.14.5.24	netvectorShort	2255
	D.14.5.25	stackNets	2256
D.14.6	phindex_lib		2256
	D.14.6.1	signatureL	2257
	D.14.6.2	signatureLqf	2257
	D.14.6.3	PH_ais	2258
	D.14.6.4	PH_nais	2259
D.14.7	polybori_lib		2259
	D.14.7.1	boolean_std	2260
	D.14.7.2	boolean_poly_ring	2261
	D.14.7.3	boolean_constant	2261
	D.14.7.4	boolean_poly	2262
	D.14.7.5	direct_boolean_poly	2262
	D.14.7.6	recursive_boolean_poly	2263
	D.14.7.7	boolean_ideal	2263
	D.14.7.8	boolean_set	2264
	D.14.7.9	from_boolean_constant	2264
	D.14.7.10	from_boolean_poly	2264
	D.14.7.11	direct_from_boolean_poly	2265
	D.14.7.12	recursive_from_boolean_poly	2265
	D.14.7.13	from_boolean_ideal	2266
	D.14.7.14	from_boolean_set	2266
	D.14.7.15	bvar	2267
	D.14.7.16	poly2zdd	2267
	D.14.7.17	zdd2poly	2267
	D.14.7.18	disp_zdd	2268
D.14.8	sets_lib		2269
	D.14.8.1	set	2269
	D.14.8.2	union	2270
	D.14.8.3	intersectionSet	2271
	D.14.8.4	complement	2271
	D.14.8.5	isElement	2272
	D.14.8.6	isSubset	2272
	D.14.8.7	isSuperset	2273
	D.14.8.8	addElement	2273
D.15	Experimental libraries		2274
	D.15.1	autgradalg_lib	2274
		D.15.1.1 autKS	2275
		D.15.1.2 autGradAlg	2279
		D.15.1.3 autGenWeights	2280
		D.15.1.4 stabilizer	2282
		D.15.1.5 autXhat	2288
		D.15.1.6 autX	2290
	D.15.2	diffform_lib	2291
		D.15.2.1 diffAlgebra	2292
		D.15.2.2 diffAlgebraStructure	2293
		D.15.2.3 diffAlgebraGens	2294
		D.15.2.4 diffAlgebraUnivDerIdeal	2294
		D.15.2.5 diffAlgebraChangeOrd	2294
		D.15.2.6 diffAlgebraListGen	2295
		D.15.2.7 diffformFromPoly	2296

D.15.2.8	difformCoef	2296
D.15.2.9	difformGenToString	2299
D.15.2.10	difformHomogDecomp	2299
D.15.2.11	difformToString	2300
D.15.2.12	difformPrint	2300
D.15.2.13	difformIsGen	2301
D.15.2.14	difformAdd	2302
D.15.2.15	difformSub	2302
D.15.2.16	difformNeg	2303
D.15.2.17	difformMul	2303
D.15.2.18	difformDiv	2304
D.15.2.19	difformEqu	2305
D.15.2.20	difformNeq	2305
D.15.2.21	difformIsBigger	2306
D.15.2.22	difformIsSmaller	2306
D.15.2.23	difformDeg	2307
D.15.2.24	difformIsHomog	2308
D.15.2.25	difformIsHomogDeg	2309
D.15.2.26	difformListCont	2310
D.15.2.27	difformListSort	2310
D.15.2.28	difformUnivDer	2312
D.15.2.29	difformDiff	2313
D.15.2.30	derivationFromList	2314
D.15.2.31	derivationCheckList	2315
D.15.2.32	derivationFromPoly	2315
D.15.2.33	derivationConstructor	2316
D.15.2.34	derivationToString	2317
D.15.2.35	derivationPrint	2317
D.15.2.36	derivationAdd	2318
D.15.2.37	derivationSub	2320
D.15.2.38	derivationNeg	2321
D.15.2.39	derivationMul	2322
D.15.2.40	derivationEqu	2323
D.15.2.41	derivationNeq	2324
D.15.2.42	derivationEval	2325
D.15.2.43	derivationContractionGen	2326
D.15.2.44	derivationContraction	2326
D.15.2.45	derivationLie	2328
D.15.3	enumpoints_lib	2329
D.15.3.1	points	2329
D.15.3.2	projPoints	2329
D.15.4	finitediff_lib	2330
D.15.4.1	visualize	2330
D.15.4.2	u	2330
D.15.4.3	scheme	2331
D.15.4.4	laxfrT	2331
D.15.4.5	laxfrX	2332
D.15.4.6	forward	2332
D.15.4.7	backward	2332
D.15.4.8	central1st	2333
D.15.4.9	central2nd	2333
D.15.4.10	trapezoid	2334
D.15.4.11	midpoint	2334

D.15.4.12	pyramid	2334
D.15.4.13	setinitials	2335
D.15.4.14	errormap	2335
D.15.4.15	matrixsystem	2336
D.15.4.16	timestep	2336
D.15.4.17	fouriersystem	2337
D.15.4.18	PartitionVar	2338
D.15.4.19	ComplexValue	2338
D.15.4.20	VarToPar	2339
D.15.4.21	ParToVar	2340
D.15.4.22	qepcad	2340
D.15.4.23	qepcadsystem	2341
D.15.5	GND_lib	2342
D.15.5.1	desingularization	2342
D.15.6	gradedModules_lib	2346
D.15.6.1	grobj	2346
D.15.6.2	grtest	2347
D.15.6.3	grdeg	2348
D.15.6.4	grview	2350
D.15.6.5	grshift	2352
D.15.6.6	grzero	2353
D.15.6.7	grtwist	2353
D.15.6.8	grtwists	2354
D.15.6.9	grsum	2354
D.15.6.10	grpowers	2357
D.15.6.11	grtranspose	2358
D.15.6.12	grgens	2359
D.15.6.13	grpres	2360
D.15.6.14	grorder	2362
D.15.6.15	grtranspose1	2364
D.15.6.16	TestGRRes	2368
D.15.6.17	KeneshlouMatrixPresentation	2390
D.15.6.18	grsyz	2392
D.15.6.19	grres	2393
D.15.6.20	grlift	2395
D.15.6.21	grprod	2396
D.15.6.22	grgroebner	2397
D.15.6.23	grconcat	2397
D.15.6.24	grrndmat	2398
D.15.6.25	grrndmap	2399
D.15.6.26	grrndmap2	2399
D.15.6.27	grlifting	2400
D.15.6.28	grlifting2	2403
D.15.6.29	mappingcone	2404
D.15.6.30	grlifting3	2406
D.15.6.31	mappingcone3	2408
D.15.6.32	grrange	2416
D.15.6.33	grneg	2417
D.15.6.34	matrixpres	2418
D.15.7	hodge_lib	2421
D.15.7.1	Vfiltration	2422
D.15.7.2	hodgeIdeals	2423
D.15.7.3	multIdeals	2424

D.15.7.4	nextHodgeIdeal .....	2425
D.15.8	lrcalc_lib .....	2425
D.15.8.1	LRinstall .....	2426
D.15.8.2	LRcoef .....	2426
D.15.8.3	LRskew .....	2426
D.15.8.4	LRmult .....	2427
D.15.8.5	LRcoprod .....	2430
D.15.8.6	LRschubmult .....	2434
D.15.9	maxlike_lib .....	2435
D.15.9.1	likeIdeal .....	2435
D.15.9.2	logHessian .....	2435
D.15.9.3	getMaxPoints .....	2436
D.15.9.4	maxPoints .....	2437
D.15.9.5	maxPointsProb .....	2438
D.15.10	modfinduni_lib .....	2439
D.15.10.1	modFinduni .....	2439
D.15.11	modwalk_lib .....	2439
D.15.11.1	modWalk .....	2440
D.15.11.2	modrWalk .....	2444
D.15.11.3	modfWalk .....	2449
D.15.11.4	modfrWalk .....	2454
D.15.12	multigrading_lib .....	2459
D.15.12.1	setBaseMultigrading .....	2460
D.15.12.2	getVariableWeights .....	2461
D.15.12.3	getGradingGroup .....	2462
D.15.12.4	getLattice .....	2464
D.15.12.5	createGroup .....	2465
D.15.12.6	createQuotientGroup .....	2466
D.15.12.7	createTorsionFreeGroup .....	2467
D.15.12.8	printGroup .....	2468
D.15.12.9	isGroup .....	2468
D.15.12.10	isGroupHomomorphism .....	2469
D.15.12.11	isGradedRingHomomorphism .....	2470
D.15.12.12	createGradedRingHomomorphism .....	2471
D.15.12.13	setModuleGrading .....	2473
D.15.12.14	getModuleGrading .....	2474
D.15.12.15	isSublattice .....	2474
D.15.12.16	imageLattice .....	2475
D.15.12.17	intRank .....	2475
D.15.12.18	kernelLattice .....	2476
D.15.12.19	latticeBasis .....	2477
D.15.12.20	preimageLattice .....	2477
D.15.12.21	projectLattice .....	2478
D.15.12.22	intersectLattices .....	2479
D.15.12.23	isIntegralSurjective .....	2480
D.15.12.24	isPrimitiveSublattice .....	2480
D.15.12.25	intInverse .....	2481
D.15.12.26	integralSection .....	2481
D.15.12.27	primitiveSpan .....	2482
D.15.12.28	factorgroup .....	2482
D.15.12.29	productgroup .....	2483
D.15.12.30	multiDeg .....	2484
D.15.12.31	multiDegBasis .....	2485



D.15.12.32	multiDegPartition	2487
D.15.12.33	isTorsionFree	2488
D.15.12.34	isPositive	2488
D.15.12.35	isZeroElement	2489
D.15.12.36	areZeroElements	2490
D.15.12.37	isHomogeneous	2491
D.15.12.38	equalMultiDeg	2494
D.15.12.39	multiDegGroebner	2495
D.15.12.40	multiDegSyzygy	2496
D.15.12.41	multiDegModulo	2497
D.15.12.42	multiDegResolution	2498
D.15.12.43	multiDegTensor	2502
D.15.12.44	multiDegTor	2505
D.15.12.45	defineHomogeneous	2505
D.15.12.46	pushForward	2506
D.15.12.47	gradiator	2507
D.15.12.48	hermiteNormalForm	2507
D.15.12.49	smithNormalForm	2508
D.15.12.50	hilbertSeries	2509
D.15.12.51	lll	2517
D.15.13	pfd_lib	2518
D.15.13.1	pfd	2518
D.15.13.2	checkpfd	2522
D.15.13.3	evaluatepfd	2524
D.15.13.4	displaypfd	2524
D.15.13.5	displaypfd_long	2525
D.15.13.6	getStringpfd	2526
D.15.13.7	getStringpfd_indexed	2527
D.15.13.8	readInputTXT	2528
D.15.13.9	pfdMat	2528
D.15.13.10	checkpfdMat	2530
D.15.14	polyclass_lib	2530
D.15.14.1	makePoly	2531
D.15.14.2	printPoly	2531
D.15.14.3	printNormalFormEquation	2531
D.15.15	puiseuxexpansions_lib	2532
D.15.15.1	puiseuxList	2533
D.15.15.2	makePuisseux	2534
D.15.15.3	makePuisseuxFromPoly	2534
D.15.15.4	printPuisseux	2534
D.15.15.5	puiseux	2535
D.15.16	ringgb_lib	2535
D.15.16.1	findZeroPoly	2535
D.15.16.2	zeroReduce	2536
D.15.16.3	testZero	2536
D.15.16.4	noElements	2537
D.15.17	rwalk_lib	2537
D.15.17.1	prwalk	2537
D.15.17.2	rwalk	2538
D.15.17.3	frandwalk	2539
D.15.18	sagbigrob_lib	2539
D.15.18.1	LTGS	2539
D.15.18.2	SGNF	2540

D.15.18.3	SPOLY .....	2540
D.15.18.4	SGB .....	2541
D.15.19	sagbiNormaliz_lib .....	2541
D.15.19.1	sagbiGeneral .....	2542
D.15.19.2	sagbiByDegree .....	2543
D.15.19.3	sagbiHilbControlled .....	2543
D.15.20	stanleyreisner_lib .....	2544
D.15.20.1	T1 .....	2544
D.15.20.2	T2 .....	2547
D.15.20.3	makeQPoly .....	2549
D.15.20.4	fPiece .....	2550
D.15.20.5	sPiece .....	2550
D.15.20.6	makeLinks .....	2550
D.15.21	stdmodule_lib .....	2551
D.15.21.1	MPGS .....	2551
D.15.21.2	SNF .....	2552
D.15.21.3	MSPOLY .....	2552
D.15.21.4	MSGB .....	2553
D.15.22	swalk_lib .....	2553
D.15.22.1	swalk .....	2554
D.15.22.2	rswalk .....	2554
D.15.23	systhreads_lib .....	2555
D.15.24	tateProdCplxNegGrad_lib .....	2555
D.15.24.1	productOfProjectiveSpaces .....	2555
D.15.24.2	truncateM .....	2557
D.15.24.3	truncateCoker .....	2558
D.15.24.4	symExt .....	2559
D.15.24.5	sufficientlyPositiveMultidegree .....	2560
D.15.24.6	tateResolution .....	2561
D.15.24.7	cohomologyMatrix .....	2563
D.15.24.8	cohomologyMatrixFromResolution .....	2564
D.15.24.9	eulerPolynomialTable .....	2564
D.15.24.10	cohomologyHashTable .....	2567
D.15.24.11	twist .....	2571
D.15.24.12	beilinsonWindow .....	2571
D.15.24.13	regionComplex .....	2572
D.15.24.14	strand .....	2574
D.15.24.15	firstQuadrantComplex .....	2575
D.15.24.16	lastQuadrantComplex .....	2575
D.15.25	tropicalEllipticCovers_lib .....	2575
D.15.25.1	cf_of .....	2576
D.15.25.2	preim .....	2576
D.15.25.3	coverTuple .....	2576
D.15.25.4	CoverMult .....	2577
D.15.25.5	Tropicalcover .....	2577
D.15.25.6	sinh .....	2578
D.15.25.7	Sfunction .....	2578
D.15.25.8	FloorDiagrams .....	2579
D.15.25.9	PropagatorFunction .....	2579
D.15.25.10	FeynmanIntegralo .....	2580
D.15.25.11	FeynmanIntegralO .....	2580
D.15.25.12	FeynmanIntegralA .....	2581
D.15.25.13	DrawCovers .....	2582

D.15.25.14	TropCovandMaps .....	2583
D.15.26	VecField_lib .....	2583
D.15.26.1	applyVecField .....	2584
D.15.26.2	changeCoordinates .....	2584
D.15.26.3	jordanVecField .....	2584
D.15.26.4	diagonalizeVecFieldLin .....	2585
D.15.26.5	SaitoBase .....	2585
D.15.26.6	diagonalizeVecField .....	2586
D.15.26.7	vecFieldToMatrix .....	2586
D.15.26.8	decomposeVecField .....	2587
D.15.26.9	diagonalizeMatrixSimul .....	2587
D.15.26.10	invertAlgebraMorphism .....	2587
<b>8</b>	<b>Release Notes .....</b>	<b>2589</b>
8.1	News and changes .....	2589
8.2	Singular 3 and Singular 4 .....	2598
8.2.1	Version schema for Singular .....	2598
8.2.2	Notes for Singular users .....	2598
8.2.3	Notes for developers .....	2600
8.2.4	Building Singular .....	2600
8.2.5	Side-by-side installation .....	2600
8.3	libSingular .....	2600
8.4	Download instructions .....	2601
8.5	Used environment variables .....	2601
8.6	Unix installation instructions .....	2602
8.7	Windows installation instructions .....	2602
8.8	Macintosh installation instructions .....	2603
<b>9</b>	<b>Index .....</b>	<b>2604</b>